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Random Witten Laplacians: traces of semigroups, L^2-Betti numbers and index

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Abstract. Random Witten Laplacians on infinite coverings of compact manifolds are considered. The probabilistic representations of the corresponding heat kernels are given. The finiteness of the von Neumann traces of the corresponding semigroups is proved, and the short-time asymptotics of the corresponding supertrace is computed. Examples associated with Gibbs measures on configuration spaces and product manifolds are considered.

Keywords. Witten Laplacian, infinite covering, von Neumann algebra, Betti numbers, configuration space, Gibbs measure

Contents

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1. Introduction

Random operators, in particular, Schrödinger operators with random potentials, play an important role in different parts of mathematics and mathematical physics (see e.g. [10], [22], [30]). A crucial role in this theory is played by the concept of metric transitivity. Let Ψ be a probability space equipped with a probability measure μ. A random operator H_γ, $\gamma \in \Psi$, acting in a Hilbert space $\mathcal{H}$, is said to be *metrically transitive* if there exists a group G of measure-preserving transformations of Ψ and a unitary representation U_g of G in $\mathcal{H}$ such that

1) its action on Ψ is ergodic,
2) for any $g \in G$ and $\gamma \in \Psi$ the following commutation relation holds:

$$U_{g^{-1}} H_\gamma U_g = H_{g\gamma}. \tag{1}$$

The importance of this notion lies in the fact that the scalar spectral characteristics of H_γ, being unitary invariants and thus invariant with respect to the action of G on Ψ, are in fact non-random. The spectral theory of operators of such type has been discussed by many authors (see a review in [30]). The concept of metric transitivity has also been used in [22] in the construction of the index theory for random pseudodifferential operators acting in vector bundles over non-compact manifolds.

In the situation where H is non-random, condition (1) means that H commutes with the action of G in $\mathcal{H}$. If H is a self-adjoint operator, the latter implies that the corresponding spectral projections $E(\lambda)$ and the semigroup e^{-tH} belong to the commutant $\mathcal{A} = \{U_g\}'_{g\in G}$ of the action of G, which is a von Neumann algebra and has a trace $\mathrm{Tr}_\mathcal{A}$ (different from the usual trace in the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$). It turns out that, in some cases, $\mathrm{Tr}_\mathcal{A}\, E(\lambda)$ and $\mathrm{Tr}_\mathcal{A}\, e^{-tH}$ are finite despite the fact that these operators are not of trace class. This approach was initiated by M. Atiyah in [9], who considered the case of H being an elliptic operator on the universal covering X of a compact manifold M with infinite fundamental group G. It has been shown in that situation that the regularized index (supertrace) of the corresponding Dirac operator is equal to the Euler characteristic $\chi(M)$ of the underlying manifold M. This approach leads to the notion of L^2-invariants (see a review in [28], [29]).

Let us observe that in the case of the random operator H_γ the corresponding semigroup

$$T_\gamma(t) = e^{-tH_\gamma} \tag{2}$$

satisfies the relation

$$U_{g^{-1}} T_\gamma(t) U_g = T_{g\gamma}(t) \tag{3}$$

and thus does not in general belong to $\mathcal{A}$, for the set of $\gamma \in \Psi$ which are fixed points of the action of G usually has measure zero. However, it is possible to understand $T_\gamma(t)$ as an element of the algebra $\mathcal{C}$ of essentially bounded maps

$$A : \Psi \to \mathcal{B}(\mathcal{H}) \tag{4}$$

such that

$$A(g\gamma) = U_g A(\gamma) U_{g^{-1}} \tag{5}$$

for any $g \in G$ and $\gamma \in \Psi$. This approach has been proposed in [3] in the case of the Witten Laplacians associated with Gibbs measures on configuration spaces. In the present work, we

a) extend this approach to the case of more general random operators,
b) investigate the short time asymptotics of the supertrace of the corresponding semigroups.

Let us note that our situation of Witten operators acting in spaces of differential forms is specifically complicated because of the structure of the corresponding potential, which is a matrix-valued function, neither positive nor bounded, in contrast to [30] and [22].

The contents of the paper are as follows. In Section 2 we introduce the main objects to be considered:

- the random measure
$$\sigma_\gamma(dx) = e^{-E_\gamma(x)}dx \tag{6}$$
on X, $\gamma \in \Psi$, where dx denotes the Riemannian volume on X, and E is a homogeneous random function,
- the corresponding Witten–Bismut Laplacian $H_\gamma^{(p)}$ in the space $L^2\Omega^p(X)$ of square integrable p-forms on X,
- the corresponding heat semigroup $T_\gamma^{(p)}(t) = e^{-tH_\gamma^{(p)}}$ in $L^2\Omega^p(X)$ and its integral kernel $K_\gamma^{(p)}(x, y; t)$,
- the θ-function
$$\theta^{(p)}(x, t) = \mathbb{E}\,\mathrm{tr}\, K_\gamma^{(p)}(x, x; t), \tag{7}$$
where tr is the usual matrix trace.

In Theorem 1, we formulate the conditions on the random function E which imply the finiteness of $\theta^{(p)}(x, t)$.

In Section 3, we develop a probabilistic representation of the semigroup $T_\gamma^{(p)}(t)$ and apply it to prove Theorem 1. In Section 4, we introduce the corresponding operator algebra $\mathcal{C} = \mathcal{C}^p$ (cf. formulae (4), (5)) and prove that it is a von Neumann algebra with the faithful normal semifinite trace $\mathbb{TR}$ given by the formula

$$\mathbb{TR}\, A = \int_{X/G} \mathbb{E}\,\mathrm{tr}\, a_\gamma(x, x)\, dx, \tag{8}$$

where $a_\gamma(x, y)$ is the integral kernel of $A(\gamma)$ (Theorem 4). Next, we consider the maps $\mathbf{T}_t^{(p)} : \gamma \mapsto T_{\gamma,t}^{(p)}$ and $\mathbf{P}^{(p)} : \gamma \mapsto P_\gamma^{(p)}$, where $P_\gamma^{(p)}$ is the orthogonal projection onto the kernel of $H_\gamma^{(p)}$. The commutation relations (71) imply that $\mathbf{T}_t^{(p)}, \mathbf{P}^{(p)} \in \mathcal{C}^p$. We prove the following theorem.

Theorem 5. 1) *For all times $t > 0$ and any $p = 0, \ldots, \dim X$,*

$$\mathbb{TR}\, \mathbf{T}_t^{(p)} = \Theta^{(p)}(t) := \int_{X/G} \theta^{(p)}(t)\, dx < \infty. \tag{9}$$

2) *For any* $p = 0, \dots, \dim X$,

$$\mathbb{TR}\,\mathbf{P}^{(p)} < \infty. \tag{10}$$

3) *The following McKean–Singer formula holds for all times* $t > 0$*:*

$$\sum_{p=0}^{\dim X} (-1)^p\, \mathbb{TR}\,\mathbf{T}_t^{(p)} = \sum_{p=0}^{\dim X} (-1)^p\, \mathbb{TR}\,\mathbf{P}^{(p)}. \tag{11}$$

Let us note that the right-hand side of the latter formula can be understood as a regularized index of the corresponding Dirac operator.

In Section 5, we study the short time asymptotics of the left-hand side of formula (11) and prove the following result.

Theorem 6.

$$\mathbb{STR}\,\mathbf{P} := \sum_p (-1)^p\, \mathbb{TR}\,\mathbf{P}^{(p)} = \chi(M), \tag{12}$$

where $\chi(M)$ *is the Euler characteristic of* $M = X/G$*.*

The latter formula allows us to discuss some properties of the spaces of harmonic forms of individual operators H_γ. We prove that, provided the G-action on Ψ is ergodic and $\chi(M) = \infty$, the latter spaces are infinite-dimensional for a.a. γ.

In Section 6, we consider two main examples which motivate our study. The first example is related to Gibbs measures on configuration spaces. In this case the probability space Ψ is the space Γ_X of locally finite configurations γ in X equipped with a Gibbs measure μ. The random field E has the form

$$E_\gamma(x) = \sum_{y \in \gamma} v(\rho(x, y)), \tag{13}$$

where ρ is the distance on X and v is a smooth function with compact support. Measures of such type appear, via the generalized Mecke identity, in the theory of configuration spaces, and in particular in the theory of Laplace operators on differential forms over Γ_X (see [5], [6], [4]). In fact, the Witten Laplacian H associated with σ is a "part" of the Hodge–de Rham operator on Γ_X associated with the Gibbs measure μ. The structure of the latter operator is very complicated in the case where μ is different from the Poisson measure. We believe that the study of spectral properties of H, which is a more realistic goal than the study of the full Hodge–de Rham operator on Γ_X, may already give interesting links between the properties of μ and geometrical and topological properties of X.

Let us note that the interest in the analysis on infinite configuration spaces has risen in recent years, because of new approaches and rich applications in statistical mechanics and quantum field theory (see [7], [8] and the review [31]). L^2-Betti numbers of configuration spaces with Poisson and Lebesgue–Poisson measures were computed in [2] and [15] respectively (see also [1], [14], [16]).

In our second example, the role of Ψ is played by the infinite product space X^G, equipped with a Gibbs measure μ which is invariant with respect to the G-action T_g given by

$$X^G \ni \overline{\xi} = (\xi_{g'})_{g' \in G} \mapsto T_g(\xi_{g'})_{g' \in G} = (\xi_{gg'})_{g' \in G}. \tag{14}$$

The random field E is defined in the following way:

$$E_{\overline{\xi}}(x) = \sum_{g \in G} v(\rho(x, g\xi_g)). \tag{15}$$

In both examples, the operator H is related to the following model from statistical mechanics. Let us consider a particle with position x performing a random motion in X and interacting with a random medium (gas in the first example and crystal in the second) described by the Gibbs measure μ. The distribution of the particle is given by the random measure $\sigma_\gamma(dx)$, where $E_\gamma(x)$ is the energy of interaction of the particle x and the configuration γ of gas particles or crystal vertices, respectively.

2. Random Witten Laplacian

Let X be a complete connected, oriented, C^∞ Riemannian manifold of infinite volume with a lower bounded curvature. We assume that there exists an infinite discrete group G of isometries of X such that X/G is a compact Riemannian manifold. For instance, X can be the universal cover of a compact oriented C^∞ Riemannian manifold M with infinite fundamental group G.

Let E be a random homogeneous field on X defined on a probability space $(\Psi, \mathcal{P}, \mu)$. That is, there exists a representation

$$G \ni g \mapsto T_g \tag{16}$$

of the group G by measure preserving transformations of Ψ such that

$$E : X \times \Psi \to \mathbb{R} \tag{17}$$

satisfies the relation

$$E(gx, T_g\gamma) = E(x, \gamma) \tag{18}$$

for all $g \in G$, $x \in X$ and a.a. $\gamma \in \Psi$. We assume in addition that $E_\gamma := E(\cdot, \gamma) \in C^\infty(X)$. In what follows, we will use the notation $g\gamma := T_g\gamma$.

For any $\gamma \in \Psi$ we introduce the measure

$$\sigma_\gamma(dx) = e^{-E_\gamma(x)}\, dx \tag{19}$$

on X, where dx denotes the Riemannian volume on X.

In what follows, we will use the following notations:

- $L^2\Omega^p(X)$ – the space of p-forms on X which are square-integrable with respect to the volume measure;

- $L^2_{\sigma_\gamma}\Omega^p(X)$ – the space of p-forms on X which are square-integrable with respect to σ_γ;
- d^p – the de Rham differential on p-forms on X;
- $H^{(p)}$ – the de Rham Laplacian on p-forms on X;
- ∇ – the Levi-Civita covariant derivative;
- $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ – the space of bounded linear operators $\mathcal{H}_1 \to \mathcal{H}_2$, $\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces;
- $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$.

Let us consider the Witten–Bismut Laplacian $H^{(p)}_{\sigma_\gamma}$ in $L^2_{\sigma_\gamma}\Omega^p$,

$$H^{(p)}_{\sigma_\gamma} := d^{p-1}(d^{p-1})^*_\gamma + (d^p)^*_\gamma d^p, \tag{20}$$

where $(d^k)^*_\gamma : L^2_{\sigma_\gamma}\Omega^{k+1} \to L^2_{\sigma_\gamma}\Omega^k$ is the adjoint of $d^k : L^2_{\sigma_\gamma}\Omega^k \to L^2_{\sigma_\gamma}\Omega^{k+1}$. It follows from the results of [11] that $H^{(p)}_{\sigma_\gamma}$ is essentially self-adjoint on the space of smooth forms with compact support (for a general definition and discussion of properties of Witten Laplacians see e.g. [13], [19]).

On smooth forms, $H^{(p)}_{\sigma_\gamma}$ is given by the expression

$$H^{(p)}_{\sigma_\gamma} = H^{(p)} + \frac{1}{2}(\nabla E_\gamma, \nabla)_{T^pX} + (\nabla^2 E_\gamma)^{\wedge p}, \tag{21}$$

where

$$(\nabla^2 E_\gamma)^{\wedge p} = \sum_{k=1}^{p} \overbrace{I \otimes I \otimes \cdots \otimes \nabla^2 E_\gamma}^{k} \otimes \cdots \otimes I. \tag{22}$$

Let

$$U : L^2_{\sigma_\gamma}\Omega^p(X) \to L^2\Omega^p(X) \tag{23}$$

be the unitary isomorphism defined by multiplication by $e^{-\frac{1}{2}E_\gamma(x)}$. Then the operator

$$H^{(p)}_\gamma := U H^{(p)}_{\sigma_\gamma} U^{-1} \tag{24}$$

in $L^2\Omega^p(X)$ has the form

$$H^{(p)}_\gamma = H^{(p)} + W^{(p)}_\gamma, \tag{25}$$

where

$$W^{(p)}_\gamma = \|\nabla E_\gamma\|^2 + \Delta E_\gamma + (\nabla^2 E_\gamma)^{\wedge p} \tag{26}$$

(see [19]). Let us remark that $W^{(p)}_\gamma$ is G-invariant in the sense that

$$W^{(p)}_\gamma(x) = (dg)^{-1} W^{(p)}_{g\gamma}(gx) \tag{27}$$

for any $g \in G$, $\gamma \in \Psi$ and $x \in X$. Here $dg \in \mathcal{B}((T_xX)^{\wedge p}, (T_{gx}X)^{\wedge p})$ is the corresponding group translation in the fibers of the tensor bundle $(TX)^{\wedge p}$.

Let us consider the corresponding heat semigroup $T_\gamma(t) = e^{-tH^{(p)}_\gamma}$, $t > 0$, in $L^2\Omega^p(X)$ and let

$$K^{(p)}_\gamma(x, y; t) \in \mathcal{B}((T_xX)^{\wedge p}, (T_yX)^{\wedge p}) \tag{28}$$

be its integral kernel. We introduce the function

$$\theta^{(p)}(x,t) = \mathbb{E}\,\mathrm{tr}\, K_\gamma^{(p)}(x,x;t), \tag{29}$$

where tr is the usual matrix trace. Our first aim is to prove the following result.

Theorem 1. *Assume that there exists a random homogeneous function $f : X \to \mathbb{R}$ such that*

$$-(W_\gamma^{(p)}(x)h, h) \le f_\gamma(x)\|h\|^2 \tag{30}$$

for all $x \in X$ and $h \in (T_x X)^{\wedge p}$. Moreover, assume that for any $t > 0$,

$$\sup_{z\in X} \mathbb{E}e^{tf(z)} < \infty. \tag{31}$$

Then, for any $t > 0$ and $p = 0, 1, \ldots, \dim X$,

$$\sup_{x\in X} \theta^{(p)}(x,t) < \infty. \tag{32}$$

Remark 1. The statement of the theorem does not rely on the particular form (26) of the potential $W_\gamma^{(p)}$ and is valid for any potential which satisfies conditions (27), (30), (31) and such that the operator $H_\gamma^{(p)}$ is essentially self-adjoint on the space $\Omega_0^p(X)$ of smooth p-forms with compact support.

The proof of the theorem will be given in the next section. In what follows, we will always assume that the conditions (30)–(31) hold. In Section 6 we will show that the latter is true in particular examples.

The G-invariance (27) of the potential $W^{(p)}$ implies the G-invariance of the kernel $K_\gamma^{(p)}(x,x;t)$, and consequently of the function $\theta^{(p)}(x,t)$ (the latter follows from the G-invariance of μ). Thus, $\theta^{(p)}(\cdot,t)$ defines a function $\widetilde{\theta}^{(p)}(\cdot,t)$ on X/G, and we can define the theta-function

$$\Theta^{(p)}(t) = \int_{X/G} \widetilde{\theta}^{(p)}(x,t)\,dx. \tag{33}$$

The next statement follows immediately from the theorem above and compactness of X/G.

Corollary 1. *For any $t > 0$ and $p = 0, 1, \ldots, \dim X$, we have*

$$\Theta^{(p)}(t) < \infty. \tag{34}$$

3. Probabilistic representation of the heat kernels

In this section we give a probabilistic representation of the semigroup $e^{-tH^{(p)}}$, $t > 0$, and apply it to prove Theorem 1.

According to the Weitzenböck formula, the Witten Laplacian $H_\gamma^{(p)}$ has the form

$$H_\gamma^{(p)} = \Delta^{(p)} + \mathcal{R}_\gamma^{(p)}, \tag{35}$$

where $\Delta^{(p)}$ is the Bochner Laplacian in $L^2\Omega^p(X)$, $\mathcal{R}_\gamma^{(p)} = R^{(p)} + W_\gamma^{(p)}$, and $R^{(p)}(x) \in \mathcal{B}((T_xX)^{\wedge p})$ is the corresponding Weitzenböck term (see e.g. [19], [13]).

Let $\xi_x(s)$, $s \in [0, t]$, be the Brownian motion on X in the time interval $[0, t]$ starting at x, defined on its own probability space (independent of the random field E), and consider the differential equation

$$\frac{D}{ds}\omega(s) = -\mathcal{R}_\gamma^{(p)}(\xi_x(s))\omega(s), \qquad \omega(0) \in (T_xX)^{\wedge p}, \tag{36}$$

in the tensor bundle $(TX)^{\wedge p}$, where D/ds is the covariant derivative along the trajectories of $\xi_x(s)$ (see [18]).

Let f be as in Theorem 1, that is,

$$-(W_\gamma^{(p)}(x)h, h) \le f_\gamma(x)\|h\|^2 \tag{37}$$

for all $x \in X$ and $h \in (T_xX)^{\wedge p}$, and for any t,

$$\mathcal{F}(z, t) := \mathbb{E}\, e^{tf(z)} \le a(t) < \infty \tag{38}$$

for some function $a(t)$ uniformly in $z \in X$. In what follows, we denote by $\mathbb{W}$ the expectation with respect to the Brownian motion ξ_x.

Theorem 2. *Assume that the estimates* (37) *and* (38) *hold. Then:*

1) *For all times $t > 0$, any $x \in X$ and a.a. $\gamma \in \Psi$ we have*

$$\int_0^t \mathbb{W}[e^{tf_\gamma(\xi_x(s))}]\, ds < \infty. \tag{39}$$

2) *A solution $\omega(t) = \omega_\gamma(t)$ of equation* (36) *exists for all times $t > 0$ and a.a. $\gamma \in \Psi$ and satisfies the estimate*

$$\mathbb{W}\|\omega(t)\|_{(T_{\xi_x(t)}X)^{\wedge p}} \le \|\omega(0)\|_{(T_xX)^{\wedge p}} e^{tc_p} \frac{1}{t} \int_0^t \mathbb{W}[e^{tf_\gamma(\xi_x(s))}]\, ds, \tag{40}$$

where $c_p = -\inf_{x \in X} \|R^{(p)}(x)\|$, $t \in \mathbb{R}$.

Proof. 1) It follows from (38) that

$$\mathbb{W}\mathbb{E}\, e^{tf_\gamma(\xi_x(s))} \le a(t). \tag{41}$$

Then, by Fubini's theorem,

$$\mathbb{E}\left[\int_0^t \mathbb{W}[e^{tf_\gamma(\xi_x(s))}]\, ds\right] = \int_0^t \mathbb{W}[\mathbb{E}[e^{tf_\gamma(\xi_x(s))}]]\, ds \le ta(t), \tag{42}$$

which implies (39).

2) We use arguments similar to [27, Th. 5.1] (in fact, our situation is simpler). We have

$$\frac{d}{dt}\|\omega(t)\|^2 = -2(\mathcal{R}_\gamma^{(p)}(\xi_x(t))\omega(t), \omega(t)), \tag{43}$$

or

$$\frac{d}{dt}\|\omega(t)\|^2 = -2\frac{(\mathcal{R}_\gamma^{(p)}(\xi_x(t))\omega(t), \omega(t))}{\|\omega(t)\|^2}\|\omega(t)\|^2, \tag{44}$$

which implies

$$\begin{aligned} \|\omega(t)\|^2 &= \|\omega(0)\|^2 \exp\left(-\int_0^t 2\frac{(\mathcal{R}_\gamma^{(p)}(\xi_x(s))\omega(t), \omega(s))}{\|\omega(s)\|^2}\, ds\right) \\ &\le \|\omega(0)\|^2 e^{tc_p} \exp\int_0^t 2f_\gamma(\xi_x(s))\, ds. \end{aligned} \tag{45}$$

Then

$$\begin{aligned} \mathbb{W}\|\omega(t)\| &\le \|\omega(0)\| e^{tc_p}\mathbb{W}\exp\int_0^t f_\gamma(\xi_x(s))\, ds \\ &\le \|\omega(0)\| e^{tc_p}\frac{1}{t}\int_0^t \mathbb{W}[e^{tf_\gamma(\xi_x(s))}]\, ds \end{aligned} \tag{46}$$

by Jensen's inequality, which together with (39) implies the result. □

Thus, for a.a. $\gamma \in \Psi$, the equation (36) defines the evolution operator family

$$U_{\xi_x,\gamma}^{(p)}(s) \in \mathcal{B}((T_x X)^{\wedge p}, (T_{\xi_x(s)}X)^{\wedge p}) \tag{47}$$

by the formula

$$U_{\xi_x,\gamma}^{(p)}(s)\omega(0) = \omega(s). \tag{48}$$

It satisfies the estimate

$$\mathbb{W}\|U_{\xi_x,\gamma}^{(p)}(t)\| \le \frac{1}{t}\int_0^t \mathbb{W}[e^{tf_\gamma(\xi_x(s))}]\, ds, \qquad t > 0. \tag{49}$$

Remark 2. In the case where $E \equiv 0$ the operator $U_{\xi_x}^{(p)}(s)$ coincides with the parallel translation along ξ_x (see [18]).

Remark 3. Let $\xi_{x,y}(s)$, $s \in [0, t]$, be the Brownian bridge from x to y. Then

$$U^{(p)}_{\xi_{x,y},\gamma}(t) \in \mathcal{B}((T_x X)^{\wedge p}, (T_y X)^{\wedge p}) \tag{50}$$

and

$$\mathbb{W}\|U^{(p)}_{\xi_{x,y},\gamma}(t)\| \le e^{tc_p} \frac{1}{t} \int_0^t \mathbb{W}[e^{tf_\gamma(\xi_{x,y}(s))}]\, ds, \tag{51}$$

where W is the bridge expectation.

Let us consider the semigroup $e^{-tH^{(p)}_\gamma}$, $t > 0$, in $L^2\Omega^p(X)$. Let $K(x, y; t)$ be the heat kernel on X. It is known [17] that K is a strictly positive C^∞ function on $X \times X \times [0, \infty)$.

Theorem 3. *For any $t > 0$ and $\gamma \in \Psi$ the semigroup $e^{-tH^{(p)}_\gamma}$ has the integral kernel*

$$K^{(p)}_\gamma(x, y; t) \in \mathcal{B}((T_x X)^{\wedge p}, (T_y X)^{\wedge p}), \tag{52}$$

which satisfies the relation

$$K^{(p)}_\gamma(x, y; t) = K(x, y; t)\mathbb{W}(U^{(p)}_{\xi_{x,y},\gamma}(t))^*, \qquad x, y \in X. \tag{53}$$

Proof. Let us recall that $H^{(p)}_\gamma$ is essentially self-adjoint on Ω^p_0. Thus, for $\omega \in \Omega^p_0$, we have the following probabilistic representation of the semigroup $e^{-tH^{(p)}_\gamma}$, $t > 0$:

$$\langle e^{-tH^{(p)}_\gamma}\omega(x), \nu\rangle = \mathbb{W}\langle\omega(\xi_x(t)), U^{(p)}_{\xi_x,\gamma}(t)\nu\rangle \tag{54}$$

for all $\nu \in (T_x X)^{\wedge p}$ (see [18]). This can be rewritten as

$$\begin{aligned}\langle e^{-tH^{(p)}_\gamma}\omega(x), \nu\rangle &= \int_X K(x, y; t)\mathbb{W}\langle\omega(\xi_{x,y}(t)), U^{(p)}_{\xi_{x,y},\gamma}(t)\nu\rangle\, dy \\ &= \int_X K(x, y; t)\mathbb{W}\langle\omega(y), U^{(p)}_{\xi_{x,y},\gamma}(t)\nu\rangle\, dy \\ &= \left\langle \int_X K(x, y; t)\mathbb{W}(U^{(p)}_{\xi_{x,y},\gamma}(t))^*\omega(y)\, dy, \nu\right\rangle,\end{aligned} \tag{55}$$

which implies (53). □

Proof of Theorem 1. Formulae (51) and (53) imply that for any $\gamma \in \Psi$ and all $x \in X$, $t > 0$,

$$\begin{aligned}\operatorname{tr} K^{(p)}_\gamma(x, x; t) &\le \binom{\dim X}{p} K(x, x; t)\mathbb{W}\|U^{(p)}_{\xi_{x,x}}(t)\| \\ &\le \binom{\dim X}{p} K(x, x; t)e^{tc_p}\frac{1}{t}\int_0^t \mathbb{W}[e^{tf_\gamma(\xi_x(s))}]\, ds,\end{aligned} \tag{56}$$

where tr denotes the usual matrix trace. Note that $k(x) := K(x, x; t)$ is G-invariant and C^∞, which together with compactness of X/G implies that it is bounded. Formula (42) immediately implies that the θ-function

$$\theta^{(p)}(x, t) = \mathbb{E}\operatorname{tr} K^{(p)}_\gamma(x, x; t) \tag{57}$$

is bounded in $x \in X$ for any $t > 0$ and $p = 0, 1, \ldots, \dim X$. □

Remark 4. More precisely, the θ-function satisfies the estimate

$$\theta^{(p)}(x,t) \le \binom{\dim X}{p} e^{tc_p}\, K(x,x;t)\, a(t), \tag{58}$$

where a is defined by (38).

Remark 5. We can also give a lower bound of $\theta^{(p)}(x,t)$. Indeed, let $g : X \times X \to \mathbb{R}$ be such that for

$$g_\gamma(x) := \langle g_x, \gamma\rangle = \sum_{y\in\gamma} g(x,y), \tag{59}$$

and any $\gamma \in \Psi$ and $x \in X$, the estimate

$$-(W_\gamma^{(p)}(x)h, h) \ge g_\gamma(x)\|h\|^2 \tag{60}$$

holds for all $h \in (T_x X)^{\wedge p}$. It then follows from (45) that

$$\mathbb{W}\|U^{(p)}_{\xi_{x,y}}(t)\| \ge e^{tb_p}\mathbb{W}e^{\int_0^t g_\gamma(\xi_{x,y}(s))\,ds} \tag{61}$$

for all $t > 0$, and consequently

$$\theta^{(p)}(x,t) \ge \binom{d}{p} e^{tb^{(p)}}\, K(x,x;t)\, \mathbb{W}\mathbb{E}\, e^{\int_0^t g_\gamma(\xi_{x,y}(s))\,ds}, \tag{62}$$

where $b_p = -\sup_{x\in X}\|R^{(p)}(x)\|$.

Remark 6. If X is a symmetric space, that is, there exists a group of isometries acting on X transitively, then both $K(x,x;t)$ and $\mathcal{F}(x,t)$ (defined by (38)) do not depend on x, and the estimate (58) gets the form

$$\theta^{(p)}(x,t) \le \binom{\dim X}{p} e^{tc_p}\, k(t), \tag{63}$$

where $k(t) := K(x,x;t)\mathcal{F}(x,t)$.

Example 1 (Euclidean space). Let $X = \mathbb{R}^d$, $G = \mathbb{Z}^d$. Then $R^{(p)}(x) = 0$ and $K_t(x,x) = (4\pi t)^{-d/2}$, $t > 0$. Formula (63) can be rewritten in the form

$$\theta^{(p)}(x,t) \le (4\pi t)^{-d/2}\binom{d}{p}\mathcal{F}(0,t). \tag{64}$$

Then

$$\Theta^{(p)}(t) = \int_{\mathbb{T}^d} \theta^{(p)}(x,t)\,dx \le (4\pi t)^{-d/2}\binom{d}{p}\mathcal{F}(0,t). \tag{65}$$

Here $\mathbb{T}^d$ is the d-dimensional torus.

Example 2 (Hyperbolic space). Let $X = \mathbb{H}^d$. Then $R^{(p)}(x) = -p(d-p)$ and we have, for $t > 0$,

$$\theta^{(p)}(x,t) \leq \binom{d}{p} e^{tp(d-p)} K(x,x;t) \frac{1}{t} \mathbb{E} \int_0^t \mathbb{W}[e^{tf_\gamma(\xi_x(s))}]\, ds, \tag{66}$$

where K is the heat kernel on $\mathbb{H}^d$. It is known that the group $SL(d,\mathbb{R})$ acts transitively on $\mathbb{H}^d$ by isometries. Thus, according to Remark 6, the latter estimate takes the form

$$\theta^{(p)}(x,t) \leq \binom{d}{p} e^{tp(d-p)} k(t). \tag{67}$$

4. Von Neumann algebras associated with random Laplacians

In this section, we construct a W^* (von Neumann) algebra containing the semigroup

$$T^{(p)}_{\gamma,t} := e^{-tH^{(p)}_\gamma} \tag{68}$$

and interpret the theta-function $\Theta^{(p)}(t)$ as its trace. We refer to [34] for general notions of the theory of von Neumann algebras.

Let $U_g,\ g \in G$, be the action of G in $L^2\Omega^p(X)$. It follows from (27) that $H^{(p)}_\gamma$ satisfies the commutation relation

$$U_g H^{(p)}_\gamma U_g^{-1} = H^{(p)}_{g\gamma} \tag{69}$$

for any $g \in G$ and $\gamma \in \Psi$. Obviously, the semigroup $T^{(p)}_{\gamma,t}$ and the orthogonal projection

$$P^{(p)}_\gamma : L^2\Omega^p(X) \to \operatorname{Ker} H^{(p)}_\gamma \tag{70}$$

satisfy similar relations:

$$U_g T^{(p)}_{\gamma,t} U_g^{-1} = T^{(p)}_{g\gamma,t}, \qquad U_g P^{(p)}_\gamma U_g^{-1} = P^{(p)}_{g\gamma}. \tag{71}$$

Remark 7. If $H^{(p)}_\gamma$ commuted with U_g, $g \in G$, then both $T^{(p)}_{\gamma,t}$ and $P^{(p)}_\gamma$ would belong to the commutant $\mathcal{U}^p := \{U_g\}'_{g\in G}$, and we would have the equality

$$\operatorname{Tr}_{\mathcal{U}} e^{-tH^{(p)}_\gamma} = \int_{X/G} \operatorname{tr} K^{(p)}_\gamma(x,x;t)\, dx. \tag{72}$$

This, however, holds only for γ such that $g\gamma = \gamma$ for all $g \in G$. Such γ form a μ-zero set.

Let us consider the space

$$L^2_\mu\Omega^p := L^2(\Psi \times X \to T^{\wedge p}X, d\mu \otimes dx) = L^2(\Psi, d\mu) \otimes L^2\Omega^p(X). \tag{73}$$

The diagonal action

$$\Psi \times X \ni (\gamma, x) \mapsto g(\gamma, x) := (g\gamma, gx) \tag{74}$$

of G on $\Psi \times X$ generates the action $G \ni g \mapsto \mathbf{U}_g$ on the space of forms $L^2_\mu\Omega^p$. We denote by $\mathcal{A}^p := \{\mathbf{U}_g\}'_{g\in G} \subset \mathcal{B}(L^2_\mu\Omega^p)$ the commutant of $\mathbf{U}_g$.

Next, we introduce the algebra $\mathcal{C}^p$ of μ-essentially bounded maps

$$A : \Psi \to \mathcal{B}(L^2\Omega^p(X)) \tag{75}$$

such that

$$A(g\gamma) = U_g A(\gamma) U_{g^{-1}} \tag{76}$$

for any $g \in G$ and $\gamma \in \Psi$. The algebra $\mathcal{C}^p$ can be naturally identified with a subalgebra of $\mathcal{A}^p$. Moreover,

$$\mathcal{C}^p = \mathcal{A}^p \cap L^\infty_\mu(\Psi \to \mathcal{B}(L^2\Omega^p(X))) \tag{77}$$

and thus is a W^*-algebra.

Let $A \in \mathcal{C}^p$ and, for any $\gamma \in \Psi$, denote by $a_\gamma(x, y)$ the integral kernel of $A(\gamma)$. Let us remark that, because of the commutation relation (76), the kernel $a_\gamma(x, y)$ is G-invariant in the sense that

$$a_{g\gamma}(gx, gy) = a_\gamma(x, y) \tag{78}$$

for all $g \in G$, $\gamma \in \Psi$ and $x, y \in X$. The latter relation together with the G-invariance of the expectation $\mathbb{E}$ imply that the function $f(x) := \mathbb{E}\,\mathrm{tr}\, a_\gamma(x, x)$, $x \in X$, is constant on each orbit of the action of G on X. Therefore it defines a function ϕ_a on X/G such that $f(x) = \phi_a(\pi(x))$, where $\pi : X \to X/G$ is the canonical projection.

Thus we can define the functional

$$\mathbb{TR}\, A = \int_{X/G} \mathbb{E}\,\mathrm{tr}\, a_\gamma(x, x)\, dx := \int_{X/G} \phi_a(x)\, dx, \tag{79}$$

where dx is the volume measure of the compact manifold X/G.

Theorem 4. *$\mathbb{TR}$ is a faithful normal semifinite trace on the W^*-algebra $\mathcal{C}^p$.*

Proof. 1) Let us prove that $\mathbb{TR}$ is cyclic, i.e. for any $A, B \in \mathcal{C}^p$ such that $\mathbb{TR}\, AB$ is finite we have

$$\mathbb{TR}\, AB = \mathbb{TR}\, BA. \tag{80}$$

Assume without loss of generality that A and B are symmetric. Then their integral kernels a_γ and b_γ satisfy the relations $a_\gamma(x, y)^+ = a_\gamma(y, x)$ and $b_\gamma(x, y)^+ = b_\gamma(y, x)$

respectively, where $n^+ : T_yX \to T_xX$ is the adjoint of an operator $n : T_xX \to T_yX$ with respect to the Riemannian structure of X. Then

$$\begin{aligned}\mathbb{TR}\,AB &= \int_{X/G} \mathbb{E}\,\mathrm{tr}\left(\int_X a_\gamma(x,y)b_\gamma(y,x)\,dy\right)dx\\ &= \int_{X/G} \mathbb{E}\,\mathrm{tr}\left(\int_X a_\gamma(x,y)b_\gamma(y,x)\,dy\right)^+ dx\\ &= \int_{X/G} \mathbb{E}\,\mathrm{tr}\left(\int_X b_\gamma(y,x)^+a_\gamma(x,y)^+\,dy\right)dx\\ &= \int_{X/G} \mathbb{E}\,\mathrm{tr}\left(\int_X b_\gamma(x,y)a_\gamma(y,x)\,dy\right)dx = \mathbb{TR}\,BA. \end{aligned}\tag{81}$$

2) Let us show that $\mathbb{TR}$ is faithful. Assume that $\mathbb{TR}\,A^*A = 0$. The integral kernel c_γ of A^*A has the form $c_\gamma(x,z) = \int_X a_\gamma(y,x)^+a_\gamma(y,z)\,dy$. We have

$$\mathbb{TR}\,A^*A = \int_{X/G} \mathbb{E}\,\mathrm{tr}\,c_\gamma(x,x)\,dx = \int_{X/G} \phi_c(x)\,dx = 0 \tag{82}$$

(cf. (79)), which implies that $\phi_c(x) = 0$ for a.a. $x \in X/G$, and consequently $\mathbb{E}\,\mathrm{tr}\,c_\gamma(x,x) = \phi_c(\pi(x)) = 0$ for a.a. $x \in X$. This implies that

$$|a_\gamma(y,x)|^2 := \mathrm{tr}(a_\gamma(y,x)^+a_\gamma(y,x)) = 0 \tag{83}$$

and $a_\gamma(y,x) = 0$ for almost all $\gamma \in \Psi$ and $x, y \in X$. Thus we have $A = 0$.

3) Let us show that $\mathbb{TR}$ is normal. We define the operator

$$\mathbb{P} : A \mapsto \mathbb{E}\,A, \quad A \in \mathcal{C}^p. \tag{84}$$

Because of G-invariance of μ we have $\mathbb{P}(A) \in \mathcal{U}^p$. It is known that $\mathcal{U}^p$ has a faithful normal semifinite trace $\mathrm{Tr}_\mathcal{U}$ defined by the formula

$$\mathrm{Tr}_\mathcal{U}\,B = \int_{X/G} \mathrm{tr}\,b(x,x)\,dx, \tag{85}$$

where b is the integral kernel of $B \in \mathcal{U}$ (see [9], [21]). Thus, for $A \in \mathcal{C}^p$, we have obviously

$$\mathbb{TR}\,A = \mathrm{Tr}_\mathcal{U}\,\mathbb{P}(A). \tag{86}$$

The normality of $\mathbb{TR}$ now follows from the normality of $\mathrm{Tr}_\mathcal{U}$ and the lemma below. □

Lemma 1. *The mapping $\mathbb{P}$ is normal.*

Proof. Let us first show that $\mathbb{P}$ is a *Schwarz mapping*, i.e.

$$\mathbb{P}(A)^*P(A) \le \mathbb{P}(A^*A) \tag{87}$$

for all $A \in \mathcal{C}^p$. We remark that

$$\|\mathrm{id} \otimes \mathbb{P}(A)\|_{\mathcal{C}^p} = \|\mathbb{E}\,A\| \le \operatorname{ess\,sup} \|A(\gamma)\| = \|A\|_{\mathcal{C}^p}. \tag{88}$$

Thus id $\otimes \mathbb{P}$ is a projection of norm one [35] from the W^*-algebra $\mathcal{C}^p$ onto its W^*-subalgebra $\mathbf{1} \otimes \mathcal{U}^p$ consisting of the constant maps

$$\Psi \ni \gamma \mapsto \mathbf{1} \otimes B \in \mathbf{1} \otimes \mathcal{U}^p \tag{89}$$

(here $\mathbf{1}$ is the identity operator in the Hilbert space $L^2(\Psi, d\mu)$), which implies the estimate (87) (see [35, Th. 1]).

It is known that any Schwarz mapping between W^*-algebras is continuous in the σ-weak topology (that is, it is normal) if it is continuous in the strong topology. Moreover, a stronger statement is true: for Schwarz mappings continuity in the weak topology is equivalent to continuity in the σ^*-strong topology (see [33]).

Thus we only need to prove that $\mathbb{P}$ is strongly continuous, which follows from the estimate

$$\begin{aligned} \|\mathbb{P}(A)f\|^2_{L^2\Omega^p(X)} &= \int_X |\mathbb{E}A_\gamma f(x)|^2\, dx \le \int_X \mathbb{E}|A_\gamma f(x)|^2\, dx \\ &= \int_X \mathbb{E}|A_\gamma \widetilde{f}(\gamma, x)|^2\, dx = \|A\widetilde{f}\|^2_{L^2_\mu\Omega^p}, \end{aligned} \tag{90}$$

where $f \in L^2\Omega^p(X)$ and $\widetilde{f} \in L^2_\mu\Omega^p$, $\widetilde{f}(\gamma, x) = f(x)$. □

Let us now consider the maps $\mathbf{T}_t^{(p)} : \gamma \mapsto T_{\gamma,t}^{(p)}$ and $\mathbf{P}^{(p)} : \gamma \mapsto P_\gamma^{(p)}$. The commutation relations (71) imply that $\mathbf{T}_t^{(p)}, \mathbf{P}^{(p)} \in \mathcal{C}^p$.

Theorem 5. 1) *For all times* $t > 0$ *and any* $p = 0, \dots, \dim X$*,*

$$\mathbb{TR}\, \mathbf{T}_t^{(p)} = \Theta^{(p)}(t) < \infty. \tag{91}$$

2) *For any* $p = 0, \dots, \dim X$*,*

$$\mathbb{TR}\, \mathbf{P}^{(p)} < \infty. \tag{92}$$

3) *The following McKean–Singer formula holds for all times* $t > 0$*:*

$$\sum_{p=0}^{\dim X} (-1)^p\, \mathbb{TR}\, \mathbf{T}_t^{(p)} = \sum_{p=0}^{\dim X} (-1)^p\, \mathbb{TR}\, \mathbf{P}^{(p)}. \tag{93}$$

Proof. 1) Formula (91) follows immediately from (79) and Corollary 1.

2) We have obviously

$$T_{\gamma,t}^{(p)}(I - P_\gamma^{(p)}) = T_{\gamma,t}^{(p)} - P_\gamma^{(p)}, \tag{94}$$

or

$$\mathbf{T}_t^{(p)}(I - \mathbf{P}^{(p)}) = \mathbf{T}_t^{(p)} - \mathbf{P}^{(p)}. \tag{95}$$

Thus

$$\mathbb{TR}\, \mathbf{P}^{(p)} = \mathbb{TR}\, \mathbf{T}_t^{(p)} - \mathbb{TR}\, \mathbf{T}_t^{(p)}(I - \mathbf{P}^{(p)}) < \infty. \tag{96}$$

3) Formula (93) follows from the McKean–Singer formula in von Neumann algebras (see [12, (5.1.10)]), applied to the algebra $\mathcal{C} = \bigoplus_p \mathcal{C}^p$ and the operators $D := \sum d^p$, $D^* = U(\sum (d^p)^*_\gamma)U^{-1}$ in $\bigoplus_p L^2_\mu\Omega^p$ (cf. (24)). □

Remark 8. The right-hand side of formula (93) can be understood as a regularized index of the Dirac operator $D + D^*$.

5. Stability of the index

Let us recall the framework of the proof of the third part of Theorem 1 and remark that the right-hand side of formula (93) can be understood as a regularized index of the Dirac operator $D + D^*$ acting in the space $L^2_\mu \Omega := \bigoplus_p L^2_\mu \Omega^p$. We will prove that it depends neither on the choice of the potential V nor on the measure μ on Ψ.

We will use the following notations:

- $L^2_\mu \Omega := \bigoplus_p L^2_\mu \Omega^p$,
- $\bigwedge T_x X := \bigoplus_p T_x^{\wedge p} X$,
- $\mathbf{P} := \sum_p \mathbf{P}^{(p)}, \ P_\gamma := \sum_p P_\gamma^{(p)}$,
- $p_\gamma(x, y, t)$ – the integral kernel of P_γ,
- $\mathbf{T}_t := \sum_p \mathbf{T}_t^{(p)}, \ T_{\gamma,t} := \sum_p T_{\gamma,t}^{(p)}$,
- $K_\gamma(x, y; t)$ – the integral kernel of $T_{\gamma,t}$,
- $U_{\xi_{x,y},\gamma} = \sum_p U^{(p)}_{\xi_{x,y},\gamma}$.

Thus we have $p_\gamma(x, y, t)$, $K_\gamma(x, y, t)$, $U_{\xi_{x,y},\gamma} \in \mathcal{B}(\bigwedge T_x X, \bigwedge T_y X)$ and $\mathbf{P} \in \mathcal{B}(L^2_\mu \Omega)$.

Theorem 6.

$$\mathbb{STR}\, \mathbf{P} := \sum_p (-1)^p \operatorname{Tr} \mathbf{P}^{(p)} = \chi(M), \tag{97}$$

where $\chi(M)$ is the Euler characteristic of $M = X/G$.

Proof. According to formula (93),

$$\mathbb{STR}\, \mathbf{P} = \mathbb{E} \int_M \operatorname{str} K_\gamma(x, x; t)\, dx \tag{98}$$

for any $t > 0$. Here str denotes the usual matrix supertrace of an operator acting in $\bigwedge T_x X$. In particular,

$$\mathbb{STR}\, \mathbf{P} = \lim_{t \to 0} \mathbb{E} \int_M \operatorname{str} K_\gamma(x, x; t)\, dx. \tag{99}$$

In order to find the latter asymptotics, we will need the probabilistic representation of the kernel K similar to the one introduced in [18], which is different from (53).

Let $z_t(s)$, $0 \le s \le 1$, $z_t(0) = x$, $z_t(1) = y$, be the semiclassical bridge, that is, the process in X with the time-dependent generator H_t,

$$H_t = t\Delta + \nabla Y_s, \qquad Y_s(x) = -\rho(x, y)^2/(1 - s) - tF(x), \tag{100}$$

where $F(x)$ is determined by the geometry of X. We do not need the explicit form of F. It is known [18] that, almost surely, z_t converges to a geodesic from x to y (as $t \to 0$).

The following formula for the heat kernel holds:

$$K_\gamma^{(p)}(x, y; t) = (2\pi t)^{-d} \omega(x)^{-1/2} \exp\left(-\frac{\rho(x, y)^2}{2t}\right) \times \mathbb{W}\left[\exp\left(t \int_0^1 V_{\text{eff}}(z_t(s))\, ds\right) U^{(p)}_{z_t,\gamma}(1)\right], \tag{101}$$

where $U_{z_t,\gamma}(s)$ is the evolution family generated by equation (36) with the process z instead of ξ and the operator $t\mathcal{R}^{(p)}$ instead of $\mathcal{R}^{(p)}$. Like F, the potential V_{eff} is determined entirely by the geometry of X, and we will not use its explicit form (see [18] for more details). The existence of the family $U_{z_t,\gamma}$ can be shown by similar methods to those used in the previous section.

Remark 9. In general, we have to assume that y is a pole. However, if $x \notin \mathrm{Cut}(y)$, we can always choose a compact domain D inside $X \setminus \mathrm{Cut}(y)$ which contains the shortest geodesic from x to y and modify X outside D in such a way that y is a pole. This modification does not affect the short time asymptotics of $K_\gamma^{(p)}$ (see [18] for more details).

According to formula (53) we have, for $t > 0$,

$$\begin{aligned}\operatorname{str} K_\gamma(x, y; t) = (2\pi t)^{-d}\omega(x)^{-1/2}\exp\left(-\frac{\rho(x,y)^2}{2t}\right)\\ \times \mathbb{W}\left[\exp\left(t\int_0^1 V_{\text{eff}}(z_t(s))\,ds\right)\operatorname{str} U_{z_t,\gamma}^{(p)}(1)\right].\end{aligned} \tag{102}$$

For any fixed trajectory z_t and $\gamma \in \Psi$, the operator $U_t(s) := (/\!/_s)^{-1}U_{z_t,\gamma}(s)$ belongs to $\mathcal{B}(\bigwedge T_xX)$, where $/\!/_s$ denotes the parallel translation along $z_t(s)$, $0 \le s \le 1$. It satisfies the equation

$$\frac{d}{ds}U_t(s) = tU_t(s)\circ(\mathcal{R}(s) + \mathcal{W}(s)), \tag{103}$$

where

$$\mathcal{R}(s) = (/\!/_s)^{-1}\sum_p R^{(p)}(z_t(s)), \tag{104}$$

$$\mathcal{W}(s) = (/\!/_s)^{-1}\sum_p W_\gamma^{(p)}(z_t(s)). \tag{105}$$

Then

$$U_t(1) = \mathrm{id} + Z_1 + \cdots + Z_l + O(t^{l+1}), \tag{106}$$

where

$$\begin{aligned}&Z_l = t^l\int_0^1\ldots\int_0^{s_2}\mathcal{S}(s_l)\circ\cdots\circ\mathcal{S}(s_2)\circ\mathcal{S}(s_1)\,ds_1\,ds_2\ldots ds_l,\\ &\mathcal{S} = \mathcal{R} + \mathcal{W}.\end{aligned} \tag{107}$$

Obviously

$$\operatorname{str} U_t(1) = \sum_{l=1}^{n/2} t^l \operatorname{str} Z_l + O(t), \tag{108}$$

and

$$\operatorname{str} K_\gamma(x, x; t) = \sum_{l=1}^{n/2-1} t^{l-n/2}\operatorname{str} Z_l + \operatorname{str} Z_{n/2} + O(t), \tag{109}$$

because

$$\operatorname{str}(U_{z_t,\gamma}(1) - U_t(1)) = O(t) \tag{110}$$

(see [18]).

The following result is well-known (see e.g. [13]).

Proposition 1. *Let V be a d-dimensional vector space, fix an orthonormal basis (e_k) in V, and let a^k be the corresponding annihilation operators on the exterior algebra $\bigwedge V$ of V. Then:*

1) *any operator $A \in \mathcal{B}(\bigwedge V)$ can be uniquely represented in the form*

$$A = \sum_{I,J\subset(1,\dots,d)} A_{IJ}(a^I)^* a^J, \tag{111}$$

where $a^I = \prod_{i\in I} a^i$, and

$$\operatorname{str} A = (-1)^d A_{(1,\dots,d)(1,\dots,d)} \tag{112}$$

(Berezin–Patodi formula);

2) *for any $B \in \mathcal{B}(V)$ given by the matrix (B_{ij}) we have*

$$\bigwedge B = \sum_{i,j} B_{ij}(a^i)^* a^j, \tag{113}$$

where $\bigwedge B := \sum_p B^{\wedge p}$.

It follows from (21), (113) and the well-known representation of the Weitzenböck term $R^{(p)}$ in terms of creation-annihilation operators (see e.g. [13]) that

$$\mathcal{R}(s) + \mathcal{W}(s) = R_{ijkl}(s)(a^i)^* a^j (a^k)^* a^l + E_{ij}(s)(a^i)^* a^j + e(s)\mathrm{id} \tag{114}$$

for some coefficients $R_{ijkl}(s)$, $E_{ij}(s)$ and $e(s)$. The basis which defines the creation-annihilation operators $(a^i)^*$, a^j can be chosen arbitrarily in $T_x X$ and then transported to $T_{z_t(s)} X$ by parallel translation along $z_t(s)$.

It is clear from (112) that

$$\operatorname{str} Z_l = \begin{cases} 0, & l < d/2, \\ \operatorname{str} \widetilde{Z}_{d/2}, & l = d/2, \end{cases} \tag{115}$$

where $\widetilde{Z}$ is defined by (107) with $\mathcal{W} = 0$. Thus the potential $\mathcal{W}$ just does not have any influence on the leading term of the decomposition (109). Therefore

$$\operatorname{str} K_\gamma(x, x; t) = \operatorname{str} \widetilde{K}(x, x; t) + O(t), \tag{116}$$

where $\widetilde{K}$ is the heat kernel on X, and consequently

$$\mathbb{STR}\,\mathbf{P} = \mathbb{E} \int_M \operatorname{str} K_\gamma(x, x, t)\, dx = \int_M \operatorname{str} \widetilde{K}(x, x, t)\, dx = \chi(M). \tag{117}$$

The latter equality follows from the index theorem for coverings (see [9]). □

Theorem 6 gives us a possibility to study the spaces of harmonic forms of individual operators H_γ. We have the following result.

Theorem 7. *Assume that the action of G on Ψ is ergodic, and let $\chi(M) \neq 0$. Then*

$$\dim \operatorname{Ker} H_\gamma = \infty \quad \textit{for } \mu\textit{-a.a. } \gamma \in \Psi. \tag{118}$$

Proof. Consider the integral kernel $p_\gamma(x, y)$ of the operator $\mathbf{P}_\gamma$. Then, for any $g \in G$,

$$p_\gamma(gx, gx) = p_{g^{-1}\gamma}(x, x), \tag{119}$$

and thus the function

$$F(\gamma) := \dim \operatorname{Ker} H_\gamma = \operatorname{Tr} \mathbf{P}_\gamma = \int_X \operatorname{tr} p_\gamma(x, x)\, dx \tag{120}$$

is G-invariant. Because of the ergodicity of the action of G on Ψ we have

$$F(\gamma) = C \tag{121}$$

for some constant C. On the other hand,

$$F(\gamma) = \sum_{g \in G} \int_{g\widetilde{X}} \operatorname{tr} p_\gamma(x, x)\, dx = \sum_{g \in G} \int_{\widetilde{X}} \operatorname{tr} p_{g^{-1}\gamma}(x, x)\, dx, \tag{122}$$

where $\widetilde{X}$ is a fundamental domain of the action of G on X. Then

$$\begin{aligned} C &= \mathbb{E} \int_X \operatorname{tr} p_\gamma(x, x)\, dx = \sum_{g \in G} \int_{\widetilde{X}} \mathbb{E} \operatorname{tr} p_{g^{-1}\gamma}(x, x)\, dx = \sum_{g \in G} \int_{\widetilde{X}} \mathbb{E} \operatorname{tr} p_\gamma(x, x)\, dx \\ &\geq \sum_{g \in G} \mathbb{STR}\, \mathbf{P} = \sum_{g \in G} \chi(M) = \infty, \end{aligned} \tag{123}$$

because G is infinite and $\chi(M) \neq 0$. □

6. Examples

6.1. Gases

We will consider the situation where $\Psi = \Gamma_X$, the space of locally finite configurations in X.

6.1.1. Configuration spaces and measures. The configuration space Γ_X over X is defined as the set of all locally finite subsets (*configurations*) in X:

$$\Gamma_X := \{\gamma \subset X : |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset X\}. \tag{124}$$

Here, $|A|$ denotes the cardinality of a set A.

We can identify any $\gamma \in \Gamma_X$ with the positive, integer-valued Radon measure

$$\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(X), \tag{125}$$

where ε_x is the Dirac measure with mass at x, $\sum_{x \in \emptyset} \varepsilon_x :=$ zero measure, and $\mathcal{M}(X)$ denotes the set of all positive Radon measures on the Borel σ-algebra of X. The space Γ_X is endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ_X with respect to which all maps

$$\Gamma_X \ni \gamma \mapsto \langle f, \gamma \rangle := \int_X f(x)\, \gamma(dx) \equiv \sum_{x \in \gamma} f(x) \tag{126}$$

are continuous. Here, $f \in C_0(X)$ (:= the set of all continuous functions on X with compact support).

The action of G on X can be lifted to a diagonal action on Γ_X:

$$\Gamma_X \ni \gamma = \{\dots, x, y, z, \dots\} \mapsto g\gamma = \{\dots, gx, gy, gz, \dots\} \in \Gamma_X, \qquad g \in G. \tag{127}$$

Let μ be a Gibbs measure on Γ_X (see Appendix). We assume that:

(i) μ satisfies the *Ruelle bound*, that is,

$$|k_\mu^{(n)}| \le a^n \tag{128}$$

for some constant a, where $k_\mu^{(n)}$ is the n-th correlation function of μ;

(ii) μ is invariant with respect to to the G-action (127).

A class of Gibbs measures with these properties is described in the Appendix (see Remark 10).

6.1.2. Probabilistic representations of Laplacians. Let $v \in C_0^2(\mathbb{R})$ with $\operatorname{supp} v \subset [-r, r]$, where $r > 0$ is the injectivity radius of X, and define the function $V : X \times X \to \mathbb{R}$ by

$$V(x, y) = v(\rho(x, y)), \qquad x, y \in X, \tag{129}$$

where ρ is the Riemannian distance on X. Let

$$E_\gamma(x) = \sum_{y \in \gamma} V(x, y), \tag{130}$$

and consider the Witten Laplacian

$$H_\gamma^{(p)} = \Delta^{(p)} + R^{(p)} + W_\gamma^{(p)} \tag{131}$$

(cf. (35)).

Theorem 8. *The operator $H_\gamma^{(p)}$ satisfies the conditions of Theorem* 1.

Proof. Choose a function $F : X \times X \to \mathbb{R}$ which satisfies the following conditions:

1) F is bounded, and for some $r \in \mathbb{R}$ and any $x \in X$, $\operatorname{supp} F(x, \cdot) \subset B(x, r)$, where $B(x, r)$ is the ball of radius r centered at x;

2) the function

$$f_\gamma(x) := \langle F(x,\cdot), \gamma\rangle = \sum_{y\in\gamma} F(x,y) \tag{132}$$

satisfies the estimate

$$-(W_\gamma^{(p)}(x)h, h) \le f_\gamma(x)\|h\|^2 \tag{133}$$

for any $\gamma \in \Gamma_X,\ x \in X$ and $h \in (T_x X)^{\wedge p}$.

Such an F always exists: for instance, we can set

$$F(x,y) := -\Delta_x V(x,y) + \|(\nabla_x^2 V(x,y))^{\wedge p}\|_{\mathcal{B}(T_x X^{\wedge p})}. \tag{134}$$

The following result is general.

Lemma 2. *Assume that F satisfies condition* 1)*. Then, for all $t > 0$, the estimate* (39) *holds, that is,*

$$\sup_{z\in X} \mathbb{E}\, e^{tf(z)} < \infty. \tag{135}$$

The lemma together with condition 2) imply the result. □

Corollary 2. *All results of the previous sections can be applied to the operator $H_\gamma^{(p)}$.*

Proof of Lemma 2. For any measurable function g on X with compact support, the Laplace transform of μ has the form

$$\int_{\Gamma_X} e^{t\langle g,\gamma\rangle}\,\mu(d\gamma)$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\int_{X^n} (e^{tg(y_1)}-1)\dots(e^{tg(y_n)}-1)k_\mu^{(n)}(y_1,\dots,y_n)\,dy_1\dots dy_n, \tag{136}$$

which follows from formula (172) in the Appendix. Here $k_\mu^{(n)}$ is the n-th correlation function of μ. According to the Ruelle bound (128),

$$\int_{\Gamma_X} e^{t\langle g,\gamma\rangle}\,\mu(d\gamma) \le \exp\left(a\int_X (e^{tg(y)}-1)\,dy\right). \tag{137}$$

The right-hand side is finite because g has compact support. Thus

$$\mathbb{E}\, e^{tF(z)} = \int_{\Gamma_X} e^{tf_\gamma(z)}\,\mu(d\gamma) \le \exp\left(a\int_X (e^{tF(z,y)}-1)\,dy\right) < \infty. \tag{138}$$

Moreover, for any $z \in X$,

$$\int_X (e^{tF(z,y)}-1)\,dy = \int_{B(z,r)} (e^{tF(z,y)}-1)\,dy$$
$$\le \max_{y,z\in X} (\operatorname{vol} B(z,r)|e^{tF(z,y)}-1|) =: C(t) < \infty. \tag{139}$$

This implies the estimate

$$\mathbb{E}\, e^{tf(z)} \le e^{aC(t)} \tag{140}$$

for any $z \in X$. □

6.2. Crystals

Another type of examples can be constructed in the following way. Let $\Psi = \mathbf{X}$ be the infinite product of identical copies $X_g,\ g \in G$, of the manifold X:

$$\mathbf{X} = X^G := \prod_{g\in G} X_g \ni \overline{\xi} = (\xi_g)_{g\in G}, \qquad \xi_g \in X. \tag{141}$$

$\mathbf{X}$ is endowed with the product topology and the corresponding Borel σ-algebra $\mathrm{Bor}(\mathbf{X})$. Let μ be a translation invariant probability measure on $\mathbf{X}$. That is, μ is invariant with respect to the following action T of G:

$$T_{g'}(\xi_g)_{g\in G} = (\xi_{g'g})_{g\in G}, \qquad g' \in G. \tag{142}$$

We define the random field E on the probability space $(\mathbf{X}, \mathrm{Bor}(\mathbf{X}), \mu)$ in the following way:

$$E_{\overline{\xi}}(x) = \sum_{g\in G} V(x, g\xi_g), \tag{143}$$

where V is given by formula (129), and consider the corresponding Witten Laplacian $H^{(p)}_{\overline{\xi}}$.

Let $\mathcal{F}(X^2)$ be the set of all bounded functions $F : X \times X \to \mathbb{R}$ such that

$$\operatorname{supp} F(x, \cdot) \subset B(x, r) \tag{144}$$

for some $r \in \mathbb{R}$ and any $x \in X$, and set

$$f_{\overline{\xi}}(x) := \sum_{g\in G} F(x, g\xi_g). \tag{145}$$

Let us assume that the measure μ satisfies the following condition:

(C) for any $F \in \mathcal{F}(X^2)$,

$$\sup_{z\in X} \mathbb{E}\, e^{tf(z)} < \infty \qquad \text{for all } t > 0. \tag{146}$$

Theorem 9. *The operator $H^{(p)}_{\overline{\xi}}$ satisfies the conditions of Theorem 1.*

Proof. The proof is quite similar to the proof of Theorem 8. Choose $F : X \times X \to \mathbb{R}$ which satisfies (144) and

$$-(W^{(p)}_{\overline{\xi}}(x)h, h) \le f_{\overline{\xi}}(x)\|h\|^2 \tag{147}$$

for any $\overline{\xi} \in \mathbf{X},\ x \in X$ and $h \in (T_x X)^{\wedge p}$. As in the proof of Theorem 8, we can set

$$F(x, y) := -\Delta_x V(x, y) + \|(\nabla^2_x V(x, y))^{\wedge p}\|_{\mathcal{B}(T_x X^{\wedge p})}. \tag{148}$$

The statement of the theorem now follows from (146). $\square$

Corollary 3. *All results of the previous sections can be applied to the operator $H^{(p)}_{\overline{\xi}}$.*

We will consider two examples of measures which satisfy the above condition (C): the product measures and Gibbs measures with compact support.

6.2.1. Product measures. Let us consider a probability measure

$$\nu(d\xi) = \phi(\xi)d\xi \tag{149}$$

on X and define

$$\mu(d\bar{\xi}) := \bigotimes_{g\in G} \nu(d\xi_g). \tag{150}$$

Lemma 3. *μ satisfies condition* (C).

Proof. We have

$$\begin{aligned}\mathcal{F}(z,t) &:= \int_{X^G} e^{tf_{\bar{\xi}}(z)}\,\mu(d\xi) = \prod_{g\in G}\int_X e^{tF(z,g\xi)}\,\nu(d\xi)\\ &= \prod_{g\in G}\int_X 1 + (e^{tF(z,g\xi)} - 1)\,\nu(d\xi)\\ &= \prod_{g\in G}\left(1 + \int_X (e^{tF(z,g\xi)} - 1)\,\nu(d\xi)\right),\end{aligned} \tag{151}$$

and

$$\sum_{g\in G}\int_{gB(z,r)} (e^{tF(z,g\xi)} - 1)\,\nu(d\xi) \le c\sum_{g\in G}\nu(gB(z,r)). \tag{152}$$

Let us prove that the right-hand side is finite. Let $\widetilde{X}$ be a fundamental domain of the action of G on X, and let

$$G_z = \{g \in G : B(z,r) \cap g\widetilde{X} \ne \emptyset\}. \tag{153}$$

Set $N = |G_z|$ (obviously $N < \infty$). Then

$$\begin{aligned}\sum_{g\in G}\nu(gB(z,r)) &\le \sum_{g\in G}\nu\Big(\bigcup_{f\in G_z} gf\widetilde{X}\Big) \le \sum_{f\in G_z}\sum_{g\in G}\nu(gf\widetilde{X})\\ &= \sum_{f\in G_z}\nu\Big(\bigcup_{g\in G} gf\widetilde{X}\Big) = \sum_{f\in G_z}\nu(X) = N.\end{aligned} \tag{154}$$

Thus

$$\mathcal{F}(z,t) < \infty \tag{155}$$

uniformly in z. □

6.2.2. Gibbs measures. Let $\mathcal{G}$ be the collection of all finite subsets of G and denote by $\mathcal{G}(g)$ the family of all sets $\Lambda \in \mathcal{G}$ containing $g \in G$. Let us consider a family of potentials $\mathcal{U} = (U_\Lambda)_{\Lambda\in\mathcal{G}}$, $U_\Lambda \in C(X^\Lambda)$, satisfying the condition

$$\sum_{\Lambda\in\mathcal{G}(g)}\sup_{x\in\mathbf{X}}|U_\Lambda(x)| < \infty, \qquad g \in G. \tag{156}$$

Let μ be the Gibbs measure on $\mathbf{X}$ defined by the family $\mathcal{U}$ and the reference measure

$$\mathbf{v}(d\bar{\xi}) := \bigotimes_{g\in G}\nu(d\xi_g), \tag{157}$$

where ν is a probability measure on X. Heuristically μ can be given by the expression

$$\mu(d\bar{\xi}) = \frac{1}{Z} e^{-E(\bar{\xi})} \mathbf{v}(d\bar{\xi}), \qquad E(\bar{\xi}) = \sum_{\Lambda \in \mathcal{G}} U_\Lambda(\bar{\xi}). \tag{158}$$

We refer to the Appendix for the general definition of Gibbs measures on $\mathbf{X}$.

We assume that the following conditions hold:

(M1) μ is G-invariant (for this, it is sufficient to assume that the family $\mathcal{U}$ is G-invariant, that is, $U_{g\Lambda}(gx) = U_\Lambda(x)$ for all $g \in G,\ \Lambda \in \mathcal{G},\ x \in \mathbf{X}$);
(M2) ν has compact support (this, together with (156), guarantees the existence of μ, and will also be needed as a technical condition in what follows).

For a compact set $K \subset X$, define $G_K \subset G$ as the set of all $g \in G$ such that $g\widetilde{X} \cap K \neq \emptyset$. Let us remark that G_K is finite. We set

$$X_K := \bigcup_{g \in G_K} g\widetilde{X}. \tag{159}$$

Lemma 4. *μ satisfies condition* (C).

Proof. Let $S = \operatorname{supp}\mu$. Assume without loss of generality that $z \in \widetilde{X}$ and denote by $U(r)$ the r-neighborhood of $\widetilde{X}$. We have

$$\begin{aligned}\mathcal{F}(z,t) &:= \int_{X^G} \exp t f_{\bar{\xi}}(z)\, \mu(d\xi) = \int_{X^G} \exp t \sum_g F(z, g\xi_g)\, \mu(d\xi) \\ &= \int_{S^G} \exp t \sum_g F(z, g\xi_g)\, \mu(d\xi) = \int_{S^G} \exp t \sum_{g \in \widetilde{G}} F(z, g\xi_g)\, \mu(d\xi),\end{aligned} \tag{160}$$

where

$$\widetilde{G} = \{g \in G : X_{U(r)} \cap gX_S \neq \emptyset\}. \tag{161}$$

We have obviously

$$N := |\widetilde{G}| \leq |X_{U(r)}| \cdot |X_S|. \tag{162}$$

Then

$$\mathcal{F}(z,t) \leq \int_{S^G} e^{tN \sup \phi}\, \mu(d\xi) = e^{tN \sup \phi}. \tag{163}$$

$\square$

7. Appendix: Gibbs measures

7.1. Gibbs measures on configuration spaces

Here we briefly discuss the definition and some properties of Gibbs measures on Γ_X, associated with pair potentials. For a detailed exposition see e.g. [7].

A *pair potential* is a measurable symmetric function $\phi\colon X \times X \to \mathbb{R} \cup \{+\infty\}$. We will also suppose that $\phi(x, y) \in \mathbb{R}$ for $x \neq y$. For a compact set $\Lambda \subset X$, the conditional energy $E^{\phi}_{\Lambda}\colon \Gamma_X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$E^{\phi}_{\Lambda}(\gamma) := \begin{cases} \sum\limits_{\{x,y\}\subset\gamma,\, \{x,y\}\cap\Lambda\neq\emptyset} \phi(x, y) & \text{if } \sum\limits_{\{x,y\}\subset\gamma,\, \{x,y\}\cap\Lambda\neq\emptyset} |\phi(x, y)| < \infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{164}$$

Given Λ, we define for $\gamma \in \Gamma$ and $\Delta \in \mathrm{Bor}(\Gamma_X)$ (the Borel σ-algebra of Γ_X) the function

$$\begin{aligned} \Pi^{z,\phi}_{\Lambda}(\gamma, \Delta) := {} & \mathbf{1}_{\{Z^{z,\phi}_{\Lambda}<\infty\}}(\gamma)\, [Z^{z,\phi}_{\Lambda}(\gamma)]^{-1} \\ & \times \int_{\Gamma_X} \mathbf{1}_{\Delta}(\gamma_{\Lambda^c} + \gamma'_{\Lambda}) \exp[-E^{\phi}_{\Lambda}(\gamma_{\Lambda^c} + \gamma'_{\Lambda})]\, \pi_z(d\gamma'), \end{aligned} \tag{165}$$

where

$$Z^{z,\phi}_{\Lambda}(\gamma) := \int_{\Gamma_X} \exp[-E^{\phi}_{\Lambda}(\gamma_{\Lambda^c} + \gamma'_{\Lambda})]\, \pi_z(d\gamma'). \tag{166}$$

A probability measure μ on $(\Gamma_X, \mathrm{Bor}(\Gamma_X))$ is called a *grand canonical Gibbs measure* with interaction potential ϕ if it satisfies the Dobrushin–Lanford–Ruelle equation

$$\int_{\Gamma_X} \Pi^{z,\phi}_{\Lambda}(\gamma, \Delta)\mu(d\gamma) = \mu(\Delta) \tag{167}$$

for all compact subsets $\Lambda \subset X$ and $\Delta \in \mathrm{Bor}(\Gamma_X)$. Let $\mathcal{G}(z, \phi)$ denote the set of all such probability measures μ.

It can be shown [23] that the unique grand canonical Gibbs measure corresponding to the free case, $\phi = 0$, is the Poisson measure π_z.

We suppose that the interaction potential ϕ satisfies the following conditions:

(S) (*Stability*) There exists $B \geq 0$ such that, for any compact $\Lambda \subset X$ and for all $\gamma \in \Gamma_\Lambda$,

$$E^{\phi}_{\Lambda}(\gamma) := \sum_{\{x,y\}\subset\gamma} \phi(x, y) \geq -B|\gamma|. \tag{168}$$

(I) (*Integrability*) We have

$$C := \operatorname*{ess\,sup}_{x\in X} \int_X |e^{-\phi(x,y)} - 1|\, dy < \infty. \tag{169}$$

(F) (*Finite range*) There exists $R > 0$ such that

$$\phi(x, y) = 0 \quad \text{if } \rho(x, y) \geq R. \tag{170}$$

Theorem 10 ([24–26]).

(1) *Assume that* (S), (I), *and* (F) *hold, and let* $z > 0$ *be such that*

$$z < \frac{1}{2e}\,(e^{2B}C)^{-1}, \tag{171}$$

where B and C are as in (S) *and* (I), *respectively. Then, there exists a Gibbs measure* $\mu \in \mathcal{G}(z, \phi)$ *such that for any* $n \in \mathbb{N}$ *and any measurable symmetric function* $f^{(n)} : X^n \to [0, \infty]$, *we have*

$$\begin{aligned}\int_{\Gamma_X} \sum_{\{x_1,\dots,x_n\}\subset\gamma} & f^{(n)}(x_1,\dots,x_n)\,\mu(d\gamma) \\ &= \frac{1}{n!}\int_{(\mathbb{R}^d)^n} f^{(n)}(x_1,\dots,x_n)k_\mu^{(n)}(x_1,\dots,x_n)\,dx_1\cdots dx_n,\end{aligned} \tag{172}$$

where $k_\mu^{(n)}$ *is a non-negative measurable symmetric function on* $(\mathbb{R}^d)^n$, *called the n-th correlation function of the measure* μ, *and this function satisfies the following estimate*

$$\forall (x_1,\dots,x_n) \in (\mathbb{R}^d)^n : \quad k_\mu^{(n)}(x_1,\dots,x_n) \le a^n, \tag{173}$$

where $a > 0$ *is independent of n (the* Ruelle bound*).*

(2) *Let* ϕ *be a non-negative potential which satisfies* (I) *and* (F). *Then, for each* $z > 0$, *there exists a Gibbs measure* $\mu \in \mathcal{G}(z, \phi)$ *such that the correlation functions* $k_\mu^{(n)}$ *of the measure* μ *satisfy the Ruelle bound* (173).

Remark 10. Let us assume that the potential $\phi(x, y)$ has the form

$$\phi(x, y) = \Phi(\rho(x, y)) \tag{174}$$

and $\Phi \in C^2(\mathbb{R}^+ \to \mathbb{R}^+)$ is such that $\operatorname{supp}\Phi \subset [0, r]$, where $r > 0$ is the injectivity radius of X. Then conditions (S), (I) and (F) are satisfied. Thus, under the condition (171), the corresponding measure μ exists and satisfies conditions (i), (ii) of Section 6.1.1.

For $X = \mathbb{R}^d$, the existence of Gibbs measures satisfying the Ruelle bound is known for arbitrary $z > 0$ under the additional conditions of superstability and lower regularity (Ruelle measures [32]). We present two classical examples of potentials $\phi(x, y) = \Phi(x - y)$ satisfying these conditions.

Example 1 (Lennard–Jones type potentials). $\Phi \in C^2(\mathbb{R}^d \setminus \{0\})$, $\Phi \ge 0$ on $\mathbb{R}^d$, $\Phi(x) = c|x|^{-\alpha}$ for $x \in B(r_1)$, $\Phi(x) = 0$ for $x \in B(r_2)^c$, where $c > 0$, $\alpha > 0$, $0 < r_1 < r_2 < \infty$.

Example 2 (Lennard–Jones 6-12 potentials). $d = 3$, $\Phi(x) = c(|x|^{-12} - |x|^{-6})$, $c > 0$.

7.2. Gibbs measures on product manifolds

Let us recall the definition of the Gibbs measure on the Borel σ-algebra $\mathrm{Bor}(\mathbf{X})$, associated with $\mathcal{U}$. For any $\Lambda \in \mathcal{G}$ we introduce the energy of the interaction in the volume Λ with fixed boundary condition $\xi \in \mathbf{X}$ as

$$V_\Lambda(x_\Lambda|\xi) = \sum_{\Lambda' \cap \Lambda \neq \emptyset} U_{\Lambda'}(y), \tag{175}$$

where $y = (x_\Lambda, \xi_{\Lambda^c}) \in \mathbf{X}$, $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$. We define the Gibbs measure in the volume Λ with boundary condition ξ as the following measure on $\mathrm{Bor}(X^\Lambda)$:

$$d\mu_\Lambda(x_\Lambda|\xi) = \frac{1}{Z_\Lambda(\xi)} e^{-V_\Lambda(x_\Lambda|\xi)}\, dx_\Lambda, \tag{176}$$

where $dx = \bigotimes_{k\in\Lambda} dx_k$ is the product of the Riemannian volume measures dx_k on X_k and

$$Z_\Lambda(\xi) = \int_{M^\Lambda} e^{-V_\Lambda(x_\Lambda|\xi)}\, dx_\Lambda. \tag{177}$$

These measures are well-defined for any finite volume Λ and all boundary conditions $\xi \in \mathbf{X}$.

$\mathrm{Bor}(\mathbf{X})$ is called a *Gibbs measure* (for given $\mathcal{U}$) if

$$\int \mathsf{E}_\Lambda f\, d\mu = \int f\, d\mu \tag{178}$$

for each $\Lambda \in \mathcal{G}$ and any continuous cylinder function f on $\mathbf{X}$, where

$$(\mathsf{E}_\Lambda f)(\xi) = \int f(x_\Lambda, \xi_{\Lambda^c})\, d\mu_\Lambda(x_\Lambda|\xi). \tag{179}$$

Remark 11. Condition (178) is equivalent to the assumption that $\mu_\Lambda(\cdot|\xi)$ is the conditional measure associated with μ under the condition ξ_{Λ^c}.

Remark 12. Heuristically μ can be given by the expression

$$d\mu(x) = \frac{1}{Z} e^{-E(x)}\, dx, \qquad E(x) = \sum_{\Lambda\in\mathcal{G}} U_\Lambda(x), \tag{180}$$

where $dx = \bigotimes_k dx_k$ is the product of the Riemannian volume measures on X_k.

Let $\mathrm{Gibbs}(\mathcal{U})$ be the family of all such Gibbs measures. If X is compact, $\mathrm{Gibbs}(\mathcal{U})$ is non-empty under the condition (156) (see e.g. [23], [20]).

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