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Scales, fields, and a problem of Hurewicz

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Abstract. Menger's basis property is a generalization of σ -compactness and admits an elegant combinatorial interpretation. We introduce a general combinatorial method to construct non- σ -compact sets of reals with Menger's property. Special instances of these constructions give known counterexamples to conjectures of Menger and Hurewicz. We obtain the first explicit solution to the Hurewicz 1927 problem, that was previously solved by Chaber and Pol on a dichotomic basis.

The constructed sets generate nontrivial subfields of the real line with strong combinatorial properties, and most of our results can be stated in a Ramsey-theoretic manner.

Since we believe that this paper is of interest to a diverse mathematical audience, we have made a special effort to make it self-contained and accessible.

Keywords. Menger property, Hurewicz property, filter covers, topological groups, selection principles

Whenever you can settle a question by explicit construction, be not satisfied with purely existential arguments.

Hermann Weyl, Princeton Conference 1946

1. Introduction and summary

Menger's property (1924) is a generalization of σ -compactness. Menger conjectured that his property actually characterizes σ -compactness. Hurewicz found an elegant combinatorial interpretation of Menger's property, and introduced a formally stronger property (1925, 1927). Hurewicz's property is also implied by σ -compactness, and Hurewicz conjectured that his formally stronger property characterizes σ -compactness. He posed the question whether his property is *strictly* stronger than Menger's. We will call this question the *Hurewicz problem*.

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In Section 2 we define the Menger and Hurewicz properties, and show that they are extremal cases of a large family of properties. We treat this family in a unified manner and obtain, using a combinatorial approach, many counterexamples to the above mentioned conjectures of Menger and Hurewicz.

In Section 3 we show that a theorem of Chaber and Pol implies a positive solution to the Hurewicz problem. In fact, it establishes the existence of a set of reals X without the Hurewicz property, such that all finite powers of X have the Menger property. However, this solution does not point out a concrete example. We construct a *concrete* set having Menger's but not Hurewicz's property, yielding a more elegant and direct solution.

Chaber and Pol's proof is topological. In Section 4 we show how to obtain Chaber and Pol's result and extensions of it using the combinatorial approach. In Section 6 we use these results to generate fields (in the algebraic sense) which are counterexamples to the Hurewicz and Menger Conjectures and examples for the Hurewicz problem. In Section 7 it is shown that some of our examples are very small, both in the sense of measure and in the sense of category.

Section 8 reveals the underlying connections with the field of selection principles, where our main results are extended further. In Section 9 we explain how to extend some of the results further, and in Section 10 we translate our results into the language of Ramsey theory, and indicate an application to the undecidable notion of strong measure zero.

2. The Menger property

2.1. Menger's property and bounded images

In 1924 Menger introduced the following basis property for a metric space X [25]:

For each basis \mathcal{B} of X , there exists a sequence $\{B_n\}_{n \in \mathbb{N}}$ in \mathcal{B} such that $\lim_{n \rightarrow \infty} \text{diam}(B_n) = 0$ and $X = \bigcup_n B_n$.

Each σ -compact metric space has this property, and Menger conjectured that this property characterizes σ -compactness. The task of disproving this conjecture without special hypotheses was first achieved in Fremlin and Miller's 1988 paper [14], alas in an existential manner. Concrete counterexamples were given much later [4]. In Section 2.3, we describe a general method to produce counterexamples to this conjecture.

In 1927 Hurewicz obtained the following characterization of Menger's property. Let \mathbb{N} denote the (discrete) space of natural numbers, including 0, and endow the *Baire space* $\mathbb{N}^{\mathbb{N}}$ with the Tikhonov product topology. Define a partial order¹ \leq^* on $\mathbb{N}^{\mathbb{N}}$ by

$$f \leq^* g \quad \text{if} \quad f(n) \leq g(n) \text{ for all but finitely many } n.$$

A subset D of $\mathbb{N}^{\mathbb{N}}$ is *dominating* if for each $g \in \mathbb{N}^{\mathbb{N}}$ there exists $f \in D$ such that $g \leq^* f$.

¹ By *partial order* we mean a reflexive and transitive relation. We do not require its being anti-symmetric.

Theorem 2.1 (Hurewicz [18]). *A set of reals X has Menger's property if, and only if, no continuous image of X in $\mathbb{N}^{\mathbb{N}}$ is dominating.*

Menger's property is a specific instance of a general scheme of properties.

Definition 2.2. *For $A, B \subseteq \mathbb{N}$, $A \subseteq^* B$ means that $A \setminus B$ is finite. Let $[\mathbb{N}]^{\aleph_0}$ denote the collection of all infinite sets of natural numbers. A nonempty family $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ is a semifilter if for each $A \in \mathcal{F}$ and each $B \subseteq \mathbb{N}$ such that $A \subseteq^* B$, also $B \in \mathcal{F}$. (Note that all elements of \mathcal{F} are infinite, and \mathcal{F} is closed under finite modifications of its elements.) \mathcal{F} is a filter if it is a semifilter and it is closed under finite intersections (this is often called a free filter). For $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ and $f, g \in \mathbb{N}^{\mathbb{N}}$, define*

$$[f \leq g] = \{n : f(n) \leq g(n)\}, \quad f \leq_{\mathcal{F}} g \quad \text{if} \quad [f \leq g] \in \mathcal{F}.$$

Fix a semifilter \mathcal{F} . A set of reals X satisfies $\mathbf{B}(\mathcal{F})$ if each continuous image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded with respect to $\leq_{\mathcal{F}}$, that is, there is $g \in \mathbb{N}^{\mathbb{N}}$ such that for each f in the image of X , $f \leq_{\mathcal{F}} g$.

Thus, Menger's property is the same as $\mathbf{B}([\mathbb{N}]^{\aleph_0})$, and it is the weakest among the properties $\mathbf{B}(\mathcal{F})$ where \mathcal{F} is a semifilter.

Hurewicz also considered the following property (the *Hurewicz property*) [17]: Each continuous image of X in $\mathbb{N}^{\mathbb{N}}$ is bounded with respect to \leq^* . This is also a special case of $\mathbf{B}(\mathcal{F})$, obtained when \mathcal{F} is the *Fréchet filter* consisting of all cofinite sets of natural numbers. The Hurewicz property is the strongest among the properties $\mathbf{B}(\mathcal{F})$ where \mathcal{F} is a semifilter, and Hurewicz conjectured that it characterizes σ -compactness. This was first disproved by Just, Miller, Scheepers and Szeptycki in [19], and will also follow from the results below.

The following is easy to verify.

Lemma 2.3. *For each semifilter \mathcal{F} , $\mathbf{B}(\mathcal{F})$ is preserved by continuous images and is hereditary for closed subsets.* \square

This allows us to work in any separable, zero-dimensional metric space instead of working in \mathbb{R} . For brevity, we will refer to any space of this kind as a *set of reals*. We consider several canonical spaces which carry a convenient combinatorial structure.

2.2. The many faces of the Baire space and the Cantor space

The Baire space $\mathbb{N}^{\mathbb{N}}$ and the *Cantor space* $\{0, 1\}^{\mathbb{N}}$ are equipped with the product topology. These spaces will appear under various guises in this paper, according to the required combinatorial structure. $P(\mathbb{N})$, the collection of all subsets of \mathbb{N} , is identified with $\{0, 1\}^{\mathbb{N}}$ via characteristic functions, and inherits its topology (so that by definition $P(\mathbb{N})$ and $\{0, 1\}^{\mathbb{N}}$ are homeomorphic). $[\mathbb{N}]^{\aleph_0}$ is a subspace of $P(\mathbb{N})$ and is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. In turn, $[\mathbb{N}]^{\aleph_0}$ is homeomorphic to its subspace $[\mathbb{N}]^{(\aleph_0, \aleph_0)}$ consisting of the infinite coinfinite sets of natural numbers. Similarly, $\mathbb{N}^{\uparrow \mathbb{N}}$, the collection of all *increasing* elements of $\mathbb{N}^{\mathbb{N}}$, is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

The following compactification of $\mathbb{N}^{\uparrow\mathbb{N}}$ appears in [3]: Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of \mathbb{N} . Let $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ be the collection of all nondecreasing elements f of $\overline{\mathbb{N}}^{\mathbb{N}}$ such that $f(n) < f(n + 1)$ whenever $f(n) < \infty$. For each nondecreasing finite sequence s of natural numbers, define $q_s \in \overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ by $q_s(n) = s(n)$ if $n < |s|$, and $q_s(n) = \infty$ otherwise. Let Q be the collection of all these elements q_s . Then Q is dense in $\overline{\mathbb{N}}^{\uparrow\mathbb{N}} = Q \cup \mathbb{N}^{\uparrow\mathbb{N}}$. $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ is another guise of the Cantor space. Let $[\mathbb{N}]^{<\aleph_0}$ denote the finite subsets of \mathbb{N} .

Lemma 2.4. Define $\Psi : \overline{\mathbb{N}}^{\uparrow\mathbb{N}} \rightarrow P(\mathbb{N})$ by

$$\Psi(f) = \begin{cases} \text{im}(f), & f \in \mathbb{N}^{\uparrow\mathbb{N}}, \\ \text{im}(s), & f = q_s \in Q, \end{cases}$$

(in short, $\Psi(f) = \text{im}(f) \setminus \{\infty\}$). Then Ψ is a homeomorphism mapping Q onto $[\mathbb{N}]^{<\aleph_0}$ and $\mathbb{N}^{\uparrow\mathbb{N}}$ onto $[\mathbb{N}]^{\aleph_0}$. □

Lemma 2.4 says that we can identify sets of natural numbers with their increasing enumerations, and obtain $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ (where a finite increasing sequence s is identified with q_s). This identification will be used throughout the paper. When using it, we will denote elements of $[\mathbb{N}]^{\aleph_0}$ by lowercase letters to indicate that we are also treating them as increasing functions. (Otherwise, we use uppercase letters.) E.g., for $a, b \in [\mathbb{N}]^{\aleph_0}$, $a \leq_{\mathcal{F}} b$ is an assertion concerning the increasing enumerations of a and b . Similarly for \leq^* , $\not\leq^*$, etc. Also, $\min\{a, b\}$ denotes the function $f(n) = \min\{a(n), b(n)\}$ for each n , and similarly for $\max\{a, b\}$, etc.

We will need the following lemma from [4]. For the reader’s convenience, we reproduce its proof.

Lemma 2.5. Assume that $Q^k \subseteq X^k \subseteq (\overline{\mathbb{N}}^{\uparrow\mathbb{N}})^k$, and $\Psi : X^k \rightarrow \mathbb{N}^{\mathbb{N}}$ is continuous on Q^k . Then there exists $g \in \mathbb{N}^{\mathbb{N}}$ such that for all $x_1, \dots, x_k \in X$,

$$[g < \min\{x_1, \dots, x_k\}] \subseteq [\Psi(x_1, \dots, x_k) \leq g].$$

Proof. For each $A \subseteq \overline{\mathbb{N}}^{\uparrow\mathbb{N}}$, let $A \upharpoonright n = \{x \upharpoonright n : x \in A\}$. For each n , let $\overline{\mathbb{N}}^{\uparrow n} = \overline{\mathbb{N}}^{\uparrow\mathbb{N}} \upharpoonright n$. For $\sigma \in \overline{\mathbb{N}}^{\uparrow n}$, write q_σ for $q_{\sigma \upharpoonright m}$ where $m = 1 + \max\{i < n : \sigma(i) < \infty\}$.

If $\sigma \in \overline{\mathbb{N}}^{\uparrow n}$ and I is a basic open neighborhood of q_σ , then there exists a natural number N such that for each $x \in \overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ with $x \upharpoonright n \in I \upharpoonright n$ and $x(n) > N$, $x \in I$.

Fix n . Use the continuity of Ψ on Q^k to choose, for each $\vec{\sigma} = (\sigma_1, \dots, \sigma_k) \in (\overline{\mathbb{N}}^{\uparrow n})^k$, a basic open neighborhood

$$I_{\vec{\sigma}} = I_{\sigma_1} \times \dots \times I_{\sigma_k} \subseteq (\overline{\mathbb{N}}^{\uparrow\mathbb{N}})^k$$

of $q_{\vec{\sigma}} = (q_{\sigma_1}, \dots, q_{\sigma_k})$ such that for all $(x_1, \dots, x_k) \in I_{\vec{\sigma}} \cap X^k$, $\Psi(x_1, \dots, x_k)(n) = \Psi(q_{\vec{\sigma}})(n)$. For each $i = 1, \dots, k$, choose N_i such that for all $x \in \overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ with $x \upharpoonright n \in I_{\sigma_i} \upharpoonright n$ and $x(n) > N_i$, $x \in I_{\sigma_i}$. Define $N(\vec{\sigma}) = \max\{N_1, \dots, N_k\}$.

The set $I_{\vec{\sigma}}^{(n)} = \{(x_1 \upharpoonright n, \dots, x_k \upharpoonright n) : (x_1, \dots, x_k) \in I_{\vec{\sigma}}\}$ is open in $(\overline{\mathbb{N}}^{\uparrow n})^k$ and the family $\{I_{\vec{\sigma}}^{(n)} : \vec{\sigma} \in (\overline{\mathbb{N}}^{\uparrow n})^k\}$ is a cover of the compact space $(\overline{\mathbb{N}}^{\uparrow n})^k$. Take a finite subcover $\{I_{\vec{\sigma}_1}^{(n)}, \dots, I_{\vec{\sigma}_m}^{(n)}\}$ of $(\overline{\mathbb{N}}^{\uparrow n})^k$. Let $N = \max\{N(\vec{\sigma}_1), \dots, N(\vec{\sigma}_m)\}$, and define

$$g(n) = \max\{N, \Psi(q_{\vec{\sigma}_1})(n), \dots, \Psi(q_{\vec{\sigma}_m})(n)\}.$$

For all $x_1, \dots, x_k \in X$, let i be such that $(x_1 \upharpoonright n, \dots, x_k \upharpoonright n) \in I_{\vec{\sigma}_i}^{(n)}$. If $x_1(n), \dots, x_k(n) > N$, then $\Psi(x_1, \dots, x_k)(n) = \Psi(q_{\vec{\sigma}_i})(n) \leq g(n)$. \square

2.3. Sets of reals satisfying $\mathbf{B}(\mathcal{F})$

Definition 2.6. For a semifilter \mathcal{F} , let $\mathfrak{b}(\mathcal{F})$ denote the minimal cardinality of a family $Y \subseteq \mathbb{N}^{\mathbb{N}}$ which is unbounded with respect to $\leq_{\mathcal{F}}$.

The most well known instances of Definition 2.6 are $\mathfrak{d} = \mathfrak{b}([\mathbb{N}]^{\aleph_0})$ (the minimal cardinality of a dominating family), and $\mathfrak{b} = \mathfrak{b}(\mathcal{F})$ where \mathcal{F} is the Fréchet filter (the minimal cardinality of an unbounded family with respect to \leq^*). For a collection (or property) \mathcal{I} of sets of reals, the *critical cardinality* of \mathcal{I} is

$$\text{non}(\mathcal{I}) = \min\{|X| : X \subseteq \mathbb{R} \text{ and } X \notin \mathcal{I}\}.$$

Lemma 2.7. For each semifilter \mathcal{F} , $\text{non}(\mathbf{B}(\mathcal{F})) = \mathfrak{b}(\mathcal{F})$. \square

The following notion is our basic building block.

Definition 2.8. $S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\}$ is a $\mathfrak{b}(\mathcal{F})$ -scale if $S \subseteq \mathbb{N}^{\mathbb{N}}$, S is unbounded with respect to $\leq_{\mathcal{F}}$, and for any $\alpha < \beta < \mathfrak{b}(\mathcal{F})$, $f_\alpha \leq_{\mathcal{F}} f_\beta$.

Lemma 2.9. For each semifilter \mathcal{F} , there exists a $\mathfrak{b}(\mathcal{F})$ -scale.

Proof. Let $B = \{b_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\} \subseteq \mathbb{N}^{\mathbb{N}}$ be unbounded with respect to $\leq_{\mathcal{F}}$. By induction on $\alpha < \mathfrak{b}(\mathcal{F})$, let g be a witness that $\{f_\beta : \beta < \alpha\}$ is bounded with respect to $\leq_{\mathcal{F}}$, and take $f_\alpha = \max\{b_\alpha, g\}$. Then $S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\}$ is a $\mathfrak{b}(\mathcal{F})$ -scale. \square

For a semifilter \mathcal{F} , define

$$\mathcal{F}^+ = \{A \subseteq \mathbb{N} : A^c \notin \mathcal{F}\}.$$

Note that $\mathcal{F}^{++} = \mathcal{F}$.

Remark 2.10. Let R be a binary relation on a set P . A subset S of P is *cofinal* with respect to R if for each $p \in P$ there is $s \in S$ such that pRs . A transfinite sequence $\{p_\alpha : \alpha < \kappa\}$ in P is *nondecreasing* with respect to R if $p_\alpha R p_\beta$ for all $\alpha \leq \beta$.

Recall that a set of reals X is *meager* (has Baire *first category*) if it is a countable union of nowhere dense sets. Since the autohomeomorphism of $P(\mathbb{N})$ defined by $A \mapsto A^c$ carries \mathcal{F}^+ to $\mathcal{F}^c = P(\mathbb{N}) \setminus \mathcal{F}$, we see that \mathcal{F} is meager if, and only if, \mathcal{F}^+ is comeager.

Lemma 2.11. *Assume that \mathcal{F} is a semifilter. Then $A \in \mathcal{F}^+$ if, and only if, $A \cap B$ is infinite for each $B \in \mathcal{F}$.*

Proof. (\Leftarrow) Assume that $A \cap B$ is infinite for each $B \in \mathcal{F}$. Since $A \cap A^c = \emptyset$, necessarily $A^c \notin \mathcal{F}$.

(\Rightarrow) If $B \in \mathcal{F}$ and $A \cap B$ is finite, then $B \subseteq^* A^c$; thus $A^c \in \mathcal{F}$. \square

Definition 2.12. *For a semifilter \mathcal{F} and $A \in \mathcal{F}^+$, define*

$$\mathcal{F} \upharpoonright A = \{B \cap A : B \in \mathcal{F}\}, \quad \mathcal{F}_A = \{C \subseteq \mathbb{N} : (\exists B \in \mathcal{F}) B \cap A \subseteq C\}.$$

Lemma 2.13. *For each semifilter \mathcal{F} and each $A \in \mathcal{F}^+$,*

- (1) \mathcal{F}_A is the smallest semifilter containing $\mathcal{F} \upharpoonright A$.
- (2) $\mathcal{F} \subseteq \mathcal{F}_A$, and if \mathcal{F} is a filter, then $\mathcal{F}_A \subseteq \mathcal{F}^+$. \square

Theorem 2.14. *Assume that \mathcal{F} is a semifilter, and $S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\}$ is a $\mathfrak{b}(\mathcal{F})$ -scale. Let $X = S \cup Q$. Then for each continuous $\Psi : X \rightarrow \mathbb{N}^{\mathbb{N}}$, there exists $A \in \mathcal{F}^+$ such that $\Psi[X]$ is bounded with respect to $\leq_{\mathcal{F}_A}$.*

Proof. Let $g \in \mathbb{N}^{\mathbb{N}}$ be as in Lemma 2.5. Since S is unbounded with respect to $\leq_{\mathcal{F}}$, there exists $\alpha < \mathfrak{b}(\mathcal{F})$ such that $f_\alpha \not\leq_{\mathcal{F}} g$, that is, $A := [g < f_\alpha] \in \mathcal{F}^+$. For each $\beta \geq \alpha$, $[f_\alpha \leq f_\beta] \in \mathcal{F}$. By Lemma 2.5,

$$[\Psi(f_\beta) \leq g] \supseteq [g < f_\beta] \supseteq A \cap [f_\alpha \leq f_\beta] \in \mathcal{F} \upharpoonright A.$$

Let $Y = \Psi[\{f_\beta : \beta < \alpha\} \cup Q]$. Since $|Y| < \mathfrak{b}(\mathcal{F})$, Y is $\leq_{\mathcal{F}}$ -bounded by some $h \in \mathbb{N}^{\mathbb{N}}$, and we may require that $[g \leq h] = \mathbb{N}$. Then for each $x \in X$, $\Psi(x) \leq_{\mathcal{F}_A} h$. \square

Corollary 2.15. *In the notation of Theorem 2.14, if \mathcal{F} is a filter, then X satisfies $\mathbf{B}(\mathcal{F}^+)$. \square*

In many cases (including the classical ones), Theorem 2.14 implies the stronger assertion that X satisfies $\mathbf{B}(\mathcal{F})$.

Corollary 2.16. *In the notation of Theorem 2.14, suppose that:*

- (1) \mathcal{F} is an ultrafilter, or
- (2) $\mathcal{F} = [\mathbb{N}]^{\aleph_0}$ (Menger property), or
- (3) \mathcal{F} is the Fréchet filter (Hurewicz property).

Then X satisfies $\mathbf{B}(\mathcal{F})$.

Proof. (1) If \mathcal{F} is an ultrafilter, then $\mathcal{F}^+ = \mathcal{F}$, and by Corollary 2.15, X satisfies $\mathbf{B}(\mathcal{F})$.

(2) If $\mathcal{F} = [\mathbb{N}]^{\aleph_0}$, then for each $A \in \mathcal{F}^+$, A is cofinite and therefore $\mathcal{F}_A = \mathcal{F}$, so X satisfies $\mathbf{B}(\mathcal{F})$.

(3) If \mathcal{F} is the Fréchet filter, then each continuous image of X in $\mathbb{N}^{\mathbb{N}}$ is \leq^* -bounded when restricted to the infinite set A . To complete the proof, we make the following easy observations.

Lemma 2.17. *The mapping $\Psi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\uparrow\mathbb{N}}$ defined by $\Psi(f)(n) = n + f(0) + \dots + f(n)$ is a homeomorphism and preserves \leq^* -unboundedness. \square*

Lemma 2.18. *If a subset of $\mathbb{N}^{\uparrow\mathbb{N}}$ is \leq^* -bounded when restricted to some infinite set $a \subseteq \mathbb{N}$, then it is \leq^* -bounded.*

Proof. If $a \subseteq^* [f \leq g]$ for each $f \in Y$ and g is increasing, then $f \leq^* f \circ a \leq^* g \circ a$ for each $f \in Y$. \square

It follows that each continuous image of X is \leq^* -bounded, so X satisfies $\mathbf{B}(\mathcal{F})$, completing the proof of Corollary 2.16. \square

Items (2) and (3) in Corollary 2.16 were first proved in [4], using two specialized proofs.

None of the examples provided by Theorem 2.14 is trivial: Each of them is a counterexample to the Menger conjecture, and some of them are counterexamples to the Hurewicz conjecture (see also Section 2.5). Let κ be an infinite cardinal. A set of reals X is κ -concentrated on a set Q if, for each open set U containing Q , $|X \setminus U| < \kappa$. Recall that a set of reals is *perfect* if it is nonempty, closed, and has no isolated points.

Lemma 2.19. *Assume that a set of reals X is \mathfrak{c} -concentrated on a countable set Q . Then X does not contain a perfect set.*

Proof. Assume that X contains a perfect set P . Then $P \setminus Q$ is Borel and uncountable, and thus contains a perfect set C . Then $U = \mathbb{R} \setminus C$ is open and contains Q , and $C = P \setminus U \subseteq X \setminus U$ has cardinality \mathfrak{c} . Thus, X is not \mathfrak{c} -concentrated on Q . \square

Theorem 2.20. *In the notation of Theorem 2.14, X does not contain a perfect subset. In particular, X is not σ -compact.*

Proof. If U is an open set containing Q , then $K = (\overline{\mathbb{N}^{\uparrow\mathbb{N}}}) \setminus U$ is a closed and therefore compact subset of $\overline{\mathbb{N}^{\uparrow\mathbb{N}}}$. Thus, K is a compact subset of $\mathbb{N}^{\uparrow\mathbb{N}}$, and therefore it is bounded with respect to \leq^* . Thus, $|S \cap K| < \mathfrak{b}(\mathcal{F})$. This shows that X is $\mathfrak{b}(\mathcal{F})$ -concentrated (in particular, \mathfrak{c} -concentrated) on Q . Use Lemma 2.19. \square

Remark 2.21. In fact, the proof of Theorem 2.20 gives more: Since $\mathfrak{b}(\mathcal{F}) \leq \mathfrak{d}$, X is \mathfrak{d} -concentrated on Q , and therefore [4] X has the property $\mathbf{S}_1(\Gamma, \mathcal{O})$ defined in [19] (see the forthcoming Section 8). By [19], $\mathbf{S}_1(\Gamma, \mathcal{O})$ is preserved under continuous images and implies that there are no perfect subsets. It follows that no continuous image of X contains a perfect subset.

2.4. Cofinal scales

In some situations the following is useful.

Definition 2.22. *For a semifilter \mathcal{F} , we say that $S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\}$ is a cofinal $\mathfrak{b}(\mathcal{F})$ -scale if:*

- (1) *For all $\alpha < \beta < \mathfrak{b}(\mathcal{F})$, $f_\alpha \leq_{\mathcal{F}} f_\beta$.*
- (2) *For each $g \in \mathbb{N}^{\mathbb{N}}$, there is $\alpha < \mathfrak{b}(\mathcal{F})$ such that for each $\beta \geq \alpha$, $g \leq_{\mathcal{F}} f_\beta$.*

If $\mathcal{F} \subseteq \mathcal{F}^+$, then every cofinal $\mathfrak{b}(\mathcal{F})$ -scale is a $\mathfrak{b}(\mathcal{F})$ -scale. If \mathcal{F}^+ is a filter, then every $\mathfrak{b}(\mathcal{F})$ -scale is a cofinal $\mathfrak{b}(\mathcal{F})$ -scale. In particular, if \mathcal{F} is an ultrafilter, then the two notions coincide.

Lemma 2.23. *Assume that \mathcal{F} is a semifilter and $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$. Then there exists a cofinal $\mathfrak{b}(\mathcal{F})$ -scale.*

Proof. Fix a dominating family $\{d_\alpha : \alpha < \mathfrak{d}\} \subseteq \mathbb{N}^{\mathbb{N}}$. At step $\alpha < \mathfrak{d}$, choose $f_\alpha \in \mathbb{N}^{\mathbb{N}}$ which is an upper bound of $\{f_\beta, d_\beta : \beta < \alpha\}$ with respect to $\leq_{\mathcal{F}}$ (this is possible because $\mathfrak{d} = \mathfrak{b}(\mathcal{F})$). Take $S = \{f_\alpha : \alpha < \mathfrak{d}\}$.

Let $g \in \mathbb{N}^{\mathbb{N}}$. Take $\alpha < \mathfrak{d}$ such that $g \leq^* d_\alpha$. For each $\beta \geq \alpha$, $g \leq^* d_\alpha \leq_{\mathcal{F}} f_\beta$, and therefore $g \leq_{\mathcal{F}} f_\beta$. \square

Theorem 2.24. *Assume that \mathcal{F} is a semifilter. Then for each cofinal $\mathfrak{b}(\mathcal{F})$ -scale $S = \{f_n : n < \mathfrak{b}(\mathcal{F})\}$, $X = S \cup Q$ satisfies $\mathbf{B}(\mathcal{F})$.*

Proof. Assume that $\Psi : X \rightarrow \mathbb{N}^{\mathbb{N}}$ is continuous. Let $g \in \mathbb{N}^{\mathbb{N}}$ be as in Lemma 2.5. Take $\alpha < \mathfrak{b}(\mathcal{F})$ such that for each $\beta \geq \alpha$, $g \leq_{\mathcal{F}} f_\beta$. Then for each $\beta \geq \alpha$, $\Psi(f_\beta) \leq_{\mathcal{F}} g$.

The cardinality of $\Psi[\{f_\beta : \beta \leq \alpha\} \cup Q]$ is smaller than $\mathfrak{b}(\mathcal{F})$, and is therefore bounded with respect to $\leq_{\mathcal{F}}$. It follows that $\Psi[X]$ is bounded with respect to $\leq_{\mathcal{F}}$. \square

2.5. Many counterexamples to the Hurewicz conjecture

Recall that Hurewicz conjectured that for sets of reals, the Hurewicz property is equivalent to σ -compactness. In the previous section we gave one type of counterexample, derived from a $\mathfrak{b}(\mathcal{F})$ -scale where \mathcal{F} is the Fréchet filter. We extend this construction to a family of semifilters.

Definition 2.25. *A family $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ is feeble if there exists $h \in \mathbb{N}^{\mathbb{N}}$ such that for each $A \in \mathcal{F}$, $A \cap [h(n), h(n+1)) \neq \emptyset$ for all but finitely many n .*

By a result of Talagrand (see [1, 5.4.1]), a semifilter \mathcal{F} is feeble if, and only if, it is a meager subset of $[\mathbb{N}]^{\aleph_0}$.

Lemma 2.26. *Assume that $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ is feeble. If $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is bounded with respect to $\leq_{\mathcal{F}}$, then Y is bounded with respect to \leq^* .*

Proof. Take $h \in \mathbb{N}^{\mathbb{N}}$ witnessing that \mathcal{F} is feeble, and $g \in \mathbb{N}^{\mathbb{N}}$ witnessing that Y is bounded with respect to $\leq_{\mathcal{F}}$. Define $\tilde{g} \in \mathbb{N}^{\mathbb{N}}$ by $\tilde{g}(k) = g(h(n+2))$ for each $k \in [h(n), h(n+1))$. It is easy to see that for each $f \in Y$, $f \leq^* \tilde{g}$. \square

Corollary 2.27. *Assume that \mathcal{F} is a feeble semifilter. Then $\mathfrak{b}(\mathcal{F}) = \mathfrak{b}$.* \square

Theorem 2.28. *Assume that \mathcal{F} is a feeble semifilter, and $S = \{f_\alpha : \alpha < \mathfrak{b}\}$ is a $\mathfrak{b}(\mathcal{F})$ -scale. Then $X = S \cup Q$ has the Hurewicz property.*

Proof. This is a careful modification of the proof of Theorem 2.14. Let $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ witness the feebleness of \mathcal{F} . Assume that $\Psi : X \rightarrow \mathbb{N}^{\mathbb{N}}$ is continuous. We may assume that all elements in $\Psi[X]$ are increasing (see Lemma 2.17). Let $g \in \mathbb{N}^{\uparrow\mathbb{N}}$ be as in Lemma 2.5. Define $\tilde{g} \in \mathbb{N}^{\uparrow\mathbb{N}}$ by $\tilde{g}(k) = g(h(n+2))$ for each $k \in [h(n), h(n+1))$.

As S is unbounded with respect to $\leq_{\mathcal{F}}$, there exists $\alpha < \mathfrak{b}$ such that $A := [\tilde{g} < f_{\alpha}] \in \mathcal{F}^+$. In particular, A is infinite. Let $C = \{n : A \cap [h(n-1), h(n)) \neq \emptyset\}$. For each $\beta \geq \alpha$, $[f_{\alpha} \leq f_{\beta}] \in \mathcal{F}$. For all but finitely many $n \in C$, there are $m \in [f_{\alpha} \leq f_{\beta}] \cap [h(n), h(n+1))$ and $l \in A \cap [h(n-1), h(n))$, and therefore

$$g(h(n+1)) = \tilde{g}(l) < f_{\alpha}(l) \leq f_{\alpha}(m) \leq f_{\beta}(m) \leq f_{\beta}(h(n+1)).$$

In particular, $[g < f_{\beta}] \cap [h(n+1), h(n+2)) \neq \emptyset$. By Lemma 2.5,

$$[\Psi(f_{\beta}) \leq g] \supseteq [g < f_{\beta}] \supseteq^* \{h(n+1) : n \in C\}.$$

Thus, $Y = \{\Psi(f_{\beta}) : \beta \geq \alpha\}$ is \leq^* -bounded on an infinite set and therefore \leq^* -bounded.

Let $Z = \Psi[\{f_{\beta} : \beta < \alpha\} \cup Q]$. Since $|Z| < \mathfrak{b}$, Z is \leq^* -bounded, and therefore $\Psi[X] = Y \cup Z$ is \leq^* -bounded. \square

By Theorem 2.20, each of the sets X of Theorem 2.28 is a counterexample to the Hurewicz conjecture.

2.6. Coherence classes

We make some order in the large family of properties of the form $\mathbf{B}(\mathcal{F})$.

Definition 2.29. For $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ and $A \subseteq \mathbb{N}$ let

$$\text{cl}_h(A) = \bigcup \{[h(n), h(n+1)) : A \cap [h(n), h(n+1)) \neq \emptyset\}.$$

A semifilter \mathcal{S} is strictly subcoherent to a semifilter \mathcal{F} if there exists $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ such that for each $A \in \mathcal{S}$, $\text{cl}_h(A) \in \mathcal{F}$ (equivalently, there is a monotone surjection $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\{\varphi[A] : A \in \mathcal{S}\} \subseteq \{\varphi[A] : A \in \mathcal{F}\}$). \mathcal{S} is strictly coherent to \mathcal{F} if each of them is strictly subcoherent to the other.

The Fréchet filter is strictly subcoherent to any semifilter, so that a semifilter is feeble exactly when it is strictly coherent to the Fréchet filter.

Lemma 2.30. Each comeager semifilter \mathcal{S} is strictly coherent to $[\mathbb{N}]^{\aleph_0}$.

Proof. Clearly, any semifilter is strictly subcoherent to $[\mathbb{N}]^{\aleph_0}$. We prove the other direction. Since \mathcal{S}^+ is homeomorphic to \mathcal{S}^c , it is meager and thus feeble. Let $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ be a witness for that. Fix $A \in [\mathbb{N}]^{\aleph_0}$ and let $B = \text{cl}_h(A)$. Then $B^c \notin \mathcal{S}^+$, and therefore $B \in (\mathcal{S}^+)^+ = \mathcal{S}$. \square

Lemma 2.31. Let $h \in \mathbb{N}^{\uparrow\mathbb{N}}$. Define a mapping $\Phi_h : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ by $\Phi_h(f) = \tilde{f}$, where for each n and each $k \in [h(n), h(n+1))$,

$$\tilde{f}(k) = \max\{f(i) : i \in [h(n), h(n+1))\}.$$

Then:

- (1) Φ_h is continuous.
- (2) For each $f \in \mathbb{N}^{\mathbb{N}}$, if $\tilde{f} = \Phi_h(f)$, then $[f \leq \tilde{f}] = \mathbb{N}$.
- (3) For each $f, g \in \mathbb{N}^{\mathbb{N}}$, if $\tilde{f} = \Phi_h(f)$, $\tilde{g} = \Phi_h(g)$, and $A = [\tilde{f} \leq \tilde{g}]$, then $A = \text{cl}_h(A)$. \square

Theorem 2.32. Assume that \mathcal{S} and \mathcal{F} are semifilters such that \mathcal{S} is strictly subcoherent to \mathcal{F} . Then $\mathbf{B}(\mathcal{S})$ implies $\mathbf{B}(\mathcal{F})$. In particular, the properties $\mathbf{B}(\mathcal{F})$ depend only on the strict-coherence class of \mathcal{F} .

Proof. Assume that X satisfies $\mathbf{B}(\mathcal{S})$, and let $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ be a witness for \mathcal{S} being strictly subcoherent to \mathcal{F} . Let Y be any continuous image of X in $\mathbb{N}^{\mathbb{N}}$. Then Y satisfies $\mathbf{B}(\mathcal{S})$, and therefore so does $\tilde{Y} = \Phi_h[Y]$ (where Φ_h is as in Lemma 2.31), a continuous image of Y . Let $g \in \mathbb{N}^{\mathbb{N}}$ be a witness for that, and take $\tilde{g} = \Phi_h(g)$. For each $f \in Y$,

$$[f \leq \tilde{g}] \supseteq [\tilde{f} \leq \tilde{g}] \supseteq [\tilde{f} \leq g] \in \mathcal{S},$$

so taking $A = [\tilde{f} \leq \tilde{g}]$, we find that $[f \leq \tilde{g}] \supseteq A = \text{cl}_h(A) \in \mathcal{F}$. \square

Note that if \mathcal{S} is a feeble semifilter, then by Theorem 2.32, $\mathbf{B}(\mathcal{S}) = \mathbf{B}(\mathcal{F})$ where \mathcal{F} is the Fréchet filter (thus, each set of reals satisfying $\mathbf{B}(\mathcal{S})$ is a counterexample to the Hurewicz conjecture). In particular, $\mathfrak{b}(\mathcal{S}) = \text{non}(\mathbf{B}(\mathcal{S})) = \text{non}(\mathbf{B}(\mathcal{F})) = \mathfrak{b}$. Thus, Theorem 2.32 can be viewed as a structural counterpart of Corollary 2.27.

3. A problem of Hurewicz

3.1. History

In his 1927 paper [18], Hurewicz writes (page 196, footnote 1):

Aus der Eigenschaft E^{**} folgt offenbar die Eigenschaft E^* . Die Frage nach der Existenz von Mengen mit der Eigenschaft E^* ohne Eigenschaft E^{**} bleibt hier offen.

In our language and terminology this reads: “The Menger property obviously follows from the Hurewicz property. The question about the existence of sets with the Menger property and without the Hurewicz property remains open.”

At the correction stage, Hurewicz added there that Sierpiński proved that the answer is positive if we assume the Continuum Hypothesis. Thus, the answer is consistently positive. But it remained open whether the answer is provably positive.

This problem of Hurewicz also appears twice in Lelek's 1969 paper [24] (pages 210 and 211), as well as in several recent accounts, for example: Problem 3 in Just, Miller, Scheepers and Szeptycki's [19].

An existential solution to the Hurewicz problem was essentially established by Chaber and Pol at the end of 2002.

Theorem 3.1 (Chaber–Pol [13]). *There exists $X \subseteq \mathbb{N}^{\mathbb{N}}$ such that all finite powers of X have the Menger property, and X is not contained in any σ -compact subset of $\mathbb{N}^{\mathbb{N}}$.*

In Theorem 5.7 of [19] it is proved that a set of reals X has the Hurewicz property if, and only if, for each G_δ set G containing X , there is a σ -compact set K such that $X \subseteq K \subseteq G$. Consequently, Chaber and Pol's result implies a positive answer to the Hurewicz problem, even when all finite powers of X are required to have the Menger property.

Prior to the present investigation, it was not observed that the Chaber–Pol Theorem 3.1 solves the Hurewicz problem, and the Hurewicz problem continued to be raised, e.g.: Problem 1 in Bukovský and Haleš' [11]; Problem 2.1 in Bukovský's [9]; Problem 1 in Bukovský's [10]; Problem 5.1 in the first author's [38].

Chaber and Pol's solution is existential in the sense that their proof does not point out a specific example for a set X , but instead gives one example if $\mathfrak{b} = \mathfrak{d}$ (the interesting case), and another if $\mathfrak{b} < \mathfrak{d}$ (the trivial case). In the current context, this approach was originated in Fremlin and Miller's [14], improved in Just, Miller, Scheepers and Szeptycki's [19] and exploited further in Chaber and Pol's argument.

We will give an explicit solution to the Hurewicz problem.

3.2. A solution of the Hurewicz problem by direct construction

A continuous metrizable image of the Baire space $\mathbb{N}^{\mathbb{N}}$ is called *analytic*.

Lemma 3.2. *Assume that \mathcal{A} is an analytic subset of $[\mathbb{N}]^{\aleph_0}$. Then the smallest semifilter \mathcal{F} containing \mathcal{A} is analytic.*

Proof. For a finite subset F of \mathbb{N} , define $\Phi_F : [\mathbb{N}]^{\aleph_0} \rightarrow [\mathbb{N}]^{\aleph_0}$ by $\Phi_F(A) = A \setminus F$ for each $A \in [\mathbb{N}]^{\aleph_0}$. Then Φ_F is continuous, and therefore $\Phi_F(\mathcal{A})$ is analytic. Let

$$\mathcal{B} = \bigcup_{\text{finite } F \subseteq \mathbb{N}} \Phi_F(\mathcal{A}).$$

Then \mathcal{B} is analytic, and therefore so is $\mathcal{B} \times P(\mathbb{N})$. Since the mapping $\Phi : P(\mathbb{N}) \times P(\mathbb{N}) \rightarrow P(\mathbb{N})$ defined by $(A, B) \mapsto A \cup B$ is continuous, we see that $\mathcal{F} = \Phi[\mathcal{B} \times P(\mathbb{N})]$ is analytic. \square

Lemma 3.3. *Assume that \mathcal{F} is a nonmeager semifilter, and $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is analytic. If Y is bounded with respect to $\leq_{\mathcal{F}^+}$, then Y is bounded with respect to \leq^* .*

Proof. Let g be a $\leq_{\mathcal{F}^+}$ -bound of Y . Define $\Phi : Y \rightarrow [\mathbb{N}]^{\aleph_0}$ by

$$\Phi(f) = [f \leq g].$$

Then Φ is continuous. Thus, $\Phi[Y]$ is analytic and by Lemma 3.2, the smallest semifilter \mathcal{S} containing it is also analytic. Since \mathcal{S} is closed under finite modifications of its elements, the topological 0-1 law [20, 8.47] shows that \mathcal{S} is either meager or comeager.

Note that $\mathcal{S} \subset \mathcal{F}^+$. Since \mathcal{F} is not meager, \mathcal{F}^+ is not comeager, hence \mathcal{S} is meager (and therefore feeble). As Y is bounded with respect to $\leq_{\mathcal{S}}$ (as witnessed by g), it follows from Lemma 2.26 that Y is bounded with respect to \leq^* . \square

Corollary 3.4. *Assume that \mathcal{F} is a nonmeager semifilter. Then for each $f \in \mathbb{N}^{\uparrow\mathbb{N}}$, the set $\{g \in \mathbb{N}^{\uparrow\mathbb{N}} : f \leq_{\mathcal{F}} g\}$ is nonmeager.*

Proof. Assume that $\{g \in \mathbb{N}^{\uparrow\mathbb{N}} : f \leq_{\mathcal{F}} g\}$ is meager. Then there exists a dense G_δ set $G \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ such that $g \leq_{\mathcal{F}^+} f$ for all $g \in G$. By Lemma 3.3, G is bounded with respect to \leq^* , and therefore meager; a contradiction. \square

Definition 3.5. *A semifilter \mathcal{S} is nonmeager-bounding if for each family $Y \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ with $|Y| < \mathfrak{b}(\mathcal{S})$, the set $\{g \in \mathbb{N}^{\uparrow\mathbb{N}} : (\forall f \in Y) f \leq_{\mathcal{S}} g\}$ is nonmeager.*

We will use the following generalization of Definition 2.29.

Definition 3.6. *For $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ and $A \subseteq \mathbb{N}$ let*

$$\text{cl}_h^+(A) = \bigcup \{[h(n), h(n+3)) : A \cap [h(n+1), h(n+2)) \neq \emptyset\}.$$

A semifilter \mathcal{S} is subcoherent to a semifilter \mathcal{F} if there exists $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ such that for each $A \in \mathcal{S}$, $\text{cl}_h^+(A) \in \mathcal{F}$. \mathcal{S} is coherent to \mathcal{F} if each of them is subcoherent to the other.

It is often, but not always, the case that subcoherence coincides with strict subcoherence—see Chapter 5 of [1].

Proposition 3.7. *Assume that \mathcal{S} is a semifilter. If any of the following holds, then \mathcal{S} is nonmeager-bounding:*

- (1) \mathcal{S} is a nonmeager filter, or
- (2) $\mathcal{S} = [\mathbb{N}]^{\aleph_0}$, or
- (3) \mathcal{S} is coherent to a nonmeager filter, or
- (4) \mathcal{S} is comeager.

Proof. (1) Assume that $Y \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ and $|Y| < \mathfrak{b}(\mathcal{S})$. Let $f \in \mathbb{N}^{\uparrow\mathbb{N}}$ be a $\leq_{\mathcal{S}}$ -bound of Y . By Corollary 3.4, $\{g \in \mathbb{N}^{\uparrow\mathbb{N}} : f \leq_{\mathcal{S}} g\}$ is nonmeager. Since \mathcal{S} is a filter, $\leq_{\mathcal{S}}$ is transitive, and therefore each member in this nonmeager set is a $\leq_{\mathcal{S}}$ -bound of Y .

(2) Assume that $Y \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ and $|Y| < \mathfrak{d}$. We may assume that Y is closed under pointwise maxima. Let $g \in \mathbb{N}^{\uparrow\mathbb{N}}$ be a witness for the fact that Y is not dominating. Then $\{[f \leq g] : f \in Y\}$ is closed under taking finite intersections. Let \mathcal{U} be an ultrafilter

extending it. By (1), $Z = \{h \in \mathbb{N}^{\uparrow\mathbb{N}} : g \leq_{\mathcal{U}} h\}$ is nonmeager. For each $h \in Z$, $f \leq_{\mathcal{U}} g \leq_{\mathcal{U}} h$ (in particular, $f \leq_{[\mathbb{N}]^{\aleph_0}} g$) for each $f \in Y$.

(3) Any semifilter coherent to a *filter* is actually strictly coherent to it [1, 5.5.3]. Thus, assume that \mathcal{S} is *strictly* coherent to a nonmeager filter \mathcal{F} . Then there is a monotone surjection $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\{\varphi[A] : A \in \mathcal{S}\} = \{\varphi[A] : A \in \mathcal{F}\}$ [1, 5.5.2]. The filter \mathcal{G} generated by $\{\varphi^{-1}[\varphi[A]] : A \in \mathcal{S}\}$ is contained in \mathcal{S} . Since \mathcal{G} is coherent to \mathcal{S} , it is nonmeager and $\mathfrak{b}(\mathcal{G}) = \mathfrak{b}(\mathcal{S})$ [1, 5.3.1 and 10.1.13]. Thus, \mathcal{G} is nonmeager-bounding and since $\mathfrak{b}(\mathcal{G}) = \mathfrak{b}(\mathcal{S})$ and $\leq_{\mathcal{S}}$ extends $\leq_{\mathcal{G}}$, \mathcal{S} is nonmeager-bounding.

(4) Using Lemma 2.30, let $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ be a witness for $[\mathbb{N}]^{\aleph_0}$ being strictly subcoherent to \mathcal{S} . Assume that $Y \subseteq \mathbb{N}^{\mathbb{N}}$ and $|Y| < \mathfrak{b}(\mathcal{S})$. For each $f \in Y$ define $\tilde{f} \in \mathbb{N}^{\mathbb{N}}$ by $\tilde{f}(n) = \max\{f(k) : k \in [h(n), h(n+1)]\}$. By (2), $Z = \{g \in \mathbb{N}^{\uparrow\mathbb{N}} : (\forall f \in Y) \tilde{f} \leq_{[\mathbb{N}]^{\aleph_0}} g\}$ is nonmeager. Fix any g in this nonmeager set. Let $f \in Y$, and $A = [\tilde{f} \leq g]$. Then $A \in [\mathbb{N}]^{\aleph_0}$, and for each $n \in A$ and each $k \in [h(n), h(n+1))$,

$$f(k) \leq \tilde{f}(n) \leq g(n) \leq g(h(n)) \leq g(k),$$

that is, $[f \leq g] \supseteq \bigcup_{n \in A} [h(n), h(n+1))$. By Lemma 2.30, the last set is in \mathcal{S} . \square

Remark 3.8. Under some set-theoretic hypotheses, e.g., $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{u} < \mathfrak{g}$, all nonmeager semifilters are nonmeager-bounding.

The assumptions on \mathcal{F} in the following theorem hold for $\mathcal{F} = [\mathbb{N}]^{\aleph_0}$. Thus, this theorem implies the promised solution to the Hurewicz problem.

Theorem 3.9. *Assume that \mathcal{F} is a nonmeager-bounding semifilter with $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$. Then there is a cofinal $\mathfrak{b}(\mathcal{F})$ -scale $S = \{f_\alpha : \alpha < \mathfrak{d}\}$ such that the set $X = S \cup Q$ satisfies $\mathbf{B}(\mathcal{F})$ but does not have the Hurewicz property.*

Proof. We will identify $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ with $P(\mathbb{N})$, identifying Q with $[\mathbb{N}]^{<\aleph_0}$ and $\mathbb{N}^{\uparrow\mathbb{N}}$ with $[\mathbb{N}]^{\aleph_0}$ (see Lemma 2.4 and the discussion following it). Recall that $[\mathbb{N}]^{(\aleph_0, \aleph_0)}$ is the collection of infinite coinfinite subsets of \mathbb{N} . For each $g \in \mathbb{N}^{\mathbb{N}}$, $\{a \in [\mathbb{N}]^{(\aleph_0, \aleph_0)} : a \leq^* g\}$ is meager, and therefore so is $M_g := \{a \in [\mathbb{N}]^{(\aleph_0, \aleph_0)} : a^c \leq^* g\}$ (since $A \mapsto A^c$ is an autohomeomorphism of $[\mathbb{N}]^{(\aleph_0, \aleph_0)}$).

Fix a dominating family $\{d_\alpha : \alpha < \mathfrak{d}\} \subseteq \mathbb{N}^{\mathbb{N}}$. Define $a_\alpha \in [\mathbb{N}]^{(\aleph_0, \aleph_0)}$ by induction on $\alpha < \mathfrak{d}$, as follows: At step α use the fact that \mathcal{F} is nonmeager-bounding to find $a_\alpha \in [\mathbb{N}]^{(\aleph_0, \aleph_0)} \setminus M_{d_\alpha}$ which is a bound for $\{d_\beta, a_\beta : \beta < \alpha\}$ with respect to $\leq_{\mathcal{F}}$. Take $S = \{a_\alpha : \alpha < \mathfrak{d}\}$.

By Theorem 2.24, $X = S \cup Q$ satisfies $\mathbf{B}(\mathcal{F})$. But $\{x^c : x \in X\}$ is a homeomorphic image of X in $\mathbb{N}^{\mathbb{N}}$, and is unbounded (with respect to \leq^*), since for each $\alpha < \mathfrak{d}$, $a_\alpha^c \not\leq^* d_\alpha$. Thus, X does not have the Hurewicz property. \square

The methods that Chaber and Pol used to prove their Theorem 3.1 are topological. We proceed to show that Chaber and Pol's theorem can also be obtained using the combinatorial approach.

4. Finite powers and the Chaber–Pol theorem

Having the property $\mathbf{B}(\mathcal{F})$ in all finite powers is useful for the generation of (nontrivial) groups and other algebraic objects satisfying $\mathbf{B}(\mathcal{F})$. In this section we restrict attention to filters.

Theorem 4.1. *Assume that \mathcal{F} is a filter, and $S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\}$ is a $\mathfrak{b}(\mathcal{F})$ -scale. Let $X = S \cup Q$. Then for each k and each continuous $\Psi : X^k \rightarrow \mathbb{N}^{\mathbb{N}}$, there exist elements $A_1, \dots, A_k \in \mathcal{F}^+$ such that $\Psi[X^k]$ is bounded with respect to $\leq_{\mathcal{F}_{A_1} \cup \dots \cup \mathcal{F}_{A_k}}$.*

The proof of Theorem 4.1 is by induction on k . To make the induction step possible, we strengthen its assertion.

Proposition 4.2. *For each k and each family \mathcal{C} of fewer than $\mathfrak{b}(\mathcal{F})$ continuous functions from X^k to $\mathbb{N}^{\mathbb{N}}$, there exist elements $A_1, \dots, A_k \in \mathcal{F}^+$ such that $\bigcup\{\Psi[X^k] : \Psi \in \mathcal{C}\}$ is bounded with respect to $\leq_{\mathcal{F}_{A_1} \cup \dots \cup \mathcal{F}_{A_k}}$.*

Proof. For each $\Psi \in \mathcal{C}$, let $g_\Psi \in \mathbb{N}^{\mathbb{N}}$ be as in Lemma 2.5. Since $|\mathcal{C}| < \mathfrak{b}(\mathcal{F})$, there is $g_0 \in \mathbb{N}^{\mathbb{N}}$ such that $g_\Psi \leq_{\mathcal{F}} g_0$ for each $\Psi \in \mathcal{C}$. Choose $\alpha < \mathfrak{b}(\mathcal{F})$ such that $[g_0 < f_\alpha] \in \mathcal{F}^+$. Then $A_k := [g_0 < f_\alpha] \in \mathcal{F}^+$. We continue by induction on k .

$k = 1$: By Lemma 2.5, for each $\beta \geq \alpha$ and each $\Psi \in \mathcal{C}$, $[\Psi(f_\beta) \leq g_0] \in \mathcal{F}_{A_1}$. Since the cardinality of the set

$$\{\Psi(f) : \Psi \in \mathcal{C}, f \in \{f_\beta : \beta < \alpha\} \cup Q\}$$

is smaller than $\mathfrak{b}(\mathcal{F})$, this set is bounded with respect to $\leq_{\mathcal{F}}$, by some function $h \in \mathbb{N}^{\mathbb{N}}$. Take $g = \max\{g_0, h\}$.

$k = m + 1$: For all $\alpha_1, \dots, \alpha_k \geq \alpha$, by Lemma 2.5 we have

$$[\Psi(f_{\alpha_1}, \dots, f_{\alpha_k}) \leq g_0] \supseteq [g_0 < \min\{f_{\alpha_1}, \dots, f_{\alpha_k}\}] \supseteq A_k \cap \bigcap_{i=1}^k [f_\alpha \leq f_{\alpha_i}] \in \mathcal{F} \upharpoonright A_k.$$

For each $f \in \{f_\beta : \beta < \alpha\} \cup Q$ and each $i = 1, \dots, k$ define $\Psi_{i,f} : X^m \rightarrow \mathbb{N}^{\mathbb{N}}$ by

$$\Psi_{i,f}(x_1, \dots, x_m) = \Psi(x_1, \dots, x_{i-1}, f, x_i, \dots, x_m).$$

Since there are fewer than $\mathfrak{b}(\mathcal{F})$ such functions, the induction hypothesis yields $A_1, \dots, A_m \in \mathcal{F}^+$ such that

$$\bigcup\{\Psi_{i,f}[X^m] : i = 1, \dots, k, f \in \{f_\beta : \beta < \alpha\} \cup Q, \Psi \in \mathcal{C}\}$$

is bounded with respect to $\leq_{\mathcal{F}_{A_1} \cup \dots \cup \mathcal{F}_{A_m}}$. Let $h \in \mathbb{N}^{\mathbb{N}}$ be such a bound, and take $g = \max\{g_0, h\}$. Then $\bigcup\{\Psi[X^k] : \Psi \in \mathcal{C}\}$ is bounded with respect to $\leq_{\mathcal{F}_{A_1} \cup \dots \cup \mathcal{F}_{A_k}}$. □

This completes the proof of Theorem 4.1. □

Corollary 4.3. *In the notation of Theorem 4.1, all finite powers of X satisfy $\mathbf{B}(\mathcal{F}^+)$.* □

Item (2) in Corollary 4.4 below was first proved in [4], using a specialized proof.

Corollary 4.4. *In the notation of Theorem 4.1, suppose that:*

- (1) \mathcal{F} is an ultrafilter, or
- (2) \mathcal{F} is the Fréchet filter (Hurewicz property).

Then all finite powers of X satisfy $\mathbf{B}(\mathcal{F})$.

Proof. (1) If \mathcal{F} is an ultrafilter, then $\mathcal{F}^+ = \mathcal{F}$.

(2) Fix k and a continuous $\Psi : X^k \rightarrow \mathbb{N}^{\mathbb{N}}$. We may assume that $\Psi[X^k] \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$. Apply Theorem 4.1, and let $g \in \mathbb{N}^{\mathbb{N}}$ be a witness for $\Psi[X^k]$ being bounded with respect to $\leq_{\mathcal{F}_{A_1} \cup \dots \cup \mathcal{F}_{A_k}}$. For each $i = 1, \dots, k$ let $Y_i = \{f \in \Psi[X^k] : f \leq_{\mathcal{F}_{A_i}} g\}$. Then each Y_i is bounded, and therefore so is

$$\bigcup_{i=1}^k Y_i = \Psi[X^k]. \quad \square$$

For later use, we point out the following.

Theorem 4.5. *Assume that \mathcal{F} is a filter and $S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\}$ is a cofinal $\mathfrak{b}(\mathcal{F})$ -scale. Then all finite powers of the set $X = S \cup Q$ satisfy $\mathbf{B}(\mathcal{F})$.*

Proof. This is a part of the proof of Theorem 4.1, with each \mathcal{F}^+ replaced by \mathcal{F} (in this case the proof can be simplified). \square

The cardinal \mathfrak{d} is not provably regular. However, in most of the known models of set theory it is regular. In Theorem 16 of [4], a weaker version of Theorem 4.6 below is established using various hypotheses, all of which imply that \mathfrak{d} is regular.

Theorem 4.6. *Assume that \mathfrak{d} is regular. Then there is an ultrafilter \mathcal{U} with $\mathfrak{b}(\mathcal{U}) = \mathfrak{d}$, and a $\mathfrak{b}(\mathcal{U})$ -scale $S = \{f_\alpha : \alpha < \mathfrak{d}\}$ such that all finite powers of the set $X = S \cup Q$ satisfy $\mathbf{B}(\mathcal{U})$, but X does not have the Hurewicz property.*

Proof. There always exists an ultrafilter \mathcal{U} with $\mathfrak{b}(\mathcal{U}) = \text{cf}(\mathfrak{d})$ [12]. Since ultrafilters are not meager, Proposition 3.7(1) shows that \mathcal{U} is nonmeager-bounding. Take a $\mathfrak{b}(\mathcal{U})$ -scale $S = \{f_\alpha : \alpha < \mathfrak{d}\}$ as in Theorem 3.9, so that the set $X = S \cup Q$ does not have the Hurewicz property. By Corollary 4.4(1), all finite powers of X satisfy $\mathbf{B}(\mathcal{U})$. \square

Remark 4.7. In particular, we obtain Chaber and Pol's Theorem 3.1:

- (1) If \mathfrak{d} is regular, use Theorem 4.6. Otherwise, let X be any unbounded subset of $\mathbb{N}^{\mathbb{N}}$ of cardinality $\text{cf}(\mathfrak{d})$. This proof is still on a dichotomic basis, but the dichotomy here puts more weight on the interesting case (since $\mathfrak{b} < \text{cf}(\mathfrak{d}) = \mathfrak{d}$ is consistent).
- (2) The sets in this argument are of cardinality $\text{cf}(\mathfrak{d})$, while Chaber and Pol's sets are of cardinality \mathfrak{b} . To get sets of cardinality \mathfrak{b} , use the dichotomy " $\mathfrak{b} = \mathfrak{d}$ (which implies that \mathfrak{d} is regular) or $\mathfrak{b} < \mathfrak{d}$ " instead.

Remark 4.8. Just as in Chaber and Pol's [13], our constructions can be carried out in any nowhere locally compact Polish space P : Fix a countable dense subset E of P . Since E and our Q are both countable metrizable with no isolated points, they are both homeomorphic to the space \mathbb{Q} of rational numbers, and hence are homeomorphic via some map $\varphi : Q \rightarrow E$. According to Lavrent'ev's theorem [20, 3.9], φ can be extended to a homeomorphism between two (dense) G_δ sets containing Q and E , respectively. Now, every G_δ set G in $\overline{\mathbb{N}^{\uparrow\mathbb{N}}}$ containing Q contains the set $\{f \in \mathbb{N}^{\uparrow\mathbb{N}} : f \not\leq^* g\}$ for some fixed $g \in \mathbb{N}^{\uparrow\mathbb{N}}$, in which our constructions can be carried out.

5. Finite powers for arbitrary feeble semifilters

We extend Theorem 2.28 and Corollary 4.4(2).

Theorem 5.1. *Assume that $\mathcal{F}_1, \dots, \mathcal{F}_k$ are feeble semifilters, and for each $i = 1, \dots, k$, $S_i = \{f_\alpha^i : \alpha < \mathfrak{b}\}$ is a $\mathfrak{b}(\mathcal{F}_i)$ -scale and $X_i = S_i \cup Q$. Then $\prod_{i=1}^k X_i$ has the Hurewicz property.*

Proof. The proof is by induction on k . The case $k = 1$ is Theorem 2.28, so assume that the assertion holds for $k - 1$ and let us prove it for k .

Let $h_1, \dots, h_k \in \mathbb{N}^{\uparrow\mathbb{N}}$ witness the feebleness of $\mathcal{F}_1, \dots, \mathcal{F}_k$. Take $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ such that for each n and each $i = 1, \dots, k$, $[h(n), h(n+1))$ contains some interval $[h_i(j), h_i(j+1))$. Clearly, h witnesses the feebleness of all semifilters $\mathcal{F}_1, \dots, \mathcal{F}_k$.

Assume that $\Psi : \prod_{i=1}^k X_i \rightarrow \mathbb{N}^{\uparrow\mathbb{N}}$ is continuous. We may assume that all elements in $\Psi[X]$ are increasing. Let $g \in \mathbb{N}^{\uparrow\mathbb{N}}$ be as in Lemma 2.5, and define $\tilde{g} \in \mathbb{N}^{\uparrow\mathbb{N}}$ by $\tilde{g}(m) = g(h(n+2))$ for each $m \in [h(n), h(n+1))$.

Lemma 5.2. *Assume that $Y_1, \dots, Y_k \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ are unbounded (with respect to \leq^*) and $g \in \mathbb{N}^{\uparrow\mathbb{N}}$. Then there exist $f_i \in Y_i$, $i = 1, \dots, k$, such that $[g < \min\{f_1, \dots, f_k\}]$ is infinite.*

Proof. Take $f_1 \in Y_1$ such that $A_1 = [g < f_1]$ is infinite. As all members of Y_2 are increasing and Y_2 is unbounded, Y_2 is not bounded on A_1 , thus there is $f_2 \in Y_2$ such that $A_2 = [g < \min\{f_1, f_2\}] = A_1 \cap [g < f_2]$ is infinite. Continue inductively. \square

Use Lemma 5.2 to choose $\alpha_1, \dots, \alpha_k < \mathfrak{b}$ such that $A = [g < \min\{f_{\alpha_1}^1, \dots, f_{\alpha_k}^k\}]$ is infinite. Let $C = \{n : A \cap [h(n-1), h(n)) \neq \emptyset\}$. Take $\alpha = \max\{\alpha_1, \dots, \alpha_k\}$.

As in the proof of Theorem 2.28, we infer that for each $\beta \geq \alpha$ and each $i = 1, \dots, k$, $g(h(n+1)) < f_\beta^i(h(n+1))$ for all but finitely many $n \in C$. Thus, for all $\beta_1, \dots, \beta_k \geq \alpha$,

$$g(h(n+1)) < \min\{f_{\beta_1}^1(h(n+1)), \dots, f_{\beta_k}^k(h(n+1))\}$$

for all but finitely many $n \in C$. By Lemma 2.5,

$$[\Psi(f_{\beta_1}^1, \dots, f_{\beta_k}^k) \leq g] \supseteq [g < \min\{f_{\beta_1}^1, \dots, f_{\beta_k}^k\}] \supseteq^* \{h(n+1) : n \in C\},$$

that is, $\{\Psi(f_{\beta_1}^1, \dots, f_{\beta_k}^k) : \beta_1, \dots, \beta_k \geq \alpha\}$ is \leq^* -bounded on an infinite set and therefore \leq^* -bounded.

It follows, as at the end of the proof of Proposition 4.2, that the image of Ψ is a union of fewer than \mathfrak{b} \leq^* -bounded sets, and is therefore \leq^* -bounded. \square

6. Adding an algebraic structure

In this section we show that most of our examples can be chosen to have an algebraic structure.

A classical result of von Neumann [27] asserts that there exists a subset C of \mathbb{R} which is homeomorphic to the Cantor space and is algebraically independent over \mathbb{Q} . Since the properties $\mathbf{B}(\mathcal{F})$ are preserved under continuous images, we may identify $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ with such a set $C \subseteq \mathbb{R}$, and for each $X \subseteq \overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ consider the subfield $\mathbb{Q}(X)$ of \mathbb{R} generated by $\mathbb{Q} \cup X$. The following theorem extends Theorem 1 of [40] significantly.

Theorem 6.1. *Assume that \mathcal{F} is a filter, $S = \{f_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\}$ is a $\mathfrak{b}(\mathcal{F})$ -scale, and $X = S \cup \mathbb{Q}$. Then:*

- (1) *All finite powers of $\mathbb{Q}(X)$ satisfy $\mathbf{B}(\mathcal{F}^+)$.*
- (2) *If \mathcal{F} is an ultrafilter, then all finite powers of $\mathbb{Q}(X)$ satisfy $\mathbf{B}(\mathcal{F})$.*
- (3) *If \mathcal{F} is a feeble filter, then all finite powers of $\mathbb{Q}(X)$ have the Hurewicz property.*

On the other hand,

- (4) *For each property P of sets of reals which is hereditary for closed subsets, if X does not have the property P , then neither does $\mathbb{Q}(X)$.*
- (5) *$\mathbb{Q}(X)$ is not σ -compact.*

Proof. (1) Denote by $\mathbb{Q}(t_1, \dots, t_n)$ the field of all rational functions in the indeterminates t_1, \dots, t_n with coefficients in \mathbb{Q} . For each n ,

$$\mathbb{Q}_n(X) = \{r(x_1, \dots, x_n) : r \in \mathbb{Q}(t_1, \dots, t_n), x_1, \dots, x_n \in X\}$$

is a union of countably many continuous images of X^n , thus for each k , $(\mathbb{Q}_n(X))^k$ is a union of countably many continuous images of X^{nk} , which by Corollary 4.3 satisfy $\mathbf{B}(\mathcal{F}^+)$.

For a family \mathcal{I} of sets of reals with $\bigcup \mathcal{I} \notin \mathcal{I}$, let

$$\text{add}(\mathcal{I}) = \min\left\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} \notin \mathcal{I}\right\}.$$

Lemma 6.2. *For each semifilter \mathcal{F} , $\text{add}(\mathbf{B}(\mathcal{F})) \geq \mathfrak{b}$. If \mathcal{F} is a filter, then $\text{add}(\mathbf{B}(\mathcal{F})) = \mathfrak{b}(\mathcal{F})$.* \square

Since $\mathbf{B}(\mathcal{F}^+)$ is preserved under taking continuous images and countable unions, we conclude that each set $(\mathbb{Q}_n(X))^k$ satisfies $\mathbf{B}(\mathcal{F}^+)$, and therefore so does $(\mathbb{Q}(X))^k = \bigcup_n (\mathbb{Q}_n(X))^k$.

(2) and (3) are obtained similarly, as consequences of Corollary 4.4 and Theorem 5.1, respectively.

(4) Since $X \subseteq C$ and C is algebraically independent, we deduce that $\mathbb{Q}(X) \cap C = X$ and therefore X is a closed subset of $\mathbb{Q}(X)$.

(5) Use (4) and apply Theorem 2.20. \square

The following theorem extends Theorem 5 of [40], and shows that even fields can witness that the Hurewicz property is stronger than Menger's.

Theorem 6.3. *If \mathfrak{d} is regular, then for the set X of Theorem 4.6, all finite powers of $\mathbb{Q}(X)$ have the Menger property, but $\mathbb{Q}(X)$ does not have the Hurewicz property.*

Proof. This follows from Theorems 4.6 and 6.1(4). \square

The following solves Hurewicz's problem for subfields of \mathbb{R} .

Corollary 6.4. *There exists $X \subseteq \mathbb{R}$ of cardinality $\text{cf}(\mathfrak{d})$ such that all finite powers of $\mathbb{Q}(X)$ have Menger's property, but $\mathbb{Q}(X)$ does not have the Hurewicz property.*

Proof. Take the dichotomic examples of Remark 4.7. \square

Remark 6.5. Problem 6 of [40] and Problem 1.3 of [41] ask (see Section 8 below) whether there exists a subgroup G of \mathbb{R} such that $|G| = \mathfrak{d}$ and G has Menger's property. Theorem 6.3 answers this question in the affirmative under the additional weak assumption that \mathfrak{d} is regular. Corollary 6.4 answers affirmatively the analogous question where \mathfrak{d} is replaced by $\text{cf}(\mathfrak{d})$.

Remark 6.6. We can make all of our examples subfields of any nondiscrete, separable, completely metrizable field \mathbb{F} . Examples for such fields are, in addition to \mathbb{R} , the complex numbers \mathbb{C} and the p -adic numbers \mathbb{Q}_p . More examples involving meromorphic functions or formal Laurent series are available in [28]. To this end, we use Mycielski's extension of von Neumann's theorem, asserting that for each countable dense subfield \mathbb{Q} of \mathbb{F} , \mathbb{F} contains an algebraically independent (over \mathbb{Q}) homeomorphic copy of the Cantor space (see [28] for a proof).

7. Smallness in the sense of measure and category

A set of reals X is *null* if it has Lebesgue measure zero. X is *universally null* if every Borel isomorphic image of X in \mathbb{R} is null. Equivalently, for each finite σ -additive measure μ on the Borel subsets of X such that $\mu\{x\} = 0$ for each $x \in X$, $\mu(X) = 0$. A classical result of Marczewski asserts that each product of two universally null sets of reals is universally null.

A set X of reals is *perfectly meager* if for each perfect set P , $X \cap P$ is meager in the relative topology of P . The set X is *universally meager* if each Borel isomorphic image of X in \mathbb{R} is meager. Zakrzewski [43] proved that each product of two universally meager sets is universally meager.

As in Section 6, we identify the Cantor space with a subset of \mathbb{R} which is algebraically independent over \mathbb{Q} .

Theorem 7.1. *Let \mathcal{F} be the Fréchet filter, $S = \{f_\alpha : \alpha < \mathfrak{b}\}$ be any $\mathfrak{b}(\mathcal{F})$ -scale, and $X = S \cup \mathbb{Q}$. Then all finite powers of $\mathbb{Q}(X)$ have the Hurewicz property and are universally null and universally meager.*

Proof. Theorem 6.1 deals with the first assertion.

Plewik [29] proved that every set S as above is both universally null and universally meager.² Since both properties are preserved under taking countable unions and are satisfied by singletons, we see that X is universally null and universally meager. Consequently, so too are all finite powers of X .

We should now understand why these properties would also hold for $\mathbb{Q}(X)$ and its finite powers. To this end, we use some results of Pfeffer and Prikry. The presentation is mutatis mutandis the one from Pfeffer's [28], in which full proofs are supplied.

Let

$$\mathbb{Q}'(t_1, \dots, t_n) = \mathbb{Q}(t_1, \dots, t_n) \setminus \bigcup_{i=1}^n \mathbb{Q}(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n).$$

The usual order in the Cantor set induces an order \leq in C , which is closed in C^2 . For $X \subseteq C$ and each m and k , define

$$X_{m,k} = \{(x_1, \dots, x_m) \in X^m : x_1 \leq \dots \leq x_m, (\forall i \neq j) |x_i - x_j| > 1/k\}.$$

Each $X_{m,k}$ is closed in X^m , in particular, each $C_{m,k}$ is compact. Since C is algebraically independent, each $r \in \mathbb{Q}'(t_1, \dots, t_m)$ defines a continuous map $(a_1, \dots, a_m) \mapsto r(a_1, \dots, a_m)$ from $\bigcup_k C_{m,k}$ to \mathbb{R} . It follows that r is a homeomorphism into $\mathbb{Q}(X)$, and

$$\mathbb{Q}(X) = \mathbb{Q} \cup \bigcup_{m,k \in \mathbb{N}} \bigcup_{r \in \mathbb{Q}'(t_1, \dots, t_m)} r[X_{m,k}].$$

For any m_1, m_2, k_1, k_2 , $X_{m_1, k_1} \times X_{m_2, k_2} \subseteq X^{m_1+m_2}$ and is therefore universally null and universally meager. Thus, so is each homeomorphic copy $r_1[X_{m_1, k_1}] \times r_2[X_{m_2, k_2}]$ of $X_{m_1, k_1} \times X_{m_2, k_2}$, where $r_1 \in \mathbb{Q}'(t_1, \dots, t_{m_1})$, $r_2 \in \mathbb{Q}'(t_1, \dots, t_{m_2})$. A similar assertion holds for products of any finite length. Consequently, each finite power of $\mathbb{Q}(X)$ is a countable union of sets which are universally null and universally meager, and is therefore universally null and universally meager. \square

8. Connections with selection principles

8.1. Selection principles

In his 1925 paper [17], Hurewicz introduced two properties of the following type. For collections \mathcal{A}, \mathcal{B} of covers of a space X , define

² The latter assertion also follows from Corollary 2.16 and Theorem 2.20, by a result of Zakszewski [43] which asserts that every set of reals having the Hurewicz property and not containing perfect sets is universally meager.

$\mathbf{U}_{\text{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $\{U_n\}_{n \in \mathbb{N}}$ of members of \mathcal{A} which do not contain a finite subcover, there exist finite subsets $\mathcal{F}_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

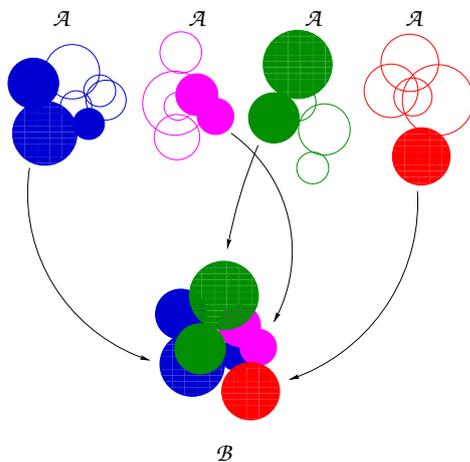


Fig. 1. $\mathbf{U}_{\text{fin}}(\mathcal{A}, \mathcal{B})$.

Hurewicz (essentially) proved that if X is a set of reals and \mathcal{O} is the collection of all open covers of X , then $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ is equivalent to $\mathbf{B}([\mathbb{N}]^{\aleph_0})$ (the Menger property). He also introduced the following property: Call an open cover \mathcal{U} of X a γ -cover if \mathcal{U} is infinite, and each $x \in X$ belongs to all but finitely many members of \mathcal{U} . Let Γ denote the collection of all open γ -covers of X . Hurewicz proved that for sets of reals, $\mathbf{U}_{\text{fin}}(\mathcal{O}, \Gamma)$ is the same as $\mathbf{B}(\mathcal{F})$ where \mathcal{F} is the Fréchet filter (the Hurewicz property).

Here too, the properties $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ and $\mathbf{U}_{\text{fin}}(\mathcal{O}, \Gamma)$ are specific instances of a general scheme of properties.

Definition 8.1. (1) Let \mathcal{U} be a cover of X enumerated bijectively as $\vec{\mathcal{U}} = \{U_n : n \in \mathbb{N}\}$. The Marczewski characteristic function of $\vec{\mathcal{U}}$, $h_{\vec{\mathcal{U}}} : X \rightarrow P(\mathbb{N})$, is defined by

$$h_{\vec{\mathcal{U}}}(x) = \{n : x \in U_n\} \quad \text{for each } x \in X.$$

(2) Let \mathcal{F} be a semifilter.

- (a) \mathcal{U} is an \mathcal{F} -cover of X if there is a bijective enumeration $\vec{\mathcal{U}} = \{U_n : n \in \mathbb{N}\}$ such that $h_{\vec{\mathcal{U}}}[X] \subseteq \mathcal{F}$.
- (b) $\mathcal{O}_{\mathcal{F}}$ is the collection of all open \mathcal{F} -covers of X .

If $\mathcal{F} = [\mathbb{N}]^{\aleph_0}$, then it is easy to see that $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathcal{F}}) = \mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ [19]. If \mathcal{F} is the Fréchet filter, then $\mathcal{O}_{\mathcal{F}} = \Gamma$ and therefore $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathcal{F}}) = \mathbf{U}_{\text{fin}}(\mathcal{O}, \Gamma)$. The families of covers $\mathcal{O}_{\mathcal{F}}$ were first introduced and studied in a similar context by García-Ferreira and Tamariz-Mascarúa [15, 16].

Definition 8.2. $S_{\mathbb{N}}$ is the collection of all permutations σ of \mathbb{N} . For a semifilter \mathcal{F} and $\sigma \in S_{\mathbb{N}}$, write $\sigma\mathcal{F} = \{\sigma[A] : A \in \mathcal{F}\}$.

Observe that if \mathcal{F} is $[\mathbb{N}]^{\aleph_0}$ or the Fréchet filter, then $\sigma\mathcal{F} = \mathcal{F}$ for all σ . Consequently, the following theorem generalizes Hurewicz's theorem.

Theorem 8.3. Assume that \mathcal{F} is a semifilter. For a set X of reals, the following are equivalent:

- (1) X satisfies $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathcal{F}})$.
- (2) For each continuous image Y of X in $\mathbb{N}^{\mathbb{N}}$, there is $\sigma \in S_{\mathbb{N}}$ such that Y is bounded with respect to $\leq_{\sigma\mathcal{F}}$.

In particular, $\mathbf{B}(\mathcal{F})$ implies $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathcal{F}})$.

Proof. (2) \Rightarrow (1): Assume that \mathcal{U}_n , $n \in \mathbb{N}$, are open covers of X , which do not contain a finite subcover of X . For each n , let $\tilde{\mathcal{U}}_n$ be a countable cover refining \mathcal{U}_n such that all elements of $\tilde{\mathcal{U}}_n$ are clopen and disjoint. Enumerate $\tilde{\mathcal{U}}_n$ bijectively as $\{C_m^n : m \in \mathbb{N}\}$. Then the function $\Psi : X \rightarrow \mathbb{N}^{\mathbb{N}}$ defined by

$$\Psi(x)(n) = m \text{ such that } x \in C_m^n$$

is continuous, and therefore $\Psi[X]$ is bounded with respect to $\leq_{\sigma\mathcal{F}}$ for some $\sigma \in S_{\mathbb{N}}$. Let $g \in \mathbb{N}^{\mathbb{N}}$ be a witness for that. By induction on n , choose finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{m \leq g(n)} C_m^n \subseteq \bigcup \mathcal{F}_n$, and such that $\bigcup \mathcal{F}_n$ is not equal to any $\bigcup \mathcal{F}_k$ for $k < n$.³ Consequently,

$$\left\{n : x \in \bigcup \mathcal{F}_n\right\} \supseteq \left\{n : x \in \bigcup_{m \leq g(n)} C_m^n\right\} = \{\Psi(x) \leq g\} \in \sigma\mathcal{F}$$

for each $x \in X$. Consequently, the (bijective!) enumeration $\{\bigcup \mathcal{F}_{\sigma^{-1}(n)} : n \in \mathbb{N}\}$ witnesses that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\}$ is an \mathcal{F} -cover.

(1) \Rightarrow (2): Assume that X satisfies $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathcal{F}})$, and let Y be a continuous image of X in $\mathbb{N}^{\mathbb{N}}$. We may assume that each $f \in Y$ is increasing. It is easy to see that the following holds.

Lemma 8.4. $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathcal{F}})$ is preserved under taking continuous images. \square

Thus, Y satisfies $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathcal{F}})$. Consider the open covers $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$ of Y defined by $U_m^n = \{f \in Y : f(n) \leq m\}$ (note that the elements U_m^n are increasing with m). There are two cases to consider.

Case 1: There is a strictly increasing sequence $\{k_n\}_{n \in \mathbb{N}}$ of natural numbers such that each $U_{k_n}^{k_n}$ contains an element $U_{m_n}^{k_n}$ which is equal to Y . Define $g(n) = m_n$ for each n . Then for each $f \in Y$ and each n , $f(n) \leq f(k_n) \leq m_n = g(n)$, that is, $[f \leq g] = \mathbb{N} \in \mathcal{F}$.

³ Since no \mathcal{U}_n contains a finite cover of X , we may achieve this as follows: Choose a finite $\mathcal{A} \subseteq \mathcal{U}_n$ such that $\bigcup_{m \leq g(n)} C_m^n \subseteq \bigcup \mathcal{A}$. For each $k < n$ take $x_k \in X \setminus \bigcup \mathcal{F}_k$. Choose a finite $\mathcal{B} \subseteq \mathcal{U}_n$ such that $\{x_1, \dots, x_{n-1}\} \subseteq \bigcup \mathcal{B}$, and take $\mathcal{F}_n = \mathcal{A} \cup \mathcal{B}$.

Case 2: There is n_0 such that for each $n \geq n_0$, \mathcal{U}_n does not contain Y as an element. Then by (1), there are finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \geq n_0$, such that $\mathcal{U} = \{\bigcup \mathcal{F}_n : n \geq n_0\}$ is an \mathcal{F} -cover of X . Let $h \in \mathbb{N}^{\uparrow \mathbb{N}}$ be such that $\{\bigcup \mathcal{F}_{h(n)} : n \in \mathbb{N}\}$ is a bijective enumeration of \mathcal{U} . Define $g(n) = \max\{m : U_m^{h(n)} \in \mathcal{F}_{h(n)}\}$ for each n . Then there is $\sigma \in S_{\mathbb{N}}$ such that for each $f \in Y$, $\{n : f \in \bigcup \mathcal{F}_{h(\sigma(n))}\} \in \mathcal{F}$. For each n with $f \in \bigcup \mathcal{F}_{h(\sigma(n))}$,

$$f(\sigma(n)) \leq f(h(\sigma(n))) \leq g(\sigma(n)).$$

Thus $f \leq_{\sigma \mathcal{F}} g$ for all $f \in Y$. \square

Remark 8.5. (1) By the methods of [44], Theorem 8.3 actually holds for arbitrary (not necessarily zero-dimensional) subsets of \mathbb{R} .

(2) One can characterize $\mathbf{B}(\mathcal{F})$ by: For each sequence \mathcal{U}_n of open covers of X , there exist finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that for each $x \in X$, $\{n : x \in \bigcup \mathcal{F}_n\} \in \mathcal{F}$.

Let X be a set of reals. In addition to γ -covers, the following type of cover plays a central role in the field: An open cover \mathcal{U} of X is an ω -cover of X if X is not in \mathcal{U} and for each finite subset F of X , there is $U \in \mathcal{U}$ such that $F \subseteq U$. Let Ω denote the collection of all countable open ω -covers of X . For each filter \mathcal{F} , $\mathcal{O}_{\mathcal{F}} \subseteq \Omega$. Consequently, $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathcal{F}})$ implies $\mathbf{U}_{\text{fin}}(\mathcal{O}, \Omega)$, which is strictly stronger than Menger's property $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ [19]. In light of Theorem 8.3, all of the examples shown to satisfy $\mathbf{B}(\mathcal{F})$ for a filter \mathcal{F} , satisfy $\mathbf{U}_{\text{fin}}(\mathcal{O}, \Omega)$.

8.2. Finer distinction

We now reveal the remainder of the framework of selection principles, and apply the combinatorial approach to obtain a new result concerning these, which further improves our earlier results.

This framework was introduced by Scheepers in [32, 19] as a unified generalization of several classical notions, and studied since in a long series of papers by many mathematicians (see the surveys [35, 21, 41]). Let X be a set of reals. An open cover \mathcal{U} of X is a τ -cover of X if each member of X is covered by infinitely many members of \mathcal{U} , and for any $x, y \in X$, at least one of the sets $\{U \in \mathcal{U} : x \in U \text{ and } y \notin U\}$ or $\{U \in \mathcal{U} : y \in U \text{ and } x \notin U\}$ is finite. Let \mathbf{T} denote the collection of all countable open τ -covers of X . It is easy to see that

$$\Gamma \subseteq \mathbf{T} \subseteq \Omega \subseteq \mathcal{O}.$$

Let \mathcal{A} and \mathcal{B} be collections of covers of X . In addition to $\mathbf{U}_{\text{fin}}(\mathcal{A}, \mathcal{B})$, we have the following selection hypotheses.

$\mathbf{S}_1(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of members of \mathcal{A} , there exist members $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_n : n \in \mathbb{N}\} \in \mathcal{B}$.

$\mathbf{S}_{\text{fin}}(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of members of \mathcal{A} , there exist finite (possibly empty) subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_n \mathcal{F}_n \in \mathcal{B}$.

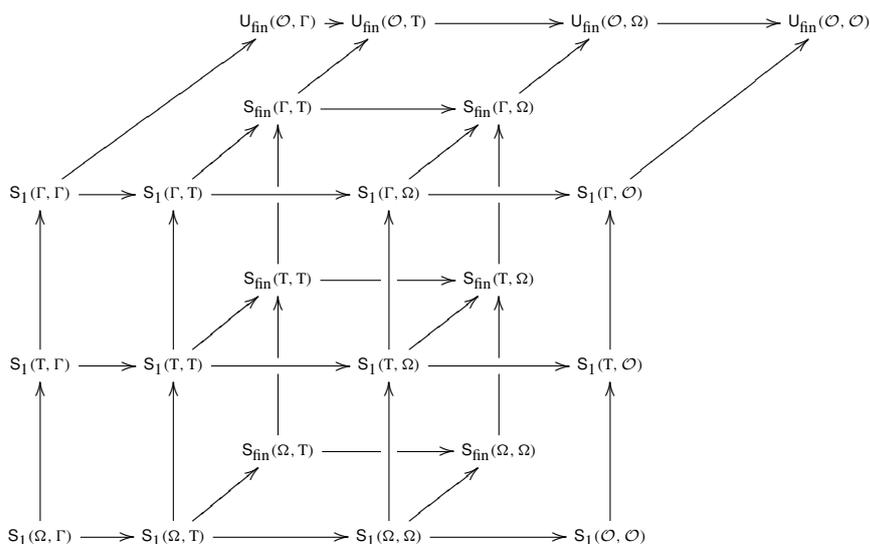


Fig. 2. The surviving properties.

In addition to the Menger ($U_{\text{fin}}(\mathcal{O}, \mathcal{O})$) and Hurewicz ($U_{\text{fin}}(\mathcal{O}, \Gamma)$) properties, several other properties of this form were studied in the past by Rothberger ($S_1(\mathcal{O}, \mathcal{O})$), Arkhangel'skiĭ ($S_{\text{fin}}(\Omega, \Omega)$),⁴ Gerlits and Nagy ($S_1(\Omega, \Gamma)$), and Sakai ($S_1(\Omega, \Omega)$). Many equivalences hold among these properties, and the surviving ones appear in Figure 2 (where an arrow denotes implication) [32, 19, 37].

In [19] it is proved that a set of reals X satisfies $S_{\text{fin}}(\Omega, \Omega)$ if, and only if, all finite powers of X have the Menger property $U_{\text{fin}}(\mathcal{O}, \mathcal{O})$. By Corollary 4.3, the examples involving filters (including those from Section 6) satisfy $S_{\text{fin}}(\Omega, \Omega)$. In Theorem 4.6 the example did not satisfy $U_{\text{fin}}(\mathcal{O}, \Gamma)$. We will improve that to find such an example which does not satisfy $U_{\text{fin}}(\mathcal{O}, T)$. Since it is consistent that $U_{\text{fin}}(\mathcal{O}, \Gamma)$ is equivalent to $U_{\text{fin}}(\mathcal{O}, T)$ [45] and that $U_{\text{fin}}(\mathcal{O}, \Omega)$ is equivalent to $U_{\text{fin}}(\mathcal{O}, \mathcal{O})$ [46], our result is the best possible with regard to Figure 2.

Again, we will identify $\bar{\mathbb{N}}^{\uparrow \mathbb{N}}$ with $P(\mathbb{N})$. We will use the following notion. A family $Y \subseteq P(\mathbb{N})$ is *splitting* if for each $A \in [\mathbb{N}]^{\aleph_0}$ there is $B \in Y$ such that $A \cap B$ and $A \setminus B$ are both infinite. Recall that if \mathfrak{d} is regular then there is an ultrafilter (necessarily nonmeager) \mathcal{F} satisfying $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$.

Theorem 8.6. *Assume that \mathfrak{d} is regular. Then for each nonmeager filter \mathcal{F} with $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$, there is a cofinal $\mathfrak{b}(\mathcal{F})$ -scale $S = \{a_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^{(\aleph_0, \aleph_0)}$ such that:*

- (1) *All finite powers of the set $X = S \cup Q$ satisfy $\mathbf{B}(\mathcal{F})$, but*

⁴ Arkhangel'skiĭ studied "Menger property in all finite powers", which was proved to be equivalent to $S_{\text{fin}}(\Omega, \Omega)$ in [19].

(2) The homeomorphic copy $\tilde{X} = \{x^c : x \in X\}$ of X is a splitting and unbounded (with respect to \leq^*) subset of $[\mathbb{N}]^{\aleph_0}$.

Proof. For each $h \in \mathbb{N}^{\uparrow\mathbb{N}}$, let

$$\mathcal{A}_h = \left\{ \bigcup_{n \in A} [h(n), h(n+1)) : A \in [\mathbb{N}]^{(\aleph_0, \aleph_0)} \right\}.$$

Then \mathcal{A}_h is homeomorphic $[\mathbb{N}]^{(\aleph_0, \aleph_0)}$, and is therefore analytic.

Lemma 8.7. For each $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ and each $f \in \mathbb{N}^{\uparrow\mathbb{N}}$, there is $a \in \mathcal{A}_h$ such that $f \leq_{\mathcal{F}} a$.

Proof. Clearly, \mathcal{A}_h is not \leq^* -bounded. Apply Lemma 3.3. □

Lemma 8.8. For all $h, f, g \in \mathbb{N}^{\uparrow\mathbb{N}}$, there is $a \in \mathcal{A}_h$ such that $f \leq_{\mathcal{F}} a$ and $a^c \not\leq^* g$.

Proof. Let $q \in \mathbb{N}^{\uparrow\mathbb{N}}$ be such that for each $a \in [\mathbb{N}]^{(\aleph_0, \aleph_0)}$ with $a \leq^* g$, a intersects all but finitely many of the intervals $[q(n), q(n+1))$. (E.g., define inductively $q(0) = g(0)$ and $q(n+1) = g(q(n)) + 1$.) We may assume that $\text{im } q \subseteq \text{im } h$, and therefore $\mathcal{A}_q \subseteq \mathcal{A}_h$. By Lemma 8.7, there is $a \in \mathcal{A}_q$ such that $f \leq_{\mathcal{F}} a$. Since $a^c \in \mathcal{A}_q$, it misses infinitely many intervals $[q(n), q(n+1))$, and therefore $a^c \not\leq^* g$. □

Let $\{d_\alpha : \alpha < \mathfrak{d}\} \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ be such that for each $A \in [\mathbb{N}]^{\aleph_0}$ there is $\alpha < \mathfrak{d}$ such that $|A \cap [d_\alpha(n), d_\alpha(n+1))| \geq 2$ for all but finitely many n [6]. In particular, $\{d_\alpha : \alpha < \mathfrak{d}\}$ is dominating.

For each $\alpha < \mathfrak{b}(\mathcal{F}) = \mathfrak{d}$ inductively, do the following: Choose $f \in \mathbb{N}^{\uparrow\mathbb{N}}$ which is a $\leq_{\mathcal{F}}$ -bound of $\{a_\beta : \beta < \alpha\}$. Use Lemma 8.8 to choose $a_\alpha \in \mathcal{A}_{d_\alpha}$ such that $\max\{f, d_\alpha\} \leq_{\mathcal{F}} a_\alpha$ and $a_\alpha^c \not\leq^* d_\alpha$. Since \mathcal{F} is a filter, $a_\beta \leq_{\mathcal{F}} a_\alpha$ for each $\beta < \alpha$.

Thus $S = \{a_\alpha : \alpha < \mathfrak{d}\}$ is a cofinal $\mathfrak{b}(\mathcal{F})$ -scale, and so by Theorem 4.5, all finite powers of $X = S \cup Q$ satisfy $\mathbf{B}(\mathcal{F})$. As for each $\alpha < \mathfrak{d}$ we have $a_\alpha^c \not\leq^* d_\alpha$, \tilde{X} is unbounded. To see that it is splitting, let $b \in [\mathbb{N}]^{\aleph_0}$ and choose α such that b intersects $[d_\alpha(n), d_\alpha(n+1))$ for all but finitely n . Since $a_\alpha \in \mathcal{A}_{d_\alpha}$, a_α splits b . □

According to [37], a subset Y of $\mathbb{N}^{\mathbb{N}}$ has the *excluded-middle property* if there exists $g \in \mathbb{N}^{\mathbb{N}}$ such that:

- (1) for each $f \in Y$, the set $[f < g]$ is infinite,
- (2) for all $f, h \in Y$ at least one of the sets $[f < g \leq h]$ and $[h < g \leq f]$ is finite.

In Theorem 3.11 and Remark 3.12 of [37] it is proved that if Y satisfies $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathbf{T})$, then all continuous images of Y in $\mathbb{N}^{\mathbb{N}}$ have the excluded-middle property.

Corollary 8.9. Assume that \mathfrak{d} is regular. Then for each nonmeager filter \mathcal{F} with $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$, there is a set of reals $Y \subseteq \mathbb{N}^{\mathbb{N}}$ such that:

- (1) All finite powers of Y satisfy $\mathbf{B}(\mathcal{F})$, but
- (2) Y does not have the excluded-middle property. In particular, Y does not satisfy $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathbf{T})$.

Proof. Let \tilde{X} be the set from Theorem 8.6(2). Define continuous functions $\Psi_\ell : \tilde{X}^2 \rightarrow \mathbb{N}^{\mathbb{N}}$, $\ell = 0, 1$, by

$$\Psi_0(x, y)(n) = \begin{cases} x(n), & n \in y, \\ 0, & n \notin y, \end{cases} \quad \Psi_1(x, y)(n) = \begin{cases} 0, & n \in y, \\ x(n), & n \notin y. \end{cases}$$

Take $Y = \Psi_0[\tilde{X}^2] \cup \Psi_1[\tilde{X}^2]$. Each finite power of Y is a finite union of continuous images of finite powers of \tilde{X} . Consequently, all finite powers of Y satisfy $\mathbf{B}(\mathcal{F})$.

The argument in the proof of Theorem 9 of [36] shows that Y does not have the excluded-middle property. \square

We obtain the following.

Theorem 8.10. *There exists a set of reals X satisfying $\mathbf{S}_{\text{fin}}(\Omega, \Omega)$ but not $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathbb{T})$.*

Proof. The proof is dichotomic. If $\text{cf}(\mathfrak{d}) = \mathfrak{d}$, use Corollary 8.9. Otherwise, $\text{cf}(\mathfrak{d}) < \mathfrak{d}$. As $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \text{cf}(\mathfrak{d})$ ($\mathfrak{s} \leq \text{cf}(\mathfrak{d})$ is proved in [26]), $\max\{\mathfrak{b}, \mathfrak{s}\} < \mathfrak{d}$. As the critical cardinalities of $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathbb{T})$ and $\mathbf{S}_{\text{fin}}(\Omega, \Omega)$ are $\max\{\mathfrak{b}, \mathfrak{s}\}$ [36] and \mathfrak{d} [19], respectively, we can take Y to be any witness for the first of these two assertions. \square

By the arguments of Section 6, we have the following.

Corollary 8.11. *Assume that \mathbb{F} is a nondiscrete, separable, completely metrizable field, and \mathbb{Q} is a countable dense subfield of \mathbb{F} .*

- (1) *If \mathfrak{d} is regular, then for each nonmeager filter \mathcal{F} with $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$, there is $X \subseteq \mathbb{F}$ such that:*
 - (a) *All finite powers of $\mathbb{Q}(X)$ satisfy $\mathbf{B}(\mathcal{F})$, but*
 - (b) *$\mathbb{Q}(X)$ does not satisfy $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathbb{T})$.*
- (2) *There exists $X \subseteq \mathbb{F}$ such that $\mathbb{Q}(X)$ satisfies $\mathbf{S}_{\text{fin}}(\Omega, \Omega)$ but not $\mathbf{U}_{\text{fin}}(\mathcal{O}, \mathbb{T})$.* \square

Readers not familiar with forcing can safely skip the following remark.

Remark 8.12. The constructions in this section can be viewed as an extraction of the essential part in the forcing-theoretic construction obtained by adding \mathfrak{c} Cohen reals to a model of set theory, and letting X be the set of the added Cohen reals. Since Cohen reals are not dominating, all finite powers of X will have Menger's property. It is also easy to see that X will not satisfy the excluded-middle property, e.g., using the reasoning in [36]. See [8] for these types of construction, but note that they only yield consistency results.

9. Towards semifilters again

We strengthen the solution to the Hurewicz problem as follows.

Theorem 9.1. *Assume that P is a nowhere locally compact Polish space, and \mathcal{S} is a nonmeager bounding semifilter such that $\mathfrak{b}(\mathcal{S}) = \mathfrak{d}$. Then there is a subspace X of P such that:*

- (1) *All finite powers of X have Menger's property.*
- (2) *X satisfies $\mathbf{B}(\mathcal{S})$.*
- (3) *X does not have the Hurewicz property.*

Proof. As pointed out in Remark 4.8, it suffices to consider the case $P = [\mathbb{N}]^{(\aleph_0, \aleph_0)} \cup [\mathbb{N}]^{< \aleph_0}$, in a disguise of our choice. We give an explicit construction in the case where \mathfrak{d} is regular. The remaining case, being “rare” but consistent, is trivial.

A family $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ is *centered* if each finite subset of \mathcal{F} has an infinite intersection. Centered families generate filters by taking finite intersections and closing upwards. We will denote the generated filter by $\langle \mathcal{F} \rangle$. For $Y \subseteq \mathbb{N}^{\mathbb{N}}$, let $\text{maxfin } Y$ denote its closure under pointwise maxima of finite subsets.

We construct, by induction on $\alpha < \mathfrak{d}$, a filter \mathcal{F} with $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$ and a $\mathfrak{b}(\mathcal{F})$ -scale $\{a_\alpha : \alpha < \mathfrak{d}\} \subseteq [\mathbb{N}]^{(\aleph_0, \aleph_0)}$ which is also a cofinal $\mathfrak{b}(\mathcal{S})$ -scale.

Let $\{d_\alpha : \alpha < \mathfrak{d}\} \subseteq \mathbb{N}^{\mathbb{N}}$ be dominating, and assume that a_β are defined for each $\beta < \alpha$. Let

$$\mathcal{A}_\alpha = \text{maxfin}\{d_\beta, a_\beta : \beta < \alpha\}, \quad \tilde{\mathcal{F}}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta, \quad \mathcal{G}_\alpha = \{f \circ b : f \in \mathcal{A}_\alpha, b \in \tilde{\mathcal{F}}_\alpha\}.$$

We inductively assume that $\mathcal{F}_\beta, \beta < \alpha$, is an increasing chain of filters such that $|\mathcal{F}_\beta| \leq |\beta|$ for each $\beta < \alpha$. This implies that $|\mathcal{G}_\alpha| \leq |\alpha| < \mathfrak{d}$.

As \mathcal{S} is nonmeager-bounding, there exists a $\leq_{\mathcal{S}}$ -bound a_α of \mathcal{G}_α such that $a_\alpha^c \not\leq^* d_\alpha$. Define

$$\mathcal{F}_\alpha = \langle \tilde{\mathcal{F}}_\alpha \cup \{[f \leq a_\alpha] : f \in \mathcal{A}_\alpha\} \rangle.$$

We must show that \mathcal{F}_α remains a filter. First, assume that there are $b \in \tilde{\mathcal{F}}_\alpha$ and $f \in \mathcal{A}_\alpha$ such that $b \cap [f \leq a_\alpha]$ is finite. Then $a_\alpha \leq a_\alpha \circ b <^* f \circ b \in \mathcal{G}_\alpha$, a contradiction. Now, for each $b \in \mathcal{F}_\alpha$ and $f_1, \dots, f_k \in \mathcal{A}_\alpha$, we have $f = \max\{f_1, \dots, f_k\} \in \mathcal{A}_\alpha$, and therefore

$$b \cap \bigcap_{i=1}^k [f_i \leq a_\alpha] = b \cap [f \leq a_\alpha]$$

is infinite.

Take $S = \{a_\alpha : \alpha < \mathfrak{d}\}$ and $\mathcal{F} = \bigcup_{\alpha < \mathfrak{d}} \mathcal{F}_\alpha$. By the construction, S is a cofinal $\mathfrak{b}(\mathcal{S})$ -scale. By Theorem 2.24, $X = S \cup Q$ satisfies $\mathbf{B}(\mathcal{S})$. For all $\alpha < \beta < \mathfrak{d}$, $a_\alpha \leq_{\mathcal{F}} a_\beta$. We claim that if \mathfrak{d} is regular, then $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$. Indeed, assume that $Y \subseteq \mathbb{N}^{\mathbb{N}}$ has cardinality less than \mathfrak{d} . As \mathfrak{d} is regular, there exists $\alpha < \mathfrak{d}$ such that each $f \in Y$ is \leq^* -bounded by some $d_\beta, \beta < \alpha$. As a_α is a $\leq_{\mathcal{F}}$ -bound of $\{d_\beta : \beta < \alpha\}$, it is a $\leq_{\mathcal{F}}$ -bound of Y . We deduce that S is also a cofinal $\mathfrak{b}(\mathcal{F})$ -scale. By Theorem 4.5, all finite powers of X satisfy $\mathbf{B}(\mathcal{F})$ (and, in particular, Menger’s property).

Finally, since $\{x^c : x \in X\}$ is an unbounded subset of $\mathbb{N}^{\mathbb{N}}$, X does not have the Hurewicz property. □

10. Topological Ramsey theory

Most of our constructions can be viewed as examples in topological Ramsey theory. We explain this briefly. The following partition relation, motivated by a study of Baumgartner and Taylor in Ramsey theory [5], was introduced by Scheepers in [32]:

$\mathcal{A} \rightarrow \lceil \mathcal{B} \rceil_k^2$: For each $U \in \mathcal{A}$ and each $f : [U]^2 \rightarrow \{0, \dots, k-1\}$, there exists $\mathcal{V} \subseteq U$ such that $\mathcal{V} \in \mathcal{B}$, $j \in \{0, \dots, k-1\}$, and a partition $\mathcal{V} = \bigcup_n \mathcal{F}_n$ of \mathcal{V} into finite sets, such that for each $\{A, B\} \in [\mathcal{V}]^2$ such that A and B are not from the same \mathcal{F}_n , $f(\{A, B\}) = j$.

Menger's property is equivalent to $(\forall k) \Omega \rightarrow \lceil \mathcal{O} \rceil_k^2$ [33], and having the Menger property in all finite powers is equivalent to $(\forall k) \Omega \rightarrow \lceil \Omega \rceil_k^2$ [34].

A cover \mathcal{U} of X which does not contain a finite subcover is γ -groupable if there is a partition of \mathcal{U} into finite sets, $\mathcal{U} = \bigcup_n \mathcal{F}_n$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\}$ is a γ -cover of X . Denote the collection of γ -groupable open covers of X by $G(\Gamma)$.

The Hurewicz property is equivalent to $(\forall k) \Omega \rightarrow \lceil G(\Gamma) \rceil_k^2$, and having the Hurewicz property in all finite powers is equivalent to $(\forall k) \Omega \rightarrow \lceil \Omega^{\text{sp}} \rceil_k^2$, where Ω^{sp} denotes covers with partition $\bigcup_n \mathcal{F}_n$ into finite sets such that for each finite $F \subseteq X$ and all but finitely many n , there is $U \in \mathcal{F}_n$ such that $F \subseteq U$ [22, 30, 31].

Clearly, $\Omega^{\text{sp}} \subseteq \Omega \cap G(\Gamma)$.

We state only three of our results using this language, leaving the statement of the remaining ones to the reader.

- Theorem 10.1.** (1) *The sets X constructed in Theorem 3.9 satisfy $(\forall k) \Omega \rightarrow \lceil \mathcal{O} \rceil_k^2$ but not $(\forall k) \Omega \rightarrow \lceil G(\Gamma) \rceil_k^2$.*
 (2) *The fields $\mathbb{Q}(X)$ constructed in Theorem 7.1 satisfy $(\forall k) \Omega \rightarrow \lceil \Omega^{\text{sp}} \rceil_k^2$ but are not σ -compact.*
 (3) *The fields $\mathbb{Q}(X)$ constructed in Theorem 8.11 satisfy $(\forall k) \Omega \rightarrow \lceil \Omega \rceil_k^2$ but not $(\forall k) \Omega \rightarrow \lceil \Omega^{\text{sp}} \rceil_k^2$ (or even $(\forall k) \Omega \rightarrow \lceil G(\Gamma) \rceil_k^2$).*

10.1. Strong measure zero and Rothberger fields

We can move beyond the decidable case. According to Borel [7], a set of reals X has *strong measure zero* if for each sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ of positive reals, there exists a cover $\{I_n : n \in \mathbb{N}\}$ of X such that for each n , the diameter of I_n is smaller than ϵ_n . This is a very strong property, and Borel conjectured that every strong measure zero set of reals is countable. This was proved consistent by Laver [23].

Rothberger's property $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$ implies strong measure zero, and its critical cardinality is $\text{cov}(\mathcal{M})$, the minimal cardinality of a cover of the real line by meager sets. By known combinatorial characterizations [2], if \mathfrak{b} is not greater than the minimal cardinality of a set of reals which is not of strong measure zero, then $\mathfrak{b} \leq \text{cov}(\mathcal{M})$. In the following theorem, any embedding of $\overline{\mathbb{N}}^{\uparrow \mathbb{N}}$ in \mathbb{R} can be used.

Theorem 10.2. *If $\mathfrak{b} \leq \text{cov}(\mathcal{M})$, then the fields $\mathbb{Q}(X)$ constructed in Theorem 7.1 satisfy $\mathbf{S}_1(\Omega, \Omega^{\text{sp}})$ (equivalently, all finite powers of $\mathbb{Q}(X)$ satisfy the Hurewicz property as well as Rothberger's property [22]).*

Proof. Since $X = S \cup Q$ is \mathfrak{b} -concentrated on the countable set Q , it satisfies—by the assumption on \mathfrak{b} —Rothberger's property $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$. As all finite powers of X have the

Hurewicz property, Theorem 4.3 of [42] implies that X satisfies $\mathbf{S}_1(\Omega, \Omega^{\text{sp}})$. By the arguments in the proof of Theorem 6.1(1), all finite powers of $\mathbb{Q}(X)$ satisfy $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$. Theorem 6.1(2) tells us that all finite powers of $\mathbb{Q}(X)$ also satisfy the Hurewicz property, so we are done. \square

The following partition relation [32] is a natural extension of Ramsey's.

$\mathcal{A} \rightarrow (\mathcal{B})_k^n$: For each $\mathcal{U} \in \mathcal{A}$ and $f : [\mathcal{U}]^n \rightarrow \{1, \dots, k\}$, there exist j and $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{B}$ and $f \upharpoonright [\mathcal{V}]^n \equiv j$.

Using this notation, Ramsey's theorem is $(\forall n, k) [\mathbb{N}]^{\aleph_0} \rightarrow ([\mathbb{N}]^{\aleph_0})_k^n$.

In [22] it is proved that $\mathbf{S}_1(\Omega, \Omega^{\text{sp}})$ is equivalent to $(\forall n, k) \Omega \rightarrow (\Omega^{\text{sp}})_k^n$.

Corollary 10.3. *If $\mathfrak{b} \leq \text{cov}(\mathcal{M})$, then the fields $\mathbb{Q}(X)$ constructed in Theorem 7.1 satisfy $(\forall n, k) \Omega \rightarrow (\Omega^{\text{sp}})_k^n$.*

11. Some concluding remarks

Using filters in the constructions allowed avoiding some of the technical aspects of earlier constructions and naturally obtain examples for the Menger and Hurewicz conjectures which possess an algebraic structure. The extension to semifilters is essential for the consideration of the Menger and Hurewicz properties in terms of boundedness on "large" sets of natural numbers. While making some of the proofs more difficult, it seems to have provided the natural solution of the Hurewicz problem, and allowed its strengthening in several manners.

Chaber and Pol asked us about the difference in strength between the construction in [4] (corresponding to item (2) in Corollary 4.4) and their dichotomic construction [13] (Theorem 3.1). The answer is now clear: The set from [4] has the Hurewicz property, and the (dichotomic) set from [13] has the Menger property but not the Hurewicz property.

Previous constructions (dichotomic or using additional hypotheses) which made various assumptions on the cardinal \mathfrak{d} can now be viewed as a "projection" of the constructions which only assume that \mathfrak{d} is regular. While giving rise (in a dichotomic manner) to ZFC theorems, the possibility to eliminate the dichotomy in Theorem 4.6 and its consequences without making any additional hypotheses remains open. It may be impossible.

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