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Estimates for L^1 vector fields under higher-order differential conditions

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Abstract. We prove that an L^1 vector field whose components satisfy some condition on k-th order derivatives induce linear functionals on the Sobolev space $\mathrm{W}^{1,n}(\mathbb{R}^n)$. Two proofs are provided, relying on the two distinct methods developed by Bourgain and Brezis (J. Eur. Math. Soc., 2007) and by the author (C. R. Math. Acad. Sci. Paris, 2004) to prove the same result for divergence-free vector fields and partial extensions to higher-order conditions.

Keywords. Critical Sobolev spaces, compensation, Sobolev inequality, Korn–Sobolev inequality

1. Introduction

1.1. Known L^1 *estimates for vector fields*

The classical Sobolev embedding theorem states that the Sobolev space $\mathrm{W}^{1,p}(\mathbb{R}^n)$ is continuously embeddeded in $\mathrm{L}^{np/(n-p)}(\mathbb{R}^n)$ if $p < n$ and in the space of Hölder continuous functions $C^{0,1-n/p}(\mathbb{R}^n)$ if $p > n$. The case $p = n$ is more delicate. When $n > 1$, there is no embedding of $\mathrm{W}^{1,n}(\mathbb{R}^n)$ in $\mathrm{L}^\infty(\mathbb{R}^n)$. By duality, a function $f \in \mathrm{L}^1(\mathbb{R}^n)$ need not be in the dual Sobolev space $\mathrm{W}^{-1,n/(n-1)}(\mathbb{R}^n)$. However, in a recent work, Bourgain and Brezis established that if $f \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^n)$ is a *divergence-free vector field,* then $f \in \mathrm{W}^{-1,n/(n-1)}(\mathbb{R}^n;\mathbb{R}^n)$:

Theorem 1 (Bourgain and Brezis [3, 4]). *For every vector field* $f \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^n)$ *and* $u \in (\mathrm{W}^{1,n} \cap \mathrm{L}^\infty)(\mathbb{R}^n;\mathbb{R}^n)$*, if* $\operatorname{div} f = 0$ *in the sense of distributions, then*

$$\left| \int_{\mathbb{R}^n} f \cdot u \right| \le C \|f\|_{\mathrm{L}^1} \, \|\nabla u\|_{\mathrm{L}^n},$$

where the constant C only depends on the dimension of the space n.

When $n = 2$, this estimate is a dual statement of the classical Gagliardo–Nirenberg–Sobolev inequality

$$\|u\|_{\mathrm{L}^2} \le C \|\nabla u\|_{\mathrm{L}^1}.$$

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In higher dimensions, this estimate implies the classical Gagliardo–Nirenberg–Sobolev inequality

$$\|u\|_{\mathrm{L}^{n/(n-1)}} \leq C\|\nabla u\|_{\mathrm{L}^1}, \tag{1}$$

as well as the inequality

$$\|U\|_{\mathrm{L}^{n/(n-1)}} \leq C\|\operatorname{curl} U\|_{\mathrm{L}^1} \tag{2}$$

for every divergence-free vector field U. When $n \geq 4$, there are intermediate interesting inequalities for k-forms with $2 \leq k \leq n-2$ [6, 4]. While the inequality (1) is still a consequence of (2) by duality, there is no direct way to deduce (2) from (1). However Theorem 1 and inequality (2) can be easily deduced from each other.

Theorem 1 was obtained by Bourgain and Brezis by a Littewood–Paley decomposition. It also has an elementary proof based on the Sobolev–Morrey embedding [10].

A natural question is whether the condition on the divergence can be replaced by conditions on higher-order derivatives. In a previous work, we obtained

Theorem 2 (Van Schaftingen [11]). *For every vector field $f = (f_{11}, f_{12}, f_{22}) \in \mathrm{L}^1(\mathbb{R}^2; \mathbb{R}^3)$ and $u \in (\mathrm{W}^{1,n} \cap \mathrm{L}^\infty)(\mathbb{R}^2; \mathbb{R}^3)$, if*

$$\partial_{11} f_{11} + \partial_{12} f_{12} + \partial_{22} f_{22} = 0$$

in the sense of distributions, then

$$\left| \int_{\mathbb{R}^2} f \cdot u \right| \leq C\|f\|_{\mathrm{L}^1} \|\nabla u\|_{\mathrm{L}^n}.$$

This inequality is dual to the Korn–Sobolev inequality of Strauss [8]: For every $u \in \mathrm{W}^{1,1}(\mathbb{R}^2; \mathbb{R}^2)$,

$$\|u\|_{\mathrm{L}^2} \leq C\|Du + Du^t\|_{\mathrm{L}^1},$$

where Du^t denotes Du transposed. Theorem 2 was obtained with the same strategy as the elementary proof of Theorem 1 in [10]. The same method could also handle some vector fields $(f_{ij})_{1 \leq i \leq 2,\, i \leq j \leq n} \in \mathrm{L}^1(\mathbb{R}^n; \mathbb{R}^{2n-1})$ satisfying the second-order condition

$$\sum_{\substack{1 \leq i \leq 2 \\ i \leq j \leq n}} \partial_i \partial_j f_{ij} = 0.$$

When $n \geq 3$, this condition is not at all natural since there are $n(n+1)/2$ distinct second-order partial derivatives, and since the condition does not have any property of invariance under the isometries of $\mathbb{R}^n$.

Theorem 1 was also extended by Bourgain and Brezis to higher-order conditions:

Theorem 3 (Bourgain and Brezis [4]). *For every vector field $f \in \mathrm{L}^1(\mathbb{R}^n; \mathbb{R}^n)$ and $u \in (\mathrm{W}^{1,n} \cap \mathrm{L}^\infty)(\mathbb{R}^n; \mathbb{R}^n)$, if*

$$\sum_{i=1}^n \partial_i^k f_i = 0$$

in the sense of distributions, then

$$\left| \int_{\mathbb{R}^n} f \cdot u \right| \leq C \|f\|_{L^1} \, \|\nabla u\|_{L^n},$$

where the constant C only depends on the dimension n of the space and on k.

When $k > 1$, the condition of Theorem 3 is not invariant under rotations of $\mathbb{R}^n$.

1.2. New estimates under higher-order conditions

In this paper, we generalize Theorem 1 to vector fields satisfying a natural and invariant condition on higher-order derivatives:

Theorem 4. *Let $k \geq 1$. For every vector field $f = (f_\alpha)_{|\alpha|=k} \in L^1(\mathbb{R}^n; \mathbb{R}^m)$ with $m = \binom{n+k-1}{k}$ and $u = (u_\alpha)_{|\alpha|=k} \in (W^{1,n} \cap L^\infty)(\mathbb{R}^n; \mathbb{R}^m)$, if*

$$\sum_{|\alpha|=k} \partial^\alpha f_\alpha = 0 \tag{3}$$

in the sense of distributions, then

$$\left| \int_{\mathbb{R}^n} f \cdot u \right| \leq C \|f\|_{L^1} \, \|\nabla u\|_{L^n},$$

where the constant C only depends on the dimension of the space n and on the order k.

The condition (3) is invariant: For any change of coordinates of $\mathbb{R}^n$, there is a change of coordinates in $\mathbb{R}^m$ such that the transformed vector field still satisfies (3). Standard linear algebra manipulations show that any translation-invariant condition on k-th order derivatives ensuring that vector fields are in $W^{-1,n/(n-1)}$ can be reduced to condition (3).

Theorem 4 generalizes Theorems 2 and 3. It can be proved by the method developed by Bourgain and Brezis [4] to prove Theorem 3 and by the elementary method of [11, 10].

With their method, Bourgain and Brezis [4] have obtained in fact a very nice result, much stronger than Theorem 3: If $f \in L^1(\mathbb{R}^n; \mathbb{R}^n)$, then one has $f \in W^{-1,n/(n-1)}$ if and only if

$$\sum_{i=1}^{n} \partial_i^k f_i \in W^{-(k+1),n/(n-1)}.$$

Applying their method, we deduce similarly that, for $f \in L^1(\mathbb{R}^n; \mathbb{R}^m)$, one has $f \in W^{-1,n/(n-1)}$ if and only if

$$\sum_{|\alpha|=k} \partial^\alpha f_\alpha \in W^{-(k+1),n/(n-1)}$$

(see Theorem 9 below).

On the other hand, the elementary method of [10] gives the estimate for a wider range of critical Sobolev spaces: If f satisfies the assumptions of Theorem 4, then

$$\left| \int_{\mathbb{R}^n} f \cdot u \right| \le C \|f\|_{\mathrm{L}^1} \, |u|_{\mathrm{W}^{s,p}} \tag{4}$$

for $0 < s < 1$ and $p > n$ such that $sp = n$, where the constant C only depends on n, k and s and where

$$|u|^p_{\mathrm{W}^{s,p}} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy$$

is the fractional Sobolev seminorm. As Bourgain and Brezis explain [4], it is not known whether their method extends to fractional Sobolev spaces. This leads to

Open Problem 1. Let $0 < s < 1$ and $q = n/(n - s)$. Does one have $f \in \mathrm{W}^{-s,q}$ if and only if

$$\sum_{|\alpha|=k} \partial^\alpha f_\alpha \in \mathrm{W}^{-(s+k),q} \ ?$$

As explained in Section 3.4, the elementary method also allows a slight perturbation of the condition (3).

A crucial elementary observation in both proofs consists in rephrasing the statement as

Theorem 5. *Let $k \ge 1$ and let $(a_i)_{1 \le i \le n+k-1} \subset \mathbb{R}^n$ be such that every n-element subset of $\{a_i\}_{i \in \{1,\dots,n-k+1\}}$ is a basis of $\mathbb{R}^n$. For every vector field*

$$f = (f^{i_1 \cdots i_k})_{1 \le i_1 < \cdots < i_k \le n+k-1} \in \mathrm{L}^1(\mathbb{R}^n; \mathbb{R}^m),$$

with $m = \binom{n+k-1}{k}$ and

$$u = (u_{i_1 \dots i_k})_{1 \le i_1 < \cdots < i_k \le n+k-1} \in (\mathrm{W}^{1,n} \cap \mathrm{L}^\infty)(\mathbb{R}^n; \mathbb{R}^m),$$

if

$$\sum_{1 \le i_1 < \cdots < i_k \le n+k-1} \frac{\partial^k f^{i_1 \cdots i_k}}{\partial a_{i_1} \cdots \partial a_{i_k}} = 0 \tag{5}$$

in the sense of distributions, then

$$\left| \int_{\mathbb{R}^n} f \cdot u \right| \le C \|f\|_{\mathrm{L}^1} \, \|\nabla u\|_{\mathrm{L}^n},$$

where the constant C only depends on the dimension n of the space and on k.

This formulation allows one either to perform suitable integrations by parts or to apply a powerful lemma of Bourgain and Brezis [4, Theorem 23] (see Theorem 7 below).

1.3. Organization of the paper

Section 2 gives some handy notations to handle condition (5) and shows how Theorems 4 and 5 can be deduced from each other.

Sections 3 and 4 are completely independent and give proofs of Theorem 5 using either an elementary method or the tools of Bourgain and Brezis.

Section 3 gives a proof in the spirit of [9–11]. It also shows how the arguments go on to fractional critical Sobolev spaces and to the case where the condition (5) is perturbed. The crucial novelty is the integration by parts formula for vector fields satisfying a higher-order condition of Lemma 3.2.

The proof of Section 4 uses the tools of [3, 4] that trace back to [2, 5]. The new arguments that we introduce consist in the definition of a suitable projector and the proof of its properties in Theorem 8.

2. Notations and equivalence between formulations

2.1. Notations

The set of compactly supported smooth functions on $\mathbb{R}^n$ is denoted by $C_c^\infty(\mathbb{R}^n)$. The directional derivative with respect to the direction a is

$$\partial_a v = \lim_{t\to 0} \frac{v(x+ta)-v(x)}{t}$$

(and the corresponding distribution when v is merely a distribution).

We also need some notations in order to alleviate manipulations of condition (5). Let

$$\begin{aligned}\mathcal{I}(n,k) &= \{I \subseteq \{1,\dots,n+k-1\} : I \text{ has } k \text{ elements}\},\\ \mathcal{S}(n,k) &= \{\alpha \in \mathbb{N}^n : |\alpha| = k\},\end{aligned}$$

and $I^c = \{1,\dots,n+k-1\} \setminus I$.

If $I \subseteq J$ are finite sets, we identify $\mathbb{R}^I$ with the following subspace of $\mathbb{R}^J$:

$$\{x \in \mathbb{R}^J : x_j = 0 \text{ if } j \notin I\};$$

we also identify $\mathbb{R}^m$ and $\mathbb{R}^{\{1,\dots,m\}}$.

The index I used as a subscript will always indicate that some formal product is performed over the set I: If $I = \{i_1,\dots,i_k\}$, then

$$\partial_{a_I} v = \partial_{a_{i_1}} \cdots \partial_{a_{i_k}} v, \qquad (a_I|\xi) = (a_{i_1}|\xi)\cdots(a_{i_k}|\xi).$$

2.2. Representation of the k-th order derivative

The main idea behind the equivalence between Theorems 4 and 5 is that both $(\partial^\alpha f)_{|\alpha|=k}$ and $(\partial_{a_I} f)_{I\in\mathcal{I}(n,k)}$ completely characterize the k-th order derivative when the family of vectors $\{a_i\}_{1\le i\le n+k-1}$ is suitably chosen.

Lemma 2.1. *If every n-element subset of $\{a_i\}_{i\in\{1,\dots,n-k+1\}} \subset \mathbb{R}^n$ is a basis of $\mathbb{R}^n$, then there exists an invertible linear operator $M : \mathbb{R}^{\mathcal{S}(n,k)} \to \mathbb{R}^{\mathcal{I}(n,k)}$ such that, for every $\alpha \in \mathcal{S}(n,k)$, for every $u \in C^k(\mathbb{R}^n;\mathbb{R})$ and $x \in \mathbb{R}^n$,*

$$(\partial^\alpha u(x))_{\alpha\in\mathcal{S}(n,k)} = M((\partial_{a_I} u(x))_{I\in\mathcal{I}(n,k)}).$$

In particular, if $f \in C^k(\mathbb{R}^n;\mathbb{R}^{\mathcal{S}(n,k)})$, then

$$\sum_{\alpha\in\mathcal{S}(n,k)} \partial^\alpha f_\alpha(x) = \sum_{I\in\mathcal{I}(n,k)} \partial_{a_I}(M^* f(x))^I,$$

where $M^* : \mathbb{R}^{\mathcal{I}(n,k)} \to \mathbb{R}^{\mathcal{S}(n,k)}$ is the adjoint of M.

Proof. For a fixed x the mapping

$$(\partial^\alpha u(x))_{\alpha\in\mathcal{S}(n,k)} \mapsto (\partial_{a_I} u(x))_{I\in\mathcal{I}(n,k)}$$

clearly gives a well-defined linear operator from $\mathbb{R}^{\mathcal{S}(n,k)}$ to $\mathbb{R}^{\mathcal{I}(n,k)}$. We need to prove that it is one-to-one and onto. Since $\mathbb{R}^{\mathcal{S}(n,k)}$ and $\mathbb{R}^{\mathcal{I}(n,k)}$ have the same dimension $\binom{n+k-1}{k}$, it is sufficient to prove that the mapping is injective.

Assume that, for every $I \in \mathcal{I}(n,k)$, $\partial_{a_I} u = 0$. Fix $J \in \mathcal{I}(n,k-1)$. Since any subset of n elements of $\{a_1,\dots,a_{n+k-1}\}$ forms a basis of $\mathbb{R}^n$, one has, for every $J \in \mathcal{I}(n,k-1)$, $\partial_{a_J} Du = 0$. Thus, by induction, $D^k u = 0$, so that $\partial^\alpha u = 0$ for every $\alpha \in \mathcal{S}(n,k)$. This proves the first claim; the second follows by standard linear operators theory. □

Remark 1. Lemma 2.1 merely states in the language of differential operators that the family $\{(a_I|\xi)\}_{I\in\mathcal{I}(n,k)}$ is a basis of the space of homogeneous polynomials in ξ of degree k.

Until the end of this paper, we fix $(a_i)_{1\le i\le n+k-1} \subset \mathbb{R}^n$ such that every n-element subset of $\{a_i\}_{i\in\{1,\dots,n-k+1\}}$ forms a basis of $\mathbb{R}^n$. Also set, for $x \in \mathbb{R}^{n+k-1}$,

$$Ax = \sum_{1\le i\le n+k-1} a_i x_i,$$

and note that

$$\partial_{a_i} f \circ A = \partial_i (f \circ A). \tag{6}$$

3. Elementary method

3.1. Strategy of proof

In the previous elementary proofs of Theorems 1 and 2 [9–11], the key observation was that a function in $W^{1,n}(\mathbb{R}^n)$ is Hölder continuous on almost every hyperplane. This allowed one to obtain good estimates on hyperplanes which could then be integrated to obtain the conclusion by Hölder's inequality.

In the current setting, the estimate on hyperplanes is given by

Lemma 3.1 (Hölder estimate). *Let* $0 < \gamma < 1$ *and* $f \in L^1(\mathbb{R}^n; \mathbb{R}^{\mathcal{I}(n,k)})$. *If*

$$\sum_{J \in \mathcal{I}(n,k)} \partial_{a_J} f^J = 0, \tag{7}$$

then for every $I \in \mathcal{I}(n,k)$ *and for every* $\varphi \in C^{0,\gamma}(A(\mathbb{R}^{I^c}))$,

$$\left| \int_{A(\mathbb{R}^{I^c})} f^I \varphi \right| \le C \|f\|_{L^1}^{\gamma} \|f^I\|_{L^1(A(\mathbb{R}^{I^c}))}^{1-\gamma} |\varphi|_{C^{0,\gamma}}.$$

Here

$$|\varphi|_{C^{0,\gamma}} = \sup_{x,y \in \mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{\gamma}}.$$

Before proving Lemma 3.1 in Section 3.3, let us see how the estimate on the space follows from the estimate on the hyperplanes.

Proof of Theorem 5. It is sufficient to estimate, for every $I \in \mathcal{I}(n,k)$,

$$\int_{\mathbb{R}^n} f^I u^I.$$

Up to a change of variables and a permutation, we can assume that $I = \{n, \dots, n+k-1\}$ and that, for $1 \le i \le n-1$, the vector a_i is the i-th element of the canonical basis of $\mathbb{R}^n$. We thus have

$$\int_{\mathbb{R}^n} f^I u^I = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f^I(y,t) u^I(y,t)\, dy\, dt.$$

For almost every $t \in \mathbb{R}$, the inner integral can be estimated by Lemma 3.1 together with the Sobolev–Morrey embedding $W^{1,n}(\mathbb{R}^{n-1}) \subset C^{0,1/n}(\mathbb{R}^{n-1})$:

$$\left| \int_{\mathbb{R}^{n-1}} f^I(y,t) u^I(y,t)\, dy \right| \le C \|f\|_{L^1}^{1/n} \|f(\cdot,t)\|_{L^1}^{1-1/n} \|\nabla u(\cdot,t)\|_{L^n}.$$

One concludes by Hölder's inequality and Fubini's theorem. □

Remark 2. The proof of the estimate (4) is similar: the embedding $W^{1,n}(\mathbb{R}^{n-1}) \subset C^{0,1/n}(\mathbb{R}^{n-1})$ should be replaced by the embedding $W^{s,p}(\mathbb{R}^{n-1}) \subset C^{0,\gamma}(\mathbb{R}^{n-1})$ with $\gamma = s - (n-1)/p$ and one should recall that

$$\int_{\mathbb{R}} |u(\cdot,t)|_{W^{s,p}}^p\, dt \le C |u|_{W^{s,p}}^p$$

(see e.g. [1]).

3.2. Integration by parts

The formula

$$\int_{\mathbb{R}^{n-1}} f^n(x,0)\psi(x,0)\,dx = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^+} f(x,t)\cdot\nabla\psi(x,t)\,dt\,dx, \tag{8}$$

when $f \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^n)$ is divergence-free and $\psi \in \mathrm{W}^{1,\infty}(\mathbb{R}^n)$, played a crucial role in the counterpart of Lemma 3.1 of the elementary proof of Theorem 1 in [10]. The treatment of second-order operators required a similar formula [11, Lemma 3]. In this section, we establish a counterpart of (8) under higher-order conditions.

Lemma 3.2. *Assume $f \in (\mathrm{L}^1 \cap C)(\mathbb{R}^n;\mathbb{R}^{\mathcal{I}(n,k)})$, let $I \in \mathcal{I}(n,k)$ and let $\psi \in \mathrm{L}^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n) \cap C^k(\mathbb{R}^n \setminus A(\mathbb{R}^I))$ be such that for every $1 \le j \le k$,*

$$\sup_{x\in\mathbb{R}^n} \operatorname{dist}(x, A(\mathbb{R}^I))^{j-1}|D^j\psi(x)| < \infty.$$

If (7) *holds, then*

$$\int_{\mathbb{R}^{I^c}} (f^I\psi)\circ A = -\sum_{L\in\mathcal{I}(n,k)} \sum_{\substack{L\setminus I\subseteq J\subseteq L\\ J\neq\emptyset}} (-1)^{|J|} \int_{\mathbb{R}^{I^c}\times\mathbb{R}_+^{(I\cap J)\cup(I\setminus L)}} (f^L\partial_{a_J}\psi)\circ A. \tag{9}$$

In particular,

$$\left|\int_{A(\mathbb{R}^{I^c})} f^I\psi\right| \le C\|f\|_{\mathrm{L}^1} \max_{1\le j\le k} \sup_{x\in\mathbb{R}^n} \operatorname{dist}(x, A(\mathbb{R}^I))^{j-1}|D^j\psi(x)|, \tag{10}$$

where the constant C only depends on the dimension n of the space and on the order k.

Remark 3. Lemma 3.2 allows us to *define* $f|_{A(\mathbb{R}^{I^c})}$ by (9) as a distribution of order k when $f \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^{\mathcal{I}(n,k)})$ satisfies condition (7).

Lemma 3.3. *For every $u \in C^\infty(\mathbb{R}^I)$ and $v \in C_c^\infty(\mathbb{R}^I)$, one has*

$$\int_{\mathbb{R}_+^I} v\partial_{e_I}u = \sum_{J\subseteq I}(-1)^{|J|}\int_{\mathbb{R}_+^J} u\partial_{e_J}v,$$

where $(e_i)_{i\in I}$ is the canonical basis of $\mathbb{R}^I$.

Proof. This is proved by integration by parts and by induction on the number of elements of I. □

The other ingredient will be

Lemma 3.4. *Let $k \geq 0$, $H \subseteq \mathbb{R}^n$ be a vector subspace, $d(x) = \operatorname{dist}(x, H)$ and $\psi \in \mathrm{L}^\infty(\mathbb{R}^n) \cap C^1(\mathbb{R}^n) \cap C^k(\mathbb{R}^n \setminus H)$. If, for every $1 \leq j \leq k$,*

$$\sup_{x \in \mathbb{R}^n} d(x)^{j-1} |D^j \psi(x)| < \infty,$$

then there exists a sequence $(\psi_m) \subseteq C_c^k(\mathbb{R}^n)$ such that, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \psi_m(x) &\to \psi(x) \quad \text{for every } x \in \mathbb{R}^n, \\ D^j \psi_m(x) &\to D^j \psi(x) \quad \text{for every } 1 \leq j \leq k \text{ and } x \in \mathbb{R}^n \setminus A(\mathbb{R}^{I^c}), \end{aligned}$$

$$\sup_{m \in \mathbb{N}} \|\psi_m\|_{\mathrm{L}^\infty} < \infty, \qquad \sup_{m \in \mathbb{N}} \sup_{x \in \mathbb{R}^n} d(x)^{j-1} |D^j \psi_m(x)| < \infty \quad \text{for every } 1 \leq j \leq k.$$

Proof. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \rho = 1$, $\operatorname{supp} \rho \subset B(0,1)$, and set $\rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon)$. Also let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 1$ when $|x| \leq 1$ and $\eta(x) = 0$ when $|x| \geq 2$. Set $\eta_\varepsilon(x) = \eta(\varepsilon x)$ and define

$$\psi_\varepsilon = \eta_\varepsilon (\rho_\varepsilon * \psi).$$

The convergences $\psi_\varepsilon(x) \to \psi(x)$ and $D^j \psi_\varepsilon(x) \to D^j \psi(x)$ follow immediately.

If $d(x) \leq 2\varepsilon$, one has, for every $1 \leq i \leq k$,

$$|D^i (\rho_\varepsilon * \psi)(x)| = |D^{i-1} (\rho_\varepsilon * D\psi)(x)| \leq \frac{C}{\varepsilon^{i-1}} \|D\psi\|_{\mathrm{L}^\infty} \leq \frac{2^{i-1} C}{d(x)^{i-1}} \|D\psi\|_{\mathrm{L}^\infty},$$

while, if $d(x) \geq 2\varepsilon$,

$$|D^i (\rho_\varepsilon * \psi)(x)| \leq \sup_{d(y) \geq d(x) - \varepsilon} |D^i \psi(y)| \leq \frac{2^{i-1}}{d(x)^{i-1}} \sup d(y)^{i-1} |D^i \psi(y)|.$$

Hence,

$$\sup_{x \in \mathbb{R}^n \setminus H} d(x)^{i-1} |D^i (\rho_\varepsilon * \psi)(x)| \leq C < \infty. \tag{11}$$

On the other hand, for $i \geq 0$,

$$|D^i \eta_\varepsilon(x)| \leq \frac{C}{|x|^i} \leq \frac{C}{d(x)^i} \tag{12}$$

and, for $i \geq 1$ and $\varepsilon \leq 1$,

$$|D^i \eta_\varepsilon(x)| \leq \frac{C}{|x|^{i-1}} \leq \frac{C}{d(x)^{i-1}}. \tag{13}$$

Since

$$|D^j \psi_\varepsilon(x)| \leq C \sum_{0 \leq i \leq j} |D^i (\rho_\varepsilon * \psi)(x)|\, |D^{j-i} \eta_\varepsilon(x)|,$$

one concludes with (11), (12), (13) and the boundedness of ψ. □

Proof of Lemma 3.2. Let us first assume that $f \in C^\infty(\mathbb{R}^n)$ and $\psi \in C_c^\infty(\mathbb{R}^n)$. By condition (7), we have

$$\sum_{L \in \mathcal{I}(n,k)} \int_{\mathbb{R}^{I^c} \times \mathbb{R}^I_+} (\psi \, \partial_{a_L} f^L) \circ A = 0. \tag{14}$$

Integrating by parts and developing each term according to Lemma 3.3, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{I^c} \times \mathbb{R}^I_+} (\psi \partial_{a_L} f^L) \circ A &= (-1)^{|L \setminus I|} \int_{\mathbb{R}^{I^c} \times \mathbb{R}^I_+} (\partial_{a_{L \cap I}} f^L \, \partial_{a_{L \setminus I}} \psi) \circ A \\
&= (-1)^{|L \setminus I|} \sum_{K \subseteq I \cap L} (-1)^{|K|} \int_{\mathbb{R}^{I^c} \times \mathbb{R}_+^{K \cup (I \setminus L)}} (f^L \, \partial_{a_{K \cup (L \setminus I)}} \psi) \circ A \\
&= \sum_{L \setminus I \subseteq J \subseteq L} (-1)^{|J|} \int_{\mathbb{R}^{I^c} \times \mathbb{R}_+^{(I \cap J) \cup (I \setminus L)}} (f^L \, \partial_{a_J} \psi) \circ A.
\end{aligned}$$

Putting this into (14), we obtain (9).

In the case where f is merely continuous, one obtains (9) by approximation by convolution and by Lebesgue's dominated convergence theorem. In the general case, note that since $a_i \notin A(\mathbb{R}^{I^c})$ for $i \in I$ and since J and $(I \cap J) \cup (I \setminus L)$ have the same number of elements, one has

$$\int_{\mathbb{R}^{I^c} \times \mathbb{R}^J_+} \frac{|f \circ A|}{\operatorname{dist}(Ax, A(\mathbb{R}^{I^c}))^{|J|-1}} \leq C \|f\|_{\mathrm{L}^1}.$$

Approximating ψ by Lemma 3.4 with $H = A(\mathbb{R}^{I^c})$, we conclude by Lebesgue's dominated convergence theorem.

The estimate (10) follows immediately. $\square$

3.3. The Lipschitz and Hölder estimates

Using the integration by parts formula of Lemma 3.2 we can now go on to the proof of Lemma 3.1.

Lemma 3.5. *Let $f \in \mathrm{L}^1(\mathbb{R}^n; \mathbb{R}^{\mathcal{I}(n,k)})$. If (7) holds, then for every $I \in \mathcal{I}(n,k)$ and for every $\varphi \in \mathrm{W}^{1,\infty}(A(\mathbb{R}^{I^c}))$,*

$$\left| \int_{A(\mathbb{R}^{I^c})} f^I \varphi \right| \leq C \|f\|_{\mathrm{L}^1} \|\nabla \varphi\|_{\mathrm{L}^\infty}.$$

Lemma 3.6. *Let $H \subset \mathbb{R}^n$ be a hyperplane. If $\varphi \in C^1(\mathbb{R}^{n-1})$ is such that $\nabla\varphi$ is bounded, then there exists $\psi \in C^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \mathbb{R}^{n-1})$ such that $\psi(x, 0) = \varphi(x)$, $\|\psi\|_{\mathrm{L}^\infty} = \|\varphi\|_{\mathrm{L}^\infty}$, and for every $k \geq 1$,*

$$\sup_{x \in \mathbb{R}^n} \operatorname{dist}(x, H)^{k-1} |D^k \psi(x)| \leq C_k \|\nabla \varphi\|_{\mathrm{L}^\infty}.$$

Proof. Choose the coordinate axes in such a manner that $H = \mathbb{R}^{n-1} \times \{0\}$. Let $\rho \in C_c^\infty(\mathbb{R}^{n-1})$ be such that $\int_{\mathbb{R}^n} \rho = 1$ and let $\rho_t(x) = \rho(x/t)/t^{n-1}$. Define ψ as

$$\psi(x,t) = (\rho_t * \varphi)(x).$$

The estimates then follow directly (see e.g. similar estimates in [7, Chapter V, §4]). $\square$

We are now in a position to obtain the Lipschitz estimate and to deduce therefrom the Hölder estimate.

Proof of Lemma 3.5. Extend φ to ψ according to Lemma 3.6 and apply the estimate (10) of Lemma 3.2. $\square$

Proof of Lemma 3.1. The conclusion is obtained by interpolation between the elementary inequality

$$\left|\int_{A(\mathbb{R}^{I^c})} f^I \varphi\right| \leq C\|f^I\|_{\mathrm{L}^1(A(\mathbb{R}^{I^c}))}\|\varphi\|_{\mathrm{L}^\infty}$$

and the estimate

$$\left|\int_{A(\mathbb{R}^{I^c})} f^I \varphi\right| \leq C\|f\|_{\mathrm{L}^1(\mathbb{R}^n)}\|\nabla\varphi\|_{\mathrm{L}^\infty}$$

that was obtained in Lemma 3.5. For every $\varepsilon > 0$, there exists $\varphi_\varepsilon \in C^1(\mathbb{R}^I)$, constructed e.g. by standard mollification, such that

$$\|\varphi - \varphi_\varepsilon\|_{\mathrm{L}^\infty} \leq C\varepsilon^\gamma |\varphi|_{C^{0,\gamma}}, \qquad \|\nabla\varphi_\varepsilon\|_{\mathrm{L}^\infty} \leq C\varepsilon^{\gamma-1}|\varphi|_{C^{0,\gamma}}.$$

Taking $\varepsilon = \|f\|_{\mathrm{L}^1}/\|f\|_{\mathrm{L}^1(A(\mathbb{R}^{I^c}))}$ yields the conclusion. $\square$

3.4. *Estimates under perturbations*

The elementary proof of Theorem 1 given in [10] allows some perturbation on the divergence-free condition. Indeed, in [10] it was proved that if $f \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^n)$, $\operatorname{div} f \in \mathrm{L}^1(\mathbb{R}^n)$ and $u \in (\mathrm{W}^{1,n} \cap \mathrm{L}^\infty)(\mathbb{R}^n;\mathbb{R}^n)$, then

$$\left|\int_{\mathbb{R}^n} f \cdot u\right| \leq C(\|f\|_{\mathrm{L}^1}\|\nabla u\|_{\mathrm{L}^n} + \|\operatorname{div} f\|_{\mathrm{L}^1}\|u\|_{\mathrm{L}^n}).$$

Similar results can be obtained for higher-order operators.

Performing the same computations as in Lemma 3.2, one has

Lemma 3.7. *Let $f \in (L^1 \cap C)(\mathbb{R}^n;\mathbb{R}^{\mathcal{I}(n,k)})$, $g_l \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^{\mathcal{I}(n,l)})$ for $0 \leq l \leq k-1$, $I \in \mathcal{I}(n,k)$, and let $\psi \in \mathrm{L}^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n) \cap C^k(\mathbb{R}^n \setminus A(\mathbb{R}^I))$ be such that for every $1 < l < k$,*

$$\sup_{x\in\mathbb{R}^n} \operatorname{dist}(x, A(\mathbb{R}^I))^{l-1}|D^l\psi(x)| < \infty.$$

If

$$\sum_{J\in\mathcal{I}(n,k)} \partial_{a_J} f^J = \sum_{l=0}^{k-1} \sum_{J\in\mathcal{I}(n,l)} \partial_{a_J} g_l^J,$$

then

$$\begin{aligned}\int_{\mathbb{R}^{I^c}} (f^I\psi)\circ A = &-\sum_{L\in\mathcal{I}(n,k)} \sum_{\substack{L\setminus I\subseteq J\subseteq L\\ J\neq\emptyset}} (-1)^{|J|} \int_{\mathbb{R}^{I^c}\times\mathbb{R}_+^{(I\cap J)\cup(I\setminus L)}} (f^L\,\partial_{a_J}\psi)\circ A\\ &+\sum_{l=0}^{k-1}\sum_{L\in\mathcal{I}(n,l)}\sum_{L\setminus I\subseteq J\subseteq L} (-1)^{|J|}\int_{\mathbb{R}^{I^c}\times\mathbb{R}_+^{(I\cap J)\cup(I\setminus L)}} (g_l^L\,\partial_{a_J}\psi)\circ A.\end{aligned}$$

In particular,

$$\begin{aligned}\left|\int_{A(\mathbb{R}^{I^c})} f^I\psi\right| \le C\Big[&\|f\|_{\mathrm{L}^1} \max_{1\le j\le k}\sup_{x\in\mathbb{R}^n} \operatorname{dist}(x, A(\mathbb{R}^I))^{j-1}|D^j\psi(x)|\\ &+\sum_{0\le l\le k-1}\|g_l\|_{\mathrm{L}^1}\max_{0\le j\le l}\sup_{x\in\mathbb{R}^n}\operatorname{dist}(x, A(\mathbb{R}^I))^{k-l+j-1}|D^j\psi(x)|\Big],\end{aligned}$$

where the constant C only depends on the dimension n of the space and on the order k, and where $g_l = (g_l^L)_{L\in\mathcal{I}(n,l)}$.

The proof of Lemma 3.7 is similar to that of Lemma 3.2 and allows us to extend Theorem 5 to

Theorem 6. *Assume $f \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^{\mathcal{I}(n,k)})$, $g_j \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^{\mathcal{I}(n,l)})$ for $\max(0, k-n) \le l \le k-1$, and let $u \in (\mathrm{W}^{1,n}\cap\mathrm{L}^\infty)(\mathbb{R}^n;\mathbb{R}^{\mathcal{I}(n,k)})$. If*

$$\sum_{L\in\mathcal{I}(n,k)} \partial_{a_I} f = \sum_{\substack{0\le l\le k-1\\ l\ge k-n}}\sum_{L\in\mathcal{I}(n,l)} \partial_{a_L} g_l^L, \tag{15}$$

then

$$\left|\int_{\mathbb{R}^n} f\cdot u\,dx\right| \le C\Big[\|f\|_{\mathrm{L}^1}\|\nabla u\|_{\mathrm{L}^n} + \sum_{\substack{0\le l\le k-1\\ l\ge k-n}} \|g_l\|_{\mathrm{L}^1}\|u\|_{\mathrm{L}^{n/(k-l)}}\Big], \tag{16}$$

where the constant C only depends on the dimension n of the space and on k.

Remark 4. As for Theorem 5, $\|\nabla u\|_{\mathrm{L}^n}$ can be replaced by $|u|_{\mathrm{W}^{s,p}}$ in (16).

Theorem 6 is proved just as Theorem 5 once one has the following estimate:

Lemma 3.8. *Let* $f \in \mathrm{L}^1(\mathbb{R}^n; \mathbb{R}^{\mathcal{I}(n,k)})$ *and* $g_j \in \mathrm{L}^1(\mathbb{R}^n; \mathbb{R}^{\mathcal{I}(n,l)})$ *for* $\max(0, k-n) \leq l \leq k-1$. *If* (15) *holds, then for every* $I \in \mathcal{I}(n,k)$ *and for every* $\varphi \in (C^{0,\gamma} \cap \bigcap_{\substack{0 \leq l \leq k-1 \\ l \geq k-n}} \mathrm{L}^{n/(k-l)})(A(\mathbb{R}^{I^c}))$,

$$\left| \int_{A(\mathbb{R}^{I^c})} f^I \varphi \right| \leq C \Big[\|f\|_{\mathrm{L}^1}^{\gamma} \|f^I\|_{\mathrm{L}^1(A(\mathbb{R}^{I^c}))}^{1-\gamma} |\varphi|_{C^{0,\gamma}} + \sum_{\substack{0 \leq l \leq k-1 \\ l \geq k-n}} (\|f^I\|_{\mathrm{L}^1(A(\mathbb{R}^{I^c}))} / \|f\|_{\mathrm{L}^1})^{1-(k-l)/n} \|g_l\|_{\mathrm{L}^1} \|\varphi\|_{\mathrm{L}^{n/(k-l)}} \Big].$$

Proof. The proof goes as the proof of Lemma 3.1. Replacing Lemma 3.2 by Lemma 3.7, one obtains the counterpart of (3.5):

$$\left| \int_{A(\mathbb{R}^{I^c})} f^I \varphi \right| \leq C \Big[\|f\|_{\mathrm{L}^1} \|\nabla \varphi\|_{\mathrm{L}^\infty} + \sum_{\substack{0 \leq l \leq k-1 \\ l \geq k-n}} \|g_l\|_{\mathrm{L}^1} \|\varphi\|_{\mathrm{L}^{(n-1)/(k-l-1)}} \Big].$$

The parameter ε is then chosen exactly in the same way and the additional terms are controlled by Young's convolution inequality. □

Remark 5. When $k \geq n+1$, an unnatural restriction $l \geq k-n$ appears in the statement of Theorem 6. This restriction does not come from the integration by parts of Lemma 3.7, but from the estimate of Lemma 3.8. In the latter lemma, one needs to find a norm on φ that plays the role of $\|\varphi\|_{\mathrm{L}^{(n-1)/(k-l-1)}}$ when $(n-1)/(k-l-1) < 1$. In order to make the proof work this norm should satisfy some kind of Hölder-type inequality and some kind of Fubini theorem. While the Lebesgue spaces $\mathrm{L}^{(n-1)/(k-l-1)}$ and the real Hardy space $\mathrm{H}^{(n-1)/(k-l-1)}$ have the right homogeneity, they do not seem to have properties that could play the role of Hölder's inequality or Fubini's theorem.

4. The Bourgain–Brezis approach

4.1. *Estimate on the torus*

The proof of Theorem 3 by Bourgain and Brezis was based on the following result:

Theorem 7 (Bourgain and Brezis [4]). *Let* $\mathcal{X} \subseteq \mathrm{L}^2(\mathbb{T}^n; \mathbb{R}^r)$ *be an invariant function space and assume that the orthogonal projection* P *on* $\mathcal{X}$ *satisfies*

$$\|Pf\|_{\mathrm{L}^p} \leq C_p \sum_{s=1}^{r} \|A_s \mathcal{R} f_s\|_{\mathrm{L}^p} \quad \textit{for all } 1 < p < \infty$$

for some fixed singular matrices $A_s \in \mathbb{Q}^{n \times n}$ $(1 \leq s \leq r)$ *and where* $\mathcal{R}$ *denotes the vector-valued Riesz transform. Then, for every* $u \in \mathrm{W}^{-1,n/(n-1)}(\mathbb{T}^n, \mathbb{R}^r)$,

$$\|u\|_{\mathrm{W}^{-1,n/(n-1)}} \leq C(\|u\|_{\mathrm{L}^1} + \operatorname{dist}(u, X)),$$

where dist *denotes the distance in* $\mathrm{W}^{-1,n/(n-1)}$.

Remark 6. Theorem 7 is an easy variant of Theorem 23 in [4]. In the spirit of the remarks preceding Theorem 10′ therein, Theorem 10 can be replaced in the proof of Theorem 23 by a variant of Theorem 10′ where $\mathbb{R}^n$ would be replaced by the torus $\mathbb{T}^n$ and A_s would be assumed to be *rational* singular matrices.

In order to state the higher-order estimate on the torus, define, for $a_1, \dots, a_{n+k-1} \in \mathbb{R}^n$ and $u \in \mathrm{L}^1(\mathbb{T}^n; \mathbb{R}^{\mathcal{I}(n,k)})$, the operator

$$Tf = \sum_{I \in \mathcal{I}(n,k)} \partial_{a_I} f^I.$$

Theorem 8. *Assume that $a_i \in \mathbb{Q}^n$ and every n-element subset of $\{a_i\}_{1 \le i \le n+k-1}$ is a basis of $\mathbb{Q}^n$. If $f \in \mathrm{L}^1(\mathbb{T}^n; \mathbb{R}^{\mathcal{I}(n,k)})$ and $Tf \in \mathrm{W}^{-(k+1),n/(n-1)}(\mathbb{T}^n)$, then $f \in \mathrm{W}^{-1,n/(n-1)}(\mathbb{T}^n; \mathbb{R}^{\mathcal{I}(n,k)})$ and*

$$\|f\|_{\mathrm{W}^{-1,n/(n-1)}} \le C(\|f\|_{\mathrm{L}^1} + \|Tf\|_{\mathrm{W}^{-(k+1),n/(n-1)}}).$$

Remark 7. If $f \in \mathrm{W}^{-1,n/(n-1)}(\mathbb{T}^n; \mathbb{R}^{\mathcal{I}(n,k)})$, one has $Tf \in \mathrm{W}^{-(k+1),n/(n-1)}(\mathbb{T}^n)$. The condition $Tf \in \mathrm{W}^{-(k+1),n/(n-1)}(\mathbb{T}^n)$ is thus necessary and sufficient.

Proof of Theorem 8. Consider the invariant space

$$\mathcal{X} = \{f \in \mathrm{L}^2(\mathbb{T}^n; \mathbb{R}^{\mathcal{I}(n,k)}) : Tf = 0\}.$$

The orthogonal projection $P : \mathrm{L}^2(\mathbb{T}^n; \mathbb{R}^{\mathcal{I}(n,k)}) \to \mathcal{X}$ is

$$(\widehat{Pf})^I(\xi) = \widehat{f}^I(\xi) - \frac{(a_I|\xi)}{\Lambda(\xi)} \sum_{J \in \mathcal{I}(n,k)} (a_J|\xi)\widehat{f}^J(\xi)$$

for $I \in \mathcal{I}(n,k)$, where

$$\Lambda(\xi) = \sum_{J \in \mathcal{I}(n,k)} (a_J|\xi)^2.$$

One also has

$$(\widehat{Pf})^I(\xi) = \sum_{J \in \mathcal{I}(n,k) \setminus \{I\}} \frac{(a_J|\xi)}{\Lambda(\xi)} ((a_J|\xi)\widehat{f}^I(\xi) - (a_I|\xi)\widehat{f}^J(\xi)).$$

Since every n-element subset of $\{a_i\}_{1 \le i \le n+k-1}$ is a basis of $\mathbb{Q}^n$, for every $\xi \in \mathbb{R}^n \setminus \{0\}$ there is $I \in \mathcal{I}(n,k)$ such that $(a_i|\xi) \neq 0$ for every $i \in I$. Therefore $\Lambda(\xi) \neq 0$. Setting

$$m^J(\xi) = \frac{(a_J|\xi)|\xi|^k}{\Lambda(\xi)},$$

one sees that m^J is dilation-invariant and $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and therefore acts boundedly on $\mathrm{L}^p(\mathbb{R}^n)$ (see e.g. [7, Theorem 6 in Chapter 3, §3.5, together with Theorem 3 in Chapter 2, §4.2]). Recalling moreover that $\mathcal{R}$ is a bounded operator on $\mathrm{L}^p(\mathbb{R}^n)$, we obtain

$$\|Pf\|_{\mathrm{L}^p} \le C \sum_{\substack{I,J \in \mathcal{I}(n,k) \\ I \neq J}} \big\|(a_J|\mathcal{R}) f^I\big\|_{\mathrm{L}^p} \le C' \sum_{I \in \mathcal{I}(n,k)} \sum_{\substack{1 \le s \le n+k-1 \\ s \notin I}} \big\|(a_s|\mathcal{R}) f^I\big\|_{\mathrm{L}^p}$$

(where $(a|\mathcal{R})v = (a|\mathcal{R}v)$). Therefore P satisfies the assumptions of Theorem 7. Hence

$$\|f\|_{W^{-1,n/(n-1)}} \leq C(\|f\|_{L^1} + \operatorname{dist}(f, \mathcal{X})).$$

Since

$$\widehat{(f - Pf)}^I(\xi) = m^I(\xi)\frac{\widehat{Tf}(\xi)}{|\xi|^k},$$

recalling that m^I acts boundedly on L^p, one concludes that

$$\operatorname{dist}(f, \mathcal{X}) \leq \|f - Pf\|_{W^{-1,n/(n-1)}} \leq C\|Tf\|_{W^{-(k+1),n/(n-1)}}. \qquad \square$$

Remark 8. The choice of the condition (5) instead of (3) has given to the space of vector fields a norm for which the orthogonal projector satisfies the assumptions of Theorem 7. Since M given by Lemma 2.1 need not be an isometry, the projection on

$$\tilde{\mathcal{X}} = \Big\{f \in L^2(\mathbb{T}^n; \mathbb{R}^{\mathcal{S}(n,k)}) : \sum_{|\alpha|=k} \partial^\alpha f_\alpha = 0\Big\}$$

is not related to the projection on $\mathcal{X}$ and need not have its good properties.

4.2. *Estimates on the whole space*

As in [4], Theorem 8 can be transported from the torus $\mathbb{T}^n$ to the euclidean space $\mathbb{R}^n$.

Theorem 9. *Assume that $a_i \in \mathbb{R}^n$ and every n-element subset of $\{a_i\}_{1\leq i\leq n+k-1}$ is a basis of $\mathbb{R}^n$. If $f \in L^1(\mathbb{R}^n; \mathbb{R}^{\mathcal{I}(n,k)})$ and $Tf \in W^{-(k+1),n/(n-1)}(\mathbb{R}^n)$, then $f \in W^{-1,n/(n-1)}(\mathbb{R}^n; \mathbb{R}^{\mathcal{I}(n,k)})$ and*

$$\|f\|_{W^{-1,n/(n-1)}} \leq C(\|f\|_{L^1} + \|Tf\|_{W^{-(k+1),n/(n-1)}}).$$

Proof. By Lemma 2.1, it suffices to prove the result for $a_i \in \mathbb{Q}^n$.

The proof is the same as in [4, Corollary 24′]. We just sketch the idea. Fix $\varphi \in C_c(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset]-1, 1[^n \cong \mathbb{T}^n$ and let $u \in (W^{1,n} \cap L^\infty)(\mathbb{R}^n; \mathbb{R}^{\mathcal{I}(n,k)})$. Defining, for $R \geq 1$, $f_R \in L^1(\mathbb{T}^n; \mathbb{R}^{\mathcal{I}(n,k)})$ and $u_R \in W^{1,n}(\mathbb{T}^n; \mathbb{R}^{\mathcal{I}(n,k)})$ by

$$f_R(x) = \varphi(Rx)f(Rx), \qquad u_R(x) = u(Rx)\varphi(Rx),$$

one has, by Theorem 8,

$$\Big|\int_{\mathbb{T}^n} f_R u_R\Big| \leq C(\|f_R\|_{L^1} + \|Tf_R\|_{W^{-(k+1),n/(n-1)}})\|u\|_{W^{1,n}}.$$

Since

$$R^n \int_{\mathbb{T}^n} f_R u_R \to \int_{\mathbb{R}^n} fu, \qquad R^n\|f_R\|_{L^1(\mathbb{T}^n)} \to \|f\|_{L^1(\mathbb{R}^n)},$$

$$R^n\|Tf_R\|_{W^{-(k+1),n/(n-1)}(\mathbb{T}^n)} \to \|Tf\|_{W^{-(k+1),n/(n-1)}(\mathbb{R}^n)}, \qquad \|u_R\|_{W^{1,n}(\mathbb{T}^n)} \to \|Du\|_{L^n(\mathbb{R}^n)},$$

as $R \to \infty$, the conclusion follows. $\square$

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