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Three-space problems for the approximation property

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Abstract. It is shown that there is a subspace Z_q of ℓ_q for $1 < q < 2$ which is isomorphic to ℓ_q and such that ℓ_q/Z_q does not have the approximation property (AP). On the other hand, for $2 < p < \infty$ there is a subspace Y_p of ℓ_p such that Y_p does not have AP but ℓ_p/Y_p is isomorphic to ℓ_p. The result is obtained by defining random "Enflo–Davie spaces" Y_p which with full probability fail to have AP for all $2 < p \le \infty$ and have AP for all $1 \le p \le 2$. For $1 < p \le 2$, Y_p is isomorphic to ℓ_p.

Keywords. Quotients of Banach spaces, approximation property

In this paper we prove the result stated in the abstract. In particular, it solves the following problem which has been around since the 1970's:

(♣) *Does there exist a reflexive Banach space X and a subspace $Y \subset X$ such that both X and Y have AP but X/Y does not have AP?*

Let us recall that a Banach space X is said to have the *approximation property* (AP) if for every compact set K in X and for every $\varepsilon > 0$, there is a finite rank operator $T = T_K$ on X such that $\|Tx - x\| \le \varepsilon$ for every $x \in K$. When all these T_K's are uniformly bounded, we say that X has the *bounded approximation property* (BAP). If a Banach space has a (Schauder) basis, then it has BAP.

In 1972 P. Enflo gave the first example of a Banach space without AP.

A thorough discussion of approximation properties can be found in [4].

The problem (♣) is a sort of a "three space problem". These are problems of the following type:

Let X, Y, Z be Banach spaces with $Y \subset X$ and $Z = X/Y$, or putting it in a fancier way, let $0 \to Y \to X \to Z \to 0$ be a short exact sequence. Suppose that two of the spaces X, Y, Z have a certain property $\mathcal{P}$. Does this imply that the third one has $\mathcal{P}$?

Thus for every property $\mathcal{P}$ we have three different three-space problems; let us call them, respectively, the X, Y, Z-problems (e.g., if X, Y have $\mathcal{P}$, we have the Z-problem, etc.).

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In the context of AP these three problems are mutually inequivalent and none is trivial:

(i) By [2], every separable Banach space X contains a subspace Y such that both Y and X/Y have BAP. Taking for X a separable Banach space without AP, we obtain a counterexample to the X-problem.
(ii) It follows from our result that the Y-problem has a negative solution, i.e. there exist X with AP and $Y \subset X$ without AP so that X/Y has AP. We do not know whether this follows from older results.
(iii) By [3], for every separable Banach space Z there is a Banach space Y with basis such that Y^{**} has a basis and Y^{**}/Y is isomorphic to Z. Taking for Z a separable space without AP, we obtain a counterexample to the Z-problem. Let us observe that one cannot dualize this example in order to obtain a negative solution to the Y-problem.

Evidently, in (iii) $X = Y^{**}$ is inherently a nonreflexive space and this approach is unlikely to provide a reflexive example. This makes the question (♣) quite natural.

To prove our main result (Theorem 2) we of course use the Enflo–Davie machinery for constructing subspaces of ℓ_p, $p > 2$, without AP. The most important new tool is Theorem 3, which is a delicate refinement of Kashin's splitting theorem and quite likely will have other applications.

1. Preliminaries

Some notation

For $\alpha = (\alpha(j))_{j=1}^n \in \mathbb{R}^n$ and $1 \le s \le \infty$ we define

$$|\alpha|_s = \Big(n^{-1}\sum_{j=1}^n |\alpha(j)|^s\Big)^{1/s} \qquad (|\alpha|_\infty = \max|\alpha(j)|)$$

and for $\alpha, \beta \in \mathbb{R}^n$ we define

$$\langle \alpha, \beta\rangle = n^{-1}\sum_{j=1}^n \alpha(j)\beta(j).$$

Let X denote the (real) vector space of all sequences $a = (a_k)_{k=0}^\infty$ with $a_k \in \mathbb{R}^{3\cdot 2^k}$ and $a_k = 0$ for almost all k. For $a, b \in X$ we define

$$\langle a, b\rangle = \sum_{k=0}^\infty \langle a_k, b_k\rangle.$$

The norms $\|\cdot\|_{s,p}$ on X are defined by

$$\|(a_k)_{k=0}^\infty\|_{s,p} = \Big(\sum_{k=0}^\infty |a_k|_s^p\Big)^{1/p}.$$

We shall identify $\mathbb{R}^{3\cdot 2^k}$ with the subspace $\{(0,\dots,u,0,\dots) : u \in \mathbb{R}^{3\cdot 2^k}\}$ of X. Notice that if $u \in \mathbb{R}^{3\cdot 2^k}$, then $\|u\|_{s,p} = |u|_s$ for every s, p.

Let $X_{s,p}$ be the completion of X in the norm $\|\cdot\|_{s,p}$. It is obvious that $X_{p,p}$ is isometric to ℓ_p. By Pełczyński's decomposition method (cf. [4, p. 54]), we know that $X_{2,p}$ is isomorphic to ℓ_p for $1 < p < \infty$.

Let $E \subset \mathbb{R}^m$ and $F \subset \mathbb{R}^p$ with $\dim F \leq \dim E$. Denote by $O(F,E)$ the set of all orthogonal (with respect to the above inner products in $\mathbb{R}^m$, $\mathbb{R}^p$, respectively) transformations from F to E and let $\nu_{F,E}$ denote the normalized invariant measure on $O(F,E)$. We also set $O(n) = O(\mathbb{R}^n,\mathbb{R}^n)$ and $\nu_n = \nu_{\mathbb{R}^n,\mathbb{R}^n}$.

Let $E \subset \mathbb{R}^m$. Let $S_E = \{x \in E : |x|_2 = 1\}$, let σ_E denote the normalized surface measure on S_E, and set $S_m = S_{\mathbb{R}^m}$ and $\sigma_m = \sigma_{\mathbb{R}^m}$.

The normalized unit vectors $e_i^m \in \mathbb{R}^m$ are defined by $e_i^m = m^{1/2}(\delta_{ij})_{j=1}^m$.

Throughout this paper C and c are fixed positive numbers.

Enflo's construction (cf. [4, Ch. 2.d])

Let us briefly recall this classical construction, in the form due to Davie. The following Theorem 1 is a "genericized" formulation.

Given a sequence $\{T_k\}_{k=0}^\infty$ with $T_k \in O(3\cdot 2^k)$ for $k = 0,1,\dots$, define

$$\begin{aligned}
u_i^k &= T_k^*(e_i^{3\cdot 2^k}) \qquad \text{for } i = 1,\dots,2^k,\\
v_i^k &= T_k^*(e_{2^k+i}^{3\cdot 2^k}) \qquad \text{for } i = 1,\dots,2^{k+1},\\
y_i^k &= v_i^{k-1} + u_i^k \quad \text{and} \quad z_i^k = v_i^{k-1} - u_i^k \qquad \text{for } i = 1,\dots,2^k,\ k = 1,2,\dots.
\end{aligned} \tag{1}$$

Let $Y = \operatorname{span}\{y_i^k \in X : k = 1,2,\dots,\ i = 1,\dots,2^k\}$ and let $Y_p = Y_p(\{T_k\})$ be the completion of Y in the norm $\|\cdot\|_{p,p}$.

Theorem 1. *There is a sequence $\{T_k\}_{k=0}^\infty$ such that $Y_p = Y_p(\{T_k\})$ does not have AP for all $2 < p \leq \infty$. Moreover, this is a generic fact, i.e. denoting by ν the product measure $\nu = \bigotimes \nu_{3\cdot 2^k}$ we have*

$$\nu(\{\{T_k\} : Y_p(\{T_k\}) \textit{ fails to have AP for all } 2 < p \leq \infty\}) = 1.$$

In this and in the next theorem we ignore the issues of measurability. What we really mean is that the above set contains a set of full measure.

We shall not prove this theorem; we think it belongs to the folklore of Banach space theory, although probably it has never been formulated this way. On demand, we are ready to supply the details of its proof.

Some estimates

Let us first recall two "volumetric" estimates (cf. [6, Ch. 6, (6.3)]). For $0 < r \leq 1$ define $A_m(r) = \{x \in S_m : |x|_1 \leq r\}$. We have

$$\sigma_m(A_m(r)) \leq (Cr)^m \tag{2}$$

and

$$N(A_m(r), \delta) \leq (3e/r)^{3rm/\delta}. \tag{3}$$

Here for $A \subset \mathbb{R}^m$, by $N(A, \delta)$ we denote the covering number by euclidean balls of radius δ:

$$N(A, \delta) = \min\Big\{N : \exists x_1, \ldots, x_N \in A \text{ such that } A \subset \bigcup_{i=1}^{N} B(x_i, \delta)\Big\}.$$

Notice also that if B is an l-dimensional euclidean unit ball, then

$$N(B, \delta) \leq (3/\delta)^l \tag{4}$$

(cf. [6, Lemma 4.16]).

We shall frequently use the rotational invariance of the measures σ_E and $\nu_{F,E}$. Let us here observe that for every measurable set $A \subset S_E$ and every $x \in S_F$ we have

$$\sigma_E(A) = \nu_{F,E}(\{T \in O(F, E) : Tx \in A\}). \tag{5}$$

2. The construction

We shall prove the following theorem.

Theorem 2.

$$\nu(\{\{T_k\} : Y_q(\{T_k\}) \textit{ is isomorphic to } \ell_q \textit{ for all } 1 < q \leq 2\}) = 1.$$

This, together with Theorem 1, yields all the results stated in the abstract:

With the notation of (1), let $Z = \operatorname{span}\{z_i^k \in X : k = 1, 2, \ldots,\ i = 1, \ldots, 2^k\}$ and let Z_p be the completion of Z in the norm $\|\cdot\|_{p,p}$. Observe that the linear map $R : Y \to Z$ defined by $R(y_i^k) = (-1)^k(z_i^k)$ is a $\|\cdot\|_{p,p}$ isometry for all p, thus Y_p and Z_p are isometric. In particular, Z_p fails to have AP for $2 < p \leq \infty$ and is isomorphic to ℓ_p for $1 < p \leq 2$ (generically).

We have the natural identification $(X_{p,p}/Y_p)^* = Y_p^{\perp} \subset X_{q,q}$ $(1/p + 1/q = 1)$. We see that $X = \mathbb{R} \cdot u_1^0 \oplus Y \oplus Z$ (orthogonal sum in the sense of $\langle\ ,\ \rangle$). Therefore $(X_{p,p}/Y_p)^*$ is isometric to $\mathbb{R} \cdot u_1^0 \oplus Z_q$, hence isomorphic to Z_q.

Therefore $X_{p,p}/Y_p$ is isomorphic to ℓ_p for $2 \leq p < \infty$ (generically). Since AP is a self-dual property for reflexive spaces, we deduce that (generically) $X_{p,p}/Y_p$ fails to have AP for $1 < p < 2$ (and for $p = 1$ as well, but this requires a slightly more delicate argument).

Let E be a subspace of $\mathbb{R}^m$. We define the norm $|\cdot|_E^{\perp}$ on E by

$$|x|_E^{\perp} = \min\{|x+y|_1 : y \in E^{\perp}\}.$$

Set $A_E^{\perp}(r) = \{x \in S_E : |x|_E^{\perp} \le r\}$ (notice that $A_{\mathbb{R}^m}^{\perp} = A_m(r)$). It is important to realize that the estimates (2) and (3) hold for $A_E^{\perp}(r)$ as well:

Lemma 1. *Let $E \subset \mathbb{R}^m$ with $n = \dim E \ge m/3$, and let $0 \le r < \delta/2 \le 1$ with $\delta \le 3rm$. Then*

$$\sigma_E(A_E^{\perp}(r)) \le (Cr)^n \tag{6}$$

and

$$N(A_E^{\perp}(r), \delta) \le (3e/r)^{3rm/\delta}. \tag{7}$$

Proof. To prove (6), let D denote the unit ball of $|\cdot|_E^{\perp}$, thus $D = \operatorname{absconv}\{m^{1/2}Pe_i^m : 1 \le i \le m\}$ where $P : \mathbb{R}^m \to E$ denotes the orthogonal projection. Therefore

$$D = \bigcup_{A \subset \{1,\dots,m\},\ \#A = n} \operatorname{absconv}\{m^{1/2}Pe_i^m : i \in A\}.$$

By Hadamard's inequality, if $\#A = n$, then

$$\operatorname{vol}(\operatorname{absconv}\{m^{1/2}Pe_i^m : i \in A\}) \le \operatorname{vol} B_1^n \le C^n \operatorname{vol} B_2^n$$

(B_p^n denotes the unit ball of $|\cdot|_p$ in $\mathbb{R}^n$). Therefore

$$\operatorname{vol} D \le C^n \binom{m}{n} \operatorname{vol} B_2^n \le C^n 2^m \operatorname{vol} B_2^n \le (8C)^n \operatorname{vol} B_2^n,$$

i.e. the volume ratio of D is bounded by $8C$. By (6.3) in [6], (6) holds.

The proof of (7) is a modification of a classical proof of (3) (cf. [7]): $|\cdot|_E^{\perp}$ is the Minkowski functional of the set $D = \operatorname{absconv}\{m^{1/2}Pe_i^m : 1 \le i \le m\}$. Thus if $x \in A_E^{\perp}(r)$, then $x = m^{1/2}\sum_{i=1}^m t_i Pe_i^m$ with $\sum_{i=1}^m |t_i| \le r$.

Let π be a permutation of $\{1,\dots,m\}$ such that $|t_{\pi(1)}| \ge |t_{\pi(2)}| \ge \cdots$. For every $1 \le l \le m$, we have

$$r \ge \sum_{i=1}^m |t_i| \ge l \cdot |t_{\pi(l)}| \ge \frac{l}{\sqrt{m-l}} \Big(\sum_{i \ge l} t_{\pi(i)}^2\Big)^{1/2}. \tag{8}$$

Let $b = m^{1/2}\sum_{i \ge l} t_{\pi(i)} Pe_i^m = Py$ where $y = m^{1/2}\sum_{i \ge l} t_{\pi(i)} e_i^m$. We have, by (8),

$$|b|_2 \le |y|_2 = \sqrt{m}\Big(\sum_{i \ge l} t_{\pi(i)}^2\Big)^{1/2} \le \frac{r}{l} \cdot \sqrt{m(m-l)} \le \frac{rm}{l}.$$

For $c \in E$ we shall write $l(c) \le l$ if c is a linear combination of at most l of the points Pe_i^m, $1 \le i \le m$. Let now l be such that $rm/l \le \delta/2$.

We see that every $x \in A_E^{\perp}(r)$ can be written as $x = b + c$ where $|b|_2 \leq \delta/2$ and $l(c) \leq l$, $|c|_2 \leq 1$. Therefore we have

$$N(A_E^{\perp}(r), \delta) \leq N(G_l, \delta/2) \quad \text{where} \quad G_l = \{c \in \mathbb{R}^m : l(c) \leq l,\ |c|_2 \leq 1\}.$$

It is clear that G_l can be covered by $\binom{m}{l}$ sets, each isometric to the l-dimensional euclidean unit ball. Therefore, by (4), we have

$$N\left(G_l, \frac{\delta}{2}\right) \leq \binom{m}{l}\left(\frac{6}{\delta}\right)^l \leq \frac{m^l}{l!}\left(\frac{6}{\delta}\right)^l \leq \left(\frac{6em}{l\delta}\right)^l,$$

by Stirling's formula. Substituting $l = [2rm/\delta] + 1$, we obtain (7), because $2rm/\delta \leq l \leq 3rm/\delta$. □

The next theorem is a refinement of Kashin's splitting theorem (cf. [6, Corollary 6.4]). We think that it is of independent interest.

Theorem 3. *Let $E \subset \mathbb{R}^m$ and $F \subset \mathbb{R}^p$ with $\dim E = \dim F = n$ and suppose $m \leq 3n$ and $p \leq 3n$. Then there is $Q \in O(F, E)$ such that*

$$\max\{|x|_F^{\perp}, |Qx|_E^{\perp}\} \geq c|x|_2 \quad \textit{for all } x \in F.$$

Moreover,

$$\nu_{F,E}(\{Q : \max\{|x|_F^{\perp}, |Qx|_E^{\perp}\} \geq c|x|_2 \textit{ for all } x \in F\}) \geq 1 - 2^{-n}. \tag{9}$$

Remark 1. Although conceptually close to Kashin's theorem, apparently this theorem neither implies nor is implied by it.

Remark 2. We have of course

$$\max\{|x|_F^{\perp}, |Qx|_E^{\perp}\} \geq (3n)^{-1/2}|x|_2 \quad \text{for all } x \in F; \tag{10}$$

the point is that the c in Theorem 3 does not depend on n.

Proof of Theorem 3. Let δ, r be positive numbers to be determined later. Let

$$B = \{T \in O(F, E) : |Tx|_E^{\perp} \geq \delta \text{ for every } x \in A_F^{\perp}(r)\}.$$

We see that if $T \in B$, then for every $x \in S_F$ we have

$$\max\{|x|_F^{\perp}, |Tx|_E^{\perp}\} \geq \min(\delta, r).$$

Thus (9) holds with $c = \min(\delta, r)$ provided we can show that $\nu_{F,E}(B) \geq 1 - 2^{-n}$.

Let $N = N(A_F^{\perp}(r), \delta)$ and let $x_1, \ldots, x_N \in S_F$ be such that

$$A_F^{\perp}(r) \subset \bigcup_{i=1}^{N} B(x_i, \delta). \tag{11}$$

Let us observe that $B \supset B'$ where

$$B' = \{T \in O(F, E) : |Tx_i|_E^\perp \geq 2\delta \text{ for } i = 1, \ldots, N\}.$$

Indeed, suppose that $T \in B'$. If $x \in B(x_i, \delta)$, then $Tx \in B(Tx_i, \delta)$, i.e. $Tx = Tx_i + y$ with $|y|_2 \leq \delta$, hence

$$|Tx|_E^\perp \geq |Tx_i|_E^\perp - |y|_E^\perp \geq 2\delta - |y|_2 \geq \delta.$$

Hence, by (11), $|Tx|_E^\perp \geq \delta$ for every $x \in A_F^\perp(r)$, thus $B \supset B'$.

By (5) and (6), we have, for every i,

$$\nu_{F,E}(\{T \in O(F, E) : Tx_i \in A_E^\perp(2\delta)\}) = \sigma_E(A_E^\perp(2\delta)) \leq (2C\delta)^n.$$

Therefore $\nu_{F,E}(B') \geq 1 - N \cdot (2C\delta)^n$, and thus, by (7),

$$\nu_{F,E}(B') \geq 1 - (3e/r)^{3rm/\delta} \cdot (2C\delta)^n. \tag{12}$$

Let now $r = \delta^2$ and let $\delta < 1$ be such that

$$2C\delta \cdot (3e/\delta^2)^{9\delta} \leq 1/2. \tag{13}$$

Since $m \leq 3n$, by (12), $\nu_{F,E}(B') \geq 1 - 2^{-n}$, thus, a fortiori, $\nu_{F,E}(B) \geq 1 - 2^{-n}$. $\square$

It will now be convenient to rephrase Theorem 3 in a setting of orthogonal transformations. Let us first observe that in the situation of Theorem 3, Q determines F and E. More precisely, Q must be a partial isometry of rank n from $\mathbb{R}^m$ to $\mathbb{R}^p$, i.e. a map of the form

$$Q = \sum_{i=1}^{n} v_i \otimes u_i,$$

where $\{v_i\}_{i=1}^n$ is an orthonormal system in $\mathbb{R}^m$ and $\{u_i\}_{i=1}^n$ is an orthonormal system in $\mathbb{R}^p$. Then $F = Q^*(\mathbb{R}^p)$ and $E = Q(\mathbb{R}^m)$. We can therefore define

$$\eta(Q) = \min_{x \in S_F} \max\{|x|_F^\perp, |Qx|_E^\perp\},$$

with F and E as above.

Given an orthonormal system $v = \{v_i\}_{i=1}^n$ in $\mathbb{R}^m$, set $F = \operatorname{span}\{v_1, \ldots, v_n\}$ and define a map $Q(\cdot, v) : O(p) \to O(F, \mathbb{R}^p)$ by

$$Q(T, v) = \sum_{i=1}^{n} v_i \otimes T^* e_i^n.$$

Lemma 2. *Let $m \leq 3n$ and $p \leq 3n$. Then for every orthonormal system $v = \{v_i\}_{i=1}^n$ in $\mathbb{R}^m$,*

$$\nu_p(\{T \in O(p) : \eta(Q(T, v)) \geq c\}) \geq 1 - 2^{-n}. \tag{14}$$

Proof. By the invariance properties of the measures $\nu_{F,E}$, for every measurable set $A \subset O(F, \mathbb{R}^p)$ we have the following identities:

$$\nu_p(\{T \in O(p) : Q(T, v) \in A\}) = \nu_{F,\mathbb{R}^p}(A),$$

$$\nu_{F,\mathbb{R}^p}(A) = \int \nu_{F,E}(A \cap E)\, dE,$$

where we integrate over the invariant probability measure on the Grassmannian of all n-dimensional subspaces of $\mathbb{R}^p$.

Let $A = \{T \in O(p) : \eta(Q(T, v)) \geq c\}$. By (9), $\nu_{F,E}(A \cap E) \geq 1 - 2^{-n}$ for every $E \subset \mathbb{R}^m$ with $\dim E = n$, and hence (14) follows. □

Given a sequence $\{T_k\}_{k=0}^{\infty}$ with $T_k \in O(3 \cdot 2^k)$ for $k = 0, 1, \ldots$, let us define, with the notation of (1):

$$E_k = \operatorname{span}\{u_i^k : i = 1, \ldots, 2^k\}, \qquad F_k = \operatorname{span}\{v_i^k : i = 1, \ldots, 2^k\},$$

$$Q_k = \sum_{i=1}^{2^k} v_i^k \otimes u_i^{k+1}.$$

Lemma 3. *Let $B_n = \{\{T_k\} : \eta(Q_n) \geq c\}$ for $n = 1, 2, \ldots$. Then*

$$\nu(B_n) \geq 1 - 2^{-2^n}. \tag{15}$$

Proof. Given $(T_1, \ldots, T_{n-1}) \in \prod_{k=1}^{n-1} O(3 \cdot 2^k)$, let $v = (T_{n-1}^* e_i^{3\cdot 2^{n-1}})_{i=2^{n-1}+1}^{3\cdot 2^{n-1}}$. Observe that $Q(T_n, v) = Q_n$, and therefore by (14),

$$\nu_{3\cdot 2^n}(\{T_n \in O(3 \cdot 2^n) : \eta(Q_n) \geq c\}) \geq 1 - 2^{-2^n}.$$

Integrating over all $(T_1, \ldots, T_{n-1}) \in \prod_{k=1}^{n-1} O(3\cdot 2^k)$ gives (15), since B_n does not depend on $T_{n+1}, T_{n+2}, \ldots$. □

Let now $A_n = \{\{T_k\} : \eta(Q_k) \geq c \text{ for } k = n, n+1, \ldots\} = \bigcap_{k \geq n} B_k$. By (15),

$$\nu(A_n) \geq 1 - 2 \cdot 2^{-2^n}.$$

Lemma 4. *For every $\{T_k\} \in A_n$, $Y_q(\{T_k\})$ is isomorphic to ℓ_q for all $1 < q \leq 2$.*

This lemma clearly implies Theorem 2.

Proof. By (10), for every $\{T_k\} \in A_n$,

$$\eta(Q_k) \geq c' = \min\{c, (3 \cdot 2^n)^{-1/2}\} \quad \text{for every } k = 1, 2, \ldots.$$

This means that for every $v \in F_k$, and all $k = 1, 2, \ldots$,

$$\max\{|v|_{F_k^{\perp}}, |Q_k v|_{E_{k+1}}^{\perp}\} \geq c'|v|_2. \tag{16}$$

Let

$$E = \{(0, a_1, a_2, \dots) : a_k \in E_k,\ k = 1, 2, \dots\} \subset X,$$
$$F = \{(b_0, b_1, b_2, \dots) : b_k \in F_k,\ k = 0, 1, 2, \dots\} \subset X,$$

both equipped with the norm $\|\cdot\|_{2,q}$. It is clear that Q defined by

$$Q(b_0, b_1, b_2, \dots) = (0, Q_0 b_0, Q_1 b_1, \dots)$$

is an isometry of F onto E and that their $\|\cdot\|_{2,q}$-completions are isometric to the ℓ_q-sum of $\ell_2^{2^k}$, $k = 0, 1, 2, \dots$, hence isomorphic to ℓ_q. We just need to exhibit an isomorphism S from Y_q onto the $\|\cdot\|_{2,q}$-completion of F.

Every $x \in Y$ has a unique representation

$$x = \sum_{k=0}^{\infty} (v^k + u^{k+1})$$

where $v^k \in F_k$, $u^{k+1} \in E_{k+1}$ and evidently $u^{k+1} = Q_k v^k$. Let

$$v = \sum v^k, \qquad u = \sum u^{k+1}.$$

Define $Tx \in F$ by $Tx = (v^0, v^1, v^2, \dots)$. It is clear that T is a surjective map from Y onto F. We will show that

(∗) $2\|Tx\|_{2,q} \geq \|x\|_{q,q}$,
(∗∗) $(c'/4)\|Tx\|_{2,q} \leq \|x\|_{q,q}$,

thus T can be extended to an isomorphism S of Y_q onto the $\|\cdot\|_{2,q}$-completion of F.

For (∗) we have $x = u + v$, thus

$$\|x\|_{q,q} \leq \|u\|_{q,q} + \|v\|_{q,q} \leq \|u\|_{2,q} + \|v\|_{2,q} = 2\|v\|_{2,q} = 2\|Tx\|_{2,q}.$$

For (∗∗), x has a unique representation $x = \sum x^k$ with $x^k \in \mathbb{R}^{3\cdot 2^k}$ (to wit $x^k = v^k + u^k$ for $k \geq 1$, $x^0 = v^0$). We have, for $k \geq 1$,

$$\begin{aligned}
|x^k|_q^q + |x^{k+1}|_q^q &= |v^k + u^k|_q^q + |u^{k+1} + v^{k+1}|_q^q \\
&\geq \frac{1}{2}(|v^k + u^k|_q + |u^{k+1} + v^{k+1}|_q)^q \\
&\geq \frac{1}{2}(|v^k + u^k|_1 + |u^{k+1} + v^{k+1}|_1)^q \geq \frac{1}{2}(|v^k|_{F_k}^{\perp} + |u^{k+1}|_{E_{k+1}}^{\perp})^q \\
&= \frac{1}{2}(|v^k|_{F_k}^{\perp} + |Q_k v^k|_{E_{k+1}}^{\perp})^q \geq \frac{(c')^q}{2}|v^k|_2^q,
\end{aligned}$$

by (16). Summing over k we obtain

$$\begin{aligned}\|x\|_{q,q}^q &= \sum_{k=0}^{\infty} |x^k|_q^q = |v^0|_q^q + \frac{1}{2}\Big(\sum_{k=1}^{\infty} |x^k|_q^q + \sum_{k=1}^{\infty} |x^k|_q^q\Big) \\ &\geq \frac{1}{3}|v^0|_2^q + \frac{1}{2}\sum_{k=1}^{\infty}(|x^k|_q^q + |x^{k+1}|_q^q) \geq \frac{1}{3}|v^0|_2^q + \frac{(c')^q}{4}\sum_{k=1}^{\infty}|v^k|_2^q \\ &\geq \frac{(c')^q}{4}\sum_{k=0}^{\infty}|v^k|_2^q = \frac{(c')^q}{4}\|Tx\|_{2,q}^q.\end{aligned}$$

This proves Lemma 4 and Theorem 2. □

A historical remark. The original argument that the "Enflo–Davie spaces" Y_p fail to have AP for $2 < p < \infty$ evidently breaks down for $1 \leq p < 2$ (otherwise it would imply that Y_2, which is a Hilbert space, does not have AP). However, it has not been clear whether another approach would imply that Y_p fails to have AP for $1 \leq p < 2$. Here we have shown that this is not the case: at least "generically", Y_p does have AP for $1 \leq p < 2$.

Remark. We do not know whether the roles of p and q in the abstract of this paper can be reversed. However, for $X = \ell_\infty$ an answer is known by the results of [1]: if X is an $\mathcal{L}_\infty$ space and if $Y \subset X$ has BAP, then X/Y has BAP as well (cf. also [5]). "Interpolating" between ℓ_∞ and ℓ_2, this suggests the following question:

Question. Let $2 < p < \infty$, and let Y be a subspace of ℓ_p which has BAP. Does ℓ_p/Y necessarily have BAP?

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