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Cambrian fans

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Abstract. For a finite Coxeter group W and a Coxeter element c of W, the c-Cambrian fan is a coarsening of the fan defined by the reflecting hyperplanes of W. Its maximal cones are naturally indexed by the c-sortable elements of W. The main result of this paper is that the known bijection cl_c between c-sortable elements and c-clusters induces a combinatorial isomorphism of fans. In particular, the c-Cambrian fan is combinatorially isomorphic to the normal fan of the generalized associahedron for W. The rays of the c-Cambrian fan are generated by certain vectors in the W-orbit of the fundamental weights, while the rays of the c-cluster fan are generated by certain roots. For particular ("bipartite") choices of c, we show that the c-Cambrian fan is linearly isomorphic to the c-cluster fan. We characterize, in terms of the combinatorics of clusters, the partial order induced, via the map cl_c, on c-clusters by the c-Cambrian lattice. We give a simple bijection from c-clusters to c-noncrossing partitions that respects the refined (Narayana) enumeration. We relate the Cambrian fan to well-known objects in the theory of cluster algebras, providing a geometric context for $\mathbf{g}$-vectors and quasi-Cartan companions.

Contents

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1. Introduction

Recent research in combinatorics has focused on the relationship between various objects counted by the W-Catalan number, $\mathrm{Cat}(W)$, for W a finite Coxeter group. This number, which has a simple formula in terms of fundamental numerical invariants of W, has arisen separately in a wide variety of the many fields with connections to Coxeter groups. These unexplained numerical coincidences have led to efforts to discover deeper mathematical connections between the different fields.

One set counted by $\mathrm{Cat}(W)$ is the set of *clusters* [13, 14] in the root system Φ associated to W. A cluster is a collection of roots in Φ that are "compatible" in a sense which will be made precise in Section 5. The positive linear spans of clusters are the maximal cones in a complete simplicial fan which we refer to as the *cluster fan* and whose dual polytope is called the *generalized associahedron* for W. Clusters of roots get their name from cluster algebras. Although it is surprising *a priori* that a cluster algebra should have anything to do with a Coxeter group, cluster algebras of finite type turn out to have a classification [14] that exactly matches the classification of finite crystallographic root systems Φ. Clusters of roots in Φ turn out to encode the combinatorics of the corresponding cluster algebra. For a very gentle introduction to cluster algebras and to "W-Catalan" combinatorics, see [11]. For a more advanced survey, see [12].

Another set counted by the W-Catalan number [4, 13, 22, 24, 30] is the set of *noncrossing partitions* associated to W. Both the name and the earliest examples of noncrossing partitions come from algebraic combinatorics (see e.g. [21, 30]), while both the general definition and important applications arise from geometric group theory. Specifically, noncrossing partitions are a powerful tool in the theory of Artin groups [4, 7]. For an accessible introduction to this application, focusing on the special case of the symmetric group (and thus the braid group), see [23], which also discusses other applications of noncrossing partitions to free probability and combinatorics. The definitions of both noncrossing partitions and clusters involve the choice of a Coxeter element c for W, and to emphasize this fact we will refer to them as c-noncrossing partitions and c-clusters.

A third combinatorial set counted by $\mathrm{Cat}(W)$ is the set of *nonnesting partitions* (antichains in the root poset of Φ), which will not play a role in the current paper. These objects arose in several closely related contexts, including double affine Hecke algebras (rational Cherednik algebras), two-sided cells and coinvariant rings. See [11, Lecture 5] for a gentle introduction and for references.

One of the main results of [28] is a bijective proof that c-clusters and c-noncrossing partitions are equinumerous. The proof begins with the definition of a fourth set counted by $\mathrm{Cat}(W)$, the set of *c-sortable elements* of W. Bijections are then given from c-sortable elements to c-noncrossing partitions and from c-sortable elements to c-clusters. Sortable elements and the bijections are defined simply without reference to the classification of finite Coxeter groups, but the proofs that these are bijections rest on several lemmas which are proved type by type using the classification.

Sortable elements have their origins in the lattice theory of the weak order. Specifically, the *c-Cambrian congruence* is a certain lattice congruence Θ_c on the weak order on W whose congruence classes are counted [27, 29] by $\mathrm{Cat}(W)$. The c-Cambrian congruence classes are (by a general fact about congruences of a finite lattice) intervals in the weak order. The quotient lattice W/Θ_c (the *Cambrian lattice*) is isomorphic to the restriction of the weak order to the minimal elements of the c-Cambrian congruence classes. These minimal elements turn out to be exactly the c-sortable elements [29].

As a special case of a construction given in [26], the congruence Θ_c defines a complete fan $\mathcal{F}_c$, called the *c-Cambrian fan*, whose maximal cones correspond to c-Cambrian congruence classes. The fan $\mathcal{F}_c$ is a coarsening of the fan defined by the reflecting hyperplanes of W. The goal of this paper is to understand in detail the polyhedral geometry of the bijection between the maximal cones of $\mathcal{F}_c$ (indexed by c-sortable elements) and the maximal cones of the c-cluster fan (defined by c-clusters). The bijection cl_c from c-sortable elements to c-clusters was defined in [28] without reference to the fan $\mathcal{F}_c$ or the c-cluster fan. The key result of this paper is the following theorem, a natural strengthening of the statement that cl_c is a bijection.

Theorem 1.1. *Let W be a finite Coxeter group and let c be a Coxeter element of W. Then the c-Cambrian fan $\mathcal{F}_c$ is simplicial and the map cl_c induces a combinatorial isomorphism from $\mathcal{F}_c$ to the c-cluster fan.*

By a combinatorial isomorphism of simplicial fans, we mean a combinatorial isomorphism of the simplicial complexes obtained by intersecting with the unit sphere. Because the fans are simplicial, this is equivalent to requiring that there be a piecewise linear homeomorphism from $\mathbb{R}^n$ to itself, linear on each face of $\mathcal{F}_c$, carrying the cones of $\mathcal{F}_c$ to the cones of the c-cluster fan. The isomorphism of Theorem 1.1 is typically only *piecewise* linear. However, we show that for any W, there exists a special "bipartite" choice of c such that the c-Cambrian fan and the c-cluster fan are linearly isomorphic. This result (Theorem 9.1) verifies the first statement of [27, Conjecture 1.4].

Theorem 1.1 shows that c-sortable elements are not simply in bijection with c-clusters, but define the same underlying combinatorial structure. This is a particularly surprising result as the Cambrian fan and cluster fan are defined in very different ways: the Cambrian fan is defined by removing walls of the fan defined by the reflecting hyperplanes while the cluster fan is defined by choosing certain rays, all of which are normal to reflecting hyperplanes, and specifying which rays lie in common cones. Moreover, the Cambrian fan contributes combinatorial structure which is not present in the cluster fan, including a poset (in fact lattice) structure which interacts well with the fan structure, as well as a notion of projection to standard parabolic subgroups.

The definition of sortable elements is valid for infinite Coxeter groups. The theory of sortable elements and Cambrian lattices/fans can be extended to infinite Coxeter groups and we will describe this program in detail in a future paper. In particular, this future paper will provide uniform proofs, valid for all finite and infinite Coxeter groups, of the results which were proved using type by type analysis in [28] and [29]. Cluster algebras of infinite type are not as well understood as cluster algebras of finite type, and one of the key motivations of this paper is to establish connections between Cambrian lattices/fans which can be generalized to give new insights into cluster algebras of infinite type.

Theorem 1.1 leads to further results which we now describe. The c-cluster fan is combinatorially isomorphic to the normal fan of the generalized associahedron for W. Thus, Theorem 1.1 implies that the c-Cambrian fan is combinatorially isomorphic to the normal fan of the generalized associahedron. This isomorphism, combined with the close structural relationship which exists between the c-Cambrian lattice and the c-Cambrian fan (see Section 4), implies that the Hasse diagram of the c-Cambrian lattice is combinatorially isomorphic to the 1-skeleton of the generalized associahedron (Corollary 8.1). This confirms [27, Conjecture 1.2.a].

The c-Cambrian lattice induces a partial order (the *c-cluster lattice*) on c-clusters via the map cl_c. We characterize this partial order in terms of the combinatorics of clusters (Theorem 8.4), generalizing and proving the second statement of [27, Conjecture 1.4]. The c-cluster lattice inherits many useful properties from the c-Cambrian lattice (see Corollary 8.5), including the property that any linear extension of the c-cluster poset is a shelling of the c-cluster fan. As a consequence of this shelling property we obtain, in Section 8, a bijective proof of the fact that the k^{th} entry of the h-vector of the c-cluster fan coincides with the number of c-noncrossing partitions of rank k. (This number is called the k^{th} Narayana number associated to W.)

In Section 11 we give a purely geometric description (Theorem 11.4) of a bijection between c-clusters and c-noncrossing partitions, in the case where c is a bipartite Coxeter element. This result draws on a "twisted" version of the c-cluster poset as well as the linear isomorphism, mentioned above, between the c-Cambrian fan and the c-cluster fan. Another connection between noncrossing partitions and clusters has arisen recently. Brady and Watt [9] construct a simplicial fan associated to c-noncrossing partitions (for bipartite c) and extend their construction to produce the c-cluster fan. Athanasiadis, Brady, McCammond and Watt [1] use the construction of [9] to give a bijection between clusters and noncrossing partitions. Their proof uses no type by type arguments and provides a different bijective proof that the k^{th} entry of the h-vector of the c-cluster fan coincides with the number of c-noncrossing partitions of rank k. The bijection of [1] incorporates elements which are similar in appearance to the constructions of the present paper (see Remark 11.5), but many details of the relation between the two theories remain unclear.

The results described above resolve all conjectures from [27] except for Conjecture 1.1. This last result has been established by Hohlweg and Lange [17] for types A and B and will be proven for all types in a future paper by Hohlweg, Lange and Thomas [18].

The relationship between the c-Cambrian fan and the c-cluster fan has several consequences for the theory of cluster algebras, which we describe in more detail in Section 10. When W admits a crystallographic root system Φ, Fomin and Zelevinsky associate a cluster algebra $\mathrm{Alg}(\Phi)$ to W. This is a commutative algebra with certain specified elements, called cluster variables, and certain specified subsets of these variables, ordinarily called clusters. We will call these subsets of variables *algebraic clusters*, to distinguish them from the combinatorially defined clusters which are certain sets of roots.

The root system Φ and a choice of Coxeter element c specify a certain algebraic cluster t_c of $\mathrm{Alg}(\Phi)$. There is a bijection between cluster variables and almost positive roots, such that the elements of t_c are taken to the negative simple roots and such that algebraic clusters are taken to c-clusters. For any cluster variable $x \in \mathrm{Alg}(\Phi)$ and any cluster t, Fomin and Zelevinsky associate to the pair (x, t) two vectors in $\mathbb{Z}^n$: the *denominator vector* and the $\mathbf{g}$-*vector* of x with respect to t. It is shown in [14] and [10] that when $t = t_c$ the denominator vector is found by expressing the corresponding root in the basis of simple roots. We show, in the case where c is bipartite, that the **g**-vector is given by expressing the corresponding ray of the c-Cambrian fan in the basis of fundamental weights; this result would follow for other c if we knew Conjecture 7.12 of [15]. We also provide a geometric context for the notion of *quasi-Cartan companions*, defined in [2].

In the following four sections, we lay out the necessary background concerning Coxeter groups, sortable elements and Cambrian congruences. We also give more precise statements of several of the results described in this introduction. In Section 6, we begin presenting our proofs.

2. The weak order

This section covers preliminary results on finite lattices and in particular on the weak order on a finite Coxeter group. We assume that the reader is familiar with the most basic definitions of Coxeter groups and lattices. Details about lattices are found in [16] and details about Coxeter groups are found in [5, 6, 19].

A *join-irreducible* element of a finite lattice L is an element which covers exactly one other element. A *meet-irreducible* element of L is an element which is covered by exactly one other element. A *homomorphism* from the lattice L_1 to the lattice L_2 is a map $\eta : L_1 \to L_2$ with $\eta(x \wedge y) = \eta(x) \wedge \eta(y)$ and $\eta(x \vee y) = \eta(x) \vee \eta(y)$ for every $x, y \in L_1$. The condition that η be a lattice homomorphism is strictly stronger than the condition that η be order preserving.

The fibers of a lattice homomorphism from L to another lattice determine an equivalence relation $\equiv$ on L. An equivalence relation which arises in this way is called a *lattice congruence* on L. More directly, an equivalence relation $\equiv$ on L is a lattice congruence if and only if $a_1 \equiv a_2$ and $b_1 \equiv b_2$ implies $a_1 \vee b_1 \equiv a_2 \vee b_2$ and $a_1 \wedge b_1 \equiv a_2 \wedge b_2$. It is an easy exercise to show that an equivalence relation Θ on a finite lattice L is a lattice congruence if and only if it satisfies the following three conditions (where $[x]_\Theta$ denotes the Θ-equivalence class of x):

(i) Each equivalence class $[x]_\Theta$ is an interval in L.
(ii) The map $\pi_\downarrow^\Theta$ taking x to the minimal element of $[x]_\Theta$ is order preserving.
(iii) The map $\pi^\uparrow_\Theta$ taking x to the maximal element of $[x]_\Theta$ is order preserving.

Given a lattice congruence Θ on L, the *quotient lattice* L/Θ is the lattice whose elements are the congruence classes, with join and meet defined by $[x]_\Theta \vee [y]_\Theta = [x \vee y]_\Theta$ and $[x]_\Theta \wedge [y]_\Theta = [x \wedge y]_\Theta$. Equivalently, L/Θ is the partial order on congruence classes which sets $[x]_\Theta \leq [y]_\Theta$ if and only if there exist $x' \in [x]_\Theta$ and $y' \in [y]_\Theta$ such that $x' \leq y'$ in L. It is easy to check that L/Θ is isomorphic to the subposet of L induced by the set $\pi_\downarrow^\Theta(L) = \{x \in L : \pi_\downarrow^\Theta(x) = x\}$. Note that $\pi_\downarrow^\Theta(L)$ need not be a sublattice of L. The following is [26, Proposition 2.2].

Proposition 2.1. *Let L be a finite lattice, Θ a congruence on L and $x \in L$. Then the map $y \mapsto [y]_\Theta$ restricts to a one-to-one correspondence between elements of L covered by $\pi_\downarrow^\Theta(x)$ and elements of L/Θ covered by $[x]_\Theta$.*

We now remind the reader of some basic facts about Coxeter groups. We also establish notation for what follows. Throughout the paper, W denotes a finite Coxeter group of rank n with simple generators S. For $J \subseteq S$, let W_J be the subgroup of W generated by J, called a *standard parabolic subgroup* of W. Most often J will be $S \setminus \{s\}$ for some $s \in S$; we write $\langle s \rangle$ for $S \setminus \{s\}$. The *reflections* of W are those elements which are conjugate to elements of S. The set of reflections is written T. An element $w \in W$ can be written as a word in S. A word for w which is minimal in length among words for w is called *reduced*, and the *length* of w, written $\ell(w)$, is the length of a reduced word for w. An *inversion* of w is a reflection $t \in T$ such that $\ell(tw) < \ell(w)$. If $s_1 s_2 \cdots s_\ell$ is a reduced word for w then the inversions of w are $s_1, s_1 s_2 s_1, \ldots, s_1 s_2 \cdots s_\ell \cdots s_2 s_1$. The set of inversions is written $I(w)$, and w is uniquely determined by $I(w)$. The *(right) weak order* on W is the partial order on W induced by containment of inversion sets. Equivalently, the weak order on W is the transitive closure of the cover relations $w \lessdot ws$ whenever $s \in S$ and $\ell(w) < \ell(ws)$. This is further equivalent to the partial order defined by $v \leq w$ if and only if there is a reduced word a for w such that some prefix (initial subword) of a is a word for v. The weak order is known to be a lattice when W is finite.

The symbol W will now denote both the group W and the set W viewed as a poset (lattice). All references to a partial order on W will refer to the weak order. The phrase "join-irreducible elements of the weak order on W" will be abbreviated to "join-irreducibles of W" and similarly we will refer to "meet-irreducibles of W."

The unique maximal element of W is called w_0. Conjugation by w_0 is an automorphism of the weak order and in particular permutes the simple generators S. The map $w \mapsto ww_0$ is an antiautomorphism of the weak order on W, and in particular $I(ww_0) = T \setminus I(w)$.

A simple reflection s is called a *descent* of w if $\ell(ws) < \ell(w)$ and an *ascent* of w if $\ell(ws) > \ell(w)$. A *cover reflection* of $w \in W$ is a reflection t such that $tw \lessdot w$; the set of cover reflections of w can also be described as those reflections of the form wsw^{-1} for s a descent of w. The set of cover reflections is denoted by $\mathrm{cov}(w)$. There is one cover reflection of w for each element of W covered by w. Thus the join-irreducibles of W are the elements with exactly one cover reflection, or equivalently one descent. The following is [29, Lemma 2.8].

Lemma 2.2. *For $x \in W_{\langle s \rangle}$, $\mathrm{cov}(s \vee x) = \mathrm{cov}(x) \cup \{s\}$.*

The map $x \mapsto sx$ is an involutive isomorphism between the intervals $[s, w_0]$ and $[1, sw_0]$. In particular, we have the following observation, which we record now to avoid giving the simple argument repeatedly later.

Lemma 2.3. *Let w be join-irreducible with $s < w$. Then sw is join-irreducible and if t is the unique cover reflection of w then sts is the unique cover reflection of sw.*

Proof. Since w is join-irreducible, w covers at most one element of $[s, w_0]$. If w does not cover any element of $[s, w_0]$ then sw does not cover any element of $[1, sw_0]$ and must thus be 1. But this contradicts the assumption that $w \neq s$. Thus, w covers exactly one element of $[s, w_0]$, say $w \gtrdot wr$. Then sw covers swr and no other element of W. The unique cover reflections of w and sw are wrw^{-1} and $swrw^{-1}s$ respectively. □

For each $w \in W$ and each subset J of S there is a unique factorization $w = w_J \cdot {}^J w$ such that $w_J \in W_J$ and ${}^J w$ satisfies $s \not\leq {}^J w$ for every $s \in J$. The element w_J appearing in this factorization is the unique element w_J such that $I(w_J) = I(w) \cap W_J$. For a fixed $w \in W$, the set of elements x such that $x_J = w_J$ is an interval in W, specifically the interval $[w_J, w_J \cdot {}^J(w_0)]$. The map $w \mapsto w_J$ is a lattice homomorphism.

For more on the factorization $w = w_J \cdot {}^J w$, see Section 2.4 of [5]. All the claims of the preceding paragraph except for the last one are either in [5] or are easy consequences of results proved there. The fact that $w \mapsto w_J$ is a lattice homomorphism is proven for example in [20] or [25, Proposition 6.3].

3. Sortable elements and Cambrian congruences

In this section we review definitions and quote or prove some preliminary results about sortable elements and Cambrian congruences. For more details, see [28, 29].

A *Coxeter element* of W is an element of W of the form $s_1 \cdots s_n$, where $s_1, \ldots, s_n$ are the simple generators S, listed in any order. (Recall that $n = |S|$.) Two orderings of the generators produce the same Coxeter element if and only if they are related by a sequence of transpositions of adjacent generators which commute in W. A generator $s \in S$ is *initial* in c, or is an *initial letter* of c, if c can be written $s_1 \cdots s_n$ with $s_1 = s$. *Final letters* are defined similarly.

Given $w \in W$, the half-infinite word

$$c^\infty = s_1 \cdots s_n s_1 \cdots s_n s_1 \cdots s_n \cdots$$

contains infinitely many subwords which are reduced words for w. The *c-sorting word* for $w \in W$ is the lexicographically leftmost subword of c^∞ which is a reduced word for w. Inserting *dividers* "|" into c^∞,

$$c^\infty = s_1 \cdots s_n | s_1 \cdots s_n | s_1 \cdots s_n | \cdots,$$

we view the c-sorting word for w as a sequence of subsets of S, namely the sets of letters of the c-sorting word which occur between adjacent dividers.

An element $w \in W$ is *c-sortable* if its c-sorting word defines a sequence of subsets which is weakly decreasing under inclusion. Formally, this definition requires a choice of reduced word for c. However, for a given w, the c-sorting words for w arising from different reduced words for c are related by commutations of letters, with no commutations across dividers. Thus in particular, the set of c-sortable elements does not depend on the choice of reduced word for c.

Example 3.1. Consider the Coxeter group B_2 with simple generators s_0 and s_1. When W is B_2 and $c = s_0 s_1$, the c-sortable elements of W are 1, s_0, $s_0 s_1$, $s_0 s_1 s_0$, $s_0 s_1 s_0 s_1$ and s_1. The non-c-sortable elements are $s_1 s_0$ and $s_1 s_0 s_1$.

The definition of sortability in terms of c^∞ is intuitive but is not always the most helpful definition. The following two lemmas, which are [28, Lemmas 2.4 and 2.5], give a recursive description of c-sortability.

Lemma 3.2. *Let s be an initial letter of c and let $w \in W$ with $s \not\leq w$. Then w is c-sortable if and only if it is an sc-sortable element of $W_{\langle s \rangle}$.*

Lemma 3.3. *Let s be an initial letter of c and let $w \in W$ with $s \leq w$. Then w is c-sortable if and only if sw is scs-sortable.*

In [28], the two lemmas above appear with the hypothesis $\ell(sw) > \ell(w)$ (resp. $\ell(sw) < \ell(w)$) instead of $s \not\leq w$ (resp. $s \leq w$). The characterization of the weak order in terms of inversion sets reconciles these two ways of stating the hypothesis. In Lemma 3.2, $W_{\langle s \rangle}$ is a Coxeter group of rank $n-1$ and in Lemma 3.3, $\ell(sw) < \ell(w)$, so these two lemmas characterize the c-sortable elements by induction on rank and length. (The identity element 1 is c-sortable for any c.)

For each c, we define a map $\pi_\downarrow^c$ from W to the c-sortable elements of W. The notation $\pi_\downarrow^c$ suggests the order-theoretic characterization of lattice congruences given in Section 2. For any Coxeter element c, let $\pi_\downarrow^c(1) = 1$ and for s an initial letter of c, define

$$\pi_\downarrow^c(w) = \begin{cases} s \cdot \pi_\downarrow^{scs}(sw) & \text{if } s \leq w, \\ \pi_\downarrow^{sc}(w_{\langle s \rangle}) & \text{if } s \not\leq w. \end{cases}$$

In [29, Section 3], it is shown that $\pi_\downarrow^c(w)$ is the unique maximal c-sortable element weakly below w. Furthermore, it is shown that the fibers of $\pi_\downarrow^c$ are a lattice congruence on W, denoted by Θ_c. In particular, $\pi_\downarrow^c$ is order preserving. In [29, Section 5], Θ_c is identified as the *c-Cambrian congruence* on W in the sense of [27]. (Although we leave the lattice-theoretic details to [27] and [29], we will adopt the name "c-Cambrian congruence" for Θ_c.) We use the abbreviation $[w]_c$ for $[w]_{\Theta_c}$.

The *c-Cambrian lattice* is the quotient W/Θ_c of W modulo the c-Cambrian congruence. This quotient is isomorphic to the restriction of the weak order to the c-sortable elements of W. Despite this isomorphism, to avoid confusion the notation of this paper will describe the c-Cambrian lattice as a partial order on congruence classes $[w]_c$, while comparisons of elements w will refer to the weak order on W. Thus $[w]_c \lessdot [w']_c$ is a cover relation in the quotient W/Θ_c, while $w \lessdot w'$ says that w is covered by w' in the weak order on W.

Example 3.4. Figure 1(a) shows the $s_1s_2s_3$-Cambrian congruence on the weak order for W of type A_3. The gray shading indicates congruence classes of cardinality greater than one, and each unshaded vertex is a singleton congruence class. The $s_1s_2s_3$-Cambrian lattice is the partial order on the congruence classes, as explained in Section 2. Equivalently, the $s_1s_2s_3$-Cambrian lattice is the restriction of the weak order to $s_1s_2s_3$-sortable elements (bottom elements of congruence classes), as indicated in Figure 1(b).

The c-Cambrian congruence has an upward projection map $\pi_c^\uparrow$ which takes each $w \in W$ to the top element of its Θ_c-congruence class. This map is given by $\pi_c^\uparrow(w) = (\pi_\downarrow^{(c^{-1})}(ww_0))w_0$, and satisfies the following recursion when s is *final* in c:

$$\pi_c^\uparrow(w) = \begin{cases} s \cdot \pi_{scs}^\uparrow(sw) & \text{if } s \not\leq w, \\ \pi_{cs}^\uparrow(w_{\langle s \rangle}) \cdot {}^{\langle s \rangle}w_0 & \text{if } s \leq w. \end{cases}$$

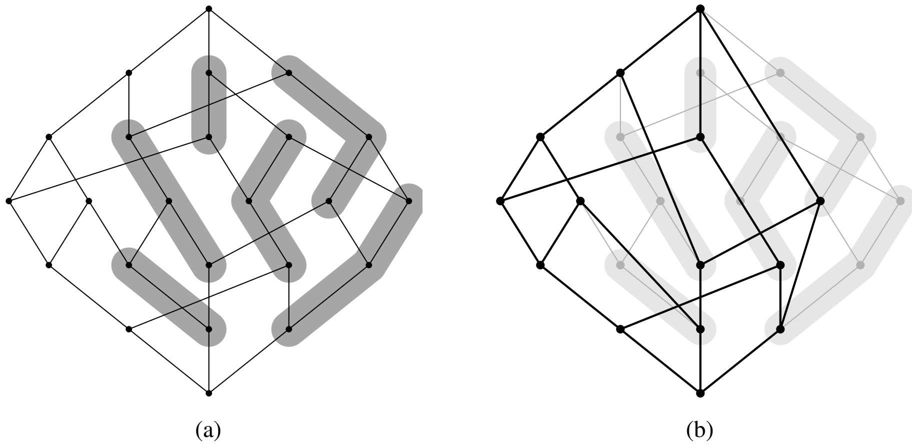


Fig. 1. A Cambrian congruence and the associated Cambrian lattice.

In particular, the antiautomorphism $w \mapsto ww_0$ takes c-Cambrian congruence classes to c^{-1}-Cambrian congruence classes. An element $w \in W$ is called *c-antisortable* if $\pi_c^{\uparrow}(w) = w$. Equivalently, w is c-antisortable if and only if ww_0 is c^{-1}-sortable. The map $\pi_c^{\uparrow}$ takes w to the unique minimal c-antisortable element above w.

The elements 1 and w_0 are alone in their c-Cambrian congruence classes. An initial letter s of c also constitutes a singleton congruence class.

We now record three simple lemmas about $\pi_{\downarrow}^c$, $\pi_c^{\uparrow}$ and c-antisortable elements.

Lemma 3.5. *Let c be a Coxeter element, let r be a simple reflection and let $w \in W$. Then $r \leq w$ if and only if $r \leq \pi_{\downarrow}^c(w)$.*

Proof. First, suppose that $\pi_{\downarrow}^c(w) \geq r$. Since $\pi_{\downarrow}^c(w) \leq w$, we also have $w \geq r$. Now, suppose $w \geq r$. Then, since $\pi_{\downarrow}^c$ is order preserving, $\pi_{\downarrow}^c(w) \geq \pi_{\downarrow}^c(r) = r$. □

Lemma 3.6. *Let s be an initial letter of c and let $w \in W$ with $s \leq w$. Then w is c-antisortable if and only if sw is scs-antisortable.*

Proof. The element w is c-antisortable if and only if ww_0 is c^{-1}-sortable. Observe that s is initial in $sc^{-1}s$ and that $s \leq sww_0$. Thus, by Lemma 3.3, sww_0 is $sc^{-1}s$-sortable if and only if ww_0 is c^{-1}-sortable. Now sww_0 is $sc^{-1}s$-sortable if and only if sw is scs-antisortable. □

Lemma 3.7. *If s is final in c then $\pi_c^{\uparrow}(s) = w_0 \cdot ((w_0)_{\langle s' \rangle})$ for $s' = w_0 s w_0$.*

Proof. The reflection s' is a simple reflection because w_0 permutes S. By the recursive characterization, $\pi_c^{\uparrow}(s)$ is equal to

$$\pi_{cs}^{\uparrow}(1) \cdot {}^{\langle s \rangle}w_0 = {}^{\langle s \rangle}w_0 = (w_0)_{\langle s \rangle} \cdot w_0 = w_0 \cdot (w_0 (w_0)_{\langle s \rangle} w_0) = w_0 \cdot ((w_0)_{\langle s' \rangle}). \quad \square$$

The proofs in this paper will also rely on nontrivial properties of sortable elements which we now quote. We begin with [28, Theorem 6.1]. In the present paper, it will not be necessary to define noncrossing partitions or the map nc_c. Full details, including citations, are found in [28, Sections 5 and 6]. Following the statement of the theorem we discuss how it applies to the present context.

Theorem 3.8. *For any Coxeter element c, the map $w \mapsto \mathrm{nc}_c(w)$ is a bijection from the set of c-sortable elements to the set of noncrossing partitions with respect to c. Furthermore, nc_c maps c-sortable elements with k descents to c-noncrossing partitions of rank k.*

The noncrossing partitions (with respect to W and c) of rank 1 are exactly the reflections in W. For the purposes of this section, all we need to know about nc_c is Theorem 3.8 and the following fact: If w is a c-sortable element with 1 descent (i.e. a c-sortable join-irreducible) then $\mathrm{nc}_c(w)$ is the unique cover reflection of w. Thus we have the following corollary to Theorem 3.8.

Corollary 3.9. *For each reflection t of W, there is exactly one c-sortable join-irreducible whose unique cover reflection is t.*

The number of noncrossing partitions of rank $n-1$ is also equal to the number $|T|$ of reflections in W. Thus c-sortable meet-irreducibles are also counted by $|T|$. (Recall that an element of W is meet-irreducible if and only if it is covered by exactly one element, or equivalently, covers exactly $n-1$ elements.) Because the map $w \mapsto ww_0$ is an anti-automorphism of the weak order on W and takes c-sortable elements to c^{-1}-antisortable elements, the same is true of c-antisortable join-irreducible or meet-irreducible elements. We summarize in the following corollary to Theorem 3.8.

Corollary 3.10. *For W a finite Coxeter group and c a Coxeter element of W, the following numbers are all equal:*

(i) *the number of c-sortable join-irreducibles of W;*
(ii) *the number of c-sortable meet-irreducibles of W;*
(iii) *the number of c-antisortable join-irreducibles of W;*
(iv) *the number of c-antisortable meet-irreducibles of W;*
(v) *the number of reflections in W.*

We conclude the section by quoting [29, Theorem 1.2] and using it to prove a lemma.

Theorem 3.11. *Let c be a Coxeter element of a finite Coxeter group W. The c-sortable elements constitute a sublattice of the weak order on W.*

Lemma 3.12. *Let s be an initial letter of c. If w is c-antisortable and $s \not\leq w$ then*

(i) $sw > w$,
(ii) $s \vee w = sw$.
(iii) *w is scs-antisortable.*

Proof. Let w be c-antisortable with $s \not\leq w$. The set of elements x such that $x_{\langle s\rangle} = w_{\langle s\rangle}$ is an interval I in W. Now, if $x \in I$ and $x \not\geq s$ then $\pi_\downarrow^c(x) = \pi_\downarrow^{sc}(x_{\langle s\rangle}) = \pi_\downarrow^{sc}(w_{\langle s\rangle}) = \pi_\downarrow^c(w)$ so, by the assumption that w is c-antisortable, we have $w \geq x$ for such an x.

On the other hand, since $(s \vee w)_{\langle s\rangle} = s_{\langle s\rangle} \vee w_{\langle s\rangle} = w_{\langle s\rangle}$, I contains elements which are greater than or equal to s. In particular, w is not maximal in I so we take w' to be some element covering w with $(w')_{\langle s\rangle} = w_{\langle s\rangle}$. By the observation in the first paragraph, $w' \geq s$. But since $w' \gtrdot w$ with $w \not\geq s$ and $w' \geq s$, we must have $sw = w' \gtrdot w$. Furthermore, $sw = s \vee w$ because $s \leq sw$. By the dual of Theorem 3.11, sw is c-antisortable, and by Lemma 3.6, w is scs-antisortable. □

4. Cambrian fans

In this section we define the c-Cambrian fan for each finite Coxeter group W and Coxeter element c of W. We also prove a few preliminary results. In Section 6 we make a careful study of the rays of the c-Cambrian fan.

An *arrangement* of hyperplanes in a vector space V is a collection of hyperplanes (codimension 1 subspaces). A central arrangement is called *central* if all of the hyperplanes pass through the origin. That is, the hyperplanes are linear subspaces rather than affine subspaces. A central arrangement is called *essential* if the intersection of the hyperplanes is the origin.

We continue to let (W, S) be a finite Coxeter system of rank n with reflections T and longest element w_0. Fix some root system Φ for W and let V be the geometric representation of W; we write $V(W)$ when it is necessary for clarity. This is the representation of W on the real vector space spanned by the root system of W. A reflection t of W acts by the orthogonal reflection

$$v \mapsto v - 2\frac{\langle \alpha_t, v\rangle}{\langle \alpha_t, \alpha_t\rangle}\alpha_t,$$

where α_t is the positive root corresponding to t and $\langle \cdot, \cdot \rangle$ is the usual inner product. The hyperplane fixed by the reflection t is denoted by H_t. The *Coxeter arrangement* $\mathcal{A}$ for W is the collection of all such hyperplanes; we will write $\mathcal{A}(W)$ when necessary. The complement $V \setminus \bigcup \mathcal{A}$ of $\mathcal{A}$ is composed of open cones whose closures are called *regions*. The regions are in canonical bijective correspondence with the elements of W, and each region has n facet hyperplanes. More specifically, the *dominant chamber* $D := \bigcap_{s\in S}\{v : \langle v, \alpha_s\rangle \geq 0\}$ corresponds to the identity and wD corresponds to w.

A subset U of V is *below* a hyperplane $H \in \mathcal{A}$ if every point in U is either on H or on the same side of H as D. The subset is *strictly below* H if it is below H and does not intersect H. Similarly, U can be *above* or *strictly above* H. The inversions of an element $w \in W$, defined in Section 2 to be those reflections t for which $\ell(tw) < \ell(w)$, can also be described as the reflections t such that wD is above H_t. This result is perhaps more frequently quoted in its dual form: a reflection t is an inversion of w if and only if $w^{-1}(\alpha_t)$ is a negative root [5, Proposition 4.4.6]. In the case of a simple reflection $s \in S$, $\ell(sw) < \ell(w)$ if and only if $s \leq w$ in the weak order. Thus deciding whether wD is above H_s or below H_s is a weak order comparison.

The Coxeter arrangement $\mathcal{A}$ is a central, essential arrangement. If $J \subset S$, let $\mathcal{A}_J$ be the subset of $\mathcal{A}$ consisting of those hyperplanes $H_t \in \mathcal{A}(W)$ for which t is in W_J. Then $\mathcal{A}_J$ is a central arrangement but it is not essential. If $I_J = \bigcap_{H \in \mathcal{A}_J} H$ then we have $V(W) \cong I_J \times V(W_J)$ and each hyperplane H_t in $\mathcal{A}_J$ is the direct product of I_J with the hyperplane H_t in $\mathcal{A}(W_J)$. We write Proj_J for the linear projection $V(W) \twoheadrightarrow V(W)/I_J \cong V(W_J)$. This projection will be used in Lemma 6.3 and the proof of Theorem 1.1. The following proposition relates the geometric projection Proj_J to the combinatorial projection $w \mapsto w_J$.

Proposition 4.1. *For $w \in W$, we have* $\mathrm{Proj}_J(wD) \subseteq w_J D_J$*, where D_J is the dominant chamber for $\mathcal{A}(W_J)$.*

A *fan* $\mathcal{F}$ is a family of nonempty closed polyhedral (and in particular convex) cones in V such that

(i) for any cone in $\mathcal{F}$, all faces of that cone are also in $\mathcal{F}$;
(ii) the intersection of two cones in $\mathcal{F}$ is a face of both.

A fan is *complete* if its union is all of V. It is *essential* (or *pointed*) if the intersection of all of the cones of $\mathcal{F}$ is the origin. For more information about fans, see [31, Lecture 7].

Let $\mathcal{F}$ be the fan consisting of the regions of $\mathcal{A}$ and all of their faces. The fan $\mathcal{F}$ is complete and essential (see [19, Sections 1.12–1.15]). The faces of $\mathcal{F}$ have an elegant description: they are in bijection with pairs (w, J) where w is an element of W and J is a subset of the ascents of w. The pair (w, J) corresponds to $C(w, J) := w \cdot (D \cap \bigcap_{s \in J} \{v : \langle v, \alpha_s \rangle = 0\})$. We may recover w from $C(w, J)$ by the fact that w is the smallest element of W (in weak order) such that wD contains $C(w, J)$. It is then easy to recover J. The cone $C(w, J)$ has dimension $n - |J|$.

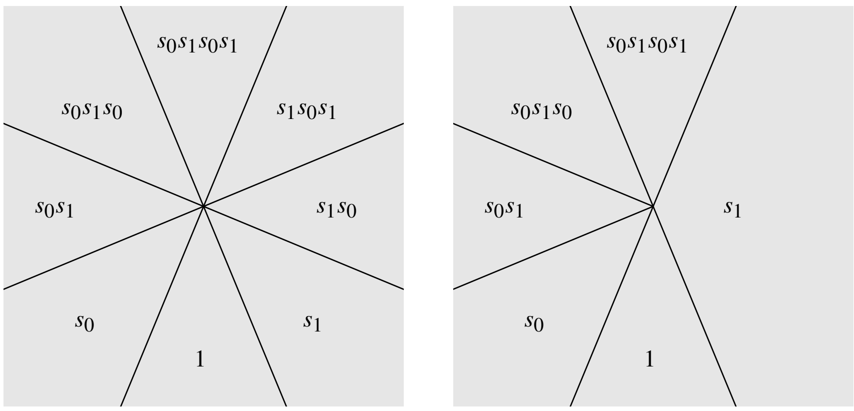
Let $\mathcal{G}$ be a complete fan in $\mathbb{R}^n$ and let v be a generic vector in $\mathbb{R}^n$. Suppose that the intersection of two maximal cones C and C' spans a hyperplane H with v on the same side of H as C. We put $C' > C$. In general, it is possible that there is a sequence $C_1, \ldots, C_r$ of maximal cones of $\mathcal{G}$ such that $C_1 > \cdots > C_r > C_1$. If this does not occur, then we define a poset on the maximal cones of $\mathcal{G}$ by taking the transitive closure of all relations $C' > C$ and we say that this poset is *induced*[1] on $\mathcal{G}$ by v; roughly speaking, going "down" in the poset means moving in the direction of v. So, for example, the weak order is induced on $\mathcal{F}$ by any v in the interior of D. If C is a maximal cone of $\mathcal{G}$ and if C is simplicial, then we define the *bottom face* of C with respect to v to be the minimal (under containment) face F of C such that for any vector x in the relative interior of F, there exists an $\epsilon > 0$ such that $x - \epsilon v$ is in C. In other words, the bottom face of C is the intersection of the facets F of C which separate C from a face lower than C in the poset induced by v.

We now review a construction from [26] which, given an arbitrary lattice congruence Θ on the weak order on W, constructs a complete fan $\mathcal{F}_\Theta$ which coarsens $\mathcal{F}$ (in the sense that every cone of $\mathcal{F}_\Theta$ is a union of cones of $\mathcal{F}$). The maximal cones of $\mathcal{F}_\Theta$ correspond to congruence classes of Θ. Specifically, each maximal cone is the union of the regions

[1] An analogous construction in [26] featured posets induced on fans by linear functionals. The vector v occurring here points in the direction which minimizes the linear functional of [26].

of $\mathcal{A}$ corresponding to the elements of the congruence class. In [26, Section 5] it is shown that the collection $\mathcal{F}_\Theta$ consisting of these maximal cones together with all of their faces is indeed a complete fan. In what follows, we identify a congruence class with the corresponding maximal cone of $\mathcal{F}_\Theta$.

Example 4.2. For $W = B_2$ with $c = s_0s_1$, the fan $\mathcal{F}$ is shown in Figure 2(a), with maximal cones labeled by elements of W. Figure 2(b) shows $\mathcal{F}_c$, with maximal cones labeled by c-sortable elements. (cf. Example 3.1). The weak order on B_2 is the poset on the regions of $\mathcal{F}$ such that one moves up in the partial order by passing to an adjacent region which is "higher" on the page. The c-Cambrian lattice is the poset on the maximal cones of $\mathcal{F}_c$ with a similar description.



(a) (b)

Fig. 2. The fans $\mathcal{F}$ and $\mathcal{F}_c$.

The lattice W/Θ is a partial order on the maximal cones of $\mathcal{F}_\Theta$. In fact, the pair $(\mathcal{F}_\Theta, W/\Theta)$ is a *fan poset* [26, Theorem 1.1], and thus by [26, Proposition 3.3], we have the following.

Proposition 4.3. *Let $[w]_\Theta$ and $[w']_\Theta$ be maximal cones of $\mathcal{F}_\Theta$. Then $[w]_\Theta$ and $[w']_\Theta$ are a covering pair in W/Θ if and only if they intersect in a common facet.*

The proof of the following lemma is essentially contained in the proof of [26, Proposition 5.5]. However, since that result is stated quite differently and in broader generality, we give a proof here. The dual statement about the upward projection $\pi^\uparrow_\Theta$ also holds.

Lemma 4.4. *Let $\pi_\downarrow^\Theta$ be the downward projection map associated to a lattice congruence on the weak order on a finite Coxeter group. A hyperplane H separates a congruence class $[w]_\Theta$ from a congruence class $[x]_\Theta \lessdot [w]_\Theta$ if and only if H separates $\pi_\downarrow^\Theta(w)$ from an element of W covered by $\pi_\downarrow^\Theta(w)$.*

Proof. Let $\mathcal{L}$ be the set of hyperplanes H in $\mathcal{A}$ such that H separates the congruence class $[w]_\Theta$ from a congruence class covered by $[w]_\Theta$. Since the congruence classes are convex cones, no two hyperplanes in $\mathcal{L}$ separate $[w]_\Theta$ from the same congruence class covered by $[w]_\Theta$. By Proposition 2.1, congruence classes covered by $[w]_\Theta$ are in one-to-one correspondence with elements x covered by $\pi_\downarrow^\Theta(w)$. Each such x is separated from $\pi_\downarrow^\Theta(w)$ by a distinct hyperplane in $\mathcal{L}$, so $\mathcal{L}$ is the set of hyperplanes H such that H separates $\pi_\downarrow^\Theta(w)$ from an element covered by $\pi_\downarrow^\Theta(w)$. □

The *c-Cambrian fan* $\mathcal{F}_c$ is the essential fan $\mathcal{F}_{\Theta_c}$ arising from this construction, where Θ_c is the c-Cambrian congruence described in Section 3. The c-Cambrian fan $\mathcal{F}_c$ and the c-Cambrian lattice W/Θ_c have many pleasant properties following from a general theorem [26, Theorem 1.1] which applies to fans $\mathcal{F}_\Theta$ and quotients W/Θ for general lattice congruences Θ on W. We list some of these properties here for emphasis.

(i) Any linear extension of the c-Cambrian lattice is a shelling order of $\mathcal{F}_c$.
(ii) The c-Cambrian lattice is the order induced on the maximal cones of $\mathcal{F}_c$ by any vector lying in the interior of D.
(iii) For any interval in the c-Cambrian lattice, the union of the corresponding cones of $\mathcal{F}_c$ is a convex cone.
(iv) For any cone F in $\mathcal{F}_c$, the set of maximal cones in $\mathcal{F}_c$ containing F is an interval in the c-Cambrian lattice.
(v) A closed interval I in the c-Cambrian lattice has proper part homotopy equivalent to an $(n-k-2)$-dimensional sphere if and only if there is some k-dimensional cone F of $\mathcal{F}_c$ such that I is the set of all maximal cones of $\mathcal{F}_c$ containing F.
(vi) A closed interval I has proper part homotopy equivalent to a $(k-2)$-dimensional sphere if and only if I has k atoms and the join of the atoms of I is the top element of I.
(vii) If the proper part of a closed interval I is not homotopy-spherical then it is contractible.

By applying Lemma 4.4 to the case of the c-Cambrian fan and appealing to Corollary 3.10, we obtain the following useful fact about c-sortable and c-antisortable join-irreducibles. Dually, the analogous statement for meet-irreducibles also holds.

Proposition 4.5. *The upward projection $\pi_c^\uparrow$ restricts to a bijection from c-sortable join-irreducibles to c-antisortable join-irreducibles. The inverse is the restriction of $\pi_\downarrow^c$.*

Proof. Let v be a c-sortable join-irreducible. Then $v = \pi_\downarrow^c(v)$ and since v covers exactly one element, by Lemma 4.4 there is exactly one hyperplane separating $[v]_c$ from a congruence class covered by $[v]_c$. Since $[v]_c$ is a full-dimensional cone in an essential fan, it must have at least n facet hyperplanes. Thus (using Proposition 4.4) there must be at least $n-1$ congruence classes covering $[v]_c$ and by the dual of Lemma 4.4 this means that $\pi_c^\uparrow(v)$ is covered by at least $n-1$ distinct elements. The only element of W covered by n elements is 1, and every other element of W is covered by fewer elements. If $\pi_c^\uparrow(v)$ is 1 then v is 1, contradicting the assumption that v is join-irreducible. Therefore

$\pi_c^{\uparrow}(v)$ is covered by exactly $n-1$ elements and covers exactly one element. We have shown the $\pi_c^{\uparrow}$ maps c-sortable join-irreducibles to c-antisortable join-irreducibles. Since $v = \pi_{\downarrow}^c(\pi_c^{\uparrow}(v))$, the map is one-to-one and thus by Corollary 3.10 it is a bijection with inverse $\pi_{\downarrow}^c$. □

We now describe the faces of $\mathcal{F}_\Theta$ (cf. [26, Proposition 5.10]). We will use this description in Section 6 when we discuss the rays of $\mathcal{F}_c$. Let w be maximal in its Θ-equivalence class, in other words, let $w = \pi_\Theta^{\uparrow}(w)$, and let J be a collection of ascents of w. In particular, for all $s \in J$, $[ws]_\Theta \neq [w]_\Theta$. Recall that $C(w, J)$ is an $(n - |J|)$-dimensional face of the Coxeter fan $\mathcal{F}$, and define $C_\Theta(w, J)$ to be the unique $(n - |J|)$-dimensional face of $\mathcal{F}_\Theta$ containing $C(w, J)$.

To see that $C_\Theta(w, J)$ is well-defined, notice that by the dual of Lemma 4.4, there is a collection of $|J|$ facets of $[w]_\Theta$ all of which contain $C(w, J)$. Furthermore, since each of these facets of $[w]_\Theta$ contains a facet of the region for w (a simplicial cone), the intersection of these facets is $(n - |J|)$-dimensional. This is a face of $\mathcal{F}_\Theta$ because it is an intersection of faces of $\mathcal{F}_\Theta$. Uniqueness is ensured because no two distinct k-dimensional faces of a fan have a k-dimensional intersection.

Proposition 4.6. *The map $(w, J) \mapsto C_\Theta(w, J)$ is a bijection from ordered pairs (w, J) with $w = \pi_\Theta^{\uparrow}(w)$ and $[ws]_\Theta > [w]_\Theta$ for every $s \in J$ to faces of $\mathcal{F}_\Theta$.*

Proof. We only give full details of the proof for the restriction of the map to pairs such that $|J| = n-1$, i.e. $C_\Theta(w, J)$ is a ray. Only that restriction is used in this paper. For more general J, we sketch how a proof can be constructed using ideas, terminology and results of [26].[2] The full proof is not difficult but would require quoting [26] in more detail than is desirable.

We first show that the map $(w, J) \mapsto C_\Theta(w, J)$ is surjective. Let C be a k-dimensional face in $\mathcal{F}_\Theta$. By [26, Theorem 1.1] (cf. property (iv) of c-Cambrian fans, above), the set of maximal cones of $\mathcal{F}_\Theta$ which contain C is an interval I in W/Θ. Thus I has a unique minimal element; this minimal element is a Θ-congruence class $[w]_\Theta$, where w is chosen to be the maximal element in the class. (In other words, $\pi_\Theta^{\uparrow}(w) = w$.)

Since $[w]_\Theta$ is lowest in W/Θ among congruence classes containing C, every facet of $[w]_\Theta$ containing C separates $[w]_\Theta$ from a class that is higher in W/Θ. Thus, by Proposition 4.3, every facet of $[w]_\Theta$ containing C separates $[w]_\Theta$ from a class $[ws]_\Theta > [w]_\Theta$. The hyperplanes separating w from those elements of W which cover it are transverse, and by Lemma 4.4 these are exactly the hyperplanes separating $[w]_\Theta$ from the classes that cover it. Thus, since C is k-dimensional, there are exactly $n-k$ facets of $[w]_\Theta$ containing C. Let J be the set of generators s such that $[ws]_\Theta > [w]_\Theta$ and such that the facet separating the two contains C. Then $C(w, J)$ is k-dimensional and is contained in C, so that $C = C_\Theta(w, J)$. We have shown that the map is surjective.

[2] In fact, this proposition, and the proof sketched here, is valid in the more general setting of [26, Section 5]. We can replace the weak order W with a poset of regions of a simplicial hyperplane arrangement. Instead of pairs (w, J), we take pairs (w, P) where w is a region maximal in $[w]_\Theta$ and P is a set of facet hyperplanes of w separating $[w]_\Theta$ from classes above $[w]_\Theta$.

The restriction of the map to pairs (w, J) with $|J| = n-1$ is injective because a ray of $\mathcal{F}_\Theta$ cannot contain two distinct rays of $\mathcal{F}$. We now sketch a proof that the unrestricted map is injective, continuing the notation of the previous two paragraphs. Suppose (w', J') obeys the conditions that $w' = \pi_\Theta^\uparrow(w')$, $[w's]_\Theta > [w']_\Theta$ for every $s \in J'$, $|J'| = |J|$ and $C(w', J') \subseteq C$. We will show that $(w', J') = (w, J)$. If $w' = w$ then since J was defined by considering the set of all facets of $[w]_\Theta$ containing C, we must have $J' = J$.

If $w' \neq w$ then $[w']_\Theta$ is in particular not minimal in I with respect to W/Θ. Arguing as in the proof of [26, Proposition 5.3], one shows that the interval I is isomorphic to the quotient of a facial interval of $\mathcal{F}$ modulo the restriction of Θ. This restriction is bisimplicial by [26, Proposition 5.5], leading to the conclusion that no element of I (except the minimal element) is covered by $n-k$ or more elements of I. Thus there do not exist $n-k$ distinct facets of $[w']_\Theta$ containing C and separating $[w']_\Theta$ from classes higher than $[w']_\Theta$ in W/Θ. In particular, the intersection of the facets separating $[w']_\Theta$ from $[w's]_\Theta$ for $s \in J'$ is a k-dimensional face of $[w']_\Theta$ distinct from C. This contradicts the supposition that $C(w', J') \subseteq C$, thus proving that $w' = w$ and thus that $(w', J') = (w, J)$. □

5. The cluster complex

In this section we review the definition of the c-cluster complex, describe the map cl_c from the c-Cambrian fan to the c-cluster fan and give examples. We begin by reviewing the definition of clusters in the sense of Fomin and Zelevinsky [13] (as extended by Marsh, Reineke and Zelevinsky [22] and extended slightly further in [28]). Let Φ be a root system for W with positive roots Φ_+ and simple roots Π. For any reflection t of W, let α_t denote the positive root associated to t. The roots in $\Phi_{\geq -1} = \Phi_+ \cup (-\Pi)$ are called *almost positive roots*. For any $J \subseteq S$, the set $(\Phi_J)_{\geq -1}$ is the intersection of $\Phi_{\geq -1}$ with the subset of Φ corresponding to the parabolic subgroup W_J.

For each $s \in S$, define an involution $\sigma_s : \Phi_{\geq -1} \to \Phi_{\geq -1}$ by

$$\sigma_s(\alpha) := \begin{cases} \alpha & \text{if } \alpha \in (-\Pi) \text{ and } \alpha \neq -\alpha_s, \\ s(\alpha) & \text{otherwise.} \end{cases}$$

The *c-compatibility* $\|_c$ relation on $\Phi_{\geq -1}$ is defined by the following properties:

(i) For any $s \in S$, $\beta \in \Phi_{\geq -1}$ and Coxeter element c,

$$-\alpha_s \;\|_c\; \beta \quad \text{if and only if} \quad \beta \in (\Phi_{\langle s \rangle})_{\geq -1}.$$

(ii) For any $\alpha_1, \alpha_2 \in \Phi_{\geq -1}$ and any initial letter s of c,

$$\alpha_1 \;\|_c\; \alpha_2 \quad \text{if and only if} \quad \sigma_s(\alpha_1) \;\|_{scs}\; \sigma_s(\alpha_2).$$

The relations $\|_c$ and $\|_{c^{-1}}$ coincide. (See [22, Proposition 3.1] and [28, Proposition 7.4].)

A *c-compatible subset* of $\Phi_{\geq -1}$ is a set of pairwise c-compatible almost positive roots. A *c-cluster* is a maximal c-compatible subset. All c-clusters have cardinality n. Since each element of a c-cluster is a vector, the positive real span of the elements of a c-cluster is a well-defined cone. In fact each c-cluster defines an n-dimensional cone, and these cones are the maximal cones in a complete fan (defined on the linear span of Φ). This is the *c-cluster fan*. A set $\{\alpha_1, \ldots, \alpha_n\}$ is a c-cluster if and only if $\{\sigma_s(\alpha_1), \ldots, \sigma_s(\alpha_n)\}$ is an scs-cluster. Thus, there is a continuous piecewise linear isomorphism between the c-cluster fan and the scs-cluster fan which is linear on each cone and sends each α to $\sigma_s(\alpha)$.

Example 5.1. The reflections in $W = B_2$ are s_0, s_1, $s_0s_1s_0$ and $s_1s_0s_1$. For $c = s_0s_1$, the c-cluster fan is shown in Figure 3, with each ray labeled by the corresponding almost positive root.

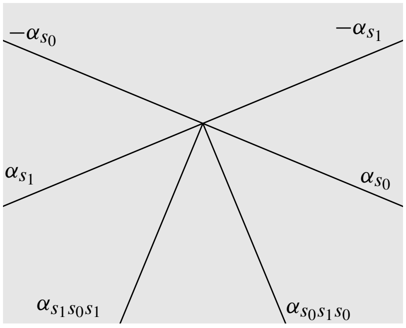


Fig. 3. The c-cluster fan.

The map cl_c takes a c-sortable element w to a set of n almost positive roots. Let $a = a_1a_2\cdots a_k$ be the c-sorting word for w. If $s \in S$ occurs in a then the *last reflection* for s in w is $a_1a_2\cdots a_{j-1}a_ja_{j-1}\cdots a_2a_1$, where a_j is the rightmost occurrence of s in a. The set $\mathrm{cl}_c(w)$ is obtained by taking the set of all positive roots for last reflections of w, together with negative simple roots $-\alpha_s$ for each $s \in S$ not appearing in a. This set does not depend on the choice of reduced word for c, because any two c-sorting words for w are related by commutations of simple generators. One of the main results of [28] is the following theorem, which is an abbreviated form of [28, Theorem 8.1].

Theorem 5.2. *The map $w \mapsto \mathrm{cl}_c(w)$ is a bijection from the set of c-sortable elements to the set of c-clusters.*

Example 5.3. In the case of $W = B_2$ and $c = s_0s_1$, $\mathrm{cl}_c(1) = \{-\alpha_{s_0}, -\alpha_{s_1}\}$ and the table below shows $\mathrm{cl}_c(w)$ for the other c-sortable elements w.

w	s_0	s_0s_1	$s_0s_1s_0$	$s_0s_1s_0s_1$	s_1
$\mathrm{cl}_c(w)$	$\alpha_{s_0}, -\alpha_{s_1}$	$\alpha_{s_0}, \alpha_{s_0s_1s_0}$	$\alpha_{s_1s_0s_1}, \alpha_{s_0s_1s_0}$	$\alpha_{s_1s_0s_1}, \alpha_{s_1}$	$-\alpha_{s_0}, \alpha_{s_1}$

The key result of this paper (Theorem 1.1) strengthens Theorem 5.2 by asserting that cl_c induces a combinatorial isomorphism from $\mathcal{F}_c$ to the c-cluster fan. Via this combinatorial isomorphism, the c-Cambrian lattice induces a partial order on c-clusters which we call the *c-cluster lattice*. The covering pairs of the c-cluster lattice are adjacent maximal cones of the c-cluster fan. One moves down in the partial order by exchanging an almost positive root for another almost positive root which is "closer" to being a negative simple root, in a sense that is made precise in Section 8.

Example 5.4. For $W = B_2$ with $c = s_0 s_1$, the fan $\mathcal{F}_c$ is shown in Figure 4(a), with maximal cones labeled by c-sortable elements. Figure 4(b) shows the c-cluster fan in the same coordinate system. Each maximal cone in the c-cluster fan corresponds to the c-cluster composed of the extreme rays of the cone. These maximal cones are labeled $\mathrm{cl}_c(w)$ (with the subscript c suppressed) for appropriate c-sortable elements w. The labeling of the rays is given in Figure 3. Observe that the obvious linear isomorphism between $\mathcal{F}_c$ and the c-cluster fan is not induced by the bijection cl_c. However, this linear isomorphism is an instance of a general result (Theorem 9.1), which constructs, for special Coxeter elements called *bipartite* Coxeter elements, a linear isomorphism from the c-cluster fan to $\mathcal{F}_c$.

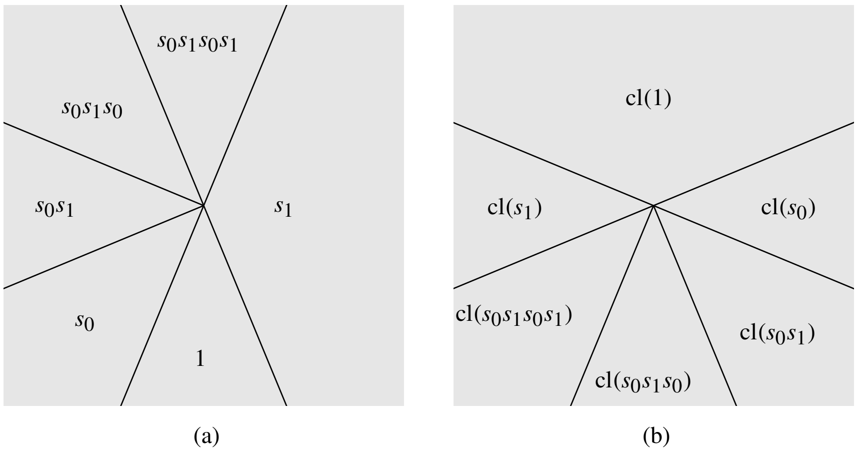


Fig. 4. The c-Cambrian fan and the c-cluster fan.

The following simple lemma, which is [28, Lemma 8.5], is a key ingredient in the proof (in [28]) of Theorem 5.2 and in the results of this paper.

Lemma 5.5. *Let s be initial in c and let w be c-sortable. If $s \not\leq w$ then $w \in W_{\langle s \rangle}$ and* $\mathrm{cl}_c(w) = \{-\alpha_s\} \cup \mathrm{cl}_{sc}(w)$. *If $s \leq w$ then* $\mathrm{cl}_c(w) = \sigma_s(\mathrm{cl}_{scs}(sw))$.

We have now surveyed the relevant background material. In Section 6, we undertake a detailed study of the rays of the Cambrian fan and prove the main technical lemmas underlying our main results. In Section 7, we prove Theorem 1.1. We spend the remaining sections developing the further results described in Section 1.

6. Rays in the c-Cambrian fan

In this section we prove the key lemmas which are used in the proof of Theorem 1.1. These key lemmas, together with certain facts established in previous sections, can be loosely summarized as follows: For s initial in c, all of the objects relevant to Theorem 1.1 are well-behaved under the operation of replacing c by scs or, in some cases, replacing c by sc and passing to the standard parabolic subgroup $W_{\langle s \rangle}$. In particular, we define a map ζ_s which describes how rays of the c-Cambrian fan transform under replacing c by scs. We show that this map is compatible with the map σ_s on almost positive roots (defined in Section 5).

The results of this section rely (indirectly through results proved or quoted in Section 3) on nontrivial results from [28] and [29]. We now specialize the description of the faces of $\mathcal{F}$ and $\mathcal{F}_\Theta$ in the preceding section in order to describe the rays of $\mathcal{F}_c$.

Rays in the Coxeter fan $\mathcal{F}$ are in bijection with pairs (w, J), where $w \in W$ and $J \subseteq S$ satisfy $|J| = n - 1$ and $\ell(ws) > \ell(w)$ for every $s \in J$. In particular, w is either 1 or a join-irreducible element of W. The correspondence is as follows: For any $s \in S$, let ρ_s be the ray in the Coxeter fan which is fixed by $W_{\langle s \rangle}$ and which is an extreme ray of the region for 1. Note that ρ_s is usually not α_s. Given (w, J), the corresponding ray is $w\rho_{s'}$, where s' is the unique element of $S \setminus J$. We write $\rho(w, J)$ for the ray associated to (w, J). Starting with a ray ρ in the Coxeter fan, we recover (w, J) as follows: The elements of W whose regions contain ρ form an interval in W, and w is the minimal element of that interval. The set J is uniquely defined by specifying that the elements covering w in that interval are $\{ws : s \in J\}$.

The following alternative description of $\rho(w, J)$ is also useful: $\rho(w, J)$ is half of the line I defined as the intersection of the hyperplanes associated to the reflections $\{wsw^{-1} : s \in J\}$. Note that any reflecting hyperplane in $\mathcal{A}$ either contains I or intersects I only at the origin. For $w > 1$, $\rho(w, J)$ is the half of I consisting of points weakly separated from D (the region for 1) by any hyperplane in $\mathcal{A}$ which separates wD from D. For $w = 1$, $\rho(w, J)$ is the half of I which is contained in D.

Proposition 4.6 implies that rays in the c-Cambrian fan are the rays of the form $\rho(w, J)$ where w is c-antisortable. By Corollary 3.10, there are $|T|$ such pairs with $w \neq 1$. There are also n such pairs with $w = 1$, namely $(1, \langle r \rangle)$ for each $r \in S$. We now proceed to define and then motivate a bijection ϕ_c from rays of the c-Cambrian fan to almost positive roots. An example is given below (Example 6.1).

Given a ray $\rho(w, J)$ of the c-Cambrian fan, define $v = \pi^c_\downarrow(w)$. In the case $w = 1$ let $\phi_c(\rho) = -\alpha_{s'}$ where $J = \langle s' \rangle$. If $w \neq 1$ then w is join-irreducible, so by Proposition 4.5, v is join-irreducible as well. Define $\phi_c(\rho) = \alpha_t$, where t is the unique cover reflection of v.

We now motivate the definition of ϕ_c by showing that it is forced on us by the requirement that $\phi_c^{-1}(\mathrm{cl}_c(v))$ be the set of rays of the cone $[v]_c$. If $w = 1$ then $v = 1$ and $\mathrm{cl}_c(v)$ consists of all the negative simple roots for W. In this case we must set $\phi_c(\rho) = -\alpha_s$ for some simple reflection $s \in S$. If instead we take $s \neq s'$ then $\phi_c^{-1}(-\alpha_s) = \rho_{s'}$ is a ray both of $[1]_c$ and $[s]_c$. This is inconsistent with the fact that $-\alpha_s \notin \mathrm{cl}_c(s)$.

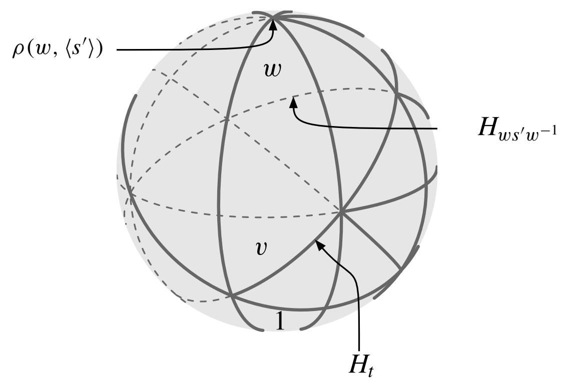


Fig. 5. An illustration of the definition of ϕ_c.

To motivate the definition of ϕ_c in the case $w \neq 1$, let v and t be as defined two paragraphs earlier. Then any reduced word for v must end in the unique letter $r \in S$ such that $t = vrv^{-1}$. In particular, the c-sorting word for v must end in r, so that t is the last reflection for r in v. The hyperplane H_t separates $[v]_c$ from the unique congruence class $[x]_c$ covered by $[v]_c$. Since every region in $[x]_c$ is below the hyperplane H_t, in particular t is not an inversion of $\pi_\downarrow^c(x)$ and so $\alpha_t \notin \mathrm{cl}_c(\pi_\downarrow^c(x))$. If cl_c is to induce a combinatorial isomorphism from the c-Cambrian fan to the c-cluster fan (which is simplicial) then $\mathrm{cl}_c(\pi_\downarrow^c(x))$ and $\mathrm{cl}_c(v)$ should have $n-1$ roots in common and α_t should be the only root in $\mathrm{cl}_c(v)$ which is not in $\mathrm{cl}_c(\pi_\downarrow^c(x))$. Furthermore, the ray ρ associated to (w, J) should be the only ray of $[v]_c$ which is not a ray of $[x]_c$. Thus we are forced to map ρ to α_t.

To see that the map ϕ_c is a bijection, note first that the n rays $\rho(1, J)$ map to the n negative simple roots. The remaining rays are $\rho(w, J)$ where w is a c-antisortable join-irreducible, with a unique J appearing for each such w. Thus Proposition 4.5 shows that there is a unique v (equal to $\pi_\downarrow^c(w)$) for each such pair and so by Corollary 3.9, ϕ_c is a bijection.

Example 6.1. Figure 5 illustrates the definition of ϕ_c in a typical instance of the case $w \neq 1$. The solid lines show the intersection of the c-Cambrian fan with a unit hemisphere. The dotted lines indicate how each maximal cone of the c-Cambrian fan is partitioned into maximal cones (regions) of the Coxeter fan. Here s' is the unique element of $S \setminus J$ and $\phi_c(\rho(w, J))$ is the positive root associated to the hyperplane H_t, the reflecting hyperplane for the unique cover reflection of v.

The following three lemmas constitute a recursive characterization of the rays of Cambrian fans. They play a key role in the proof of Theorem 1.1 in Section 7.

Lemma 6.2. *If s is initial in c then ρ_s is the only ray of the c-Cambrian fan which is below H_s but not contained in H_s.*

Proof. Suppose ρ is a ray of the c-Cambrian fan which is on or below H_s and suppose $\rho \neq \rho_s$. We will show that ρ is on H_s. Let $\rho = \rho(w, J)$. The claim is easy when

$(w, J) = (1, \langle r\rangle)$ for $r \neq s$ and we have excluded the case $(w, J) = (1, \langle s\rangle)$, so we may assume that $w \neq 1$. Then since ρ is on or below H_s, w is also below H_s, or in other words $s \not\leq w$. By Lemma 3.12, $sw \gtrdot w$. The ray ρ is contained in the intersection of all hyperplanes separating w from an element covering w, so in particular, ρ is contained in H_s. □

Lemma 6.3. *Let s be initial in c. For any almost positive root $\alpha \in (\Phi_{\langle s\rangle})_{\geq -1}$,*

$$\phi_c^{-1}(\alpha) = \mathrm{Proj}_{\langle s\rangle}^{-1}(\phi_{sc}^{-1}(\alpha)) \cap H_s,$$

where H_s is the hyperplane associated to the reflection s.

Recall that Proj_J is the linear projection $V(W) \twoheadrightarrow V(W_J)$; in this case, the kernel of $\mathrm{Proj}_{\langle s\rangle}$ is $\mathrm{Span}_{\mathbb{R}}\, \rho_s$. Here $\phi_c^{-1}(\alpha)$ is a ray of the c-Cambrian fan, and thus in particular a ray of the Coxeter fan for W. The ray $\phi_{sc}^{-1}(\alpha)$ is the ray in the sc-Cambrian fan mapped to α by ϕ_{sc}. Thus $\mathrm{Proj}_{\langle s\rangle}^{-1}(\phi_{sc}^{-1}(\alpha))$ is two-dimensional and $\mathrm{Proj}_{\langle s\rangle}^{-1}(\phi_{sc}^{-1}(\alpha)) \cap H_{\langle s\rangle}$ is a ray again.

Here is another description of Lemma 6.3: The projection $\mathrm{Proj}_{\langle s\rangle}$ restricts to an isomorphism from H_s to $V_{\langle s\rangle}$. If we identify $V_{\langle s\rangle}$ and H_s by this isomorphism, Lemma 6.3 is the statement that $\phi_c^{-1}(\alpha) = \phi_{sc}^{-1}(\alpha)$. However, this identification of $V_{\langle s\rangle}$ with H_s is not norm preserving, and we will therefore not pursue this viewpoint.

Proof of Lemma 6.3. Let $\rho(w, J) = \phi_c^{-1}(\alpha)$ and let $\rho(w', J') = \phi_{sc}^{-1}(\alpha)$. Let $v = \pi_\downarrow^c(w)$, let $v' = \pi_\downarrow^{sc}(w')$ and let $\tilde{s}$ be the unique element of $S \setminus J$. If $v = 1$ then $\alpha = -\alpha_{\tilde{s}}$ and thus $v' = 1$ as well, with $J' = J \setminus \{s\}$. Now $\phi_c^{-1}(\alpha) = \rho_{\tilde{s}}$, where $\rho_{\tilde{s}}$ is a ray in $V(W)$, and $\phi_{sc}^{-1}(\alpha)$ is also $\rho_{\tilde{s}}$, interpreted as a ray in $V(W_{\langle s\rangle})$. The desired conclusion now follows from the definition of $\rho_{\tilde{s}}$ in each case.

If $v > 1$ then α is not a negative simple root, so $v' > 1$ as well. Thus v' is join-irreducible, and since v' is sc-sortable, it is in particular c-sortable. For both v and v', the positive root associated to the unique cover reflection is α. We have (by Corollary 3.9) $v = v'$ and thus $w = \pi_c^\uparrow(v)$ and $w' = \pi_{sc}^\uparrow(v)$. The ray $\rho(w, J)$ is half of the line I defined as the intersection of the hyperplanes $\{H_{wrw^{-1}}(V(W)) : r \in J\}$, and the ray $\rho(w', J')$ is half of the line I' defined as the intersection of the hyperplanes $\{H_{w'r(w')^{-1}}(V(W_{\langle s\rangle})) : r \in J'\}$. Here we have written $H_t(V(W))$ for the hyperplane in $V(W)$ associated to the reflection t. We have $\mathrm{Proj}_{\langle s\rangle}^{-1}(H_t(V(W_{\langle s\rangle}))) = H_t(V(W))$.

We next show that $I = \mathrm{Proj}_{\langle s\rangle}^{-1}(I') \cap H_s$ by proving the stronger statement

$$\{wrw^{-1} : r \in J\} = \{w'r(w')^{-1} : r \in J'\} \cup \{s\}.$$

Recall that the set $\{wrw^{-1} : r \in J\}$ is the set of all reflections t such that H_t separates w from an adjacent region covering w in weak order. By the dual of Lemma 4.4, these are the reflections t such that H_t separates $[w]_c = [w']_c$ from an adjacent region covering it in $\mathcal{F}_c$. Similarly, $\{w'r(w')^{-1} : r \in J'\}$ is the set of reflections $t \in W_{\langle s\rangle}$ such that H_t separates $[w']_{sc}$ from an adjacent region covering it in $\mathcal{F}_{sc}$. But the c-Cambrian congruence restricted to $W_{\langle s\rangle}$ is simply the sc-Cambrian congruence, so any region covering $[w']_{sc}$

in W/Θ_{sc} also corresponds to a cover of $[w']_c$ in W/Θ_c with a separating hyperplane corresponding to the same reflection. Thus, $\{w'r(w')^{-1} : r \in J'\} \subseteq \{wrw^{-1} : r \in J\}$.

Now, the set $\{wrw^{-1} : r \in J\}$ contains exactly one additional element not contained in $\{w'r(w')^{-1} : r \in J'\}$. Since $s \not\leq v$, Lemma 3.5 says that $s \not\leq w$. Thus Lemma 3.12 says that sw covers w, so that $s \in \{wrw^{-1} : r \in J\}$, completing the proof that $\{wrw^{-1} : r \in J\} = \{w'r(w')^{-1} : r \in J'\} \cup \{s\}$.

Having established that $\phi_c^{-1}(\alpha)$ and $\mathrm{Proj}_{\langle s\rangle}^{-1}(\phi_{sc}^{-1}(\alpha)) \cap H_s$ each constitute half of the line I, it remains to show that they are the same half of I. Since $v' > 1$, there is a hyperplane H in $\mathcal{A}_{\langle s\rangle}$ which separates $w'D$ from D, and since $w \geq w'$, H separates wD from D as well. Thus both $\phi_c^{-1}(\alpha)$ and $\mathrm{Proj}_{\langle s\rangle}^{-1}(\phi_{sc}^{-1}(\alpha)) \cap H_s$ are the half of I weakly separated from D by H. □

For a ray ρ of the Cambrian fan, define

$$\zeta_s(\rho) = \begin{cases} s\rho & \text{if } \rho \neq \rho_s, \\ -\rho & \text{if } \rho = \rho_s. \end{cases}$$

Since the Coxeter fan is preserved by the action of s and by the antipodal map, $\zeta_s(\rho)$ is a ray of the Coxeter fan.[3] The following lemma states that $\zeta_s(\rho)$ is a ray of the scs-Cambrian fan and establishes the compatibility of ζ_s with ϕ_c, σ_s and ϕ_{scs}. The proof is not difficult, but it has many cases.

Lemma 6.4. *Let s be an initial letter of c and let ρ be a ray in the c-Cambrian fan. Then $\zeta_s(\rho)$ is a ray in the scs-Cambrian fan and $\phi_{scs}(\zeta_s(\rho)) = \sigma_s(\phi_c(\rho))$.*

Proof. Let s, c and ρ be as in the statement of the lemma and let $\rho = \rho(w, J)$. Further, let s' be such that $J = \langle s'\rangle$ and let $v = \pi_\downarrow^c(w)$.

Case 1: $w = 1$. This case splits into two subcases:

Case 1a: $(w, J) = (1, \langle s\rangle)$. This is the exceptional case in the definition of ζ_s, where $\rho = \rho_s$. In this case $\phi_c(\rho) = -\alpha_s$, so $\sigma_s(\phi_c(\rho)) = \alpha_s$. Since $\zeta_s(\rho) = -\rho_s$, we need to show that $-\rho_s$ is a ray in the scs-Cambrian fan and that $\phi_{scs}(-\rho_s) = \alpha_s$. Let $\rho' = \rho(w', J')$ be the unique ray of the scs-Cambrian fan with $\phi_{scs}(\rho') = \alpha_s$. Since s is the unique scs-sortable join-irreducible whose associated reflection is s, we must have $\pi_\downarrow^{scs}(w') = s$ so that $\pi^\uparrow_{scs}(s) = w'$. Thus $w' = w_0 \cdot ((w_0)_{\langle w_0 s w_0\rangle})$ by Lemma 3.7. The ascents of w' are $J' = \langle w_0 s w_0\rangle$, so $\rho' = w'\rho_{w_0 s w_0} = w_0\rho_{w_0 s w_0}$, the latter equality holding because $(w_0)_{\langle w_0 s w_0\rangle}$ fixes $\rho_{w_0 s w_0}$. But $w_0\rho_{w_0 s w_0} = -\rho_s$.

Case 1b: $(w, J) = (1, \langle r\rangle)$ for $r \neq s$. In this case $\rho = \rho_r$ and since ρ_r is on the reflecting hyperplane for s, $\zeta_s(\rho) = s\rho = \rho$. Since 1 is also scs-sortable and J is a set of $n-1$ ascents of 1, ρ is also a ray of the scs-Cambrian fan with $\phi_{scs}(\rho) = -\alpha_r$. Also, $\sigma_s(\phi_c(\rho)) = \sigma_s(-\alpha_r) = -\alpha_r$.

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3 One should note that the inverse map ζ_s^{-1} from rays of the scs-Cambrian fan to rays of the c-Cambrian fan is *not* given by the same formula.

Case 2: $s \leq w$. By Lemmas 2.3 and 3.6, sw is scs-antisortable and sw is either 1 or join-irreducible. Furthermore, J is a set of elements which lengthen not only w but also sw on the right. We consider two subcases, depending on whether or not $w = s$. Notice that since s is initial in c, the sole element in the c-Cambrian equivalence class of s is s itself. Thus $w = s$ if and only if $v = s$.

Case 2a: $w = s$. In this case $(sw, J) = (1, \langle s \rangle)$, with associated ray $\rho_s = \zeta_s(\rho)$. Also $v = s$ and $\rho = s\rho_s$. So we have $\sigma_s(\phi_c(\rho)) = \sigma_s(\alpha_s) = -\alpha_s = \phi_{scs}(\rho_s) = \phi_{scs}(\zeta_s(\rho))$.

Case 2b: $w \neq s$. In this case $\phi_c(\rho)$ is a positive root α_t for some reflection $t \neq s$ and thus $\sigma_s(\phi_c(\rho))$ is α_{sts}. The ray $\rho(sw, J)$ of the scs-Cambrian fan is $\rho' := sw\rho_{s'} = \zeta_s(\rho)$. Also, $\pi_\downarrow^c(w) = s \cdot \pi_\downarrow^{scs}(sw)$, so that $\pi_\downarrow^{scs}(sw) = s \cdot \pi_\downarrow^c(w) = sv$. The reflection t is the unique cover reflection of v and, by Lemma 2.3, the element sv is join-irreducible with unique cover reflection sts. In particular, $\phi_{scs}(\rho') = \alpha_{sts}$. This concludes the proof for the case $s \leq w$.

Case 3: $s \not\leq w$ and $w \neq 1$. By Lemma 3.12, w is scs-antisortable and $sw = s \vee w$. Since w is scs-antisortable, the pair (w, J) defines ρ not only as a ray of the c-Cambrian fan but also as a ray of the scs-Cambrian fan. Since $w \neq 1$ but w is below H_s, ρ is contained in H_s by Lemma 6.2, so $\rho = \zeta_s(\rho)$.

Let $v' = \pi_\downarrow^{scs}(w)$. By definition, $\pi_\downarrow^c(sw) = s \cdot \pi_\downarrow^{scs}(w) = sv'$. Since $\pi_\downarrow^c$ is a lattice homomorphism, $\pi_\downarrow^c(sw) = \pi_\downarrow^c(s \vee w) = s \vee \pi_\downarrow^c(w) = s \vee v$. Thus $v' = s \cdot (s \vee v)$. Because $s \not\leq w$, v is in $W_{\langle s \rangle}$, so by Lemma 2.2, the set of cover reflections of $s \vee v$ is $\{s, t\}$, where t is the unique cover reflection of v. Let t' be the unique cover reflection of v'; by definition, $\phi_{scs}(w') = \alpha_{t'}$. Since the interval $[1, sw_0]$ is isomorphic to the interval $[s, w_0]$ by the map $x \mapsto sx$, the reflection $st's$ is a cover reflection of $sv' = s \vee v$. But $v' \not\geq s$, so $t' \neq s$ and thus $st's = t$. Therefore $\phi_{scs}(\rho) = \alpha_{sts} = \sigma_s(\alpha_t) = \sigma_s(\phi_c(\rho))$. □

7. Proof of the combinatorial isomorphism

In this section we prove the main theorem, Theorem 1.1, which states that, for W finite, the map cl_c induces a combinatorial isomorphism from the c-Cambrian fan to the c-cluster fan. We also discuss some first consequences of Theorem 1.1.

The c-cluster fan is simplicial. That is, each of its maximal faces is the positive linear span of a collection of linearly independent vectors. Specifically, this collection of vectors is a c-cluster of almost positive roots. In contrast, we do not even know that the c-Cambrian fan is simplicial. However, we know that the maximal cones of the c-Cambrian fan are, by definition, unions of regions of the Coxeter arrangement (see Section 4). Specifically, each maximal cone of the c-Cambrian fan is the union over a fiber of the map $\pi_\downarrow^c$. Showing that the c-Cambrian fan is simplicial means showing the following: For each fiber of $\pi_\downarrow^c$, there is a collection E of n rays of the c-Cambrian fan such that a given region is a member of the fiber if and only if that region is contained in the positive linear span of E.

The stronger statement that cl_c induces a combinatorial isomorphism between the c-Cambrian fan and the c-cluster fan is equivalent to the additional condition that there is a bijection ϕ between the rays of the c-Cambrian fan and the rays of the c-cluster fan such that the collection E of rays used to determine membership in $(\pi_\downarrow^c)^{-1}(x)$ obeys $\phi(E) = \mathrm{cl}_c(x)$.

As was shown in Section 6, the map ϕ_c is a bijection from rays of the c-Cambrian fan to almost positive roots—that is, to rays of the c-cluster fan. Thus the proof of Theorem 1.1 is completed by Proposition 7.1 below. Recall that, if W_J is a standard parabolic subgroup, then the dominant chamber of $\mathcal{A}(W_J)$ is denoted by D_J.

Proposition 7.1. *Let x be c-sortable. Then the following are equivalent for any $w \in W$.*

(i) $\pi_\downarrow^c(w) = x$.
(ii) *The interior of the region wD intersects the positive span of $\phi_c^{-1}(\mathrm{cl}_c(x))$.*
(iii) *The region wD is contained in the positive span of $\phi_c^{-1}(\mathrm{cl}_c(x))$.*

Here, since $\mathrm{cl}_c(x)$ is a cluster of almost positive roots, $\phi_c^{-1}(\mathrm{cl}_c(x))$ represents the set of rays obtained by applying ϕ_c^{-1} to each member of the cluster.

Proof. The fact that (iii) implies (ii) is trivial. We prove that (i) implies (iii) and that (ii) implies (i) by induction on the length of w and the rank of W. Let s be initial in c. For each implication we will consider two cases: $s \not\leq w$ and $s \leq w$.

First, assume (i). If $s \not\leq w$ then $\pi_\downarrow^{sc}(w_{\langle s \rangle}) = x$, so that in particular $x \in W_{\langle s \rangle}$. By Lemma 5.5, $\mathrm{cl}_c(x) = \mathrm{cl}_{sc}(x) \cup \{-\alpha_s\}$. By Lemma 6.3, each ray in $\phi_c^{-1}(\mathrm{cl}_{sc}(x))$ is obtained from the corresponding ray ρ in $\phi_{sc}^{-1}(\mathrm{cl}_{sc}(x))$ by intersecting $\mathrm{Proj}_{\langle s \rangle}^{-1}(\rho)$ with the hyperplane H_s. Since $\phi_c^{-1}(-\alpha_s) = \rho_s$ is the half of the intersection of the hyperplanes in $\mathcal{A}_{\langle s \rangle}$ which is below the hyperplane H_s, the positive span of $\phi_c^{-1}(\mathrm{cl}_c(x))$ is the part of the positive span of $\mathrm{Proj}_{\langle s \rangle}^{-1}(\phi_{sc}^{-1}(\mathrm{cl}_{sc}(x)))$ which is below the hyperplane H_s. Now, wD is contained in $\mathrm{Proj}_{\langle s \rangle}^{-1}(w_{\langle s \rangle} D_{\langle s \rangle})$ (Proposition 4.1), which is, by induction on rank, contained in the positive span of $\mathrm{Proj}_{\langle s \rangle}^{-1}(\phi_{sc}^{-1}(\mathrm{cl}_{sc}(x)))$. Since $s \not\leq w$, wD is below the hyperplane H_s. Thus we see that wD is in the positive span of $\phi_c^{-1}(\mathrm{cl}_c(x))$.

If $s \leq w$ then (i) implies $\pi_\downarrow^{scs}(sw) = sx$. By Lemma 3.5, $s \leq x$, and thus $x \notin W_{\langle s \rangle}$, so that in particular $\rho_s \notin \phi_c^{-1}(\mathrm{cl}_c(x))$. By induction on length, swD is completely in the positive span of $\phi_{scs}^{-1}(\mathrm{cl}_{scs}(sx))$. By Lemma 5.5, $\phi_{scs}^{-1}(\mathrm{cl}_{scs}(sx))$ equals $\phi_{scs}^{-1}(\sigma_s(\mathrm{cl}_c(x)))$, which by Lemma 6.4 equals $\zeta_s(\phi_c^{-1}(\mathrm{cl}_c(x))) = s\phi_c^{-1}(\mathrm{cl}_c(x))$. Since swD is completely in the positive span of $s\phi_c^{-1}(\mathrm{cl}_c(x))$, wD is completely in the positive span of $\phi_c^{-1}(\mathrm{cl}_c(x))$.

Now suppose (ii). If $s \not\leq w$ then every point in the interior of wD is strictly below H_s. By Lemma 6.2, $\phi_c^{-1}(\mathrm{cl}_c(x))$ must contain the ray ρ_s, so that $\mathrm{cl}_c(x)$ contains $-\alpha_s$. In particular, $x \in W_{\langle s \rangle}$, and furthermore by Lemma 5.5, $\mathrm{cl}_c(x) = \mathrm{cl}_{sc}(x) \cup \{-\alpha_s\}$. By Proposition 4.1, $\mathrm{Proj}_{\langle s \rangle}(wD) \subseteq w_{\langle s \rangle} D_{\langle s \rangle}$ and the interior of wD is taken into the interior of $w_{\langle s \rangle} D_{\langle s \rangle}$ by $\mathrm{Proj}_{\langle s \rangle}$ (by considerations of dimension). Since ρ_s is in the kernel of $\mathrm{Proj}_{\langle s \rangle}$, the interior of $w_{\langle s \rangle} D_{\langle s \rangle}$ intersects the positive span of $\phi_{sc}^{-1}(\mathrm{cl}_{sc}(x))$. By induction on rank, $\pi_\downarrow^{sc}(w_{\langle s \rangle}) = x$ and thus $\pi_\downarrow^c(w) = x$.

If $s \leq w$ then we claim that $s \leq x$. Supposing to the contrary that $s \not\leq x$, by Lemma 5.5, $\mathrm{cl}_c(x) = \{-\alpha_s\} \cup \mathrm{cl}_{sc}(x)$. Thus by Lemma 6.3, $\phi_c^{-1}(\mathrm{cl}_c(x))$ consists of rays which are weakly below H_s. But the interior of wD is strictly above H_s, contradicting the supposition that (ii) holds. This contradiction proves the claim that $s \leq x$. In particular, ρ_s is not in $\phi_c^{-1}(\mathrm{cl}_c(x))$, so that $\zeta_s(\phi_c^{-1}(\mathrm{cl}_c(x))) = s\phi_c^{-1}(\mathrm{cl}_c(x))$. Thus the interior of swD meets the positive span of $\zeta_s(\phi_c^{-1}(\mathrm{cl}_c(x)))$, which by Lemma 6.4 equals $\phi_{scs}^{-1}(\sigma_s(\mathrm{cl}_c(x)))$. Since $s \leq x$, Lemma 5.5 says that the latter is $\phi_{scs}^{-1}(\mathrm{cl}_{scs}(sx))$. By induction on length, $\pi_\downarrow^{scs}(sw) = sx$, so that $\pi_\downarrow^c(w) = x$. □

This completes the proof of Theorem 1.1. In fact, we have proven the following more detailed version of Theorem 1.1.

Theorem 7.2. *The c-Cambrian fan $\mathcal{F}_c$ is simplicial and the bijection ϕ_c between the rays of the Cambrian fan and the almost positive roots induces a combinatorial isomorphism of fans between the c-Cambrian fan and the c-cluster fan. Under this isomorphism, the maximal cone $[w]_c$ is taken to the cluster $\mathrm{cl}_c(w)$.*

If w is c-sortable and $x \lessdot w$ then the maximal cones $[w]_c$ and $[\pi_\downarrow^c(x)]_c$ intersect in a facet of dimension $n-1$ and so have exactly $n-1$ rays in common. Thus Theorem 1.1 has the following corollary.

Corollary 7.3. *Let w be c-sortable and let $x \lessdot w$. Then the c-clusters $\mathrm{cl}_c(w)$ and $\mathrm{cl}_c(\pi_\downarrow^c(x))$ have exactly $n-1$ almost positive roots in common.*

In Section 5, we noted that the action of σ_s on almost positive roots induces a combinatorial isomorphism between the c-cluster fan and the scs-cluster fan. Thus, by Theorem 1.1, the map $\phi_{scs}^{-1} \circ \sigma_s \circ \phi_c$ induces a combinatorial isomorphism between $\mathcal{F}_c$ and $\mathcal{F}_{scs}$. But Lemma 6.4 implies that $\phi_{scs}^{-1} \circ \sigma_s \circ \phi_c$ coincides with ζ_s. Since the combinatorial isomorphism is determined by its action on rays, we have the following.

Proposition 7.4. *The action of ζ_s on the rays of the c-Cambrian fan $\mathcal{F}_c$ induces a combinatorial isomorphism between $\mathcal{F}_c$ and $\mathcal{F}_{scs}$.*

In particular, $\mathcal{F}_c$ and $\mathcal{F}_{scs}$ are related by a piecewise linear map that is only a slight deformation of the linear map s. On and above H_s the map agrees with s. Below H_s the map agrees with the linear map that fixes H_s and takes ρ_s to $-\rho_s$.

We now describe the isomorphism between $\mathcal{F}_c$ and $\mathcal{F}_{scs}$ directly in terms of sortable elements (cf. [29, Remark 3.8]). For s initial in c, define a map Z_s from the set of c-sortable elements of W to the set of scs-sortable elements by

$$Z_s(w) = \begin{cases} sw & \text{if } s \leq w, \\ s \vee w & \text{if } s \not\leq w. \end{cases}$$

We now check that Z_s maps c-sortable elements to scs-sortable elements. If $s \leq w$, this is Lemma 3.3. If $s \not\leq w$ then Lemma 3.2 states that $w \in W_{\langle s \rangle}$ and w is sc-sortable. The sc-sorting word for w is identically equal to the scs-sorting word for w, so that w is scs-sortable. The set of scs-sortable elements forms a sublattice of W (Theorem 3.11), so $s \vee w$ is also scs-sortable.

The inverse of Z_s is

$$Z_s^{-1}(w) = \begin{cases} w_{\langle s \rangle} & \text{if } s \leq w, \\ sw & \text{if } s \not\leq w. \end{cases}$$

There are only two nontrivial assertions in the statement that this map is indeed the inverse of Z_s: first, that any c-sortable element w with $s \not\leq w$ obeys the condition $(s \vee w)_{\langle s \rangle} = w$; and second, that an scs-sortable element w with $s \leq w$ obeys the condition $(w_{\langle s \rangle} \vee s) = w$. Recall that $x \mapsto x_{\langle s \rangle}$ is a lattice homomorphism, so that $(s \vee w)_{\langle s \rangle} = s_{\langle s \rangle} \vee w_{\langle s \rangle} = w_{\langle s \rangle}$. Thus the first assertion follows from Lemma 3.2. The second assertion is exactly [29, Lemma 2.10].

The following lemma states that $[w]_c \mapsto [Z_s(w)]_{scs}$ is the isomorphism between $\mathcal{F}_c$ and $\mathcal{F}_{scs}$ corresponding to the isomorphism σ_s of cluster fans.

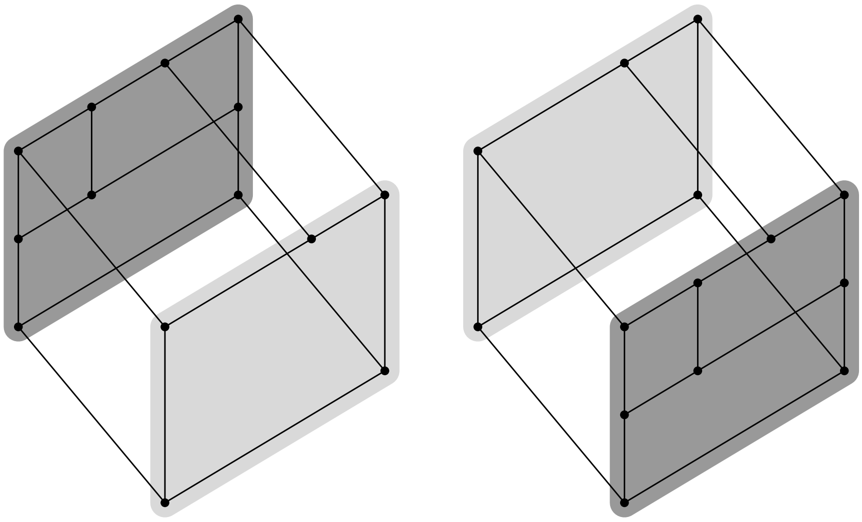
Lemma 7.5. *For a c-sortable element w,* $\mathrm{cl}_{scs}(Z_s(w)) = \sigma_s \, \mathrm{cl}_c(w)$.

Proof. If $s \leq w$ then Lemma 5.5 is the desired statement.

If $s \not\leq w$ then the desired equality is $\mathrm{cl}_{scs}(s \vee w) = \sigma_s \, \mathrm{cl}_c(w)$. Since ϕ_{scs} is a bijection, this is equivalent to checking that $\phi_{scs}^{-1} \, \mathrm{cl}_{scs}(s \vee w) = \phi_{scs}^{-1} \sigma_s \, \mathrm{cl}_c(w)$, which can be rewritten, using Lemma 6.4, as $\phi_{scs}^{-1} \, \mathrm{cl}_{scs}(s \vee w) = \zeta_s \phi_c^{-1} \, \mathrm{cl}_c(w)$. In other words, the requirement is that the rays $\rho_1, \ldots, \rho_n$ of the c-Cambrian cone $[w]_c$ are mapped by ζ_s to the rays of the scs-Cambrian cone $[s \vee w]_{scs}$. By Lemma 3.5 all the ρ_i are below H_s. By Lemma 6.2, all of the ρ_i are in H_s except for possibly one, which is ρ_s. We know that $\rho_1, \ldots, \rho_n$ are linearly independent, so one of the ρ_i must be ρ_s; without loss of generality let $\rho_n = \rho_s$. Then $\zeta_s(\rho_n) = -\rho_s$ and $\zeta_s(\rho_i) = \rho_i$ for $i < n$. Now, for any $u \in W$, u is in the positive span of $-\rho_s$ and $\rho_1, \ldots, \rho_{n-1}$ if and only if the following conditions hold: $u \geq s$ and $u_{\langle s \rangle}$, considered as a region of $V(W_{\langle s \rangle})$, is in the positive span of $\rho_1, \ldots, \rho_{n-1}$.

We have $s \vee w \geq s$ and also $(s \vee w)_{\langle s \rangle} = s_{\langle s \rangle} \vee w_{\langle s \rangle} = w_{\langle s \rangle} = w$. Our hypothesis is that w, when considered as a region of $V(W)$, is in the positive span of $\rho_1, \ldots, \rho_n$. This implies that w considered as a region of $V(W_{\langle s \rangle})$ is in the positive span of $\rho_1, \ldots, \rho_{n-1}$. So we conclude that $s \vee w$ is in the positive span of the $\zeta_s(\rho_i)$ as desired. □

Example 7.6. The map Z_s is perhaps more easily visualized as a map from the c-Cambrian lattice to the scs-Cambrian lattice. Figure 6(a) shows the $s_1s_2s_3$-Cambrian lattice for W of type A_3; this is also the lattice depicted in Figure 1(b). The light gray shading indicates (congruence classes of) $s_1s_2s_3$-sortable elements not above s_1, while dark gray shading indicates $s_1s_2s_3$-sortable elements above s_1. Figure 6(b) shows the $s_2s_3s_1$-Cambrian lattice for the same W. The map Z_{s_1} takes the $s_1s_2s_3$-sortable elements not above s_1 to the $s_2s_3s_1$-sortable elements above s_1, which are shaded light gray in Figure 6(b). The $s_1s_2s_3$-sortable elements above s_1 are taken to $s_2s_3s_1$-sortable elements not above s_1, shaded dark gray in Figure 6(b). Notice that Z_{s_1} restricted to light-shaded elements in Figure 6(a) is a poset isomorphism to light-shaded elements in Figure 6(b), and similarly for dark-shaded elements.



(a) (b)

Fig. 6. The c- and scs-Cambrian lattices.

8. The cluster lattice

We have now established in great detail the combinatorial isomorphism between the c-Cambrian fan and the c-cluster fan. The maximal cones of the c-Cambrian fan are partially ordered by the Cambrian lattice W/Θ_c, so we obtain an induced poset (in fact, lattice) structure on the set of c-clusters. In this section we will apply our results to describe this poset directly in terms of cluster combinatorics.

The *exchange graph* on c-clusters is the adjacency graph on maximal cones of the c-cluster complex. In other words, the vertices are the c-clusters, with an edge between C and C' if and only if $|C \cap C'| = n-1$. The exchange graph is isomorphic to the 1-skeleton of the (simple) generalized associahedron for W as defined in [13]. We have the following corollary of Theorem 1.1.

Corollary 8.1. *The undirected Hasse diagram of the c-Cambrian lattice* W/Θ_c *is isomorphic to the exchange graph on c-clusters and hence isomorphic to the* 1*-skeleton of the generalized associahedron for W.*

Proof. Proposition 4.3 implies that the Hasse diagram of W/Θ_c is the adjacency graph of maximal cones of $\mathcal{F}_c$, which, according to Theorem 1.1, is mapped by cl_c to the exchange graph. □

In light of Corollary 8.1, to describe the poset induced on c-clusters by the c-Cambrian lattice, it is sufficient to give the correct orientation of the exchange graph. Two almost positive roots α and α' are said to be *c-exchangeable* if they are distinct and if there is a set $B \subseteq \Phi_{\geq -1} \setminus \{\alpha, \alpha'\}$ such that both $B \cup \{\alpha\}$ and $B \cup \{\alpha'\}$ are c-clusters. Note that c-exchangeable roots α and α' are never c-compatible. We will show that the correct orientation of an edge $B \cup \{\alpha\}$ — $B \cup \{\alpha'\}$ depends only on c and the pair (α, α') of c-exchangeable roots. Specifically, the orientation is given by comparing α and α' using a function R_c from almost positive roots to natural numbers which we now proceed to define.

Recall from the introduction the involution $\sigma_s : \Phi_{\geq -1} \to \Phi_{\geq -1}$ for each $s \in S$. Let $s_1 \cdots s_n$ be a reduced word for c and define $\sigma_c = \sigma_{s_1} \cdots \sigma_{s_n}$. Any two reduced words for c differ only by interchanging commuting reflections and $\sigma_s \sigma_t = \sigma_t \sigma_s$ whenever $st = ts$, so σ_c is a well defined permutation of $\Phi_{\geq -1}$. Note that $(\sigma_c)^{-1} = \sigma_{c^{-1}}$.

Proposition 8.2. *For any almost positive root α and any Coxeter element c, there exists a nonnegative integer R such that $\sigma_c^{-R}(\alpha)$ is a negative simple root.*

Proposition 8.2 will be proved later in the section. We write $R_c(\alpha)$ for the smallest such R. Assuming the proposition, we define the *c-cluster lattice* Clust_c to be the partial order on c-clusters whose cover relations are $B \cup \{\alpha\} \lessdot B \cup \{\alpha'\}$ if and only if $R_c(\alpha) < R_c(\alpha')$ (cf. [27, Section 8]). It is not obvious from this definition that these relations are in fact cover relations of the partial order they generate. However, in light of the preceding discussion, the following proposition implies that the relations above are in fact cover relations.

Proposition 8.3. *Suppose α and α' are c-exchangeable almost positive roots and let B be a subset of $\Phi_{\geq -1} \setminus \{\alpha, \alpha'\}$ such that $B \cup \{\alpha\}$ and $B \cup \{\alpha'\}$ are c-clusters. Then $[\mathrm{cl}_c^{-1}(B \cup \{\alpha\})]_c$ is covered by $[\mathrm{cl}_c^{-1}(B \cup \{\alpha'\})]_c$ in the c-Cambrian lattice if and only if $R_c(\alpha) < R_c(\alpha')$.*

Notice that the case $R_c(\alpha) = R_c(\alpha')$ is impossible for α and α' as in the proposition. This is because if $R_c(\alpha) = R_c(\alpha')$, one can iterate the definition of c-compatibility (see Section 5) to show that $\alpha \parallel_c \alpha'$, contradicting the fact that α and α' are c-exchangeable. Proposition 8.3 and Corollary 8.1 immediately imply the main theorem of this section, which is a generalization of the second statement of [27, Conjecture 1.4].

Theorem 8.4. *The map cl_c is an isomorphism of lattices from the c-Cambrian lattice W/Θ_c to the c-cluster lattice.*

Before proving Propositions 8.2 and 8.3, we mention some consequences of Theorem 8.4. The following corollary is immediate from Theorem 1.1, Theorem 8.4 and the properties of $\mathcal{F}_c$ and W/Θ_c listed in Section 4.

Corollary 8.5. *The c-cluster lattice Clust_c has the following properties.*

(i) Clust_c *is a lattice.*
(ii) *Any linear extension of* Clust_c *is a shelling order of the c-cluster complex.*
(iii) *For any set F of c-compatible almost positive roots, the set of c-clusters containing F forms an interval in* Clust_c.
(iv) *A closed interval I in* Clust_c *has proper part homotopy equivalent to an $(n-k-2)$-dimensional sphere if and only if there is some set F of k c-compatible almost positive roots such that I is the set of all c-clusters containing F.*

(v) *A closed interval I in* Clust_c *has proper part homotopy equivalent to an* $(k-2)$*-dimensional sphere if and only if I has k atoms and the join of the atoms of I is the top element of I.*
(vi) *If the proper part of a closed interval I is not homotopy-spherical then it is contractible.*

Theorem 8.4 also has important enumerative consequences. Let $C \subset \Phi_{\geq -1}$ be a c-cluster. For each $\alpha \in C$, there is a unique $\alpha' \in \Phi_{\geq -1}$ such that $(C \setminus \{\alpha\}) \cup \{\alpha'\}$ is also a c-cluster. Call α an *upper root* of C if $R_c(\alpha) > R_c(\alpha')$ and a *lower root* of C if $R_c(\alpha) < R_c(\alpha')$. Equivalently, α is an upper root if $C \gtrdot (C \setminus \{\alpha\}) \cup \{\alpha'\}$ or a lower root if $C \lessdot (C \setminus \{\alpha\}) \cup \{\alpha'\}$. Note that the identification of α as a lower or upper root depends on the c-cluster C. A root may be an upper root in one c-cluster and a lower root in another c-cluster.

Corollary 8.6. *The map* cl_c *takes c-sortable elements with k descents to c-clusters with k upper roots.*

Since any linear extension of Clust_c is a shelling order, by standard arguments the number of c-clusters with k upper roots is the k^{th} entry in the h-vector of c-cluster fan, or equivalently the h-vector of the generalized associahedron for W. Thus Corollary 8.6 combines with the second sentence of Theorem 3.8 to give a bijective proof of the following.

Corollary 8.7. *The number of noncrossing partitions (with respect to c) of rank k equals the k^{th} entry in the h-vector of the generalized associahedron for W.*

This number is known as the k^{th} *Narayana number* associated to W. Corollary 8.6 can be viewed as a direct combinatorial interpretation of the Narayana numbers in terms of c-clusters.

We now proceed to prove Propositions 8.2 and 8.3. We begin by proving a strengthening of Proposition 8.2. This argument follows a suggestion of a referee.

Proposition 8.8. *For any almost positive root α and any reduced word $s_1 \cdots s_n$ for a Coxeter element c, there exists a nonnegative integer r with the property that $\sigma_{s_r}\sigma_{s_{r-1}} \cdots \sigma_{s_2}\sigma_{s_1}\alpha$ is a negative simple root.*

Here, the subscripts are interpreted cyclically, so that $s_{n+1} = s_1$, etc.

Proof. We first claim that the c-orbit of α contains a negative root. Suppose to the contrary that every root in the c-orbit of α is positive. Then in particular the sum of the roots in the c-orbit is a nonzero vector fixed by c. (The assumption that W is finite is critical here. If W is infinite then the c-orbit of α may be infinite, so it may not have a well-defined sum.) However, it is well-known that c acts without fixed points (see for example [6, Section V.6.2] or [19, Lemma 3.16]). This contradiction proves the claim.

If α is a negative simple root then take $r = 0$. Otherwise let R be the smallest nonnegative integer such that $c^{-R}(\alpha)$ is a negative root and let β be the positive root $c^{-R+1}(\alpha)$. We claim that for r between 0 and $(-R+1)n$, the root $s_r s_{r-1} \cdots s_1\alpha$ is positive. To prove the claim by contradiction, take r to be the smallest exception and let r' be the smallest multiple of n greater than r. Since each $s \in S$ only changes the positive/negative status of the roots $\pm\alpha_s$, necessarily $s_r s_{r-1} \cdots s_1 \alpha = -\alpha_{s_r}$. Furthermore, $s_{r'} s_{r'-1} \cdots s_1 \alpha = s_{r'} s_{r'-1} \cdots s_{r+1}(-\alpha_{s_r})$ is a negative root. But $s_{r'} s_{r'-1} \cdots s_1 \alpha = c^{-R'}(\alpha)$ for some R' with $0 < R' < R$, contradicting the choice of R. This proves the claim, implying in particular that $\beta = \sigma_c^{-R+1}(\alpha)$.

Since $c\beta$ is negative and since each $s \in S$ only changes the positive/negative status of the roots $\pm\alpha_s$, there is some $i \in [n]$ such that $s_j s_{j-1} \cdots s_1 \beta$ is positive for all $j < i$ and $s_{i-1} s_{i-2} \cdots s_1 \beta = \alpha_{s_i}$. Setting $r = (-R+1)n + i$ we have $\sigma_{s_r} \sigma_{s_{r-1}} \cdots \sigma_{s_2} \sigma_{s_1} \alpha = s_r s_{r-1} \cdots s_1 \alpha = -\alpha_{s_i}$. □

Let $r_{s_1 \cdots s_n}(\alpha)$ be the smallest nonnegative integer r such that $\sigma_{s_r} \sigma_{s_{r-1}} \cdots \sigma_{s_1} \alpha$ is a negative simple root. The information given by $r_{s_1 \cdots s_n}$ is more refined than that of R_c and the behavior of $r_{s_1 \cdots s_n}$ is simpler to describe. However, as the notation suggests, r depends not only on c but on a choice of a reduced word for c. The following lemma shows how Proposition 8.8 implies Proposition 8.2 and describes the relationship between R_c and $r_{s_1 \cdots s_n}$. Its proof follows immediately from the proof of Proposition 8.8.

Lemma 8.9. *For any almost positive root α, the integer $R_c(\alpha)$ exists and equals $\lceil r_{s_1 \cdots s_n}(\alpha)/n \rceil$.*

We now proceed with the proof of Proposition 8.3, which states that if $B \cup \{\alpha\}$ and $B \cup \{\alpha'\}$ are clusters then $\mathrm{cl}_c^{-1}(B \cup \{\alpha\}) \lessdot \mathrm{cl}_c^{-1}(B \cup \{\alpha'\})$ if and only if $R_c(\alpha) < R_c(\alpha')$.

Proof of Proposition 8.3. We shall in fact prove that, for any reduced word $s_1 \cdots s_n$ for c, $\mathrm{cl}_c^{-1}(B \cup \{\alpha\}) \lessdot \mathrm{cl}_c^{-1}(B \cup \{\alpha'\})$ if and only if $r_{s_1 \cdots s_n}(\alpha) < r_{s_1 \cdots s_n}(\alpha')$. By Lemma 8.9 and the fact that $R_c(\alpha)$ cannot equal $R_c(\alpha')$, this implies Proposition 8.3.

Let $w = \mathrm{cl}_c^{-1}(B \cup \{\alpha\})$, let $w' = \mathrm{cl}_c^{-1}(B \cup \{\alpha'\})$ and let $[w]_c$ and $[w']_c$ be the corresponding maximal cones in the c-Cambrian fan. By Corollary 8.1, either $[w]_c \lessdot [w']_c$ or $[w]_c \gtrdot [w']_c$. Since the possibility $r_{s_1 \cdots s_n}(\alpha) = r_{s_1 \cdots s_n}(\alpha')$ is also ruled out, it suffices by symmetry to prove one direction of implication. Thus we will prove that if $w \lessdot w'$ then $r_{s_1 \cdots s_n}(\alpha) < r_{s_1 \cdots s_n}(\alpha')$. We will use Lemma 6.2 repeatedly. Let $r = \min(r_{s_1 \cdots s_n}(\alpha), r_{s_1 \cdots s_n}(\alpha'))$. Our proof is by induction on r.

First, suppose that $r = 0$, so either α or α' is a negative simple root. Let H be the hyperplane separating $[w']_c$ from $[w]_c$ in $\mathcal{F}_c$. Then $\phi_c^{-1}(\alpha')$ is strictly above H. The rays ρ_{s_i} are not strictly above any hyperplane in $\mathcal{A}(W)$, so $\phi_c^{-1}(\alpha') \neq \rho_{s_i}$ and $\alpha' \neq \phi_c(\rho_{s_i}) = -\alpha_{s_i}$. In other words, α' is not a negative simple root and $r_{s_1 \cdots s_n}(\alpha') > r_{s_1 \cdots s_n}(\alpha) = 0$ as desired.

Now, we consider the situation where $r \geq 1$. There are three cases. For brevity, set $s = s_1$. The first case is that $s \leq w$ and $s \leq w'$. Then $Z_s(w') = sw' > sw = Z_s(w)$. By Lemma 7.5, $\mathrm{cl}_{scs}(Z_s(w)) = \sigma_s(B) \cup \{\sigma_s \alpha\}$ and $\mathrm{cl}_{scs}(Z_s(w')) = \sigma_s(B) \cup \{\sigma_s \alpha'\}$. By Proposition 8.1, one of $[Z_s(w)]_{scs}$ and $[Z_s(w')]_{scs}$ covers the other in W/Θ_{scs}, but, by the isomorphism $[1, sw_0] \cong [s, w_0]$, we have $Z_s(w) \leq Z_s(w')$ in weak order, so $[Z_s(w)]_{scs} \lessdot [Z_s(w')]_{scs}$. By induction, $r_{s_2 \cdots s_n s_1}(\sigma_s(\alpha)) < r_{s_2 \cdots s_n s_1}(\sigma_s(\alpha'))$ and thus $r_{s_1 \cdots s_n}(\alpha) < r_{s_1 \cdots s_n}(\alpha')$.

The second case, $s \not\leq w$ and $s \not\leq w'$, is very similar to the preceding one. By Lemma 7.5, $\mathrm{cl}_{scs}(w)$ and $\mathrm{cl}_{scs}(w')$ differ only by the exchange of $\sigma_s(\alpha)$ for $\sigma_s(\alpha')$.

By Proposition 8.1, $[Z_s(w)]_{scs}$ and $[Z_s(w')]_{scs}$ are a covering pair in W/Θ_{scs}. Since $Z_s(w) = s \vee w \leq s \vee w' = Z_s(w')$, the cover must be $[Z_s(w)]_{scs} \lessdot [Z_s(w')]_{scs}$. As in the previous case, we conclude that $r_{s_1\cdots s_n}(\alpha) < r_{s_1\cdots s_n}(\alpha')$.

The case $s \leq w$ and $s \not\leq w'$ is impossible because $w' > w$. So we complete the proof by considering the case $s \not\leq w$ and $s \leq w'$. Then $\phi_c^{-1}(\alpha)$ and $\phi_c^{-1}(B)$ are on or below H_s, and $\phi_c^{-1}(\alpha')$ and $\phi_c^{-1}(B)$ are on or above H_s. Thus all the rays in $\phi_c^{-1}(B)$ are contained in H_s. Since $\phi_c^{-1}(\alpha)$ is not in the linear span of $\phi_c^{-1}(B)$, $\phi_c^{-1}(\alpha)$ must be strictly below H_s. But then by Proposition 6.2, $\phi_c^{-1}(\alpha) = \rho_s$ and $\alpha = -\alpha_s$. This is the case $r = r_{s_1\cdots s_n}(\alpha) = 0$, which we have already described. □

For any Coxeter element c of W and any $J \subseteq S$, let c' be the Coxeter element of W_J obtained by deleting the letters $S \setminus J$ from any reduced word for c. Since the c'-Cambrian lattice $W_J/\Theta_{c'}$ is a lower interval in W/Θ_c, we have the following combinatorial fact about clusters which appears to be difficult to prove directly:

Proposition 8.10. *For c and c' as above, if α and α' are c-exchangeable almost positive roots then $R_{c'}(\alpha) < R_{c'}(\alpha')$ if and only if $R_c(\alpha) < R_c(\alpha')$. Thus a root in a c'-cluster C is an upper root in C if and only if it is an upper root in the c-cluster $C \cup \{-\alpha_s : s \in S \setminus J\}$, and the same is true for lower roots.*

Remark 8.11. It is known that every face of an associahedron is combinatorially isomorphic to another associahedron. Equivalently, the link of any cone in the cluster complex is combinatorially isomorphic to a cluster complex. One can prove a stronger version of this result in the Cambrian setting, showing that the star of a face in the Cambrian fan is not only combinatorially a Cambrian fan, but has the polyhedral and lattice structure of a Cambrian fan as well. Specifically, for w any c-antisortable element and J a set of ascents of w, there is a choice of Coxeter element $\gamma(w, J, c)$ such that the following proposition holds.

Proposition 8.12. *Let $C_{\Theta_c}(w, J)$ be a face of the c-Cambrian fan. Identify[4] the star of $C(w, J)$ (in the W-Coxeter fan) with the star of $C(e, J)$, and hence with the W_J-Coxeter fan, by the map w^{-1}. Then $\mathcal{F}_{\gamma(w,J,c)}$ and the star of $C_{\Theta_c}(w, J)$ coincide as coarsenings of the W_J-Coxeter fan.*

Defining $\gamma(w, J, c)$ means deciding, for each $r_1, r_2 \in J$ with $r_1r_2 \neq r_2r_1$, whether the reflection r_1 comes before r_2 in every reduced word for $\gamma(w, J, c)$ or *vice versa*. In [28, Section 3], a directed graph is defined on the set T of reflections of W, with arrows $\xrightarrow{c}$. We put r_1 before r_2 in $\gamma(w, J, c)$ if and only if $wr_1w^{-1} \xrightarrow{c} wr_2w^{-1}$.

To prove Proposition 8.12, one first reduces to the case that $C_{\Theta_c}(w, J)$ is a ray $\rho(w, J)$. For s initial in c, one analyzes the effect of ζ_s and Z_s on the star of $\rho(w, J)$. When $w \geq s$ the star is unaltered. When $w \not\geq s$ and $w \neq 1$, the star of $\rho(w, J)$ is partly below H_s and partly above. Passing from $\mathcal{F}_c$ to $\mathcal{F}_{scs}$ has the effect of swapping the part above with the part below, as explained in Example 7.6 and illustrated in Figure 6. In

[4] Note that we identify a cone κ in the (W_J)-Coxeter fan with a cone isomorphic to $\kappa \times \mathbb{R}^{n-|J|}$ in the star of $C(w, J)$.

either case, the effect is compatible with the properties of $\xrightarrow{c}$ established in [28, Proposition 3.1]. By Proposition 8.8, one eventually reaches a ray of the dominant chamber, where the proposition is straightforward.

9. A linear isomorphism

In this section we show that for a special choice of c, the c-Cambrian fan is linearly isomorphic to the c-cluster fan. We also describe, for a special choice of c, a "twisted" version of the c-cluster lattice which is induced on the c-cluster fan by any vector in a certain cone in $\mathbb{R}^n$.

Recall that Φ is a fixed root system for W. For $\alpha \in \Phi$, the corresponding *coroot* is $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$, so that the reflection of a vector v in the hyperplane perpendicular to a root α is $v - \langle v, \alpha^\vee\rangle\alpha$. The simple coroots $\alpha_s^\vee$ for $s \in S$ are a basis for V, and the *fundamental weights* ω_s are the dual basis vectors to the simple coroots.[5] We have

$$\alpha_s = \sum_{r\in S} \langle \alpha_s, \alpha_r^\vee\rangle \omega_r. \tag{9.1}$$

We define the *Cartan matrix* of Φ to be the $n \times n$ square matrix A where $A_{ij} = \langle \alpha_{s_i}^\vee, \alpha_{s_j}\rangle$. A root system is called *crystallographic* if all the entries of A are integers.[6]

Fix a bipartition $S = S_+ \sqcup S_-$ of the Coxeter diagram for W, let c_+ be the product of the elements of S_+ (which commute pairwise) and let c_- be the product of the elements of S_- (which likewise commute). The Coxeter element c_+c_- is called a *bipartite* Coxeter element. For each $s \in S$, let ϵ_s be $+1$ if $s \in S_+$ and let ϵ_s be -1 if $s \in S_-$.

Let L be the linear map that sends a simple root α_s to $-\epsilon_s\omega_s$ (cf. [27, Conjecture 1.4]). The map L depends on the choice of bipartition, but we suppress this dependence in our notation. Following the notation of [13], for $\epsilon \in \{+,-\}$ let $\tau_\epsilon = \prod_{s\in S_\epsilon} \sigma_s$, where again the order of composition is unimportant. Thus $\tau_+\tau_- = \sigma_c$ (in the sense of Section 8) for $c = c_+c_-$. The main result of this section is the following:

Theorem 9.1. *For $c = c_+c_-$, the map L is a linear isomorphism from the c-cluster fan to the c-Cambrian fan. As a map on rays, the map L coincides with $\phi_c^{-1} \circ \tau_-$.*

We begin the proof of Theorem 9.1 with a simple lemma.

Lemma 9.2. *The linear maps c, c_+, c_- and L on V satisfy the following equalities.*

(i) $c_+L = -Lc_-$.
(ii) $c_-L = -Lc_+$.
(iii) $c^{-1}L = Lc$.

[5] We abuse terminology slightly by calling these "weights" even in the noncrystallographic case, where there is no "weight lattice."

[6] Our convention for A_{ij} is the convention used in [13], [14] and [15]; some references use the transpose of this choice.

Proof. We prove equality (i) by evaluating each side on the basis elements α_s. If $s \in S_-$ then $c_+ L\alpha_s = c_+\omega_s = \omega_s$, with the latter equality holding because ω_s is orthogonal to α_r for each $r \neq s$. On the other hand, $-Lc_-\alpha_s = -L(-\alpha_s) = \omega_s$.

If $s \in S_+$ then $c_+ L\alpha_s = c_+(-\omega_s) = -s\omega_s = -\omega_s + \alpha_s$. On the other hand,

$$-Lc_-\alpha_s = -L\Big(\alpha_s - \sum_{r\in S_-}\langle\alpha_s, \alpha_r^\vee\rangle\alpha_r\Big) = \omega_s + \sum_{r\in S_-}\langle\alpha_s, \alpha_r^\vee\rangle\omega_r.$$

To see that these two sides are equal, we must show that

$$\alpha_s = 2\omega_s + \sum_{r\in S_-}\langle\alpha_s, \alpha_r^\vee\rangle\omega_r.$$

We have $\langle\alpha_s, \alpha_s^\vee\rangle = 2$ and $\langle\alpha_s, \alpha_r^\vee\rangle = 0$ for $r \in S_+ \setminus \{s\}$, so the right hand side is $\sum_{r\in S}\langle\alpha_s, \alpha_r^\vee\rangle\omega_r$, which, as already noted, equals α_s.

Now (ii) follows by reversing the roles of "+" and "−", and (iii) follows by combining (i) and (ii), keeping in mind that $c^{-1} = c_-c_+$. □

We now prove a version of equality (iii) in the previous lemma which is more complicated in the sense that it involves maps which are not linear. Specifically, it uses the maps ζ_s and σ_s which appear in Lemma 6.4. In what follows, we apply the maps ζ_s to vectors rather than rays. To do this, we define the *fundamental vector* in a ray ρ of the Coxeter fan to be the unique ω in the W-orbit of the fundamental weights $\{\omega_s : s \in S\}$ such that $\omega \in \rho$. Notice that Lemma 6.4 applies even when rays are replaced by fundamental vectors. For $\epsilon \in \{+, -\}$ let $\zeta_{c_\epsilon} = \prod_{s\in S_\epsilon}\zeta_s$. For $c = c_+c_-$, let $\zeta_c = \zeta_{c_+}\zeta_{c_-}$ and $\zeta_{c^{-1}} = \zeta_{c_-}\zeta_{c_+}$.

Lemma 9.3. *For $c = c_+c_-$, if α is a positive root then $L\sigma_c\alpha = \zeta_{c^{-1}}L\alpha$.*

Proof. If α is a simple root α_s for $s \in S_-$ then

$$L\sigma_c\alpha = L\tau_+\tau_-\alpha = L\tau_+(-\alpha_s) = L(-\alpha_s) = -\omega_s.$$

On the other hand,

$$\zeta_{c^{-1}}L\alpha = \zeta_{c_-}\zeta_{c_+}\omega_s = \zeta_{c_-}\omega_s = -\omega_s.$$

If α is a positive root not of the form α_s for $s \in S_-$ then

$$L\sigma_c\alpha = L\tau_+\tau_-\alpha = L\tau_+c_-\alpha = Lc_+c_-\alpha.$$

(The second equality holds because α is a positive root. The only positive roots which are sent to negative roots by c_- are roots of the form α_s for $s \in S_-$. Thus $c_-\alpha$ is a positive root and therefore the third equality holds as well.) On the other hand, since $L^{-1}(\omega_s) = -\alpha_s$ when $s \in s_+$, the vector $L\alpha$ is not of the form ω_s for $s \in S_+$. Thus $\zeta_{c^{-1}}L\alpha = \zeta_{c_-}\zeta_{c_+}L\alpha = \zeta_{c_-}c_+L\alpha$. If $c_+L\alpha$ is ω_s for some $s \in S_-$ then $L\alpha = \omega_s$ as well, so that $\alpha = \alpha_s$. Since we are currently in the case which excludes such an α, we can write $\zeta_{c_-}c_+L\alpha = c_-c_+L\alpha$. Thus in this case the requirement is that $Lc\alpha = c^{-1}L\alpha$, which was proved in Lemma 9.2. □

The map ϕ_c^{-1}, as defined in Section 6, takes almost positive roots to rays. In what follows, we continue to identify each ray ρ with the fundamental vector in ρ.

Proposition 9.4. *For $c = c_+c_-$, the map L takes almost positive roots to rays of the c-Cambrian fan. Specifically, L restricted to almost positive roots is $\phi_c^{-1} \circ \tau_-$.*

Proof. Let α be an almost positive root. We show by induction on $R_{c^{-1}}(\alpha)$ that $L\alpha$ is a ray of the c-Cambrian fan and that $\phi_c L\alpha = \tau_-\alpha$. First suppose that $R_{c^{-1}}(\alpha) = 0$, so that α is a negative simple root $-\alpha_s$. In this case, $L\alpha = \pm\omega_s$, which in either case is a ray of the c-Cambrian fan. If $s \in S_+$ then $\tau_-\alpha = \alpha$ and $\phi_c L\alpha = \phi_c\omega_s = -\alpha_s = \alpha$. If $s \in S_-$ then

$$\phi_c L\alpha = \phi_c(-\omega_s) = \phi_c(\zeta_{c_-}\omega_s) = \tau_-\phi_{c^{-1}}\omega_s = \tau_-(-\alpha_s).$$

Here the next-to-last equality follows from Lemma 6.4, applied several times, and the fact that $c^{-1} = c_-c_+$.

Next suppose that $R_{c^{-1}}(\alpha) > 0$ so that α is a positive root and $\sigma_c\alpha = \alpha'$ for some α' with $R_{c^{-1}}(\alpha') = R_{c^{-1}}(\alpha) - 1$. By induction, $L\alpha'$ is a ray of the c-Cambrian fan and $\phi_c L\alpha' = \tau_-\alpha'$. To evaluate $\phi_c L\alpha$, first note that by Lemma 9.3,

$$L\alpha = \zeta_{c^{-1}}^{-1} L\sigma_c\alpha = \zeta_{c^{-1}}^{-1} L\alpha'.$$

In particular, $L\alpha$ is a ray in the c-Cambrian fan and $\phi_c L\alpha = \phi_c\zeta_{c^{-1}}^{-1} L\alpha'$. Repeated applications of Lemma 6.4 give the identity $\phi_c\zeta_{c^{-1}} = \sigma_{c^{-1}}\phi_c$, so that $\phi_c\zeta_{c^{-1}}^{-1} = \sigma_{c^{-1}}^{-1}\phi_c = \sigma_c\phi_c$. Thus

$$\phi_c L\alpha = \sigma_c\phi_c L\alpha' = \sigma_c\tau_-\alpha' = \tau_+\alpha' = \tau_+\sigma_c\alpha = \tau_-\alpha. \qquad \square$$

The map τ_- induces a combinatorial isomorphism between the (c_+c_-)-cluster fan and the (c_-c_+)-cluster fan. These fans in fact coincide, so that τ_- is a combinatorial automorphism of the (c_+c_-)-cluster fan. By Theorem 1.1, $\phi_{c_+c_-}^{-1}$ induces a combinatorial isomorphism as well. This completes the proof of Theorem 9.1.

We conclude the section with an application of Theorem 9.1. Proposition 9.4 suggests the definition of a "twisted" cluster lattice on c-clusters, where $c = c_+c_-$. Namely, for c-clusters C and C', set $C \leq_{\mathrm{tw}} C'$ in the twisted c-cluster lattice if and only if $\tau_-C \leq \tau_-C'$ in the c-cluster lattice. In particular, the cover relations in the twisted c-cluster lattice are $B \cup \{\alpha\} \lessdot B \cup \{\alpha'\}$ if and only if $R_c(\tau_-\alpha) < R_c(\tau_-\alpha')$.

The twisted c-cluster lattice can be described in terms of a quantity $\epsilon(\alpha, \alpha')$ which plays an important role in [14], where cluster algebras of finite type are constructed in terms of the combinatorics of clusters of almost positive roots. Let $\tau_-^{(k)}$ denote the k-fold composition $\tau_{(-1)^k}\tau_{(-1)^{k-1}}\cdots\tau_-\tau_+\tau_-$. For each almost positive root α, let $k_-(\alpha)$ be the smallest nonnegative integer such that $\tau_-^{(k)}(\alpha)$ is a negative simple root and $\tau_-^{(k)}(\alpha) = \tau_-^{(k+1)}(\alpha)$. Given two c-clusters $B \cup \{\alpha\}$ and $B \cup \{\alpha'\}$, define $\epsilon(\alpha, \alpha')$ to be -1 if $k_-(\alpha) < k_-(\alpha')$ or 1 if $k_-(\alpha') < k_-(\alpha)$. (see [14, Lemma 4.1]). As with R_c and $r_{s_1\cdots s_n}$, the case $k_-(\alpha) = k_-(\alpha')$ is impossible.

The following proposition says that the twisted c-cluster lattice is analogous to the ordinary c-cluster lattice, except that k_- plays the role of $R_{c_+c_-}$. The proof is a straightforward induction on $k_-(\alpha)$, and we omit the details.

Proposition 9.5. *For $c = c_+c_-$, if α and α' are c-exchangeable then*

$$R_c(\tau_-\alpha) < R_c(\tau_-\alpha') \quad \textit{if and only if} \quad \epsilon(\alpha, \alpha') = -1.$$

In particular, the cover relations of the twisted c-cluster lattice are of the form $B \cup \{\alpha\} \lessdot B \cup \{\alpha'\}$ for $\epsilon(\alpha, \alpha') = -1$.

Since the twisted c-cluster lattice is isomorphic to the ordinary (c_-c_+)-cluster lattice by a map which also induces a combinatorial isomorphism of fans, the twisted c-cluster lattice inherits all of the properties listed in Corollary 8.5. (These properties are all combinatorial.) Since the isomorphism between the twisted c-cluster lattice and the c-Cambrian lattice is given by a linear map of fans, the following property of the c-Cambrian lattice (see Section 4) carries over to the twisted c-cluster lattice:

Proposition 9.6. *The twisted c-cluster lattice is the order induced on the maximal cones of the c-cluster fan by any vector in the interior of the cone spanned by the c-cluster $\{-\epsilon_s\alpha_s : s \in S\}$.*

10. Connections to cluster algebras

In this section we connect our results to the theory of cluster algebras. Rather than give the lengthy definition of a cluster algebra, we merely describe the properties of cluster algebras and refer the reader to [15] for definitions.

Let $\mathbb{F}$ be a field isomorphic to $\mathbb{Q}(x_1, \ldots, x_n)$ and let B be an $n \times n$ integer matrix that is *skew-symmetrizable*, meaning that there exists an invertible diagonal matrix D such that DB is skew-symmetric. The combinatorial data for a cluster algebra is the matrix B and an n-tuple $(x_1, \ldots, x_n)$ of rational functions generating $\mathbb{F}$ as a field. The cluster algebra $\mathrm{Alg}(B, (x_1, \ldots, x_n))$ is a certain subalgebra of the Laurent-polynomial ring $\mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]$, which is, in turn, a subring of $\mathbb{F}$. The data $(B, (x_1, \ldots, x_n))$ also determines a collection of transcendence bases of $\mathrm{Alg}(B, (x_1, \ldots, x_n))$, known as *algebraic clusters*.[7] The elements of the algebraic clusters are known as *cluster variables*. One algebraic cluster is $(x_1, \ldots, x_n)$ and the others are defined by a certain recursive procedure. The recursive procedure also associates a skew-symmetrizable matrix B^t to each algebraic cluster $t = (y_1, \ldots, y_n)$ so that $\mathrm{Alg}(B, (x_1, \ldots, x_n)) = \mathrm{Alg}(B^t, (y_1, \ldots, y_n))$ and so that $(B, (x_1, \ldots, x_n))$ and $(B^t, (y_1, \ldots, y_n))$ each give the same collection of algebraic clusters.

A cluster algebra is *of finite type* if it has finitely many cluster variables. We now briefly describe the connection between cluster algebras of finite type and finite Coxeter groups/root systems. For more details, see [14]. Let Φ be a crystallographic root system for the Coxeter group W. We refer the reader to the beginning of Section 9 for our conventions regarding roots, coroots, Cartan matrices and fundamental weights. Let c be a Coxeter element of W. If r and s are two simple reflections of W which do not commute, then either r comes before s in every reduced word for c or *vice versa*. We write $r \to s$ to indicate that r comes before s in every reduced word for c.

[7] Typically, these are simply called "clusters," but we use the adjective "algebraic" here to avoid confusion with c-clusters.

Define a square matrix B^c by

$$B^c_{jk} = \begin{cases} 0 & \text{if } s_j s_k = s_k s_j, \\ -A_{jk} = -\langle \alpha_{s_j}^\vee, \alpha_{s_k} \rangle & \text{if } s_j \to s_k, \\ A_{jk} = \langle \alpha_{s_j}^\vee, \alpha_{s_k} \rangle & \text{if } s_j \leftarrow s_k. \end{cases}$$

Then the matrix B^c (together with any choice of $(x_1, \ldots, x_n)$) defines a cluster algebra of finite type. Furthermore, cluster algebras arising from different choices of c and $(x_1, \ldots, x_n)$ are isomorphic; we thus suppress the choice of c and $(x_1, \ldots, x_n)$ and write $\mathrm{Alg}(\Phi)$ for a cluster algebra arising in this manner.

Conversely, given any cluster algebra of finite type, there exists[8] a finite Coxeter group W (with root system Φ), a Coxeter element c in W and an algebraic cluster $t_c = (x_1^c, \ldots, x_n^c)$ such that the given cluster algebra is $\mathrm{Alg}(\Phi) = \mathrm{Alg}(B^c, (x_1^c, \ldots, x_n^c))$. Thus the cluster algebras of finite type are precisely the cluster algebras of the form $\mathrm{Alg}(\Phi)$, so that the following theorem applies to any cluster algebra of finite type.

Theorem 10.1. *Given a specific representation of* $\mathrm{Alg}(\Phi)$ *as* $\mathrm{Alg}(B^c, t_c)$*, there is a bijection* $\alpha \mapsto x^c(\alpha)$ *between* $\Phi_{\geq -1}$ *and the cluster variables of* $\mathrm{Alg}(\Phi)$ *such that:*

(i) $x^c(-\alpha_{s_j}) = x_j^c$ *for all* $j \in [n]$*;*
(ii) *c-clusters are mapped to algebraic clusters;*
(iii) *for positive roots* $\alpha = \sum a_i \alpha_{s_i}$*, the rational function* $x^c(\alpha)$ *can be written in reduced form with denominator* $\prod x^c(-\alpha_{s_i})^{-a_i}$*.*

Furthermore, if s is initial in c then t_{scs} *can be chosen so that* $x^c(\alpha) = x^{scs}(\sigma_s(\alpha))$*.*

Proof. In the case of bipartite c, the first assertion is [14, Theorem 1.9]. For general c, the entire theorem was proven for simply laced root systems (i.e. $A_{ij} = 0$ or -1 for all $i \neq j$) in [10], relying on previous work cited therein. The result for non-simply laced root systems can be established by folding arguments. □

In rough terms, Theorem 10.1 says that the cluster variables of $\mathrm{Alg}(\Phi)$ correspond to almost positive roots by assigning a variable to its *denominator vector* $\prod x^c(-\alpha_{s_i})^{-a_i}$. There is another natural way to encode cluster variables by integer vectors, namely the **g**-vector, defined in [15]. The **g**-vector of a cluster variable x depends on a fixed algebraic cluster t and is written $g^t(x)$, with components $g_j^t(x)$. In [15, Proposition 11.3], Fomin and Zelevinsky compute the **g**-vector when (in the language of the current paper) t is of the form t_c for c a bipartite Coxeter element. They encode the **g**-vector as an element of V by the sum $\mathbf{g}^t_{\mathrm{root}}(x) := \sum_{j=1}^n g_j^t(x)\alpha_{s_j}$. (In [15], this sum is also denoted by $g^t(x)$. However, it is important here to distinguish between the integer vector $g^t(x)$ and the vector $\mathbf{g}^t_{\mathrm{root}}(x)$ lying in the root lattice.) They establish the formula

$$\mathbf{g}^{t_c}_{\mathrm{root}}(x^c(\alpha)) = (E \circ \tau_-)(\alpha).$$

Here τ_- has the same meaning as in Section 9, and E is the linear map such that $E(\alpha_s) = -\epsilon(s)\alpha_s$.

[8] Most often, c does not uniquely determine t_c. Here we assume that some choice of t_c has been made. On the other hand, outside of rank two, not every algebraic cluster can serve as t_c.

The **g**-vector has no obvious connection to the geometry of the c-cluster fan, but remarkably, it arises naturally in the geometry of the c-Cambrian fan. To see this, we encode the **g**-vector in the weight lattice by $\mathbf{g}^t_{\text{weight}}(x) := \sum_{j=1}^n g^t_j(x)\omega_{s_j}$. Let U denote the linear map which takes α_s to ω_s, so that $\mathbf{g}^t_{\text{weight}}(x) = U(\mathbf{g}^t_{\text{root}}(x))$. Thus when c is bipartite, Theorem 9.1 implies that

$$\mathbf{g}^{t_c}_{\text{weight}}(x^c(\alpha)) = (U \circ E \circ \tau_-)(\alpha) = (L \circ \tau_-)(\alpha) = \phi_c^{-1}(\alpha).$$

Theorem 10.2. *If c is a bipartite Coxeter element, with t_c a corresponding cluster, and if α is an almost positive root, then*

$$\phi_c^{-1}(\alpha) = \mathbf{g}^{t_c}_{\text{weight}}(x^c(\alpha)).$$

Thus **g**-vectors arise naturally from the correspondence between cluster variables and rays in the Cambrian fan: the **g**-vector associated to a ray is recovered by computing the fundamental-weight coordinates of the fundamental vector in the ray. This is precisely analogous to the situation in the cluster fan, where the denominator vector associated to a ray is recovered by taking the simple-root coordinates of the root in the ray.

We conjecture that Theorem 10.2 is true without assuming that c is bipartite. In [15, Conjecture 7.12], Fomin and Zelevinsky give a conjectured recurrence for $g^t(x)$ as t varies. By a straightforward but lengthy computation, one can verify that the more general version of Theorem 10.2 follows from [15, Conjecture 7.12].

We now sketch an additional connection between Cambrian fans and cluster algebras. Choose a Coxeter element c of W and a cluster t_c of $\text{Alg}(\Phi)$ as above. Let $t = (x_1, \ldots, x_n)$ be an arbitrary algebraic cluster in $\text{Alg}(\Phi)$. (In particular, we do *not* assume that $B^t = B^{c'}$ for some c'.) Then $x_i = x^c(\alpha_i)$ for some c-cluster $(\alpha_1, \ldots, \alpha_n)$. Let $[w]_c$ be the cone of $\mathcal{F}_c$ represented by a c-sortable element w with $\text{cl}_c(w) = (\alpha_1, \ldots, \alpha_n)$. There is another collection of roots, besides the α_i, naturally associated to $[w]_c$, namely the roots $(\beta_1, \ldots, \beta_n)$ orthogonal to the walls of $[w]_c$. More specifically, let β_i be the root determined by the requirements that $\langle \phi_c^{-1}(\alpha_i), \beta_j \rangle = 0$ for $i \neq j$ and $\langle \phi_c^{-1}(\alpha_i), \beta_i \rangle < 0$. Let $\beta_i^\vee$ be the coroot corresponding to the root β_i and let Q^t be the $n \times n$ matrix $\langle \beta_i^\vee, \beta_j \rangle$.

The matrix Q^t depends on the choice of c and t_c above. This dependence is not as bad as one might suspect. If s is initial in c and t_{scs} is the cluster referred to in Theorem 10.1, then the cone corresponding to t changes from $[w]_c$ to $[Z_s(w)]_{scs}$. Either $[w]_c$ and $[Z_s(w)]_{scs}$ are related by an isometry or else they are two regions among the 2^n regions defined by the same set of n hyperplanes. In the first case, Q^t is preserved, in the second it is conjugated by a diagonal matrix all of whose diagonal entries are ± 1. So changing t_c in this manner any number of times simply conjugates Q^t by such a matrix. We will see soon that it follows from results of [2] that Q^t is well defined up to such conjugation independent of any of our choices.[9]

[9] In a previous version of this paper, we argued that this independence could be established by a sequence of steps, each changing from t_c to t_{scs} for s initial. A comment by one of the referees has led us to doubt this argument. It follows from [3, Theorem 1.2(1)] that we may change c to any other Coxeter element c' by such a sequence of steps. What is not clear is whether we may change any cluster t_c corresponding to c to any cluster $t_{c'}$ corresponding to c'.

Proposition 10.3. *Let Q^t be as above and let B^t be the matrix associated to the algebraic cluster t. Then $Q^t_{ij} = \pm B^t_{ij}$ for $i \neq j$.*

Sketch of proof. For $i \neq j$, the quantity $B^t_{ij} B^t_{ji}$ is encoded in the combinatorics of the algebraic cluster complex: One counts the number of algebraic clusters containing $t \setminus \{x^t_i, x^t_j\}$. This number is 4, 5, 6 or 8, corresponding (in order) to $B^t_{ij} B^t_{ji} = 0, -1, -2$ or -3. To prove Proposition 10.3, we verify that $-Q^t_{ij} Q^t_{ji}$ takes only the values $0, -1, -2$ or -3 and that the value of $-Q^t_{ij} Q^t_{ji}$ corresponds to the number (4, 5, 6 or 8) of c-clusters containing $(\alpha_1, \ldots, \alpha_n) \setminus \{\alpha_i, \alpha_j\}$. Once this is verified, we have

$$B^t_{ij} B^t_{ji} = -Q^t_{ij} Q^t_{ji}$$

by the isomorphism between the c-cluster complex and the algebraic cluster complex. The matrices Q^t and B^t are (respectively) symmetrizable and skew-symmetrizable. One can check that the same diagonal matrix D makes both DQ^t symmetric and DB^t skew symmetric so we conclude from $B^t_{ij} B^t_{ji} = -Q^t_{ij} Q^t_{ji}$ that $Q^t_{ij} = \pm B^t_{ij}$.

Let F be the face of $[w]_c$ spanned by $\phi_c^{-1}\left((\alpha_1, \ldots, \alpha_n) \setminus \{\alpha_i, \alpha_j\}\right)$. In other words, F is $[w]_c \cap \beta_i^\perp \cap \beta_j^\perp$. By Theorem 1.1, the number of c-clusters containing $(\alpha_1, \ldots, \alpha_n) \setminus \{\alpha_i, \alpha_j\}$ is equal to the number of maximal faces of the c-Cambrian fan containing F. By Proposition 8.12, the star of F is a Cambrian fan for a (crystallographic) Coxeter group of rank 2, of which there are only four types. Moreover, β_i and β_j are roots in a rank 2 root subsystem of corresponding type. (Specifically, if $F = C_{\Theta_c}(w, J)$ then β_i and $\beta_j \in w\Phi_J$.) By inspection of Cambrian lattices of rank 2, we see that either (β_i, β_j) or $(\beta_i, -\beta_j)$ form a simple system for this root subsystem. Thus $-\langle \beta_i^\vee, \beta_j \rangle \langle \beta_j^\vee, \beta_i \rangle$ is 0, -1, -2 or -3 according to whether the root subsystem is $A_1 \times A_1$, A_2, B_2 or G_2; this in turn corresponds to whether the star of F has 4, 5, 6 or 8 maximal cones. □

Rephrased in the language of [2], Proposition 10.3 says that Q^t is a *quasi-Cartan companion* for B^t. The matrix Q^t is positive definite[10] because it is (essentially) a matrix of inner products between n linearly independent vectors. One direction of [2, Theorem 1.2] states that B^t has a positive definite quasi-Cartan companion which is (by [2, Propositions 1.4 and 1.5]) unique up to conjugation by diagonal matrices with diagonal entries ± 1. Thus Q^t is unique up to such conjugation. One of the virtues of this manner of obtaining Q^t is that this uniqueness occurs for a geometrically natural reason, as described above.

In this section we have suggested two new geometric approaches to the study of cluster algebras. First, to encode **g**-vectors as linear combinations of the fundamental weights. Second, to view quasi-Cartan companions as matrices of inner products between normal vectors to a simplicial cone in the hyperplane arrangement. We hope that both of these ideas will have wider applications, including applications beyond finite type.

[10] More accurately, DQ^t is positive definite, but we follow the convention of [2] of saying that Q^t is positive definite in this case.

11. Clusters and noncrossing partitions

In light of Theorems 3.8 and 5.2, the map $\mathrm{nc}_c \circ \mathrm{cl}_c^{-1}$ is a bijection from c-clusters to c-noncrossing partitions. In this section we describe this composition as a direct map, eliminating the intermediate c-sortable elements. For brevity, we continue to leave out the precise details about noncrossing partitions. The c-noncrossing partitions are certain elements of W. Brady and Watt showed [8, Lemma 5] that a c-noncrossing partition can be recovered (among the set of all c-noncrossing partitions) from its fixed point set. The fixed point set of a c-noncrossing partition is called a *c-noncrossing subspace*. The map nc_c takes the cover reflections of a c-sortable element w and multiplies them in a certain specific order such that the result is a c-noncrossing partition. Let NC_c be the map taking a c-sortable element w to the fixed points of $\mathrm{nc}_c(w)$; this is the intersection of the reflecting hyperplanes associated to cover reflections of w. The map NC_c is a bijection between c-sortable elements and c-noncrossing subspaces.

The composition $\mathrm{NC}_c \circ \mathrm{cl}_c^{-1}$ takes a c-cluster C to the intersection I of the set of hyperplanes separating $[\mathrm{cl}_c^{-1}(C)]_c$ from equivalence classes which it covers in W/Θ_c. The subspace I equals the linear span of the rays of $[\mathrm{cl}_c^{-1}(C)]_c$ contained in I. A ray is in I if and only if it is not an upper root of C, i.e. if and only if it is a lower root of C. Thus

Theorem 11.1. *The bijection* $\mathrm{NC}_c \circ \mathrm{cl}_c^{-1}$ *maps a c-cluster C to the c-noncrossing subspace* $\mathrm{Span}_{\mathbb{R}}\{\phi_c^{-1}(\alpha) : \alpha$ *is a lower root in* $C\}$. *In particular, the c-cluster C is uniquely identified by this subspace.*

This description of the bijection has the disadvantage of depending on the recursively defined function ϕ_c and on a notion of lower roots in clusters which is also defined recursively. We conjecture the following description of I, which would eliminate the map ϕ_c.

Conjecture 11.2. *Let $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_{n-k})$ be a c-cluster of W, with α_i the lower roots and β_i the upper roots. Then*

$$\mathrm{Span}_{\mathbb{R}}(\phi_c^{-1}(\alpha_1), \ldots, \phi_c^{-1}(\alpha_k)) = \beta_1^{\perp} \cap \cdots \cap \beta_{n-k}^{\perp}.$$

It is easy to see that $\beta_1^{\perp} \cap \cdots \cap \beta_{n-k}^{\perp}$ and $\mathrm{Span}_{\mathbb{R}}(\phi_c^{-1}(\alpha_1), \ldots, \phi_c^{-1}(\alpha_k))$ have the same dimension, so in order to prove Conjecture 11.2 it is enough to show that the former contains the latter, i.e. that $\phi_c^{-1}(\alpha_i) \perp \beta_j$ for all i and j. This orthogonality has been verified computationally for all choices of c in all Coxeter groups whose rank is at most 7. Combined with Theorem 11.1, Conjecture 11.2 would immediately imply the following conjecture.

Conjecture 11.3. *Consider the map taking a c-cluster C to the intersection of the hyperplanes orthogonal to the upper roots of C. This map is a bijection from c-clusters to c-noncrossing subspaces. It coincides with* $\mathrm{NC}_c \circ \mathrm{cl}_c^{-1}$.

In the case of bipartite $c = c_+c_-$, the related bijection $\mathrm{NC}_c \circ \mathrm{cl}_c^{-1} \circ \tau_-$ can be described in a completely geometric manner, as follows. Recall from Section 4 the definition of the bottom face, with respect to a generic vector, of a maximal cone in a simplicial fan.

Choosing a vector v as in Proposition 9.6, we map each maximal cone C to the subspace $\mathrm{Span}_{\mathbb{R}}(L(F))$, where L is the linear map of Section 9 and F is the bottom face of C with respect to v. In light of Propositions 9.4 and 9.6, $\mathrm{Span}_{\mathbb{R}}(L(F))$ is the span of $\{\phi_c^{-1}\tau_-\alpha : \tau_-\alpha$ is a lower root in $\tau_- C\}$. Thus

Theorem 11.4. *The map $C \mapsto \mathrm{Span}_{\mathbb{R}}(L(F))$ is the bijection $\mathrm{NC}_c \circ \mathrm{cl}_c^{-1} \circ \tau_-$ from (c_+c_-)-clusters to (c_+c_-)-noncrossing subspaces. In particular, a maximal cone C in the (c_+c_-)-cluster complex is uniquely determined by $\mathrm{Span}_{\mathbb{R}}(F)$.*

Remark 11.5. In [1], Athanasiadis, Brady, McCammond and Watt give another bijection between c-clusters and c-noncrossing partitions for the case where c is bipartite. A key element of their bijection is a labeling of the roots of each cluster as "left" or "right" roots [1, Section 4]. The cluster is then mapped to the product of the reflections corresponding to its right roots, in some specified order. Although the connection is not immediately obvious, it is natural to suspect that the left-right dichotomy of [1] corresponds to the upper-lower dichotomy of the present paper. In particular, it seems quite likely that the map of [1] coincides, in the bipartite case, with the bijection of Conjecture 11.3.

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