JEMS

Ed Swartz

Face enumeration—from spheres to manifolds

Abstract. We prove a number of new restrictions on the enumerative properties of homology manifolds and semi-Eulerian complexes and posets. These include a determination of the affine span of the fine h-vector of balanced semi-Eulerian complexes and the toric h-vector of semi-Eulerian posets.

The lower bounds on simplicial homology manifolds, when combined with higher dimensional analogues of Walkup's 3-dimensional constructions [47], allow us to give a complete characterization of the f-vectors of arbitrary simplicial triangulations of $S^1 \times S^3$, $\mathbb{C}P^2$, K3 surfaces, and $(S^2 \times S^2) \# (S^2 \times S^2)$. We also establish a principle which leads to a conjecture for homology manifolds which is almost logically equivalent to the g-conjecture for homology spheres. Lastly, we show that with sufficiently many vertices, every triangulable homology manifold without boundary of dimension three or greater can be triangulated in a 2-neighborly fashion.

1. Introduction

The fundamental *combinatorial* invariant of a $(d-1)$-dimensional triangulated space is its f-vector, $(f_0, \ldots, f_{d-1})$, where f_i counts the number of i-dimensional faces. After the Euler–Poincaré formula, the Dehn–Sommerville equations for simplicial polytopes are the best known restrictions on the f-vectors of manifolds. While algebraic topology in general, and the topology of manifolds in particular, made great strides in the first half of the twentieth century, it was not until 1964 that Klee published the manifold equivalent of the Dehn–Sommerville equations.

In the 70's, the introduction of commutative algebra in the form of the face ring by Hochster [43, Theorems 4.1 and 4.8], Reisner [32] and Stanley [38], and the connection between toric varieties and rational polytopes (see, for instance, [8]), led to dramatic advances in the understanding of the enumerative properties of polytopes and spheres. By 1980, McMullen's conjectured characterization of the f-vectors of simplicial convex polytopes [28] was verified by Stanley [40] (necessity), and Billera and Lee [3] (sufficiency). Since then, one of the most important problems in understanding the combinatorics of triangulations has become known as the g-conjecture (cf. Conjecture 4.22): Do f-vectors of simplicial spheres, or more generally homology spheres, also satisfy McMullen's conditions?

E. Swartz: Department of Mathematics, Cornell University, Ithaca, NY 14853, USA;
e-mail: ebs@math.cornell.edu

Motivated by a desire to understand the face posets of polytopes, the 1980's and 90's saw the introduction of balanced complexes [39], the cd-index [1], [2], and the toric h-vector [44, Section 3.14]. All of these invariants make sense and were studied in the context of Eulerian posets, which include the face posets of regular cell decompositions of spheres and odd-dimensional compact manifolds without boundary. Section 3 is devoted to extending these ideas to semi-Eulerian posets and complexes. These include the face posets of regular cell decompositions of compact even-dimensional manifolds without boundary. The main results determine the affine span of each of these invariants.

The great variety and complications possible in the topology of manifolds has made the study of their f-vectors a daunting task. At present there is not even a guess as to what the set of all possible f-vectors of manifolds (without boundary) would look like in dimensions greater than three. The two most comprehensive conjectures in print are due to Kalai, [31, Conjecture 7.5] and Kühnel, [26, Conjecture 18]. While these conjectures would have far reaching consequences for f-vectors of manifolds, they only concern the rational Betti numbers. It is not an exaggeration to say that at this point there is no understanding whatsoever of the impact on the combinatorics of triangulations of many of the classical manifold invariants such as the cohomology ring structure, characteristic classes, or even torsion Betti numbers! Perhaps it is appropriate that as of the beginning of the twenty-first century it is still an open question in dimensions five and above whether or not every compact topological manifold without boundary has a triangulation. For information on what *is* known, especially concerning combinatorial manifolds, see the recent surveys by Datta [9] and Lutz [26].

One of the main results in Section 4, Theorem 4.26, can roughly be interpreted to mean that the distance between what we know about spheres and manifolds, while still substantial, is not as great as it might seem. It turns out that there is a conjecture for homology manifolds which is almost logically equivalent to the g-conjecture for spheres. The rest of the section contains a number of restrictions on the f-vectors of homology manifolds. All of our proofs work for arbitrary triangulations, not just combinatorial ones. The main new feature is the use of the face ring to produce lower bounds for the number of vertices *and edges*. One consequence is that Kühnel's triangulations of sphere bundles over the circle [21] minimize the f-vector over all homology manifolds without boundary and nonzero first Betti number (Theorem 4.7).

The last section contains several constructions, most of which are higher dimensional analogues of those introduced by Walkup in dimension three [47]. In combination with our previous results, these techniques allow us to give complete characterizations of the f-vectors of $S^1 \times S^3$, $\mathbb{C}P^2$, any K3 surface, and $(S^2 \times S^2) \# (S^2 \times S^2)$. In addition, many partial results are possible, such as a description of all possible pairs (f_0, f_1) which can occur in triangulations of $S^3 \times S^3$. We end with another extension to higher dimensions of a result of Walkup's in dimension three. This theorem says that for any boundaryless homology manifold M^{d-1} which can be triangulated, there exists $\gamma(M^{d-1})$ such that if $f_1 - df_0 \geq \gamma(M^{d-1})$ and $f_1 \leq \binom{f_0}{2}$, then there is a triangulation of M^{d-1} with f_0 vertices and f_1 edges. In particular, for sufficiently many vertices, M^{d-1} has a 2-neighborly triangulation.

We have covered all of the manifolds for which we know necessary and sufficient conditions on the f-vectors of all possible triangulations. Otherwise, we have not attempted to be encyclopedic in listing all possible applications of our methods to the large number of currently known triangulations. Rather, we have given a sample of the ways these techniques might be employed.

Note: Since this paper was originally written the set of all possible f-vectors of the nonorientable S^3-bundle over S^1 was determined in [7] using Theorem 4.30.

2. Notations and conventions

Throughout, Δ is a connected, pure, $(d-1)$-dimensional simplicial complex with n vertices and vertex set $V = \{v_1, \ldots, v_n\}$. A simplicial complex is *pure* if all of its facets (maximal faces) have the same dimension. In addition, we will always assume that $d \geq 4$. The *geometric realization* of Δ, $|\Delta|$, is the union in $\mathbb{R}^n$ over all faces $\{v_{i_1}, \ldots, v_{i_j}\}$ of Δ of the convex hull of $\{e_{i_1}, \ldots, e_{i_j}\}$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. We say Δ is *homeomorphic* to another space whenever $|\Delta|$ is. A *triangulation* of a topological space M is any simplicial complex Δ such that Δ is homeomorphic to M.

The *link* of a face $\rho \in \Delta$ is

$$\mathrm{lk}\,\rho = \bigcup_{\substack{\tau \cup \rho \in \Delta \\ \tau \cap \rho = \emptyset}} \tau.$$

The *closed star* of a face $\rho \in \Delta$ is

$$\overline{\mathrm{st}}\,\rho = \bigcup_{\substack{\sigma \subseteq \tau, \tau \supseteq \rho \\ \sigma \in \Delta}} \sigma.$$

The *join* of Δ and Δ', where the vertex set V' of Δ' is disjoint from V, is

$$\Delta * \Delta' = \{\rho \cup \rho' : \rho \in \Delta, \rho' \in \Delta'\}.$$

For any poset $(P, \leq)$, the *order complex* of P is the simplicial complex whose vertices are the elements of P and whose faces are the chains of P. If P contains a greatest element $\hat{1}$ and/or a least element $\hat{0}$, then the *reduced* order complex of P is the order complex of $P - \{\hat{0}, \hat{1}\}$.

Homology manifolds are a natural generalization of topological manifolds. Fix a field k. If for all $x \in |\Delta|$, $\tilde{H}_i(|\Delta|, |\Delta| - x; k) = 0$ when $i < d-1$, and either k or 0 when $i = d-1$, then Δ is a *k-homology manifold*. Equivalently, for every nonempty face $\rho \in \Delta$, $H_\star(\mathrm{lk}\,\rho; k)$ is isomorphic to the k-homology of either $S^{d-|\rho|-1}$ or $B^{d-|\rho|-1}$, where $B^{d-|\rho|-1}$ is the $(d-|\sigma|-1)$-dimensional ball. The *boundary* of a homology manifold, denoted $\partial\Delta$, is all of the faces ρ such that $H_{d-|\rho|-1}(\mathrm{lk}\,\rho; k) = 0$. If $H_{d-1}(\Delta, \partial\Delta; k) \simeq k$, then Δ is *orientable* over k. We say Δ is a *closed* homology manifold over k if Δ has no boundary and is orientable over k. If the boundary of Δ is not empty and Δ is a homology manifold over all fields, then $\partial\Delta$ is a $(d-2)$-dimensional k-homology manifold without boundary [30]. Otherwise, we assume the boundary is a $(d-2)$-dimensional k-homology manifold without boundary.

The *f-vector* of Δ is $(f_0, \ldots, f_{d-1})$, where f_i is the number of i-dimensional faces in Δ. Sometimes it is convenient to set $f_{-1} = 1$ corresponding to the empty set. The *face polynomial* of Δ is

$$f_\Delta(x) = f_{-1}x^d + f_0x^{d-1} + \cdots + f_{d-2}x + f_{d-1}.$$

The *h-vector* of Δ is $(h_0, \ldots, h_d)$ and is defined so that the corresponding h-polynomial, $h_\Delta(x) = h_0x^d + h_1x^{d-1} + \cdots + h_{d-1}x + h_d$, satisfies $h_\Delta(x + 1) = f_\Delta(x)$. Equivalently,

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}. \tag{1}$$

Each f_i is a *nonnegative* linear combination of $h_0, \ldots, h_{i+1}$. Specifically,

$$f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{d-i} h_j. \tag{2}$$

A simplicial complex Δ is *i-neighborly* if every subset of vertices of cardinality i is a face of Δ.

A *stacked* polytope is the following inductively defined class of polytopes. The simplex is a stacked polytope and any polytope obtained from a stacked polytope by adding a pyramid to a facet is a stacked polytope. Stacked polytopes are simplicial and the boundary of a stacked polytope is a *stacked sphere*. A purely combinatorial characterization of stacked spheres is due to Kalai. Let $\phi_i(n, d)$ be the number of i-dimensional faces in a $(d-1)$-dimensional stacked sphere with n vertices. Equivalently,

$$\phi_i(n, d) = \begin{cases} \binom{d}{i}n - \binom{d+1}{i+1}i & \text{for } 1 \le i \le d-2, \\ (d-1)n - (d+1)(d-2) & \text{for } i = d-1. \end{cases} \tag{3}$$

Theorem 2.1 ([16, Theorem 1.1]). *Let Δ be a homology manifold without boundary. Then $f_i(\Delta) \ge \phi_i(n, d)$. If $f_i(\Delta) = \phi_i(n, d)$ for any $1 \le i \le d-1$, then Δ is a stacked sphere.*

Corollary 2.2 ([16]). *Let Δ be a homology manifold without boundary. Then Δ is a stacked sphere if and only if $h_1(\Delta) = h_2(\Delta)$.*

Let Δ' be another $(d-1)$-dimensional complex and let σ' be a facet of Δ'. Let σ be a facet of Δ and choose a bijection between the vertices of σ' and the vertices of σ. The *connected sum* of Δ and Δ', $\Delta \# \Delta'$, is the complex obtained by identifying the vertices (and corresponding faces) of Δ and Δ' by the chosen bijection, and then removing the facet corresponding to $\sigma (= \sigma')$. If both complexes are $(d-1)$-dimensional homology manifolds without boundary, then any connected sum is also a homology manifold without boundary. However, the homeomorphism type of $\Delta \# \Delta'$ may depend on the chosen bijection. Direct calculation shows that $h_d(\Delta \# \Delta') = h_d(\Delta) + h_d(\Delta') - 1$ and for $0 < i < d, h_i(\Delta \# \Delta') = h_i(\Delta) + h_i(\Delta')$.

Another method for forming new complexes out of old is handle addition. Let σ and σ' be disjoint facets of Δ. Also, let ϕ be a bijection between the vertices of the two facets. Identify each pair of vertices $(v, \phi(v))$ and any corresponding faces. As long as v and $\phi(v)$ are not neighbors and there are no vertices which have both v and $\phi(v)$ as neighbors, the resulting space will still be a simplicial complex and we say it is obtained by *handle addition*. If the original complex is a homology manifold without boundary, then so is the new complex. As before, the homeomorphism type of the new complex may depend on the choice of bijection.

In [47] Walkup introduced $\mathcal{H}^{d-1}$, the set of simplicial complexes that can be obtained from $(d-1)$-dimensional stacked spheres by repeated handle addition. As we will see (cf. Theorem 4.10 and Theorem 4.30), the triangulations in $\mathcal{H}^{d-1}$ are minimal in a certain sense.

Theorem 2.3 ([47], [16]). *$\Delta \in \mathcal{H}^{d-1}$ if and only if the link of every vertex of Δ is a stacked sphere.*

3. Linear relations

After the Euler–Poincaré formula, the Dehn–Sommerville equations for simplicial polytopes ([10], [36]) were one of the first known restrictions on the f-vectors of a class of manifolds. These relations for polytopes were generalized to semi-Eulerian complexes by Klee. We say Δ is a *semi-Eulerian* complex if for every nonempty face ρ of Δ, the Euler characteristic of its link, $\chi(\text{lk}\,\rho)$, equals $\chi(S^{d-|\rho|-1})$. Homology manifolds without boundary are a motivating example. If in addition $\chi(\Delta) = \chi(S^{d-1})$, then we say Δ is an *Eulerian* complex.

Theorem 3.1 ([19]). *Let Δ be a semi-Eulerian complex. Then*

$$h_{d-i} - h_i = (-1)^i \binom{d}{i} (\chi(\Delta) - \chi(S^{d-1})). \tag{4}$$

Our semi-Eulerian complexes were called Eulerian manifolds in [19]. Related equations were discovered earlier by Vaccaro [46].

If Δ is an odd-dimensional semi-Eulerian complex, then setting $i = d/2$ shows that the Euler characteristic of Δ is zero and hence Δ is Eulerian. We will refer to the above equations as the *generalized Dehn–Sommerville equations*.

Under certain conditions there is a refinement of the generalized Dehn–Sommerville equations. Let $\mathbf{a} = (a_1, \ldots, a_m)$ be a sequence of positive integers. Define $|\mathbf{a}| = a_1 + \cdots + a_m$. Let $\phi : V \to [m]$, with $[m] = \{1, \ldots, m\}$, be a surjective function and set $V_j = \phi^{-1}(j)$.

Definition 3.2. *Suppose $|\mathbf{a}| = d$. The pair (Δ, ϕ) is a* balanced complex of type $\mathbf{a}$ *if for every facet $\sigma \in \Delta$ and j, $1 \le j \le m$,*

$$|\sigma \cap V_j| = a_j.$$

Balanced complexes of type $(1, \ldots, 1)$ are called *completely balanced*. The canonical example of a completely balanced complex is the order complex of a graded poset with $\phi(v) = \mathrm{rk}(v)$. If $\mathbf{b} = (b_1, \ldots, b_m)$ is a sequence of nonnegative integers with $b_j \leq a_j$, then we write $\mathbf{b} \leq \mathbf{a}$. When Δ is completely balanced we can identify sequences $\mathbf{b} \leq (1, \ldots, 1)$ with subsets of $[d]$ in the usual way, $\mathbf{b} \leftrightarrow \{i \in [d] : b_i = 1\}$.

One way to produce examples of balanced complexes is to start with a completely balanced complex (Δ, ϕ) and specialize. Given $\mathbf{a}$ with $|\mathbf{a}| = d$, let $\psi : [d] \to [m]$ be the map such that $\psi^{-1}(j) = [a_1 + \cdots + a_{j-1} + 1, a_1 + \cdots + a_j]$. Then $(\Delta, \psi \circ \phi)$ is a balanced complex of type $\mathbf{a}$. Under these conditions we write $S \to \mathbf{b}$ if $S \subseteq [d]$ and $|S \cap [a_1 + \cdots + a_{j-1} + 1, a_1 + \cdots + a_j]| = b_j$ for each j. If Δ is any $(d-1)$-dimensional pure complex and $|\mathbf{a}| = d$, then we can construct a balanced complex of type $\mathbf{a}$ which is homeomorphic to Δ. Indeed, the (reduced) order complex of the face poset of Δ is a completely balanced complex homeomorphic to Δ which can then be specialized to $\mathbf{a}$. For a simple example of a balanced complex which is not the specialization of a completely balanced complex, see Figure 1.

Let (Δ, ϕ) be a balanced complex of type $\mathbf{a}$. For $\mathbf{b} \leq \mathbf{a}$ define $f_\mathbf{b}$ (if necessary, $f_\mathbf{b}(\Delta)$) to be the number of faces ρ such that for all j, $|\rho \cap \phi^{-1}(j)| = b_j$. The collection $\{f_\mathbf{b}\}_{\mathbf{b} \leq \mathbf{a}}$ is the *fine f-vector* of Δ and is a refinement of the f-vector in the sense that

$$\sum_{|\mathbf{b}| = i} f_\mathbf{b} = f_{i-1}(\Delta).$$

The *fine h-vector* of Δ is defined for $\mathbf{b} \leq \mathbf{a}$ by

$$h_\mathbf{b} = \sum_{\mathbf{c} \leq \mathbf{b}} f_\mathbf{c} \prod_{i=1}^{m} (-1)^{b_i - c_i} \binom{a_i - c_i}{b_i - c_i}. \tag{5}$$

Example 3.3. The bipyramid in Figure 1 is a balanced complex of type $(1, 2)$. The fine f- and h-vectors are

$$
\begin{aligned}
f_{0,0} &= 1, & h_{0,0} &= 1, \\
f_{1,0} &= 2, & h_{1,0} &= 1, \\
f_{0,1} &= 5, & h_{0,1} &= 3, \\
f_{1,1} &= 10, & h_{1,1} &= 3, \\
f_{0,2} &= 5, & h_{0,2} &= 1, \\
f_{1,2} &= 10, & h_{1,2} &= 1.
\end{aligned}
$$

Theorem 3.4 ([39]). *Let (Δ, ϕ) be a balanced complex of type $\mathbf{a}$. Then*

$$h_i = \sum_{|\mathbf{b}| = i} h_\mathbf{b}. \tag{6}$$

If Δ is completely balanced and we have identified $\mathbf{b} \leq (1, \ldots, 1)$ with subsets of $[d]$ as above, then the collections f_S and h_S, $S \subseteq [d]$, are called the *flag f-vector* and *flag h-vector* respectively. Here f_S is the number of faces ρ such that the image of the vertices of ρ under ϕ is S. In this case equation (5) becomes

$$h_S = \sum_{T \subseteq S} (-1)^{|S - T|} f_S. \tag{7}$$

An equivalent way to define h and $h_{\mathbf{b}}$ is through the face ring. Let k be a field and set $R = k[x_1, \ldots, x_n]$.

Definition 3.5. *The* face ring (*also known as the* Stanley–Reisner ring) *of Δ is $k[\Delta] = R/I_\Delta$, where*

$$I_\Delta = \langle \{x_{i_1}, \ldots, x_{i_k} : \{v_{i_1}, \ldots, v_{i_k}\} \notin \Delta\} \rangle.$$

The Hilbert function of $k[\Delta]$ encodes the h-vector of Δ in a nice way. Let $k[\Delta]_i$ be the degree i component of $k[\Delta]$. Define

$$F(\Delta, \lambda) = \sum_{i=0}^{\infty} \dim_k k[\Delta]_i \, \lambda^i.$$

Theorem 3.6 (see, for instance, [43, II.2]).

$$F(\Delta, \lambda) = \sum_{i=0}^{d} \frac{h_i \lambda^i}{(1 - \lambda)^d}.$$

When Δ is balanced, $k[\Delta]$ has a natural $\mathbb{N}^m$ grading by assigning x_i to $\lambda_{\phi(v_i)}$. For instance, let Δ be the boundary of a bipyramid over a pentagon in $\mathbb{R}^3$ as in Figure 1. With ϕ as given, Δ is a balanced complex of type $\{1, 2\}$. The fine Hilbert function for $k[\Delta]$ is

$$1 + 2\lambda_1 + 5\lambda_2 + 2\lambda_1^2 + 10\lambda_2^2 + 10\lambda_1\lambda_2 + 2\lambda_1^3 + 10\lambda_1^2\lambda_2 + 20\lambda_1\lambda_2^2 + 15\lambda_2^3 + \cdots.$$
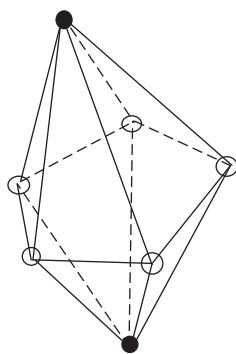

Fig. 1. Balanced bipyramid, $\phi(\bullet) = 1$, $\phi(\circ) = 2$.

Theorem 3.7 ([39]). *Let Δ be a balanced complex of type $\mathbf{a} = (a_1, \ldots, a_m)$. Then*

$$F(k[\Delta], \lambda) = \prod_{j=1}^{m} \frac{1}{(1 - \lambda_j)^{a_j}} \sum_{\mathbf{b} \leq \mathbf{a}} h_{\mathbf{b}} \lambda^{\mathbf{b}}. \tag{8}$$

As usual $\lambda^{\mathbf{b}} = \lambda_1^{b_1} \cdots \lambda_m^{b_m}$. When $\mathbf{b} \leq \mathbf{a}$, denote by $\mathbf{a} - \mathbf{b}$ the m-tuple $(a_1 - b_1, \ldots, a_m - b_m)$. For the special case of balanced Eulerian complexes, the following theorem was first stated in [4].

Theorem 3.8. *If Δ is a balanced semi-Eulerian complex of type $\mathbf{a}$, then for all $\mathbf{b} \le \mathbf{a}$,*

$$h_{\mathbf{a}-\mathbf{b}} - h_{\mathbf{b}} = (-1)^{|b|}[\chi(\Delta) - \chi(S^{d-1})] \prod_{j=1}^{m} \binom{a_j}{b_j}. \tag{9}$$

Proof. The strategy of the proof is not new and follows the ideas of [43, II.7]. We compute the fine Hilbert function $F(k[\Delta], 1/\lambda)$ in two different ways. From equation (8),

$$F(k[\Delta], 1/\lambda) = \prod_{j=1}^{m} \frac{1}{(1-1/\lambda_j)^{a_j}} \sum_{\mathbf{b} \le \mathbf{a}} h_{\mathbf{b}}/\lambda^{\mathbf{b}} = \prod_{j=1}^{m} \frac{1}{(\lambda_j - 1)^{a_j}} \sum_{\mathbf{b} \le \mathbf{a}} h_{\mathbf{b}} \lambda^{\mathbf{a}-\mathbf{b}}$$

$$= (-1)^d \prod_{j=1}^{m} \frac{1}{(1-\lambda_j)^{a_j}} \sum_{\mathbf{b} \le \mathbf{a}} h_{\mathbf{b}} \lambda^{\mathbf{a}-\mathbf{b}}.$$

For a face ρ, define $\rho(l) = |\{v \in \rho : \phi(v) = l\}|$. By [43, Corollary 7.2],

$(-1)^d F(k[\Delta], 1/\lambda)$

$$= (-1)^{d-1} \tilde{\chi}(\Delta) + \sum_{\rho \ne \emptyset} \prod_{v_i \in \rho} \frac{\lambda_{\phi(v_i)}}{1 - \lambda_{\phi(v_i)}}$$

$$= (-1)^{d-1} \tilde{\chi}(\Delta) + \prod_{j=1}^{m} \frac{1}{(1-\lambda_j)^{a_j}} \sum_{\rho \ne \emptyset} \prod_{v_i \in \rho} \lambda_{\phi(v_i)} \prod_{l=1}^{m} (1-\lambda_l)^{a_l - \rho(l)}$$

$$= (-1)^{d-1} \tilde{\chi}(\Delta) + \prod_{j=1}^{m} \frac{1}{(1-\lambda_j)^{a_j}} \sum_{\mathbf{b} \le \mathbf{a}} \sum_{\substack{\mathbf{c} \le \mathbf{b} \\ |\mathbf{c}| \ne 0}} (-1)^{|\mathbf{b}-\mathbf{c}|} f_{\mathbf{c}} \prod_{l=1}^{m} \binom{a_l - c_l}{b_l - c_l} \lambda^{\mathbf{b}}$$

$$= (-1)^{d-1} \tilde{\chi}(\Delta) + \prod_{j=1}^{m} \frac{1}{(1-\lambda_j)^{a_j}} \left\{ \sum_{\mathbf{b} \le \mathbf{a}} h_{\mathbf{b}} - (-1)^{|\mathbf{b}|} \prod_{l=1}^{m} \binom{a_l}{b_l} \right\} \lambda^{\mathbf{b}}.$$

Multiplying both equations by $\prod_{j=1}^{m} (1-\lambda_j)^{a_j}$ leaves

$$\sum_{\mathbf{b} \le \mathbf{a}} h_{\mathbf{b}} \lambda^{\mathbf{a}-\mathbf{b}} = \sum_{\mathbf{b} \le \mathbf{a}} \left\{ h_{\mathbf{b}} + (-1)^{|\mathbf{b}|}[(-1)^{d-1} \tilde{\chi}(\Delta) - 1] \prod_{j=1}^{m} \binom{a_j}{b_j} \right\} \lambda^{\mathbf{b}}.$$

Since $(-1)^{d-1} \tilde{\chi}(\Delta) - 1 = \chi(\Delta) - \chi(S^{d-1})$, comparing the coefficients of $\lambda^{\mathbf{b}}$ finishes the proof. $\qquad\square$

As far as we know, the only other place where semi-Eulerian balanced (as opposed to completely balanced) complexes are considered is Magurn [27], where balanced compact 2-manifolds are analyzed. Equation (9) for completely balanced semi-Eulerian posets appears in [41, Proposition 2.2]. Balanced complexes of type $\mathbf{a} = (d)$ are just pure complexes, and in this case (9) recovers the generalized Dehn–Sommerville equations.

Corollary 3.9. *If Δ is a completely balanced semi-Eulerian complex, then*

$$h_{[d]-S} - h_S = (-1)^{|S|}[\chi(\Delta) - \chi(S^{d-1})]. \tag{10}$$

For Eulerian complexes the relations in the above corollary are also called the generalized Dehn–Sommerville equations. For the history of these equations see the discussion in [1].

Let $H_E(d)$ be the affine span of fine h-vectors of balanced $(d-1)$-dimensional Eulerian complexes of type $\mathbf{a}$. Billera and Magurn determined the dimension of $H_E(d)$ in [4]. Their answer was in terms of the number of $\mathbf{b} \leq \mathbf{a}$, $n(\mathbf{a}) = \prod_{j=1}^{m}(a_j + 1)$. Equation (9) allows us to extend their result to semi-Eulerian complexes.

Theorem 3.10. *Let Δ be a semi-Eulerian complex. Fix $\mathbf{a}$ with $|\mathbf{a}| = d$. Let H_Δ be the affine span of $\{h_\mathbf{b}(\Delta')\}$, where Δ' ranges over all balanced complexes of type $\mathbf{a}$ homeomorphic to Δ. Then*

$$\dim H_\Delta = \begin{cases} \frac{1}{2}(n(\mathbf{a}) - 1) & \text{if every } a_i \text{ is even,} \\ \frac{1}{2}(n(\mathbf{a}) - 2) & \text{otherwise.} \end{cases} \tag{11}$$

Proof. If $\mathbf{b} \leq \mathbf{a}$, then $\mathbf{b} \neq \mathbf{a} - \mathbf{b}$ unless each a_i is even and $b_i = a_i/2$ for every i. Also, $h_{\{0,\ldots,0\}} = 1$ for any balanced complex. Hence, Theorem 3.8 implies that H_Δ satisfies $1 + (n(\mathbf{a})-1)/2$ linearly independent equations if every a_i is even, and $1+(n(\mathbf{a})/2)$ otherwise. Therefore, the dimension of H_Δ is bounded above by the right-hand side of (11).

In order to prove the opposite inequality, we first construct the requisite number of balanced spheres of type $\mathbf{a}$ whose fine h-vectors affinely span H_Δ for $\Delta = S^{d-1}$. This is accomplished in [4, Section 5]. Denote by $\{\mathcal{P}^\mathbf{c}\}_{\mathbf{c}\in\mathcal{C}}$ the corresponding collection of balanced spheres.

Now let Δ be an arbitrary semi-Eulerian complex. As noted before, there exists Δ' homeomorphic to Δ with Δ' a balanced complex of type $\mathbf{a}$. For $\mathbf{c} \in \mathcal{C}$, $\mathbf{b} \neq (0,\ldots,0)$, $\mathbf{b} \neq \mathbf{a}$ we have $h_\mathbf{b}(\Delta' \# \mathcal{P}^\mathbf{c}) = h_\mathbf{b}(\Delta') + h_\mathbf{b}(\mathcal{P}^\mathbf{c})$, where the connected sum identifies vertices so that the resulting complex is still balanced of type $\mathbf{a}$. In addition, $h_{\{0,\ldots,0\}}(\Delta' \# \mathcal{P}^\mathbf{c}) = 1$ and $h_\mathbf{a}(\Delta'\#\mathcal{P}^\mathbf{c}) = h_\mathbf{a}(\Delta')$. So, the affine span of $\{h_\mathbf{b}(\Delta'\#\mathcal{P}^\mathbf{c})\}_{\mathbf{c}\in\mathcal{C}}$ is a translation of the affine span of $\{h_\mathbf{b}(\mathcal{P}^\mathbf{c})\}_{\mathbf{c}\in\mathcal{C}}$ and hence has the same dimension. $\square$

When Δ is the order complex of a poset there are further restrictions on the flag h-vector of Δ. A finite graded poset P with a least element $\hat{0}$ and greatest element $\hat{1}$ is semi-Eulerian if $\mu(x, y) = (-1)^{\text{rk}(x)-\text{rk}(y)}$ for all $x \leq y$, $(x, y) \neq (\hat{0}, \hat{1})$. Equivalently, the reduced order complex of P is a semi-Eulerian complex. If in addition, $\mu(\hat{0}, \hat{1}) = (-1)^{\text{rk}(P)}$, then we say P is *Eulerian*.

Let $F_E(d)$ be the affine span of flag f-vectors of reduced order complexes of rank d Eulerian posets. Bayer and Billera determined $F_E(d)$ explicitly [1]. For the purposes of stating the linear equations satisfied by the elements of $F_E(d)$, we temporarily extend the definition of f_S to subsets $S \subseteq \{0, 1, \ldots, d-1\}$. If $0 \in S$, then define $f_S = f_{S-\{0\}}$.

Theorem 3.11 ([1]). *Let P be an Eulerian poset of rank d, let Δ be the order complex of P and let $S \subseteq [d-1]$. If $\{i, k\} \subseteq S \cup \{-1, d\}$, $i < k-1$, and S contains no j such*

that $i < j < k$, then

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup j} = f_S (1 - (-1)^{k-i-1}). \tag{12}$$

Bayer and Billera proved that the affine span of the set of flag f-vectors which satisfy (12) has dimension $e_d - 1$, where e_d is the d-th Fibonacci number. Then they constructed a family, $\mathcal{P}^d$, of polytopes whose flag f-vectors were affinely independent with $|\mathcal{P}^d| = e_d$, thus proving that $F_E(d)$ consists of all $\{f_S\}$ which satisfy (12).

J. Fine gave a basis for $F_E(d)$ which we now describe. The coefficients with respect to this basis have come to be known as the **cd**-*index* of P. Encode the flag h-vector of P (or more accurately, of the reduced order complex of P) as a polynomial $h_P(\mathbf{a}, \mathbf{b})$ in noncommuting variables a and b (not to be confused with the indices in the previous section) by

$$S \leftrightarrow \begin{cases} \mathbf{a}, & i \notin S, \\ \mathbf{b}, & i \in S. \end{cases}$$

For instance, if P is the face poset of the bipyramid in Figure 1, then

$$h_P(\mathbf{a}, \mathbf{b}) = \mathbf{aaa} + 6\mathbf{baa} + 14\mathbf{aba} + 9\mathbf{aab} + 6\mathbf{abb} + 14\mathbf{bab} + 9\mathbf{bba} + \mathbf{bbb}.$$

Now let $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. In the above example, $h_P(\mathbf{a}, \mathbf{b}) = \mathbf{ccc} + 5\mathbf{dc} + 8\mathbf{cd}$. Let $F(\mathbf{c}, \mathbf{d})$ be the linear subspace spanned by all monomials in $\mathbf{c}$ and $\mathbf{d}$ of degree $d - 1$ other than $\mathbf{c}^{d-1}$, where the degree of $\mathbf{c}$ is one and the degree of $\mathbf{d}$ is two. Bayer and Klapper proved that $F_E(d) = \mathbf{c}^{d-1} + F(\mathbf{c}, \mathbf{d})$ [2, Theorem 4].

The results for flag f-vectors of semi-Eulerian posets are similar. Suppose P is a rank d semi-Eulerian poset and let Δ_P be the reduced order complex of P. Note that the dimension of Δ_P is $d - 2$. In order to describe F_P, the affine span of flag f-vectors of posets whose order complexes are homeomorphic to Δ_P, set $X = \chi(\Delta_P) - \chi(S^{d-2})$.

Theorem 3.12. *Let $f^X(d)$ be the flag f-vector which is zero for all $S \subseteq [d-1]$ except $f_{\{d-1\}} = X$. Then $F_P = F_E(d) + f^X(d)$.*

Proof. If d is odd, then P is Eulerian and $X = 0$. Hence, $F_P \subseteq F_E(d)$. So assume that d is even. Since P is semi-Eulerian the flag f-vector of Δ_P satisfies all of the equations in (12) except the one equivalent to Euler's formula for the sphere,

$$\sum_{j=0}^{d-1} (-1)^j f_{\{j\}} = f_\emptyset (1 - (-1)^{d-1}).$$

Let f' be the flag f-vector defined by $f'_S = f_S(\Delta_P) - f^X_S(d)$. The only expressions of (12) which are different for f' are the one above, which by the definition of X is now valid, and

$$\sum_{j=0}^{d-2} (-1)^j f'_{\{d-1\} \cup j} = f'_{\{d-1\}} (1 - (-1)^d).$$

Comparing this expression with the corresponding expression for f shows that the left hand side is unchanged, while the right hand side in both cases is zero since d is even. Hence f' is in $F_E(d)$. Therefore, $F_P \subseteq F_E(d) + f^X(d)$.

To establish the opposite inclusion, consider the family of flag f-vectors given by the (reduced) order complex of $\{\Delta_P \# \partial \mathcal{P}_t\}$, where $\mathcal{P}_t$ is the collection of $(d-1)$-polytopes given by Bayer and Billera whose flag f-vectors (affinely) span $F_E(d)$ [1]. As this set of flag f-vectors is a translation of the flag f-vectors of the reduced order complex of $\{\partial \mathcal{P}_t\}$, its affine dimension is the same. Since each $\Delta_P \# \partial \mathcal{P}_t$ is homeomorphic to Δ_P we have $\dim F_P \geq \dim F_E(d)$. □

The containment $F_P \subseteq F_E(d) + f^X(d)$ is a special case of [11, Theorem 4.2], where Ehrenborg considers posets whose intervals of varying lengths are Eulerian.

In view of Karu's proof that the *cd*-index of any Gorenstein* poset has nonnegative coefficients [18], and the fact that the flag f-vectors of semi-Eulerian posets with the same Euler characteristic and dimension lie in the same affine subspace of flag f-vectors, it seems natural to ask the following question.

Problem 3.13. *For a fixed semi-Eulerian poset P, describe the cone of flag f-vectors of posets P' such that Δ_P is homeomorphic to $\Delta_{P'}$.*

An alternative approach to the combinatorics of semi-Eulerian posets is through the toric h-vector. Originally introduced to correspond to the Betti numbers of the intersection cohomology of toric varieties associated to rational polytopes, the toric h-vector can be defined for any finite graded poset with a minimum element $\hat{0}$ and a maximum element $\hat{1}$. With the exception noted below, we follow Stanley's presentation [44, Section 3.14] and refer the reader to [42] for background on the motivation and history behind its definition.

Let P be a finite graded poset with $\hat{0}$ and $\hat{1}$ and let ρ be the rank function of P. Let $\tilde{P}$ be the set of all intervals $[\hat{0}, z]$ in P ordered by inclusion. The map $z \mapsto [\hat{0}, z]$ is a poset isomorphism from P to $\tilde{P}$. Define two polynomials $\tilde{h}$ and $\tilde{g}$ inductively as follows.

- $\hat{h}(\mathbf{1}, x) = \hat{g}(\mathbf{1}, x) = 1$. Here $\mathbf{1}$ is the poset with only one element $\hat{1} = \hat{0}$.
- If the rank of P is $d + 1$, then $\hat{h}(P, x)$ has degree d. Write $\hat{h}(P, x) = \hat{h}_d + \hat{h}_{d-1}x + \hat{h}_{d-2}x^2 + \cdots + \hat{h}_0 x^d$. Then define

$$\hat{g}(P, x) = \hat{h}_d + (\hat{h}_{d-1} - \hat{h}_d)x + (\hat{h}_{d-2} - \hat{h}_{d-1})x^2 + \cdots + (\hat{h}_{d-m} - \hat{h}_{d-m+1})x^m,$$

where $m = \lfloor d/2 \rfloor$.
[NOTE: Our $\hat{h}_i$ is $\hat{h}_{d-i}$ in [44].]
- If the rank of P is $d + 1$, then define

$$\hat{h}(P, x) = \sum_{\substack{Q \in \tilde{P} \\ Q \neq P}} \hat{g}(Q, x)(x - 1)^{d - \rho(Q)}. \tag{13}$$

Induction shows that if B_d is the face poset of the $(d - 1)$-simplex, then $\hat{h}(B_d, x) = 1 + x + \cdots + x^{d-1}$ and $\hat{g}(B_d, x) = 1$. From this it follows that if P is the face poset of a simplicial complex Δ with $\hat{1}$ adjoined, then $\hat{h}_i(P, x) = h_i(\Delta)$.

Example 3.14. Figure 3 shows the Hasse diagram, P, of the face poset of the cell decomposition of the torus depicted in Figure 2. The rank one and rank two elements correspond to simplices. The four rank two elements, which correspond to the four rectangles of the cell decomposition have $\hat{g} = 1 + x$. So,

$$\hat{h}_P(x) = (x-1)^3 + 4(x-1)^2 + 8(x-1) + 4(x+1) = x^3 + x^2 + 7x - 1.$$

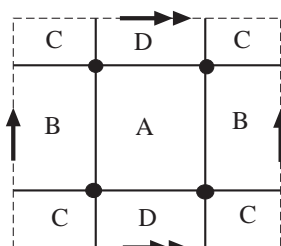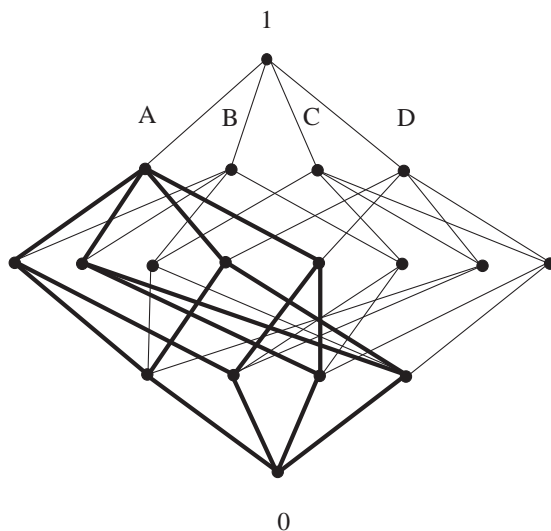**Fig. 2.** Cell decomposition of the torus.

Fig. 3. Hasse diagram of P.

Theorem 3.15. *Let P be a semi-Eulerian poset of rank $d + 1$ and let Δ_P be the reduced order complex of P. Then*

$$\hat{h}_{d-i} - \hat{h}_i = (-1)^i \binom{d}{i} [\chi(\Delta_P) - \chi(S^{d-1})]. \tag{14}$$

Proof. The proof is a small variation of Stanley's proof of this equation in the special case that P is Eulerian [44, p. 139]. Write $\hat{h}(P)$ for $\hat{h}(P, x)$ and $\hat{g}(P)$ for $\hat{g}(P, x)$. Let

$y = x - 1$. Mutliply (13) by y and add $\hat{g}(P)$ to obtain for $P \neq \mathbf{1}$,

$$\hat{g}(P) + y\hat{h}(P) = \sum_{Q \in \tilde{P}} g(Q)y^{\rho(P)-\rho(Q)}.$$

Hence for $P \neq \mathbf{1}$,

$$y^{-\rho(P)}(\hat{g}(P) + y\hat{h}(P)) = \sum_{Q} \hat{g}(Q)y^{-\rho(Q)}.$$

Since $\sum_{Q \in \mathbf{1}} \hat{g}(Q)y^{-\rho(Q)} = 1$, Möbius inversion implies

$$\hat{g}(P)y^{-\rho(P)} = \mu_P(\hat{0}, \hat{1}) + \sum_{\substack{Q \in \tilde{P} \\ Q \neq \mathbf{1}}} (\hat{g}(Q) + y\,\hat{h}(Q))y^{-\rho(Q)}\mu_{\tilde{P}}(Q, P).$$

Since $\tilde{P}$ is semi-Eulerian, $\mu_{\tilde{P}}(Q, P) = (-1)^{\rho(P)-\rho(Q)}$. So,

$$\hat{g}(P) = y^{\rho(P)}\mu_P(\hat{0}, \hat{1}) + \sum_{Q \neq \mathbf{1}} (\hat{g}(Q) + y\hat{h}(Q))(-y)^{\rho(P)-\rho(Q)}. \tag{15}$$

Let $\hat{h}(Q) = a_0 + a_1 x + \cdots + a_r x^r$, where $\rho(Q) = r + 1$. Then

$$\hat{g}(Q) + y\hat{h}(Q) = (a_s - a_{s+1})x^{s+1} + (a_{s+1} - a_{s+2})x^{s+2} + \cdots,$$

where $s = \lfloor r/2 \rfloor$. Since each Q is neither $\mathbf{1}$ nor P, it is Eulerian, so we may assume by induction on the rank of Q that $a_i = a_{r-i}$, where $r + 1 = \rho(Q)$, $r < d$. In this case,

$$\hat{g}(Q) + y\,\hat{h}(Q) = (a_s - a_{s-1})x^{s+1} + (a_{s-1} - a_{s-2})x^{s+2} + \cdots = x^{\rho(Q)}\hat{g}(Q, 1/x). \tag{16}$$

Now subtract $y\hat{h}(P) + \hat{g}(P)$ from both sides of (15) and use (16) to obtain

$$-y\hat{h}(P) = y^{\rho(P)}\mu_P(\hat{0}, \hat{1}) + \sum_{\hat{0} < Q < \hat{1}} x^{\rho(Q)}\hat{g}(Q, 1/x)(-y)^{\rho(P)-\rho(Q)}$$

and so

$$\hat{h}(P) = -(y^d)[\mu_P(\hat{0}, \hat{1}) - (-1)^{d+1}] + \sum_{Q < \hat{1}} x^{\rho(Q)}\hat{g}(Q, 1/x)(-y)^{d-\rho(Q)}$$

$$= -(y^d)[\mu_P(\hat{0}, \hat{1}) - (-1)^{d+1}] + x^d\hat{h}(P, 1/x).$$

Comparing like terms of the last equation gives

$$\hat{h}_{d-i} - \hat{h}_i = (-1)^{d-i-1}\binom{d}{i}[\mu_P(\hat{0}, \hat{1}) - (-1)^{d+1}]. \tag{17}$$

When d is even, P is Eulerian, so the right hand side of (17) is zero and the equality agrees with (14). If d is odd, then, since $\mu_P(\hat{0}, \hat{1}) = \chi(\Delta_P) - 1$ and $(-1)^{d+1} = \chi(S^{d-1}) - 1$, (17) also agrees with (14). $\qquad\square$

As the toric h-vector agrees with the usual h-vector for simplicial complexes, it is easy to see that if we fix the order complex homeomorphism type of a semi-Eulerian poset, then (14) spans all of the linear relations among the $\hat{h}_i$.

4. Inequalities

There are two very general inequalities for h-vectors of homology manifolds. One is due to Schenzel (Theorem 4.14 below), and the following *rigidity inequality* is due to Kalai and, independently, Gromov.

Theorem 4.1 (Rigidity inequality, [13, 2.4.10], [16]). *Suppose Δ is a homology manifold without boundary and $d \geq 3$. Then $h_0 \leq h_1 \leq h_2$.*

The rigidity inequality has a strong implication when the fundamental group of Δ is nontrivial. Suppose $\tilde{\Delta}$ is a t-sheeted covering of Δ. Then the triangulation of Δ lifts to a triangulation of $\tilde{\Delta}$ with $f_i(\tilde{\Delta}) = t f_i(\Delta)$ for $i \geq 0$.

Proposition 4.2. *If $\tilde{\Delta}$ is a t-sheeted covering of Δ, then*

- $h_1(\tilde{\Delta}) = t \cdot h_1(\Delta) + d(t - 1)$.
- $h_2(\tilde{\Delta}) = t \cdot h_2(\Delta) - (t - 1)\binom{d}{2}$.

Proof. This is a straightforward application of $f_i(\tilde{\Delta}) = t f_i(\Delta)$ and the definition of h-vectors in terms of f-vectors. $\square$

Theorem 4.3. *Let Δ be a closed homology manifold. If $\pi_1(\Delta)$ has a subgroup of index t, then*

$$\frac{t-1}{t}\binom{d+1}{2} \leq h_2 - h_1 \leq \binom{h_1}{2}.$$

In particular, if $|\pi_1(\Delta)|$ is finite and greater than $\binom{d+1}{2}$, or if $\beta_1 > 0$, then

$$\binom{d+1}{2} \leq h_2 - h_1 \leq \binom{h_1}{2}.$$

Proof. The inequality $h_2 - h_1 \leq \binom{h_1}{2}$ holds for *any* pure complex. Let $\tilde{\Delta}$ be a t-sheeted covering space of Δ corresponding to a subgroup of $\pi_1(\Delta)$ of index t. By the rigidity inequality, $0 \leq h_2(\tilde{\Delta}) - h_1(\tilde{\Delta})$. But, by the above proposition,

$$0 \leq h_2(\tilde{\Delta}) - h_1(\tilde{\Delta}) = t \cdot h_2(\Delta) - (t-1)\binom{d}{2} - t \cdot h_1(\Delta) - d(t-1).$$

If $\beta_1 \geq 0$, then π_1 has subgroups of arbitrarily large index, so the second inequality follows from the first. $\square$

The inequality involving only h_1 in Theorem 4.3 can be improved if Δ is a combinatorial manifold. A *combinatorial $(d-1)$-manifold* is a simplicial complex in which the link of every vertex is PL-homeomorphic to the boundary of the $(d-1)$-simplex.

Theorem 4.4 ([5]). *Let Δ be a combinatorial manifold. If $\pi_1(\Delta)$ is not trivial, then $d + 1 \leq h_1$.*

Problem 4.5. *Do there exist triangulated manifolds with nontrivial fundamental group and $h_2 - h_1 < \binom{d+1}{2}$ or $h_1 < d + 1$?*

One application of Theorem 4.3 is a proof that a family of triangulations of spherical bundles over S^1 given by Kühnel have the minimum possible f-vector for homology manifolds without boundary and nonzero first Betti number. This family of complexes has the following properties.

Theorem 4.6 ([21]). *For every $d \geq 3$ there is a simplicial complex M^d with the following properties.*

- M^d *has $2d + 1$ vertices.*
- M^d *is 2-neighborly, i.e. $f_1(M^d) = \binom{2d+1}{2}$.*
- *If d is odd, then M^d is homeomorphic to $S^1 \times S^{d-2}$. If d is even, then M^d is homeomorphic to the nonorientable S^{d-2}-bundle over S^1.*
- M^d *is vertex transitive with dihedral symmetry group.*
- *The link of every vertex of M^d is a stacked sphere.*

Our M^d is called M^{d-1} in [21]. These triangulations were generalized by Kühnel and Lassmann [23]. While we will consider all of the Kühnel–Lassmann triangulations of $S^1 \times S^{2m-1}$ in Section 5, we refer the reader to [23] for details on the others.

Theorem 4.7. *If Δ is a homology manifold without boundary and nonzero first Betti number, then for all i, $f_i(\Delta) \geq f_i(M^d)$.*

Proof. By Theorem 4.3, $f_0(\Delta) \geq 2d + 1 = f_0(M^d)$ and $f_1(\Delta) \geq \binom{2d+1}{2} = f_1(M^d)$. Define

$$\tilde{f}_i(\Delta) = \sum_{j=1}^{n} f_i(\operatorname{lk}_\Delta v_j). \tag{18}$$

Since $f_{i+1} = \tilde{f}_i/(i+2)$, it suffices to prove that

$$\tilde{f}_i(\Delta) \geq \tilde{f}_i(M^d). \tag{19}$$

Recall that $\phi_i(n, d)$ is the minimal number of i-faces in a $(d-1)$-dimensional homology manifold without boundary which has n vertices. Define

$$\Phi_i(N, n, d) = \sum_{j=1}^{n} \phi_i(N_j, d),$$

where $N = N_1 + \cdots + N_n$ is any composition of N into n nonzero parts. The formula for ϕ, equation (3), implies that this definition is independent of the choice of the N_j.

Now let N_j be the number of vertices in the link of v_j. So, $N_1 + \cdots + N_n = \tilde{f}_0(\Delta) = 2f_1(\Delta) \geq 2f_1(M^d) = 2d(2d + 1)$. Theorem 2.1 tells us that

$$\tilde{f}_i(\Delta) \geq \sum_{j=1}^{n} \phi_i(N_j, d-1) = \Phi_i(2f_1(\Delta), f_0(\Delta), d-1).$$

Theorem 4.3 says that $f_1(\Delta) \geq df_0(\Delta)$. As $\Phi_i(N, n, d)$ is monotonically increasing for fixed n, d and i, $\tilde{f}_i(\Delta)$ will be minimized by the least value of $\Phi_i((d-1)n, n, d-1)$. However, for fixed $d \geq 4$ and $1 \leq i \leq d-1$ this function is strictly increasing as a function of n. Since M^d minimizes n and $\tilde{f}_i(M^d) = \Phi_i((d-1)n, n, d-1)$, where $n = 2d + 1$, we are done.   □

Another way to use the rigidity inequality is to sum it over the links of all the faces of a fixed dimension. For this purpose we consider a generalization of the short simplicial h-vector introduced by Hersh and Novik in [14].

Definition 4.8.
$$\tilde{h}_i^{(m)}(\Delta) = \sum_{|\rho|=m} h_i(\text{lk } \rho).$$

The vector $(\tilde{h}_0^{(1)}, \ldots, \tilde{h}_{d-1}^{(1)})$ was called the *short simplicial h-vector* in [14].

Proposition 4.9 ([45]).
$$(m+1)\,\tilde{h}_{i-1}^{(m+1)} = i\tilde{h}_i^{(m)} + (d - m - i + 1)\tilde{h}_{i-1}^{(m)}. \tag{20}$$

As long as all of the links in question are homology manifolds of dimension at least three, the rigidity inequality implies $\tilde{h}_0^{(m)} \leq \tilde{h}_1^{(m)} \leq \tilde{h}_2^{(m)}$. Here are two examples of this principle.

Theorem 4.10. *Suppose the link of every vertex of Δ is a $(d-2)$-dimensional homology manifold without boundary. Then*

$$(d-1)h_1 \leq 3h_3 + (d-4)h_2. \tag{21}$$

Furthermore, when $d \geq 5$, equality occurs if and only if $\Delta \in \mathcal{H}^{d-1}$. In this case, the h-vector of Δ is determined by h_1 and h_2.

Proof. From the previous proposition, $\tilde{h}_1^{(1)} = 2h_2 + (d-1)h_1$ and $\tilde{h}_2^{(1)} = 3h_3 + (d-2)h_2$. The rigidity inequality applied to these two equations gives (21). In addition, equality occurs if and only if for each vertex v of Δ, $h_1(\text{lk } v) = h_2(\text{lk } v)$. By Corollary 2.2 each such link must be a stacked sphere and thus $h_1(\text{lk } v) = h_2(\text{lk } v) = \cdots = h_{d-2}(\text{lk } v)$. Hence $\tilde{h}_1^{(1)} = \tilde{h}_2^{(1)} = \cdots = \tilde{h}_{d-2}^{(1)}$. Since h_1 and h_2 determine $\tilde{h}_1^{(1)}$ they determine all of the $\tilde{h}_i^{(1)}$. It is not difficult to see that (20) ensures that this determines the entire h-vector. Finally, the link of every vertex of Δ is a stacked sphere if and only if $\Delta \in \mathcal{H}^{d-1}$.   □

The above theorem is optimal in the following sense. When $d = 4$, (21) reduces to $h_1 \leq h_3$. Both $h_1 = h_3$ (any homology manifold without boundary) and $h_1 < h_3$ (for instance, the suspension of $\mathbb{R}P^2$) can occur. For $d > 4$, any triangulation in $\mathcal{H}^{d-1}$ satisfies equality, including, for instance, the Kühnel–Lassmann triangulations of $S^1 \times S^{2m+1}$ and the nonorientable S^{2m} bundle over S^1 [23], stacked spheres, and connected sums along facets of any of these spaces.

When $d = 5$, (21) becomes

$$4h_1 \leq 3h_3 + h_2, \quad \text{i.e.} \quad 4(h_1 - h_2) \leq 3(h_3 - h_2).$$

Any three-manifold without boundary has Euler characteristic zero, so Δ is semi-Eulerian and we can substitute $10(\chi(\Delta) - 2)$ for $h_3 - h_2$. Hence,

$$h_2 - h_1 \geq -\frac{15}{2}(\chi(\Delta) - 2). \tag{22}$$

This formula first appears in [47], as does the characterization of equality. As Walkup's proof is logically equivalent to the one above, (21) can be viewed as a higher dimensional analog of (22). For another example, we consider $m = 2$.

Theorem 4.11. *If the link of every edge is a homology manifold without boundary and $d \geq 5$, then*

$$12h_4 + 6(d - 4)h_3 + (d - 2)(d - 7)h_2 - (d - 1)(d - 2)h_1 \geq 0. \tag{23}$$

Proof. From (20), $2h_1^{(2)} = 2h_2^{(1)} + (d - 2)h_1^{(1)}$ and $2h_2^{(2)} = 3h_3^{(1)} + (d - 3)h_2^{(1)}$. Applying (20) again to the right hand side of these equations, we obtain

$$2h_1^{(2)} = 2[3h_3 + (d - 2)h_2] + (d - 2)[2h_2 + (d - 1)h_1],$$
$$2h_2^{(2)} = 3[4h_4 + (d - 3)h_3] + (d - 3)[3h_3 + (d - 2)h_2].$$

The rigidity inequality implies $2h_2^{(2)} \geq 2h_1^{(2)}$.   □

Corollary 4.12. *If Δ is a 6-dimensional homology manifold without boundary, then*

$$\chi(\Delta) \leq 2 + \frac{1}{14}(h_3 - h_1).$$

Furthermore, $\chi(\Delta) = 2 + \frac{1}{14}(h_3 - h_1)$ if and only if $\Delta \in \mathcal{H}^{d-1}$.

Proof. For $d = 7$, (23) is equivalent to

$$12(h_4 - h_3) \geq 30(h_1 - h_3).$$

By the generalized Dehn–Sommerville equations, $h_4 - h_3 = -35(\chi(\Delta) - 2)$. So,

$$\chi(\Delta) - 2 \leq \frac{30}{12 \cdot 35}(h_3 - h_1).$$

Now suppose $\chi(\Delta) = 2 + \frac{1}{14}(h_3 - h_1)$. Let Δ_v be the link of a vertex of Δ. The proof of (23) shows that for every vertex $w \in \Delta_v$, $h_1(\mathrm{lk}_{\Delta_v} w) = h_2(\mathrm{lk}_{\Delta_v} w)$. Indeed, equality holds if and only if this is true. Since Δ is a homology manifold, $H_1(\Delta_v; \mathbb{Q}) = 0$. Kalai's first proof of [16, Theorem 1.1] shows that Δ_v is a stacked sphere and thus $\Delta \in \mathcal{H}^{d-1}$. Conversely, suppose $\Delta \in \mathcal{H}^{d-1}$. Since the link of every vertex of a stacked sphere is a stacked sphere, $h_1^{(2)} = h_2^{(2)}$ and equality in (23) follows.   □

Corollary 4.13. *If* Δ *is a* 6-*dimensional Eulerian homology manifold, then the h-vector of* Δ *is positive.*

For further estimates we turn to the face ring. By introducing a linear system of parameters, the face ring can be a powerful tool in understanding the combinatorics of Δ. Let $\Theta = \{\theta_1, \ldots, \theta_d\}$ be a set of one-forms in $R = k[x_1, \ldots, x_n]$. For each i, write $\theta_i = \theta_{i,1} x_1 + \cdots + \theta_{i,n} x_n$ and for each facet $\sigma \in \Delta$ let T_σ be the $d \times d$ matrix whose entries are $\{\theta_{i,j}\}_{v_j \in \sigma}$. We say Θ is a *linear system of parameters* (l.s.o.p.) for $k[\Delta]$ if T_σ has rank d for every facet σ of Δ.

Theorem 4.14 (Schenzel's formula, [35]). *Let* Θ *be a l.s.o.p. for* $k[\Delta]$, *and let* $h'_i = \dim_k (k[\Delta]/\langle\Theta\rangle)_i$. *If* Δ *is a k-homology manifold (with or without boundary), then*

$$h'_i = h_i + \binom{d}{i} \sum_{j=2}^{i-1} (-1)^{i-j-1} \beta_{j-1}, \qquad (24)$$

where the β_j *are the k-Betti numbers of* Δ.

Schenzel's proof of the above formula applies to the much more general class of connected Buchsbaum complexes. As an application of (24), we note that if Δ is a homology manifold without boundary, then $h'_d(\Delta) = 1$ if Δ is orientable, and $h'_d(\Delta) = 0$ if Δ is not orientable.

Schenzel's formula frequently allows us to move back and forth between the commutative algebra of $k[\Delta]/\langle\Theta\rangle$ and the combinatorics of Δ. As an example, the rigidity inequality has an interpretation in $k[\Delta]/\langle\Theta\rangle$ due to Lee.

Theorem 4.15 ([24]). *Let* Δ *be a homology manifold without boundary and let k be a field of characteristic zero. For generic pairs* (ω, Θ), ω *a one-form in R and* Θ *a l.s.o.p. for* $k[\Delta]$, *multiplication*

$$\omega : (k[\Delta]/\langle\Theta\rangle)_1 \rightarrow (k[\Delta]/\langle\Theta\rangle)_2$$

is an injection.

In view of Theorem 4.14, the study of f-vectors, h-vectors and h'-vectors, where we define h'_i by (24), are all equivalent for homology manifolds. The value of the h'-vectors is that for homology manifolds, $(h'_0, h'_1, \ldots, h'_d)$ is the Hilbert function of $k[\Delta]/\langle\Theta\rangle$, and Hilbert functions of homogeneous quotients of polynomial rings were characterized by Macaulay.

Given a and i positive integers there is a unique way to write

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. Define

$$a^{\langle i \rangle} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j + 1}.$$

Theorem 4.16 ([43, II.2.2]). *Let $(h_0, \ldots, h_d)$ be a sequence of nonnegative integers. Then the following are equivalent.*

- *$(h_0, \ldots, h_d)$ is the Hilbert function of a homogeneous quotient of a polynomial ring.*
- *$h_0 = 1$ and $h_{i+1} \leq h_i^{\langle i \rangle}$ for all $1 \leq i \leq d - 1$.*

Any sequence $(h_0, \ldots, h_d)$ which satisfies the above conditions is called an *M-vector*.

In combination with our previous results and the following theorem, Macaulay's formula leads to restrictions on n and $h_2 - h_1$ for $2m$-dimensional homology manifolds without boundary.

Theorem 4.17 ([31]). *Let Δ be a homology manifold, Θ a l.s.o.p. for $k[\Delta]$, and $\omega \in R_1$ (one-forms in R). Then the kernel of multiplication $\omega : (k[\Delta]/\langle \Theta \rangle)_i \to (k[\Delta]/\langle \Theta \rangle)_{i+1}$ has dimension greater than or equal to $\binom{d-1}{i} \beta_{i-1}$.*

Kalai has conjectured that the correct lower bound for homology manifolds without boundary is $\binom{d}{i} \beta_{i-1}$ [17, Conjecture 36].

For connected $2m$-dimensional homology manifolds without boundary, $h'_{m+1} - h'_m$ does not depend on the triangulation. Define

$$G(\Delta) = (-1)^m \binom{2m + 1}{m} [(\beta_1 - \beta_{2m-1}) - (\beta_2 - \beta_{2m-2}) + \cdots$$

$$\cdots + (-1)^{m-1}(\beta_{m-2} - \beta_{m+2}) + (-1)^m(\beta_m - \beta_{m+1})].$$

By the generalized Dehn–Sommerville equations and Schenzel's formula, $h'_{m+1} - h'_m = G(\Delta)$ [31]. If Δ is orientable, then $G(\Delta)$ reduces to $\binom{2m+1}{m}(\beta_m - \beta_{m-1})$.

Theorem 4.18. *Let Δ be a $2m$-dimensional connected homology manifold without boundary and suppose $G(\Delta) > 0$, where $G(\Delta)$ is computed using rational coefficients. Write $h_2 - h_1 = \binom{a}{2} + \binom{b}{1}$ with $a > b$. Then*

(a) $G(\Delta) + \binom{2m}{m} \beta_{m-1} \leq \binom{n-m-2}{m+1}$.

(b) $G(\Delta) + \binom{2m}{m} \beta_{m-1} \leq \binom{a+m-1}{m+1} + \binom{b+m-1}{m}$.

Proof. Let Θ be a l.s.o.p. for $\mathbb{Q}[\Delta]$ and $\omega \in \mathbb{Q}[\Delta]_1$ satisfy the conclusion of Theorem 4.15. Define $g'_i = \dim_{\mathbb{Q}} (\mathbb{Q}[\Delta]/\langle \Theta, \omega \rangle)_i$. Then $g'_{m+1} \geq h'_{m+1} - h'_m + \binom{2m}{m} \beta_{m-1}$, with equality if and only if the dimension of the kernel of multiplication $\omega : (\mathbb{Q}[\Delta])/\langle \Theta \rangle)_m \to (\mathbb{Q}[\Delta])/\langle \Theta \rangle)_{m+1}$ is $\binom{2m}{m} \beta_{m-1}$. The choice of Θ and ω implies that $g'_2 = h'_2 - h'_1$ and by Schenzel's formula this is $h_2 - h_1$. The inequalities now follow from Macaulay's arithmetic criterion for Hilbert functions. □

If $\beta_{m-1} = 0$ and Δ is orientable, then the left hand side of both inequalities reduces to $\binom{2m+1}{2m} \beta_m$. If Kalai's conjecture concerning the lower bound for the dimension of the kernel of multiplication by a one-form is correct, then again the left hand sides of both inequalities reduce to $\binom{2m+1}{2m} \beta_m$ whenever Δ is orientable.

While the second inequality is new and a key ingredient to the complete characterization of f-vectors in the next section, the first inequality is neither new nor best. See [31, Theorem 5.7] for a related stronger inequality and a discussion.

When Δ is a homology sphere or ball, $h_i = h_i'$. In the special case of Δ equal to the boundary of a simplicial polytope even more can be said. In [28] P. McMullen conjectured the following characterization of h-vectors of simplicial polytopes.

Conjecture 4.19 ([28]). *A sequence $(h_0, h_1, \ldots, h_d)$ is the h-vector of the boundary of a simplicial d-polytope if and only if*

(a) $h_0 = 1$.
(b) $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$.
(c) $(h_0, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ *is an M-vector.*

The sequence $(g_0, \ldots, g_{\lfloor d/2 \rfloor}) = (h_0, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is usually called the *g-vector* of Δ. The correctness of McMullen's conjecture was proved in two separate papers. In their 1981 paper [3] Billera and Lee showed how to construct a simplicial d-polytope with a given h-vector whenever it satisfied McMullen's conditions. Stanley's proof of the necessity of McMullen's conditions used a hard Lefschetz theorem for toric varieties associated to rational polytopes [40]. The main point is that, generically, $k[\Delta]/\langle \Theta \rangle$ has Lefschetz elements.

Definition 4.20. *Let Δ be a homology sphere. A* Lefschetz element *for $k[\Delta]/\langle \Theta \rangle$ is a one-form $\omega \in R$ such that for all $i \leq \lfloor d/2 \rfloor$ multiplication*

$$\omega^{d-2i} : (k[\Delta]/\langle \Theta \rangle)_i \to (k[\Delta]/\langle \Theta \rangle)_{d-i}$$

is an isomorphism.

Suppose that ω is a Lefschetz element for $k[\Delta]/\langle \Theta \rangle$. Then for $i \leq \lfloor d/2 \rfloor$ multiplication $\omega : (k[\Delta]/\langle \Theta \rangle)_{i-1} \to (k[\Delta]/\langle \Theta \rangle)_i$ must be an injection. Hence, for such i, $h_{i-1} \leq h_i$. In addition, we can see that $g_i = \dim_k (k[\Delta]/\langle \Theta, \omega \rangle)_i$, so the g-vector of Δ must be an M-vector. Hence, if $L(\Delta) = \{(\omega, \Theta) : \omega$ is a Lefschetz element for $k[\Delta]/\langle \Theta \rangle\}$ is nonempty, then Δ satisfies McMullen's conditions.

Theorem 4.21 ([40], [29]). *If Δ is the boundary of a simplicial polytope, then $L(\Delta)$ is nonempty.*

Perhaps the most important problem involving f-vectors is whether or not McMullen's conditions extend to simplicial spheres or even homology spheres. This question is sometimes referred to as the *g-conjecture*. As the above discussion shows, the following *algebraic g-conjecture* would imply the g-conjecture.

Conjecture 4.22. *If Δ is a homology sphere, then $L(\Delta) \neq \emptyset$.*

For a related, even stronger conjecture, see [45].

Definition 4.23. *A* Lefschetz sphere *is a homology sphere* Δ *such that* $L(\Delta) \neq \emptyset$. *A* Lefschetz ball *is a homology ball* Δ *which is a full dimensional subcomplex of a Lefschetz sphere.*

Unlike homology spheres, for an arbitrary homology manifold Δ we do not expect multiplication by a one-form in $k[\Delta]/\langle\Theta\rangle$ to be an injection in degrees larger than two. Indeed, by Theorem 4.17 the dimension of the kernel of multiplication $\omega : (k[\Delta]/\langle\Theta\rangle)_i \to (k[\Delta]/\langle\Theta\rangle)_{i+1}$ is at least $\binom{d-1}{i}\beta_{i-1}$. However, as we will now show, if "enough" links of the vertices of Δ are Lefschetz spheres or balls, then multiplication by a generic one-form is a surjection in the higher degrees.

Let $\rho = \{v_{m_1}, \ldots, v_{m_{|\rho|}}\}$ be an ordered face of Δ, and let $\sigma = \{v_{m_1}, \ldots, v_{m_{|\rho|}}, v_{m_{|\rho|+1}}, \ldots, v_{m_d}\}$ be a facet containing ρ. Suppose Θ is a l.s.o.p. for $k[\Delta]$ with corresponding matrix $T = \theta_{i,j}$. Then there is a unique set $\Theta' = \{\theta'_1, \ldots, \theta'_d\}$ of one-forms such that $\langle\Theta\rangle = \langle\Theta'\rangle$ and for any i, $1 \le i \le d$,

$$\theta'_{i,j} = \begin{cases} 1, & j = m_i, \\ 0, & j = m_l,\ 0 \le l \le d,\ l \ne i. \end{cases}$$

Indeed, T' corresponds to the reduced row echelon form of T with pivot columns $\{m_1, \ldots, m_d\}$. For future reference, we note that the $\theta'_{i,j}$ are rational functions of the $\theta_{i,j}$.

In order to use Θ' as a l.s.o.p. for $k[\text{lk }\rho]$, let R^ρ be the polynomial ring over k with variables $\{x_i\}_{v_i \notin \rho}$. For each θ'_i, let θ^ρ_i be the one-form in R^ρ obtained from θ'_i by removing all the variables corresponding to vertices in ρ. Equivalently, θ^ρ_i is the image of θ'_i under the natural surjection from R to R^ρ. It is now easy to check that $\Theta^\rho = \{\theta^\rho_{|\rho|+1}, \ldots, \theta^\rho_d\}$ is a l.s.o.p. for $k[\text{lk }\rho]$. While these definitions depend on the choice of facet $\sigma \supseteq \rho$, we will suppress this dependence as it will not matter.

We intend to analyze ideals of the form $\langle x_\rho\rangle \subseteq k[\Delta]/\langle\Theta\rangle$, where $x_\rho = x_{m_1} \cdots x_{m_{|\rho|}}$, by using $k[\text{lk }\rho]/\langle\Theta^\rho\rangle$. This requires us to give $k[\text{lk }\rho]/\langle\Theta^\rho\rangle$ an R-module structure. It is sufficient to describe $x_{m_i} \cdot q(\mathbf{x})$ for $q(\mathbf{x}) \in k[\text{lk }\rho]/\langle\Theta^\rho\rangle$. For each $v_{m_i} \in \rho$ the construction of Θ' forces θ'_{m_i} to be of the form

$$\theta'_{m_i} = x_{m_i} + \sum_{v_j \notin \sigma} \theta'_{m_i,j} x_j.$$

So we define

$$x_{m_i} \cdot q(\mathbf{x}) = \left(-\sum_{v_j \notin \sigma} \theta'_{m_i,j} x_j\right) \cdot q(\mathbf{x}). \tag{25}$$

As θ'_i is in Θ, this definition ensures that multiplication by x_ρ is an R-module homomorphism of degree $|\rho|$ from $k[\text{lk }\rho]/\langle\Theta^\rho\rangle$ to the ideal $\langle x_\rho\rangle$ in $k[\Delta]/\langle\Theta\rangle$. Indeed, if R^ρ is given an R-module structure in the same way as $k[\text{lk }\rho]$, then there is a commutative diagram of R-modules:

$$
\begin{array}{ccc}
R^\rho/\langle\Theta^\rho\rangle & \xrightarrow{\cdot x_\rho} & \langle x_\rho\rangle(R/\langle\Theta'\rangle) \\
\downarrow & & \downarrow \\
k[\text{lk }\rho]/\langle\Theta^\rho\rangle & \xrightarrow{\cdot x_\rho} & \langle x_\rho\rangle(k[\Delta]/\langle\Theta\rangle)
\end{array}
$$

Proposition 4.24. *Let Δ be a homology manifold. Then the multiplication map*

$$x_\rho : k[\text{lk}\,\rho]/\langle\Theta^\rho\rangle \to \langle x_\rho\rangle(k[\Delta]/\langle\Theta\rangle)$$

is a surjective graded R-module homomorphism of degree $|\rho|$. If Δ is a homology manifold without boundary, then x_ρ is an isomorphism unless $i = d$ and Δ is not orientable.

Proof. Evidently, the map has degree equal to the cardinality of ρ. To see that the map is surjective, let $x_\rho \cdot q(\mathbf{x}) \in \langle x_\rho\rangle$. Replace each occurrence of a variable x_{m_i} in $q(\mathbf{x})$ using (25). This leaves a polynomial which is clearly in the image of multiplication by x_ρ.

In order to show that multiplication is an isomorphism when Δ is a homology manifold without boundary and either Δ is orientable or $i \neq d$, it is sufficient to show that the dimensions over k agree. Since Δ has no boundary, the link of ρ is a homology sphere, so $\dim_k(k[\text{lk}\,\rho]/\langle\Theta^\rho\rangle)_i$ is $h_i(\text{lk}\,\rho)$. To compute $\dim_k\langle x_\rho\rangle_{i+|\rho|}$, consider the exact sequence

$$0 \to \langle x_\rho\rangle \to k[\Delta]/\langle\Theta\rangle \to k[\Delta - \rho]/\langle\Theta\rangle \to 0,$$

where $\Delta - \rho$ is Δ with ρ and any incident faces removed. Since $\Delta - \rho$ is a homology manifold with boundary, Schenzel's formula says

$$
\begin{aligned}
\dim_k\langle x_\rho\rangle_{i+|\rho|} &= \dim_k(k[\Delta]/\langle\Theta\rangle)_{i+|\rho|} - \dim_k(k[\Delta - \rho]/\langle\Theta\rangle)_{i+|\rho|} \\
&= h'_{i+|\rho|}(\Delta) - h'_{i+|\rho|}(\Delta - \rho).
\end{aligned}
$$

The Mayer–Vietoris sequence for $\Delta = (\Delta - \rho) \cup \overline{\text{st}}\,\rho$ shows that if Δ is orientable or $j \neq d - 2$, $\beta_j(\Delta) = \beta_j(\Delta - \rho)$. Hence, $\dim_k\langle x_\rho\rangle_{i+|\rho|} = h_{i+|\rho|}(\Delta) - h_{i+|\rho|}(\Delta - \rho)$. This difference is the coefficient of $t^{d-i-|\rho|}$ in

$$\sum_{\substack{\phi\in\overline{\text{st}}\,\rho \\ \phi\notin\partial\,\overline{\text{st}}\,\rho}} (t - 1)^{d-|\phi|}.$$

This is known to be $h_{d-i-|\rho|}(\overline{\text{st}}\,\rho)$ [37, Lemma 2.3]. Since the h-vector of a cone is the h-vector of the original space, $h_{d-i-|\rho|}(\overline{\text{st}}\,\rho) = h_{d-i-|\rho|}(\text{lk}\,\rho)$. As the dimension of the link of ρ is $d - 1 - |\rho|$, the generalized Dehn–Sommerville equations show that this is $h_i(\text{lk}\,\rho)$. $\qquad\square$

Define $L_s^i(\Delta)$ to be the set of all pairs (ω, Θ) such that $\omega \in R_1$, Θ is a l.s.o.p. for Δ, and multiplication

$$\omega : (k[\Delta]/\langle\Theta\rangle)_i \to (k[\Delta]/\langle\Theta\rangle)_{i+1}$$

is a surjection. If Δ is a homology sphere, then $L(\Delta) \subseteq L_s^{\lfloor d/2\rfloor}(\Delta)$.

Proposition 4.25. *If Δ is a homology manifold and $L_s^i(\Delta) \neq \emptyset$, then for all j with $i \leq j \leq d - 1$,*

$$h'_j \geq h'_{j+1} + \binom{d-1}{j}\beta_{j-1}.$$

Proof. Let $(\omega, \Theta) \in L_s^i(\Delta)$. Then for any $j, i \leq j \leq d - 1$, multiplication $\omega : (k[\Delta]/\langle\Theta\rangle)_j \to (k[\Delta]/\langle\Theta\rangle)_{j+1}$ is a surjection with a kernel whose dimension is at least $\binom{d-1}{j}\beta_{j-1}$. $\square$

Theorem 4.26. *Suppose Δ is a k-homology manifold with k an infinite field. If for at least $n - d$ of the vertices v of Δ, $L_s^i(\mathrm{lk}\, v) \neq \emptyset$, then $L_s^{i+1}(\Delta) \neq \emptyset$.*

Proof. Let $V_s = \{v_j\}_{j=1}^{n-d}$ be vertices of Δ such that for every j, $L_s^i(\mathrm{lk}\, v_j)$ is nonempty. For each such vertex v, consider the set of pairs

$$L_s^i(v) = \{(\omega, \Theta) : \Theta \text{ is a l.s.o.p. for } k[\Delta], \text{ and } (\omega, \Theta^{\{v\}}) \in L_s^i(\mathrm{lk}\, v)\}.$$

Since $\Theta \mapsto \Theta^{\{v\}}$ is a rational map and $L_s^i(\mathrm{lk}\, v)$ is a nonempty Zariski open set [45], $L_s^i(v)$ is a nonempty Zariski open set. We call Θ *generic* if every $d \times d$ minor of the associated matrix T is nonsingular. To finish the proof we show that

$$L = \{(\omega, \Theta) : \Theta \text{ is generic}\} \cap \bigcap_{j=1}^{n-d} L_s^i(v_j)$$

is a nonempty subset of $L_s^{i+1}(\Delta)$.

Since each of the intersecting subsets in L is a nonempty open Zariski subset of $k^{(d+1)n}$, L is nonempty. So let $(\omega, \Theta) \in L$. In order to see that multiplication $\omega : (k[\Delta]/\langle\Theta\rangle)_{i+1} \to (k[\Delta]/\langle\Theta\rangle)_{i+2}$ is surjective it is sufficient to show that every monomial in $(k[\Delta]/\langle\Theta\rangle)_{i+2}$ is in the image. We consider two cases.

Case 1: The monomial can be written in the form $x_j \cdot x^\alpha$, where $v_j \in V_s$. Using the fact that x^α is in the image of multiplication by ω in $k[\mathrm{lk}\, v_j]/\langle\Theta^{\{v_j\}}\rangle$, and Proposition 4.24, we see that this monomial is in the image of multiplication by ω in $\langle x_j\rangle \subseteq k[\Delta]/\langle\Theta\rangle$.

Case 2: All of the variables in the monomial correspond to vertices not in V_s. Write the monomial $x_l \cdot x^\alpha$. Since Θ is generic, it contains an element θ of the form

$$\theta = x_l + \sum_{j, v_j \in V_s} \theta_j x_j.$$

This implies that the monomial is equivalent to a sum of monomials from Case 1 and hence is in the image of multiplication by ω. $\square$

The above results suggest the following conjectures for homology manifolds (with or without boundary).

Conjecture 4.27 (Manifold algebraic g-conjecture). *If Δ is a homology manifold, then $L_s^i(\Delta) \neq \emptyset$ for $i \geq \lceil d/2 \rceil$.*

Conjecture 4.28 (Manifold g-conjecture). *If Δ is a homology manifold, then $h_i' \geq h_{i+1}' + \binom{d-1}{i}\beta_{i-1}$ for $i \geq \lceil d/2 \rceil$.*

If Kalai's conjecture concerning the kernel of multiplication by a one-form is correct (see discussion following Theorem 4.17), then the manifold g-conjecture would be $h'_i \geq h'_{i+1} + \binom{d}{i}\beta_{i-1}$. Figure 4 shows the interrelationships among the various g-conjectures. The dotted arrows indicate partial implications. The manifold algebraic g-conjecture does not imply the existence of Lefschetz elements for homology spheres. However, the surjective maps promised by the manifold algebraic g-conjecture, combined with the Gorenstein property of face rings of homology spheres, is enough to establish the injective maps needed to establish the g-conjecture. The manifold g-conjecture would ensure that g-vectors of homology spheres are nonnegative, but would not show that they are M-vectors.
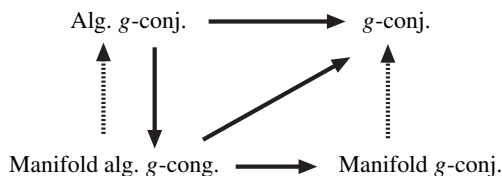

Fig. 4. Various g-conjectures.

Even without an affirmation of the algebraic g-conjecture, Theorem 4.26 can be used effectively to limit the possible h-vectors of homology manifolds.

Corollary 4.29. *If k has characteristic zero and Δ is a k-homology manifold, then $L_s^{d-2}(\Delta) \neq \emptyset$.*

Proof. Every two-dimensional homology sphere Δ' is the boundary of a simplicial 3-polytope, hence by Theorem 4.21, $L(\Delta') \neq \emptyset$. By [37], L_s^1 is nonempty for two-dimensional homology balls. Now apply induction and Theorem 4.26. $\square$

Theorem 4.30. *If Δ is a homology manifold and β_i are the rational Betti numbers of Δ, then*

$$h'_{d-2} \geq h'_{d-1} + (d-1)\beta_{d-3}. \tag{26}$$

If, in addition, Δ is closed, then

$$h_2 \geq h_1 + \binom{d+1}{2}\beta_1 - \binom{d-1}{2}\beta_2. \tag{27}$$

Furthermore, if Δ is closed, $d \geq 5$, $\beta_2 = 0$ and $h_2 = h_1 + \binom{d+1}{2}\beta_1$, then $\Delta \in \mathcal{H}^{d-1}$.

Proof. The first formula is an immediate consequence of Proposition 4.25 and the above corollary. So assume that Δ is closed and orientable. By Schenzel's formula, $h_1 = h'_1$ and $h_2 = h'_2$. On the other hand, by [31], $h'_{d-2} = h'_2 + \binom{d}{2}(\beta_2 - \beta_1)$ and $h'_{d-1} = h'_1 + d\beta_1$. Combining this with (26) and Poincaré duality gives (27).

Now suppose $\beta_2 = 0$, $h_2 = h_1 + \binom{d+1}{2}\beta_1$, $d \geq 5$ and Δ is closed. These conditions imply that $h'_{d-2} = h'_{d-1}$. This means that for $(\omega, \Theta) \in L_s^{d-2}(\Delta)$ multiplication $\omega : (k[\Delta]/\langle\Theta\rangle)_{d-2} \to (k[\Delta]/\langle\Theta\rangle)_{d-1}$ is a bijection. So it must be an injection

when considered as a map $\omega : \langle x_i \rangle_{d-2} \rightarrow \langle x_i \rangle_{d-1}$. By Proposition 4.24, multiplication $\omega : (k[\mathrm{lk}\, v_i]/\langle \Theta^{x_i} \rangle)_{d-2} \rightarrow (k[\mathrm{lk}\, v_i]/\langle \Theta^{x_i} \rangle)_{d-1}$ must be an injection. Hence, for any vertex v_i of Δ, $h_{d-2}(\mathrm{lk}\, v_i) \leq h_{d-1}(\mathrm{lk}\, v_i)$. Each such link is a homology sphere, so the generalized Dehn–Sommerville equations imply $h_2 \leq h_1$ in each vertex link. The rigidity inequality and Corollary 2.2 imply that every vertex link is a stacked sphere. Finally, Theorem 2.3 says that $\Delta \in \mathcal{H}^{d-1}$. $\qquad\square$

Kalai conjectured in [16] that for homology manifolds without boundary,

$$h_2 - h_1 \geq \binom{d+1}{2} \beta_1. \qquad (28)$$

The above theorem verifies this conjecture when $\beta_2 = 0$ and Δ is closed. Theorem 4.3 confirms (28) when $\beta_1 = 1$. If Kalai's conjecture concerning the lower bound for the dimension of the kernel of multiplication by a one-form is correct (see comment following Theorem 4.17), the suitably altered statement of Proposition 4.25 and proof of Theorem 4.30 would prove (28) for closed homology manifolds. In dimension four with $\beta_2 = 0$, (28) is equivalent to (22).

5. Constructions

In order to completely characterize the f-vectors of all possible triangulations of a given space, we will need ways of constructing new triangulations from old ones which preserve homeomorphism type and alter the f-vector in a predictable fashion. Two such techniques are bistellar moves and central retriangulations.

Let F and G be disjoint subsets of the vertices of Δ such that $|F| + |G| = d + 1$. Suppose that the vertex induced subcomplex of Δ with respect to $F \cup G$ is $F * \partial G$. Removing $F * \partial G$ and replacing it with $\partial F * G$ is a $(|G| - 1)$-*bistellar move*. A 0-bistellar move is also called subdividing a facet. If Δ' is obtained from Δ by a bistellar move, then Δ' is homeomorphic to Δ. The effect of a bistellar move on the h-vector is contained in the proposition below. We omit its elementary proof.

Proposition 5.1. *Suppose Δ' is obtained from Δ by an m-bistellar move. Then*

$$h_i(\Delta') = \begin{cases} h_i(\Delta), & i \leq m \text{ or } i \geq d - m, \\ h_i(\Delta) + 1, & m < i < d - m. \end{cases}$$

There are very few manifolds for which a complete characterization of all possible f-vectors of triangulations is known. Aside from S^1 and closed 2-manifolds [15] [33], the only other spaces for which this question has been solved are S^3, $S^1 \times S^2$, $\mathbb{R}P^3$, the nonorientable S^2 bundle over S^1, and S^4. The four 3-manifolds and S^4 were done by Walkup [47].

The first manifold we will consider in detail is $S^1 \times S^3$. For every $n \geq 11$, Kühnel and Lassmann constructed a vertex-transitive triangulation of $S^1 \times S^3$ with n vertices and dihedral symmetry [23]. We use $\Delta_{S^1 \times S^3}(n)$ to denote these complexes. (In [23]

they used $M_3^4(n)$ for these triangulations.) Identify the vertices with $[n] = \{1, \ldots, n\}$. Since the triangulation is invariant under the action of $\mathbb{Z}_n$, it is sufficient to specify for which 4-tuples (y_1, y_2, y_3, y_4) there are simplices of the form $\{x_1, x_2, x_3, x_4, x_5\}$, with $x_{i+1} - x_i = y_i \bmod n$, $1 \leq i \leq 4$. The 4-tuples which generate $\Delta_{S^1 \times S^3}(n)$ are $[1, 1, 1, 2]$, $[1, 1, 2, 1]$, $[1, 2, 1, 1]$, and $[2, 1, 1, 1]$. The link of each vertex has 10 vertices, so $f_1 = 5n$ and $h_2 = 5n - 10 - 4(n - 5) = n + 10$. So, $g_2 = h_2 - h_1 = 15$. By the generalized Dehn–Sommerville equations, the g-vector of a homology manifold without boundary determines its h-vector, and hence its f-vector.

Theorem 5.2. *The following are equivalent.*

  (i) $(1, g_1, g_2)$ *is the g-vector of a triangulation of* $S^1 \times S^3$.
 (ii) $(1, g_1, g_2)$ *is the g-vector of a closed four-dimensional homology manifold* Δ *with* $\beta_1 = 1$ *and* $\beta_2 = 0$.
(iii) $15 \leq g_2 \leq \binom{g_1 + 1}{2}$.

Proof. Evidently (i)$\Rightarrow$(ii). *All* simplicial complexes satisfy $g_2 \leq \binom{g_1 + 1}{2}$. For $S^1 \times S^3$ with $\beta_1 = 1$, Theorem 4.3 implies $15 \leq g_2$, hence (ii)$\Rightarrow$(iii).

For (iii)$\Rightarrow$(i), assume $h_2 \leq \binom{h_1 + 1}{2}$, $h_2 - h_1 \geq 15$, and $h_1 \geq 6$. As $h_1(\Delta_{S^1 \times S^3}(n)) = n - 5$ and $h_2(\Delta_{S^1 \times S^3}(n)) = n + 10$, it is sufficient to show that for each n, beginning with $\Delta_{S^1 \times S^3}(n)$, it is possible to perform consecutive 1-bistellar moves, each such move adding exactly one edge to the 1-skeleton, until the 1-skeleton is the complete graph on n vertices.

What are the nonedges of $\Delta_{S^1 \times S^3}(n)$? From the description of the facets, the link of $x \in [n]$ consists of all y within $\pm 5 \bmod n$ of x. Hence (x, y) is a nonedge of $\Delta_{S^1 \times S^3}(n)$ if and only if x and y are separated by at least 6 modulo n.

First, group the nonedges of $\Delta_{S^1 \times S^3}(n)$ by the value of $y - x \bmod n$, where we insist this difference be between 6 and $n - 1$. For instance, if $n = 14$, then the first group contains the pairs

$$(1, 7), (2, 8), \ldots, (7, 13), (8, 14), (9, 1), \ldots, (14, 6).$$

Similarly, the second group would contain

$$(1, 8), (2, 9), \ldots, (6, 13), (7, 14), (8, 1), \ldots, (14, 7).$$

In general, if n is odd, then the nonedges will partition into blocks each of which has cardinality n, while if n is even, the last block will have cardinality $n/2$.

Starting with the first group, for each pair $(x, x + 6)$ perform a 1-bistellar move using the facets $\{x, x + 1, x + 2, x + 4, x + 5\}$ and $\{x + 1, x + 2, x + 4, x + 5, x + 6\}$. Now consider the pairs $(x, x + 7)$ in the second group. From the bistellar move applied to the pair $(x, x + 6)$, x is contained in a facet $\{x, x + 1, x + 2, x + 5, x + 6\}$, while the bistellar move applied to $(x + 1, x + 7)$ puts $x + 7$ in the facet $\{x + 1, x + 2, x + 5, x + 6, x + 7\}$. Hence, we can now perform a 1-bistellar move for each pair in the second group. Similarly, a bistellar move corresponding to $(x, x+8)$ in the third group can use the facets $\{x, x + 1, x + 2, x + 6, x + 7\}$ and $\{x + 1, x + 2, x + 6, x + 7, x + 8\}$ obtained via the bistellar move from the second group. Continuing in this way, it is possible to perform bistellar moves until the 1-skeleton is the complete graph on n vertices. $\square$

An examination of the proof shows that after kn 1-bistellar moves the resulting complex once again has dihedral vertex-transitive symmetry.

Corollary 5.3. *If $n \geq 11$, $k \geq 5$, and $kn \leq \binom{n}{2}$, then there is a vertex-transitive triangulation of $S^1 \times S^3$ with n vertices, kn edges and dihedral symmetry.*

When n is odd, this is the best result possible since any vertex-transitive triangulation will have kn edges for some k. For even n, any vertex-transitive triangulation must have $n + (kn/2)$ edges, and the catalog of small vertex-transitive triangulations by Köhler and Lutz [20] suggests that this may be possible once $n \geq 14$. The manifolds $^413_1^2$, $^413_2^2$, $^414_2^3$, $^414_3^3$, $^415_1^2$, $^415_2^2$, and $^415_3^2$ in [20] come from the above construction.

As indicated previously, the complexes $\Delta_{S^1 \times S^3}(n)$ are part of a much larger family of triangulations introduced by Kühnel and Lassmann in [23]. All of their spaces are disk or sphere bundles over tori. They include a collection $\Delta_{S^1 \times S^{2m-1}}(n)$, denoted by $M_{2m-1}^{2m}(n)$ in [23], of triangulations of $S^1 \times S^{2m-1}$ for $m \geq 2$ and $n \geq 4m + 3$. When $n = 4m + 3$, $\Delta_{S^1 \times S^{2m-1}}(n) = M^{2m}$, which we recall from Theorem 4.7, is a minimum triangulation of $S^1 \times S^{2m-1}$. The construction of $\Delta_{S^1 \times S^{2m-1}}(n)$ is along the same lines as $\Delta_{S^1 \times S^3}(n)$. The facets are specified by the $2m$ difference vectors (modulo n) obtained by all possible permutations of $[1, 1, \ldots, 1, 2]$. The resulting complex has dihedral symmetry. Using 1-bistellar moves in a fashion similar to above, and Theorem 4.3, it is possible to prove the following.

Proposition 5.4. *Let $m \geq 2$. There exists a triangulation of $S^1 \times S^{2m-1}$ with n vertices and e edges if and only if $n \geq 4m + 3$ and $e - (2m + 1)n \geq 0$. If in addition e is a multiple of n, then there exists a triangulation which is vertex-transitive with dihedral symmetry.*

A second technique for creating new triangulations out of old ones is the central retriangulation of a simple $(d - 1)$-tree. Let B be a subcomplex of Δ which is a simplicial ball. Remove all of the interior faces of B and replace them with the interior faces of the cone on the boundary of B, where the cone point is a new vertex. We call this new complex the *central retriangulation* of B. See Figure 5 for a simple example in dimension two. If Δ' is obtained from Δ by a retriangulation of B, then Δ and Δ' are homeomorphic.
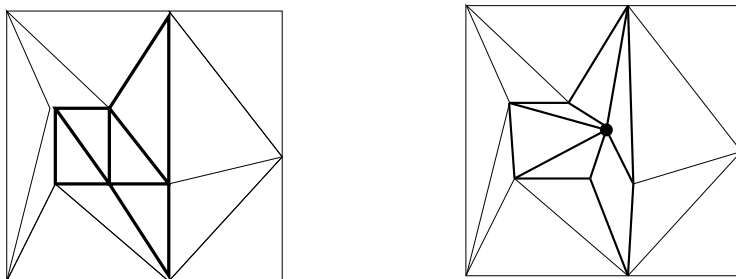

Fig. 5. Central retriangulation of a simple 2-tree.

Let T be a pure $(d-1)$-dimensional complex. We say T is a *simple $(d-1)$-tree* if the facets of T can be ordered, $\sigma_1, \ldots, \sigma_m$, so that for each $i \geq 2$ the intersection of σ_i with the union of all previous facets is a codimension one face of σ_i which is on the boundary of $\bigcup_{j=1}^{i-1} \sigma_j$. A simple 1-tree is a path. The bold face subcomplex on the left hand side of Figure 5 is a simple 2-tree. The *length* of a simple $(d-1)$-tree is the number of facets in the tree. Each facet, other than the first, adds exactly one new vertex to the tree. An ordering of the vertices, $(v_1, \ldots, v_{d+m-1})$, of a simple $(d-1)$-tree T is *natural* if there exists an ordering of the facets of T such that the vertices of σ_1 are $(v_1, \ldots, v_d)$ and for $i \geq 2$, the new vertex introduced by σ_i is v_{d+i-1}. Any simple $(d-1)$-tree is a simplicial ball and its boundary is a stacked sphere.

Proposition 5.5. *If Δ' is obtained from Δ by central retriangulation of a simple $(d-1)$-tree of length m, then $h_1(\Delta') = h_1(\Delta) + 1$, and $h_2(\Delta') = h_2(\Delta) + m$.*

The following idea is due to Walkup. Indeed, our statement and proof are just the $(d-1)$-dimensional analogue of [47, Lemma 7.3]. A simple $(d-1)$-tree in Δ is *spanning* if it contains all of the vertices of Δ.

Proposition 5.6. *Let Δ be a 2-neighborly triangulation of a homology manifold without boundary. Suppose Δ contains a spanning $(d-1)$-tree T such that every facet of T contains a fixed set of distinct vertices $\{v_1, \ldots, v_{d-3}\}$. Equivalently, there is a codimension three face ρ, and a spanning simple 2-tree in the link of ρ. Then for every pair (a, b) satisfying $a \geq h_1(\Delta)$ and $g_2(\Delta) + a \leq b \leq \binom{a+1}{2}$ there exists a complex Δ' which is homeomorphic to Δ with $h_1(\Delta') = a$ and $h_2(\Delta') = b$.*

Proof. Let $\Delta = \Delta_0$ and let Δ_1 be the complex obtained by a central retriangulation of T and let w_1 be the new vertex in Δ_1. As Δ was neighborly and T is spanning, Δ' is neighborly. For each $i > d-3$, $\{v_1, \ldots, v_{d-3}, v_i\}$ is a face of the boundary of T. Hence $\{w_1, v_1, \ldots, v_{d-3}, v_i\}$ is a face of Δ_1, so the link of $\{w_1, v_1, \ldots, v_{d-3}\}$ contains all of the other vertices of Δ_1. Since Δ_1 is a closed homology manifold, this link must be a circle. This implies that Δ_1 has a spanning simple $(d-1)$-tree T_1 and distinct vertices $\{v'_1, \ldots, v'_{d-2}\}$ which are contained in every facet of T_1. Repeating this process we obtain an infinite family of complexes Δ_k. For each k, Δ_k is homeomorphic to Δ, 2-neighborly, and $h_1(\Delta_k) = h_1(\Delta) + k$. In addition, each Δ_k has a spanning simple $(d-1)$-tree T_k. Given j with $1 \leq j \leq h_1(\Delta) + k + 1$, define $\Delta_{k,j}$ to be the complex obtained from Δ_k by a central retriangulation of the simple $(d-1)$-tree consisting of the the first j facets of T_k. By Proposition 5.5 the collection of pairs $(h_1(\Delta_{k,j}), h_2(\Delta_{k,j}))$ is exactly the pairs (a, b) guaranteed by the theorem. $\qquad\square$

Theorem 5.7. *The following are equivalent.*

(i) $(1, g_1, g_2)$ *is the g-vector of a triangulation of* $\mathbb{C}P^2$.
(ii) $(1, g_1, g_2)$ *is the g-vector of a triangulation of a closed four-dimensional homology manifold with* $\beta_1 = 0$ *and* $\beta_2 = 1$.
(iii) $6 \leq g_2 \leq \binom{g_1+1}{2}$.

Proof. As before, (i)⇒(ii) is trivial and any complex satisfies $g_2 \leq \binom{g_1+1}{2}$. Theorem 4.18 implies $6 \leq g_2$, and hence (ii)⇒(iii).

Table 1. $\Delta(\mathbb{C}P^2)$, a triangulation of $\mathbb{C}P^2$ with $h_1 = 4, h_2 = 10$ [22].

[1, 2, 3, 4, 5], [1, 2, 3, 4, 7], [1, 2, 3, 5, 8], [1, 2, 3, 7, 8],
[1, 2, 4, 5, 6], [1, 2, 4, 6, 7], [1, 2, 5, 6, 8], [1, 2, 6, 7, 9],
[1, 2, 6, 8, 9] [1, 2, 7, 8, 9] [1, 3, 4, 5, 9], [1, 3, 4, 7, 8],
[1, 3, 4, 8, 9], [1, 3, 5, 6, 8], [1, 3, 5, 6, 9], [1, 3, 6, 8, 9],
[1, 4, 5, 6, 7], [1, 4, 5, 7, 9], [1, 4, 7, 8, 9], [1, 5, 6, 7, 9],
[2, 3, 4, 5, 9], [2, 3, 4, 6, 7], [2, 3, 4, 6, 9], [2, 3, 5, 7, 8],
[2, 3, 5, 7, 9], [2, 3, 6, 7, 9], [2, 4, 5, 6, 8], [2, 4, 5, 8, 9],
[2, 4, 6, 8, 9], [2, 5, 7, 8, 9], [3, 4, 6, 7, 8], [3, 4, 6, 8, 9],
[3, 5, 6, 7, 8], [3, 5, 6, 7, 9], [4, 5, 6, 7, 8], [4, 5, 7, 8, 9]

Table 2. A spanning simple 2-tree in the link of [1, 2] in $\Delta(\mathbb{C}P^2)$.

[3, 4, 7], [3, 4, 5], [4, 5, 6], [5, 6, 8], [6, 8, 9]

For (iii)⇒(i), the previous proposition shows that it is sufficient to find a triangulation of $\mathbb{C}P^2$ with g-vector $(1, 3, 6)$ and simple 4-tree which satisfies the conditions of the previous proposition. Table 1 contains such a triangulation, originally due to Kühnel [22], and Table 2 shows an appropriate simple 2-tree in the link of an edge. □

A similar technique can be used to characterize g-vectors of triangulations of K3 surfaces. Any nonsingular quartic in $\mathbb{C}P^3$ is a *K3 surface*. While different quartics can lead to distinct complex structures, all K3 surfaces are homeomorphic. In fact, they are diffeomorphic (see, for instance, [12, Theorem 3.4.9]). As a closed four-manifold, every K3 surface is simply connected with second Betti number equal to twenty-two. Hence any triangulation satisfies $h_3 - h_2 = 220 = \binom{12}{3}$.

Theorem 5.8. *The following are equivalent.*

(i) $(1, g_1, g_2)$ *is the g-vector of a triangulation of a K3 surface.*
(ii) $(1, g_1, g_2)$ *is the g-vector of a triangulation of a closed four-dimensional homology manifold with $\beta_1 = 0, \beta_2 = 22$.*
(iii) $55 \leq g_2 \leq \binom{g_1+1}{2}$.

Proof. The proof of (i)⇒(ii) and (ii)⇒(iii) follows the same reasoning as in Theorem 5.7. In addition, as in that proof, (iii)⇒(i) will be established with the existence of a triangulation Δ_{K3} of a K3 surface with g-vector $(1, 10, 55)$ and an appropriate simple 4-tree. Such a triangulation was given by Casella and Kühnel [6]. We refer the reader to this reference for the triangulation and a verification of its g-vector. Figure 6, which is [6, Figure 1], shows the link of an edge in this triangulation and contains a spanning simple 2-tree in this link. The labeled vertices on the boundary of the hexagons are identified and the numbering of the triangles is an ordering for the facets of the tree. Coning this tree with the edge provides the desired 4-tree. □
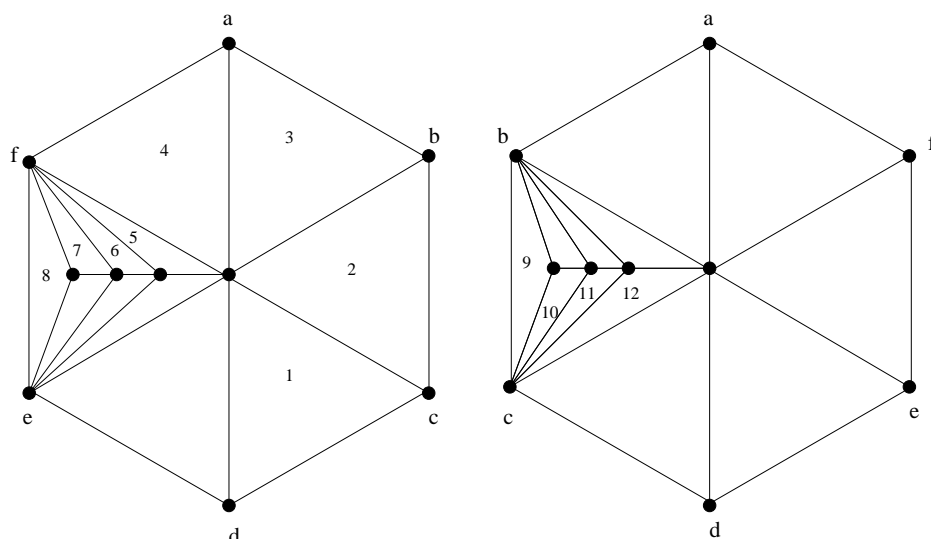
Fig. 6. Spanning 2-tree in the link of an edge of Δ_{K3}.

Theorem 5.9. *The following are equivalent.*

(i) $(1, g_1, g_2)$ *is the g-vector of a triangulation of the connected sum of $S^2 \times S^2$ with itself.*

(ii) $(1, g_1, g_2)$ *is the g-vector of a triangulation of a closed four-dimensional homology manifold with $\beta_1 = 0$ and $\beta_2 = 4$.*

(iii) $18 \leq g_2 \leq \binom{g_1+1}{2}$.

Proof. Using the same logic as the previous two theorems, $g_2 \geq 18$. Table 3 shows a triangulation with *g*-vector $(1, 6, 18)$. This triangulation comes from [25]. Every pair of vertices in this triangulation span an edge except for $\{1, 5\}$, $\{5, 6\}$ and $\{1, 6\}$. For g_2 equal to 18,19 or 20, first perform zero, one or two 1-bistellar moves from Table 4. Then subdivide facets until the desired number of vertices are obtained. As before, for $g_2 \geq 21$ Proposition 5.6 tells us that it is sufficient to find a spanning simple 2-tree in the link of some edge of a 2-neighborly triangulation of $(S^2 \times S^2) \# (S^2 \times S^2)$ with 12 vertices. After the three 1-bistellar moves in Table 4, the triangulation in Table 3 is 2-neighborly. Table 5 lists such a tree in the link of $[1, 2]$.   □

While the methods we have introduced are not sufficient to completely characterize the *h*-vectors of higher dimensional spaces, many partial results are possible. For instance, consider $S^3 \times S^3$.

Proposition 5.10. *The componentwise minimum h-vector for triangulations of $S^3 \times S^3$ is $(1, 6, 21, 56, 126, -21, 20, -1)$. There exists a triangulation Δ of $S^3 \times S^3$ with $h_1(\Delta) = a$ and $h_2(\Delta) = b$ if and only if $15 \leq b - a \leq \binom{a}{2}$.*

Table 3. $\Delta((S^2 \times S^2) \# (S^2 \times S^2))$, a triangulation of $(S^2 \times S^2) \# (S^2 \times S^2)$ with $h_1 = 7$ and $g_2 = 18$ [25].

[1, 2, 3, 4, 7],	[1, 2, 3, 4, 10],	[1, 2, 3, 7, 10],	[1, 2, 4, 7, 8],
[1, 2, 4, 8, 11],	[1, 2, 4, 9, 10],	[1, 2, 4, 9, 12],	[1, 2, 4, 11, 12],
[1, 2, 7, 8, 10],	[1, 2, 8, 9, 10],	[1, 2, 8, 9, 12],	[1, 2, 8, 11, 12],
[1, 3, 4, 7, 11],	[1, 3, 4, 10, 11],	[1, 3, 7, 8, 11],	[1, 3, 7, 8, 12],
[1, 3, 7, 9, 10],	[1, 3, 7, 9, 12],	[1, 3, 8, 11, 12],	[1, 3, 9, 10, 12],
[1, 3, 10, 11, 12],	[1, 4, 7, 8, 11],	[1, 4, 9, 10, 11],	[1, 4, 9, 11, 12],
[1, 7, 8, 9, 10],	[1, 7, 8, 9, 12],	[1, 9, 10, 11, 12],	[2, 3, 4, 6, 9],
[2, 3, 4, 6, 10],	[2, 3, 4, 7, 12],	[2, 3, 4, 9, 12],	[2, 3, 5, 7, 9],
[2, 3, 5, 7, 10],	[2, 3, 5, 8, 10],	[2, 3, 5, 8, 11],	[2, 3, 5, 9, 11],
[2, 3, 6, 9, 11],	[2, 3, 6, 10, 11],	[2, 3, 7, 9, 12],	[2, 3, 8, 10, 11],
[2, 4, 5, 7, 8],	[2, 4, 5, 7, 12],	[2, 4, 5, 8, 11],	[2, 4, 5, 11, 12],
[2, 4, 6, 9, 10],	[2, 5, 7, 8, 10],	[2, 5, 7, 9, 11],	[2, 5, 7, 11, 12],
[2, 6, 7, 9, 11],	[2, 6, 7, 9, 12],	[2, 6, 7, 11, 12],	[2, 6, 8, 9, 10],
[2, 6, 8, 9, 12],	[2, 6, 8, 10, 12],	[2, 6, 10, 11, 12],	[2, 8, 10, 11, 12],
[3, 4, 5, 8, 9],	[3, 4, 5, 8, 12],	[3, 4, 5, 9, 12],	[3, 4, 6, 7, 11],
[3, 4, 6, 7, 12],	[3, 4, 6, 8, 9],	[3, 4, 6, 8, 12],	[3, 4, 6, 10, 11],
[3, 5, 7, 9, 10],	[3, 5, 8, 9, 11],	[3, 5, 8, 10, 12],	[3, 5, 9, 10, 12],
[3, 6, 7, 8, 11],	[3, 6, 7, 8, 12],	[3, 6, 8, 9, 11],	[3, 8, 10, 11, 12],
[4, 5, 7, 8, 10],	[4, 5, 7, 10, 12],	[4, 5, 8, 9, 11],	[4, 5, 8, 10, 12],
[4, 5, 9, 11, 12],	[4, 6, 7, 10, 11],	[4, 6, 7, 10, 12],	[4, 6, 8, 9, 10],
[4, 6, 8, 10, 12],	[4, 7, 8, 9, 10],	[4, 7, 8, 9, 11],	[4, 7, 9, 10, 11],
[5, 7, 9, 10, 11],	[5, 7, 10, 11, 12],	[5, 9, 10, 11, 12],	[6, 7, 8, 9, 11],
[6, 7, 8, 9, 12],	[6, 7, 10, 11, 12]		

Table 4. Three 1-bistellar moves on $\Delta((S^2 \times S^2) \# (S^2 \times S^2))$.

([1, 2, 3, 7, 10], [2, 3, 5, 7, 10])

([2, 3, 5, 9, 11], [2, 3, 6, 9, 11])

([1, 2, 4, 9, 10], [2, 4, 6, 9, 10])

Table 5. A spanning simple 2-tree in the link of $\{1, 2\}$ in $\Delta((S^2 \times S^2) \# (S^2 \times S^2))$ after three 1-bistellar moves.

{[3, 5, 10], [5, 7, 10], [7, 8, 10], [8, 9, 10], [8, 9, 12], [4, 9, 12], [4, 6, 9], [4, 11, 12]}

Proof. Let Δ be a triangulation of $S^3 \times S^3$. By Schenzel's formula (see Theorem 4.14) $h_i = h_i'$ for $i = 1, 2, 3$ or 4. The generalized Dehn–Sommerville equations imply that $h_4' - h_3' = 70 = \binom{8}{4}$. Hence, $h_4' = h_4 \geq \binom{9}{4}$, $h_3' = h_3 \geq \binom{8}{3}$, $h_2' = h_2 \geq \binom{7}{2}$ and $h_1' = h_1 \geq \binom{6}{1}$. In [26, Section 7], Lutz gives a triangulation of $S^3 \times S^3$ with this h-vector. In addition, the link of the face denoted by [1, 2, 3, 4] in that triangulation has a spanning simple 2-tree, so Proposition 5.6 applies. □

In [47] Walkup proved that for any closed three-manifold M there exists $\gamma(M)$ such that for any pair (a, b) with $\gamma(M) \leq b \leq \binom{a+1}{2}$, there exists a triangulation Δ of M with

$g_1(\Delta) = a$ and $g_2(\Delta) = b$. In fact, this is true for all closed homology manifolds of dimension at least three which can be triangulated. We prove this in a series of lemmas which are an adaptation of Walkup's proof to higher dimensions.

Lemma 5.11. *Let M be a connected homology manifold without boundary of dimension $d - 1$. If M has a triangulation, then M has a triangulation which contains a spanning simple $(d - 1)$-tree.*

Proof. Let Δ be a triangulation of T. Since Δ is connected, there exists a simple $(d-1)$-tree T and a dimension preserving simplicial map $\phi : T \to \Delta$ which maps surjectively onto the vertices of Δ. Indeed, T and ϕ can be constructed inductively by beginning with a facet of Δ and attaching new facets to T along codimension one faces corresponding to free codimension one faces of the image of T until all of the vertices of Δ are in the image of ϕ.

If ϕ is one-to-one on the vertices of T, then the image of T satisfies the conclusion of the lemma. Otherwise, let $y_1, \ldots, y_s$ be a natural ordering of the vertices of T. Let y_t be the last vertex of T such that $|\phi^{-1}(\phi(y))| \geq 2$. The definition of y_t implies that ϕ is a simplicial isomorphism when restricted to the closed star of y_t. Hence B, the image of $\overline{\mathrm{st}}(y_t)$, is a ball. Now let Δ' be the complex obtained from Δ by the central retriangulation of B. Define a new map ϕ' by $\phi'(y_i) = \phi(y_i)$, except $\phi(y_t) = w$, where w is the new vertex of Δ'. To see that ϕ' induces a simplicial map, it is sufficient to note that if $\phi(y_t) \cup \rho$ is a facet of Δ which includes $\phi(y_t)$, then ρ is a face on the boundary of B, so $w \cup \rho$ is a face of Δ'. Thus, $\phi' : T \to \Delta'$ is a simplicial map which also maps surjectively onto the vertices of Δ', is one-to-one on all the vertices after y_t, and $|\phi'^{-1}(\phi'(y_t))| = |\phi^{-1}(\phi'(y_t))| - 1$. Repeating this procedure enough times gives the desired triangulation and $(d - 1)$-tree. See Figure 7 for a two-dimensional portrayal. $\square$

Lemma 5.12. *Let M be a connected homology manifold without boundary of dimension $d - 1$. If M has a triangulation, then M has a triangulation Δ which contains a spanning simple $(d - 1)$-tree T in which every facet of T contains a fixed $(d - 3)$-dimensional face.*

Proof. By the previous lemma we can choose a triangulation Δ_0 of M and a spanning simple $(d - 1)$-tree T_0 of Δ_0. Let Δ_1' be the complex obtained from the central retriangulation of T_0 and let w_1 be the new vertex. The link of w_1 contains all of the other vertices of Δ_1'. Now we repeat the procedure used in the proof of the previous lemma. However, the original tree T and ϕ are constructed by first choosing a simple $(d - 2)$-tree $\tilde{T}_1$ and simplicial map $\tilde{\phi}_1 : \tilde{T}_1 \to \mathrm{lk}\, w_1$ such that all of the vertices in the link of w_1 are in the image of $\tilde{\phi}_1$. Then let $T_1 = \{w_1\} * \tilde{T}_1$ and ϕ_1 be $\tilde{\phi}_1$ extended to T_1 by setting $\phi_1(w_1) = w_1$. Now each facet of the image of ϕ_1 contains w_1 and $|\phi_1^{-1}(w_1)| = 1$. Hence, while retriangulating Δ_1' for the purposes of forcing ϕ_1 to be one-to-one on the vertices, w_1 will remain in all of the facets in the image. At the end of this process we will have a triangulation Δ_1 and spanning simple $(d - 1)$-tree T_1 all of whose facets contain w_1. If $d = 4$ we are done. Otherwise, let Δ_2' be the complex obtained from Δ_1 by the central retriangulation of T_1 and let w_2 be the new vertex. Repeat this process beginning with a simple $(d - 3)$-tree $\tilde{T}_2$ and a vertex spanning simplicial map $\tilde{\phi}_2$ from $\tilde{T}_2$ into the link of
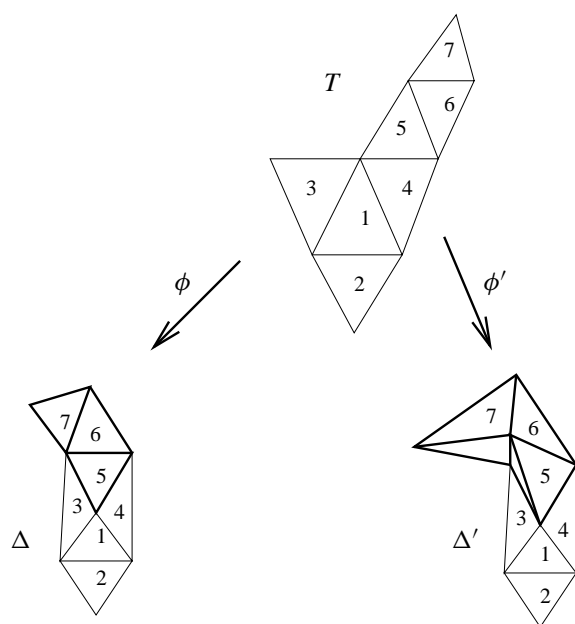
Fig. 7. Creating a spanning simple $(d-1)$-tree.

$\{w_1, w_2\}$. Arguing as before we will end up with a simple $(d-1)$-tree T_2 in Δ_2 all of whose facets contain $\{w_1, w_2\}$. This process can be repeated until we obtain the promised triangulation and spanning simple $(d-1)$-tree. $\square$

Lemma 5.13. *Let M be a connected homology manifold without boundary of dimension $d-1$. If M has a triangulation, then M has a 2-neighborly triangulation which contains a spanning simple $(d-1)$-tree T and a codimension three face σ such that σ is in every facet of T.*

Proof. Using the previous lemma, we begin with a triangulation Δ_0 and spanning simple $(d-1)$-tree T_0 every facet of which contains the codimension three face ρ_0. If Δ_0 is 2-neighborly we are done. So suppose Δ_0 has m pairs of vertices which do not have an edge between them. Let Δ_1 be the complex obtained by the central retriangulation of T_0 and let w_1 be the new vertex. As in the proof of Proposition 5.6, the link of $\rho_0 \cup \{w_1\}$ contains all of the other vertices of Δ_1. So Δ_1 has the same m pairs of vertices without edges. Induction will complete the proof if we can construct a triangulation Δ and face $\rho \in \Delta$ which satisfy the hypothesis of the lemma such that Δ has only $m-1$ pairs of vertices with no edge between them.

Let x, y be vertices in Δ_1 with no edge between them. Since $\rho_0 \cup \{w_1\}$ has codimension two, its link is a circle. As x and y do not have an edge between them, they must be in the link of $\rho_0 \cup \{w_1\}$ and be separated by at least one vertex as one travels around the circle. The construction of Δ and ρ consists of three steps.

Step 1: Retriangulate so that x and y are only separated by one vertex in the link of a codimension two face whose link contains all of the other vertices. Write the vertices in the link of $\rho_0 \cup w_1$ in cyclic order, $(x, v_1, \ldots, v_s, y, u_1, \ldots, u_t)$. If either s or t is one, then Step 1 is complete. If not, let P_1 be the path with ordered vertices $(x, v_1, \ldots, v_s, y)$ and let P_2 be the path with ordered vertices $(v_s, y, u_1, \ldots, u_t)$. Choose $w_0 \in \rho_0$ and set ρ_0' to be ρ_0 with w_0 removed. If $d = 4$, then $\rho_0' = \emptyset$. Set $S = ((\{w_1\} \cup \rho_0') * P_1) \cup ((\{w_0\} \cup \rho_0') * P_2)$ and $T = (\rho_0 \cup w_1) * (P_1 \cup P_2)$. Now T is a spanning simple $(d-1)$-tree in Δ_1 and S is a spanning simple $(d-2)$-tree in the boundary of T. Every facet of S contains ρ_0'. Figure 8 shows S in the link of ρ_0'.
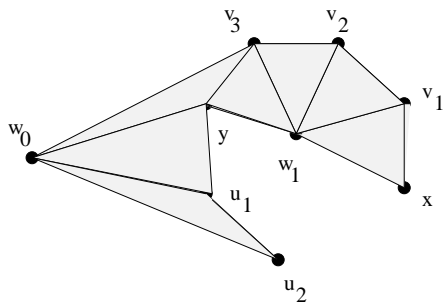

Fig. 8. The tree S in the link of ρ_0'.

Let Δ_2 be the complex obtained from Δ_1 by the central retriangulation of T and let w_2 be the new vertex. In this complex $S * \{w_2\}$ is a spanning simple $(d-1)$-tree. So now we let Δ_3 be the complex obtained from Δ_2 by the central retriangulation of $S * \{w_2\}$ and call the new vertex w_3. Using Figure 8, we can see that the link of $\{w_2, w_3\} * \rho_0'$ in Δ_3 is the circle $(y, w_1, x, v_1, \ldots, v_s, w_0, u_t, u_{t-1}, \ldots, u_1)$. Note that in each of the retriangulations the pairs of vertices without edges have not changed.

Step 2: Perform a 1-bistellar move on $\partial(\{x, y\}) * (\{w_1, w_2, w_3\} * \rho_0')$ and call the resulting complex Δ_4. This introduces an edge between x and y and leaves $m - 1$ pairs of vertices without edges. However, w_1 is no longer in the link of $\{w_2, w_3\} * \rho_0'$.

Step 3: Let P_4 be the path whose vertices in order are $(x, y, u_1, \ldots, u_t, w_0, v_s, \ldots, v_1)$. Form a new $(d-2)$-tree, $\tilde{S}_4 = \rho_0' * ((\{w_2\} * P_4) \cup \{x, w_1, y\})$. Now $T_4 = \{w_3\} * \tilde{S}_4$ is a spanning simple $(d-1)$-tree in Δ_4. In addition, $S_4 = \tilde{S}_4 \cup \{\{y, w_1, w_3\} * \rho_0'\}$ is a spanning simple $(d-2)$-tree in the boundary of T_4. Two more triangulations will finish the job. First, Δ_5 is the complex obtained from Δ_4 by the central retriangulation of T_4 with new vertex w_4. In this complex $T_5 = \{w_4\} * S_4$ is a spanning simple $(d-1)$-tree. Finally, set Δ to be the complex obtained from the central retriangulation of T_5 with new vertex w_5, and let $\rho = \{w_4, w_5\} \cup \rho_0'$. Here we see that the link of the codimension two face ρ contains all the other vertices of Δ and there are only $m - 1$ pairs of vertices that do not span an edge. $\square$

Theorem 5.14. *Let M be a connected homology manifold without boundary of dimension $d-1$. Then there exists $\gamma(M)$ such that for every pair (a, b) with $\gamma(M) \leq b \leq \binom{a+1}{2}$ there exists a triangulation Δ of M with $g_1(\Delta) = a$ and $g_2(\Delta) = b$.*

Proof. Apply Proposition 5.6 to the triangulation guaranteed by Lemma 5.13. □

Corollary 5.15. *Let M be a closed homology manifold of dimension at least three. If M has a triangulation, then for n sufficiently large there exist 2-neighborly triangulations of M with n vertices.*

As mentioned before, our proof of the existence of 2-neighborly triangulations is based on Walkup's proof of this for 3-manifolds [47]. A very different approach in dimension three is Sarkaria's proof of the existence of 2-neighborly triangulations of 3-manifolds with or without boundary [34].

References

[1] Bayer, M., Billera, L.: Generalized Dehn–Sommerville relations for polytopes, spheres and Eulerian partially ordered sets. Invent. Math. **79**, 143–157 (1985)   Zbl 0543.52007   MR 0774533

[2] Bayer, M., Klapper, A.: A new index for polytopes. Discrete Comput. Geom. **6**, 33–47 (1991)   Zbl 0761.52009   MR 1073071

[3] Billera, L., Lee, C.: A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial polytopes. J. Combin. Theory Ser. A **31**, 237–255 (1981)   Zbl 0479.52006   MR 0635368

[4] Billera, L., Magurn, K.: Balanced subdivision and enumeration in balanced spheres. Discrete Comput. Geom. **2**, 297–317 (1987)   Zbl 0633.05023   MR 0892174

[5] Brehm, U., Kühnel, W.: Combinatorial manifolds with few vertices. Topology **26**, 465–473 (1987)   Zbl 0681.57009   MR 0919730

[6] Casella, M., Kühnel, W.: A triangulated K3 surface with the minimum number of vertices. Topology **40**, 753–772 (2001)   Zbl 0981.57011   MR 1851562

[7] Chestnut, J., Sapir, J., Swartz, E.: Enumerative properties of triangulations of spherical bundles over S^1. Eur. J. Combin. **29**, 662–671 (2008)   Zbl 1137.51011   MR 2397347

[8] Danilov, V. I.: The geometry of toric varieties. Russian Math. Surveys **33**, 97–154 (1978)   Zbl 0425.14013   MR 0495499

[9] Datta, B.: Minimal triangulations of manifolds. J. Indian Inst. Sci. **87**, 429–449 (2007)   MR 2392107

[10] Dehn, M.: Die Eulersche Formel im Zusammenhang mit dem Inhalt in der nicht-Euklidischen Geometrie. Math. Ann. **61**, 561–568 (1905) JFM 37.0492.01   MR 1511363

[11] Ehrenborg, R.: k-Eulerian posets. Order **18**, 227–236 (2001)   Zbl 1010.06005   MR 1867234

[12] Gompf, R., Stipsicz, A.: 4-Manifolds and the Kirby Calculus. Grad. Stud. Math. 20, Amer. Math. Soc. (1999)   Zbl 0933.57020   MR 1707327

[13] Gromov, M.: Partial Differential Relations. Springer (1986)   Zbl 0651.53001   MR 0864505

[14] Hersh, P., Novik, I.: A short simplicial *h*-vector and the upper bound theorem. Discrete Comput. Geom. **28**, 283–289 (2002)   Zbl 1015.52007   MR 1923232

[15] Jungerman, M., Ringel, G.: Minimal triangulations of orientable surfaces. Acta Math. **145**, 121–154 (1980)   Zbl 0451.57005   MR 0586595

[16] Kalai, G.: Rigidity and the lower bound theorem. I. Invent. Math. **88**, 125–151 (1987)   Zbl 0624.52004   MR 0877009

[17] Kalai, G.: Algebraic shifting. In: Computational Commutative Algebra and Combinatorics, Adv. Stud. Pure Math. 33, Math. Soc. Japan, Tokyo, 121–163 (2002)   Zbl 1034.57021   MR 1890098

[18] Karu, K.: The cd-index of fans and posets. Compos. Math. **142**, 701–718 (2006)   Zbl 1103.14029   MR 2231198

[19] Klee, V.: A combinatorial analogue of Poincaré's duality theorem. Canad. J. Math. **16**, 517–531 (1964)   Zbl 0134.42403   MR 0189039

[20] Köhler, E. G., Lutz, F.: Triangulated manifolds with few vertices: Vertex-transitive triangulations I. 2005. arXiv:math.GT/0506520.

[21] Kühnel, W.: Higher dimensional analogues of Császár's torus. Result. Math. **9**, 95–106 (1986)   Zbl 0552.52003   MR 0841439

[22] Kühnel, W., Lassmann, G.: The unique 3-neighborly 4-manifold with few vertices. J. Combin. Theory Ser. A **35**, 173–184 (1983)   Zbl 0526.52008   MR 0712103

[23] Kühnel, W., Lassmann, G.: Permuted difference cycles and triangulated sphere bundles. Discrete Math. **162**, 215–227 (1996)   Zbl 0866.52011   MR 1425789

[24] Lee, C.: Generalized stress and motions. In: T. Bisztricky et al. (eds.), Polytopes: Abstract, Convex and Computational, Kluwer, 249–271 (1994)   Zbl 0809.52014   MR 1322066

[25] Lutz, F.: The manifold page, 1999–2005. http://www.math.tu-berlin.de/diskregeom/stellar/.

[26] Lutz, F.: Triangulated manifolds with few vertices: Combinatorial manifolds, 2005. arXiv: math.CO/0506372.

[27] Magurn, K.: Enumeration in balanced 2-manifolds. Geom. Dedicata **37**, 167–181 (1991)   Zbl 0717.52012   MR 1094907

[28] McMullen, P.: The number of faces of simplicial polytopes. Israel J. Math. **9**, 559–571 (1971)   Zbl 0209.53701   MR 0278183

[29] McMullen, P.: On simple polytopes. Invent. Math. **113**, 419–444 (1993)   Zbl 0803.52007   MR 1228132

[30] Mitchell, W. J. R.: Defining the boundary of a homology manifold. Proc. Amer. Math. Soc. **110**, 509–513 (1990)   Zbl 0835.57011   MR 1019276

[31] Novik, I.: Upper bound theorems for homology manifolds. Israel J. Math. **108**, 45–82 (1998)   Zbl 0922.52004   MR 1669325

[32] Reisner, G.: Cohen–Macaulay quotients of polynomial rings. Adv. Math. **21**, 30–49 (1976)   Zbl 0345.13017   MR 0407036

[33] Ringel, G.: Wie man die geschlossenen nichtorientierbaren Flächen in möglichst wenig Dreiecke zerlegen kann. Math. Ann. **130**, 317–326 (1955)   Zbl 0066.41702   MR 0075591

[34] Sarkaria, K. S.: On neighbourly triangulations. Trans. Amer. Math. Soc. **277(1)**, 213–239 (1983)   Zbl 0522.57010   MR 0690049

[35] Schenzel, P.: On the number of faces of simplicial complexes and the purity of Frobenius. Math. Z. **178**, 125–142 (1981)   Zbl 0472.13012   MR 0627099

[36] Sommerville, D.: The relations connecting the angle-sums and volume of a polytope in space of *n* dimensions. Proc. Roy. Soc. London Ser. A **115**, 103–119 (1927)   Zbl 53.0578.03

[37] Stanley, R. P.: A monotonicity property of h-vectors and $h^\star$-vectors. Eur. J. Combin. **14**, 251–258 (1993)   Zbl 0799.52008   MR 1215335

[38] Stanley, R. P.: Cohen–Macaulay complexes. In: M. Aigner (ed.), Higher Combinatorics, Reidel, Dordrecht, 51–62 (1977)   Zbl 0376.55007   MR 0572989

[39] Stanley, R. P.: Balanced Cohen–Macaulay complexes. Trans. Amer. Math. Soc. **249**, 139–157 (1979)   Zbl 0411.05012   MR 0526314

[40] Stanley, R. P.: The number of faces of a simplicial convex polytope. Adv. in Math. **35**, 236–238 (1980)   Zbl 0427.52006   MR 0563925

[41] Stanley, R. P.: Some aspects of groups acting on finite posets. J. Combin. Theory Ser. A **32**, 132–161 (1982)   Zbl 0496.06001   MR 0654618

[42] Stanley, R. P.: A survey of Eulerian posets. In: T. Bisztricky et al. (eds.) Polytopes: Abstract, Convex and Computational, Kluwer, 301–333 (1994)   Zbl 0816.52004   MR 1322068

[43] Stanley, R. P.: Combinatorics and Commutative Algebra. Progr. Math. 41, Birkhäuser Boston (1996)   Zbl pre02206511   MR 1453579

[44] Stanley, R. P.: Eumerative Combinatorics, Vol. 1. Cambridge Stud. Adv. Math. 49, Cambridge Univ. Press (1997)   Zbl 0945.05006   MR 1442260

[45] Swartz, E.: g-elements, finite buildings and higher Cohen–Macaulay connectivity. J. Combin. Theory Ser. A **113**, 1305–1320 (2006)   Zbl 1102.13025   MR 2259062

[46] Vaccaro, M.: Sulla caratteristica dei complessi simpliciali χ-omogenei. Ann. Mat. Pura Appl. (4) **41**, 1–20 (1956)   Zbl 0072.40502   MR 0077936

[47] Walkup, D.: The lower bound conjecture for 3- and 4-manifolds. Acta Math. **125**, 75–107 (1970)   Zbl 0204.56301   MR 0275281