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Stability of closed characteristics on compact convex hypersurfaces in $\mathbb{R}^6$

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Abstract. Let $\Sigma \subset \mathbb{R}^6$ be a compact convex hypersurface. We prove that if Σ carries only finitely many geometrically distinct closed characteristics, then at least two of them must have irrational mean indices. Moreover, if Σ carries exactly three geometrically distinct closed characteristics, then at least two of them must be elliptic.

Keywords. Compact convex hypersurfaces, closed characteristics, Hamiltonian systems, Morse theory, mean index identity, stability

1. Introduction and main results

Let Σ be a fixed C^3 compact convex hypersurface in $\mathbb{R}^{2n}$, i.e., Σ is the boundary of a compact and strictly convex region U in $\mathbb{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose U contains the origin. We consider *closed characteristics* (τ, y) on Σ, which are solutions of the problem

$$\begin{cases} \dot{y} = JN_\Sigma(y), \\ y(\tau) = y(0), \end{cases} \qquad \text{where} \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{1.1}$$

I_n is the identity matrix in $\mathbb{R}^n$, $\tau > 0$, $y : \mathbb{R} \to \mathbb{R}^{2n}$ with $y(\mathbb{R}) \subset \Sigma$, and $N_\Sigma(y)$ is the outward normal vector of Σ at y normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbb{R}^{2n}$. A closed characteristic (τ, y) is *prime* if τ is the minimal period of y. Two closed characteristics (τ, y) and (σ, z) are *geometrically distinct* if $y(\mathbb{R}) \neq z(\mathbb{R})$. We denote by $\mathcal{J}(\Sigma)$ and $\widetilde{\mathcal{J}}(\Sigma)$ the set of all closed characteristics (τ, y) on Σ with τ being the minimal period of y and the set of all geometrically distinct ones respectively. Note that $\mathcal{J}(\Sigma) = \{\theta \cdot y \mid \theta \in S^1,\ y \text{ is prime}\}$, while $\widetilde{\mathcal{J}}(\Sigma) = \mathcal{J}(\Sigma)/S^1$, where the natural S^1-action is defined by $\theta \cdot y(t) = y(t + \tau\theta)$ for $\theta \in S^1$ and $t \in \mathbb{R}$.

Let $j : \mathbb{R}^{2n} \to \mathbb{R}$ be the *gauge function* of Σ, i.e., $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$, $j \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^0(\mathbb{R}^{2n}, \mathbb{R})$ and $\Sigma = j^{-1}(1)$. Fix a constant $\alpha \in (1, 2)$ and define

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the Hamiltonian function $H_\alpha : \mathbb{R}^{2n} \to [0, \infty)$ by

$$H_\alpha(x) = j(x)^\alpha, \quad \forall x \in \mathbb{R}^{2n}. \tag{1.2}$$

Then $H_\alpha \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. It is well known that the problem (1.1) is equivalent to the following given energy problem for the Hamiltonian system:

$$\begin{cases} \dot{y}(t) = JH'_\alpha(y(t)), \quad H_\alpha(y(t)) = 1, \quad \forall t \in \mathbb{R}, \\ y(\tau) = y(0). \end{cases} \tag{1.3}$$

Denote by $\mathcal{J}(\Sigma, \alpha)$ the set of all solutions (τ, y) of (1.3) where τ is the minimal period of y and by $\widetilde{\mathcal{J}}(\Sigma, \alpha)$ the set of all geometrically distinct solutions of (1.3). As above, $\widetilde{\mathcal{J}}(\Sigma, \alpha)$ is obtained from $\mathcal{J}(\Sigma, \alpha)$ by dividing by the natural S^1-action. Note that elements in $\mathcal{J}(\Sigma)$ and $\mathcal{J}(\Sigma, \alpha)$ are in one-to-one correspondence, and similarly for $\widetilde{\mathcal{J}}(\Sigma)$ and $\widetilde{\mathcal{J}}(\Sigma, \alpha)$.

Let $(\tau, y) \in \mathcal{J}(\Sigma, \alpha)$. The fundamental solution $\gamma_y : [0, \tau] \to \mathrm{Sp}(2n)$ with $\gamma_y(0) = I_{2n}$ of the linearized Hamiltonian system

$$\dot{w}(t) = JH''_\alpha(y(t))w(t), \quad \forall t \in \mathbb{R}, \tag{1.4}$$

is called the *associate symplectic path* of (τ, y). The eigenvalues of $\gamma_y(\tau)$ are called the *Floquet multipliers* of (τ, y). By Proposition 1.6.13 of [Eke3], the Floquet multipliers of $(\tau, y) \in \mathcal{J}(\Sigma)$ together with their multiplicities do not depend on the particular choice of the Hamiltonian function in (1.3). For any $M \in \mathrm{Sp}(2n)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane $\mathbb{C}$. Since M is symplectic, $e(M)$ is even and $0 \le e(M) \le 2n$. As usual, $(\tau, y) \in \mathcal{J}(\Sigma)$ is *elliptic* if $e(\gamma_y(\tau)) = 2n$. It is *nondegenerate* if 1 is a double Floquet multiplier of it, and *hyperbolic* if 1 is a double Floquet multiplier of it and $e(\gamma_y(\tau)) = 2$. It is well known that these concepts are independent of the choice of $\alpha > 1$.

For the existence and multiplicity of geometrically distinct closed characteristics on convex compact hypersurfaces in $\mathbb{R}^{2n}$ we refer to [Rab], [Wei], [EkL], [EkH], [Szu], [HWZ], [LoZ], [LLZ], and the references therein. Note that recently in [WHL], Wang, Hu and Long proved $\#\tilde{\mathcal{J}}(\Sigma) \ge 3$ for every $\Sigma \in \mathcal{H}(6)$.

Concerning the stability problem, Ekeland [Eke2] in 1986 and Long [Lon2] in 1998 proved, for any $\Sigma \in \mathcal{H}(2n)$, the existence of at least one non-hyperbolic closed characteristic on Σ provided $\#\tilde{\mathcal{J}}(\Sigma) < \infty$. Ekeland [Eke2] also proved the existence of at least one elliptic closed characteristic on Σ provided $\Sigma \in \mathcal{H}(2n)$ is $\sqrt{2}$-pinched. In 1992, Dell'Antonio, D'Onofrio and Ekeland [DDE] proved the existence of at least one elliptic closed characteristic on Σ provided $\Sigma \in \mathcal{H}(2n)$ satisfies $\Sigma = -\Sigma$. In 2000, Long [Lon3] proved that $\Sigma \in \mathcal{H}(4)$ and $\#\tilde{\mathcal{J}}(\Sigma) = 2$ imply that both the closed characteristics must be elliptic. In 2002, Long [LoZ] and Zhu further proved that when $\#\tilde{\mathcal{J}}(\Sigma) < \infty$, there exists at least one elliptic closed characteristic and there are at least $[n/2]$ geometrically distinct closed characteristics on Σ possessing irrational mean indices, which are then nonhyperbolic. In the recent paper [LoW], Long and Wang proved that there exist at least two nonhyperbolic closed characteristics on $\Sigma \in \mathcal{H}(6)$ when $\#\tilde{\mathcal{J}}(\Sigma) < \infty$. Motivated by these results, we prove the following results:

Theorem 1.1. *On every $\Sigma \in \mathcal{H}(6)$ satisfying $\#\tilde{\mathcal{J}}(\Sigma) < \infty$, there exist at least two geometrically distinct closed characteristics with irrational mean indices.*

Theorem 1.2. *Suppose $\#\tilde{\mathcal{J}}(\Sigma) = 3$ for some $\Sigma \in \mathcal{H}(6)$. Then there exist at least two elliptic closed characteristics in $\tilde{\mathcal{J}}(\Sigma)$.*

The proofs of Theorems 1.1 and 1.2 are given in Section 3. The main ingredients in the proofs are: the mean index identity for closed characteristics established in [WHL], the Morse inequality and the index iteration theory developed by Long and his coworkers, specially the common index jump theorem of Long and Zhu ([LoZ, Theorem 4.3], cf. [Lon4, Theorem 11.2.1]). In Section 2, we briefly review the equivariant Morse theory and the mean index identity for closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$ developed in [WHL].

In this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and positive real numbers respectively. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in $\mathbb{R}^{2n}$. Denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the standard L^2-inner product and L^2-norm. For an S^1-space X, we denote by X_{S^1} the homotopy quotient of X modulo the S^1-action, i.e., $X_{S^1} = S^\infty \times_{S^1} X$. We define the functions

$$[a] = \max\{k \in \mathbb{Z} \mid k \le a\}, \quad E(a) = \min\{k \in \mathbb{Z} \mid k \ge a\}, \quad \varphi(a) = E(a) - [a]. \quad (1.5)$$

Specifically, $\varphi(a) = 0$ if $a \in \mathbb{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbb{Z}$. In this paper we use only $\mathbb{Q}$-coefficients for all homology modules. For a $\mathbb{Z}_m$-space pair (A, B), let $H_*(A, B)^{\pm\mathbb{Z}_m} = \{\sigma \in H_*(A, B) \mid L_*\sigma = \pm\sigma\}$, where L is a generator of the $\mathbb{Z}_m$-action.

2. Equivariant Morse theory for closed characteristics

In the rest of this paper, we fix a $\Sigma \in \mathcal{H}(2n)$ and assume the following condition on Σ:

(F) *There exist only finitely many geometrically distinct closed characteristics $\{(\tau_j, y_j)\}_{1\le j\le k}$ on Σ.*

In this section, we briefly review the equivariant Morse theory for closed characteristics on Σ developed in [WHL] which will be needed in Section 3. All the details of proofs can be found in [WHL].

Let $\hat{\tau} = \inf\{\tau_j \mid 1 \le j \le k\}$. Note that here τ_j's are prime periods of y_j's for $1 \le j \le k$. Then by §2 of [WHL], for any $a > \hat{\tau}$, we can construct a function $\varphi_a \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ such that 0 is the unique critical point in $[0, \infty)$ and φ_a is strictly convex for $t \ge 0$. Moreover, $\varphi_a'(t)/t$ is strictly decreasing for $t > 0$ with $\lim_{t\to 0^+} \varphi_a'(t)/t = 1$ and $\varphi_a(0) = 0 = \varphi_a'(0)$. More precisely, we define φ_a via Propositions 2.2 and 2.4 of [WHL]. The precise dependence of φ_a on a is explained in Remark 2.3 of [WHL].

Define the Hamiltonian function $H_a(x) = a\varphi_a(j(x))$ and consider the fixed period problem

$$\begin{cases} \dot{x}(t) = JH_a'(x(t)), \\ x(1) = x(0). \end{cases} \tag{2.1}$$

Then $H_a \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R})$ is strictly convex. Solutions of (2.1) are $x \equiv 0$ and $x = \rho y(\tau t)$ with $\varphi_a'(\rho)/\rho = \tau/a$, where (τ, y) is a solution of (1.1). In particular, nonzero solutions of (2.1) are in one-to-one correspondence with solutions of (1.1) with period $\tau < a$.

In the following, we use the Clarke–Ekeland dual action principle. As usual, let G_a be the Fenchel transform of H_a defined by $G_a(y) = \sup\{x \cdot y - H_a(x) \mid x \in \mathbb{R}^{2n}\}$. Then $G_a \in C^2(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R})$ is strictly convex. Let

$$L_0^2(S^1, \mathbb{R}^{2n}) = \left\{ u \in L^2([0,1], \mathbb{R}^{2n}) \;\middle|\; \int_0^1 u(t)\, dt = 0 \right\}. \tag{2.2}$$

Define a linear operator $M : L_0^2(S^1, \mathbb{R}^{2n}) \to L_0^2(S^1, \mathbb{R}^{2n})$ by $\frac{d}{dt} Mu(t) = u(t)$ and $\int_0^1 Mu(t)\, dt = 0$. The dual action functional on $L_0^2(S^1, \mathbb{R}^{2n})$ is defined by

$$\Psi_a(u) = \int_0^1 \left(\frac{1}{2} Ju \cdot Mu + G_a(-Ju) \right) dt. \tag{2.3}$$

Then the functional $\Psi_a \in C^{1,1}(L_0^2(S^1, \mathbb{R}^{2n}), \mathbb{R})$ is bounded below and satisfies the Palais–Smale condition. Suppose x is a solution of (2.1). Then $u = \dot{x}$ is a critical point of Ψ_a. Conversely, suppose u is a critical point of Ψ_a. Then there exists a unique $\xi \in \mathbb{R}^{2n}$ such that $Mu - \xi$ is a solution of (2.1). In particular, solutions of (2.1) are in one-to-one correspondence with critical points of Ψ_a. Moreover, $\Psi_a(u) < 0$ for every critical point $u \neq 0$ of Ψ_a.

Suppose u is a nonzero critical point of Ψ_a. Then following [Eke3] the formal Hessian of Ψ_a at u is defined by

$$Q_a(v, v) = \int_0^1 (Jv \cdot Mv + G_a''(-Ju)Jv \cdot Jv)\, dt,$$

which defines an orthogonal splitting $L_0^2 = E_- \oplus E_0 \oplus E_+$ of $L_0^2(S^1, \mathbb{R}^{2n})$ into the negative, zero and positive subspaces. The *index* of u is defined by $i(u) = \dim E_-$, and the *nullity* of u is $\nu(u) = \dim E_0$. Let $u = \dot{x}$ be the critical point of Ψ_a such that x corresponds to the closed characteristic (τ, y) on Σ. Then the index $i(u)$ and the nullity $\nu(u)$ defined above coincide with the Ekeland indices defined in [Eke1] and [Eke3]. In paticular, $1 \leq \nu(u) \leq 2n - 1$ always holds.

We have a natural S^1-action on $L_0^2(S^1, \mathbb{R}^{2n})$ defined by $\theta \cdot u(t) = u(\theta + t)$ for all $\theta \in S^1$ and $t \in \mathbb{R}$. Clearly Ψ_a is S^1-invariant. For any $\kappa \in \mathbb{R}$, we define

$$\Lambda_a^\kappa = \{u \in L_0^2(S^1, \mathbb{R}^{2n}) \mid \Psi_a(u) \leq \kappa\}. \tag{2.4}$$

For a critical point u of Ψ_a, we set

$$\Lambda_a(u) = \Lambda_a^{\Psi_a(u)} = \{w \in L_0^2(S^1, \mathbb{R}^{2n}) \mid \Psi_a(w) \le \Psi_a(u)\}. \tag{2.5}$$

Clearly, both sets are S^1-invariant. Since the S^1-action preserves Ψ_a, if u is a critical point of Ψ_a, then the whole orbit $S^1 \cdot u$ is formed by critical points of Ψ_a. Denote by $\mathrm{crit}(\Psi_a)$ the set of critical points of Ψ_a. Note that by the condition (F), the number of critical orbits of Ψ_a is finite. Hence as usual we can make the following definition.

Definition 2.1. *Suppose u is a nonzero critical point of Ψ_a and $\mathcal{N}$ is an S^1-invariant open neighborhood of $S^1 \cdot u$ such that $\mathrm{crit}(\Psi_a) \cap (\Lambda_a(u) \cap \mathcal{N}) = S^1 \cdot u$. Then the S^1-*critical modules *of $S^1 \cdot u$ are defined by*

$$C_{S^1,q}(\Psi_a, S^1 \cdot u) = H_q((\Lambda_a(u) \cap \mathcal{N})_{S^1}, ((\Lambda_a(u) \setminus S^1 \cdot u) \cap \mathcal{N})_{S^1}).$$

We have the following proposition for critical modules.

Proposition 2.2 (Proposition 3.2 of [WHL]). *The critical module $C_{S^1,q}(\Psi_a, S^1 \cdot u)$ is independent of a in the sense that if x_i are solutions of* (2.1) *with Hamiltonian functions $H_{a_i}(x) \equiv a_i\varphi_{a_i}(j(x))$ for $i = 1, 2$ respectively such that both x_1 and x_2 correspond to the same closed characteristic (τ, y) on Σ, then*

$$C_{S^1,q}(\Psi_{a_1}, S^1 \cdot \dot{x}_1) \cong C_{S^1,q}(\Psi_{a_2}, S^1 \cdot \dot{x}_2), \qquad \forall q \in \mathbb{Z}.$$

Now let $u \ne 0$ be a critical point of Ψ_a with multiplicity $\mathrm{mul}(u) = m$, i.e., u corresponds to a closed characteristic $(m\tau, y) \subset \Sigma$ with (τ, y) being prime. Hence $u(t + 1/m) = u(t)$ for all $t \in \mathbb{R}$ and $S^1 \cdot u \cong S^1/\mathbb{Z}_m \cong S^1$. Let $f : N(S^1 \cdot u) \to S^1 \cdot u$ be the normal bundle of $S^1 \cdot u$ in $L_0^2(S^1, \mathbb{R}^{2n})$ and let $f^{-1}(\theta \cdot u) = N(\theta \cdot u)$ be the fiber over $\theta \cdot u$, where $\theta \in S^1$. Let $DN(S^1 \cdot u)$ be the ϱ-disk bundle of $N(S^1 \cdot u)$ for some $\varrho > 0$ sufficiently small, i.e., $DN(S^1 \cdot u) = \{\xi \in N(S^1 \cdot u) \mid \|\xi\| < \varrho\}$, and let $DN(\theta \cdot u) = f^{-1}(\theta \cdot u) \cap DN(S^1 \cdot u)$ be the disk over $\theta \cdot u$. Clearly, $DN(\theta \cdot u)$ is $\mathbb{Z}_m$-invariant and we have $DN(S^1 \cdot u) = DN(u) \times_{\mathbb{Z}_m} S^1$, where the $\mathbb{Z}_m$-action is given by

$$(\theta, v, t) \in \mathbb{Z}_m \times DN(u) \times S^1 \mapsto (\theta \cdot v, \theta^{-1}t) \in DN(u) \times S^1.$$

Hence for an S^1-invariant subset Γ of $DN(S^1 \cdot u)$, we have $\Gamma/S^1 = (\Gamma_u \times_{\mathbb{Z}_m} S^1)/S^1 = \Gamma_u/\mathbb{Z}_m$, where $\Gamma_u = \Gamma \cap DN(u)$. Since Ψ_a is not C^2 on $L_0^2(S^1, \mathbb{R}^{2n})$, we need to use a finite-dimensional approximation introduced by Ekeland in order to apply Morse theory. More precisely, we can construct a finite-dimensional submanifold $\Gamma(\iota)$ of $L_0^2(S^1, \mathbb{R}^{2n})$ which admits a $\mathbb{Z}_\iota$-action with $m \mid \iota$. Moreover, Ψ_a and $\Psi_a|_{\Gamma(\iota)}$ have the same critical points. $\Psi_a|_{\Gamma(\iota)}$ is C^2 in a small tubular neighborhood of the critical orbit $S^1 \cdot u$, and the Morse index and nullity of its critical points coincide with those of the corresponding critical points of Ψ_a. Let

$$D_\iota N(S^1 \cdot u) = DN(S^1 \cdot u) \cap \Gamma(\iota), \quad D_\iota N(\theta \cdot u) = DN(\theta \cdot u) \cap \Gamma(\iota). \tag{2.6}$$

Then we have

$$C_{S^1,*}(\Psi_a, S^1 \cdot u) \cong H_*(\Lambda_a(u) \cap D_\iota N(u), (\Lambda_a(u) \setminus \{u\}) \cap D_\iota N(u))^{\mathbb{Z}_m}. \tag{2.7}$$

Now we can apply the results of Gromoll and Meyer [GrM] to the manifold $D_{p\iota}N(u^p)$ with unique critical point u^p, where $p \in \mathbb{N}$. Then $\mathrm{mul}(u^p) = pm$ is the multiplicity of u^p and the isotropy group $\mathbb{Z}_{pm} \subseteq S^1$ of u^p acts on $D_{p\iota}N(u^p)$ by isometries. According to Lemma 1 of [GrM], we have a $\mathbb{Z}_{pm}$-invariant decomposition

$$T_{u^p}(D_{p\iota}N(u^p)) = V^+ \oplus V^- \oplus V^0 = \{(x_+, x_-, x_0)\}$$

with $\dim V^- = i(u^p)$, $\dim V^0 = \nu(u^p) - 1$ and a $\mathbb{Z}_{pm}$-invariant neighborhood $B = B_+ \times B_- \times B_0$ of 0 in $T_{u^p}(D_{p\iota}N(u^p))$ together with two $\mathbb{Z}_{pm}$-invariant diffeomorphisms

$$\Phi : B = B_+ \times B_- \times B_0 \to \Phi(B_+ \times B_- \times B_0) \subset D_{p\iota}N(u^p)$$

and

$$\eta : B_0 \to W(u^p) \equiv \eta(B_0) \subset D_{p\iota}N(u^p)$$

such that $\Phi(0) = \eta(0) = u^p$ and

$$\Psi_a \circ \Phi(x_+, x_-, x_0) = |x_+|^2 - |x_-|^2 + \Psi_a \circ \eta(x_0), \tag{2.8}$$

with $d(\Psi_a \circ \eta)(0) = d^2(\Psi_a \circ \eta)(0) = 0$. Following [GrM], we call $W(u^p)$ a *local characteristic manifold* and $U(u^p) = B_-$ a local *negative disk* at u^p. By the proof of Lemma 1 of [GrM], $W(u^p)$ and $U(u^p)$ are $\mathbb{Z}_{pm}$-invariant. Then we have

$$\begin{aligned} &H_*(\Lambda_a(u^p) \cap D_{p\iota}N(u^p), (\Lambda_a(u^p) \setminus \{u^p\}) \cap D_{p\iota}N(u^p)) \\ &= H_*(U(u^p), U(u^p) \setminus \{u^p\}) \otimes H_*(W(u^p) \cap \Lambda_a(u^p), (W(u^p) \setminus \{u^p\}) \cap \Lambda_a(u^p)), \end{aligned} \tag{2.9}$$

where

$$H_q(U(u^p), U(u^p) \setminus \{u^p\}) = \begin{cases} \mathbb{Q} & \text{if } q = i(u^p), \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Now we have the following proposition.

Proposition 2.3 (Proposition 3.10 of [WHL]). *Let $u \neq 0$ be a critical point of Ψ_a with* $\mathrm{mul}(u) = 1$. *Then for all $p \in \mathbb{N}$ and $q \in \mathbb{Z}$, we have*

$$C_{S^1,q}(\Psi_a, S^1 \cdot u^p) \cong H_{q-i(u^p)}(W(u^p) \cap \Lambda_a(u^p), (W(u^p) \setminus \{u^p\}) \cap \Lambda_a(u^p))^{\beta(u^p)\mathbb{Z}_p}, \tag{2.11}$$

where $\beta(u^p) = (-1)^{i(u^p)-i(u)}$. Thus

$$C_{S^1,q}(\Psi_a, S^1 \cdot u^p) = 0 \quad \text{if } q < i(u^p) \text{ or } q > i(u^p) + \nu(u^p) - 1. \tag{2.12}$$

In particular, if u^p is nondegenerate, i.e., $\nu(u^p) = 1$, then

$$C_{S^1,q}(\Psi_a, S^1 \cdot u^p) = \begin{cases} \mathbb{Q} & \text{if } q = i(u^p) \text{ and } \beta(u^p) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

We make the following definition:

Definition 2.4. *Let $u \neq 0$ be a critical point of Ψ_a with* $\mathrm{mul}(u) = 1$. *Then for all $p \in \mathbb{N}$ and $l \in \mathbb{Z}$, let*

$$k_{l,\pm 1}(u^p) = \dim H_l(W(u^p) \cap \Lambda_a(u^p), (W(u^p) \setminus \{u^p\}) \cap \Lambda_a(u^p))^{\pm \mathbb{Z}_p},$$
$$k_l(u^p) = \dim H_l(W(u^p) \cap \Lambda_a(u^p), (W(u^p) \setminus \{u^p\}) \cap \Lambda_a(u^p))^{\beta(u^p)\mathbb{Z}_p}.$$

The $k_l(u^p)$'s are called the critical type numbers *of u^p.*

We have the following properties for critical type numbers:

Proposition 2.5 (Proposition 3.13 of [WHL]). *Let $u \neq 0$ be a critical point of Ψ_a with* $\mathrm{mul}(u) = 1$. *Then there exists a minimal $K(u) \in \mathbb{N}$ such that*

$$\nu(u^{p+K(u)}) = \nu(u^p), \qquad i(u^{p+K(u)}) - i(u^p) \in 2\mathbb{Z},$$

and $k_l(u^{p+K(u)}) = k_l(u^p)$ for all $p \in \mathbb{N}$ and $l \in \mathbb{Z}$. We call $K(u)$ the minimal period of critical modules of iterations of the functional Ψ_a at u.

For a closed characteristic (τ, y) on Σ, we denote by $y^m \equiv (m\tau, y)$ the m-th iteration of y for $m \in \mathbb{N}$. Let $a > \tau$ and choose φ_a as above. Determine ρ uniquely by $\varphi_a'(\rho)/\rho = \tau/a$. Let $x = \rho y(\tau t)$ and $u = \dot{x}$. Then we define the *index* $i(y^m)$ and *nullity* $\nu(y^m)$ of $(m\tau, y)$ for $m \in \mathbb{N}$ by

$$i(y^m) = i(u^m), \qquad \nu(y^m) = \nu(u^m).$$

These indices are independent of a when a tends to infinity. Now the *mean index* of (τ, y) is defined by

$$\hat{i}(y) = \lim_{m \to \infty} \frac{i(y^m)}{m}.$$

Note that always $\hat{i}(y) > 2$, which was proved by Ekeland and Hofer [EkH] in 1987 (cf. Corollary 8.3.2 and Lemma 15.3.2 of [Lon4] for a different proof).

By Proposition 2.2, we can define the critical type numbers $k_l(y^m)$ of y^m to be $k_l(u^m)$, where u^m is the critical point of Ψ_a corresponding to y^m. We also define $K(y) = K(u)$. Then we have

Proposition 2.6. *We have $k_l(y^m) = 0$ for $l \notin [0, \nu(y^m) - 1]$ and it can take only values* 0 *or* 1 *when $l = 0$ or $l = \nu(y^m) - 1$. Moreover, the following properties hold* (*cf. Lemma* 3.10 *of* [BaL], [Cha] *and* [MaW])*:*

(i) $k_0(y^m) = 1$ *implies* $k_l(y^m) = 0$ *for* $1 \leq l \leq \nu(y^m) - 1$.
(ii) $k_{\nu(y^m)-1}(y^m) = 1$ *implies* $k_l(y^m) = 0$ *for* $0 \leq l \leq \nu(y^m) - 2$.
(iii) $k_l(y^m) \geq 1$ *for some* $1 \leq l \leq \nu(y^m) - 2$ *implies* $k_0(y^m) = k_{\nu(y^m)-1}(y^m) = 0$.
(iv) *If $\nu(y^m) \leq 3$, then at most one of the $k_l(y^m)$'s for $0 \leq l \leq \nu(y^m) - 1$ can be non-zero.*
(v) *If $i(y^m) - i(y) \in 2\mathbb{Z} + 1$ for some $m \in \mathbb{N}$, then $k_0(y^m) = 0$.*

Proof. By Definition 2.4 we have

$$k_l(y^m) \le \dim H_l(W(u^m) \cap \Lambda_a(u^m), (W(u^m) \setminus \{u^m\}) \cap \Lambda_a(u^m)) \equiv \eta_l(y^m).$$

Then from Corollary 1.5.1 of [Cha] or Corollary 8.4 of [MaW], (i)–(iv) hold.

If $\eta_0(y^m) = 0$, then (v) follows directly from Definition 2.4.

By Corollary 8.4 of [MaW], $\eta_0(y^m) = 1$ if and only if u^m is a local minimum in the local characteristic manifold $W(u^m)$. Hence $(W(u^m) \cap \Lambda_a(u^m), (W(u^m) \setminus \{u^m\}) \cap \Lambda_a(u^m)) = (\{u^m\}, \emptyset)$. By Definition 2.4, we have

$$\begin{aligned} k_{0,+1}(u^m) &= \dim H_0(W(u^m) \cap \Lambda_a(u^m), (W(u^m) \setminus \{u^m\}) \cap \Lambda_a(u^m))^{+\mathbb{Z}_m} \\ &= \dim H_0(\{u^m\})^{+\mathbb{Z}_m} = 1. \end{aligned}$$

This implies $k_0(u^m) = k_{0,-1}(u^m) = 0$. □

For a closed characteristic (τ, y) on Σ, we define, as in [WHL],

$$\hat{\chi}(y) = \frac{1}{K(y)} \sum_{\substack{1 \le m \le K(y) \\ 0 \le l \le 2n-2}} (-1)^{i(y^m)+l} k_l(y^m). \tag{2.14}$$

In particular, if all y^m's are nondegenerate, then by Proposition 2.3 we have

$$\hat{\chi}(y) = \begin{cases} (-1)^{i(y)} & \text{if } i(y^2) - i(y) \in 2\mathbb{Z}, \\ (-1)^{i(y)}/2 & \text{otherwise.} \end{cases} \tag{2.15}$$

We have the following mean index identity for closed characteristics.

Theorem 2.7 (Theorem 1.2 of [WHL]). *Suppose $\Sigma \in \mathcal{H}(2n)$ satisfies $\#\widetilde{\mathcal{J}}(\Sigma) < \infty$. Let $\{(\tau_j, y_j)\}_{1 \le j \le k}$ be all the geometrically distinct closed characteristics. Then*

$$\sum_{1 \le j \le k} \frac{\hat{\chi}(y_j)}{\hat{i}(y_j)} = \frac{1}{2}.$$

Let Ψ_a be the functional defined by (2.3) for some $a \in \mathbb{R}$ large enough and let $\varepsilon > 0$ be small enough such that $[-\varepsilon, \infty) \setminus \{0\}$ contains no critical values of Ψ_a. Denote by I_a the greatest integer in $\mathbb{N}_0$ such that $I_a < i(\tau, y)$ for all closed characteristics (τ, y) on Σ with $\tau \ge a$. Then by Section 5 of [WHL], we have

$$H_{S^1,q}(\Lambda_a^{-\varepsilon}) \cong H_{S^1,q}(\Lambda_a^{\infty}) \cong H_q(\mathbb{C}P^{\infty}), \quad \forall q < I_a. \tag{2.16}$$

For any $q \in \mathbb{Z}$, let

$$M_q(\Lambda_a^{-\varepsilon}) = \sum_{1 \le j \le k, 1 \le m_j < a/\tau_j} \dim C_{S^1,q}(\Psi_a, S^1 \cdot u_j^{m_j}). \tag{2.17}$$

Then the equivariant Morse inequalities for the space $\Lambda_a^{-\varepsilon}$ yield

$$M_q(\Lambda_a^{-\varepsilon}) \geq b_q(\Lambda_a^{-\varepsilon}), \tag{2.18}$$

$$M_q(\Lambda_a^{-\varepsilon}) - M_{q-1}(\Lambda_a^{-\varepsilon}) + \cdots + (-1)^q M_0(\Lambda_a^{-\varepsilon}) \tag{2.19}$$

$$\geq b_q(\Lambda_a^{-\varepsilon}) - b_{q-1}(\Lambda_a^{-\varepsilon}) + \cdots + (-1)^q b_0(\Lambda_a^{-\varepsilon}), \tag{2.20}$$

where $b_q(\Lambda_a^{-\varepsilon}) = \dim H_{S^1,q}(\Lambda_a^{-\varepsilon})$. Now we have the following Morse inequalities for closed characteristics:

Theorem 2.8. *Let $\Sigma \in \mathcal{H}(2n)$ satisfy $\#\widetilde{\mathcal{J}}(\Sigma) < \infty$. Let $\{(\tau_j, y_j)\}_{1\leq j\leq k}$ be all the geometrically distinct closed characteristics. Let*

$$M_q = \lim_{a\to\infty} M_q(\Lambda_a^{-\varepsilon}), \quad \forall q \in \mathbb{Z}, \tag{2.21}$$

$$b_q = \lim_{a\to\infty} b_q(\Lambda_a^{-\varepsilon}) = \begin{cases} 1 & \text{if } q \in 2\mathbb{N}_0, \\ 0 & \text{otherwise}. \end{cases} \tag{2.22}$$

Then

$$M_q \geq b_q, \tag{2.23}$$

$$M_q - M_{q-1} + \cdots + (-1)^q M_0 \geq b_q - b_{q-1} + \cdots + (-1)^q b_0, \quad \forall q \in \mathbb{Z}. \tag{2.24}$$

Proof. As mentioned before, $\hat{i}(y_j) > 2$ for $1 \leq j \leq k$. Hence the Ekeland index satisfies $i(y_j^m) = i(u_j^m) \to \infty$ as $m \to \infty$ for $1 \leq j \leq k$. Note that $I_a \to \infty$ as $a \to \infty$. Now fix a $q \in \mathbb{Z}$ and a sufficiently large $a > 0$. By Propositions 2.2, 2.3 and (2.17), $M_i(\Lambda_a^{-\varepsilon})$ is invariant for all $a > A_q$ and $0 \leq i \leq q$, where $A_q > 0$ is some constant. Hence (2.21) is meaningful. Now for any a such that $I_a > q$, (2.16)–(2.20) imply that (2.22)–(2.24) hold. □

3. Proofs of the main theorems

In this section, we give proofs of Theorems 1.1 and 1.2 by using the mean index identity of [WHL], Morse inequality and the index iteration theory developed by Long and his coworkers.

Following Definition 1.1 of [LoZ], we introduce

Definition 3.1. *For $\alpha \in (1, 2)$, we define a map $\varrho_n : \mathcal{H}(2n) \to \mathbb{N} \cup \{\infty\}$ by*

$$\varrho_n(\Sigma) = \begin{cases} \infty & \text{if } \#\mathcal{V}(\Sigma,\alpha) = \infty, \\ \min\left\{ \left[\dfrac{i(x,1) + 2S^+(x) - \nu(x,1) + n}{2} \right] \,\middle|\, (\tau, x) \in \mathcal{V}_\infty(\Sigma,\alpha) \right\} & \\ & \text{if } \#\mathcal{V}(\Sigma,\alpha) < \infty, \end{cases} \tag{3.1}$$

where $\mathcal{V}(\Sigma,\alpha)$ and $\mathcal{V}_\infty(\Sigma,\alpha)$ are the variationally visible and infinite variationally visible sets respectively given by Definition 1.4 *of* [LoZ] (*cf. Definition* 15.3.3 *of* [Lon4]).

Theorem 3.2 (cf. Theorem 15.1.1 of [Lon4]). *Suppose* $(\tau, y) \in \mathcal{J}(\Sigma)$. *Then*

$$i(y^m) \equiv i(m\tau, y) = i(y, m) - n, \quad \nu(y^m) \equiv \nu(m\tau, y) = \nu(y, m), \quad \forall m \in \mathbb{N}, \quad (3.2)$$

where $i(y, m)$ *and* $\nu(y, m)$ *are the Maslov-type index and nullity of* $(m\tau, y)$ *defined by Conley, Zehnder and Long* (*cf.* §5.4 *of* [Lon4]).

Recall that for a principal $U(1)$-bundle $E \to B$, the *Fadell–Rabinowitz* index (cf. [FaR]) of E is defined to be $\sup\{k \mid c_1(E)^{k-1} \neq 0\}$, where $c_1(E) \in H^2(B, \mathbb{Q})$ is the first rational Chern class. For a $U(1)$-space, i.e., a topological space X with a $U(1)$-action, the Fadell–Rabinowitz index is defined to be the index of the bundle $X \times S^\infty \to X \times_{U(1)} S^\infty$, where $S^\infty \to \mathbb{C}P^\infty$ is the universal $U(1)$-bundle.

As on p. 199 of [Eke3], choose some $\alpha \in (1, 2)$ and associate with U a convex function H such that $H(\lambda x) = \lambda^\alpha H(x)$ for $\lambda \geq 0$. Consider the fixed period problem

$$\begin{cases} \dot{x}(t) = JH'(x(t)), \\ x(1) = x(0). \end{cases} \quad (3.3)$$

Define

$$L_0^{\alpha/(\alpha-1)}(S^1, \mathbb{R}^{2n}) = \left\{ u \in L^{\alpha/(\alpha-1)}(S^1, \mathbb{R}^{2n}) \,\middle|\, \int_0^1 u\,dt = 0 \right\}. \quad (3.4)$$

The corresponding Clarke–Ekeland dual action functional is defined by

$$\Phi(u) = \int_0^1 \left(\frac{1}{2} Ju \cdot Mu + H^*(-Ju) \right) dt, \quad \forall u \in L_0^{\alpha/(\alpha-1)}(S^1, \mathbb{R}^{2n}), \quad (3.5)$$

where Mu is defined by $\frac{d}{dt} Mu(t) = u(t)$ and $\int_0^1 Mu(t)\,dt = 0$, and H^* is the Fenchel transform of H defined in §2.

For any $\kappa \in \mathbb{R}$, we set

$$\Phi^{\kappa-} = \{u \in L_0^{\alpha/(\alpha-1)}(S^1, \mathbb{R}^{2n}) \mid \Phi(u) < \kappa\}. \quad (3.6)$$

Then as on p. 218 of [Eke3], we define

$$c_i = \inf\{\delta \in \mathbb{R} \mid \hat{I}(\Phi^{\delta-}) \geq i\}, \quad (3.7)$$

where $\hat{I}$ is the Fadell–Rabinowitz index given above. Then by Proposition 3 on p. 218 of [Eke3], we have

Proposition 3.3. *Every* c_i *is a critical value of* Φ. *If* $c_i = c_j$ *for some* $i < j$, *then there are infinitely many geometrically distinct closed characteristics on* Σ.

As in Definition 2.1, we introduce

Definition 3.4. *Suppose* u *is a nonzero critical point of* Φ, *and* $\mathcal{N}$ *is an* S^1*-invariant open neighborhood of* $S^1 \cdot u$ *such that* $\mathrm{crit}(\Phi) \cap (\Lambda(u) \cap \mathcal{N}) = S^1 \cdot u$. *Then the* S^1-critical module *of* $S^1 \cdot u$ *is defined by*

$$C_{S^1,q}(\Phi, S^1 \cdot u) = H_q((\Lambda(u) \cap \mathcal{N})_{S^1}, ((\Lambda(u) \setminus S^1 \cdot u) \cap \mathcal{N})_{S^1}), \quad (3.8)$$

where $\Lambda(u) = \{w \in L_0^{\alpha/(\alpha-1)}(S^1, \mathbb{R}^{2n}) \mid \Phi(w) \leq \Phi(u)\}$.

Comparing with Theorem 4 on p. 219 of [Eke3], we have the following

Proposition 3.5. *For every $i \in \mathbb{N}$, there exists a point $u \in L_0^{\alpha/(\alpha-1)}(S^1, \mathbb{R}^{2n})$ such that*

$$\Phi'(u) = 0, \qquad \Phi(u) = c_i, \tag{3.9}$$

$$C_{S^1,2(i-1)}(\Phi, S^1 \cdot u) \neq 0. \tag{3.10}$$

Proof. By Lemma 8 on p. 206 of [Eke3], we can use Theorem 1.4.2 of [Cha] in the equivariant form to obtain

$$H_{S^1,*}(\Phi^{c_i+\epsilon}, \Phi^{c_i-\epsilon}) = \bigoplus_{\Phi(u)=c_i} C_{S^1,*}(\Phi, S^1 \cdot u), \tag{3.11}$$

for ϵ small enough such that the interval $(c_i - \epsilon, c_i + \epsilon)$ contains no critical values of Φ except c_i.

Similar to p. 431 of [EkH], we have the exact sequence

$$H^{2(i-1)}((\Phi^{c_i+\epsilon})_{S^1}, (\Phi^{c_i-\epsilon})_{S^1}) \xrightarrow{q^*} H^{2(i-1)}((\Phi^{c_i+\epsilon})_{S^1}) \xrightarrow{p^*} H^{2(i-1)}((\Phi^{c_i-\epsilon})_{S^1}), \tag{3.12}$$

where p and q are natural inclusions. Denote by $f : (\Phi^{c_i+\epsilon})_{S^1} \to \mathbb{C}P^\infty$ a classifying map and let $f^\pm = f|_{(\Phi^{c_i\pm\epsilon})_{S^1}}$. Then clearly each $f^\pm : (\Phi^{c_i\pm\epsilon})_{S^1} \to \mathbb{C}P^\infty$ is a classifying map on $(\Phi^{c_i\pm\epsilon})_{S^1}$. Let $\eta \in H^2(\mathbb{C}P^\infty)$ be the first universal Chern class.

By definition of c_i, we have $\hat{I}(\Phi^{c_i-\epsilon}) < i$, hence $(f^-)^*(\eta^{i-1}) = 0$. Note that $p^*(f^+)^*(\eta^{i-1}) = (f^-)^*(\eta^{i-1})$. Hence the exactness of (3.12) yields a $\sigma \in H^{2(i-1)}((\Phi^{c_i+\epsilon})_{S^1}, (\Phi^{c_i-\epsilon})_{S^1})$ such that $q^*(\sigma) = (f^+)^*(\eta^{i-1})$. Since $\hat{I}(\Phi^{c_i+\epsilon}) \geq i$, we have $(f^+)^*(\eta^{i-1}) \neq 0$. Hence $\sigma \neq 0$, and so

$$H_{S^1}^{2(i-1)}(\Phi^{c_i+\epsilon}, \Phi^{c_i-\epsilon}) = H^{2(i-1)}((\Phi^{c_i+\epsilon})_{S^1}, (\Phi^{c_i-\epsilon})_{S^1}) \neq 0.$$

Now the proposition follows from (3.11) and the universal coefficient theorem. □

Proposition 3.6. *Suppose u is the critical point of Φ found in Proposition 3.5. Then*

$$C_{S^1,2(i-1)}(\Psi_a, S^1 \cdot u_a) \neq 0, \tag{3.13}$$

where Ψ_a is given by (2.3) and $u_a \in L_0^2(S^1, \mathbb{R}^{2n})$ is its critical point corresponding to u in the natural sense.

Proof. Fixing u, we modify the function H only in a small neighborhood Ω of 0 as in [Eke1] so that the corresponding orbit of u does not enter Ω and the resulting function $\widetilde{H}$ has properties similar to those in Definition 1 on p. 26 of [Eke1] with $3/2$ there replaced by α. Define the dual action functional $\widetilde{\Phi} : L_0^{\alpha/(\alpha-1)}(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ by

$$\widetilde{\Phi}(v) = \int_0^1 \left(\frac{1}{2} Jv \cdot Mv + \widetilde{H}^*(-Jv)\right) dt; \tag{3.14}$$

clearly Φ and $\widetilde{\Phi}$ are C^1-close to each other. By the continuity of critical modules (cf. Theorem 8.8 of [MaW] or Theorem 1.5.6 on p. 53 of [Cha], which can be easily generalized to the equivariant case) for the u in the proposition, we have

$$C_{S^1,*}(\Phi, S^1 \cdot u) \cong C_{S^1,*}(\widetilde{\Phi}, S^1 \cdot u). \tag{3.15}$$

Using a finite-dimensional approximation as in Lemma 3.9 of [Eke1], we have

$$C_{S^1,*}(\widetilde{\Phi}, S^1 \cdot u) \cong H_*(\widetilde{\Lambda}(u) \cap D_\iota N(u), (\widetilde{\Lambda}(u) \setminus \{u\}) \cap D_\iota N(u))^{\mathbb{Z}_m}, \tag{3.16}$$

where $\widetilde{\Lambda}(u) = \{w \in L_0^{\alpha/(\alpha-1)}(S^1, \mathbb{R}^{2n}) \mid \widetilde{\Phi}(w) \le \widetilde{\Phi}(u)\}$ and $D_\iota N(u)$ is a $\mathbb{Z}_m$-invariant finite-dimensional disk transversal to $S^1 \cdot u$ at u (cf. Lemma 3.9 of [WHL]), m being the multiplicity of u.

By Lemma 3.9 of [WHL], we have

$$C_{S^1,*}(\Psi_a, S^1 \cdot u_a) \cong H_*(\Lambda_a(u_a) \cap D_\iota N(u_a), (\Lambda_a(u_a) \setminus \{u_a\}) \cap D_\iota N(u_a))^{\mathbb{Z}_m}. \tag{3.17}$$

By the construction of H_a in [WHL], $H_a = \widetilde{H}$ in an L^∞-neighborhood of $S^1 \cdot u$. We remark here that multiplying H by a constant will not affect the corresponding critical modules, i.e., the corresponding critical orbits have isomorphic critical modules. Hence we can assume $H_a = H$ in an L^∞-neighborhood of $S^1 \cdot u$. Then Ψ_a and $\widetilde{\Phi}$ coincide in an L^∞-neighborhood of $S^1 \cdot u$. Note also that by Lemma 3.9 of [Eke1], the two finite-dimensional approximations are actually the same. Hence we have

$$\begin{aligned} &H_*(\widetilde{\Lambda}(u) \cap D_\iota N(u), (\widetilde{\Lambda}(u) \setminus \{u\}) \cap D_\iota N(u))^{\mathbb{Z}_m} \\ &\qquad \cong H_*(\Lambda_a(u_a) \cap D_\iota N(u_a), (\Lambda_a(u_a) \setminus \{u_a\}) \cap D_\iota N(u_a))^{\mathbb{Z}_m}. \end{aligned} \tag{3.18}$$

Now the proposition follows from Proposition 3.5 and (3.16)–(3.18). □

Now we can give:

Proof of Theorem 1.1. By the assumption (F) at the beginning of Section 2, we let $\{(\tau_j, y_j)\}_{1 \le j \le k}$ be all the geometrically distinct closed characteristics on Σ, and denote by $\gamma_j \equiv \gamma_{y_j}$ the associated symplectic path of (τ_j, y_j) on Σ for $1 \le j \le k$. Then by Lemma 15.2.4 of [Lon4], there exist $P_j \in \mathrm{Sp}(6)$ and $M_j \in \mathrm{Sp}(4)$ such that

$$\gamma_j(\tau_j) = P_j^{-1}(N_1(1,1) \diamond M_j)P_j, \quad \forall 1 \le j \le k, \tag{3.19}$$

where $N_1(1,b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for $b \in \mathbb{R}$.

Without loss of generality, by Theorem 1.3 of [LoZ] (cf. Theorem 15.5.2 of [Lon4]), we may assume that (τ_1, y_1) has irrational mean index. Hence by Theorem 8.3.1 and Corollary 8.3.2 of [Lon4], $M_1 \in \mathrm{Sp}(4)$ in (3.19) can be connected to $R(\theta_1) \diamond Q_1$ within $\Omega^0(M_1)$ for some $\theta_1/\pi \notin \mathbb{Q}$ and $Q_1 \in \mathrm{Sp}(2)$, where $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\theta \in \mathbb{R}$. Here we use the notations from Definition 1.8.5 and Theorem 1.8.10 of [Lon4]. By Theorem 2.7, the following identity holds:

$$\frac{\hat{\chi}(y_1)}{\hat{i}(y_1)} + \sum_{2 \le j \le k} \frac{\hat{\chi}(y_j)}{\hat{i}(y_j)} = \frac{1}{2}. \tag{3.20}$$

Now we have the following four cases according to the classification of basic norm forms (cf. Definition 1.8.9 of [Lon4]).

Case 1. $Q_1 = R(\theta_2)$ *with* $\theta_2/\pi \notin \mathbb{Q}$ *or* $Q_1 = D(\pm 2) \equiv \begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 1/2 \end{pmatrix}$.

In this case, by Theorems 8.1.6 and 8.1.7 of [Lon4], we have $\nu(y_1^m) \equiv 1$, i.e., y_1^m is nondegenerate for all $m \in \mathbb{N}$. Hence it follows from (2.15) that $\hat{\chi}(y_1) \neq 0$. Now (3.20) implies that at least one of the y_j's for $2 \leq j \leq k$ must have irrational mean index. Hence the conclusion of the theorem holds.

Case 2. $Q_1 = N_1(1, b)$ *with* $b = \pm 1, 0$.

We have two subcases according to the value of $\hat{\chi}(y_1)$.

Subcase 2.1. $\hat{\chi}(y_1) \neq 0$.

In this case, (3.20) implies that at least one of the y_j's for $2 \leq j \leq k$ must have irrational mean index. Hence the desired conclusion holds.

Subcase 2.2. $\hat{\chi}(y_1) = 0$.

Note that by Theorems 8.1.4 and 8.1.7 of [Lon4] and our Proposition 2.5, we have $K(y_1) = 1$. Since $\nu(y_1) \leq 3$, it follows from Proposition 2.6 and (2.14) that

$$0 = \hat{\chi}(y_1) = (-1)^{i(y_1)}(k_0(y_1) - k_1(y_1) + k_2(y_1)). \tag{3.21}$$

By Proposition 2.6(iv), at most one of $k_l(y_1)$ for $l = 0, 1, 2$ can be nonzero. Then (3.21) yields $k_l(y_1) = 0$ for $l = 0, 1, 2$. Hence it follows from Proposition 2.3 and Definition 2.4 that

$$C_{S^1,q}(\Psi_a, S^1 \cdot u_1^p) = 0, \quad \forall p \in \mathbb{N}, q \in \mathbb{Z}, \tag{3.22}$$

where we denote by u_1 the critical point of Ψ_a corresponding to (τ_1, y_1). In other words, u_1^m is homologically invisible for all $m \in \mathbb{N}$.

By Propositions 3.5 and 3.6, we can replace the phrase *infinite variationally visible* in Definition 1.4 of [LoZ] (cf. Definition 15.3.3 of [Lon4]) by *homologically visible*, and it is easy to check that all the results in [LoZ] remain true under this change. Hence by Theorem 1.3 of [LoZ] (cf. Theorem 15.5.2 of [Lon4]), at least one of the y_j's for $2 \leq j \leq k$ must have irrational mean index, i.e., we can forget y_1 and consider only y_j's for $2 \leq j \leq k$, and then apply that theorem. This proves the desired conclusion.

Case 3. $Q_1 = N_1(-1, 1)$.

In this case, by Theorems 8.1.4, 8.1.5 and 8.1.7 of [Lon4], we have

$$i(y_1, m) = mi(y_1, 1) + 2E\left(\frac{m\theta_1}{2\pi}\right) - 2, \quad \nu(y_1, m) = 1 + \frac{1 + (-1)^m}{2}, \quad \forall m \in \mathbb{N},$$

with $i(y_1, 1) \in 2\mathbb{Z} + 1$. Hence $K(y_1) = 2$ by Proposition 2.5. Because y_1 is nondegenerate, we have $k_l(y_1) = \delta_0^l$ for all $l \in \mathbb{Z}$ by (2.11), (2.13) and Definition 2.4. By Theorem 3.2, we have $i(y_1) = i(y_1, 1) - 3 \in 2\mathbb{Z}$ and $i(y_1^2) - i(y_1) = i(y_1, 2) - i(y_1, 1) \in 2\mathbb{Z} + 1$.

Hence $k_0(y_1^2) = 0$ by Proposition 2.6(v). Because $\nu(y_1^2) = 2$, we have $k_l(y_1^2) = 0$ for $l \geq 2$. Then (2.14) implies

$$\hat{\chi}(y_1) = \frac{1 + k_1(y_1^2)}{2} \neq 0.$$

Now (3.20) implies that at least one of the y_j's for $2 \leq j \leq k$ must have irrational mean index. Hence the conclusion of the theorem holds.

Case 4. *$Q_1 = N_1(-1, b)$ with $b = 0, -1$ or $Q_1 = R(\theta_2)$ with $\theta_2/2\pi = L/N \in \mathbb{Q} \cap (0, 1)$ with $N > 1$ and $(L, N) = 1$.*

Note first that if $Q_1 = N_1(-1, b)$ with $b = 0, -1$, then Theorems 8.1.5 and 8.1.7 of [Lon4] imply that their index iteration formulae coincide with that of a rotational matrix $R(\theta)$ with $\theta = \pi$. Hence in the following we shall only consider the case $Q_1 = R(\theta_2)$ with $\theta_2/\pi \in (0, 2) \cap \mathbb{Q}$. The same argument also shows that the conclusion of the theorem is true for $Q_1 = N_1(-1, -1)$.

By Theorems 8.1.4 and 8.1.7 of [Lon4], we have

$$i(y_1, m) = m(i(y_1, 1) - 1) + 2E\left(\frac{m\theta_1}{2\pi}\right) + 2E\left(\frac{m\theta_2}{2\pi}\right) - 3, \tag{3.23}$$

$$\nu(y_1, m) = 3 - 2\varphi\left(\frac{m\theta_2}{2\pi}\right), \tag{3.24}$$

with $i(y_1, 1) \in 2\mathbb{Z} + 1$ and all $m \in \mathbb{N}$. By Proposition 2.5, we have $K(y_1) = N$. Note that because y_1^m is nondegenerate for $1 \leq m \leq N - 1$, it follows that $k_l(y_1^m) = \delta_0^l$ for $1 \leq m \leq N - 1$ by (2.11), (2.13) and Definition 2.4. By Theorem 3.2, we have $i(y_1) = i(y_1, 1) - 3 \in 2\mathbb{Z}$. Then (2.14) implies

$$\hat{\chi}(y_1) = \frac{N - 1 + k_0(y_1^N) - k_1(y_1^N) + k_2(y_1^N)}{N}. \tag{3.25}$$

This follows from $\nu(y_1^m) \leq 3$ for all $m \in \mathbb{N}$.

We have two subcases according to the value of $\hat{\chi}(y_1)$.

Subcase 4.1. $\hat{\chi}(y_1) \neq 0$.

In this subcase, (3.20) implies that at least one of the y_j's for $2 \leq j \leq k$ must have irrational mean index. Hence the conclusion holds.

Subcase 4.2. $\hat{\chi}(y_1) = 0$.

In this subcase, it follows from (3.25) and Proposition 2.6(iv) that

$$k_1(y_1^N) = N - 1 > 0. \tag{3.26}$$

Using the common index jump theorem (Theorems 4.3 and 4.4 of [LoZ], Theorems 11.2.1 and 11.2.2 of [Lon4]), we obtain some $(T, m_1, \ldots, m_k) \in \mathbb{N}^{k+1}$ such that $m_1\theta_2/\pi \in \mathbb{Z}$

(cf. (11.2.18) of [Lon4]) and the following hold by (11.2.6), (11.2.7) and (11.2.26) of [Lon4]:

$$i(y_j, 2m_j) \geq 2T - e(\gamma_j(\tau_j))/2, \tag{3.27}$$
$$i(y_j, 2m_j) + \nu(y_j, 2m_j) \leq 2T + e(\gamma_j(\tau_j))/2 - 1, \tag{3.28}$$
$$i(y_j, 2m_j + 1) = 2T + i(y_j, 1), \tag{3.29}$$
$$i(y_j, 2m_j - 1) + \nu(y_j, 2m_j - 1) = 2T - (i(y_j, 1) + 2S^+_{\gamma_j(\tau_j)}(1) - \nu(y_j, 1)). \tag{3.30}$$

By p. 340 of [Lon4], we have

$$\begin{aligned} 2S^+_{\gamma_j(\tau_j)}(1) - \nu(y_j, 1) &= 2S^+_{N_1(1,1)}(1) - \nu_1(N_1(1, 1)) + 2S^+_{M_j}(1) - \nu_1(M_j) \\ &= 1 + 2S^+_{M_j}(1) - \nu_1(M_j) \\ &\geq -1, \quad 1 \leq j \leq k. \end{aligned} \tag{3.31}$$

In the last inequality, we have used the fact that the worst case for $2S^+_{M_j}(1) - \nu_1(M_j)$ happens when $M_j = N_1(1, -1)^{\diamond 2}$, which gives the lower bound -2.

By Corollary 15.1.4 of [Lon4], we have $i(y_j, 1) \geq 3$ for $1 \leq j \leq k$. Note that $e(\gamma_j(\tau_j)) \leq 6$ for $1 \leq j \leq k$. Hence Theorem 10.2.4 of [Lon4] yields

$$\begin{aligned} i(y_j, m) + \nu(y_j, m) &\leq i(y_j, m + 1) - i(y_j, 1) + e(\gamma_j(\tau_j))/2 - 1 \\ &\leq i(y_j, m + 1) - 1, \quad \forall m \in \mathbb{N},\ 1 \leq j \leq k. \end{aligned} \tag{3.32}$$

In particular, we have

$$i(y_j, m) < i(y_j, m + 1), \quad \forall m \in \mathbb{N}, 1 \leq j \leq k.$$

Now (3.27)–(3.30) become

$$i(y_j, 2m_j) \geq 2T - 3, \tag{3.33}$$
$$i(y_j, 2m_j) + \nu(y_j, 2m_j) - 1 \leq 2T + 1, \tag{3.34}$$
$$i(y_j, 2m_j + m) \geq 2T + 3, \quad \forall m \geq 1, \tag{3.35}$$
$$i(y_j, 2m_j - m) + \nu(y_j, 2m_j - m) - 1 \leq 2T - 3, \quad \forall m \geq 1, \tag{3.36}$$

where $1 \leq j \leq k$. By Proposition 2.3, we have

$$C_{S^1,q}(\Psi_a, S^1 \cdot u_1^{2m_1}) = \delta^q_{i(u_1^{2m_1})+1} \mathbb{Q}^{k_1(y_1^N)} = \delta^q_{i(u_1^{2m_1})+1} \mathbb{Q}^{N-1}. \tag{3.37}$$

Note that by Theorem 3.2,

$$i(y_j^m) = i(y_j, m) - 3, \quad \forall m \in \mathbb{N},\ 1 \leq j \leq k. \tag{3.38}$$

Hence (3.23) implies that $i(y_1^m)$ is even for all $m \in \mathbb{N}$. This together with (3.35)–(3.38) and Proposition 2.3 yield

$$C_{S^1,2T-2}(\Psi_a, S^1 \cdot u_1^m) = 0, \quad \forall m \in \mathbb{N}, \tag{3.39}$$

$$C_{S^1,2T-4}(\Psi_a, S^1 \cdot u_1^m) = 0, \quad \forall m \in \mathbb{N}, \tag{3.40}$$

$$C_{S^1,2T-2}(\Psi_a, S^1 \cdot u_j^m) = 0, \quad \forall m \neq 2m_j,\ 2 \leq j \leq k, \tag{3.41}$$

$$C_{S^1,2T-4}(\Psi_a, S^1 \cdot u_j^m) = 0, \quad \forall m \neq 2m_j,\ 2 \leq j \leq k. \tag{3.42}$$

In fact, by (3.35), (3.36) and (3.38) for $1 \leq j \leq k$, we have $i(u_j^m) = i(y_j^m) \geq 2T$ for all $m > 2m_j$ and $i(u_j^m) + \nu(u_j^m) - 1 = i(y_j^m) + \nu(y_j^m) - 1 \leq 2T - 6$ for all $m < 2m_j$. Thus (3.41)–(3.42) hold and (3.39)–(3.40) hold for $m \neq 2m_1$ by Proposition 2.3. Since $i(y_1^{2m_1})$ is even, by (3.37), (3.39)–(3.40) also hold for $m = 2m_1$.

Thus by Propositions 3.5 and 3.6 we can find $p, q \in \{2, \ldots, k\}$ such that

$$\Phi'(u_p^{2m_p}) = 0, \quad \Phi(u_p^{2m_p}) = c_{T-1}, \quad C_{S^1,2T-4}(\Psi_a, S^1 \cdot u_p^{2m_p}) \neq 0, \tag{3.43}$$

$$\Phi'(u_q^{2m_q}) = 0, \quad \Phi(u_q^{2m_q}) = c_T, \quad C_{S^1,2T-2}(\Psi_a, S^1 \cdot u_q^{2m_q}) \neq 0, \tag{3.44}$$

where we also denote by $u_p^{2m_p}$ and $u_q^{2m_q}$ the corresponding critical points of Φ; this will not lead to confusion.

Note that by assumption (F) and Proposition 3.3, we have $c_{T-1} < c_T$. Hence $p \neq q$ by (3.43) and (3.44). Then the proof of Lemma 3.1 in [LoZ] (cf. Lemma 15.3.5 of [Lon4]) yields

$$\hat{i}(y_p, 2m_p) < \hat{i}(y_q, 2m_q). \tag{3.45}$$

Now if both $\hat{i}(y_p) \in \mathbb{Q}$ and $\hat{i}(y_q) \in \mathbb{Q}$, then the proof of Theorem 5.3 in [LoZ] (cf. Theorem 15.5.2 of [Lon4]) yields

$$\hat{i}(y_p, 2m_p) = \hat{i}(y_q, 2m_q).$$

Note that we may first choose T such that $T/M\hat{i}(y_j) \in \mathbb{N}$ for all $\hat{i}(y_j) \in \mathbb{Q}$ and then use the proof of Theorem 5.3 in [LoZ]. Here M is the least integer in $\mathbb{N}$ that satisfies $M\theta/\pi \in \mathbb{Z}$ whenever $e^{\sqrt{-1}\theta} \in \sigma(\gamma_j(\tau_j))$ and $\theta/\pi \in \mathbb{Q}$ for some $1 \leq j \leq k$. Hence either $\hat{i}(y_p) \notin \mathbb{Q}$ or $\hat{i}(y_q) \notin \mathbb{Q}$. This together with $\hat{i}(y_1) \notin \mathbb{Q}$ and $p, q \neq 1$ proves the theorem. □

Proof of Theorem 1.2. We denote by $\{(\tau_j, y_j)\}_{1 \leq j \leq 3}$ the three geometrically distinct closed characteristics on Σ, and by $\gamma_j \equiv \gamma_{y_j}$ the associated symplectic path of (τ_j, y_j) on Σ for $1 \leq j \leq 3$. Then as in the proof of Theorem 1.1, there exist $P_j \in \mathrm{Sp}(6)$ and $M_j \in \mathrm{Sp}(4)$ such that

$$\gamma_j(\tau_j) = P_j^{-1}(N_1(1,1) \diamond M_j)P_j, \quad \forall 1 \leq j \leq 3. \tag{3.46}$$

As on p. 356 of [LoZ], if there is no (τ_j, y_j) with $M_j = N_1(1,-1)^{\diamond 2}$ and $i(y_j, 1) = 3$ in $\mathcal{V}_\infty(\Sigma, \alpha)$, then $\varrho_n(\Sigma) = 3$. Hence we can use Theorem 1.4 of [LoZ] (Theorem 15.5.2

of [Lon4]) to obtain the existence of at least two elliptic closed characteristics. This proves the assertion of the theorem.

It remains to show that if there exists a (τ_j, y_j) with $M_j = N_1(1,-1)^{\diamond 2}$ and $i(y_j, 1) = 3$ in $\mathcal{V}_\infty(\Sigma, \alpha)$, then we have at least two elliptic closed characteristics. We may assume $M_1 = N_1(1,-1)^{\diamond 2}$ and $i(y_1, 1) = 3$ without loss of generality. Note that (τ_1, y_1) has rational mean index by Theorem 8.3.1 of [Lon4] and Theorem 3.2.

By Theorem 1.3 of [LoZ], we may assume that (τ_2, y_2) has irrational mean index. Hence by Theorem 8.3.1 and Corollary 8.3.2 of [Lon4], $M_2 \in \mathrm{Sp}(4)$ in (3.46) can be connected to $R(\theta_2) \diamond Q_2$ within $\Omega^0(M_2)$ for some $\theta_2/\pi \in \mathbb{R}\setminus\mathbb{Q}$ and $Q_2 \in \mathrm{Sp}(2)$, where $R(\theta) = \left(\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right)$ for $\theta \in \mathbb{R}$. Here we use the notations from Definition 1.8.5 and Theorem 1.8.10 of [Lon4]. By Theorem 2.7,

$$\frac{\hat{\chi}(y_1)}{\hat{i}(y_1)} + \frac{\hat{\chi}(y_2)}{\hat{i}(y_2)} + \frac{\hat{\chi}(y_3)}{\hat{i}(y_3)} = \frac{1}{2}. \tag{3.47}$$

Now if Q_2 is not hyperbolic, then both (τ_1, y_1) and (τ_2, y_2) are elliptic, so the conclusion of the theorem holds.

Hence it remains to consider the case where Q_2 is hyperbolic. Clearly (τ_2, y_2) is nondegenerate, so it follows from (2.15) that $\hat{\chi}(y_2) \neq 0$. Hence (3.47) implies that $\hat{i}(y_3) \in \mathbb{R}\setminus\mathbb{Q}$. Now by Theorem 8.3.1 and Corollary 8.3.2 of [Lon4], $M_3 \in \mathrm{Sp}(4)$ in (3.46) can be connected to $R(\theta_3) \diamond Q_3$ within $\Omega^0(M_3)$ for some $\theta_3/\pi \in \mathbb{R}\setminus\mathbb{Q}$ and $Q_3 \in \mathrm{Sp}(2)$. By the same reason as above, it suffices to consider the case where Q_3 is hyperbolic.

Combining all the above, the only case we need to kick off is that

$$M_1 = N_1(1,-1)^{\diamond 2}, \quad i(y_1, 1) = 3, \quad M_2 = R(\theta_2)\diamond Q_2, \quad M_3 = R(\theta_3)\diamond Q_3, \tag{3.48}$$

where both Q_2 and Q_3 are hyperbolic. Then by Theorem 8.3.1 of [Lon4] and Theorem 3.2, we have

$$i(y_1^m) = m(i(y_1, 1) + 1) - 4 = 4m - 4, \quad \nu(y_1^m) = 3, \quad \forall m \in \mathbb{N}, \tag{3.49}$$

$$i(y_j^m) = m(i(y_j) + 3) + 2E\left(\frac{m\theta_j}{2\pi}\right) - 5, \quad \nu(y_j^m) = 1, \quad \forall m \in \mathbb{N},\ j = 2, 3. \tag{3.50}$$

By Proposition 2.5, we have $K(y_1) = 1$. Note that $i(y_1) = i(y_1, 1) - 3 = 0$ by Theorem 3.2. Hence Proposition 2.6, (2.14) and (2.15) imply

$$\hat{\chi}(y_1) \le 1, \quad \hat{\chi}(y_1) \in \mathbb{Z}, \tag{3.51}$$

$$\hat{\chi}(y_j) = \begin{cases} -1 & \text{if } i(y_j) \in 2\mathbb{N}_0 + 1, \\ 1/2 & \text{if } i(y_j) \in 2\mathbb{N}_0, \end{cases} \quad j = 1, 2. \tag{3.52}$$

By (3.49) and (3.50), we have

$$\hat{i}(y_1) = 4, \tag{3.53}$$

$$\hat{i}(y_j) = i(y_j) + 3 + \theta_j/\pi > 3, \quad j = 2, 3. \tag{3.54}$$

By (3.51)–(3.54), in order to make (3.47) hold, we must have

$$\hat{\chi}(y_1) = 1, \tag{3.55}$$

$$i(y_j) \in 2\mathbb{N}_0, \quad j = 2, 3. \tag{3.56}$$

In fact, by (3.52) and (3.54), we have

$$\frac{\hat{\chi}(y_2)}{\hat{i}(y_2)} + \frac{\hat{\chi}(y_3)}{\hat{i}(y_3)} < \frac{1}{6} + \frac{1}{6} < \frac{1}{2}.$$

Thus to make (3.47) hold, we must have $\hat{\chi}(y_1)/\hat{i}(y_1) > 0$. Hence (3.55) follows from (3.51). Now if $i(y_2) \in 2\mathbb{N}_0 + 1$ or $i(y_3) \in 2\mathbb{N}_0 + 1$, then by (3.52), we have

$$\frac{\hat{\chi}(y_1)}{\hat{i}(y_1)} + \frac{\hat{\chi}(y_2)}{\hat{i}(y_2)} + \frac{\hat{\chi}(y_3)}{\hat{i}(y_3)} < \frac{1}{4} + \frac{1}{6} < \frac{1}{2}.$$

Hence (3.56) must hold.

By (2.14), (3.49) and (3.55), we have $1 = \hat{\chi}(y_1) = k_0(y_1) - k_1(y_1) + k_2(y_1)$. Since $\nu(y_1) = 3$, by Proposition 2.6, only one of $k_0(y_1), k_1(y_1), k_2(y_1)$ can be nonzero. Hence we obtain

$$k_1(y_1) = 0, \quad k_0(y_1) + k_2(y_1) = 1. \tag{3.57}$$

By Proposition 2.3, we have

$$C_{S^1,q}(\Psi_a, S^1 \cdot u_j^p) = 0, \quad \forall p \in \mathbb{N},\ q \in 2\mathbb{Z} + 1,\ 1 \le j \le 3. \tag{3.58}$$

In fact, by (3.49), we have $i(y_1^m) \in 2\mathbb{N}$ for all $m \in \mathbb{N}$. Thus (3.58) holds for $j = 1$ by (2.11), (3.57) and Definition 2.4. By (3.50), and (3.56), for $j = 2, 3$, we have $i(y_j^m) \in 2\mathbb{N}$ when $m \in 2\mathbb{N}_0 + 1$ and $i(y_j^m) \in 2\mathbb{N}_0 + 1$ when $m \in 2\mathbb{N}$. In particular, all y_j^m are nondegenerate for $m \in \mathbb{N}$ and $j = 2, 3$. Thus (3.58) holds for $j = 2, 3$ by (2.13).

Note that (3.58) implies

$$M_q = 0, \quad \forall q \in 2\mathbb{Z} + 1. \tag{3.59}$$

Together with the Morse inequality of Theorem 2.8, this yields

$$-M_{2k} - \cdots - M_2 - M_0 \ge -b_{2k} - \cdots - b_2 - b_0.$$

Thus by the Morse inequality again,

$$b_{2k} + \cdots + b_2 + b_0 \ge M_{2k} + \cdots + M_2 + M_0 \ge b_{2k} + \cdots + b_2 + b_0$$

for all $k \ge 0$. Therefore we obtain

$$M_q = b_q, \quad \forall q \in \mathbb{Z}. \tag{3.60}$$

By (3.57), we have two cases according to the values of $k_l(y_1)$s.

Case 1. $k_0(y_1) = 1$ *and* $k_2(y_1) = 0$.

In this case, by Propositions 2.3, 2.5 and Definition 2.4, we have

$$\dim C_{S^1,q}(\Psi_a, S^1 \cdot u_1^m) = \delta_{4m-4}^q, \quad \forall m \in \mathbb{N},\ q \in \mathbb{Z}. \tag{3.61}$$

Then by (3.60) and (2.22), we must have

$$C_{S^1,4m-4}(\Psi_a, S^1 \cdot u_j^p) = 0, \quad \forall p, m \in \mathbb{N},\ j = 2, 3. \tag{3.62}$$

By (3.60) and (2.22) again, $M_2 = b_2 = 1$ implies

$$C \equiv C_{S^1,2}(\Psi_a, S^1 \cdot u_j^p) = \mathbb{Q}, \tag{3.63}$$

for some $p \in \mathbb{N}$ and $j = 2$ or 3. If $p \geq 2$, by (3.50), we have

$$i(y_j^p) \geq 3p + 2E\left(\frac{p\theta_j}{2\pi}\right) - 5 \geq 3. \tag{3.64}$$

Thus $C = 0$ by Proposition 2.3. Hence $p = 1$. Without loss of generality, we assume $j = 2$. Then by Proposition 2.3 and (3.63), we have

$$i(y_2) = 2. \tag{3.65}$$

Then by (3.50), we have

$$i(y_2^m) \geq 7, \quad \forall m \geq 2. \tag{3.66}$$

By (3.60) and (2.22), $M_6 = b_6 = 1$ implies

$$C_{S^1,6}(\Psi_a, S^1 \cdot u_j^p) = \mathbb{Q} \tag{3.67}$$

for some $p \in \mathbb{N}$ and $j = 2$ or 3. By (3.65) and (3.66), we have $j \neq 2$, i.e., $j = 3$. We must have $p = 1$. In fact, by (3.61) and (3.63), y_1^m and y_2^n already contribute a 1 to M_q for $q = 0, 2, 4$. Hence by (2.22), (3.60) and (3.56), we have $i(y_3) \geq 6$, and so $i(y_3^m) \geq 15$ by (3.50) for $m \geq 2$. Thus $p = 1$ follows from Proposition 2.3. Now we have

$$i(y_3) = 6. \tag{3.68}$$

Hence by (3.53) and (3.55) for y_1, and (3.50), (3.52), (3.65) and (3.68) for y_2 and y_3, we have

$$\frac{\hat{\chi}(y_1)}{\hat{i}(y_1)} + \frac{\hat{\chi}(y_2)}{\hat{i}(y_2)} + \frac{\hat{\chi}(y_3)}{\hat{i}(y_3)} = \frac{1}{4} + \frac{1}{2(5 + \theta_2/\pi)} + \frac{1}{2(9 + \theta_3/\pi)} < \frac{1}{2}.$$

This contradicts (3.47) and proves Case 1.

Case 2. $k_0(y_1) = 0$ *and* $k_2(y_1) = 1$.

The study of this case is similar to that of Case 1. Thus we are rather sketchy here.

In this case, by Proposition 2.3 and Definition 2.4, we have

$$\dim C_{S^1,q}(\Psi_a, S^1 \cdot u_1^m) = \delta_{4m-2}^q, \quad \forall m \in \mathbb{N},\ q \in \mathbb{Z}. \tag{3.69}$$

Hence by (3.60) and (2.22), we must have

$$C_{S^1,4m-2}(\Psi_a, S^1 \cdot u_j^p) = 0, \quad \forall p, m \in \mathbb{N},\ j = 2, 3. \tag{3.70}$$

By (3.69), (3.60) and (2.22), $M_0 = b_0 = 1$ implies

$$C_{S^1,0}(\Psi_a, S^1 \cdot u_j^p) = \mathbb{Q} \tag{3.71}$$

for some $p \in \mathbb{N}$ and $j = 2$ or 3. By (3.64), we have $p = 1$. Without loss of generality, we assume $j = 2$. Then by Proposition 2.3 and (3.50), we have

$$i(y_2) = 0, \quad i(y_2^m) \geq 6, \quad \forall m \geq 3. \tag{3.72}$$

By (3.60) and (2.22), $M_4 = b_4 = 1$ implies

$$C_{S^1,4}(\Psi_a, S^1 \cdot u_j^p) = \mathbb{Q} \tag{3.73}$$

for some $p \in \mathbb{N}$ and $j = 2$ or 3. By (3.69) and (3.72), as in the verification of (3.68), we have $j = 3$ and $p = 1$. Then by Proposition 2.3, we have

$$i(y_3) = 4. \tag{3.74}$$

Hence by (3.53) and (3.55) for y_1, and (3.50), (3.52), (3.72) and (3.74) for y_2 and y_3, we have

$$\frac{\hat{\chi}(y_1)}{\hat{i}(y_1)} + \frac{\hat{\chi}(y_2)}{\hat{i}(y_2)} + \frac{\hat{\chi}(y_3)}{\hat{i}(y_3)} = \frac{1}{4} + \frac{1}{2(3+\theta_2/\pi)} + \frac{1}{2(7+\theta_3/\pi)} < \frac{1}{2}.$$

This contradicts (3.47) and proves Case 2 and therefore the whole theorem. □

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