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Critical points via Γ-convergence: general theory and applications

Abstract. It is well-known that Γ-convergence of functionals provides a tool for studying global and local minimizers. Here we present a general result establishing the existence of critical points of a Γ-converging sequence of functionals provided the associated Γ-limit possesses a nondegenerate critical point, subject to certain mild additional hypotheses. We then go on to prove a theorem that describes suitable nondegenerate critical points for functionals, involving the arclength of a limiting singular set, that arise as Γ-limits in a number of problems. Finally, we apply the general theory to prove some new results, and give new proofs of some known results, establishing the existence of critical points of the 2d Modica–Mortola (Allen–Cahn) energy and 3d Ginzburg–Landau energy with and without magnetic field, and various generalizations, all in a unified framework.

Keywords. Gamma-convergence, critical points, Allen–Cahn, Ginzburg–Landau

1. Introduction

From the time of its inception by De Giorgi in the 1970's, the notion of Γ-convergence of a family of functionals $\{E^\varepsilon\}_{\varepsilon\in(0,1]}$ to a limiting functional E has proven to be a very powerful tool in studying the relationship between the minimizers of the sequence and those of the limit. Here we argue that somewhat surprisingly, it can provide a vehicle for connecting certain unstable critical points of E to critical points of E^ε as well.

Let us recall that Γ-convergence, in its simplest form, can be characterized through two requirements: Given a Banach space U, we say a sequence of functionals $E^\varepsilon : U \to \mathbb{R}$ Γ-converges to a limiting functional $E : U \to \mathbb{R}$ as $\varepsilon \to 0$ if for every $u \in U$ one has

(i) whenever $\{u^\varepsilon\} \subset U$ converges to u, then $\liminf_{\varepsilon\to 0} E^\varepsilon(u^\varepsilon) \geq E(u)$,
(ii) there exists a sequence $\{\tilde{u}^\varepsilon\} \subset U$ such that $\tilde{u}^\varepsilon$ converges to u and $\lim_{\varepsilon\to 0} E^\varepsilon(\tilde{u}^\varepsilon) = E(u)$.

A primary motivation for introducing this topology on functionals was to characterize the weakest notion of convergence that would guarantee minimizers of E^ε converge to a

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minimizer of E. In [15] it was shown that when the Γ-limit E possesses an isolated *local* minimizer, then E^ε will also have a local minimizer. More recently, Sandier and Serfaty [29] have introduced a stronger notion that can be thought of as C^1 Γ-convergence, and they have shown that the gradient flows of C^1 Γ-convergent sequences converge to the gradient flow of the limit. In related work [30], Serfaty shows that information from the second variation of E^ε may be passed to the Γ-limit. The present article then represents an additional contribution to the expanding list of implications of this convergence.

Many of the most interesting examples of Γ-convergence concern the situation where a sequence of functionals, say E_U^ε, maps one Banach space, U, into $\mathbb{R}$ while the limit $E_V : V \to (-\infty, \infty]$ is more naturally defined on another Banach space V, which is typically weaker in its topology. This leads us to broaden our description of Γ-convergence to essentially incorporate the two properties (i) and (ii) above when composed with appropriate maps, say $P_{VU}^\varepsilon : U \to V$ and $Q_{UV}^\varepsilon : V \to U$, that mediate between the two Banach spaces. For example, (ii) is replaced by the requirement that for every $v \in V$ one has

$$P_{VU}^\varepsilon Q_{UV}^\varepsilon(v) \to v \quad \text{in } V, \quad \text{and} \quad E_U^\varepsilon(Q_{UV}^\varepsilon(v)) \to E_V(v) \quad \text{as } \varepsilon \to 0.$$

Here, in particular, one should think of $\{Q_{UV}^\varepsilon(v)\}_{\varepsilon \in (0,1]}$ as taking on the role of the "recovery sequence" $\{\tilde{u}^\varepsilon\}$. Generalizations of Γ-convergence along these lines are by now a fairly common practice; see Section 3 for a complete description.

Our main abstract result, Theorem 4.4, says roughly speaking, that if E_U^ε is a family of functionals Γ-converging to a limiting functional E_V, and if E_V has a saddle point v_s with corresponding critical value $c_* = E_V(v_s)$, then under certain mild additional hypotheses, E_U^ε has a critical point for every sufficiently small ε, and the associated critical values converge to c_* as $\varepsilon \to 0$. The additional hypotheses include a Palais–Smale condition, and a requirement that E_U^ε is in a certain sense uniformly close to E_V on a specific finite-dimensional set that corresponds roughly speaking to the unstable manifold of E_V near v_s. We do not claim that the critical points of E_U^ε converge to v_s; in the level of generality of our theorem, this is not necessarily true (see Remark 4.5).

One subtlety that must be addressed is that the functionals E_V that arise as Γ-limits are typically merely lower semicontinuous, and indeed are typically infinite on a dense subset of V. This forces us to introduce a definition of a saddle point that can be formulated without appealing to any differentiability properties of E_V (see Definition 4.1). This is the notion we use in Theorem 4.4 as described above. The second main result of this paper proves the existence of saddle points, in this sense, for an energy E_V that arises as a Γ-limit in a number of problems, and that corresponds roughly to the functional that associates to a Lipschitz curve its arclength. In various stronger topologies, a nondegenerate critical point of the arclength functional (with natural boundary conditions in a smooth domain $\Omega \subset \mathbb{R}^{n+1}$) is a line segment contained in Ω and joining two points $X_0, Y_0 \in \partial\Omega$, where (X_0, Y_0) is a nondegenerate critical point of the smooth function $(X, Y) \in \partial\Omega \times \partial\Omega \mapsto |X - Y|$. We prove in Theorem 5.1 that to any such line segment there corresponds a saddle point of the generalized arclength functional E_V in weak topologies useful for Γ-convergence. As applications of the general theory, we present in Section 6 existence results for critical points of the 2d Modica–Mortola energy

(3.9); the 3d Ginzburg–Landau energy, both without magnetic field (3.11) and with the field (6.7); and a generalized Ginzburg–Landau type energy (3.16) in higher dimensions. In all cases, no boundary conditions are specified, so critical points satisfy "natural" homogeneous Neumann boundary conditions. We emphasize that there are numerous other examples of families of functionals that are known to Γ-converge to the functional E_V considered in Section 5, and for most of these examples, one should be able to deduce the existence of critical points from Theorems 4.4 and 5.1 by arguments very similar to those given in the examples that we discuss here.

There is a large literature that uses Γ-convergence to study connections between Allen–Cahn and Ginzburg–Landau type problems and geometric problems involving minimal surfaces and minimal connections. Basic Γ-convergence results for the Modica–Mortola functional are established in [23, 22], and for the Ginzburg–Landau and related functionals in [2], [14]. These automatically yield some results describing asymptotic behavior of minimizing sequences. Existence of local minimizers for these functionals is proved using Γ-convergence arguments in [15, 24, 13].

As far as we know, the present paper is the first to use Γ-convergence to prove existence of critical points of Modica–Mortola and Ginzburg–Landau type functionals associated with unstable critical points of a limiting functional. Prior results of this sort have, however, been established via techniques that employ precise control over the spectrum of linearized operators associated with explicitly constructed approximate solutions, together with Lyapunov–Schmidt reduction or arguments in a similar spirit. An early result in this direction, due to Matano [21], establishes existence of stable critical points for the Modica–Mortola functional. More recently, existence of more general critical points for the Modica–Mortola functional, associated with nondegenerate critical points of the arclength functional has been proved by Kowalczyk [16] in two dimensions, and by Pacard and Ritoré [25] in dimensions $n \geq 2$, with critical points of arclength replaced by minimal hypersurfaces. Similar general existence results for the magnetic Ginzburg–Landau functional (6.7) in a specific (formally self-dual) scaling in dimensions $n \geq 3$ have been established in a preprint of Brendle [6]. These techniques give very precise descriptions of the solutions that are proved to exist, much more precise in fact than can be established via Γ-convergence arguments. (In particular, our results on the Modica–Mortola functional are strictly weaker than those of [16], [25].) The main drawback of these linearization techniques is that the required spectral control can be very difficult to obtain. This has limited the range of applicability of these methods. In particular, they have not yet been extended to cover critical points of the model Ginzburg–Landau functional (3.11), which has worse spectral properties than its magnetic counterpart (6.7). Thus, our results on (3.11) are the first existence results for higher-energy critical points for this functional in three dimensions; this is also true for our results on a generalized Ginzburg–Landau functional in arbitrary higher dimensions. Our results on (6.7) are new in that they consider a different scaling than that of [6], and they also incorporate some new lower-order terms involving an applied magnetic field.

For the Ginzburg–Landau functional (3.11) in two dimensions existence of both stable and unstable critical points has been established by proofs that, as in this paper, combine variational methods with arguments about convergence of energy functionals, in the

spirit of Γ-convergence (see for example [17, 8, 19, 3]); by PDE techniques [26]; and by variational reduction, which combines elements of both approaches [9]. This last paper considers homogeneous Neumann boundary conditions, whereas the others cited above treat Dirichlet data. The first paper to identify a limiting "renormalized energy" for (3.11) in two dimensions was [4]. Both global and local minimizers for the problem (6.7) in two dimensions in the presence of an applied field are investigated in depth in [28, 30]. We do not consider any of these problems in this paper. It should be noted that here the limiting energy depends on only finitely many degrees of freedom, making asymptotic variational arguments easier in some respects than the problems that we focus on.

A different family of related papers characterizes the asymptotic behavior of sequences of critical points of functionals (3.9), (3.11), (6.7). Particularly relevant to our concerns is work of Hutchinson and Tonegawa [12] for Allen–Cahn, and of various authors, including among others [27, 20, 5, 7], for Ginzburg–Landau. In [12] it is shown that the sequence of varifolds associated with critical points of Allen–Cahn (3.9) of uniformly bounded energy converge to a stationary varifold, i.e. a minimal surface in a suitably defined weak sense. The same sort of result is proved in [27, 20] for minimizers of the functionals (3.11), (6.7), and in [5, 7] for general sequences of critical points with uniformly bounded energy. In particular, Chiron [7] studies Ginzburg–Landau equations with Neumann conditions, and obtains natural boundary conditions for the corresponding limiting stationary varifolds.

Our Theorem 6.1 and Theorem 6.5 can be seen as a sort of partial converse of results from [7], proving that for certain possible limiting configurations as identified by Chiron—those consisting of a single line segment, and satisfying a nondegeneracy condition—there indeed exists an associated sequence of critical points of (3.11), (6.7). Among other possible limiting configurations from [7], our results could be extended without much difficulty to unions of nonintersecting nondegenerate critical line segments, but probably not, for example, to unions of nondegenerate critical line segments that intersect. If the segments intersect at isolated points, it is not clear how to prove that they can be identified with saddle points of the arclength functional in the sense of Definition 4.1. And a configuration consisting of two or more segments that coincide (corresponding to a vortex of multiplicity 2 or higher) can probably be shown to be a saddle point in the sense of Definition 4.1, but we do not think that other hypotheses of our abstract minmax theorem, such as the uniformity condition (4.8), can be verified near such a (conjectured) saddle point.

We have attempted to present a largely self-contained discussion by first reviewing elements of geometric measure theory and degree theory in Section 2 that are required later in the paper. Section 3 contains a description of the more general characterization of Γ-convergence we work with, and also places the by now well-known Γ-convergence and compactness results for Modica–Mortola [22, 23] and Ginzburg–Landau [2, 14] in this framework. We also recall in this section the fact that the energies we consider satisfy the Palais–Smale condition.

In Section 4 we give the definition of saddle point for a Γ-limit that we employ and then we present the statement and proof of the abstract existence theorem for critical points of a Γ-converging sequence of functionals. We emphasize that this material does

not in any way rely on the geometric measure theory machinery introduced in Section 2; it uses only some standard facts about degree theory, recalled in Section 2.3, and our definition of a Γ-limit from Section 3.1.

In Section 5 we assume a domain contains a line segment with endpoints on its boundary that is a nondegenerate critical point of arclength among competing line segments that similarly span the domain. Under this assumption we show that the Γ-limits of the 2d Modica–Mortola energy and 3d Ginzburg–Landau energy, both arclength in the weak sense of mass of rectifiable 1-currents, possess a saddle point in the flat norm topology—the sense required for our abstract framework. This is the only place where the full machinery from Section 2 is really used.

Finally, in Section 6 we verify that for 2d Modica–Mortola and 3d Ginzburg–Landau, with and without field, all of the remaining conditions of the abstract theorem are met, thus providing the existence of critical points for these energies.

2. Preliminaries

Throughout this article Ω will denote a bounded, open set in $\mathbb{R}^{n+1}$. Elements of $\mathbb{R}^n$ will be denoted by x or y, and elements of $\mathbb{R}^{n+1}$ will generally be denoted by either X or (x, x_{n+1}). We denote the k-dimensional Hausdorff measure of a set S by $\mathcal{H}^k(S)$.

2.1. Currents

We review here some notions from geometric measure theory. We refer to [10, 31] for more detail. For integers $0 \le k \le n + 1$, the space of Grassmann k-covectors is denoted by $\bigwedge^k(\mathbb{R}^{n+1})$ endowed with the usual Euclidean norm $|\cdot|$. A differential k-form ϕ on Ω is a mapping $\phi \colon \Omega \to \bigwedge^k(\mathbb{R}^{n+1})$. The space of C^∞ k-forms compactly supported within Ω is denoted by $\mathcal{D}^k(\Omega)$.

A k-*current* in Ω is a continuous linear functional on the space $\mathcal{D}^k(\Omega)$ and the space of such k-currents is denoted by $\mathcal{D}_k(\Omega)$. We recall that the boundary of a k-current T, denoted by ∂T, is the $(k-1)$-current defined by the relation

$$\partial T(\phi) = T(d\phi) \quad \text{for all } \phi \in \mathcal{D}^{k-1}(\Omega),$$

where $d\phi$ represents the k-form obtained by exterior differentiation of ϕ. In particular, we note that a k-current T has zero boundary relative to the set Ω if $T(d\phi) = 0$ for all $\phi \in \mathcal{D}^{k-1}(\Omega)$. We will denote by $\mathcal{D}'_k(\Omega)$ the elements of $\mathcal{D}_k(\Omega)$ that are boundaries, i.e.

$$\mathcal{D}'_k(\Omega) := \{T \in \mathcal{D}_k(\Omega) : T = \partial S \text{ for some } S \in \mathcal{D}_{k+1}(\Omega)\}. \tag{2.1}$$

For $T \in \mathcal{D}_k(\Omega)$, we denote the *mass* of T in Ω by

$$\mathbf{M}(T) \equiv \sup_{\{\phi \in \mathcal{D}^k(\Omega) : \|\phi\|_{L^\infty(\Omega)} \le 1\}} |T(\phi)|. \tag{2.2}$$

If T is a 0-current with finite mass, then there exists a finite Radon measure μ such that $T(\phi) = \int \phi \, d\mu$ for all smooth, compactly supported functions (= 0-forms). When this holds we will often abuse notation and write simply $T = \mu$.

If $T \in \mathcal{D}_k(\Omega)$ is a k-current with locally finite mass, then there exists a nonnegative measure $\|T\|$ and a $\|T\|$-measurable map $\vec{T} : \Omega \to \bigwedge_k(\mathbb{R}^{n+1})$ such that

$$T(\phi) = \int_\Omega \langle \phi, \vec{T} \rangle \, d\|T\|, \quad \phi \in \mathcal{D}^k(\Omega).$$

If $B \subset \Omega$ is a Borel set, the restriction of T to B, denoted $T \llcorner B$, is defined by

$$(T \llcorner B)(\phi) = \int_B \langle \phi, \vec{T} \rangle \, d\|T\|, \quad \phi \in \mathcal{D}^k(\Omega). \tag{2.3}$$

Most prominent in our approach will be the class $\mathcal{R}_1(\Omega)$ of rectifiable, integer multiplicity 1-currents. These are geometric measure theoretic generalizations of Lipschitz curves. Indeed, if $I \subset \mathbb{R}$ is an interval and $\gamma : I \to \Omega \subset \mathbb{R}^{n+1}$ is a Lipschitz curve, we can define a 1-current T corresponding to integration over γ by

$$T\Big(\sum_{i=1}^{n+1} \phi^i \, dX^i\Big) = \int_I \sum_{i=1}^{n+1} \phi^i(\gamma(t)) \frac{d}{dt} \gamma^i(t) \, dt. \tag{2.4}$$

We define an element of $\mathcal{R}_1(\Omega)$ to be a current with finite mass in Ω that can be written in the form

$$T = \sum_j T_j \tag{2.5}$$

where the sum is finite or countable, and each T_j corresponds to integration over a Lipschitz curve γ_j. Normally a different definition is given, and the above characterization is established as a theorem. It can also be shown (see [10, 4.1.25]) that the sum in (2.5) can be written in such a way that

$$\mathbf{M}(T) = \sum_j \mathbf{M}(T_j) = \sum_j \mathcal{H}^1(\gamma_j), \quad \mathbf{M}(\partial T) = \sum_j \mathbf{M}(\partial T_j). \tag{2.6}$$

In particular, if $\partial T = 0$ in Ω then $\partial T_j = 0$ in Ω for every j.

We introduce the notation $\mathcal{R}'_1(\Omega)$ to denote the finite-mass elements of $\mathcal{R}_1(\Omega)$ that are boundaries, i.e.

$$\mathcal{R}'_1(\Omega) := \{T \in \mathcal{R}_1(\Omega) : \mathbf{M}(T) < \infty, \ T = \partial S \text{ for some } S \in \mathcal{D}_2(\Omega)\}. \tag{2.7}$$

In this article, we will denote the flat norm of a k-current S by

$$\mathbf{F}(S) = \inf\{\mathbf{M}(R) : R \in \mathcal{D}_{k+1}, \ \partial R = S \text{ in } \Omega\}. \tag{2.8}$$

We set $\mathbf{F}(S) = +\infty$ if there does not exist any current R with finite mass such that $\partial R = S$ in Ω. This is a variant of the standard flat norm of geometric measure theory. We write

$$\mathcal{F}'_k(\Omega) := \{T \in \mathcal{D}'_k(\Omega) : \mathbf{F}(T) < \infty\}. \tag{2.9}$$

Remark 2.1. Note that $\mathcal{F}'_k(\Omega)$ is a Banach space when endowed with the norm **F**. This follows from two facts. First, the space of $(k+1)$-currents on Ω with finite mass, denoted $\mathcal{M}_{k+1}(\Omega)$, is a Banach space when endowed with the norm **M**; in fact, this space can be identified with the Banach space of Radon measures on Ω with values in the space $\bigwedge_{k+1}(\mathbb{R}^n)$ of $(k+1)$-vectors. And second, $\mathcal{F}'_k(\Omega)$ with the norm **F** can be identified with the quotient space $\mathcal{M}_{k+1}(\Omega)/\{T \in \mathcal{M}_{k+1}(\Omega) : \partial T = 0\}$; this follows directly from the definitions. Since $\{T \in \mathcal{M}_{k+1} : \partial T = 0\}$ is closed in $\mathcal{M}_{k+1}$, this quotient space is itself a Banach space.

If T is a k-current in $\Omega \subset \mathbb{R}^n$ such that $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$, and if $f : \Omega \to \mathbb{R}$ is a Lipschitz continuous function, then for a.e. $s \in \mathbb{R}$ there is a $(k-1)$-current denoted $\langle T, f, s \rangle$, supported in $f^{-1}(s)$, and characterized by the property that

$$T((\omega \circ f)\phi \wedge df) = \int_{\mathbb{R}} \langle T, f, s \rangle(\phi)\omega(s)\,ds$$

for every compactly supported $(k-1)$-form ϕ and every smooth function ω on $\mathbb{R}$. The currents $\langle T, f, s \rangle$ are called *slices of T by level sets of f*. In some cases there are simple explicit formulas for these slices. In particular, suppose that T is a 1-current corresponding to integration over a Lipschitz curve $\gamma : I \to \Omega$, where I is an interval. Then for any Lipschitz function $f : \Omega \to \mathbb{R}$,

$$\langle T, f, s \rangle = \sum_{t \in I : \gamma(t) \in f^{-1}(s)} \mathrm{sign}(\gamma'(t) \cdot \nabla f(\gamma(t)))\delta_{\gamma(t)} \tag{2.10}$$

for a.e. $s \in \mathbb{R}$, where we use the convention that $\mathrm{sign}(0) = 0$. This is a special case of a general result proved in [10, 4.3.8]; the proof there implies in particular that the sum on the right contains finitely many nonzero terms for a.e. $s \in \mathbb{R}$.

A useful inequality related to slices is the following (cf. [31, p. 158]):

$$\int_{-\infty}^{\infty} \mathbf{M}(\langle T, f, s \rangle)\,ds \leq \sup_{x \in \gamma} |\nabla^\gamma f(x)|\mathbf{M}(T). \tag{2.11}$$

We will need the following lemma which is a sort of isoperimetric inequality:

Lemma 2.2. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a smooth bounded domain. Let $d_\Omega = +\infty$ if $\partial\Omega$ is connected, and if not, let d_Ω be the smallest distance between any two distinct components of $\partial\Omega$. Then there exists a constant C_Ω such that if $T \in \mathcal{R}'_1(\Omega)$ satisfies $\mathbf{M}(T) < d_\Omega$, then $T \in \mathcal{F}'_1(\Omega)$ and*

$$\mathbf{F}(T) \leq C_\Omega \mathbf{M}(T)^2.$$

Proof. Given $T \in \mathcal{R}'_1(\Omega)$ with $\mathbf{M}(T) < d_\Omega$, we must find a 2-current S such that $\partial S = T$ in Ω, with $\mathbf{M}(S) \leq C\mathbf{M}(T)^2$. For use in this proof only, we introduce the notation $\mathbf{M}_{\mathbb{R}^{n+1}}(\cdot)$ to denote the mass of a current in all of $\mathbb{R}^{n+1}$ as opposed to $\mathbf{M}(\cdot)$ which refers to mass in Ω.

We first claim that there exists a current $\tilde{T}$ on $\mathbb{R}^{n+1}$ such that $\partial\tilde{T} = 0$ in $\mathbb{R}^{n+1}$, $\mathbf{M}_{\mathbb{R}^{n+1}}(\tilde{T}) \leq C_1\mathbf{M}(T)$, and $\tilde{T}(\phi) = T(\phi)$ for all smooth 1-forms ϕ with support in Ω.

To see this, write $T = \sum T_j$ as in (2.5), so that T_j is a current in Ω that corresponds to either a closed Lipschitz loop or a Lipschitz curve that connects two points on $\partial\Omega$. We will define $\tilde{T} = \sum \tilde{T}_j$, where each $\tilde{T}_j$ is a suitable extension of T_j to a current on $\mathbb{R}^{n+1}$. It suffices to show that this can be done so that

$$\partial\tilde{T}_j = 0 \quad \text{in } \mathbb{R}^{n+1}, \quad \text{and} \quad \mathbf{M}(\tilde{T}_j) \leq C_1\mathbf{M}(T_j). \tag{2.12}$$

If T_j is compactly supported in Ω (i.e., if the corresponding Lipschitz curve γ_j is a closed loop), we define $\tilde{T}_j$ to be the current on $\mathbb{R}^{n+1}$ corresponding to the same loop γ_j, so that (2.12) clearly holds.

Now consider the other case, and suppose that $\gamma_j : [0, 1] \to \bar{\Omega}$ is the Lipschitz curve corresponding to T_j. The length of γ_j is bounded by $\mathbf{M}(T)$, which is less than d_Ω by hypothesis, so both endpoints of γ_j must belong to the same component of $\partial\Omega$. Thus we can find a curve $\tilde{\gamma}_j : [1, 2] \to \mathbb{R}^{n+1} \setminus \Omega$ such that $\tilde{\gamma}_j(1) = \gamma_j(1)$, $\tilde{\gamma}_j(2) = \gamma_j(0)$, and $\int_1^2 |\tilde{\gamma}_j'| \leq C_1 \int_0^1 |\gamma_j'|$ for a constant C_1 depending on the geometry of the domain. (For example, $\tilde{\gamma}_j$ can be taken to be a length-minimizing geodesic in $\partial\Omega$ connecting the given endpoints.) Now let

$$\Gamma_j(t) = \begin{cases} \gamma_j(t) & \text{if } t \in [0, 1], \\ \tilde{\gamma}_j(t) & \text{if } t \in [1, 2]. \end{cases}$$

and let $\tilde{T}_j$ be the corresponding integral current in $\mathbb{R}^{n+1}$. Then the construction implies that (2.12) holds.

Now the isoperimetric inequality (see [10, 4.2.10 for example]) implies that there exists some 2-current on $\mathbb{R}^{n+1}$, say $\tilde{S}$, such that $\partial\tilde{S} = \tilde{T}$ and $\mathbf{M}_{\mathbb{R}^{n+1}}(\tilde{S}) \leq C_2\mathbf{M}_{\mathbb{R}^{n+1}}(\tilde{T})^2$. If we now let S be given by the restriction $\tilde{S} \llcorner \Omega$, then it is clear that $\partial S = \partial\tilde{S} = \tilde{T} = T$ in Ω, and that $\mathbf{M}(S) \leq \mathbf{M}_{\mathbb{R}^{n+1}}(\tilde{S}) \leq C_2\mathbf{M}_{\mathbb{R}^{n+1}}(\tilde{T})^2 \leq C_1^2 C_2\mathbf{M}(T)^2$. This completes the proof of the lemma. $\qquad\square$

We will also need the following simple result:

Lemma 2.3. *Suppose that Q is a 1-dimensional current in an open set $\Omega \subset \mathbb{R}^{n+1}$, and that there is an open set $\Omega_1 \subset \Omega$ and a point $p \in \Omega_1$ such that $\partial Q \llcorner \Omega_1 = \delta_p$. Then $\mathbf{M}(Q) \geq \mathrm{dist}(p, \partial\Omega_1)$.*

Proof. Let $d = \mathrm{dist}(p, \partial\Omega_1)$. Then for any $\varepsilon > 0$, there exists a smooth function f such that $f(p) > d - \varepsilon$, $\|df\|_\infty \leq 1$, and with support in Ω_1. For example, such a function can be constructed by mollifying the function $g(x) = \max\{d - \varepsilon/2 - |x - p|, 0\}$. Then

$$d - \varepsilon \leq f(p) = \partial Q(f) = Q(df) \leq \mathbf{M}(Q)\|df\|_\infty \leq \mathbf{M}(Q).$$

Since this holds for all $\varepsilon > 0$, we find that $\mathbf{M}(Q) \geq d$. $\qquad\square$

2.2. *Identification of L^1 functions and Jacobians with 1-currents*

We would next like to single out two particular types of 1-currents that are boundaries of 2-currents with finite mass, that is, particular examples of elements of $\mathcal{F}'_1(\Omega)$.

First consider the situation where $\Omega \subset \mathbb{R}^{n+1}$ and a function v lies in $L^1(\Omega)$. Then we can associate to v an n-current, denoted $\star dv$, via the formula

$$\star dv(\phi) = \int_\Omega v \, d\phi \qquad (2.13)$$

for all $\phi \in \mathcal{D}^n(\Omega)$. Note that the $(n+1)$-current $S_v(\phi) := \int_\Omega v\phi$ for all $\phi \in \mathcal{D}^{n+1}(\Omega)$ satisfies $\partial S_v = \star dv$ so that $\star dv \in \mathcal{F}'_n(\Omega)$ and one sees that[1]

$$\mathbf{F}(\star dv) \le \mathbf{M}(S_v) = \|v\|_{L^1(\Omega)}. \qquad (2.14)$$

As a special case of this association that will be relevant to the Modica–Mortola setting later in the paper, consider the case $n = 1$ with $v \in BV(\Omega; \{\pm 1\})$. Then

$$v(X) = \begin{cases} 1 & \text{if } X \in A, \\ -1 & \text{if } X \in \Omega \setminus A, \end{cases} \qquad (2.15)$$

for some set $A \subset \Omega$ of finite perimeter. If we denote by Γ the rectifiable set comprised of the reduced boundary of A in Ω one finds through an application of the Divergence Theorem that $\star dv \in \mathcal{R}'_1(\Omega)$ since

$$\star dv(\phi) = 2 \int_\Gamma \langle \phi(X), \tau(X) \rangle \, d\mathcal{H}^1(X) \qquad (2.16)$$

for any 1-form ϕ, where τ is the (approximate) unit tangent vector orienting Γ. From (2.16) it is clear that for $v \in BV(\Omega; \{\pm 1\})$, one has

$$\mathbf{M}(\star dv) = \text{total variation of } v = 2\mathcal{H}^1(\Gamma). \qquad (2.17)$$

Next, consider the situation where $\Omega \subset \mathbb{R}^3$, and where a function u lies in the Sobolev space $W^{1,2}(\Omega; \mathbb{C})$. This will be the setting of a second main application: 3d Ginzburg–Landau theory. We write $J(u)$ to denote the 2-form

$$J(u) = u^\#(dy) = du_1 \wedge du_2$$

where $u = u_1 + iu_2$, dy denotes the standard area form on the target $\mathbb{C}$ and $^\#$ denotes the pullback. This object is simply the 2-form naturally associated with the Jacobian vector of 2×2 minors,

$$(\det(u_{X_2}, u_{X_3}), \det(u_{X_3}, u_{X_1}), \det(u_{X_1}, u_{X_2})).$$

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[1] In fact, if S is any $(n+1)$-current such that $\partial S = \star dv$, and if Ω is connected, then the definitions imply that there exists some $c \in \mathbb{R}$ such that $S = S_{v+c}$. It follows that $\mathbf{F}(\star dv) = \inf_{c \in \mathbb{R}} \mathbf{M}(S_{v+c}) = \inf_{c \in \mathbb{R}} \|v + c\|_{L^1(\Omega)}$. This is a special case of the representation of $\mathcal{F}'_n(\Omega)$ as a quotient space of $\mathcal{M}_{n+1}(\Omega)$ (see Remark 2.1).

It is often convenient to identify $J(u)$ with a 1-current, which we denote $\star J(u)$, and which is defined through its action on 1-forms ϕ by

$$\star J(u)(\phi) = \int \phi \wedge J(u). \tag{2.18}$$

The current $\star J(u)$ can still be defined for u in certain Sobolev spaces below $W^{1,p}$ for $p < 2$. To this end, we define the 1-form $j(u)$ via the formula

$$j(u) = \frac{1}{2i} \sum_{k=1}^{3} (\bar{u}u_{X_k} - u\bar{u}_{X_k})dX_k = \frac{1}{2i}(\bar{u}du - ud\bar{u}) \tag{2.19}$$

where $\bar{}$ denotes complex conjugation. We also define an associated 2-current $\star j(u)$ that acts on 2-forms ϕ via $\star j(u)(\phi) = \int \phi \wedge j(u)$. Note that $|j(u)| \in L^1(\Omega)$ for $u \in W^{1,2}(\Omega; \mathbb{C})$ or $W^{1,1}(\Omega; S^1)$. Then we define $\star J(u) \in \mathcal{F}'_1(\Omega)$ through

$$\star J(u) = \frac{1}{2}\partial(\star j(u)), \quad \text{so that} \quad \star J(u)(\phi) = \frac{1}{2}\int_\Omega d\phi \wedge j(u) \tag{2.20}$$

for any $\phi \in \mathcal{D}^1(\Omega)$. One can check through integration by parts that this agrees with the previous definition (2.18) of $\star J(u)$ when $u \in W^{1,2}(\Omega)$. This is a consequence of the identity $J(u) = \frac{1}{2}dj(u)$. It follows that

$$\mathbf{F}(\star J(u)) \leq \frac{1}{2}\mathbf{M}(\star j(u)) = \frac{1}{2}\|j(u)\|_{L^1(\Omega)}. \tag{2.21}$$

2.3. Background on degree

Some of our arguments will involve topological degree. The facts we will need are summarized in the following (cf., for example, [33]):

Lemma 2.4. *For any open subset $\mathcal{O} \subset \mathbb{R}^\ell$ and any continuous $f : \bar{\mathcal{O}} \to \mathbb{R}^\ell$ such that $f \neq 0$ on $\partial\mathcal{O}$, there exists an integer called the* degree *of f in $\mathcal{O}$, and denoted $\deg(f, \mathcal{O})$, with the following properties: First, if $f \in C^1(\mathcal{O}; \mathbb{R}^\ell)$ and 0 is a regular value of f, then*

$$\deg(f, \mathcal{O}) = \sum_{\{x \in \mathcal{O} : f(x)=0\}} \text{sign}(\det \nabla f(x)). \tag{2.22}$$

Next,

$$\text{if} \quad \deg(f, \mathcal{O}) \neq 0 \quad \text{then} \quad \exists x \in \mathcal{O} \text{ such that } f(x) = 0. \tag{2.23}$$

Also,

$$\text{if} \quad \mathcal{O}' \text{ is open, } \mathcal{O}' \subset \mathcal{O}, \quad \text{then} \quad \deg(f, \mathcal{O}) = \deg(f, \mathcal{O} \setminus \bar{\mathcal{O}}') + \deg(f, \mathcal{O}') \tag{2.24}$$

whenever the right-hand side makes sense (i.e., whenever $f \neq 0$ on $\partial\mathcal{O}'$). Finally, if $h : \bar{\mathcal{O}} \times [a, b] \to \mathbb{R}^\ell$ is continuous and $h(x, t) \neq 0$ for $x \in \partial\mathcal{O}$ and $t \in [a, b]$, then

$$\deg(h(\cdot, a), \mathcal{O}) = \deg(h(\cdot, b), \mathcal{O}). \tag{2.25}$$

3. Background on Γ-limits

3.1. Definition of Γ-limit

We consider here the Γ-convergence as $\varepsilon \to 0$ of a family of functionals always denoted

$$E_U^\varepsilon : U \to (-\infty, \infty], \quad \text{where } U \text{ is a Banach space and } \varepsilon \in (0, 1], \quad (3.1)$$

to a limiting functional

$$E_V : V \to (-\infty, \infty], \quad \text{where } V \text{ is a Banach space.} \quad (3.2)$$

We will always write

$$V_0 := \{v \in V : E_V(v) < \infty\}. \quad (3.3)$$

In the situations we consider, E_V is always lower semicontinuous (see below), and so V_0 is always closed.

   We say that E_U^ε Γ-*converges* to E_V as $\varepsilon \to 0$ if for all $\varepsilon \in (0, 1]$ there exists a continuous map $P_{VU}^\varepsilon : U \to V$ and a map $Q_{UV}^\varepsilon : V_0 \to U$ (not necessarily continuous) such that

- **Lower bound**: If $v \in V_0$ and $\{u_\varepsilon\} \subset U$ is a sequence such that $\|P_{VU}^\varepsilon(u_\varepsilon) - v\|_V \to 0$ as $\varepsilon \to 0$, then

$$\liminf E_U^\varepsilon(u_\varepsilon) \geq E_V(v). \quad (3.4)$$

- **Upper bound**: For every $v \in V_0$,

$$E_U^\varepsilon(Q_{UV}^\varepsilon(v)) \to E_V(v) \quad \text{and} \quad \|P_{VU}^\varepsilon Q_{UV}^\varepsilon(v) - v\|_V \to 0 \quad \text{as } \varepsilon \to 0. \quad (3.5)$$

   When the above holds, we will sometimes write that E_U^ε Γ-converges to E_V via maps $P_{VU}^\varepsilon, Q_{UV}^\varepsilon$; this is more accurate than simply speaking about Γ-convergence, since the relationship between E_U^ε and E_V is not determined until P_{VU}^ε is specified.

   For fixed $v \in V_0$, the sequence $\{Q_{UV}^\varepsilon(v)\}_{\varepsilon \in (0,1]} \subset U$ is what is sometimes called a *recovery sequence* for v. Our later results will actually require that (3.5) hold only for v in certain finite-dimensional subsets of V_0; we will also need Q_{UV}^ε to be continuous on these subsets. In most applications, including those presented in this article, the maps P_{VU}^ε are independent of $\varepsilon \in (0, 1]$.

   We will only be interested in Γ-limits for which the following compactness condition is satisfied:

- **Compactness**: If $\sup_{\varepsilon \in (0,1]} E_U^\varepsilon(u_\varepsilon) < \infty$ then

$$\{P_{VU}^\varepsilon(u_\varepsilon)\}_{\varepsilon \in (0,1]} \text{ is precompact in } V. \quad (3.6)$$

The definitions imply that

$$E_V : V \to \mathbb{R} \text{ is lower semicontinuous} \quad (3.7)$$

and also that for every $K \in \mathbb{R}$,

$$\{v \in V_0 : E_V(v) \leq K\} \text{ is compact in } V. \quad (3.8)$$

These facts are standard and are quite easy to check.

3.2. Example 1: the 2d Modica–Mortola functional

For this family of problems, Ω is a bounded domain in $\mathbb{R}^2$, and

$$E_U^\varepsilon(u) := \frac{3}{2\sqrt{2}} \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{4\varepsilon} (u^2 - 1)^2 \right) dX \qquad (3.9)$$

is a family of functionals on $U := H^1(\Omega) (= H^1(\Omega; \mathbb{R}))$. We note that depending on context, this energy is also referred to as the Allen–Cahn energy or, in the presence of a mass constraint, the Cahn–Hilliard energy. The factor of $\frac{3}{2\sqrt{2}}$ is a convenient normalization that has the effect of setting a constant in the Γ-limit functional to 1.

In order to emphasize parallels and give a unified treatment of various problems we consider, we describe the Γ-limit in a slightly unusual way:

Theorem 3.1 (cf. [23]). *Let Ω be a bounded, open subset of $\mathbb{R}^2$. Let $U = H^1(\Omega)$ as above, and let $V = \mathcal{F}_1'(\Omega)$, endowed as usual with the flat norm. Define $P_{VU}(u) = \star du/2$ (cf. Section* 2.2*), and define*

$$E_V(T) = \begin{cases} \mathbf{M}(T) & \text{if } T \in V_0 = \{\star dv/2 : v \in BV(\Omega; \{\pm 1\})\}, \\ +\infty & \text{otherwise.} \end{cases} \qquad (3.10)$$

Then there exists a family of maps $Q_{UV}^\varepsilon : V_0 \to U$ such that the family E_U^ε given by (3.9) *Γ-converges to E_V given by* (3.10) *in the sense of* (3.4) *and* (3.5). *Furthermore, the compactness property* (3.6) *holds.*

Proof. This follows from the standard Modica–Mortola Γ-limit result, which we will describe by a functional $E_{\tilde V}$ defined on a space $\tilde V$. In this standard result, $\tilde V = L^1(\Omega)$, $P_{\tilde V U}(u) = u$, and

$$E_{\tilde V}(v) = \begin{cases} \frac{1}{2} \int_\Omega |Dv| & \text{if } v \in \tilde V_0 = BV(\Omega; \{\pm 1\}), \\ +\infty & \text{if not,} \end{cases}$$

where $\int_\Omega |Dv|$ denotes the total variation of the gradient measure.

It follows from (2.14) that if $\{v_k\}$ is precompact in $L^1(\Omega)$, then $\{\star dv_k/2\}$ is precompact in $\mathcal{F}_1'(\Omega)$, so the compactness property (3.6) in $V = \mathcal{F}_1'(\Omega)$ follows from the corresponding property in $\tilde V$, which is established in [22] or [32]. Note also that $v \in \tilde V_0$ if and only if $\star dv/2 \in V_0$, and $E_{\tilde V}(v) = 2E_V(\star dv/2)$.

To check (3.4), we may assume $\liminf E_U^\varepsilon(u_\varepsilon) < \infty$. Hence, invoking the precompactness in $L^1(\Omega)$, we can assert that $\|P_{\tilde V U}^{\varepsilon_j} Q_{U \tilde V}^{\varepsilon_j}(v) - v\|_{\tilde V} \to 0$ along a subsequence $\{\varepsilon_j\} \to 0$ for some $v \in L^1(\Omega)$. Then (3.4) for $E_V : V \to (-\infty, \infty]$ follows from the lower semicontinuity established for $E_{\tilde V}$ under L^1 convergence in [23]. Condition (3.5) also follows from the analogous construction in [23]. $\qquad \square$

The topology we have specified for the Γ-limit is slightly weaker than the more usual L^1 topology. This does not have any effect on our later applications.

In Section 6, the mappings Q_{UV}^ε satisfying (3.5) in this setting will be recalled explicitly for the case of a straight interface. The theorem above holds for $\Omega \subset \mathbb{R}^{n+1}$ with n arbitrary, as long as $\mathcal{F}_1'(\Omega)$ is replaced by $\mathcal{F}_n'(\Omega)$.

3.3. Example 2: the Ginzburg–Landau functional

For this family of problems, Ω is a bounded domain in $\mathbb{R}^3$, and

$$E_U^\varepsilon(u) := \frac{1}{\pi |\ln \varepsilon|} \int_\Omega \left(\frac{|\nabla u|^2}{2} + \frac{(|u|^2 - 1)^2}{4\varepsilon^2} \right) dX \qquad (3.11)$$

is a family of functionals defined on $U := H^1(\Omega; \mathbb{C}) \cong U := H^1(\Omega; \mathbb{R}^2)$. The result on Γ-convergence in this setting is then:

Theorem 3.2 (cf. [2, 14]). *Let Ω be a bounded, open subset of $\mathbb{R}^3$. Let $U = H^1(\Omega; \mathbb{C})$, and let $V = \mathcal{F}_1'(\Omega)$, endowed with the flat norm* (2.8). *Define*

$$P_{VU}(u) = \frac{\star Ju}{\pi} \qquad (3.12)$$

(cf. (2.18) *or* (2.20)*) and*

$$E_V(T) = \begin{cases} \mathbf{M}(T) & \text{if } T \in V_0 := \mathcal{R}_1'(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \qquad (3.13)$$

(cf. (2.7)*). Then there exists a family of maps $Q_{UV}^\varepsilon : V_0 \to U$ such that E_U^ε given by* (3.11) *Γ-converges to E_V given by* (3.13) *in the sense of* (3.4) *and* (3.5). *Furthermore, the compactness property* (3.6) *holds.*

In Section 6, the mappings Q_{UV}^ε satisfying (3.5) will be recalled explicitly for the case of a straight vortex line, that is, for the case where $T \in \mathcal{R}_1'(\Omega)$ consists of an oriented line segment with endpoints lying on $\partial\Omega$. Proofs of (3.4) and (3.6) can be found in [14] while the general recovery sequence construction (3.5) is established in [2].

In Section 6 we will also consider the case of the Ginzburg–Landau energy with magnetic field (cf. (6.7)).

We point out that the Γ-limits in the two examples above involve the mass of integral 1-currents—that is, arclength. The only differences are in the dimension of the ambient space, and the fact that V_0 in (3.10) is smaller than its counterpart in (3.13); see the proof of Corollary 5.2 at the end of Section 5 for a full discussion.

Finally, we record here the simple fact that both the Modica–Mortola functional and the Ginzburg–Landau functional satisfy the Palais–Smale condition (cf. [33]). We recall that given a C^1 functional $F : U \to \mathbb{R}$, a sequence $\{u_k\}_{k=1}^\infty$ is said to be a *Palais–Smale sequence* if

$$\|\nabla F(u_k)\|_{U^*} \to 0 \quad \text{as } k \to \infty \quad \text{and} \quad \{F(u_k)\}_{k=1}^\infty \text{ is bounded.} \qquad (3.14)$$

The functional F is said to satisfy the *Palais–Smale condition* if

$$\text{every Palais–Smale sequence is precompact in } U. \qquad (3.15)$$

Proposition 3.3. *The functionals E_U^ε given by* (3.9) *or* (3.11) *satisfy the Palais–Smale condition in the H^1 topology.*

We recall the standard proof:

Proof. Consider the 2d Modica–Mortola energy; the argument for 3d Ginzburg–Landau is almost identical. Since the argument is unrelated to the (fixed) value of ε, we set $\varepsilon = 1$ here. Suppose $\{u_k\} \subset H^1(\Omega)$ is a sequence satisfying the conditions

$$\sup_k E^\varepsilon_U(u_k) < \infty, \quad \|\delta E^\varepsilon_U(u_k)\| \to 0 \quad \text{as } k \to \infty.$$

The energy bound immediately yields a uniform H^1 bound. Hence there is a subsequence $\{k_j\} \to \infty$ such that

$$u_{k_j} \rightharpoonup u \quad \text{in } H^1(\Omega) \quad \text{and} \quad u_{k_j} \to u \quad \text{in } L^p(\Omega), \quad 1 \le p < \infty,$$

for some $u \in H^1(\Omega)$. Noting that the first variation is given by

$$\delta E^\varepsilon_U(u_k)(v) = \int_\Omega (\nabla u_k \cdot \nabla v + (u_k^3 - u_k)v) \, dX,$$

we then use the conditions $\delta E^\varepsilon_U(u_k)(u_k) \to 0$ and $\delta E^\varepsilon_U(u_k)(u) \to 0$ to see that

$$\lim_{j \to \infty} \int_\Omega |\nabla u_{k_j}|^2 \, dX = \lim_{j \to \infty} \int_\Omega (u_{k_j}^4 - u_{k_j}^2) \, dX = \int_\Omega (u^4 - u^2) \, dX = \int_\Omega |\nabla u|^2 \, dX,$$

and so the convergence is strong. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.4. Example 3: some generalizations

It is worth noting that Example 2 is a special case of the following more general fact, due to Alberti, Baldo, and Orlandi [2]:

Theorem 3.4 (cf. [2]). *Let Ω be a bounded, open subset of $\mathbb{R}^{n+1}$. Let $U = H^1(\Omega; \mathbb{R}^n)$, and let*

$$E^\varepsilon_U(u) := \frac{1}{\omega_n |\ln \varepsilon|} \int_\Omega \left(\frac{1}{n} |\nabla u|^n + \frac{1}{4\varepsilon^2} (|u|^2 - 1)^2 \right) dX, \tag{3.16}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Let $V = \mathcal{F}'_1(\Omega)$, $V_0 = \mathcal{R}'_1(\Omega)$ and let $P_{VU} : U \to V$ be given by

$$P_{VU}(u) = \frac{\star J u}{\omega_n} \tag{3.17}$$

where $J(u) = u^\#(dy) = du_1 \wedge \cdots \wedge du_n$ and $\star J(u)$ denotes the 1-current associated with the n-form $J(u)$ defined exactly as in (2.18).

Then there exists a family of maps $Q^\varepsilon_{UV} : V_0 \to U$ such that E^ε_U given by (3.11) Γ-converges to E_V given by (3.13) in the sense of (3.4) and (3.5). Furthermore, the compactness property (3.6) holds.

Note also that the proof of (3.3) is easily modified to prove that the functional defined in (3.16) satisfies the Palais–Smale condition.

4. General asymptotic saddle point theorem

In this section we define saddle points, and we prove our main result, an abstract theorem stating that if a Γ-limiting functional E_V has a saddle point at some $v_s \in V_0$, then for sufficiently small ε the approximating functional E_U^ε has a critical point whose associated critical value approaches the number $E_V(v_s)$.

Throughout this work, all saddle points are taken to have finitely many unstable directions.

4.1. Definition of saddle point

Throughout this section we assume that V is a Banach space, and that $E_V : V \to \mathbb{R}$ is a lower semicontinuous functional such that sublevel sets of E_V are compact in V. We continue to write V_0 as in (3.3).

Definition 4.1. *We say that E_V has a* saddle point *at $v_s \in V_0$ if there exists a nonnegative integer ℓ, a number $\delta_0 > 0$, a neighborhood $W \subset \mathbb{R}^\ell$ of 0, a continuous map $P_{WV} : V \to \mathbb{R}^\ell$ such that $P_{WV}(v_s) = 0$, and a continuous map $Q_{VW} : W \to V_0$ satisfying the conditions*

$$E_V(v_s) < E_V(v) \quad \text{for } v \in V \text{ with } 0 < \|v - v_s\|_V \leq \delta_0,\ P_{WV}(v) = 0, \tag{4.1}$$

$$Q_{VW}(0) = v_s, \tag{4.2}$$

$$P_{WV} \circ Q_{VW}(w) = w \quad \text{for all } w \in W, \tag{4.3}$$

$$\sup_{\{w \in W : |w| \geq r\}} E_V(Q_{VW}(w)) < E_V(v_s) \quad \text{for every } r > 0. \tag{4.4}$$

We note that the value 0 in the condition $P_{WV}(v_s) = 0$ is chosen simply for convenience. Also, if we write $E_W(w) := \inf\{E_V(v) : \|v - v_s\|_V \leq \delta_0,\ P_{WV}(v) = w\}$ for $w \in W$, then these conditions imply that E_W has a strict local maximum at $w = 0$.

Remark 4.2. The integer ℓ can be thought of as the number of unstable directions at v_s. A local minimum can be seen as a degenerate saddle point for which there are no unstable directions. Indeed, if we adopt the convention that $\mathbb{R}^0 = \{0\}$, then a local minimum $v_s \in V_0$ of E_V satisfies (4.1) with $\ell = 0$ and $P_{WV}(v) = 0$, and the conditions of Definition 4.1 hold trivially.

We will need the following

Lemma 4.3. *Suppose that $v_s \in V_0$ is a saddle point in the sense of Definition 4.1. Then for every $\gamma > 0$, there exists $r(\gamma) > 0$ such that*

$$\|v - v_s\|_V \leq \gamma$$

whenever $v \in V_0$ satisfies

$$\|v - v_s\|_V \leq \delta_0, \quad E_V(v) \leq E_V(v_s) + r(\gamma) \quad \text{and} \quad |P_{WV}(v)| \leq r(\gamma).$$

Proof. We suppose toward a contradiction that the conclusion of the lemma is false, so that there exists some $\gamma > 0$ and a sequence $\{v_n\} \subset V_0$ such that

$$\|v_n - v_s\|_V \le \delta_0, \quad E_V(v_n) \le E_V(v_s) + 1/n, \quad |P_{WV}(v_n)| \le 1/n$$

and

$$\|v_n - v_s\|_V > \gamma.$$

In view of (3.7) and (3.8) and the continuity of P_{WV}, we may assume after passing to a subsequence (still labeled v_n) that $v_n \to \bar{v} \in V_0$ with

$$\gamma \le \|\bar{v} - v_s\|_V \le \delta_0, \quad E_V(\bar{v}) \le E_V(v_s), \quad P_{WV}(\bar{v}) = 0.$$

However, (4.1) implies that this is impossible. $\square$

4.2. The asymptotic minmax theorem

In this section we prove a general theorem asserting that if a Γ-limiting functional E_V has a saddle point v_s at which $E_V(v_s) = c$, and if some other uniformity conditions are satisfied, then for every sufficiently small $\varepsilon > 0$, the approximating functional E_U^ε has a Palais–Smale sequence "near the energy level c" for every sufficiently small ε. In all the examples later in this paper in which we prove existence of critical points of concrete functionals, we will do so by verifying that the hypotheses of this abstract theorem are satisfied and then checking that the specific functional satisfies the Palais–Smale condition.

Theorem 4.4. *Suppose that U, V are Banach spaces and that $\{E_U^\varepsilon\}_{\varepsilon \in (0,1]}$ is a family of C^1 functionals mapping U to $\mathbb{R}$ that Γ-converge to a limiting functional $E_V : V_0 \to \mathbb{R}$ via maps $P_{VU}^\varepsilon : U \to V$ and $Q_{UV}^\varepsilon : V_0 \to U$. Assume also that the compactness condition (3.6) holds.*

Let $v_s \in V$ be a saddle point in the sense of Definition 4.1. Assume also (using notation from the definition of a saddle point) that

P_{WV} is uniformly continuous in $\{v \in V : \|v - v_s\|_V \le 2\delta_0\}$,	(4.5)
$Q_{UW}^\varepsilon := Q_{UV}^\varepsilon \circ Q_{VW} : W \to U$ is continuous for all ε,	(4.6)
$\|P_{VU}^\varepsilon \circ Q_{UW}^\varepsilon(w) - Q_{VW}(w)\|_V \to 0$ uniformly in $w \in W$ as $\varepsilon \to 0$,	(4.7)
$E_U^\varepsilon(Q_{UW}^\varepsilon(w)) \to E_V(Q_{VW}(w))$ uniformly in $w \in W$ as $\varepsilon \to 0$.	(4.8)

Then given $\delta_1 > 0$, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ there exists a Palais–Smale sequence $\{u_k^\varepsilon\}_{k=1}^\infty$ satisfying

$$\sup_k |E_U^\varepsilon(u_k^\varepsilon) - E_V(v_s)| \le \delta_1. \tag{4.9}$$

In particular, if E_U^ε satisfies the Palais–Smale condition (3.15) for every ε, then for every small ε there exists a critical point u^ε of E_U^ε such that $\lim_{\varepsilon \to 0} E_U^\varepsilon(u^\varepsilon) = E_V(v_s)$.

Recall that P^{ε}_{VU} and Q_{VW} are assumed to be continuous as parts of the definition of Γ-limit and saddle point respectively.

Remark 4.5. In this level of generality, it need not be true that the critical points of E^{ε}_U converge in any sense to the limiting point v_s as $\varepsilon \to 0$. This is illustrated by the following elementary example:

Fix $\lambda > 0$ and define a family of functions $f^{\varepsilon} : \mathbb{R}^2 \to \mathbb{R}$ by

$$f^{\varepsilon}(x) = x_1[\tanh(x_2) - \lambda \operatorname{sech}(x_1/\varepsilon)].$$

Note that $f^{\varepsilon} \to f = x_1 \tanh(x_2)$ uniformly as $\varepsilon \to 0$. In fact, $f(x) - f^{\varepsilon}(x) = \varepsilon g(x/\varepsilon)$ for $g(x) = \lambda x_1 \operatorname{sech} x_1$. It is clear that $\{\varepsilon g(\cdot/\varepsilon)\}$ are uniformly Lipschitz and converge uniformly to 0, so an interpolation inequality implies that $f^{\varepsilon} \to f$ in $C^{0,\alpha}$ as $\varepsilon \to 0$ for every $\alpha \in (0, 1)$.

It is easy to check that

- f has exactly one critical point, at $x = (0, 0)$. This critical point is nondegenerate in the sense that the Hessian is nonsingular.
- f^{ε} has no critical points if $\lambda \geq 1$; in this case the Palais–Smale condition is not satisfied by f^{ε}, $\varepsilon > 0$.
- For $0 < \lambda < 1$, f^{ε} has a unique critical point (independent of ε) at $x = (0, \tanh^{-1}(\lambda))$. Note that this point can be arbitrarily far from the critical point $(0, 0)$ of f, since $\tanh^{-1}(\lambda) \nearrow \infty$ as $\lambda \nearrow 1$.

Remark 4.6. An inspection of the proof shows that we do not need the full Γ-limit to hold. In particular, we do not need to construct the maps Q^{ε}_{UV} for every $v \in V_0$. It suffices that $Q^{\varepsilon}_{UV}(v)$ be defined for every v of the form $v = Q_{VW}(w)$, $w \in W$. In particular, note that the hypotheses (4.6), (4.7) and (4.8) of Theorem 4.4 only involve $Q^{\varepsilon}_{UW}(w) := Q^{\varepsilon}_{UV} \circ Q_{VW}(w)$. For our later applications to Modica–Mortola and Ginzburg–Landau, this will mean we only need recovery sequences where the limiting singular set is a line segment.

Remark 4.7. If $\delta_0 = +\infty$ in (4.1) then one can give a simpler proof of Theorem 4.4 by a direct appeal to a general minmax principle. Indeed, in this case (4.14) and (4.15) imply that E^{ε}_U satisfies the hypotheses of Theorem 2.8 of [33], and our theorem then follows from this. When δ_0 is finite, however, we do not know of any result in the literature from which Theorem 4.4 can be deduced. One issue is that for $\delta_0 < \infty$, Theorem 4.4 is a local result, unlike say the theorem from [33] cited above, which requires global knowledge of the behavior of E^{ε}_U. Here the only information we have about E^{ε}_U involves its behavior at points u such that $P_{VU}(u)$ is close to the given saddle point of E_V.

For the proof we need the following quantitative deformation lemma:

Lemma 4.8 (cf. [33, Lemma 2.3]). *Let U be a Banach space, $E_U \in C^1(U; \mathbb{R})$, $S \subset U$, $c \in \mathbb{R}$, $\delta, \rho > 0$ such that*

$$\|\nabla E_U(u)\|_{U^*} \geq 8\delta/\rho \quad \text{for all } u \in S_{2\rho} \text{ such that } E_U(u) \in [c - 2\delta, c + 2\delta],$$

where $S_{2\rho} = \{u \in U : \operatorname{dist}(u, S) \leq 2\rho\}$. Then there exists $\phi \in C([0, 1] \times U, U)$ such that

(i) $\phi(t, u) = u$ if $t = 0$ or $u \notin \{u \in S_{2\rho} : E_U(u) \in [c - 2\delta, c + 2\delta]\}$,

(ii) if $u \in S$ and $E_U(u) \le c + \delta$ then $E_U(\phi(1, u)) \le c - \delta$,

(iii) $\phi(t, \cdot)$ is a homeomorphism of U for all $t \in [0, 1]$,

(iv) $\|\phi(t, u) - u\|_U \le \rho$ for all $u \in U, t \in [0, 1]$,

(v) $t \mapsto E_U(\phi(t, u))$ is nonincreasing for all u.

Using the lemma we present the proof of Theorem 4.4.

Proof of Theorem 4.4.

Step 1. The proof will use degree theory at various points. Facts about degree that we will need are summarized in Lemma 2.4.

Let $r > 0$ be small enough that W contains the closed ball of radius r centered at the origin in $\mathbb{R}^\ell$. We will write

$$M := \{w \in W : |w| \le r\} = B(r, 0) \subset \mathbb{R}^\ell.$$

By taking r smaller if necessary, we may also assume that

$$r \le r(\delta_0/4) \quad \text{as defined in Lemma 4.3,} \tag{4.10}$$

and, since Q_{VW} is continuous, that

$$\|Q_{VW}(w) - v_s\|_V \le \delta_0/2 \quad \text{for all } w \in M. \tag{4.11}$$

We will write $P_{WU}^\varepsilon := P_{WV} \circ P_{VU}^\varepsilon$.

Step 2. We first claim that

if $\phi \in C(M; U)$ and $\phi(w) = Q_{UW}^\varepsilon(w)$ for all $w \in \partial M$ then $\deg(P_{WU}^\varepsilon \circ \phi, M) = 1$ (4.12)

for all sufficiently small ε. We write $f^\varepsilon(w) = P_{WU}^\varepsilon \circ \phi(w)$. Note that assumptions (4.5) and (4.7) imply that

$$|P_{WU}^\varepsilon \circ Q_{UW}^\varepsilon(w) - w| \to 0 \quad \text{uniformly in } W \text{ as } \varepsilon \to 0, \tag{4.13}$$

since

$$|P_{WU}^\varepsilon \circ Q_{UW}^\varepsilon(w) - w| = |P_{WV} \circ P_{VU}^\varepsilon \circ Q_{UW}^\varepsilon(w) - P_{WV} \circ Q_{VW}(w)|.$$

It follows that there exists $\varepsilon_0 > 0$ such that

$$|f^\varepsilon(w) - w| = |P_{WU}^\varepsilon \circ Q_{UW}^\varepsilon(w) - w| \le r/2$$

for all $w \in \partial M$, whenever $\varepsilon < \varepsilon_0$. We will show that (4.12) holds for such ε. Indeed, define

$$F^\varepsilon(s, w) = s f^\varepsilon(w) + (1 - s)w = w - s(f^\varepsilon(w) - w).$$

Then $|F^\varepsilon(s, w)| \geq |w| - s|f^\varepsilon(w) - w| \geq r - r/2 = r/2$ for all $s \in [0, 1]$ when $w \in \partial M$ and $\varepsilon < \varepsilon_0$. In particular, $F^\varepsilon(s, w) \neq 0$. It follows from the homotopy invariance of degree (2.25) that

$$\deg(f^\varepsilon, M) = \deg(F^\varepsilon(1, \cdot), M) = \deg(F^\varepsilon(0, \cdot), M).$$

Since $F^\varepsilon(0, \cdot) : M \to M$ is just the identity map and therefore has degree 1 by the explicit formula (2.22), this establishes (4.12).

Step 3. Next we define

$$a_\varepsilon := \sup_{w \in \partial M} E_U^\varepsilon(Q_{UW}^\varepsilon(w)), \quad c_\varepsilon := \inf\{E_U^\varepsilon(u) : P_{WU}^\varepsilon(u) = 0, \; \|P_{VU}^\varepsilon(u) - v_s\|_V \leq \delta_0\}.$$

We claim that as $\varepsilon \to 0$,

$$a_\varepsilon \to a := \sup_{w \in \partial M} E_V(Q_{VW}(w)), \tag{4.14}$$

$$c_\varepsilon \to c := E_V(v_s) > a. \tag{4.15}$$

In fact, (4.14) is an immediate consequence of (4.8). The fact that $c > a$ follows from condition (4.4) in the definition of a saddle point. It follows from (4.12) and property (2.23) of degree that if $\varepsilon < \varepsilon_0$, then there exists $\bar{w} \in M$ such that $P_{WU}^\varepsilon(Q_{UW}^\varepsilon(\bar{w})) = 0$. In addition, if ε_0 is small enough then

$$\|P_{VU}^\varepsilon \circ Q_{UW}^\varepsilon(w) - v_s\|_V \leq \|P_{VU}^\varepsilon \circ Q_{UW}^\varepsilon(w) - Q_{VW}(w)\|_V + \|Q_{VW}(w) - v_s\|_V$$

$$\leq \frac{2}{3}\delta_0 \tag{4.16}$$

for all $w \in M$, on account of (4.7) and the constraint (4.11) on the choice of the parameter r. Since (4.16) holds in particular for $\bar{w}$, it follows that

$$c_\varepsilon \leq E_U^\varepsilon(Q_{UW}^\varepsilon(\bar{w})) \leq \sup_{w \in M} E_U^\varepsilon(Q_{UW}^\varepsilon(w))$$

when $0 < \varepsilon < \varepsilon_0$. Together with (4.8) this yields

$$\limsup_{\varepsilon \to 0} c_\varepsilon \leq \lim_{\varepsilon \to 0} \sup_{w \in M} E_U^\varepsilon(Q_{UW}^\varepsilon(w)) = \sup_{w \in M} E_V(Q_{VW}(w)) = c.$$

To finish the proof of (4.15) we must show that

$$\liminf_{\varepsilon \to 0} c_\varepsilon \geq c. \tag{4.17}$$

To prove this, let ε_n, u_n be sequences such that $\varepsilon_n \to 0$, $P_{WU}^{\varepsilon_n}(u_n) = 0$ and $\lim_{n \to \infty} E_U^{\varepsilon_n}(u_n) = \liminf_{\varepsilon \to 0} c_\varepsilon$. Since $\{E_U^{\varepsilon_n}(u_n)\}$ is bounded, the compactness assumption (3.6) implies that $\{P_{VU}^{\varepsilon_n}(u_n)\}$ is precompact in V. Writing $v_n := P_{VU}^{\varepsilon_n}(u_n)$, after passing to a subsequence and relabeling if necessary, we may assume that $v_n \to \bar{v}$ in V as $n \to \infty$, and then the Γ-limit lower bound (3.4) implies that

$$\liminf c_\varepsilon = \lim E_U^{\varepsilon_n}(u_n) \geq E_V(\bar{v}).$$

However, since P_{WV} is continuous and $P_{WU}^{\varepsilon_n}(u_n) = P_{WV}(v_n) = 0$ for all n, it is clear that $P_{WV}(\bar{v}) = 0$, and then (4.1) implies that $E_V(\bar{v}) \geq E_V(v_s)$. This proves (4.17) and hence (4.15).

Step 4. We now conclude the proof of the theorem, modulo a final claim that will be established below. Recall that we are given $\delta_1 > 0$, and we must find (for every sufficiently small ε) a Palais–Smale sequence satisfying (4.9). We claim that there exists a value $\delta_2 > 0$ such that, taking ε_0 smaller if necessary, we have

$$c_\varepsilon \leq \sup_{w \in M} \{E_U^\varepsilon(Q_{UW}^\varepsilon(w))\} < c_\varepsilon + \delta_2 \qquad (4.18)$$

and

$$\max\{a_\varepsilon, c - \delta_1\} < c_\varepsilon - 2\delta_2 < c_\varepsilon + 2\delta_2 < \min\{c + \delta_1, c + r\} \qquad (4.19)$$

for all $\varepsilon \in (0, \varepsilon_0)$. To see this, just take $\delta_2 < \frac{1}{4}\min\{\delta_1, c - a, r\}$. Then (4.18), (4.19) hold for sufficiently small ε due to (4.14), (4.15), and hypothesis (4.8). We write

$$S = \{u \in U : \|P_{VU}^\varepsilon(u) - v_s\|_V \leq \delta_0\}.$$

Temporarily fix some $\rho > 0$, and assume toward a contradiction that

$$\|\nabla E_U^\varepsilon(u)\|_{U^*} \geq 8\delta_2/\rho \text{ for all } u \in S_{2\rho} \text{ such that } E_U(u) \in [c_\varepsilon - 2\delta_2, c_\varepsilon + 2\delta_2], \qquad (4.20)$$

where $S_{2\rho} = \{u \in U : \text{dist}(u, S) \leq 2\rho\}$. Then the hypotheses of Lemma 4.8 are satisfied (with c replaced by c_ε and δ replaced by δ_2). Let ϕ be a function satisfying (i)–(v) from that lemma. Let us write $\mathfrak{u}_1^\varepsilon(w) := \phi(1, Q_{UW}^\varepsilon(w))$. We will prove below that

$$\exists w \in M \text{ such that } P_{WU}^\varepsilon \circ \mathfrak{u}_1^\varepsilon(w) = 0 \text{ and } \|P_{VU}^\varepsilon \circ \mathfrak{u}_1^\varepsilon(w) - v_s\|_V \leq \delta_0. \qquad (4.21)$$

It follows from this and the definition of c_ε that $\sup_{w \in M} E_U^\varepsilon(\mathfrak{u}_1^\varepsilon(w)) \geq c_\varepsilon$ for all ε sufficiently small.

On the other hand, from (4.18) and property (ii) of ϕ, we deduce that $\sup_{w \in M} E_U^\varepsilon(\mathfrak{u}_1^\varepsilon(w)) \leq c_\varepsilon - \delta_2$. This is a contradiction, which proves that there exists some $u_\rho^\varepsilon \in S_{2\rho}$ such that

$$\|\nabla E_U^\varepsilon(u_\rho^\varepsilon)\|_{U^*} < 8\delta_2/\rho \text{ and } E_U^\varepsilon(u_\rho^\varepsilon) \in [c_\varepsilon - 2\delta_2, c_\varepsilon + 2\delta_2] \subset [c - \delta_1, c + \delta_1]. \qquad (4.22)$$

If now we consider a sequence ρ_n tending to ∞, then $\{u_{\rho_n}^\varepsilon\}_{n=1}^\infty$ gives us a sequence satisfying the conclusions of Theorem 4.4.

It remains to prove (4.21), which is the most technical part of the proof. In fact we will prove

Step 5. If $\phi \in C([0, 1] \times U; U)$ is any map with properties (i), (iii), (iv), (v) of Lemma 4.8, and if we write $\mathfrak{u}_t^\varepsilon(w) := \phi(t, Q_{UW}^\varepsilon(w))$, then for every $t \in [0, 1]$,

$$\exists \bar{w} \in M \text{ such that } P_{WU}^\varepsilon \circ \mathfrak{u}_t^\varepsilon(\bar{w}) = 0 \text{ and } \|P_{VU}^\varepsilon \circ \mathfrak{u}_t^\varepsilon(\bar{w}) - v_s\|_V \leq \frac{1}{2}\delta_0. \qquad (4.23)$$

Note that (4.23) immediately implies (4.21).

We henceforth suppress the superscript ε and write simply $\mathfrak{u}_t$. We will also use the notation

$$\mathfrak{v}_t = P_{VU}^\varepsilon \circ \mathfrak{u}_t^\varepsilon : M \to V \quad \text{and} \quad \mathfrak{w}_t = P_{WU}^\varepsilon \circ \mathfrak{u}_t^\varepsilon : M \to \mathbb{R}^\ell$$

for $w \in W$ and $t \in [0, 1]$.

First, if $w \in \partial M$, then $E_U^\varepsilon(\phi(w)) = E_U^\varepsilon(Q_{UW}^\varepsilon(w)) \leq a_\varepsilon < c_\varepsilon - 2\delta_2$ by (4.19), so property (i) of ϕ implies that

$$\mathfrak{u}_t(w) = \phi(t, Q_{UW}^\varepsilon(w)) = Q_{UW}^\varepsilon(w) \quad \text{for all } w \in \partial M \text{ and } t \in [0, 1]. \qquad (4.24)$$

Hence (4.12) implies that

$$\deg(\mathfrak{w}_t, M) = 1 \quad \text{for all } t \in [0, 1]. \qquad (4.25)$$

Step 6. We next argue that

$$\mathfrak{w}_t(w) \neq 0 \text{ for all } w \text{ such that } \tfrac{1}{2}\delta_0 < \|\mathfrak{v}_t(w) - v_s\|_V \leq \tfrac{3}{4}\delta_0 \text{ and all } t \in [0, 1]. \quad (4.26)$$

Assume that t, w are such that $\|\mathfrak{v}_t(w) - v_s\|_V \leq \tfrac{3}{4}\delta_0$ and $\mathfrak{w}_t(w) = 0$; we must show that $\|\mathfrak{v}_t(w) - v_s\|_V \leq \tfrac{1}{2}\delta_0$. Let γ_0 denote a small positive number that will be fixed in a moment. It will turn out that γ_0 depends only on δ_0 and r. Then Lemma 4.9, which is proved after this theorem, implies that if ε_0 is sufficiently small, there exists $\mathfrak{v}_t'(w) \in V$ such that

$$E_V(\mathfrak{v}_t'(w)) \leq E_U^\varepsilon(\mathfrak{u}_t(w)) + \delta_2, \quad \|\mathfrak{v}_t'(w) - \mathfrak{v}_t(w)\|_V < \gamma_0. \qquad (4.27)$$

We will require that $\gamma_0 \leq \delta_0/4$; then $\|\mathfrak{v}_t'(w) - v_s\|_V \leq \delta_0$. We also require that γ_0 be so small that

$$|P_{WV}(v') - P_{WV}(v)| \leq r$$

for all $v, v' \in V$ such that $\|v - v_s\|_V \leq \tfrac{1}{2}\delta_0$ and $\|v - v'\|_V \leq \gamma_0$. This is possible due to (4.5). In particular, it follows that

$$|P_{WV}(\mathfrak{v}_t'(w))| = |P_{WV}(\mathfrak{v}_t'(w)) - \mathfrak{w}_t(w)| = |P_{WV}(\mathfrak{v}_t'(w)) - P_{WV}(\mathfrak{v}_t(w))| \leq r. \quad (4.28)$$

Next, note that

$$\begin{aligned}
E_V(\mathfrak{v}_t'(w)) &\leq E_U^\varepsilon(\mathfrak{u}_0(w)) + \delta_2 &&\text{by (4.27) and property (v) of } \phi \\
&\leq c_\varepsilon + 2\delta_2 &&\text{by (4.18)} \\
&\leq E_V(v_s) + r &&\text{by (4.19).} &&(4.29)
\end{aligned}$$

Then (4.29), (4.28) are precisely the hypotheses of Lemma 4.3, which implies (recalling our condition (4.10) on r) that $\|\mathfrak{v}_t'(w) - v_s\|_V \leq \delta_0/4$. Then the triangle inequality and (4.27) yield $\|\mathfrak{v}_t(w) - v_s\|_V \leq \delta_0/2$, which is exactly (4.26).

Step 7. We proceed by defining

$$A_t(\sigma) := \{w \in M : \|\mathfrak{v}_t(w) - v_s\|_V > \sigma\}$$

for $\sigma \in \mathbb{R}$ and $t \in [0, 1]$. It is an immmediate consequence of (4.26) that for $\sigma \in [\frac{1}{2}\delta_0, \frac{3}{4}\delta_0]$, $\mathfrak{w}_t$ does not vanish on $\partial A_t(\sigma)$ and hence that $\deg(\mathfrak{w}_t, A_t(\sigma))$ is well-defined for such σ. Property (2.23) of degree further implies that for every $t \in [0, 1]$, there exists a number $d(t)$ such that

$$d(t) = \deg(\mathfrak{w}_t, A_t(\sigma)) \quad \text{for all } \sigma \in [\tfrac{1}{2}\delta_0, \tfrac{3}{4}\delta_0]. \tag{4.30}$$

We will prove that

$$d(t) = 0 \quad \text{for all } t \in [0, 1]. \tag{4.31}$$

Note that $A_0(\frac{3}{4}\delta_0)$ is empty, by (4.16), so $d(0) = 0$. Since $d(t)$ is an integer for every t, it now suffices to prove that $d(\cdot)$ is continuous on the interval $[0, 1]$.

To see this, fix some $t_0 \in [0, 1]$. Since $(w, t) \mapsto \|\mathfrak{v}_t(w) - v_s\|_V$ is continuous, and hence uniformly continuous on $M \times [0, 1]$, we infer that

$$A_t(\tfrac{3}{4}\delta_0) \subset A_{t_0}(\tfrac{2}{3}\delta_0) \subset A_t(\tfrac{1}{2}\delta_0) \tag{4.32}$$

for $t \in [0, 1]$ sufficiently close to t_0. (Note that the subscript in the middle term of (4.32) is t_0 rather than t.) Thus (4.26) implies that for this range of t, $\mathfrak{w}_t$ has no zeros on $\partial A_{t_0}(\frac{2}{3}\delta_0)$, and hence (by the homotopy invariance (2.25) of degree) that

$$\deg(\mathfrak{w}_t, A_{t_0}(\tfrac{2}{3}\delta_0)) = \deg(\mathfrak{w}_{t_0}, A_{t_0}(\tfrac{2}{3}\delta_0)) = d(t_0).$$

Moreover, $\deg(\mathfrak{w}_t, A_{t_0}(\frac{2}{3}\delta_0) \setminus A_t(\frac{3}{4}\delta_0)) = 0$; this is a consequence of (4.26), (4.32) and property (2.23) of degree. Thus for such t,

$$\begin{aligned} d(t) &= \deg(\mathfrak{w}_t, A_t(\tfrac{3}{4}\delta_0)) = \deg(\mathfrak{w}_t, A_{t_0}(\tfrac{2}{3}\delta_0)) - \deg\big(\mathfrak{w}_t, A_{t_0}(\tfrac{2}{3}\delta_0) \setminus A_t(\tfrac{3}{4}\delta_0)\big) \\ &= \deg(\mathfrak{w}_t, A_{t_0}(\tfrac{2}{3}\delta_0)) = d(t_0). \end{aligned}$$

Thus the proof of (4.31) is completed.

Step 8. It follows from (4.25) and (4.31) and the additivity property (2.24) of degree that

$$\deg(\mathfrak{w}_t, M \setminus A_t(\tfrac{1}{2}\delta_0)) = \deg(\mathfrak{w}_t, M) - \deg(\mathfrak{w}_t, A_t(\tfrac{1}{2}\delta_0)) = 1 \quad \text{for all } t \in [0, 1]$$

and hence via the property (2.23) of degree that, for every $t \in [0, 1]$, $\mathfrak{w}_t$ has a zero in $M \setminus A_t(\frac{1}{2}\delta_0)$. After undoing the notation, this is exactly equivalent to (4.23) and hence completes the proof of the theorem. $\qquad\square$

This lemma was used above:

Lemma 4.9. *Given $K, \gamma_0, \delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(K, \gamma_0, \delta) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the condition*

$$E_U^\varepsilon(u) < K \quad \text{for some } u \in U \tag{4.33}$$

implies the existence of $v' \in V$ such that

$$\|P_{VU}^\varepsilon(u) - v'\|_V < \gamma_0 \quad \text{and} \quad E_U^\varepsilon(u) > E_V(v') - \delta. \tag{4.34}$$

Proof. Suppose toward a contradiction that no such ε_0 exists, so that there exist sequences $\varepsilon_n \to 0$ and $u_n \in U$ satisfying (4.33) and such that

$$E_U^{\varepsilon_n}(u_n) + \delta \leq \inf\{E_V(v) : v \in V, \; \|P_{VU}^{\varepsilon_n}(u_n) - v\|_V < \gamma_0\}. \tag{4.35}$$

Let us write $v_n := P_{VU}^{\varepsilon_n}(u_n)$ and $w_n := P_{WU}^{\varepsilon_n}(u_n) = P_{WV}(v_n)$. The energy bound (4.33) and compactness assumption (3.6) imply that there exists some $\bar{v} \in V_0$ such that after passing to a subsequence (still labeled $\{v_n\}$) if necessary, $v_n \to \bar{v}$ in V. We then deduce from (3.4) that

$$\liminf_{n \to \infty} E_U^{\varepsilon_n}(u_n) \geq E_V(\bar{v}).$$

On the other hand, since $\|v_n - \bar{v}\|_V \to 0$, for sufficiently large n it must be the case that

$$\inf\{E_V(v) : \|v - v_n\|_V < \gamma_0\} \leq E_V(\bar{v}).$$

Recalling (4.35), we deduce that

$$\limsup_{n \to \infty} E_U^{\varepsilon_n}(u_n) + \delta \leq E_V(\bar{v}) \leq \liminf_{n \to \infty} E_U^{\varepsilon_n}(u_n),$$

which is a contradiction. This proves (4.34). □

5. Critical points of arclength

We wish to illustrate the use of Theorem 4.4 by applying it to produce critical points of the 2d Modica–Mortola energy (3.9) and 3d Ginzburg–Landau energy (3.11), as well as some other examples. As was discussed in Section 3, these functionals Γ-converge to the mass of 1-currents; that is, roughly speaking, they converge to the arclength of a Lipschitz continuous curve. Throughout this section, we will assume that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, is a bounded domain with smooth boundary. We will prove that given an oriented line segment joining two points on $\partial\Omega$ that is a nondegenerate saddle point of arclength in a naive sense, the associated current T_* is a saddle point in the sense of Definition 4.1 of the Γ-limit of the functional

$$E_V(T) = \begin{cases} \mathbf{M}(T) & \text{if } T \in V_0 := \mathcal{R}_1'(\Omega), \\ +\infty & \text{if not,} \end{cases} \tag{5.1}$$

where $V = \mathcal{F}_1'(\Omega)$.

To formulate these results, we introduce some notation. As usual, we will write points in $\mathbb{R}^{n+1}$ either in the form (x, x_{n+1}) with $x \in \mathbb{R}^n$, or simply as $X \in \mathbb{R}^{n+1}$. Throughout this section, B_r denotes the n-dimensional ball $\{x \in \mathbb{R}^n : |x| < r\}$. We will write

$$C_r := B_r \times \mathbb{R} \subset \mathbb{R}^{n+1}.$$

We assume that for some fixed $R > 0$, there are two C^3 functions $h^-, h^+ : B_R \to \mathbb{R}$ and a connected component Ω_R of $\Omega \cap C_R$ such that

$$\Omega_R = \{(x, x_{n+1}) : |x| < R, \; h^-(x) < x_{n+1} < h^+(x)\} \tag{5.2}$$

with

$$h^-(0) = 0 \quad \text{and} \quad L := h^+(0) > 0.$$

We will write $\partial\Omega^-, \partial\Omega^+$ for the lower and upper portions of $\partial\Omega_R$ respectively and we will write

$$\psi^{\pm}(x) := (x, h^{\pm}(x)).$$

We will also write

$$\Omega_r = \Omega_R \cap C_r \quad \text{for } r \leq R.$$

A crucial role in describing saddle points of length will be played by the distance function $d_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$d_0(x, y) := |\psi^-(x) - \psi^+(y)| = \sqrt{|x - y|^2 + (h^-(x) - h^+(y))^2}. \qquad (5.3)$$

The main result of this section is

Theorem 5.1. *Let Ω be an open, bounded set in $\mathbb{R}^{n+1}$, $n \geq 1$. Assume that* (5.2) *holds and that d_0 has a nondegenerate critical point at $(0, 0)$ in the sense that*

$$\nabla d_0(0, 0) = (-\nabla h^-(0), \nabla h^+(0)) = 0 \quad and \quad \det D^2 d_0(0, 0) \neq 0, \qquad (5.4)$$

where $D^2 d_0$ denotes the Hessian matrix of second partials. Let T_* denote the multiplicity 1-current corresponding to the oriented line segment in Ω starting at $\psi^-(0)$ and ending at $\psi^+(0)$, and assume that $T_* \in \mathcal{R}'_1(\Omega)$. Let E_V be given by (5.1) *with $V = \mathcal{F}'_1(\Omega)$. Then T_* is a saddle point of $E_V : V \to \mathbb{R}$ in the sense of Definition* 4.1.

For applications to the Modica–Mortola problem, we will also need

Corollary 5.2. *Suppose that $n = 1$, and that all assumptions of Theorem* 5.1 *hold. Then T_* is a saddle point of $E_V : V \to \mathbb{R}$ as defined in* (3.10).

The proof of the corollary appears at the end of this section.

The nondegeneracy condition states in geometric language that there are no nontrivial Jacobi fields associated with the segment connecting $\psi^-(0)$ and $\psi^+(0)$ and with natural boundary conditions; this condition appears also in [16, 25, 6].

Since T_* is by definition integer multiplicity rectifiable, the point of the assumption $T_* \in \mathcal{R}'_1(\Omega)$ is that we require T_* to be a boundary. This holds if and only if $\psi^-(0)$ and $\psi^+(0)$ belong to the same component of $\partial\Omega$.

Recall that Definition 4.1 involves maps $P_{WV} : V \to W$ and $Q_{VW} : W \to V$, where W is a subset of some Euclidean space $\mathbb{R}^\ell$. Here ℓ will turn out to be the number of negative eigenvalues of $D^2 d_0(0, 0)$.

5.1. Construction of P_{WV}

Invoking the nondegeneracy assumption, let us denote the $2n$ not necessarily distinct eigenvalues of the symmetric matrix $D^2 d_0(0, 0)$ by

$$\lambda_1 \leq \cdots \leq \lambda_\ell < 0 < \lambda_{\ell+1} \leq \cdots \leq \lambda_{2n}. \qquad (5.5)$$

Here we assume $\ell \in \{0, \ldots, 2n\}$. The case $\ell = 0$ corresponds to a local minimum of arclength. In this case, our arguments will show that the associated current T_* is a local minimizer of mass. This has been proved elsewhere when $l = 0$, under the assumption that h^- is concave and h^+ convex near $x = 0$ [15, 24]. Below we adopt the convention that when $l = 0$, $\mathbb{R}^\ell = \{0\}$.

We let A denote the $2n \times 2n$ matrix having as column vectors an orthonormal basis of eigenvectors for $D^2 d_0(0, 0)$ ordered as in (5.5). Then we introduce $w = (w_1, \ldots, w_\ell)$ and $\zeta = (\zeta_1, \ldots, \zeta_{2n-\ell})$ through the relation

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} w \\ \zeta \end{pmatrix}. \tag{5.6}$$

We also define a linear mapping $p : \mathbb{R}^{2n} \to \mathbb{R}^\ell$ by

$$p(x, y) = w. \tag{5.7}$$

Note from the construction that $AA^T = A^T A = I$. Also, from Taylor's Theorem, we have

$$d_0(x, y) = d_0\left(A^T \begin{pmatrix} w \\ \zeta \end{pmatrix}\right)$$
$$= L + \lambda_1 w_1^2 + \cdots + \lambda_\ell w_\ell^2 + \lambda_{\ell+1} \zeta_1^2 + \cdots + \lambda_{2n} \zeta_{2n-\ell}^2 + o(|w|^2, |\zeta|^2). \tag{5.8}$$

The idea of the construction of P_{WV} is as follows: Note that $p : \mathbb{R}^{2n} \to \mathbb{R}^\ell$ is a projection onto the "unstable directions" of d_0 near $(0, 0)$, that is, onto directions associated with negative eigenvalues of $D^2 d_0$. Given a current T near T_*, we would similarly like to construct a projection onto an ℓ-dimensional space of unstable directions for the functional E_V given by (5.1), which is a sort of extension of d_0 to a much larger space. Heuristically, we would like to define

$$P_{WV}(T) = p(x(T), y(T)), \tag{5.9}$$

where $x(T)$ is the "lower endpoint of T" (in the coordinates we have been using for $\partial\Omega^-$), and $y(T)$ is the "upper endpoint." Then for example, to verify condition (4.1) in the definition of a saddle point, we would have to check that if T corresponds to a curve near T_* whose endpoints $x(T)$, $y(T)$ satisfy the constraint $p(x(T), y(T)) = 0$, then this curve is longer than L. This is immediate from (5.5), (5.7), (5.8).

However, (5.9) does not in general make sense for an arbitrary rectifiable 1-current T, and even in cases when we can define what we mean by an upper and lower endpoint of T, the map

$$T \mapsto \text{lower endpoint of } T \tag{5.10}$$

for example, is certainly not continuous in the V (flat) norm. To get around this, we define $\mathbb{R}^n$-valued 1-forms Φ^-, Φ^+ such that, for example, $T \mapsto T(\Phi^-)$ is a smoothing of (5.10), constructed by averaging the "x-coordinates" of T over level sets of the function $s^-(X) = \text{dist}(X, \partial\Omega^-)$ for $X \in \Omega$ near $\partial\Omega^-$. Then we replace $x(T)$, $y(T)$ in (5.9) by

$T(\Phi^-)$, $T(\Phi^+)$ to obtain a map P_{WV} that is formally similar to (5.9) but well-defined and continuous in suitable weak topologies.

To set up the definitions of these 1-forms, we will need a bit more notation. In particular, we will write $\nu^\pm(x) := \pm(-\nabla h^\pm(x), 1)/(1 + |\nabla h^\pm|^2)^{1/2}$ for the outer unit normal to $\partial\Omega$ at $\psi^\pm(x)$, and

$$\Psi^\pm(x, s) := \psi^\pm(x) - s\nu^\pm(x). \tag{5.11}$$

For concreteness we assume that

$$h^-(x) \le |x| \le L - |x| \le h^+(x) \quad \text{in } B_R, \tag{5.12}$$

and note that $d_0(0, 0) = \mathbf{M}(T_*) = L$. It is convenient to assume that $R \le L/3$. Appealing to (5.8), we also assume, taking R smaller if necessary, that

$$d_0(x, y) \ge L - \lambda_-|w|^2 + \lambda^+|\zeta|^2 \quad \text{for all } (x, y) \in B_R \times B_R \tag{5.13}$$

for $\lambda_- = 2|\lambda_1|$ and $\lambda_+ = \frac{1}{2}\lambda_{\ell+1}$.

Next, after further shrinking R if need be, we fix $s_0 > 0$ such that

$$\Psi^-, \Psi^+ \text{ are diffeomorphisms of } B_R \times (0, s_0) \text{ onto their images}, \tag{5.14}$$

$$\{X \in \Omega_{R/2} : \text{dist}(X, \partial\Omega^\pm) < s_0\} \subset \Psi^\pm(B_{2R/3} \times (0, s_0)) \tag{5.15}$$

and

$$|(\nabla_x \Psi^\pm)v| \ge \frac{1}{2}|v| \quad \text{in } B_R \times (0, s_0) \quad \text{for all } v \in \mathbb{R}^n. \tag{5.16}$$

The last conditions are possible since $\nabla h^\pm(0) = 0$. (Here $(\nabla_x \Psi^\pm)\,v$ denotes matrix-vector multiplication.)

At this point R and s_0 are fixed once and for all.

We remark that it is easy to check that

$$\partial_{x_i}\Psi^\pm \cdot \partial_s\Psi^\pm \equiv 0 \quad \text{for } i = 1, \ldots, n \quad \text{and} \quad |\partial_s\Psi^\pm| \equiv 1 \tag{5.17}$$

in $B_R \times (0, s_0)$.

Let $X \mapsto (x^\pm(X), s^\pm(X))$ denote the inverse maps of $\Psi^\pm$, so that

$$x^-(\Psi^-(x, s)) = x, \quad s^-(\Psi^-(x, s)) = s, \quad x^+(\Psi^+(x, s)) = x, \quad s^+(\Psi^+(x, s)) = s \tag{5.18}$$

in $B_R \times (0, s_0)$. We set $s^\pm = 0$, $x^\pm = 0$ away from the set $\Psi^\pm(B_R \times (0, s_0))$. Note that on the image of $\Psi^\pm$, $s^\pm$ is just the distance from $\partial\Omega^\pm$, and also that $x^\pm(X) = x$ if and only $\Psi^\pm(x)$ is the unique closest point to X on $\partial\Omega$.

Let $\eta \in C_c^\infty(B_R)$ be a function such that $\eta(x) = 1$ if $|x| < 2R/3$ and $0 \le \eta \le 1$ for all x, and define $\xi^\pm : \Omega \to B_R \subset \mathbb{R}^n$ by

$$\xi^\pm(X) := \begin{cases} \eta(x^\pm(X))x^\pm(X) & \text{if } X \in \Psi^\pm(B_R \times (0, s_0)), \\ 0 & \text{if not.} \end{cases} \tag{5.19}$$

Observe that the definitions and (5.15) imply that

$$\xi^\pm = x^\pm \quad \text{in } \{X \in \Omega_{R/2} : \text{dist}(X, \partial\Omega^\pm) < s_0\}. \tag{5.20}$$

We next introduce a new positive parameter $s_1 < \frac{1}{2}s_0$, to be further specified later, and we fix a smooth nonnegative function $\omega : \mathbb{R} \to [0, \infty)$ satisfying

$$\operatorname{supp}(\omega) \subset\subset (0, s_1), \quad \int_0^{s_1} \omega(s)\,ds = 1, \quad \omega \leq 2/s_1, \tag{5.21}$$

and we define vector-valued 1-forms $\Phi^\pm = (\phi_1^\pm, \ldots, \phi_n^\pm)$ by

$$\Phi^\pm(X) = \mp \xi^\pm(X)\omega(s^\pm(X))\,ds^\pm. \tag{5.22}$$

Note that $\Phi^\pm$ are smooth; the discontinuities of $\xi^\pm$ and $s^\pm$ occur away from the support of $\Phi^\pm$. Finally, we define

$$P_{WV}(T) := p(T(\Phi^-), T(\Phi^+)) \quad \text{for } T \in \mathcal{F}_1'(\Omega). \tag{5.23}$$

This should be viewed as a well-defined if more complicated realization of the heuristic definition given in (5.9).

Note that if T is a current corresponding to a Lipschitz curve $\gamma : I \to \Omega$, then from (2.4) we find that

$$T(\Phi^\pm) = \mp \int_I \xi^\pm(\gamma(t))\omega(s^\pm(\gamma(t)))\frac{d}{dt}s^\pm(\gamma(t))\,dt. \tag{5.24}$$

The interpretation of $T(\Phi^-)$, for example, as an average of the "x-coordinate" of T over level sets of s^- can be extracted from (5.24) as follows: suppose for simplicity that γ can be reparametrized near $\partial\Omega^-$ in such a way that $\gamma(s) = \Psi^-(x(s), s)$ for some path $x(s) \in B_{R/2}$, $s \in (0, s_0)$. (This can be done if the original parametrization $\gamma(t)$ moves monotonically away from $\partial\Omega^-$ with increasing t, and remains close to T_*.) When this holds, it follows from (5.24) and the definitions that

$$T(\Phi^-) = \int \xi^-(\gamma(s))\omega(s)\,ds = \int x(s)\omega(s)\,ds$$

which is an average of the x-coordinates, as asserted.

We point out that for $T = T_*$, one has $T_*(\Phi^\pm) = 0$ since $x^\pm(X) = 0$ for X on the line segment joining $\psi^-(0)$ to $\psi^+(0)$. Hence,

$$P_{WV}(T_*) = 0. \tag{5.25}$$

This represents the first requirement of Definition 4.1.

We turn next to checking the continuity of the map P_{WV}. In fact, we will establish that $P_{WV} : V \to W$ is Lipschitz continuous since this will imply the extra requirement (4.5) of Theorem 4.4, to be needed later.

Lemma 5.3. *There exists a constant C (depending on R, s_0, $\partial\Omega$ but independent of s_1) such that for all $T, T' \in \mathcal{F}_1'(\Omega)$ one has*

$$|P_{WV}(T) - P_{WV}(T')| \leq \frac{C}{s_1}\mathbf{F}(T - T'). \tag{5.26}$$

One can check that (5.26) holds with $C = 3 \max\{\|\nabla\xi^-\|_\infty, \|\nabla\xi^+\|_\infty\}$.

Proof. Given T and T' in $\mathcal{F}'_1(\Omega)$, we can find S such that $\partial S = T - T'$ and $\mathbf{M}(S) \leq 2F(T - T')$. Then

$$|T(\Phi^-) - T'(\Phi^-)| = |(T - T')(\Phi^-)| = |\partial S(\Phi^-)| = |S(d\Phi^-)| \leq \|d\Phi^-\|_\infty \mathbf{M}(S).$$

Recalling (5.19), (5.21) and (5.22), one finds that

$$d\Phi^-(X) = \omega(s^-(X))\frac{\partial\xi^-(X)}{\partial X^i}\frac{\partial s^-(X)}{\partial X^j}\, dX^i \wedge dX^j,$$

so that $\|d\Phi^-\|_\infty \leq C/s_1$, with C independent of s_1. Hence,

$$|T(\Phi^-) - T'(\Phi^-)| \leq \frac{C}{s_1}\mathbf{F}(T - T').$$

The same argument applies also to Φ^+. Finally, it is clear from the definitions (5.6), (5.7) and (5.23) that

$$|P_{WV}(T) - P_{WV}(T')| \leq C(|T(\Phi^-) - T'(\Phi^-)| + |T(\Phi^+) - T'(\Phi^+)|),$$

so (5.26) follows from the above estimates. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.2. *Verification of* (4.1)

The main part of the proof of Theorem 5.1 consists in verifying that T_* is a strict local minimizer of mass in the flat norm topology among competitors in the set $\{T \in \mathcal{R}'_1(\Omega) : P_{WV}(T) = P_{WV}(T_*) = 0\}$ with P_{WV} given by (5.23); this is condition (4.1) in our definition of a saddle point.

   This is the content of the following proposition.

Proposition 5.4. *There exists a positive number δ_0 such that if $T \in \mathcal{R}'_1(\Omega)$ satisfies the conditions*

$$\mathbf{F}(T - T_*) < \delta_0 \quad P_{WV}(T) = P_{WV}(T_*) = 0 \quad and \quad \mathbf{M}(T) \leq \mathbf{M}(T_*) \qquad (5.27)$$

for P_{WV} given by (5.23), then

$$T = T_*. \qquad\qquad\qquad\qquad\qquad (5.28)$$

Given a current T satisfying (5.27), we begin the proof of Proposition 5.4 by arguing that we can modify it in order to be able to assume certain good properties. Our first lemma allows us to replace T by a current corresponding to a single Lipschitz curve that is uniformly close to T_*:

Lemma 5.5. *There exists a constant c such that, for any $r \in (0, R)$, if $\delta_0 < cr^2$ and $T \in \mathcal{R}'_1(\Omega)$ satisfies (5.27), then there exists a 1-current $T' \in \mathcal{R}'_1(\Omega)$ that consists of a single Lipschitz curve starting in $C_r \cap \partial\Omega^-$, ending in $C_r \cap \partial\Omega^+$, and satisfying*

$$\operatorname{supp} T' \subset \bar{\Omega}_r, \tag{5.29}$$

$$\mathbf{M}(T - T') = \mathbf{M}(T) - \mathbf{M}(T'). \tag{5.30}$$

Proof. We will show that the conclusions of the lemma hold with $c = 1/400$. Thus, we fix any $r \in (0, R)$, and we assume that

$$\delta_0 < \frac{1}{400} r^2. \tag{5.31}$$

Then we will prove the existence of a current T' satisfying (5.29) and (5.30). We should mention that the proof never uses the property $P_{WV}(T) = P_{WV}(T_*) = 0$ of (5.27).

Step 1. First, in view of (5.27), there exists a 2-current S such that

$$\partial S = T - T_* \quad \text{in } \Omega, \quad \mathbf{M}(S) < \delta_0.$$

For $X = (x, x_{n+1}) \in \Omega$, let $p_{n+1}(X) = x_{n+1}$. Writing $\langle S, p_{n+1}, s \rangle$ as usual to denote a slice of S by a level set of p_{n+1}, note that

$$\partial \langle S, p_{n+1}, s \rangle = \langle \partial S, p_{n+1}, s \rangle = \langle T, p_{n+1}, s \rangle - \langle T_*, p_{n+1}, s \rangle = \langle T, p_{n+1}, s \rangle - \delta_{(0,s)} \tag{5.32}$$

for a.e. $s \in (0, L)$, since it is clear from (2.10) and our geometric assumptions (see in particular (5.12)) that $\langle T_*, p_{n+1}, s \rangle = \delta_{(0,s)}$ for such s. Then (2.11) implies that

$$\int \mathbf{M}(\langle S, p_{n+1}, s \rangle) \, ds \leq \mathbf{M}(S) < \delta_0, \tag{5.33}$$

so that in particular, $\mathbf{M}(\langle S, p_{n+1}, s \rangle) < \infty$ for a.e. s.

Step 2. We define the sets

$$\Sigma_0 := \{ s \in (0, L) : \mathbf{M}(\langle T, p_{n+1}, s \rangle \llcorner \Omega_{r/2}) = 0 \},$$
$$\Sigma_1 := \{ s \in (0, L) : \mathbf{M}(\langle T, p_{n+1}, s \rangle) = \mathbf{M}(\langle T, p_{n+1}, s \rangle \llcorner \Omega_{r/2}) = 1 \},$$
$$\Sigma_2 := \{ s \in (0, L) : \mathbf{M}(\langle T, p_{n+1}, s \rangle) \geq 2 \}.$$

Recall that if T is a current and B a Borel set, then $T \llcorner B$ denotes the restriction of T to B, defined in (2.3). For a.e. $s \in \Sigma_1$, T intersects the level set $p_{n+1}^{-1}(s)$ exactly once, at a point within distance $r/2$ of T_*. This is a good property, and our goal for now is to show that Σ_1 is large.

Note that for any $s \in (0, L)$ such that $\langle T, p_{n+1}, s \rangle$ is well-defined, one has $s \notin \Sigma_1$ if and only if $s \in \Sigma_0 \cup \Sigma_2$, though Σ_0 and Σ_2 are not necessarily disjoint. Thus

$$|\Sigma_0| + |\Sigma_1| + |\Sigma_2| \geq L = \mathbf{M}(T_*) \geq \mathbf{M}(T). \tag{5.34}$$

So to prove that $|\Sigma_1|$ is large, it suffices to show that $|\Sigma_0|$ and $|\Sigma_2|$ are small.

We first consider Σ_0. For a.e. $s \in \Sigma_0$, it follows from (5.32) and the definition of Σ_0 that $\partial \langle S, p_{n+1}, s \rangle \llcorner \Omega_{r/2} = -\delta_{(0,s)}$. Then, by applying Lemma 2.3 to $Q = \langle S, p_{n+1}, s \rangle$ we find that

$$\mathbf{M}(\langle S, p_{n+1}, s \rangle) \geq \mathrm{dist}((0, s), \partial \Omega_{r/2}).$$

Assumptions (5.2) and (5.12) imply that

$$\mathrm{dist}((0, s), \partial \Omega_{r/2}) \geq \frac{1}{2} \min\{r, s, L - s\}$$

for $s \in (0, L)$. Thus

$$\mathbf{M}(\langle S, p_{n+1}, s \rangle) \geq \frac{1}{2} \min\{r, s, L - s\} \quad \text{for a.e. } s \in \Sigma_0.$$

Comparing this with (5.33), we find that $\delta_0 \geq \frac{1}{2} \int_{\Sigma_0} \min\{s, r, L - s\} \, ds$. To estimate the right-hand side from below, note that the integral decreases if we replace Σ_0 with the set $(0, |\Sigma_0|/2) \cup (L - |\Sigma_0|/2, L)$ of the same measure but concentrated near the ends of the interval $(0, L)$, where the integrand $\min\{s, r, L - s\}$ is smallest. This leads to the inequality

$$\delta_0 \geq \frac{1}{2} \int_{\Sigma_0} \min\{s, r, L - s\} \, ds \geq \int_0^{|\Sigma_0|/2} \min\{s, r, L - s\} \, ds$$

$$= \int_0^{|\Sigma_0|/2} \min\{s, r\} \, ds \geq \frac{1}{8} \min\{|\Sigma_0|^2, r^2\}.$$

Since we have assumed that $\delta_0 < r^2/400$, it must be the case that the right-hand side above is smaller than $r^2/8$, hence that

$$|\Sigma_0| = \min\{|\Sigma_0|, r\} \leq 3\sqrt{\delta_0}. \tag{5.35}$$

Step 3. Next, note that

$$\mathbf{M}(T) \geq \int_0^L \mathbf{M}(\langle T, p_{n+1}, s \rangle) \, ds \geq |\Sigma_1| + 2|\Sigma_2|.$$

Combining this with (5.34), we see that $|\Sigma_2| \leq |\Sigma_0| \leq 3\sqrt{\delta_0}$, and hence that

$$|\Sigma_1| \geq L - 9\sqrt{\delta_0}. \tag{5.36}$$

Also, from the definition of Σ_1,

$$\mathbf{M}(T \llcorner \Omega_{r/2}) \geq \int_{\Sigma_1} \mathbf{M}(\langle T, p_{n+1}, s \rangle \llcorner \Omega_{r/2}) \, ds = |\Sigma_1| \geq L - 9\sqrt{\delta_0}.$$

Thus, since $M(T) \leq L$, we have

$$\mathbf{M}(T \llcorner (\Omega \setminus \Omega_{r/2})) \leq 9\sqrt{\delta_0}. \tag{5.37}$$

Step 4. Now write T as a sum $T = \sum T_i$ of indecomposable currents (so that each is the current associated with integration over a Lipschitz curve) and

$$\sum \mathbf{M}(T_i) = \mathbf{M}(T) \leq L, \qquad \sum \mathbf{M}(\partial T_i) = \mathbf{M}(\partial T) = 0. \tag{5.38}$$

Now if for some i one has $\mathrm{supp}(T_i) \cap \Omega_{r/2} \neq \emptyset$, then we claim that our choice of δ_0 guarantees this T_i satisfies

$$\mathrm{supp}(T_i) \subset \Omega_r. \tag{5.39}$$

To see this, note that any indecomposable T_i whose support intersects both $\Omega_{r/2}$ and $\Omega \setminus \Omega_r$ must correspond to a curve that stretches between these two sets, and therefore has arclength at least $r/2$ outside $\Omega_{r/2}$. It would follow that

$$\mathbf{M}(T_i \llcorner (\Omega \setminus \Omega_{r/2})) \geq r/2,$$

which contradicts (5.37) in light of our assumption (5.31) that $\delta_0 < r^2/400$.

Step 5. Let γ_i denote the Lipschitz curves associated with the currents T_i in the decomposition $T = \sum T_i$. Then for any positive integer j, let

$$\Sigma_{1,j} := \left\{ s \in \Sigma_1 : \sum_i \mathcal{H}^0(\gamma_i \cap p_{n+1}^{-1}(s)) = j \right\}.$$

We claim that

$$|\Sigma_{1,1}| \geq L - 18\sqrt{\delta_0}. \tag{5.40}$$

(In fact $|\Sigma_{1,1}| = |\Sigma_1|$, but this is a bit harder to prove.) The explicit formula (2.10) for the slice of an indecomposable 1-current implies for a.e. s that

$$\mathbf{M}(\langle T, p_{n+1}, s \rangle) \leq \sum_i \mathbf{M}(\langle T_i, p_{n+1}, s \rangle) = \sum_i \mathcal{H}^0(\gamma_i \cap p_{n+1}^{-1}(s)).$$

It follows that $\sum_i \mathcal{H}^0(\gamma_i \cap p_{n+1}^{-1}(s)) \geq 1$ for a.e. $s \in \Sigma_1$, and hence that

$$\sum_{j=1}^{\infty} |\Sigma_{1,j}| = |\Sigma_1| \geq L - 9\sqrt{\delta_0}. \tag{5.41}$$

Now we essentially repeat arguments from Step 3: we compute

$$L \geq \mathbf{M}(T) = \sum_i \mathbf{M}(T_i) = \sum_i \mathcal{H}^1(\gamma_i) \geq \int_{\Sigma_1} \sum_i \mathcal{H}^0(\gamma_i \cap p_{n+1}^{-1}(s)) \, ds$$

$$= \sum_{j=1}^{\infty} j|\Sigma_{1,j}| \geq |\Sigma_{1,1}| + 2\sum_{j=2}^{\infty} |\Sigma_{1,j}|.$$

Together with (5.41), this implies (5.40).

Step 6. Our assumptions (5.12) about the geometry of Ω_R imply that if $s \in (R, 2R) \subset (R, L-R)$, then $p_{n+1}^{-1}(s)$ separates the two components $\partial\Omega^-$ and $\partial\Omega^+$ of $\bar{\Omega}_R \cap \partial\Omega$. Using the Lebesgue differentiation theorem, we may pick a value s to be a point of density of the set $\Sigma_{1,1} \cap (R, 2R)$; this is possible since $|(0, L) \setminus \Sigma_{1,1}| \leq 18\sqrt{\delta_0} < r \leq R$, from

Step 5 and our choice of δ_0. Let γ_i be the Lipschitz curve that intersects $p_{n+1}^{-1}(s)$; there is a *unique* such curve due to the definition of $\Sigma_{1,1}$. Since $s \in \Sigma_1$, the point at which γ_i intersects $p_{n+1}^{-1}(s)$ must be in $\Omega_{r/2}$. Therefore, by (5.39), γ_i is entirely contained in Ω_r. Since $\partial T_i = 0$, γ_i must either be a closed loop, or start and end in $\partial \Omega$. The former is impossible unless γ_i meets the hyperplane $p_{n+1}^{-1}(s)$ tangentially in view of the fact that γ_i intersects $p_{n+1}^{-1}(s)$ exactly once. But this too is impossible since $\Sigma_{1,1}$ has nonempty intersection with every neighborhood of s, forcing such a loop γ_i to intersect some nearby hyperplane $p_{n+1}^{-1}(s')$ at least twice. Thus, it follows from the choice of s that γ_i must run from $\partial \Omega^-$ through Ω_r to $\partial \Omega^+$.

Step 7. We finally define T' to be the indecomposable current T_i associated with the Lipschitz curve γ_i from the previous step. Then we have proved (5.29), and (5.30) follows from (5.38).   □

Having proved Lemma 5.5, we now turn to the proof of Proposition 5.4, which is the central part of the proof of Theorem 5.1.

Proof of Proposition 5.4.

Step 1. Recall that s_0 is a number, depending on the geometry of $\partial \Omega$, that was fixed in (5.14), (5.15), (5.16), and that the definition of P_{WV} involves a small parameter $s_1 \leq \frac{1}{2} s_0$. This parameter will be required to satisfy conditions (5.45), (5.58), (5.60), appearing in the proof below.[2]

We next select a small parameter r as follows: Note that since the map $\binom{x}{y} \mapsto \binom{w}{\zeta}$ is an isometry, it follows from (5.13) that $d_0(x, y) \geq L - 2\lambda_- r^2$ for all $(x, y) \in B_r \times B_r$. We fix r such that $r \leq \sqrt{s_1/2\lambda_-}$, so that

$$\text{dist}(C_r \cap \partial \Omega^-, C_r \cap \partial \Omega^+) \geq L - s_1. \tag{5.42}$$

We also insist that $r \leq R/2$, so that the maps $x^\pm, s^\pm$ are defined for points in Ω_r that are within distance s_0 of $\partial \Omega$, using (5.15).

Having fixed r, we take $\delta_0 < cr^2$ for the small number c from Lemma 5.5 (in fact we have shown that $c = 1/400$ is small enough). We will prove that the conclusions of the proposition hold for this δ_0.

Now let T be a current satisfying the hypotheses (5.27). Applying Lemma 5.5 for the above choice of δ_0, we obtain a current T' corresponding to a single Lipschitz curve supported in Ω_r and satisfying (5.30). Let $\gamma : [0, L'] \to \bar{\Omega}_r$ denote the corresponding Lipschitz curve, parametrized by arclength.

Step 2. We will write $\delta_1 := \mathbf{M}(T_*) - \mathbf{M}(T')$, with $\delta_1 \geq 0$ by assumption. (We will eventually show that $\delta_1 = 0$.) Note from (5.30) and (5.42) that

$$L' = \mathbf{M}(T') = \text{length}(\gamma) \geq \text{dist}(C_r \cap \partial \Omega^-, C_r \cap \partial \Omega^+) \geq L - s_1 \tag{5.43}$$

by (5.42). In particular, it follows that $\delta_1 \leq s_1$.

---

[2] One can check that $s_1 = \frac{1}{2} \min\{s_0, d_\Omega, 1/(10\lambda_-(\|\nabla \xi^\pm\|_\infty C_\Omega)^2), 1/(10^4 \lambda_-)\}$ satisfies the stated conditions, where d_Ω and C_Ω are defined in Lemma 2.2, $\xi^\pm$ are defined in (5.19), and $\|\nabla \xi^\pm\|_\infty$ denotes $\max\{\|\nabla \xi^-\|_\infty, \|\nabla \xi^+\|_\infty\}$.

Then by (5.27) and (5.30),

$$\mathbf{M}(T - T') = \mathbf{M}(T) - \mathbf{M}(T') \le \mathbf{M}(T_*) - \mathbf{M}(T') = \delta_1 \le s_1. \qquad (5.44)$$

We additionally impose the condition

$$s_1 < d_\Omega \quad \text{(defined in the statement of Lemma 2.2)}, \qquad (5.45)$$

and then it follows from Lemma 2.2 that $\mathbf{F}(T - T') \le C_\Omega \mathbf{M}(T - T')^2$. Recalling that $P_{WV}(T) = P_{WV}(T_*) = 0$, we see from Lemma 5.3 and (5.44) that

$$|P_{WV}(T')| = |P_{WV}(T' - T)| \le \frac{C}{s_1} \mathbf{F}(T - T')$$

$$\le \frac{C}{s_1} \mathbf{M}(T - T')^2 \le \frac{C}{s_1} \delta_1^2 \le C\sqrt{\delta_1}\sqrt{s_1}. \qquad (5.46)$$

Here the constant depends on $\partial\Omega$, R and s_0 but is independent of s_1.

Step 3. Recall that $\gamma(0) \in \partial\Omega^-$ and $\gamma(L') \in \partial\Omega^+$. Define

$$\tau^- := \max\{t > 0 : \mathrm{dist}(\gamma(t), \partial\Omega^-) \le s_1\}$$
$$\tau^+ := \max\{t > 0 : \mathrm{dist}(\gamma(L' - t), \partial\Omega^+) \le s_1\}.$$

Using (5.42) we see that

$$L - s_1 \le \mathrm{dist}(C_r \cap \partial\Omega^-, C_r \cap \partial\Omega^+)$$
$$\le \mathrm{dist}(C_r \cap \partial\Omega^-, \gamma(\tau^-)) + |\gamma(\tau^-) - \gamma(L' - \tau^+)|$$
$$\quad + \mathrm{dist}(\gamma(L' - \tau^+), C_r \cap \partial\Omega^+)$$
$$= |\gamma(\tau^-) - \gamma(L' - \tau^+)| + 2s_1.$$

Also, since γ is parametrized by arclength,

$$L \ge L' \ge \tau^- + |\gamma(\tau^-) - \gamma(L' - \tau^+)| + \tau^+.$$

By combining these we find that $\tau^- + \tau^+ \le 3s_1$. Again, because of the arclength parametrization, $\tau^-, \tau^+ \ge s_1$, so we conclude that

$$\tau^+, \tau^- \le 2s_1. \qquad (5.47)$$

Step 4. In particular, $\mathrm{dist}(\gamma(t), \partial\Omega) \le 2s_1 \le s_0$ for $t \in (0, \tau^-) \cup (L' - \tau^+, L')$. This together with the fact that $\gamma(t) \in \Omega_r$ for all t, our assumption $r \le R/2$, and (5.15), implies for example that $\gamma(t) \in \Psi^-(B_R \times (0, s_0))$ for $t \in (0, \tau^-)$. We will abuse notation somewhat and write

$$x^-(t) = x^-(\gamma(t)) \in B_r, \quad s^-(t) = s^-(\gamma(t)) \in [0, 2s_1],$$

for $t \in [0, \tau^-]$, and similarly

$$x^+(t) = x^+(\gamma(L' - t)), \quad s^+(t) = s^+(\gamma(L' - t))$$

for $t \in [0, \tau^+]$. These are Lipschitz functions, since $x^\pm$ and $s^\pm$ are smooth and γ is Lipschitz. We differentiate the relation $\gamma(t) = \Psi^-(x^-(t), s^-(t))$ to find that

$$\gamma'(t) = \nabla_x \Psi^- \frac{dx^-}{dt} + \frac{\partial \Psi^-}{\partial s} \frac{ds^-}{dt} = \nabla_x \Psi^- \frac{dx^-}{dt} - \nu^-(x^-) \frac{ds^-}{dt}.$$

Squaring both sides and using (5.15), (5.17), we find that

$$1 = |\gamma'|^2 = \left| \nabla_x \Psi^- \frac{dx^-}{dt} \right|^2 + \left(\frac{ds^-}{dt} \right)^2 \geq \frac{1}{4} \left| \frac{dx^-}{dt} \right|^2 + \left(\frac{ds^-}{dt} \right)^2. \qquad (5.48)$$

Step 5. Next, note that for any $t^- \in [0, \tau^-]$ and $t^+ \in [0, \tau^+]$, since γ is parametrized by arclength,

$$\mathbf{M}(T') = L' \geq t^- + t^+ + |\gamma(t^-) - \gamma(L' - t^+))|. \qquad (5.49)$$

Also, note from the definitions that for example

$$|\Psi(x^-(t^-), 0) - \gamma(t^-)| = |\Psi(x^-(t^-), 0) - \Psi(x^-(t^-), s^-(t^-))| = s^-(t^-),$$

so the triangle inequality implies that

$$d_0(x^-(t^-), x^+(t^+)) = |\Psi(x^-(t^-), 0) - \Psi(x^+(t^+), 0)|$$
$$\leq s^-(t^-) + |\gamma(t^-) - \gamma(L' - t^+))| + s^+(t^+). \qquad (5.50)$$

Combining this with (5.49) and recalling the notation $\delta_1 = \mathbf{M}(T_*) - \mathbf{M}(T')$, we find that

$$-\delta_1 \geq [t^- - s^-(t^-)] + [t^+ - s^+(t^+)] + [d_0(x^-(t^-), x^+(t^+)) - L].$$

For $(t^-, t^+) \in [0, \tau^-] \times [0, \tau^+]$, we define $w(t^-, t^+), \zeta(t^-, t^+)$ by

$$\begin{pmatrix} w(t^-, t^+) \\ \zeta(t^-, t^+) \end{pmatrix} = A \begin{pmatrix} x^-(t^-) \\ x^+(t^+) \end{pmatrix}, \qquad (5.51)$$

for A from (5.6). Then we deduce from (5.13) that

$$[t^- - s^-(t^-)] + [t^+ - s^+(t^+)] + \delta_1 \leq \lambda_- |w(t^-, t^+)|^2 - \lambda_+ |\zeta(t^-, t^+)|^2. \qquad (5.52)$$

Step 6. Now we claim that

$$s^-(t^-)^2 + \frac{1}{4} |x^-(0) - x^-(t^-)|^2 \leq (t^-)^2 \qquad (5.53)$$

and similarly $(s^+)^2 + \frac{1}{4} |x^+(0) - x^+|^2 \leq (t^+)^2$. This follows from (5.48), which can be rewritten

$$\left| \frac{d}{dt} \left(\frac{1}{2} x^-, s^- \right) \right| \leq 1$$

for all $t^- \in (0, \tau^-]$. Thus

$$\left| \left(\frac{1}{2} x^-(t^-), s^-(t^-) \right) - \left(\frac{1}{2} x^-(0), 0 \right) \right| = \left| \int_0^{t^-} \frac{d}{dt} \left(\frac{1}{2} x^-, s^- \right) dt \right|$$

$$\leq \int_0^{t^-} \left| \left(\frac{1}{2} \frac{dx^-}{dt}, \frac{ds^-}{dt} \right) \right| dt \leq t^-.$$

This is (5.53).

Step 7. Since the map $\binom{x}{y} \mapsto \binom{w}{\zeta}$ is a linear isometry,

$$|w(0, 0) - w(t^-, t^+)|^2 \leq |x^-(0) - x^-(t^-)|^2 + |x^+(0) - x^+(t^+)|^2.$$

Thus, by adding (5.53) and its counterpart for t^+, s^+ etc., we obtain

$$\frac{1}{4} |w(0, 0) - w(t^-, t^+)|^2 \leq [(t^-)^2 - (s^-)^2] + [(t^+)^2 - (s^+)^2]. \qquad (5.54)$$

Step 8. Next we will use the constraint (5.46) to show that we can find some (t^-, t^+) such that

$$|w(t^-, t^+)| \leq C |w(0, 0) - w(t^-, t^+)| \qquad (5.55)$$

for some absolute constant C. This is the key point in the proof.[3] We use (5.20) to rewrite $T'(\Phi^-)$ (see (5.24)) in the form

$$T'(\Phi^-) = \int_0^{\tau^-} x^-(t) f^-(t) dt, \quad f^-(t) = \omega(s^-(t)) \frac{d}{dt} s^-(t).$$

We define f^+ in a similar way. A change of variables and the definition (5.21) of ω show that

$$\int_0^{\tau^-} f^-(t^-) dt^- = 1,$$

and (5.47), (5.48) and (5.21) imply that

$$\int_0^{\tau^-} |f^-(t^-)| dt^- \leq \tau^- \|f^-\|_\infty \leq \tau^- \|\omega\|_\infty \leq 4.$$

The same estimates hold for f^+. Now, since $p : \mathbb{R}^{2n} \to \mathbb{R}^\ell$ is linear,

$$P_{WV}(T') = p(T'(\Phi^-), T'(\Phi^+))$$

$$= \int_0^{\tau^+} \int_0^{\tau^-} p(x^-(t^-), x^+(t^+)) f^-(t^-) f^+(t^+) \, dt^- \, dt^+$$

$$= \int_0^{\tau^+} \int_0^{\tau^-} w(t^-, t^+) f^-(t^-) \, f^+(t^+) \, dt^- \, dt^+. \qquad (5.56)$$

---

[3] To understand the idea, suppose for simplicity that $\delta_1 = 0$ and that f^+, f^- are nonnegative. The constraint then forces certain averages of $w(\cdot, \cdot)$ to vanish, as follows from (5.56). In particular, this implies that $w(\cdot, \cdot)$ must leave the halfspace $\{w' \in \mathbb{R}^\ell : w' \cdot w(0, 0) > 0\}$, and (5.55), with $C = 1$ say, can be deduced from this.

We assume that $w(0,0) \neq 0$—otherwise (5.55) is obvious—and we write $\eta := w(0,0)/|w(0,0)|$. By taking $t^- = t^+ = 0$ in (5.52) we find that

$$|w(0,0)| \geq (\delta_1/\lambda_-)^{1/2}.$$

We combine the above inequality with (5.46), to see that

$$|\eta \cdot P_{WV}(T')| \leq C\sqrt{\delta_1}\sqrt{s_1} \leq C\sqrt{s_1}\sqrt{\lambda_-}|w(0,0)| \tag{5.57}$$

where C is independent of s_1. We now require that

$$s_1 \leq \frac{1}{4C^2\lambda_-}, \quad \text{for the same } C \text{ as in (5.57).} \tag{5.58}$$

Then $|\eta \cdot P_{WV}(T')| \leq \frac{1}{2}|w(0,0)|$, and so

$$\frac{1}{2}|w(0,0)| \leq |w(0,0)| - \eta \cdot P_{WV}(T') = \eta \cdot [w(0,0) - P_{WV}(T')]$$

$$\leq \eta \cdot \int_0^{\tau^+}\int_0^{\tau^-} [w(0,0) - w(t^-,t^+)]f^-(t^-)\, f^+(t^+)\, dt^-\, dt^+$$

$$\leq \int_0^{\tau^+}\int_0^{\tau^-} |w(0,0) - w(t^-,t^+)||f^-(t^-)|\,|f^+(t^+)|\, dt^-\, dt^+$$

$$\leq 16\max\{|w(0,0) - w(t^-,t^+)| : t^- \in [0,\tau^-], t^+ \in [0,\tau^+]\}.$$

In other words,

$$|w(0,0)| \leq 32|w(0,0) - w(t^-,t^+)|$$

for some (t^-,t^+). It follows that

$$|w(t^-,t^+)| \leq |w(t^-,t^+) - w(0,0)| + |w(0,0)| \leq 33|w(0,0) - w(t^-,t^+)|$$

at the same point (t^+,t^-), proving (5.55).

Step 9. Now by combining (5.52), (5.55), and (5.54), we find that

$$(t^- - s^-) + (t^+ - s^+) \leq C\lambda_-[((t^-)^2 - (s^-)^2) + ((t^+)^2 - (s^+)^2)]$$

for some numerical constant C. From (5.47) we have $((t^-)^2-(s^-)^2) = (t^--s^-)(t^++s^-) \leq 4s_1(t^- - s^-)$ for example, so the above reduces to

$$(t^- - s^-) + (t^+ - s^+) \leq C\lambda_- s_1[(t^- - s^-) + (t^+ - s^+)]. \tag{5.59}$$

We now require that

$$s_1 \leq \frac{1}{2C\lambda_-} \quad \text{for the same } C \text{ as in (5.59).} \tag{5.60}$$

Then

$$t^- = s^-(t^-), \quad t^+ = s^+(t^+).$$

Step 10. Now (5.54) implies that $w(0,0) = w(t^-, t^+)$ for the point (t^-, t^+) considered above. Since this same point also satisfies (5.55), we conclude that $w(0,0) = w(t^-, t^+) = 0$. Then (5.52) (evaluated at $(t^-, t^+) = (0,0)$) implies that $\delta_1 = 0$ and that $\zeta(0,0) = 0$.

Then (5.51) implies that $x^-(0) = 0$ and $x^+(0) = 0$.

Undoing the notation, this means that the curve γ starts at the point $(0, 0)$ and ends at the point $(0, L)$. Since the length of the curve is at most L, it must consist of the straight segment joining these two points.

Since T_* is exactly the current corresponding to integration over this segment, it follows that $T' = T_*$.

Finally, recall that at the first step of this proof, we used Lemma 5.5 to replace a given current T by a current T' with better properties; all of our arguments since then have dealt with T'. So to finish the proof we must show that $T = T'$. This is easy, however, since from (5.30),

$$\mathbf{M}(T - T') = \mathbf{M}(T) - \mathbf{M}(T').$$

By hypothesis, $\mathbf{M}(T) \leq \mathbf{M}(T_*) = \mathbf{M}(T')$, so we conclude that $\mathbf{M}(T - T') \leq 0$, and hence that $T = T'$. □

5.3. Construction of Q_{VW}. Verification of (4.2)–(4.4)

To complete the proof of Theorem 5.1, it remains to construct a continuous map Q_{VW} satisfying the conditions (4.2)–(4.4) of Definition 4.1. For this purpose, we introduce the notation T_{xy} to denote the element of $\mathcal{R}'_1(\Omega)$ corresponding to the multiplicity one, oriented line segment joining $\psi^-(x)$ to $\psi^+(y)$. Note that with this notation, one has $T_* = T_{00}$.

For given $w \in \mathbb{R}^\ell$ with $|w|$ small, we would like to find values $x(w)$ and $y(w)$ in $\mathbb{R}^n$ such that the 1-current $T_{x(w)y(w)}$ can be used as a definition of $Q_{VW}(w)$. For this to be successful, we will in particular need to fulfill requirement (4.3), which reads

$$A \begin{pmatrix} T_{x(w)y(w)}(\Phi^-) \\ T_{x(w)y(w)}(\Phi^+) \end{pmatrix} = \begin{pmatrix} w \\ 0 \end{pmatrix}, \tag{5.61}$$

in light of the definition (5.23) of P_{WV}.

Lemma 5.6. *There exists a positive number a and C^1 functions $x(w)$ and $y(w)$ defined for $w \in B_a \subset \mathbb{R}^\ell$ such that (5.61) holds.*

Proof. We define

$$F(x, y) = \begin{pmatrix} T_{xy}(\Phi^-) \\ T_{xy}(\Phi^+) \end{pmatrix}.$$

Note that $F(0, 0) = (0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$. If we can show that F is invertible near $(0, 0)$, then (since A is nonsingular) we can define $(x(w), y(w))$ by

$$(x(w), y(w)) = F^{-1}\left(A^{-1} \begin{pmatrix} w \\ 0 \end{pmatrix} \right). \tag{5.62}$$

Then (5.61) follows immediately. We check the local invertibility of F at $(0, 0)$ using the Inverse Function Theorem. Since it is clear that F is C^2, it suffices to check that

$$D_{xy}F = \begin{pmatrix} D_x T_{xy}(\Phi^-) & D_y T_{xy}(\Phi^-) \\ D_x T_{xy}(\Phi^+) & D_y T_{xy}(\Phi^+) \end{pmatrix} \quad \text{is nonsingular} \qquad (5.63)$$

when evaluated at $x = y = 0$. A rather lengthy calculation shows that

$$\begin{pmatrix} D_x T_{xy}(\Phi^-) & D_y T_{xy}(\Phi^-) \\ D_x T_{xy}(\Phi^+) & D_y T_{xy}(\Phi^+) \end{pmatrix}$$
$$= \begin{pmatrix} \int_0^{s_1} B_-(t)(1 - t/L)\omega(t)dt & \int_0^{s_1} B_-(t)(t/L)\omega(t)dt \\ \int_0^{s_1} B_+(t)(t/L)\omega(t)dt & \int_0^{s_1} B_+(t)(1 - t/L)\omega(t)dt \end{pmatrix}, \qquad (5.64)$$

where $B_{\pm}(t) = [I - t D^2 h^{\pm}(0)]^{-1}$.

From the definition of B_- and B_+ one sees that $B_{\pm} = I + O(t)$ with a bound on the error depending on $D^2 h^{\pm}(0)$. Recalling that ω is supported in the interval $[0, s_1]$, with $\int_0^{s_1} \omega(t)dt = 1$, we infer that

$$D_{xy}F = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + O(s_1) \qquad (5.65)$$

where $O(s_1)$ denotes a matrix whose entries are all bounded above and below by Cs_1, with the constant C depending on $D^2 h^{\pm}(0)$ and L. Thus it is clear that the nullspace of $D_{xy}F(0, 0)$ is trivial if s_1 is taken to be small enough. Consequently, we have condition (5.63) satisfied so we conclude that for some $a > 0$, there do indeed exist C^1 functions $x = x(w)$ and $y = y(w)$ defined on $B_a \subset \mathbb{R}^{\ell}$ and taking values in $\mathbb{R}^n$ such that (5.61) is satisfied.   □

We are now prepared to define $Q_{VW} : W \to V = \mathcal{F}_1'$ by

$$Q_{VW}(w) := T_{x(w)y(w)}. \qquad (5.66)$$

Taken in conjunction with Lemma 5.3 and Proposition 5.4, the following proposition establishes Theorem 5.1.

Proposition 5.7. *There exists a number $a_1 > 0$ such that for $W = B_{a_1} \subset \mathbb{R}^{\ell}$ and $V = \mathcal{F}_1'(\Omega)$, the map Q_{VW} given by (5.66) is continuous. Furthermore, with P_{WV} given by (5.23) and saddle point $v_s = T_*$ $(= T_{00}$ in the notation of (5.66)), the conditions (4.2)–(4.4) are satisfied.*

Proof. We first verify the assertion of continuity. To this end, given w_1 and w_2 in $B_a \subset \mathbb{R}^{\ell}$ (with a provided by Lemma 5.6), let $\Gamma_{w_1 w_2}^-$ be the 1-current corresponding to the shortest (oriented) curve along $\partial\Omega^-$ joining the point $\psi^-(x(w_2))$ to $\psi^-(x(w_1))$ and let $\Gamma_{w_1 w_2}^+$ be the 1-current corresponding to the shortest curve along $\partial\Omega^+$ joining the point $\psi^+(y(w_1))$ to $\psi^+(y(w_2))$. Then let $S_{w_1 w_2}$ be the oriented, multiplicity one 2-current corresponding to the surface of least area such that in $\bar\Omega$ one has $\partial S_{w_1 w_2} = T_{x(w_1)y(w_1)} +$

$\Gamma^+ - T_{x(w_2)y(w_2)} + \Gamma^-$. Since $w \to x(w)$ and $w \to y(w)$ are continuous it is clear that $\mathbf{M}(S_{w_1 w_2}) \to 0$ as $w_2 \to w_1$. Hence, $\mathbf{F}(Q_{VW}(w_2) - Q_{VW}(w_1)) \to 0$ as well.

Next we note that conditions (4.2) and (4.3) follow immediately in light of (5.61), along with (5.25).

To check the last condition (4.4), we fix any $w \in B_{a_1} \setminus \{0\}$ and first note that for E_V given by (5.1) one has $E_V(Q_{VW}(w)) = d_0(x(w), y(w))$ (cf. (5.3)). Let us write $\binom{\tilde{w}}{\tilde{\zeta}} = A\binom{x(w)}{y(w)}$. Then (5.8) implies that

$$E_V(Q_{VW}(w)) - E_V(T_{00}) = d_0(x(w), y(w)) - L \le -\Lambda_- |\tilde{w}|^2 + \Lambda_+ |\tilde{\zeta}|^2 \qquad (5.67)$$

for $\Lambda_- = \frac{1}{2}|\lambda_\ell| > 0$ and $\Lambda_+ = 2|\lambda_{2n}| > 0$. Since F is C^2, (5.65) implies that $DF^{-1} = \binom{1\ 0}{0\ 1} + O(s_1)$ is a neighborhood of $\binom{0}{0}$. Recall also that $F^{-1}(\binom{0}{0}) = \binom{0}{0}$. Thus the Mean Value Theorem implies that $\left|F^{-1}(\binom{x}{y}) - \binom{x}{y}\right| \le Cs_1|\binom{x}{y}|$ for x, y in a neighborhood of $(0, 0)$. It follows from this and (5.62) that

$$\binom{\tilde{w}}{\tilde{\zeta}} = A\binom{x(w)}{y(w)} = AF^{-1}\left(A^{-1}\binom{w}{0}\right) = \binom{w}{0} + O(s_1|w|).$$

Choosing s_1 small enough, it follows that the right-hand side of (5.67) is negative, which is what we needed to check. $\square$

Finally, we prove Corollary 5.2, stated earlier, which adapts the above results to the functional arising as the Γ-limit of the Modica–Mortola functional.

Proof of Corollary 5.2. For the duration of this proof only, let us write $E_{V,MM}$ to denote the functional defined in (3.10), arising as the Γ-limit of the Modica–Mortola functional, and $E_{V,GL}$ for the 2-dimensonal case of the functional defined in (5.1). The only difference between these two functionals is that

$$V_{0,MM} := \{T : E_V(T) < \infty\} \subset V_{0,GL} := \{T : E_{V,GL}(T) < \infty\}.$$

To see this, recall that every $T \in V_{0,MM}$ has the form $T = \star dv/2$ for some $v \in BV(\Omega; \{\pm 1\})$. Then it follows from basic facts recalled in Section 2.2 (see for example (2.17), (2.14)) and the definition (2.7) of $\mathcal{R}'_1(\Omega)$ that $T \in \mathcal{R}'_1(\Omega) = V_{0,GL}$. Although we do not need it here, note also that the inclusion is strict, since for example every element of $V_{0,MM}$ is a current with constant multiplicity 1, whereas $\mathcal{R}'_1(\Omega) = V_{0,GL}$ contains currents of arbitrary integer multiplicity.

In particular, $E_{V,MM}(T) = E_{V,GL}(T)$ for $T \in V_{0,MM}$, and $E_{V,GL}(T) \le E_{V,MM}(T)$ for all T. By inspection of the definition of saddle point, we then find that to deduce the corollary from the theorem, it suffices to check that $Q_{VW}(w) \in V_{0,MM}$ for all sufficiently small w (and in particular for T_*). This is clear however from the definition (5.66) of Q_{VW}. $\square$

6. Some applications

In principle, it should be possible to combine the general asymptotic minmax result, Theorem 4.4, with the description in Theorem 5.1 of critical points of the functional E_V as defined in (5.1), to prove existence results for a very large number of examples of functionals that Γ-converge to an energy of the form of E_V, that is, an energy involving the arclength of an asymptotic singular set. In this section we carry this out for several examples, as described in the introduction.

Throughout this section, we assume that $\Omega \subset \mathbb{R}^{n+1}$ is bounded, and $\partial\Omega$ is C^3. In order to bypass some technicalities we also assume that there exists some $R > 0$ such that

$$\{(x, x_{n+1}) \in \Omega : |x| < R\} \text{ consists of a single connected component.} \quad (6.1)$$

This component agrees with Ω_R as defined in the last section. We continue to use other notation introduced in the previous section, and we also assume throughout this section that (5.4) holds, so that $(0, 0)$ is a nondegenerate critical point of the function d_0 as defined in (5.3).

We give more or less exactly the same proof in every case. (The Ginzburg–Landau functional with magnetic field requires a bit of extra work, since we must also adapt Γ-convergence results from the literature.) We begin with the 3d Ginzburg–Landau energy:

Theorem 6.1. *Assume that Ω is a bounded domain in $\mathbb{R}^3$ with C^3 boundary and that (6.1) holds. Assume that the distance function d_0 given by (5.3) has a nondegenerate critical point at $(0, 0)$ in the sense of (5.4). Let $T_* \in \mathcal{R}'_1(\Omega)$ correspond to the oriented line segment joining $\psi^-(0)$ to $\psi^+(0)$. Then there exists a value $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the Ginzburg–Landau energy E_U^ε as defined in (3.11) possesses a critical point u_ε and $E_U^\varepsilon(u_\varepsilon) \to E_V(T_*)$ as $\varepsilon \to 0$ for E_V given by (3.13).*

Remark 6.2. As noted earlier in Remark 4.5, the abstract existence result Theorem 4.4 that we are invoking cannot predict, in general, the closeness of the sequence of critical points to the saddle point of the Γ-limit. One might hope that in the particular case of the Ginzburg–Landau energy, this closeness could be established, but we do not see an easy path to such a conclusion. One can invoke Theorem 1 of [7] to assert that the sequence of varifolds associated with the critical points u_ε converges to a stationary 1-rectifiable varifold, but it does not seem to be easy to relate this limiting varifold to the given current corresponding to a saddle point of arclength. One problem is the difficulty of controlling the multiplicity of the limiting varifold, and the related issue of possible cancellations in the passage to the limit in the sense of currents.

Proof. We recall that for the Ginzburg–Landau example, $U = H^1(\Omega; \mathbb{C})$, $V = \mathcal{F}_1(\Omega)$ and again $W = B_{a_1} \subset \mathbb{R}^\ell$ for $l \in \{0, 1, 2, 3, 4\}$. The fact that E_U^ε Γ-converges to E_V is the content of Theorem 3.2. Recall that the map $P_{VU}^\varepsilon : H^1(\Omega; \mathbb{C}) \to \mathcal{F}'_1(\Omega)$ in this case is again independent of ε and is given by $P_{VU}(u) = \star Ju/\pi$ (cf. (3.12)). The mapping Q_{UW}^ε corresponding to the recovery sequence construction will be described below. The Palais–Smale condition is verified in Proposition 3.3.

We recall that for the case $l = 0$, we have adopted the convention that $\mathbb{R}^0$ denotes $\{0\}$ so that in particular, P_{WV} is trivial in this case. For $l \in \{1, 2, 3, 4\}$, we define the map P_{WV} via (5.23) and we define Q_{VW} via the formula $Q_{VW}(w) = T_{x(w)y(w)}$, where we recall that $T_{x(w)y(w)}$ is the element of $\mathcal{R}'_1(\Omega)$ corresponding to the directed line segment joining $\psi^-(x(w))$ to $\psi^+(y(w))$ and $x(w)$ and $y(w)$ are defined through the condition (5.61). We observe that $T_* \ (= T_{00})$ is a saddle point of E_V in light of Theorem 5.1. We can also appeal to Lemma 5.3 to see that the uniform continuity condition (4.5) is met.

We now verify (4.6)–(4.8). In light of Remark 4.6, we will only need to construct the mapping $Q^\varepsilon_{UW} := Q^\varepsilon_{UV} \circ Q_{VW}$ defined for $w \in W$. In other words, we only require a recovery sequence for the case of a straight interface. To define the mapping Q^ε_{UW}, we first introduce some auxiliary functions. For $w \in B_{a_1} \subset \mathbb{R}^\ell$, let $O_w : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion of $\mathbb{R}^3$ (that is, a combination of a translation and a rotation) that maps the line passing through $\psi^-(x(w))$ and $\psi^+(y(w))$ onto the x_3 coordinate axis $\{(0, x_3) : x_3 \in \mathbb{R}\}$. We always assume that $(X, w) \mapsto O_w(X)$ is C^1; this is clearly possible, since $w \mapsto (x(w), y(w))$ is C^1. We take u^ε_w to have the form

$$u^\varepsilon_w(X) = v^\varepsilon(O_w(X)) \quad \text{with } v^\varepsilon(X) = v^\varepsilon(x, x_3) = q^\varepsilon(x) \tag{6.2}$$

where

$$q^\varepsilon(x) = \begin{cases} x/\varepsilon & \text{if } |x| \le \varepsilon, \\ x/|x| & \text{if not.} \end{cases} \tag{6.3}$$

We also write $u^0_w(X) = v^0(O_w(X))$, where $v^0(X) = q^0(x) := x/|x|$.

The smoothness of $w \mapsto O_w$ implies that (4.6) holds, in other words that $w \mapsto u^\varepsilon_w$ is continuous for every ε.

The verification of (4.7) follows by noting first that the 1-current $T_{x(w)y(w)}$ satisfies the relation

$$\star J(u^0_w) = \pi T_{x(w)y(w)}$$

(cf. (2.20)). Then through an appeal to (2.21) we have

$$\|P^\varepsilon_{VU} \circ Q^\varepsilon_{UW}(w) - Q_{VW}(w)\|_V = \frac{1}{\pi} \mathbf{F}(\star J(u^\varepsilon_w) - \star J(u^0_w)) \le \frac{1}{2\pi} \|j(u^\varepsilon_w) - j(u^0_w)\|_{L^1(\Omega)}.$$

By a change of variables, since O_w is a rigid motion,

$$\|j(u^\varepsilon_w) - j(u^0_w)\|_{L^1(\Omega)} = \|j(v^\varepsilon) - j(v^0)\|_{L^1(O_w(\Omega))}. \tag{6.4}$$

Recalling the definition of (2.19) of $j(\cdot)$, we compute

$$j(v^\varepsilon)(X) - j(v^0)(X) \le C/|x|.$$

Since $v^\varepsilon = v^0$ when $|x| > \varepsilon$, it is also clear that the set $\{X \in O_w(\Omega) : j(v^\varepsilon)(X) - j(v^0)(X) \ne 0\}$ is contained in a cylinder of radius ε and length at most $\text{diam}(\Omega)$, so it is easy to see that the right-hand side of (6.4) is bounded by $C\varepsilon \, \text{diam}(\Omega)$. Thus (4.7) follows.

It remains to verify (4.8). This follows from inspection of the argument on pages 110–111 of [24]. We give a slightly different argument here, which can and will be repeated

with very few changes for every example we consider in this section. Changing variables as in (6.4), we find that

$$
\begin{aligned}
E_U^\varepsilon(u_w^\varepsilon) = {} & \frac{1}{\pi |\ln \varepsilon|} \int_{O_w(\Omega) \cap \{(x,x_3) : |x| \le |\ln \varepsilon|^{-1}\}} \left(\frac{|\nabla v^\varepsilon|^2}{2} + \frac{(|v^\varepsilon|^2 - 1)^2}{4\varepsilon^2} \right) dX \\
& + \frac{1}{\pi |\ln \varepsilon|} \int_{O_w(\Omega) \cap \{(x,x_3) : |x| > |\ln \varepsilon|^{-1}\}} \left(\frac{|\nabla v^\varepsilon|^2}{2} + \frac{(|v^\varepsilon|^2 - 1)^2}{4\varepsilon^2} \right) dX \\
= {} & E_1 + E_2.
\end{aligned}
\tag{6.5}
$$

A short calculation shows that

$$
\frac{1}{\pi |\ln \varepsilon|} \int_{\{x \in \mathbb{R}^2 \,:\, |\ln \varepsilon|^{-1} \le |x| \le \operatorname{diam}(\Omega)\}} \left(\frac{|\nabla q^\varepsilon|^2}{2} + \frac{(|q^\varepsilon|^2 - 1)^2}{4\varepsilon^2} \right) dx \ \le\ C \frac{\ln(|\ln \varepsilon|)}{|\ln \varepsilon|}
$$

for a constant depending on $\operatorname{diam}(\Omega)$. This implies $E_2 \le C \operatorname{diam}(\Omega) \frac{\ln(|\ln \varepsilon|)}{|\ln \varepsilon|}$. This bound is independent of w.

To estimate the other term E_1 from (6.5), we use the notation

$$
A_w^\varepsilon := \{x_3 : (x, x_3) \in O_w(\Omega) \text{ for all } x \text{ such that } |x| \le |\ln \varepsilon|^{-1}\},
$$
$$
B_w^\varepsilon := \{x_3 : (x, x_3) \in O_w(\Omega) \text{ for some } x \text{ such that } |x| \le |\ln \varepsilon|^{-1}\}.
$$

Note that

$$
\frac{1}{\pi |\ln \varepsilon|} \int_{\{x \in \mathbb{R}^2 \,:\, |x| \le |\ln \varepsilon|^{-1}\}} \left(\frac{|\nabla q^\varepsilon|^2}{2} + \frac{(|q^\varepsilon|^2 - 1)^2}{4\varepsilon^2} \right) dx = 1 + O\left(\frac{\ln(|\ln \varepsilon|)}{|\ln \varepsilon|} \right).
$$

Since $v^\varepsilon(x, x_3) = q^\varepsilon(x)$, the definition of A_w^ε implies that

$$
\begin{aligned}
E_1 \ge {} & \int_{A_w^\varepsilon} \left(\frac{1}{\pi |\ln \varepsilon|} \int_{\{x \in \mathbb{R}^2 : |x| \le |\ln \varepsilon|^{-1}\}} \left(\frac{|\nabla q^\varepsilon|^2}{2} + \frac{(|q^\varepsilon|^2 - 1)^2}{4\varepsilon^2} \right) dx \right) dx_3 \\
= {} & \mathcal{H}^1(A_w^\varepsilon)\left(1 + O\left(\frac{\ln(|\ln \varepsilon|)}{|\ln \varepsilon|} \right) \right).
\end{aligned}
$$

Similar considerations imply that $E_1 \le \mathcal{H}^1(B_w^\varepsilon)(1 + O(\frac{\ln(|\ln \varepsilon|)}{|\ln \varepsilon|}))$. Finally, elementary geometric arguments show that

$$
\mathcal{H}^1(A_w^\varepsilon), \mathcal{H}^1(B_w^\varepsilon) \to \mathcal{H}^1(\{x_3 : (0, x_3) \in O_w(\Omega)\}) \quad \text{as } \varepsilon \to 0,
$$

with the convergence uniform for w in a small neighborhood of the origin. Also, the definition of O_w implies that

$$
\mathcal{H}^1(\{x_3 : (0, x_3) \in O_w(\Omega)\}) = |\psi^-(x(w)) - \psi^+(y(w))| = E_V(T_{x(w)y(w)}).
$$

Thus $E_w^\varepsilon(u_w^\varepsilon) \to E_V(T_{x(w)y(w)})$ uniformly for w in a neighborhood of the origin, which is (4.8). $\qquad\square$

In fact Theorem 6.1 above is a special case of a more general result:

Theorem 6.3. *Assume that Ω is a bounded domain in $\mathbb{R}^{n+1}$, $n \geq 2$, with C^3 boundary, and that (6.1) holds. Assume that the distance function d_0 given by (5.3) has a nondegenerate critical point at $(0,0)$ in the sense of (5.4). Let $T_* \in \mathcal{R}'_1(\Omega)$ be defined as in Theorem 6.4. Then there exists a value $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the generalized Ginzburg–Landau energy E^ε_U as defined in (3.16) possesses a critical point u_ε and $E^\varepsilon_U(u_\varepsilon) \to E_V(T_*)$ as $\varepsilon \to 0$ for E_V given by (3.13).*

Proof. The proof is exactly like that of Theorem 6.1, with $(x, x_3) \in \mathbb{R}^3$ replaced by $(x, x_{n+1}) \in \mathbb{R}^{n+1}$ throughout. So for example, $u^\varepsilon_w = Q^\varepsilon_{UW}(w)$ is defined by formulas (6.2), where now q^ε is a map $\mathbb{R}^n \to \mathbb{R}^n$ defined exactly as in (6.3), and O_w is a rigid motion of R^{n+1} that maps the line passing through $x(w)$ and $y(w)$ onto the x_{n+1} coordinate axis $\{(0, x_{n+1}) : x_{n+1} \in \mathbb{R}\}$. Then the continuity (4.6) of $w \mapsto u^\varepsilon_w$ follows as before, and the estimate of $E^\varepsilon_U(u^\varepsilon_w)$ is also precisely the same as that of Theorem 6.1, once we note that for $n \geq 3$ one has

$$\frac{1}{\omega_n |\ln \varepsilon|} \int_{\{x \in \mathbb{R}^n \,:\, |\ln \varepsilon|^{-1} \leq |x| \leq \mathrm{diam}(\Omega)\}} \left(\frac{|\nabla q^\varepsilon|^n}{n} + \frac{(|q^\varepsilon|^2 - 1)^2}{4\varepsilon^2} \right) dx \leq C |\ln \varepsilon|^{-1},$$

$$\frac{1}{\omega_n |\ln \varepsilon|} \int_{\{x \in \mathbb{R}^n \,:\, |x| \leq |\ln \varepsilon|^{-1}\}} \left(\frac{|\nabla q^\varepsilon|^n}{n} + \frac{(|q^\varepsilon|^2 - 1)^2}{4\varepsilon^2} \right) dx = 1 + O(|\ln \varepsilon|^{-1}).$$

Finally, the verification of (4.7) goes precisely as in the proof of Theorem 6.1, after we recall that for a map $u = (u^1, \ldots, u^n) \in W^{1,n}(\Omega \subset \mathbb{R}^{n+1}; \mathbb{R}^n)$,

$$J(u) = \frac{1}{n} d(u^1 \wedge du^2 \wedge \ldots \wedge du^n),$$

and also that $\star J(u^0_w) = \omega_n T_{x(w)y(w)}$ (see for example [2]). $\qquad \square$

Next, we recover some known results about the 2d Modica–Mortola functional.

Theorem 6.4. *Assume that Ω is a bounded domain in $\mathbb{R}^2$ with C^3 boundary and that (6.1) holds. Assume that the distance function d_0 given by (5.3) has a nondegenerate critical point at $(0,0)$ in the sense of (5.4). Let $T_* \in \mathcal{R}'_1(\Omega)$ be as defined in Theorem 6.1. Then there exists a value $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the Modica–Mortola energy (3.9) possesses a critical point u_ε and $E^\varepsilon_U(u_\varepsilon) \to E_V(v_*)$ as $\varepsilon \to 0$ for E_V given by (3.10).*

As remarked in the introduction, stronger and more general results in the same vein are established by very different techniques in [16, 25]. Also, one may apply Theorem 1 of [12] to assert that the sequence of varifolds associated with the energy density of $\{u_\varepsilon\}$ converges to a stationary varifold.

Proof. The proof again follows very closely the argument of the proof of Theorem 6.1. Recall that we have recast the usual Modica–Mortola Γ-limit so that $U = H^1(\Omega; \mathbb{R})$, $V = \mathcal{F}'_1(\Omega)$ and $W = B_{a_1} \subset \mathbb{R}^\ell$, $\ell \in \{0, 1, 2\}$. The fact that E^ε_U Γ-converges to E_V is the content of Theorem 3.1. The map P^e_{VU} is again independent of ε and is given

by $P_{VU}(u) = \star du/2$ (cf. (2.13)). The mapping Q^ε_{UV} corresponding to the recovery sequence construction will be described below in the case needed for the present theorem. Proposition 3.3 establishes the fact that E^ε_U satisfies the Palais–Smale condition.

We again define P_{WV} and Q_{VW} via (5.23) and (5.66). With an eye towards verifying the hypotheses of Theorem 4.4, we first note that T_* is a saddle point of E_V in light of Corollary 5.2. It remains to verify (4.6)–(4.8) To this end, we now describe in detail the mapping Q^ε_{UV}. We define $u^\varepsilon_w = Q^\varepsilon_{UW}(w)$ for $X = (x, x_2)$ by

$$u^\varepsilon_w(X) = v^\varepsilon(O_w(X)) \quad \text{where} \quad v^\varepsilon(X) = v^\varepsilon(x, x_2) = q^\varepsilon(x).$$

Now $O_w : \mathbb{R}^2 \to \mathbb{R}^2$ is a rigid motion of $\mathbb{R}^2$ depending smoothly on w and mapping the line through $\psi^-(x(w))$ and $\psi^+(y(w))$ onto the x_2 coordinate axis, and $q^\varepsilon(x) = q(x/\varepsilon)$, where $q : \mathbb{R} \to \mathbb{R}$ denotes the (heteroclinic) solution to the differential equation

$$q'' = q^3 - q \quad \text{on } \mathbb{R}, \qquad q(\pm\infty) = \pm 1, q(0) = 0. \tag{6.6}$$

(In fact one can solve this explicitly to find $q(t) = \tanh(t/\sqrt{2})$.) The continuity (4.6) follows exactly as before from the smoothness of $w \mapsto O_w$.

As before we write $u^0_w(X) = v^0(O_w(x))$ where $v^0(X) = q^0(x) = x/|x|$. Regarding condition (4.7) we use (2.14) to verify that

$$\|P_{VU} \circ Q^\varepsilon_{UW}(w) - Q_{VW}(w)\|_{\mathcal{F}'_1(\Omega)} \leq \|u^\varepsilon_w - u^0_w\|_{L^1(\Omega)} = \|v^\varepsilon_w - v^0_w\|_{L^1(O_w(\Omega))}$$

$$\leq \text{diam}(\Omega)\|q^\varepsilon - q^0\|_{L^1(\mathbb{R})}.$$

A change of variables shows that $\|q^\varepsilon - q^0\|_{L^1(\mathbb{R})} = \varepsilon\|q^1 - q^0\|_{L^1(\mathbb{R})}$, proving (4.7).

Finally, (4.8) follows almost exactly as in the proof of Theorem 6.1, once we use (6.6) to see that

$$\frac{1}{2}\left(q'(t)\right)^2 = \frac{1}{4}(q(t)^2 - 1)^2.$$

This implies through equipartition of energy (i.e. $A^2 + B^2 = 2AB$) that

$$\frac{3}{2\sqrt{2}} \int_{\{x \in \mathbb{R} : |\ln \varepsilon|^{-1} \leq |x|\}} \left(\frac{\varepsilon(q^{\varepsilon\,\prime})^2}{2} + \frac{(|q^\varepsilon|^2 - 1)^2}{4\varepsilon}\right) dx \leq C e^{-1/\sqrt{\varepsilon}}$$

and

$$\frac{3}{2\sqrt{2}} \int_{\{x \in \mathbb{R} : |x| \leq |\ln \varepsilon|^{-1}\}} \left(\frac{\varepsilon(q^{\varepsilon\,\prime})^2}{2} + \frac{(|q^\varepsilon|^2 - 1)^2}{4\varepsilon}\right) dx = 1 + O(e^{-1/\sqrt{\varepsilon}}).$$

Changing x_3 to x_2 in the definitions of the sets A^ε_w and B^ε_w, the remainder of the verification follows as in Theorem 6.1. $\square$

We conclude this section on applications with a result on critical points for the full 3d Ginzburg–Landau energy modeling superconductivity, namely,

$$E^\varepsilon_U(u, A)$$

$$:= \frac{1}{|\ln \varepsilon|}\left\{\int_\Omega \left(\frac{1}{2}|(\nabla - iA)u|^2 + \frac{1}{4\varepsilon^2}(|u|^2 - 1)^2\right) dX + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \times A - H^\varepsilon_{\text{ap}}|^2 dX\right\}. \tag{6.7}$$

Here, as before, $u \in H^1(\Omega; \mathbb{C})$ for $\Omega \subset \mathbb{R}^3$, while one typically takes the vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$ to lie in the space $\mathcal{H}_0$ consisting of the completion of $C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$, zero-divergence vector fields A with respect to the norm $\|\nabla A\|_{L^2(\mathbb{R}^3)}$. Physically, A corresponds to the effective magnetic potential. The vector field $H_{\mathrm{ap}}^\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ denotes a given external magnetic field and for the result below we must assume

$$\limsup_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|^2} \int_\Omega |H_{\mathrm{ap}}^\varepsilon|^2 \, dX = 0. \tag{6.8}$$

It is under the assumption (6.8) that, roughly speaking, the Γ-limit is again given by arc-length. (We refer e.g. to [1] for a discussion of the asymptotic behavior of the Ginzburg–Landau energy in the presence of larger magnetic fields.) Using techniques very similar to those just invoked for the energy (3.11), we have

Theorem 6.5. *Assume that Ω is a bounded, simply connected domain in $\mathbb{R}^3$ with C^3 boundary. Assume that the distance function d_0 given by (5.3) has a nondegenerate critical point at $(0, 0)$ in the sense of (5.4). Let $T_* \in \mathcal{R}_1'(\Omega)$ be defined as in Theorem 6.4. Then under assumption (6.8), there exists a value $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the full Ginzburg–Landau energy (6.7) possesses a critical point $(u_\varepsilon, A_\varepsilon)$ and $E_U^\varepsilon(u_\varepsilon, A_\varepsilon) \to E_V(T_*)$ as $\varepsilon \to 0$ for E_V again given by (3.13).*

Proof. We will only sketch the proof. Following the general approach of [13], it is convenient to introduce the decomposition of any vector field $A \in H^1(\Omega; \mathbb{R}^3)$ in the form

$$A = \nabla \times B + \nabla \phi$$

where $\phi \in H^1(\Omega; \mathbb{R})$ is unique up to a constant and $B \in H^2(\Omega; \mathbb{R}^3)$ is uniquely determined by the decomposition and the requirements

$$\operatorname{div} B = 0 \text{ in } \Omega, \quad B \times \nu = 0 \text{ on } \partial\Omega, \quad \|B\|_{H^2(\Omega; \mathbb{R}^3)} \le C \|\nabla \times A\|_{L^2(\Omega; \mathbb{R}^3)}. \tag{6.9}$$

We then write

$$\mathbb{P}(A) := \nabla \times B = A - \nabla \phi. \tag{6.10}$$

It is then easy to argue that to find critical points of (6.7) it is sufficient to find critical points of the functional $\mathcal{G}_U^\varepsilon(u, A)$ given by

$$\mathcal{G}_U^\varepsilon(u, A) := E_U^\varepsilon(u e^{i(\phi + \phi_{\mathrm{ap}}^\varepsilon)}, A + A_{\mathrm{ap}}^\varepsilon)$$

$$= E_U^\varepsilon(u) + \frac{\int_\Omega (\frac{1}{2}|u|^2 |\mathbb{P}(A + A_{\mathrm{ap}}^\varepsilon)|^2 - \langle \mathbb{P}(A + A_{\mathrm{ap}}^\varepsilon), j(u) \rangle) \, dX + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times A|^2 \, dX}{|\ln \varepsilon|}. \tag{6.11}$$

Here and throughout this proof, $E_U^\varepsilon(u)$ (as distinguished from $E_U^\varepsilon(u, A)$) denotes the Ginzburg–Landau energy without field (3.11) and we have introduced $A_{\mathrm{ap}}^\varepsilon$ to denote the applied magnetic potential satisfying

$$\nabla \times A_{\mathrm{ap}}^\varepsilon = H_{\mathrm{ap}}^\varepsilon \text{ in } \mathbb{R}^3, \quad \operatorname{div} A_{\mathrm{ap}}^\varepsilon = 0 \text{ in } \Omega, \quad A_{\mathrm{ap}}^\varepsilon \cdot \nu = 0 \text{ on } \partial\Omega.$$

In (6.11), A and ϕ and $A_{\mathrm{ap}}^\varepsilon$ and $\phi_{\mathrm{ap}}^\varepsilon$ respectively are related via (6.10).

One can then apply the machinery of our Theorem 4.4 with $U = H^1(\Omega; \mathbb{C}) \times \mathcal{H}_0$, $V = \mathcal{F}'_1(\Omega)$, $\mathcal{G}^\varepsilon_U$ playing the role of E^ε_U, and E_V still given by (3.13). As in the case with no field, we define $P_{VU}(u, A) := \star J(u)/\pi$ and we define $Q^\varepsilon_{UW}(w) = (u^\varepsilon_w, 0)$ where as before, the definition of u^ε_w is given in (6.2), (6.3). The mappings P_{WV} and Q_{VW} are the same as in the 3d Ginzburg–Landau setting without field.

Verification of (4.5)–(4.7) follows as before. As regards the verification of Γ-convergence in the sense of (3.4), (3.5) and (4.8), along with the compactness requirement (3.6), it turns out that the key term to control in the expression (6.11) is the only indefinite one:

$$\frac{\int_\Omega \langle \mathbb{P}(A + A^\varepsilon_{\text{ap}}), j(u) \rangle \, dX}{|\ln \varepsilon|}.$$

Here we appeal to (6.10) to write

$$\int_\Omega \langle \mathbb{P}(A + A^\varepsilon_{\text{ap}}), j(u) \rangle \, dX = \star J(u)(B^\varepsilon)$$

where $\mathbb{P}(A + A^\varepsilon_{\text{ap}}) = \nabla \times B^\varepsilon$ defines B^ε. A technical issue that is more fully developed in [13] is the fact that in viewing B^ε as a 1-form to be acted upon by the 1-current $\star J(u)$, we must extend various estimates on weak Jacobians beyond their action on compactly supported 1-forms to the setting where they act instead upon 1-forms such as B^ε that are purely normal at the boundary (cf. (6.9)). Particularly crucial is the estimate that for any $\alpha \in (0, 1]$ there exist positive constants γ and $C(\alpha, \Omega)$ such that for any $u \in H^1(\Omega; \mathbb{C})$ one has

$$\|\star J(u)\|_{C^{0,\alpha}_T(\Omega)^*} \leq C(\alpha, \Omega) \left(\varepsilon^\gamma + \frac{E^\varepsilon_U(u)}{|\ln \varepsilon|} \right). \tag{6.12}$$

Here $C^{0,\alpha}_T(\Omega)$ denotes the space of Hölder continuous 1-forms having zero tangential component, and $C^{0,\alpha}_T(\Omega)^*$ denotes its dual.

Using (6.12) with $\alpha \in (0, 1/2)$, and invoking (6.8), (6.9) and the Sobolev embedding of H^2 in $C^{0,\alpha}$, one easily checks that whenever a sequence $\{(u^\varepsilon, A^\varepsilon)\} \subset H^1(\Omega; \mathbb{C}) \times \mathcal{H}_0$ obeys the uniform bound $\mathcal{G}^\varepsilon_U(u^e, A^\varepsilon) < C$, one has

$$\frac{|\int_\Omega \langle \mathbb{P}(A^\varepsilon + A^\varepsilon_{\text{ap}}), j(u^\varepsilon) \rangle \, dX|}{|\ln \varepsilon|} = o(1)E^\varepsilon_U(u^\varepsilon) + o(1).$$

In light of (6.11), this allows us to infer the desired lower semicontinuity and compactness properties of such a sequence $\{(u^\varepsilon, A^\varepsilon)\}$ from the corresponding properties, already discussed, enjoyed by sequences $\{u^\varepsilon\}$ satisfying a uniform bound $E^\varepsilon_U(u^\varepsilon) < C$. Property (3.5) and its strengthened version (4.8) also easily follow from the corresponding conditions already verified for E^ε_U since we then see that $\mathcal{G}^\varepsilon_U(u^\varepsilon_w, 0) = E^\varepsilon_U(u^\varepsilon_w) + o(1)$.

When combined with the verification of the Palais–Smale condition below, this proves all the requirements of Theorem 4.4. □

Lemma 6.6. *For each $\varepsilon > 0$, every Palais–Smale sequence $\{(u_k, A_k)\} \subset H^1(\Omega) \times \mathcal{H}_0$ for the functional $\mathcal{G}^\varepsilon_U$ given by (6.11) has a strongly convergent subsequence.*

Proof. We pursue an argument similar to that found in the appendix to [3], where the Palais–Smale condition for the 2d Ginzburg–Landau energy with field is verified. Since ε is fixed and plays no role in the proof, we will ignore the factor of $1/|\ln \varepsilon|$ appearing in the definition of $\mathcal{G}_U^\varepsilon$.

Assume $\{(u_k, A_k)\} \subset H^1(\Omega) \times \mathcal{H}_0$ is a Palais–Smale sequence (cf. (3.14)). The uniform energy bound

$$\mathcal{G}_U^\varepsilon(u_k, A_k) < C \tag{6.13}$$

immediately yields

$$\|\nabla \times A_k\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)} < C \tag{6.14}$$

and it then follows from (6.9) that $\|\mathbb{P}(A_k)\|_{H^1(\Omega; \mathbb{R}^3)} < C$. Hence, there exists $A_0 \in \mathcal{H}_0$ such that after passing to a subsequence (with subsequential notation here and later suppressed), $\nabla A_k \rightharpoonup \nabla A_0$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ while $A_k \to A_0$ and $\mathbb{P}(A_k) \to \mathbb{P}(A_0)$ strongly in $L^p(\Omega; \mathbb{R}^3)$ as $k \to \infty$ for all $1 \le p < 6$.

The energy bound also immediately yields a uniform bound on the $L^4(\Omega)$ norm of $\{u_k\}$. Then applying Hölder's inequality twice, we find

$$\left| \int_\Omega \langle \mathbb{P}(A_k + A_{\mathrm{ap}}^\varepsilon), j(u_k) \rangle \, dX \right|$$

$$\le 2 \left(\int_\Omega |\mathbb{P}(A_k + A_{\mathrm{ap}}^\varepsilon)|^2 |u_k|^2 \, dX \right)^{1/2} \left(\int_\Omega |\nabla u_k|^2 \, dX \right)^{1/2}$$

$$\le C \left(\int_\Omega |\mathbb{P}(A_k + A_{\mathrm{ap}}^\varepsilon)|^4 \, dX \right)^{1/2} \left(\int_\Omega |u_k|^4 \, dX \right)^{1/2} + \frac{1}{4} \int_\Omega |\nabla u_k|^2 \, dX.$$

Once again appealing to the uniform energy bound, we can absorb this last term into the left-hand side of (6.13) to conclude that

$$\|u_k\|_{H^1(\Omega)} < C.$$

Consequently, the sequence $\{u_k\}$ is also uniformly bounded in $L^6(\Omega)$ and for a subsequence, one has $\nabla u_k \rightharpoonup \nabla u_0$ in $L^2(\Omega; \mathbb{R}^3)$, and $u_k \to u_0$ in $L^p(\Omega)$, $1 \le p < 6$, for some $u_0 \in H^1(\Omega)$.

Considering variations only in the first argument of $\mathcal{G}_U^\varepsilon$, the hypothesis that

$$\|\nabla \mathcal{G}_U^\varepsilon(u_k, A_k)\|_{(H^1(\Omega) \times \mathcal{H}_0)^*} \to 0 \tag{6.15}$$

implies that

$$|L_k(v)| \le C_k \|v\|_{H^1(\Omega)} \quad \text{with} \quad C_k \to 0 \tag{6.16}$$

where L_k is the linear functional on $H^1(\Omega)$ given by

$$L_k(v) := \frac{1}{2} \int_\Omega \{ \langle \nabla u_k, \nabla v^* \rangle + \langle \nabla u_k^*, \nabla v \rangle + (|u_k|^2 - 1)(u_k v^* + u_k^* v) \} \, dX$$

$$- \frac{1}{2i} \int_\Omega \langle (u_k^* \nabla v - u_k \nabla v^* + v^* \nabla u_k - v \nabla u_k^*), \mathbb{P}(A_k + A_{\mathrm{ap}}^\varepsilon) \rangle \, dX$$

$$+ \frac{1}{2} \int_\Omega (u_k v^* + v u_k^*) |\mathbb{P}(A_k + A_{\mathrm{ap}}^\varepsilon)|^2 \, dX.$$

Choosing $v = u_k - u_0$ in (6.16), we note that all terms from the second line above of $L_k(u_k - u_0)$ will approach zero in that they involve integrals pairing strongly convergent sequences with weakly convergent ones. The last line also approaches zero in the limit. Then we can rearrange the terms coming from the first line to obtain from (6.16) an inequality of the form

$$\|\nabla u_k - \nabla u_0\|_{L^2(\Omega)}^2 \leq \int_\Omega (|\nabla u_0|^2 - |\nabla u_k|^2)\, dX$$
$$+ C_k(\|\nabla u_k - \nabla u_0\|_{L^2(\Omega)} + \|u_k - u_0\|_{L^2(\Omega)}) + o(1).$$

Since $\liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2\, dX \geq \int_\Omega |\nabla u_0|^2\, dX$, we conclude that $u_k \to u_0$ strongly in $H^1(\Omega)$.

A similar manipulation of the condition coming from the application of (6.15) to variations of the second argument of $\mathcal{G}_U^\varepsilon$ allows us to improve the convergence of A_k to A_0 from weak to strong as well. □

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