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Skeletons, bodies and generalized $E(R)$-algebras

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Abstract. In this paper we want to solve a fifty year old problem on R-algebras over cotorsion-free commutative rings R with 1. For simplicity (but only for the abstract) we will assume that R is any countable principal ideal domain, but not a field. For example R can be the ring $\mathbb{Z}$ or the polynomial ring $\mathbb{Q}[x]$. An R-algebra A is called a generalized $E(R)$-algebra if its algebra $\operatorname{End}_R A$ of R-module endomorphisms of the underlying R-module ${}_RA$ is isomorphic to A (as an R-algebra). Properties, including the existence of such algebras are derived in various papers ([5, 6, 9, 10, 20, 22, 24, 25]). The study was stimulated by Fuchs [13], and specially by Schultz [26]. But due to [26] the investigation concentrated on *ordinary* commutative $E(R)$-algebras. A substantial part of problem 45 (p. 232) in the monograph [13] (repeated in later publications, e.g. [27]), which will be answered positively for all rings R above in this paper, remained open:

Can we find non-commutative generalized $E(R)$-algebras?

In Theorem 1.5 we will show that for all countable, principal ideal domains R which are not fields and for any infinite cardinal κ there is a non-commutative R-algebra A of cardinality $|A| = \kappa^{\aleph_0}$ with $\operatorname{End}_R A \cong A$, so A is a non-commutative generalized $E(R)$-algebra, and—not too surprisingly—there is a proper class of examples.

The new strategy should be interesting and useful for other problems as well: We will first translate the heart of the algebraic question on the existence of certain monoids via model theory into geometric structures leading to a special class of (decorated) trees and solve this problem introducing products of trees etc. This can be compared with the well-known, but different process of translating group problems to small cancelations in groups via the van Kampen lemma. By small cancelation of trees we are able to find a suitable monoid and thus a non-commutative algebra A with an important non-canonical embedding $A \hookrightarrow \operatorname{End}_R A$, our $*$-scalar multiplication. In a second part of this paper we must enlarge A to get rid of all undesired endomorphisms. This can be done more easily (thus first) with the help of additional set theory (Jensen's diamond predictions), which will support the reader to understand more quickly the last steps of the proof. In a final chapter we will also give an argument (for removing unwanted endomorphisms) which is based on

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ordinary set theory of ZFC only (using our favored Black Box predictions; see [20]). Thus we get the result as stated above. Furthermore, this last chapter includes a construction for rigid systems of non-commutative generalized $E(R)$-algebras.

Keywords. Endomorphism rings, indecomposable modules, E-rings

1. Introduction

As indicated in the abstract we want to construct certain non-commutative R-algebras over commutative rings R with 1. Recall that an R-algebra A is a *generalized $E(R)$-algebra* if $\mathrm{End}_R A \cong A$ as R-algebras, where $\mathrm{End}_R M$ denotes the algebra of R-module endomorphisms of an R-module M. The question originates from Problem 45 (p. 232) in the monograph [13] by Fuchs, which (in the case $R = \mathbb{Z}$) reads as follows:

> *Characterize the rings A with* $\mathrm{End}(A^+) \cong A$, *$A^+$ considered as a group without operators.*

Despite the fact that many such commutative R-algebras (the so called *$E(R)$-algebras*, see Definition 1.3) were constructed in the last twenty years, the existence of non-commutative generalized $E(R)$-algebras remained a challenging open problem; see also Vinsonhaler [27]. In the introduction we will discuss how the existence of those algebras can be achieved. It turns out that the main case $R = \mathbb{Z}$ (for the underlying ring) can be extended to arbitrary commutative rings R which are *cotorsion-free* (or more precisely p-cotorsion-free). Thus the ring R must satisfy the following conditions:

- There is an element $p \in R$ (which pretends to be a prime) satisfying $\bigcap_{n<\omega} p^n R = 0$ such that the multiplicatively closed set $\mathbb{S} = \{p^n \mid n < \omega\}$ has no zero-divisors; this is to say that R is p-reduced and p-torsion-free, thus the p-adic topology on R (generated by $\{p^n R \mid n < \omega\}$) is Hausdorff and the p-adic completion $\widehat{R}$ of R exists.
- The p-adic completion $\widehat{R}$ of R satisfies $\mathrm{Hom}_R(\widehat{R}, R) = 0$.

By the first condition the ring R cannot be a field and by the second condition R cannot be the ring J_p of p-adic integers. It is easy to see that for these two examples the required R-algebras would not exist. Thus cotorsion-freeness is not only sufficient but also necessary for our work. Furthermore, it was shown in [18] that any p-reduced and p-torsion-free, commutative ring R with 1 (in particular any principal ideal domain with prime p) of size $|R| < 2^{\aleph_0}$ is also cotorsion-free; and it is easy to find arbitrarily large cotorsion-free principal ideal domains, which can serve as our ground ring of the algebra A. The element p will also be used to define pure submodules $U \subseteq_* M$ of R-modules M, which means that $p^n M \cap U \subseteq p^n U$ for all $n < \omega$; in particular $R \subseteq_* \widehat{R}$.

Since the rings R need not satisfy the freeness criterion by Pontryagin (see [14, p. 93]), we will use the following definition to circumvent its application.

Definition 1.1. *An R-module M is* $\aleph_0$-free *if every finite subset $S \subseteq M$ belongs to a free, p-pure submodule $F \subseteq M$ with p-reduced quotient M/F.*

Thus an $\aleph_0$-free R-module M is $\aleph_1$-free if R permits application of Pontryagin's theorem. Recall here the definition of κ-free and strongly κ-free R-modules from [8, 20].

Definition 1.2. *An R-module M is* κ-free *for some cardinal $\kappa \geq \aleph_1$ if every subset $S \subseteq M$ of size $|S| < \kappa$ belongs to a free submodule $F \subseteq M$. Moreover, M is* strongly κ-free *if we can always choose a free submodule $F \subseteq M$ with κ-free quotient M/F.*

The study of $E(R)$-algebras began with the fundamental paper [26] by Schultz, where the question about non-commutative rings answering Problem 45 is mentioned and where also the term E-rings was chosen. We recall the following

Definition 1.3. *If A is an R-algebra, then $\delta : A \to \operatorname{End}_R A$ denotes the R-algebra homomorphism which takes any $a \in A$ to the induced right multiplication $a_r \in \operatorname{End}_R A$ by a. If this homomorphism is an isomorphism, then A is called an* $E(R)$-algebra *and $_RA$ is called an* $E(R)$-module. *Furthermore, $E(\mathbb{Z})$-algebras are also called* E-rings.

The class of E-rings attracted works concerning its existence and properties (see [3, 5, 6, 9, 10, 15, 16, 17, 22, 24, 25]). Several of these results are discussed in the monograph [20]. In particular, it is easy to see that $E(R)$-algebras are necessarily commutative; see [20, Proposition 13.1.9, p. 468]. Since we will construct non-commutative R-algebras A with $\operatorname{End}_R A \cong A$, it follows that our new examples A are not $E(R)$-algebras, but *proper* generalized $E(R)$-algebras, as required. Their existence is also needed to illustrate a main result in [11]. Moreover, this fills in essential details to justify the main result of [19].

It is crucial for the construction of a proper generalized $E(R)$-algebra A to derive the existence of a well-behaving but non-canonical embedding $A \hookrightarrow \operatorname{End}_R A$. Well-behaving means that we are able to perform calculations in a reasonable way, and non-canonical means very distinct from the embedding in Definition 1.3. This is the main topic of the next five sections of this paper. Using model-theoretic arguments, we will define structures which we call *skeletons*. These are objects with a special binary function symbol $*$, the $*$-scalar product. Due to required closure properties of this $*$-multiplication (the $*$-scalar product law; Definition 2.1(iv)) further function symbols and a right identity 1 are needed satisfying certain (not very algebraic) laws of first order logic; see Section 2. If now this $*$-scalar product acts *faithfully* on a skeleton M, then M can be converted into a monoid with a new multiplication $\cdot$ derived from its $*$-multiplication. Thus the skeleton M turns into an ordinary (non-commutative) monoid $(M, \cdot, *)$ with an additional $*$-scalar product (and some further functions). From this special monoid $(M, \cdot)$ we will proceed in Section 3 to our desired non-commutative R-algebra A as the induced monoid-R-algebra, while the $*$-scalar product will be used for the non-canonical embedding $A \hookrightarrow \operatorname{End}_R A$.

It remains to establish the existence of skeletons $(M, *)$. Here we will use an idea which has a flavor of van Kampen's lemma, where relations (terms) in groups can be interpreted by diagrams of cell decompositions of surfaces. This allows one to transfer group-theoretic questions to problems of combinatorial topology, which are often easier. A consequence is small cancelation theory which opened simpler proofs for the Burnside problem and many other group-theoretic questions (see [23]). A first nice application of this transfer method is a proof by Baer–Levi [1] from 1936 of Kurosh's subgroup theorem for free products (see also [21, pp. 274–285]). Through Sections 4, 5 and 6 we now have to find new geometric objects (which are finite (decorated) trees), and must define a $*$-multiplication of these trees. This is related to products of n-fold operads (see [12, Section 3, pp. 8–10] and the references in this survey paper). And since our $*$-multiplication must satisfy the first order logic laws of Definition 2.1, we also need reductions between these trees. By putting up a carefully chosen family of permitted small cancelations (see Definition 4.3 and the subsequent Reduction Cases A to D) we finally derive the existence of arbitrarily large skeletons which extend to R-algebras with an additional $*$-scalar product (and some further functions)—the *bodies* of our construction. The nice advantage of this geometric-combinatorial approach is that now many proofs can be easily inspired by pictures (which are included).

In the second part of this paper (starting with Section 7) we will enlarge bodies A with their derived non-canonical $*$-embedding $A \hookrightarrow \mathrm{End}_R A$ in such a way that this embedding becomes an R-algebra isomorphism. The arguments here are obvious, we must kill all undesired R-module endomorphisms. However, the details need a more sophisticated, but (as we believe even) elegant Step Lemma taking care of the strays; see Section 8.

To get rid of unwanted R-module endomorphisms globally (in the construction of the final R-algebra A), we will first (in Section 9) apply Jensen's diamond principle for their prediction. This has the advantage that the proof is more transparent, which might help the reader to understand the transfinite construction of A. As usual, as an extra gift, we can conclude that the R-algebra A (being the union of a κ-filtration of free bodies) is very free in a by now obvious sense (strongly κ-free as an R-module); see Eklof–Mekler [8], for instance.

Theorem 1.4. *Let R be a cotorsion-free commutative ring with 1 and $|R| < \kappa$ be a regular, uncountable cardinal with $\diamondsuit_\kappa E$ for a non-reflecting stationary subset $E \subseteq \kappa$ of ordinals cofinal to ω. Then there is a strongly κ-free, non-commutative R-algebra A of cardinality $|A| = \kappa$ with $\mathrm{End}_R A \cong A$. Thus A is a proper generalized $E(R)$-algebra.*

Proof. In Section 9 we construct an R-algebra $A = (\mathbb{B}, +, \cdot)$ from a suitable body $\mathbb{B}$. This algebra possesses another binary operation $*$, the $*$-scalar product, and every $g \in A$ induces an R-module endomorphism $*_g : x \mapsto x * g$ $(x \in A)$ of ${}_RA$. With our Main Lemma 9.1 this correspondence $A \to \mathrm{End}_R A$ $(g \mapsto *_g)$ is an R-algebra isomorphism $A \cong \mathrm{End}_R A$. ◻

Our main theorem will take place in the ordinary set theory of ZFC (see Section 10). The final steps are based on the Black Box (see [4, 20]). This prediction principle has been adjusted for its application to bodies (over R) in Theorem 10.3. It will be applied in the same Section 10.3 to prove Main Lemma 10.7 and this will be used below to derive the existence of the R-algebras A mentioned in our Main Theorem 1.5. By this construction and the setting of the Black Box it follows that these R-algebras A are $\aleph_0$-free as R-modules (see Definition 1.1). Moreover, if R permits Pontryagin's theorem (as for all principal ideal domains), then A will automatically be an $\aleph_1$-free R-module.

Main Theorem 1.5. *Let R be a cotorsion-free commutative ring with* 1 *and* $|R| \leq \kappa$ *be an uncountable cardinal with* $\kappa = \kappa^{\aleph_0}$. *Then there is an* $\aleph_0$*-free, non-commutative* R*-algebra* A *of cardinality* $|A| = \kappa$ *with* $\operatorname{End}_R A \cong A$. *Thus* A *is a proper generalized* $E(R)$*-algebra.*

From Corollary 8.8 it follows that A in the theorem is cotorsion-free.

In Appendix 10.4 we also indicate how to construct a *rigid system* of non-commutative generalized $E(R)$-algebras. This is a maximal family of generalized $E(R)$-algebras A as in Main Theorem 1.5 with only the zero-homomorphism between its distinct members.

Proof of Main Theorem 1.5. Arguing as above for Theorem 1.4 we now apply our Main Lemma 10.7 to complete the proof. □

Finally, we would like to emphasize that the presented method will be useful for various other problems in ring and module theory. We furthermore thank Professor Charles Vinsonhaler (from the University of Connecticut) for a donation of \$45 for the solution of the problem discussed here.

2. The theory of skeletons

Let τ be a vocabulary **with no predicates** consisting of an infinite set of free variables X and a set $\mathcal{F}$ of function symbols with an arity function $\operatorname{ar} : \mathcal{F} \to \omega$ assigning to each function symbol F its arity $n = \operatorname{ar}(F)$, the number n of places of F. Furthermore, let 1 be a fixed constant symbol, $*$ a particular binary function symbol and F_{id} a special unary function belonging to τ. Apart from 1 there may be further constant symbols belonging to τ which we view as function symbols in $\mathcal{F}$ with arity 0.

As usual we define τ-terms of the language τ starting from atomic terms (elements from X and constant symbols) and deriving inductively new terms $F(t_0, \ldots, t_{n-1})$ if F is any function symbol with arity n and $t_0, \ldots, t_{n-1}$ are terms already defined.

Thus, with equations between τ-terms, a τ-theory is obtained by first order logic. We now come to our first basic

Definition 2.1. *We call the following axioms the* skeleton theory T *on* τ.

(i) $\forall x : (x * 1 = x)$ *and* $\forall x : (F_{\mathrm{id}}(x) = x)$, *where x is a variable and* $x * 1 := *(x, 1)$.

(ii) *If $F \in \tau$ is a function symbol with arity* $\operatorname{ar}(F) = n$ *and*

$$\pi : \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$$

is a permutation, then

$$\forall x_i : F(x_{0\pi}, x_{1\pi}, \ldots, x_{(n-1)\pi}) = F'(x_0, x_1, \ldots, x_{n-1})$$

for another function symbol $F' \in \tau$ of arity $\operatorname{ar}(F') = n$.

(iii) *If $F_j, F \in \tau$ is a family of function symbols with arities* $\mathrm{ar}(F) = k$*,* $\mathrm{ar}(F_j) = n_j$ *for* $j < k$ *and* $n = \sum_{j=0}^{k-1} n_j$*, then*

$$\forall x_i : F(F_0(x_0, \dots, x_{n_0-1}), F_1(x_{n_0}, \dots, x_{n_0+n_1-1}), \dots, F_{k-1}(x_{n-n_{k-1}}, \dots, x_{n-1})) = F'(x_0, \dots, x_{n-1})$$

for another function symbol $F' \in \tau$ *of arity* $\mathrm{ar}(F') = n$.

(iv) *If* $F \in \tau$ *and* $0 \le m < n = \mathrm{ar}(F)$*, then*

$$\forall x_i, y : F(x_0, x_1, \dots, x_{m-1}, y, x_{m+1}, \dots) = y * F'(x_0, x_1, \dots, x_{m-1}, x_{m+1}, \dots)$$

for another function symbol $F' \in \tau$ *of arity* $\mathrm{ar}(F') = n - 1$.

We note that 1 by (i) acts like an identity element on the right and F_{id} is the identity function. The set $\mathcal{F}$ of function symbols is closed under permutation of arguments by (ii). Moreover, (iii) says that substitutions of legal functions are legal functions, where all variables x_i must be pairwise distinct, while (iv) is a $*$-scalar product: the function $F(y)$ on the left can be viewed as $*$-multiplication of y by some function F' on the right. We will call (ii) the *permutation law*, (iii) the *substitution law* and (iv) the $*$*-scalar product law*, and M is a τ*-skeleton* or just a *skeleton* if M is a τ-structure satisfying these axioms.

As usual, we will also say that a skeleton M is a T*-model*. Furthermore, recall that the axioms of our skeleton theory T are a family of equations in first order logic only: We fix the choice of $F' \in \tau$ rather than using the quantifier $\exists F'$. Thus, strictly speaking, T depends on the exact choice of the function symbols F'. (But this important fact will not be relevant for the following general observations.) It follows immediately that T is a τ-theory and the class of all skeletons is a variety, which we also denote by T. As a variety, T is closed under taking cartesian products, epimorphic images and substructures and it has free skeletons. We are particularly interested in the existence and the description of these T-free objects and will represent them by free generators with the help of special finite trees in Section 6.

We will very often write τ-terms σ as $\sigma(x_0, \dots, x_{n-1})$ to emphasize the free variables $x_0, \dots, x_{n-1}$ appearing in σ. Furthermore, the notion of a τ-term carries over naturally to the (well-known) definition of an M-term. Next we single out particular τ-terms which will later serve as endomorphisms.

Definition 2.2. *We will call a τ-term σ a* strict τ-term *if* $\sigma = F(x_0, \dots, x_{n-1})$ *for a function symbol F of arity n and suitable variables* $x_i \in X$.

Again, recall that the variables x_i are all pairwise distinct. We note that $x_0 * (x_1 * x_2)$ itself is not a strict term, because this term is not generated by a single function symbol, but the theory allows us to show that it is equal to a strict term after using the axioms: first note that $x_1 * x_2 = *(x_1, x_2)$ and $x_0 = F_{\mathrm{id}}(x_0)$ can be expressed by using function symbols from τ. By the substitution axiom (iii), there is a function symbol F with $x_0 * (x_1 * x_2) = *(F_{\mathrm{id}}(x_0), *(x_1, x_2)) = F(x_0, x_1, x_2)$ which is strict; observe also that $F_{\mathrm{id}} \in \mathcal{F}$ was needed for substitution in the last equation. However, $x_0 * x_0 = *(x_0, x_0)$ cannot be transformed by the axioms from T into a strict term, because x_0 appears twice and substitution does not apply. Furthermore, the class of strict τ-terms is obviously closed under permutation of variables.

Next we present another simple method to find derived strict τ-terms.

Observation 2.3. *Let F be a function symbol from τ of arity n and suppose that $c_{n_1}, \ldots, c_{n-1}$ ($n_1 < n$) are constants. Then there is a strict τ-term $F'(x_0, \ldots, x_{n_1-1})$ such that*

$$F'(x_0, \ldots, x_{n_1-1}) = F(x_0, \ldots, x_{n_1-1}, c_{n_1}, \ldots, c_{n-1}).$$

Proof. Note that in the τ-term $F(x_0, \ldots, x_{n_1-1}, c_{n_1}, \ldots, c_{n-1})$ we can replace $x_i = F_{\mathrm{id}}(x_i)$ and the constant c_i is interpretable as a function symbol F_{c_i} of arity 0. This leads to the equivalent representation $F(F_{\mathrm{id}}(x_0), \ldots, F_{\mathrm{id}}(x_{n_1-1}), F_{c_{n_1}}, \ldots, F_{c_{n-1}})$, the substitution axiom applies and there is a strict τ-term $F'(x_0, \ldots, x_{n_1-1})$ as required. □

By the same argument, any τ-term, in which every variable x_i appears only once, equals in T some strict τ-term. All other τ-terms are still closely connected with strict τ-terms, as shown in the next

Observation 2.4. *For every τ-term $\sigma(x_0, \ldots, x_{n-1})$ there exists a strict τ-term $F'(y_0, \ldots, y_{m-1})$ with suitable variables $y_i \in X$ and a surjective map $\pi : \{y_0, \ldots, y_{m-1}\} \to \{x_0, \ldots, x_{n-1}\}$ with $\sigma(x_0, \ldots, x_{n-1}) = F'(y_0\pi, \ldots, y_{m-1}\pi)$.*

Proof. By induction on the complexity of σ: The initial step is trivial. We assume that $\sigma(x_0, \ldots, x_{n-1}) = F(\sigma_0, \ldots, \sigma_{k-1})$ and the observation holds for the σ_is, thus

$$\sigma_i = \sigma_i(x_0^i, \ldots, x_{n_i-1}^i) = F_i(y_0^i\pi_i, \ldots, y_{m_i-1}^i\pi_i),\ \ F_i(y_0^i, \ldots, y_{m_i-1}^i) \text{ is } \tau\text{-strict}$$

and $\pi_i : \{y_0^i, \ldots, y_{m_i-1}^i\} \to \{x_0^i, \ldots, x_{n_i-1}^i\} \subseteq \{x_0, \ldots, x_{n-1}\}$ is surjective.

Without loss of generality, the variables y_j^i can be chosen pairwise distinct. By the substitution axiom, there is a function symbol F' (with an obvious arity) such that

$$\begin{aligned} &F(F_0(y_0^0, \ldots, y_{m_0-1}^0), \ldots, F_{k-1}(y_0^{k-1}, \ldots, y_{m_{k-1}-1}^{k-1})) \\ &\qquad = F'(y_0^0, \ldots, y_{m_0-1}^0, \ldots, y_0^{k-1}, \ldots, y_{m_{k-1}-1}^{k-1}), \end{aligned}$$

which is strict. If we patch the surjective maps π_i together, then $\pi = \bigcup_{i<k} \pi_i$ and

$$\begin{aligned} \sigma &= F(\sigma_0, \ldots, \sigma_{k-1}) \\ &= F(F_0(y_0^0\pi_0, \ldots, y_{m_0-1}^0\pi_0), \ldots, F_{k-1}(y_0^{k-1}\pi_{k-1}, \ldots, y_{m_{k-1}-1}^{k-1}\pi_{k-1})) \\ &= F'(y_0^0\pi, \ldots, y_{m_{k-1}-1}^{k-1}\pi) \end{aligned}$$

is as required. □

We will often use the following

Notation 2.5. If $k < \omega$, then $\overline{s} = \langle s_0, s_1, \ldots, s_{k-1}\rangle$ denotes an arbitrary finite sequence of length k consisting of elements s_i. If s is a fixed element (e.g. $s \in X$ or s is taken from a model M), then $\overline{s}^k = \langle s, \ldots, s\rangle$ denotes the constant sequence of k copies of s.

For $m < \omega$ and M a set we denote by ${}^mM$ the set of all sequences in M of length m. We next define a notion similar to strict τ-terms for M-terms and introduce at the same time a simple, but very useful argument to get rid of redundant M-terms.

Definition 2.6. *Let M be a τ-skeleton.*

(i) *Furthermore, let $\overline{s} \in {}^mM$ and $\sigma(\overline{x}_1, \overline{x}_2)$ be a strict τ-term with $\overline{x}_1$ of length k and $\overline{x}_2$ of length m. We will write $\sigma(x, \overline{s})$ for the derived M-term $\sigma^M(\overline{x}^k, \overline{s})$ and call $\sigma(x, \overline{s})$ a* strict M-term.

(ii) *If $\sigma_1(x, \overline{s}_1)$ and $\sigma_2(x, \overline{s}_2)$ are strict M-terms, then we say that $\sigma_1(x, \overline{s}_1)$ and $\sigma_2(x, \overline{s}_2)$ are* M-equivalent *if $\sigma_1(t, \overline{s}_1) = \sigma_2(t, \overline{s}_2)$ for all $t \in M$.*

(iii) *If $\sigma = \sigma(x, \overline{s}) = \sigma^M(\overline{x}^k, \overline{s})$ is a strict M-term, then we can choose k minimal under all M-equivalent strict representations of σ and call σ a* k-strict M-term (*or just* k-strict).

Observation 2.4 has immediate consequences for strict M-terms.

Corollary 2.7. *Let M be a τ-skeleton, $\sigma(x, \overline{x})$ be a τ-term with $\overline{x}$ of length m, and $\sigma^M(x, \overline{s})$ be a derived M-term for some $\overline{s} \in {}^mM$. Then there exists a strict τ-term $F'(\overline{x}_1, \overline{x}_2)$ (with $\overline{x}_1$ of length k and $\overline{x}_2$ of length $n \geq m$) and a finite sequence $\overline{s}' \in {}^nM$ with* $\mathrm{Im}(\overline{s}') = \mathrm{Im}(\overline{s})$ *such that $\sigma^M(x, \overline{s}) = F'^M(\overline{x}^k, \overline{s}')$ is a k-strict M-term.*

Next we will derive from $*$ another multiplication on M. For its uniqueness we will require that M is faithful (with respect to $*$). If x, y, z are variables, then by the axiom of substitution there is a function symbol F_0 with $F_0(x, y, z) = (x*y)*z$ and by the $*$-scalar product law there is another function symbol F_1 such that $F_0(x, y, z) = x * F_1(y, z)$, thus $(x * y) * z = x * F_1(y, z)$ holds in T.

Definition 2.8. *Let M be a τ-skeleton.*

(i) *M is* faithful *if the $*$-multiplication is faithful, meaning that for all $a \neq b \in M$ there is $c \in M$ such that $c * a \neq c * b$.*

(ii) *Using the equation $(x * y) * z = x * F_1(y, z)$ from above, we let $y \cdot z := F_1(y, z)$, thus $(x * y) * z = x * (y \cdot z)$. The product $y \cdot z$ is uniquely determined by the last equation for any faithful τ-skeleton M.*

While M with the operation $*$ is not associative, nor is 1 a two-sided identity element, the new product is more algebraic; indeed, we have the following important

Lemma 2.9. *Let M be a faithful τ-skeleton. Then the following holds.*

(a) *The new product is associative, i.e. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in M$.*

(b) *The element $1 \in M$ of the skeleton is a two-sided identity element, $1 \cdot x = x \cdot 1 = x$ for all $x \in M$ and $(M, \cdot)$ is an associative monoid with* 1.

Proof. (a) As M is faithful, it suffices to show

$$a * ((b \cdot c) \cdot d) = a * (b \cdot (c \cdot d)) \quad \text{for all } a, b, c, d \in M.$$

Using the definition of the new product we have $a * ((b \cdot c) \cdot d) = (a * (b \cdot c)) * d = ((a * b) * c) * d$ and also $a * (b \cdot (c \cdot d)) = (a * b) * (c \cdot d) = ((a * b) * c) * d$. Thus (a) is immediate.

(b) Again using the assumption that M is faithful, it suffices to show that

$$a * (1 \cdot b) = a * (b \cdot 1) = a * b \quad \text{for all } a, b \in M.$$

From T it follows that $a*(1\cdot b) = (a*1)*b = a*b$ and also $a*(b\cdot 1) = (a*b)*1 = a*b$, thus also (b) is immediate. □

Finally, in order to have enough elements in M to control the maps derived from strict M-terms locally, we will also need the definition of a nice skeleton. This technical definition is easily achieved for T-models and will be used for a linear independence argument similar to linear algebra to relate endomorphisms and strict M-terms by $*$-multiplication; see the crucial Theorem 3.5.

We first identify every $n < \omega$ with its underlying set $\{0, 1, \dots, n-1\}$ and define for any $m \le n < \omega$ the set $\Lambda_{m,n}$ of all non-constant maps $\eta : m \to n$; thus $\Lambda_{0,n} = \Lambda_{1,n} = \emptyset$ for all $n < \omega$. If $\overline{s} = \langle s_0, \dots, s_{n-1}\rangle$ and $\eta \in \Lambda_{m,n}$, then we put $\overline{s}_\eta = \langle s_{l\eta} \mid l < m\rangle$.

Definition 2.10. *A skeleton M is* nice *if the following holds. Suppose that $2 \le k_i \le n_i$, ${}^i\overline{t} \in {}^{n_i-k_i}M$ and $\sigma_i(x, {}^i\overline{t}) = \sigma_i^M(\overline{x}^{k_i}, {}^i\overline{t})$ $(i < i_*)$ is a family of strict M-terms which are pairwise not M-equivalent. Then for $k_* = \max\{k_i \mid i < i_*\}$ there is a sequence $\overline{s} = \langle s_0, \dots, s_{k_*-1}\rangle$ of elements from M such that for all $i, j < i_*$, $\eta \in \Lambda_{k_i,k_*}$, and $\theta \in \Lambda_{k_j,k_*}$ with $(i,\eta) \ne (j,\theta)$ we have $\sigma_i^M(\overline{s}_\eta, {}^i\overline{t}) \ne \sigma_j^M(\overline{s}_\theta, {}^j\overline{t})$.*

3. Bodies

We now extend the language τ of a skeleton and also strengthen the variety T by including a commutative ring R with 1. This leads to τ_R and T_R, respectively.

Definition 3.1. *Let R be a (fixed) commutative ring with* 1.

(i) *Then we introduce the extended language τ_R of τ by including additional unary function symbols F_a $(a \in R)$ representing scalar multiplication by $a \in R$ for T_R-models B, a binary function symbol $+$ for addition on B and a constant symbol* 0*. As usual we will write $x + y$ for $+(x, y)$ and $ax = F_a(x)$ for any $a \in R$.*

(ii) *The theory T of skeletons (which remains restricted to function symbols from τ) is extended to the theory of bodies T_R by the following additional laws. (We will omit the obvious quantifiers!)*

  (1) The left R-module axioms: *The function symbols $+$, F_a and the constant symbol* 0 *satisfy the usual axioms for R-modules, i.e. for all $a, b \in R$ the following equations hold.*

   (a) $x + 0 = x$, $x + y = y + x$, $x + (y + z) = (x + y) + z$, $x + (-1)x = 0$.

   (b) $1x = x$, $a(bx) = (ab)x$, $(a + b)x = ax + bx$, $a(x + y) = ax + ay$.

  (2) Multi-linearity: *If F is a function symbol with* $\operatorname{ar}(F) = n$, *$a_i \in R$ $(i \le t)$ and $0 \le l < n$, then*

$$F\Big(x_0, \dots, x_{l-1}, \sum_{i=1}^{t} a_i x_{li}, x_{l+1}, \dots, x_{n-1}\Big) = \sum_{i=1}^{t} a_i F(x_0, \dots, x_{l-1}, x_{li}, x_{l+1}, \dots, x_{n-1}).$$

(iii) *If the τ_R-structure B is a T_R-model, then B is called a τ_R*-body (*over the ring R*) *or for short an R-body.*

The theory of bodies is defined by laws which are equations, so the class of bodies is a variety (and as the class of skeletons) closed under cartesian products, quotients and substructures. Moreover, there are free bodies, the building blocks for the construction of generalized $E(R)$-algebras. We will show the existence of non-trivial free bodies and investigate their properties in the next sections. We summarize some immediate consequences of the last definition.

Observation 3.2. *Let B be a τ_R-body. Then the following holds.*

(a) *B is an R-module and a τ-skeleton.*
(b) $t * 0 = 0$ *for all $t \in B$.*

Proof. The first statement follows by definition of a body. Moreover, by the module laws and multi-linearity we have $0 + 0 = 0$ and $t * 0 = t * (0 + 0) = t * 0 + t * 0$, which implies $t * 0 = 0$. □

Next we convert skeletons into bodies, a technique that will be central for this paper.

Definition 3.3. *If M is a τ-skeleton, then we define:*

(i) $\mathrm{lin}_R\, M := \bigoplus_{t\in M} Rt$, *the R-module freely generated by the elements of M.*
(ii) *If F is a function symbol from τ (of arity n), then, by multi-linearity, we can extend the map $F : M^n \to M$ to a unique function $\overline{F} : (\mathrm{lin}_R\, M)^n \to \mathrm{lin}_R\, M$.*
(iii) *The canonical R-module* $\mathrm{lin}_R\, M$ *together with the extended maps $\overline{F}$ from* (ii) *is a τ_R-body, denoted by* $\mathrm{Lin}_R\, M$.

Note that $\mathrm{lin}_R\, M$ is just the free R-module structure of the τ_R-body $\mathrm{Lin}_R\, M$. Furthermore, $\mathrm{lin}_R\, M$ provides a natural class of R-endomorphisms induced by the 1-strict M-terms; we state this more precisely as

Corollary 3.4. *If M is a τ-skeleton and $\sigma^M(x, \overline{s})$ is a* 1*-strict M-term, then for $K = \mathrm{Lin}_R\, M$ and $G = \mathrm{lin}_R\, M$,*

$$\sigma : G \to G\ (x \mapsto \sigma^K(x, \overline{s})) \text{ is an } R\text{-endomorphism of } G.$$

Proof. Apply the definition of bodies and note that all the functions involved are multi-linear. □

Additional induced R-endomorphisms on $\mathrm{lin}_R\, M$ can be derived from linear combination of 1-strict M-terms. Furthermore, $|\mathrm{lin}_R\, M| \le |M| + |R| + \aleph_0$ is immediate. Also recall from Lemma 2.9 that for any faithful skeleton M an associative multiplication $\cdot$ on M is induced by a multi-linear function symbol. In this particular case $(\mathrm{Lin}_R\, M, +, \cdot)$ becomes an R-algebra, the (classical) monoid-R-algebra (with multiplication as in a group ring). These are the obvious parts (a) and (c) of the following crucial

Theorem 3.5. *Let R be a commutative ring with* 1 *and M be a nice τ-skeleton. For $K = \mathrm{Lin}_R\, M$ and $G = \mathrm{lin}_R\, M$ the following holds.*

(a) *K is a τ_R-body of size $\leq |M| + |R| + \aleph_0$ with $G = {}_R K$.*
(b) *If $\sigma(x, \overline{x})$ is a τ_R-term and $\overline{t}$ is a sequence from G such that $\lg(\overline{t}) = \lg(\overline{x})$ and*

$$G \to G \ (x \mapsto \sigma^K(x, \overline{t})) \text{ is an } R\text{-endomorphism of } G,$$

then there is some $g \in G$ such that $\sigma(t, \overline{t}) = t * g$ for all $t \in G$.
(c) *If in addition M is faithful, then $(K, +, \cdot)$ with the multiplication $\cdot$ from Definition* 2.8 *is a monoid-R-algebra over the monoid $(M, \cdot)$.*

Proof. It remains to show part (b). Recall from the theorem that the R-endomorphism

$$\sigma : G \to G \ (x \mapsto \sigma^K(x, \overline{t}))$$

should be $*$-scalar multiplication by some $g \in G$. Suppose that this is not the case. Using the definition of G and the multi-linearity in K we can express the K-term $\sigma^K(x, \overline{t})$ as

$$\sigma^K(x, \overline{t}) = \sum_{i<i_*} a_i \sigma_i^K(x, \overline{s}_i) \quad \text{in reduced form}$$

with suitable $0 \neq a_i \in R$, skeleton τ-terms $\sigma_i(x, \overline{x}_i)$ and sequences $\overline{s}_i$ from M.

We may assume that $i_* > 0$, because otherwise $\sigma^K(t, \overline{t}) = 0 = t * 0$ for all $t \in G$ by Observation 3.2. Thus the choice $g = 0 \in G$ would contradict our assumption. From Observation 2.4 it follows that we can find strict τ-terms $F_i'(y_0, \dots, y_{n_i-1})$ and surjective maps $\pi_i : \{y_0, \dots, y_{n_i-1}\} \to \{x\} \cup \mathrm{Im}(\overline{x}_i)$ with $\sigma_i(x, \overline{x}_i) = F_i'(y_0\pi_i, \dots, y_{n_i-1}\pi_i)$. Let $k_i = |\pi_i^{-1}(x)|$ and $k_* = \max\{k_i \mid i < i_*\}$. Using axiom (ii) of Definition 2.1 we may assume that $\pi_i^{-1}(x) = \{y_0, \dots, y_{k_i-1}\}$, hence we can express

$$\sigma_i^K(x, \overline{s}_i) = F_i'^K(\overline{x}^{k_i}, \overline{s}_i') \text{ with a strict } \tau\text{-term and a suitable sequence } \overline{s}_i' \text{ from } M.$$

Finally, using Corollary 2.7, we can assume that $\sigma_i^M(x, \overline{s}_i) = F_i'^M(\overline{x}^{k_i}, \overline{s}_i')$ is k_i-strict.

Now we decompose

$$\sigma^K(x, \overline{t}) = \sum_{i<i_*} a_i F_i'^K(\overline{x}^{k_i}, \overline{s}_i') = t'(x) + t''(x)$$

with $t'(x) = \sum_{i<i_*, k_i=0} a_i F_i'^K(\overline{s}_i')$ and $t''(x) = \sum_{i<i_*, k_i \neq 0} a_i F_i'^K(\overline{x}^{k_i}, \overline{s}_i')$. Clearly, the first summand $t'(x) = t'$ does not depend on x and the second summand must be 0 by multi-linearity if we substitute 0, thus $t''(0) = 0$. By assumption $\sigma^K(x, \overline{t})$ represents an R-endomorphism and therefore also $\sigma^K(0, \overline{t}) = 0$; it follows that $t' = 0$. So we can assume that $k_i > 0$ for all $i < i_*$. If $k_i = 1$ for some $i < i_*$, then $F_i'^K(\overline{x}^{k_i}, \overline{s}_i') = F_i'^K(x, \overline{s}_i') = x * F_i''^K(\overline{s}_i')$ by axiom (iv) in Definition 2.1, thus this summand acts as an R-endomorphism as required (see Corollary 3.4) and can be removed from the sum $\sigma^K(x, \overline{t}) = \sum_{i<i_*} a_i F_i'^K(\overline{x}^{k_i}, \overline{s}_i')$. Hence we may also assume $k_i > 1$ for all $i < i_*$. We can reduce this sum even further, adding up those summands which are M-equivalent (and therefore also K-equivalent); see Definition 2.6. Thus we may assume that all summands $F_i'^M(\overline{x}^{k_i}, \overline{s}_i')$ $(i < i_*)$ are pairwise not M-equivalent with $k_i > 1$.

Finally, we apply the fact that M is a nice skeleton and for any sequence $\bar{t} = \langle t_0, \ldots, t_{k_*-1}\rangle$ from $M \subseteq G$ and $\eta \in \Lambda_{k_i,k_*}$ we define as in Definition 2.10 a sequence $\bar{t}_\eta = \langle t_{l\eta} \mid l < k_i\rangle$. We first claim that

$$\sum_{i<i_*} a_i \sum_{\eta\in\Lambda_{k_i,k_*}} F_i'^M(\bar{t}_\eta, \bar{s}_i') = 0 \quad \text{in } G = \operatorname{lin}_R M. \tag{3.1}$$

We use multi-linearity of the $F_i'^K$s, the definition of $\sigma^K(x, \bar{t})$ and the fact that σ represents an R-endomorphism to calculate

$$\begin{aligned}
\sum_{i<i_*} a_i \sum_{\eta\in{}^{k_i}k_*} F_i'^M(\bar{t}_\eta, \bar{s}_i') &= \sum_{i<i_*} a_i \sum_{\eta\in{}^{k_i}k_*} F_i'^K(\bar{t}_\eta, \bar{s}_i') = \sum_{i<i_*} a_i F_i'^K\Big(\overline{\sum_{l<k_*} t_l}^{\,k_i}, \bar{s}_i'\Big)\\
&= \sigma^K\Big(\sum_{l<k_*} t_l, \bar{t}\Big) = \Big(\sum_{l<k_*} t_l\Big)\sigma = \sum_{l<k_*} t_l\sigma = \sum_{l<k_*} \sigma^K(t_l, \bar{t})\\
&= \sum_{l<k_*}\sum_{i<i_*} a_i F_i'^K(\bar{t}_l^{k_i}, \bar{s}_i') = \sum_{i<i_*} a_i \sum_{l<k_*} F_i'^M(\bar{t}_l^{k_i}, \bar{s}_i'),
\end{aligned}$$

and the difference of the two ends of this equation with the help of the definition of Λ_{k_i,k_*} establishes the claim.

Since M is nice we find a sequence $\bar{t} \in {}^{k_*}M$ such that for all $i < i_*$ and $\eta \in \Lambda_{k_i,k_*}$,

$$F_i'^M(\bar{t}_\eta, \bar{s}_i') \notin \{F_j'^M(\bar{t}_\theta, \bar{s}_j') \mid j < i_*,\ \theta \in \Lambda_{k_j,k_*} \text{ and } (j,\theta) \neq (i,\eta)\}.$$

We fix $i < i_*$, $\eta \in \Lambda_{k_i,k_*}$ and consider the $F_i'^M(\bar{t}_\eta, \bar{s}_i')$-component of (3.1). Hence $a_i = 0$ by linear independence. But this means that the sum $\sigma^K(x, \bar{t}) = \sum_{i<i_*} a_i\sigma_i^K(x, \bar{s}_i)$ was not reduced, contrary to our assumption. The theorem follows. □

4. Types

After introducing the notions of skeletons and bodies in the last two sections and discussing the model-theoretic part of the construction of generalized $E(R)$-algebras we will now investigate (in Section 4 and 5) this topic from the other side, introducing a combinatorial-geometric description of the algebra which are our (decorated) trees, called types for short. This section is fairly independent of the model-theoretic part which we just completed, but we will draw attention to useful connections. Finally, in Section 6 all this will be used to construct large free skeletons and bodies.

We begin with the combinatorial-geometric part by defining types, which are finite (decorated) trees. Let ${}^{\omega>}\omega$ be the family of all finite sequences with values in ω. This set is partially ordered by inclusion: if $\eta \subseteq \nu$, then η is an initial segment of ν, and we also write $\eta \trianglelefteq \nu$. If η is **not** an initial segment of ν, we will write $\eta \ntrianglelefteq \nu$. Every non-empty subset of ${}^{\omega>}\omega$ that is closed under initial segments is called a *tree*. Naturally ${}^{\omega>}\omega$ itself is a countable tree by this definition and every other tree is a subtree of ${}^{\omega>}\omega$. The empty sequence $\perp = \langle\,\rangle$ is the minimal element of every tree. If u is a subtree of ${}^{\omega>}\omega$, then for any $\eta \in u$,

$$\operatorname{suc}(\eta) := \{\nu \in u \mid \eta \subset \nu,\ \lg(\nu) = \lg(\eta) + 1\}$$

denotes the set of direct *successors* of η in u. Furthermore, $\max u$ denotes the subset $\{\eta \in u \mid \mathrm{suc}(\eta) = \emptyset\}$ which is the family of all sequences η in u which cannot be extended inside u. We will apply standard notations for trees, call $\perp \in u$ the *root* of the tree u, and all other elements $\eta \in u$ are called *knots* or *branches*, in particular, the elements in $\max u$ are the maximal branches of this tree.

Definition 4.1. *Let $\mathfrak{T}$ be the family of* types $\mathbf{t}$ *which are octuples*

$$\mathbf{t} = (u^{\mathbf{t}}, P_l^{\mathbf{t}}, F^{\mathbf{t}}, G^{\mathbf{t}} \mid l \leq 5) = (u, P_l, F, G \mid l \leq 5)$$

satisfying several (carefully chosen) conditions.

(i) *u is a finite subtree of ${}^{\omega>}\omega$.*
(ii) *$\{P_l \mid l \leq 5\}$ is a partition of u.*
(iii) *P_0, P_3, P_4 are subsets of* $\max u$.
(iv) *If $\eta \in P_1$, then* $\mathrm{suc}(\eta) = \{\eta^\wedge\langle 0\rangle, \eta^\wedge\langle 1\rangle\}$.
(v) *$F : P_2 \to P_3$ is a bijective map.*
(vi) *If $\eta \in P_2$, then* $\mathrm{suc}(\eta) = \{\eta^\wedge\langle 1\rangle\}$ *and* $\eta \trianglelefteq F(\eta)$.
(vii) *$G : P_5 \to \tau_0 \setminus \{F_{\mathrm{id}}, 1, *\}$ and if $\eta \in P_5$, then* $\mathrm{ar}(G(\eta)) = \{n \mid \eta^\wedge\langle n\rangle \in u\}$.

Remark 4.2. By Definition 4.1(vii) of types $\mathbf{t} \in \mathfrak{T}$ we also introduce the function $G^{\mathbf{t}}$. This needs explanation: the language of skeletons τ (from Section 2) can be viewed as the closure (under the axioms (ii), (iii) and (iv) of Definition 2.1) of a set of function symbols τ_0 with $\{F_{\mathrm{id}}, 1, *\} \subseteq \tau_0$. There we did not specify the elements in $\tau_0 \setminus \{F_{\mathrm{id}}, 1, *\}$. These elements will be made precise in Section 6, when we fix the language τ suitable for free skeletons and τ_0 as a corresponding canonical system of generators. The function symbols from $\tau_0 \setminus \{F_{\mathrm{id}}, 1, *\}$ will be related to the construction of free skeletons, so they have to wait until we come to their particular tree-elements. The option $\tau_0 = \{F_{\mathrm{id}}, 1, *\}$ will be the minimal choice for τ_0 in which case we can restrict our trees $\mathbf{t}$ to sixtuples $(u^{\mathbf{t}}, P_l^{\mathbf{t}}, F^{\mathbf{t}} \mid l \leq 4)$ without any reference to $G^{\mathbf{t}}$. For the sake of additional applications we will continue to use $\mathbf{t}$ as in Definition 4.1 with the extended language coming from τ_0 and provide all the proofs (without any harm) for the more general languages τ_0s.

The basic idea of this approach is the correspondence between certain (decorated) trees from $\mathfrak{T}$ with the elements of the aimed skeleton. The relations of the skeleton must appear in $\mathfrak{T}$. This is similar to the relations in combinatorial group theory which must carry over to path-relations in 2-dimensional manifolds. In this context the goal will be taken care of by the very crucial reduction of types, which we call, parallel to the group case, small cancelation of trees (or types). We declare certain branches as cancelation points—they will be cut down.

Definition 4.3. (i) *η is a* cancelation point *of* $\mathbf{t} = (u, P_l, F, G \mid l \leq 5)$ *if one of the following conditions holds.*

(a) *$\eta \in P_1$ and $\eta^\wedge\langle 0\rangle \in P_0 \cup P_2$.*
(b) *$\eta \in P_2$ and $\eta^\wedge\langle 1\rangle \in P_3$.*
(c) *$\eta \in P_2$, $\eta^\wedge\langle 1\rangle \in P_1$ and $F(\eta) = \eta^\wedge\langle 1, 1\rangle$.*

(ii) *Let* can($\mathbf{t}$) *be the collection of cancelation points of* $\mathbf{t}$.
(iii) *A type* $\mathbf{t} \in \mathfrak{T}$ *is* reduced *if* can($\mathbf{t}$) $= \emptyset$. *Let* $\mathfrak{T}^{\text{red}}$ *be the family of all reduced types* $\mathbf{t}$.

We are mainly interested in the family $\mathfrak{T}^{\text{red}}$ of reduced (types) trees and will make it a monoid while multiplying trees. The product of two reduced trees, as in free groups, will not in general be a reduced tree, so we will have to apply small cancelation to products of trees again. Thus we must first study general small cancelations of trees from $\mathfrak{T}$.

5. Small cancelation of types

In this section we want to see that every element of $\mathfrak{T}$ can be reduced to a unique element of $\mathfrak{T}^{\text{red}}$ by small cancelation. Here we first introduce various cancelation steps; their combined efforts will lead to reduced trees. It is trivial to check that the set $P_4^{\mathbf{t}}$ of a tree $\mathbf{t} = (u^{\mathbf{t}}, P_l^{\mathbf{t}}, F^{\mathbf{t}}, G^{\mathbf{t}} \mid l \leq 5)$ can also be written as

$$P_4^{\mathbf{t}} = \{\eta \in \max u_{\mathbf{t}} \mid \eta \notin P_0^{\mathbf{t}} \cup P_5^{\mathbf{t}} \cup \operatorname{Im}(F^{\mathbf{t}})\}. \tag{5.1}$$

As a warm-up we start with two construction tools that will prove very useful later on.

Construction Tool I—“Picking a twig”: *If* $\mathbf{t} = (u^{\mathbf{t}}, P_l^{\mathbf{t}}, F^{\mathbf{t}}, G^{\mathbf{t}} \mid l \leq 5) \in \mathfrak{T}$ *and* $\nu \in u^{\mathbf{t}}$, *then we define* $\mathbf{s} = \mathbf{t}^{\geq \nu}$ *by the following components.*

(i) $u^{\mathbf{s}} = \{\rho \mid \nu^\wedge \rho \in u^{\mathbf{t}}\}$.
(ii) $P_l^{\mathbf{s}} = \{\rho \mid \nu^\wedge \rho \in P_l^{\mathbf{t}}\}$ *for* $l = 0, 1, 2, 5$.
(iii) *If* $F^{\mathbf{t}}(\nu^\wedge \rho) = \nu^\wedge \sigma$, *then* $F^{\mathbf{s}}(\rho) = \sigma$.
(iv) $P_3^{\mathbf{s}} = \operatorname{Im}(F^{\mathbf{s}})$.
(v) $P_4^{\mathbf{s}} = \{\rho \mid \nu^\wedge \rho \in P_4^{\mathbf{t}}\} \cup \{\rho \mid \nu^\wedge \rho \in P_3^{\mathbf{t}} \setminus (\nu^\wedge P_3^{\mathbf{s}})\}$.
(vi) $G^{\mathbf{s}}(\rho) = G^{\mathbf{t}}(\nu^\wedge \rho)$ *for* $\rho \in P_5^{\mathbf{s}}$.

Recall that $\nu^\wedge P_3^{\mathbf{s}}$ in (v) is the collection of all concatenations of ν with elements from $P_3^{\mathbf{s}}$. This and (v) ensure that $\{P_l^{\mathbf{s}} \mid l \leq 5\}$ is a partition of all of $u^{\mathbf{s}}$. From (iii) and (iv) it follows that $F^{\mathbf{s}}$ is a bijection. It is now easy to check that $\mathbf{s} \in \mathfrak{T}$. Figure 1 illustrates this construction tool.

Construction Tool II—“Engrafting a twig”: *If* $\mathbf{s}, \mathbf{r} \in \mathfrak{T}$ *and* $\nu \in P_4^{\mathbf{s}}$, *then we define* $\mathbf{t} = \mathbf{s}^{[\nu, \mathbf{r}]} \in \mathfrak{T}$ *componentwise as follows.*

(i) $u^{\mathbf{t}} = u^{\mathbf{s}} \cup \{\nu^\wedge \rho \mid \rho \in u^{\mathbf{r}}\}$.
(ii) *If* $\perp \notin P_l^{\mathbf{r}}$, *then* $P_l^{\mathbf{t}} = (P_l^{\mathbf{s}} \setminus \{\nu\}) \cup \{\nu^\wedge \rho \mid \rho \in P_l^{\mathbf{r}}\}$.
(iii) *If* $\perp \in P_l^{\mathbf{r}}$, *then* $P_l^{\mathbf{t}} = P_l^{\mathbf{s}} \cup \{\nu^\wedge \rho \mid \rho \in P_l^{\mathbf{r}}\}$.
(iv) $F^{\mathbf{t}}(\eta) = F^{\mathbf{s}}(\eta)$ *for* $\eta \in P_2^{\mathbf{s}}$ *and* $F^{\mathbf{t}}(\nu^\wedge \rho) = \nu^\wedge F^{\mathbf{r}}(\rho)$ *if* $\rho \in P_2^{\mathbf{r}}$.
(v) $G^{\mathbf{t}}(\eta) = G^{\mathbf{s}}(\eta)$ *for* $\eta \in P_5^{\mathbf{s}}$ *and if* $\rho \in P_5^{\mathbf{r}}$, *then* $G^{\mathbf{t}}(\nu^\wedge \rho) = G^{\mathbf{r}}(\rho)$.

From (ii) and (iii) it follows that ($\perp \in P_l^{\mathbf{r}} \Leftrightarrow \nu \in P_l^{\mathbf{t}}$). It is easy to check that $\mathbf{t} \in \mathfrak{T}$. Figure 2 illustrates the second construction tool.

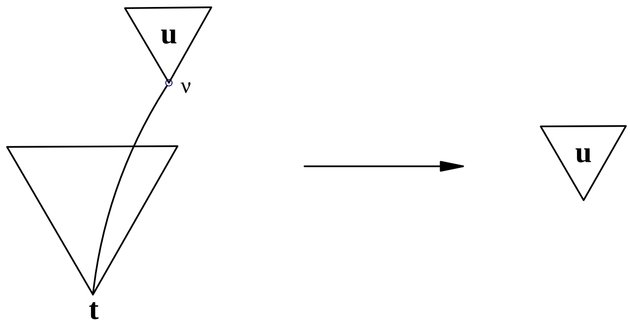


Fig. 1

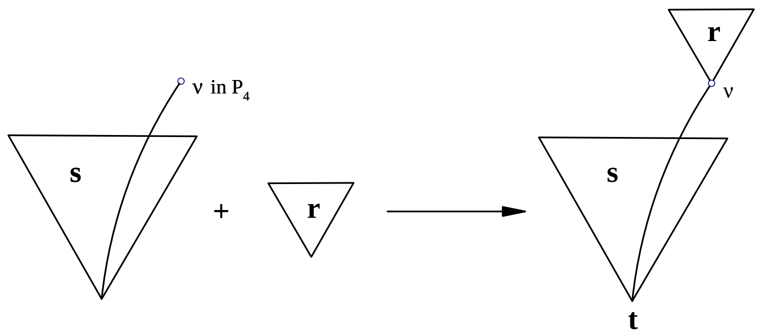


Fig. 2

Now we give details on the four small cancelation steps for types.

Reduction—Case A: *If* $\mathbf{s} \in \mathfrak{T}$, $\eta \in \mathrm{can}(\mathbf{s}) \cap P_1^{\mathbf{s}}$ *and* $\eta^\wedge\langle 0\rangle \in P_0^{\mathbf{s}}$, *then we define* $\mathbf{t} = \mathrm{red}_\eta\, \mathbf{s} \in \mathfrak{T}$ *componentwise as follows.*

(i) $u^{\mathbf{t}} = u_1^{\mathbf{s}} \cup u_2^{\mathbf{s}}$ *with* $u_1^{\mathbf{s}} = \{\nu \mid \nu \in u^{\mathbf{s}}, \eta \ntrianglelefteq \nu\}$ *and* $u_2^{\mathbf{s}} = \{\eta^\wedge\rho \mid \eta^\wedge\langle 1\rangle^\wedge\rho \in u^{\mathbf{s}}\}$.

(ii) *Next we define a bijection* $h^{\mathbf{st}} : u^{\mathbf{s}} \setminus \{\eta, \eta^\wedge\langle 0\rangle\} \to u^{\mathbf{t}}$ *by the following two cases: If* $\eta \ntrianglelefteq \nu$, *then* $h^{\mathbf{st}}(\nu) = \nu$, *and if* $\nu = \eta^\wedge\langle 1\rangle^\wedge\rho$, *then* $h^{\mathbf{st}}(\nu) = \eta^\wedge\rho$.

(iii) $P_l^{\mathbf{t}} = h^{\mathbf{st}}(P_l^{\mathbf{s}} \cap \mathrm{Dom}(h^{\mathbf{st}}))$ *for* $l \leq 5$.

(iv) *If* $g^{\mathbf{ts}} = (h^{\mathbf{st}})^{-1}$, *then* $F^{\mathbf{t}} = h^{\mathbf{st}} \circ F^{\mathbf{s}} \circ g^{\mathbf{ts}}$ *and* $G^{\mathbf{t}} = G^{\mathbf{s}} \circ g^{\mathbf{ts}}$.

Note that $\eta^\wedge\langle 0\rangle$ is maximal in $u^{\mathbf{s}}$. It is clear from the construction that $h^{\mathbf{st}}$ is a bijection and we leave it to the reader to check that $\mathbf{t} \in \mathfrak{T}$. Figure 3 illustrates this small cancelation.

Reduction—Case B: *If* $\mathbf{s} \in \mathfrak{T}$, $\eta \in \mathrm{can}(\mathbf{s}) \cap P_1^{\mathbf{s}}$ *and* $\eta^\wedge\langle 0\rangle \in P_2^{\mathbf{s}}$, *then we define* $\mathbf{t} = \mathrm{red}_\eta\, \mathbf{s} \in \mathfrak{T}$ *componentwise as follows.*

(i) $u^{\mathbf{t}} = u_1^{\mathbf{s}} \cup u_2^{\mathbf{s}} \cup u_3^{\mathbf{s}}$ *with* $u_1^{\mathbf{s}} = \{\nu \mid \nu \in u^{\mathbf{s}}, \eta \ntrianglelefteq \nu\}$, $u_2^{\mathbf{s}} = \{\eta^\wedge\rho \mid \eta^\wedge\langle 0, 1\rangle^\wedge\rho \in u^{\mathbf{s}}, \rho \neq \eta_1\}$ *and* $u_3^{\mathbf{s}} = \{\eta^\wedge\eta_1{}^\wedge\rho \mid \eta^\wedge\langle 1\rangle^\wedge\rho \in u^{\mathbf{s}}\}$, *where* η_1 *is defined by* $F^{\mathbf{s}}(\eta^\wedge\langle 0\rangle) = \eta^\wedge\langle 0, 1\rangle^\wedge\eta_1$.

(ii) *Next we define a bijection* $h^{\mathbf{st}} : u^{\mathbf{s}} \setminus \{\eta, \eta^\wedge\langle 0\rangle, F^{\mathbf{s}}(\eta^\wedge\langle 0\rangle)\} \to u^{\mathbf{t}}$ *by three cases: If* $\eta \ntrianglelefteq \nu$, *then* $h^{\mathbf{st}}(\nu) = \nu$, *if* $\nu = \eta^\wedge\langle 0, 1\rangle^\wedge\rho$, *then* $h^{\mathbf{st}}(\nu) = \eta^\wedge\rho$, *and if* $\nu = \eta^\wedge\langle 1\rangle^\wedge\rho$, *then* $h^{\mathbf{st}}(\nu) = \eta^\wedge\eta_1{}^\wedge\rho$.

(iii) $P_l^{\mathbf{t}} = h^{\mathbf{st}}(P_l^{\mathbf{s}} \cap \mathrm{Dom}(h^{\mathbf{st}}))$ *for* $l \leq 5$.
(iv) *If* $g^{\mathbf{ts}} = (h^{\mathbf{st}})^{-1}$, *then* $F^{\mathbf{t}} = h^{\mathbf{st}} \circ F^{\mathbf{s}} \circ g^{\mathbf{ts}}$ *and* $G^{\mathbf{t}} = G^{\mathbf{s}} \circ g^{\mathbf{ts}}$.

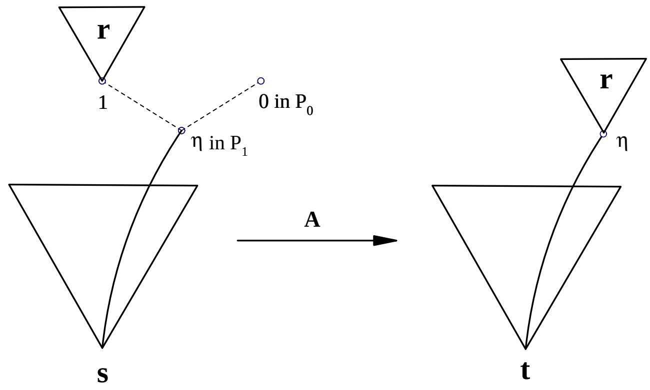


Fig. 3

Again, it must be checked that $\mathbf{t} \in \mathfrak{T}$. Figure 4 visualizes this small cancelation.

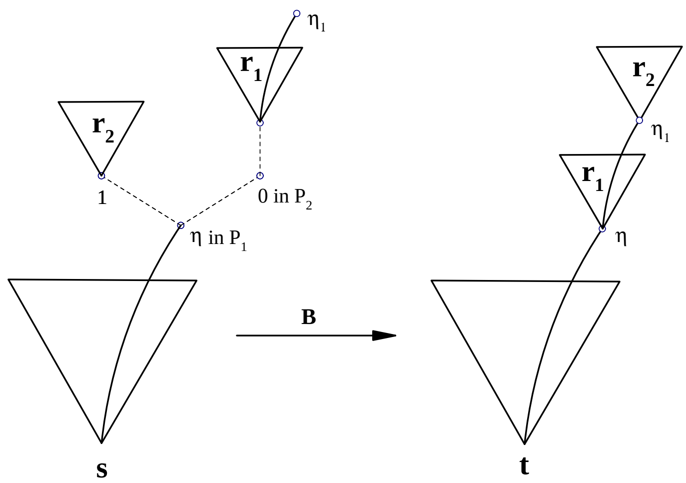


Fig. 4

Reduction—Case C: *If* $\mathbf{s} \in \mathfrak{T}$, $\eta \in \mathrm{can}(\mathbf{s}) \cap P_2^{\mathbf{s}}$ *and* $\eta^\wedge\langle 1\rangle \in P_3^{\mathbf{s}}$, *then we define* $\mathbf{t} = \mathrm{red}_\eta\, \mathbf{s} \in \mathfrak{T}$ *componentwise as follows.*

(i) $u^{\mathbf{t}} = u^{\mathbf{s}} \setminus \{\langle \eta^\wedge\langle 1\rangle\rangle\}$.
(ii) *Next we define a bijection* $h^{\mathbf{st}} : u^{\mathbf{s}} \setminus \{\eta\} \to u^{\mathbf{t}}$ *by two cases: If* $\nu \neq \eta^\wedge\langle 1\rangle$, *then* $h^{\mathbf{st}}(\nu) = \nu$, *and if* $\nu = \eta^\wedge\langle 1\rangle$, *then* $h^{\mathbf{st}}(\nu) = \eta$. *Moreover, let* $g^{\mathbf{ts}} = (h^{\mathbf{st}})^{-1}$.
(iii) *If* $0 < l \leq 5$, *then let* $P_l^{\mathbf{t}} = P_l^{\mathbf{s}} \setminus \{\eta^\wedge\langle 1\rangle, \eta\}$, *and if* $l = 0$, *then* $P_0^{\mathbf{t}} := P_0^{\mathbf{s}} \cup \{\eta\}$.
(iv) *Put* $F^{\mathbf{t}} = F^{\mathbf{s}} \restriction P_2^{\mathbf{t}}$ *and* $G^{\mathbf{t}} = G^{\mathbf{s}} \restriction P_5^{\mathbf{t}}$.

Again, it must be checked that $\mathbf{t} \in \mathfrak{T}$. Figure 5 illustrates this small cancelation.

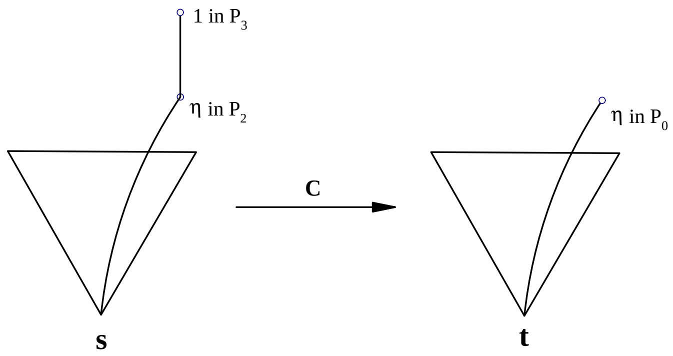


Fig. 5

Reduction—Case D: *If* $\mathbf{s} \in \mathfrak{T}$, $\eta \in \operatorname{can}(\mathbf{s}) \cap P_2^{\mathbf{s}}$, $\eta^\wedge\langle 1\rangle \in P_1^{\mathbf{s}}$ *and* $F^{\mathbf{s}}(\eta) = \eta^\wedge\langle 1, 1\rangle$, *then we define* $\mathbf{t} = \operatorname{red}_\eta \mathbf{s} \in \mathfrak{T}$ *componentwise as follows.*

(i) $u^{\mathbf{t}} = u_1^{\mathbf{s}} \cup u_2^{\mathbf{s}}$ *with* $u_1^{\mathbf{s}} = \{\nu \mid \nu \in u^{\mathbf{s}},\ \eta \not\trianglelefteq \nu\}$ *and* $u_2^{\mathbf{s}} = \{\eta^\wedge\rho \mid \eta^\wedge\langle 1, 0\rangle^\wedge\rho \in u^{\mathbf{s}}\}$.
(ii) *Next we define a bijection* $h^{\mathbf{st}} : u^{\mathbf{s}} \setminus \{\eta, \eta^\wedge\langle 1\rangle, \eta^\wedge\langle 1, 1\rangle\} \to u^{\mathbf{t}}$ *by two cases: If* $\eta \not\trianglelefteq \nu$, *then* $h^{\mathbf{st}}(\nu) = \nu$, *and if* $\nu = \eta^\wedge\langle 1, 0\rangle^\wedge\rho$, *then* $h^{\mathbf{st}}(\nu) = \eta^\wedge\rho$.
(iii) $P_l^{\mathbf{t}} = h^{\mathbf{st}}(P_l^{\mathbf{s}} \cap \operatorname{Dom}(h^{\mathbf{st}}))$ *for* $l \leq 5$.
(iv) *If* $g^{\mathbf{ts}} = (h^{\mathbf{st}})^{-1}$, *then* $F^{\mathbf{t}} = h^{\mathbf{st}} \circ F^{\mathbf{s}} \circ g^{\mathbf{ts}}$ *and* $G^{\mathbf{t}} = G^{\mathbf{s}} \circ g^{\mathbf{ts}}$.

Again, it must be checked that $\mathbf{t} \in \mathfrak{T}$. Figure 6 illustrates this small cancelation.

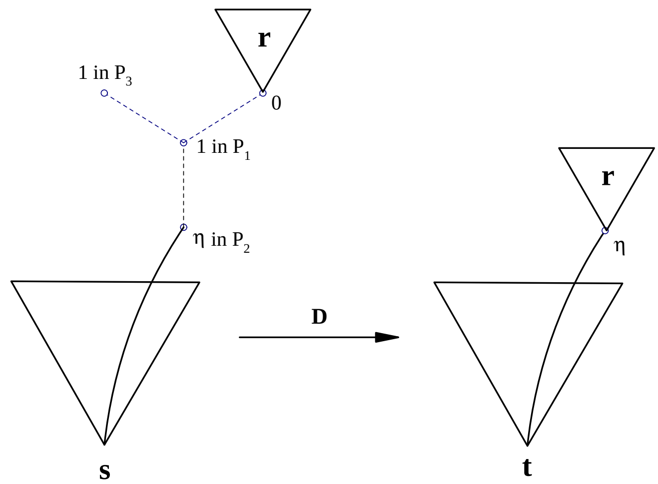


Fig. 6

Next we apply small cancelation by Case A, B, C and D, respectively, to any tree $\mathbf{s} \in \mathfrak{T}$ and will show that after a finite number of such steps we obtain a unique, reduced tree $\operatorname{red} \mathbf{s} \in \mathfrak{T}^{\mathrm{red}}$.

It is tempting to base this recursion on the number $|\operatorname{can}(\mathbf{s})|$ as we remove one cancelation point $\eta \in \operatorname{can}(\mathbf{s})$ in every small cancelation step; however, unfortunately, it can happen that $|\operatorname{can}(\mathbf{s})| = |\operatorname{can}(\operatorname{red}_\eta \mathbf{s})|$ as new cancelation points may appear. This phenomenon is possible in all Cases A, B, C and D, and we will next exemplify this for Case A.

Example 5.1. Let $u = \{\perp, \langle 1\rangle, \langle 0\rangle, \langle 0,1\rangle, \langle 0,0\rangle\}$, which is closed under initial segments, thus a finite 5-element subtree of ${}^{\omega>}\omega$. Put $P_0 = \{\langle 1\rangle, \langle 0,1\rangle, \langle 0,0\rangle\}$, $P_1 = \{\perp, \langle 0\rangle\}$, $P_2 = P_3 = P_4 = P_5 = \emptyset$ and note that $\bigcup_{l\leq 5} P_l$ is a partition of u. Moreover, define the maps $F = G = \emptyset$ (as relations). We obtain a decorated tree $\mathbf{s} = (u, P_l, F, G \mid l \leq 5)$ from $\mathfrak{T}$ with $\operatorname{can}(\mathbf{s}) = \{\langle 0\rangle\}$, hence $|\operatorname{can}(\mathbf{s})| = 1$. If we reduce $\mathbf{s}$ by Case A with $\eta = \langle 0\rangle$, then we get the tree $\mathbf{t} = \operatorname{red}_\eta \mathbf{s}$ with $u^{\mathbf{t}} = \{\perp, \langle 1\rangle, \langle 0\rangle\}$, $P_0^{\mathbf{t}} = \{\langle 1\rangle, \langle 0\rangle\}$, $P_1^{\mathbf{t}} = \{\perp\}$ and $\operatorname{can}(\mathbf{t}) = \{\perp\}$, hence $|\operatorname{can}(\mathbf{t})| = |\operatorname{can}(\mathbf{s})| = 1$.

Figure 7 illustrates the example.

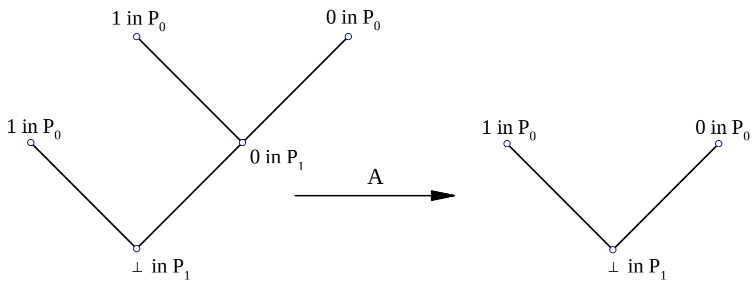


Fig. 7

In view of Example 5.1 the small cancelation of trees $\mathbf{s} \in \mathfrak{T}$ will be based on $|P_1^{\mathbf{s}} \cup P_2^{\mathbf{s}}|$. Thus we consider first the following proposition which controls the size of the P_i s during the small cancelation.

Proposition 5.2. *Let* $\mathbf{s} \in \mathfrak{T}$ *be a tree with* $\eta \in \operatorname{can}(\mathbf{s})$ *and* $\mathbf{t} = \operatorname{red}_\eta \mathbf{s}$ *be its small cancelation, depending on Cases A, B, C and D, respectively. Then the following holds.*

(a) $g^{\mathbf{ts}} : u^{\mathbf{t}} \to u^{\mathbf{s}}$ *is injective.*
(b) $g^{\mathbf{ts}}(P_4^{\mathbf{t}}) = P_4^{\mathbf{s}}$ *and* $g^{\mathbf{ts}}(P_5^{\mathbf{t}}) = P_5^{\mathbf{s}}$.
(c) *In Case A we have* $u^{\mathbf{s}} \setminus \operatorname{Im}(g^{\mathbf{ts}}) = \{\eta, \eta^\wedge\langle 0\rangle\}$.
(d) *In Case B we have* $u^{\mathbf{s}} \setminus \operatorname{Im}(g^{\mathbf{ts}}) = \{\eta, \eta^\wedge\langle 0\rangle, F^{\mathbf{s}}(\eta^\wedge\langle 0\rangle)\}$.
(e) *In Case C we have* $u^{\mathbf{s}} \setminus \operatorname{Im}(g^{\mathbf{ts}}) = \{\eta\}$.
(f) *In Case D we have* $u^{\mathbf{s}} \setminus \operatorname{Im}(g^{\mathbf{ts}}) = \{\eta, \eta^\wedge\langle 1\rangle, \eta^\wedge\langle 1,1\rangle\}$.
(g) $\eta \in \operatorname{can}(\mathbf{s}) \setminus \operatorname{Im}(g^{\mathbf{ts}})$.
(h) $|u^{\mathbf{t}}| < |u^{\mathbf{s}}|$ *and* $|P_1^{\mathbf{t}} \cup P_2^{\mathbf{t}}| < |P_1^{\mathbf{s}} \cup P_2^{\mathbf{s}}|$.

Proof. A simple application of the definitions. □

The canonical embeddings $g^{\mathbf{ts}}$ compose naturally, that is, for $\mathbf{s}_2 = \mathrm{red}_{\eta_1} \mathbf{s}_1$ and $\mathbf{s}_3 = \mathrm{red}_{\eta_2} \mathbf{s}_2$ we set $g^{\mathbf{s}_3\mathbf{s}_1} = g^{\mathbf{s}_2\mathbf{s}_1} \circ g^{\mathbf{s}_3\mathbf{s}_2}$.

Next we will show the crucial claim that the reduced form $\mathrm{red}\,\mathbf{t}$ of a type $\mathbf{t} \in \mathfrak{T}$ is uniquely determined.

Main Lemma 5.3. *For every $\mathbf{t} \in \mathfrak{T}$ the following holds.*

(a) *By a finite application of small cancelations by Cases A, B, C and D we obtain a unique reduced tree* $\mathrm{red}\,\mathbf{t} \in \mathfrak{T}^{\mathrm{red}}$. *In particular, the reduction does not depend on the path of the small cancelation steps.*
(b) *Let $g^{\mathrm{red}\,\mathbf{t},\,\mathbf{t}}$ be the canonical embedding associated to a particular path of small cancelation steps. Then the map $g^{\mathrm{red}\,\mathbf{t},\mathbf{t}}\restriction(P_4^{\mathrm{red}\,\mathbf{t}} \cup P_5^{\mathrm{red}\,\mathbf{t}})$ is uniquely determined, thus independent of the path chosen.*

Proof. We will outline the proof of claim (a); the proof of (b) is essentially the same.

First we note that $\mathrm{can}(\mathbf{t}) \subseteq P_1^{\mathbf{t}} \cup P_2^{\mathbf{t}}$, so by Proposition 5.2(h) we definitely get $\mathrm{can}(\mathbf{t}) = \emptyset$ after $|P_1^{\mathbf{t}} \cup P_2^{\mathbf{t}}|$ steps and the corresponding tree is reduced. It remains to show the uniqueness of this tree. Observe that this includes the uniqueness of the partition $\langle P_l^{\mathrm{red}\,\mathbf{t}} \mid l \leq 5\rangle$ and of the functions $F^{\mathrm{red}\,\mathbf{t}}$ and $G^{\mathrm{red}\,\mathbf{t}}$, respectively.

We apply induction on $|P_1^{\mathbf{t}} \cup P_2^{\mathbf{t}}|$. (Alternatively also induction on $|u^{\mathbf{t}}|$ is possible.)

If $|P_1^{\mathbf{t}} \cup P_2^{\mathbf{t}}| = 0$, then obviously $\mathrm{can}(\mathbf{t}) = \emptyset$ and $\mathbf{t} = \mathrm{red}\,\mathbf{t}$ is uniquely determined and reduced. We put $g^{\mathrm{red}\,\mathbf{t},\,\mathbf{t}} = \mathrm{id}$.

Given $n < \omega$, we assume that the claim holds for $\mathbf{t}$'s with $|P_1^{\mathbf{t}} \cup P_2^{\mathbf{t}}| \leq n$, and we consider some $\mathbf{t} \in \mathfrak{T}$ with $|P_1^{\mathbf{t}} \cup P_2^{\mathbf{t}}| = n + 1$ and must distinguish various cases.

Case I: $\mathrm{can}(\mathbf{t}) = \emptyset$. In this case $\mathbf{t} = \mathrm{red}\,\mathbf{t}$ is already reduced and there is nothing to show. Set again $g^{\mathrm{red}\,\mathbf{t},\,\mathbf{t}} = \mathrm{id}$.

Case II: $\mathrm{can}(\mathbf{t}) \neq \emptyset$. This time we must apply small cancelation and consider subcases. Suppose that there are two distinct paths for small cancelation, one beginning with $\eta_1 \in \mathrm{can}(\mathbf{t})$, the other with $\eta_2 \in \mathrm{can}(\mathbf{t})$.

Subcase 1: Suppose that $\eta_1 = \eta_2 = \eta$. Then by induction hypothesis $\mathrm{red}\,\mathbf{t} = \mathrm{red}(\mathrm{red}_\eta\,\mathbf{t})$ for both paths, and the claim follows.

Subcase 2: Suppose that $\eta_1 \not\trianglelefteq \eta_2$ and $\eta_2 \not\trianglelefteq \eta_1$, thus η_1, η_2 are incomparable. We put $\mathbf{s}_i = \mathrm{red}_{\eta_i}\,\mathbf{t}$ for $i = 1, 2$. In this case the cancelations of the starting tree $\mathbf{t}$ take place in disjoint areas of the tree, which immediately gives the commutativity

$$\mathrm{red}_{h^{\mathbf{ts}_1}(\eta_2)}\,\mathbf{s}_1 = \mathrm{red}_{h^{\mathbf{ts}_2}(\eta_1)}\,\mathbf{s}_2, \tag{5.2}$$

and the claim follows again by induction hypothesis.

Thus without loss of generality we may focus on the remaining case $\eta_2 \neq \eta_1 \trianglelefteq \eta_2$.

Subcase 3: Suppose that $\eta_2 \neq \eta_1 \trianglelefteq \eta_2$, and η_1 does not reduce by Case B. Now we must distinguish further subcases depending on the small cancelations A, B, C and D used.

Note that η_1 cannot reduce by Case C, as follows from the hypothesis $\eta_1 \trianglelefteq \eta_2$. In most cases small cancelations by η_1 and η_2, respectively, are performed again on disjoint areas of $\mathbf{t}$, and (5.2) will apply. Hence we will restrict attention to those interesting subcases, which need more sophisticated arguments.

Subcase 3DA: Suppose that $\mathbf{t}$ reduces with η_1 by Case D and with $\eta_2 = \eta_1{}^\wedge\langle 1\rangle$ by Case A. Small cancelation by η_2 is illustrated in Figure 8.

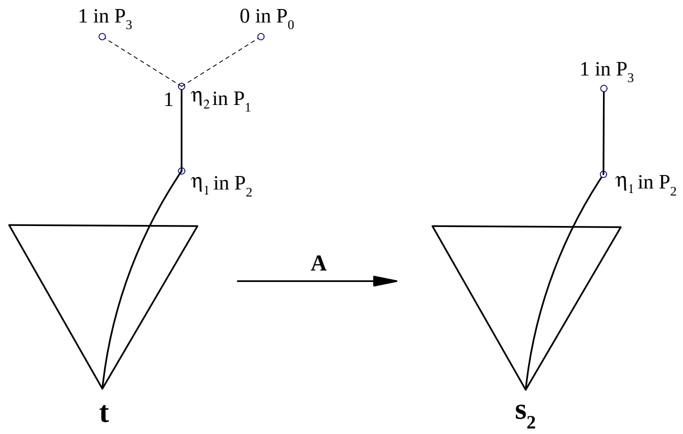


Fig. 8

The resulting tree $\mathbf{s}_2 = \mathrm{red}_{\eta_2}\,\mathbf{t}$ is not yet reduced and reduces further using small cancelation of $\eta_1 \in \mathrm{can}(\mathbf{s}_2)$ by Case C. We obtain the tree $\mathbf{s}_1$ as illustrated in Figure 9.

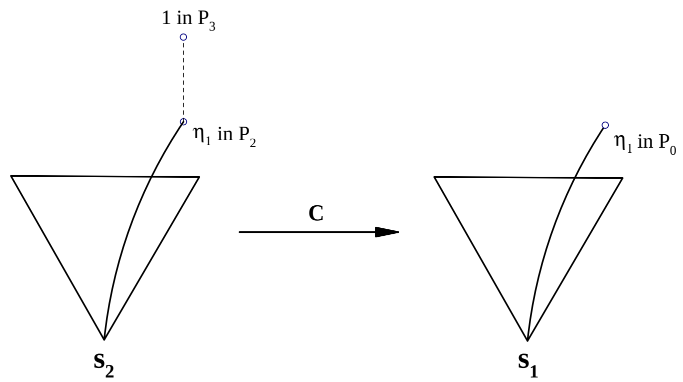


Fig. 9

If we apply small cancelation by η_1 to $\mathbf{t}$ (this is Case D), we obtain directly the last tree $\mathbf{s}_1$. Despite the fact that the paths of small cancelation may differ in length, application of the induction hypothesis shows that $\mathrm{red}\,\mathbf{t} = \mathrm{red}\,\mathbf{s}_1 = \mathrm{red}\,\mathbf{s}_2$.

Subcase 3DB: Suppose that $\mathbf{t}$ reduces with η_1 by Case D and with $\eta_2 = \eta_1{}^\wedge\langle 1\rangle$ by Case B. Recall that $F(\eta_1) = \eta_1{}^\wedge\langle 1, 1\rangle$ and observe that $\text{red}_{\eta_1}\,\mathbf{t} = \text{red}_{\eta_2}\,\mathbf{t} = \mathbf{s}$. The small cancelation is illustrated in Figure 10.

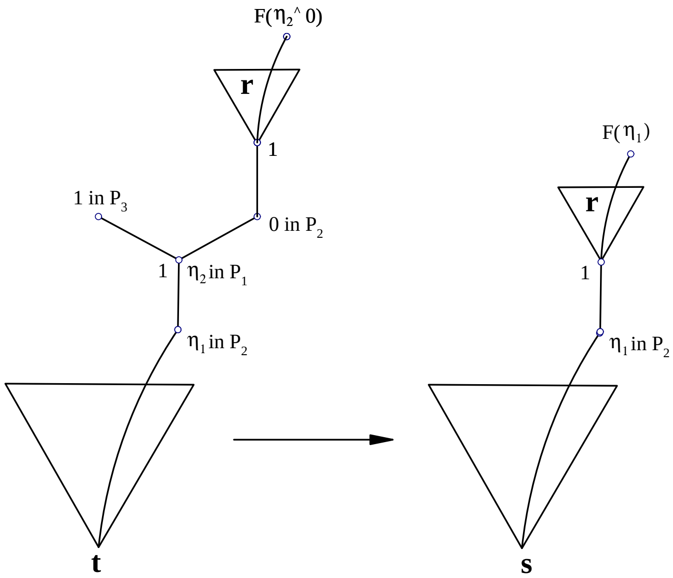


Fig. 10

Subcase 4: Suppose that $\eta_2 \neq \eta_1 \trianglelefteq \eta_2$, and $\mathbf{t}$ reduces with η_1 by Case B. In this subcase the arguments are particularly complex and deserve more details. We begin with the possibility that $\eta_2 = \eta_1{}^\wedge\langle 0\rangle$.

Subcase 4A: Suppose that $\mathbf{t}$ reduces with $\eta_2 = \eta_1{}^\wedge\langle 0\rangle$ by Case C. Small cancelation by η_2 is illustrated in Figure 11. The resulting tree $\mathbf{s}_2 = \text{red}_{\eta_2}\,\mathbf{t}$ can be further reduced

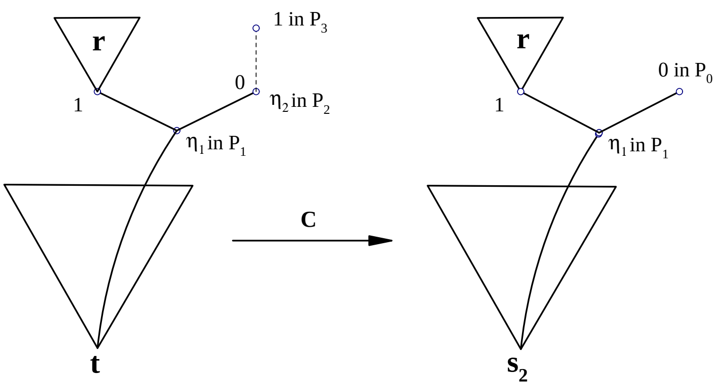


Fig. 11

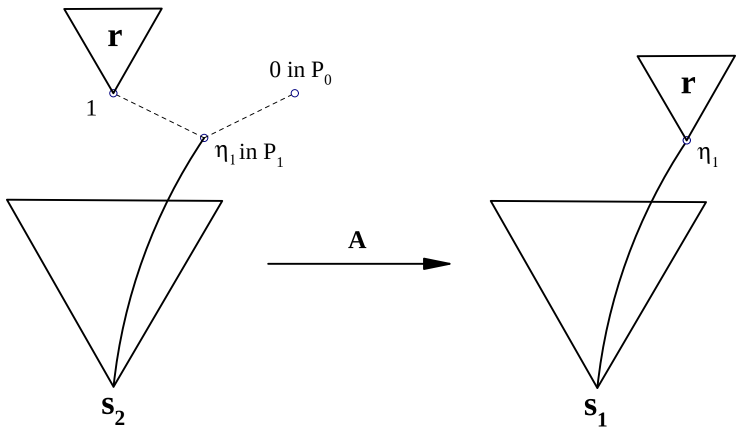


Fig. 12

using small cancelation of $\eta_1 \in \operatorname{can}(\mathbf{s}_2)$ by Case A. We obtain the tree $\mathbf{s}_1$ as illustrated in Figure 12. If we apply small cancelation by η_1 to $\mathbf{t}$ (this is Case B), we obtain directly this last tree $\mathbf{s}_1$. Application of the induction hypothesis shows that $\operatorname{red} \mathbf{t} = \operatorname{red} \mathbf{s}_1 = \operatorname{red} \mathbf{s}_2$.

Subcase 4B: Suppose that $\mathbf{t}$ reduces with $\eta_2 = \eta_1{}^\wedge\langle 0\rangle$ by Case D. Recall that $F(\eta_2) = \eta_2{}^\wedge\langle 1, 1\rangle$ and observe that $\operatorname{red}_{\eta_1} \mathbf{t} = \operatorname{red}_{\eta_2} \mathbf{t} = \mathbf{s}$. The small cancelation is illustrated in Figure 13.

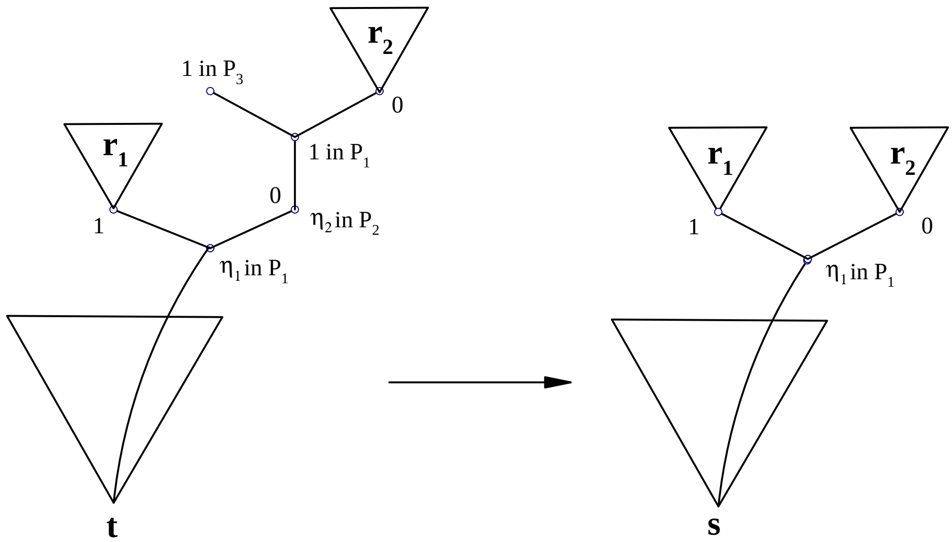


Fig. 13

After inspecting these two special cases we can exclude the hypothesis $\eta_2 = \eta_1{}^\wedge\langle 0\rangle$.

Subcase 4C: Suppose that $\eta_1{}^\wedge\langle 1\rangle \trianglelefteq \eta_2$.

Subcase 4D: Suppose that $\eta_2 \neq \eta_1{}^\wedge\langle 0\rangle$ and $\eta_1{}^\wedge\langle 0\rangle \trianglelefteq \eta_2 \ntrianglelefteq F(\eta_1{}^\wedge\langle 0\rangle)$.

In these last two cases commutativity (5.2) follows, because the action takes again place on disjoint areas of **t**.

Subcase 4E: Suppose that $\eta_2 \neq \eta_1{}^\wedge\langle 0\rangle$ and $\eta_1{}^\wedge\langle 0\rangle \trianglelefteq \eta_2 \trianglelefteq F(\eta_1{}^\wedge\langle 0\rangle)$, and η_2 does not reduce by Case B. Then η_2 does not reduce by Case C, as can be seen in Figure 14. From $\eta_2 \trianglelefteq F(\eta_1{}^\wedge\langle 0\rangle)$ it follows that $F(\eta_1{}^\wedge\langle 0\rangle) = \eta_2{}^\wedge\langle 1\rangle$, while also $F(\eta_2) = \eta_2{}^\wedge\langle 1\rangle$, contradicting the injectivity of the function F.

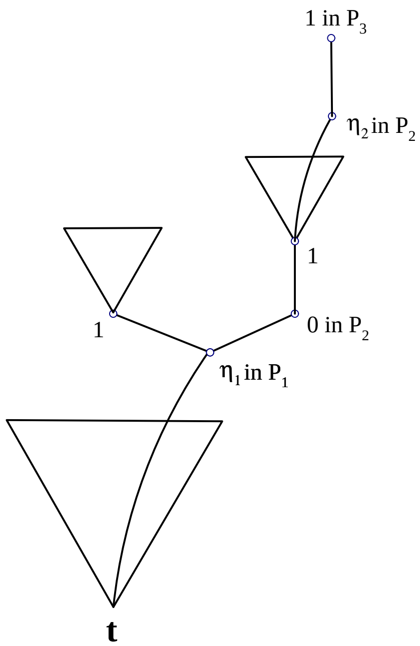


Fig. 14

If η_2 reduces by Case A or D, then by disjointness of the areas of action we again derive commutativity (5.2) and the claim follows.

Subcase 4F: Suppose that $\eta_2 \neq \eta_1{}^\wedge\langle 0\rangle$ and $\eta_1{}^\wedge\langle 0\rangle \trianglelefteq \eta_2 \trianglelefteq F(\eta_1{}^\wedge\langle 0\rangle)$, and **t** reduces with η_2 by Case B. In this subcase the following diagrams will explain the commutativity (5.2) of the small cancelation steps showing the resulting tree $\mathbf{s} = \mathrm{red}_{h^{\mathbf{ts}_1}(\eta_2)}\,\mathbf{s}_1 = \mathrm{red}_{h^{\mathbf{ts}_2}(\eta_1)}\,\mathbf{s}_2$, respectively. We must distinguish two cases depending on the position of $F(\eta_1{}^\wedge\langle 0\rangle)$ in the tree **t**.

For $\eta_2{}^\wedge\langle 0\rangle \trianglelefteq F(\eta_1{}^\wedge\langle 0\rangle)$ small cancelation is illustrated in Figure 15.

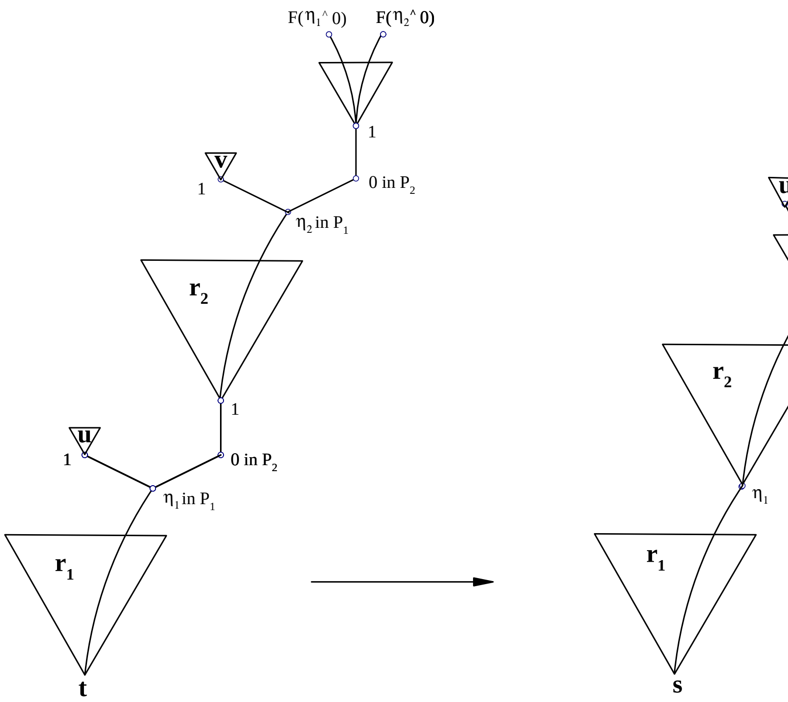

F(η1^0)
F(η2^0)
1
v
1
0 in P2
η2 in P1
r2
1
u
1
0 in P2
η1 in P1
r1
t
u
v
η'2
r2
η1
r1
s

Fig. 15

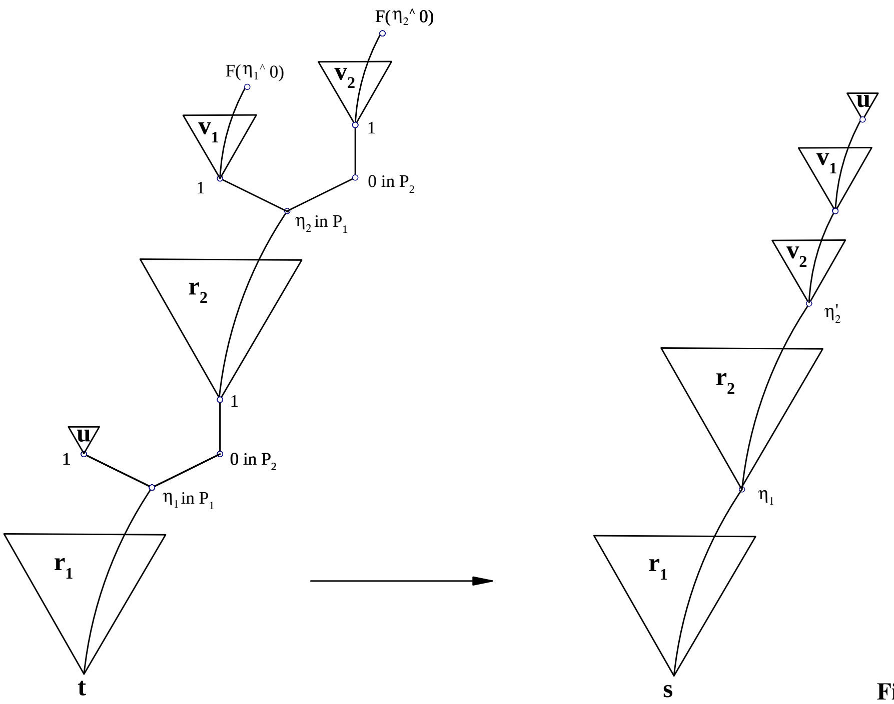

F(η2^0)
F(η1^0)
v2
v1
1
1
0 in P2
η2 in P1
r2
1
u
1
0 in P2
η1 in P1
r1
t
u
v1
v2
η'2
r2
η1
r1
s

Fig. 16

Finally, small cancelation for $\eta_2{}^\wedge\langle 1\rangle \trianglelefteq F(\eta_1{}^\wedge\langle 0\rangle)$ is illustrated in Figure 16.

This subcase completes the proof of the main lemma. □

6. Arbitrarily large free skeletons

In this section we will show how to modify the set $\mathfrak{T}^{\mathrm{red}}$ of reduced trees (from Section 4) to get $\mathfrak{T}^{\mathrm{red}}_Y$. The small cancelation theory from Section 5 will be extended to arbitrarily large canonical τ-skeletons. Thus, we will first apply $\mathfrak{T}^{\mathrm{red}}$ to get the universes (the underlying sets) $\mathfrak{T}^{\mathrm{red}}_Y$ of our actual desired τ-skeletons.

Definition 6.1. *Let $\{F_{\mathrm{id}}, 1, *\} \subseteq \tau_0$ be a set of function symbols and Y be a non-empty set.*

(i) *The family of (decorated) Y-*colored trees *$\mathfrak{T}_Y$ consists of all pairs $\mathbf{t}_Y = (\mathbf{t}, f^{\mathbf{t}})$ of trees $\mathbf{t} \in \mathfrak{T}$ with a (coloring) map $f^{\mathbf{t}} : P_4^{\mathbf{t}} \to Y$. Moreover, we define the subset $\mathfrak{T}^{\mathrm{red}}_Y$ of all* reduced *Y-colored trees, and let $\mathbf{t}_Y = (\mathbf{t}, f^{\mathbf{t}}) \in \mathfrak{T}^{\mathrm{red}}_Y \Leftrightarrow \mathbf{t} \in \mathfrak{T}^{\mathrm{red}}$.*

(ii) *We now define the reduction of an arbitrary element $\mathbf{t}_Y = (\mathbf{t}, f^{\mathbf{t}}) \in \mathfrak{T}_Y$ by letting*

$$\operatorname{red} \mathbf{t}_Y = (\operatorname{red} \mathbf{t}, f^{\mathbf{t}} \circ g^{\operatorname{red} \mathbf{t}, \mathbf{t}})$$

with a canonical embedding

$$g^{\operatorname{red} \mathbf{t}, \mathbf{t}} : \operatorname{red} \mathbf{t} \to \mathbf{t}$$

which comes from the Main Lemma 5.3 *related to a fixed path of small cancelation steps. It is clear by this lemma that* $\operatorname{red} \mathbf{t}_Y \in \mathfrak{T}^{\mathrm{red}}_Y$ *does not depend on the particular choice of the path, thus is uniquely determined by $\mathbf{t}_Y$.*

We will next turn the set $\mathfrak{T}^{\mathrm{red}}_Y$ into a τ-skeleton by adjoining a suitable family of functions, in particular adding the $*$-scalar product. To do this, we use the same set $\mathfrak{T}^{\mathrm{red}}$ of trees (from Section 5) to define a family of function symbols (Definition 6.3) representing concrete functions acting on $\mathfrak{T}^{\mathrm{red}}_Y$ (Definition 6.2) and thus make the following two definitions. The map σ takes care of the permutation law in Definition 2.1(ii) of the skeleton. Observe that a well-ordering on ${}^{\omega >}\omega$ is defined comparing two elements $\eta, \mu \in {}^{\omega >}\omega$ by inclusion and then (if that fails) by lexicographic order, thus we can compare both (finite) sequences componentwise. And without loss of generality we will always assume that the list $P_4^{\mathbf{s}} = \langle \eta_0, \dots, \eta_{n-1}\rangle$ is given in decreasing order for any $\mathbf{s} \in \mathfrak{T}^{\mathrm{red}}$.

Definition 6.2. *For every $\mathbf{s} \in \mathfrak{T}^{\mathrm{red}}$ with $|P_4^{\mathbf{s}}| = n$ we fix an enumeration $P_4^{\mathbf{s}} = \langle \eta_0, \dots, \eta_{n-1}\rangle$. Moreover, let $\sigma : n \to n$ be any permutation and define an n-ary function symbol $F^{\mathbf{s}\sigma}$ to represent the following function on $\mathfrak{T}^{\mathrm{red}}_Y$:*

$$F_Y^{\mathbf{s}\sigma} : (\mathfrak{T}^{\mathrm{red}}_Y)^n \to \mathfrak{T}^{\mathrm{red}}_Y, \qquad (\mathbf{t}_Y^0, \dots, \mathbf{t}_Y^{n-1}) \mapsto \operatorname{red} \mathbf{t}_Y,$$

where for $\mathbf{t}^l_Y = (\mathbf{t}^l, f^l) \in \mathfrak{T}^{\mathrm{red}}_Y$ $(l < n)$ *we define* $\mathbf{t}_Y = (\mathbf{t}, f) \in \mathfrak{T}_Y$ *by iterated application of Construction Tool II:*

$$\mathbf{t} = (\dots((\mathbf{s}^{[\eta_0,\,\mathbf{t}^{0\sigma}]})^{[\eta_1,\,\mathbf{t}^{1\sigma}]})\dots)^{[\eta_{n-1},\,\mathbf{t}^{(n-1)\sigma}]} \quad \textit{and} \quad f(\eta_l{}^\wedge\rho) = f^{l\sigma}(\rho) \textit{ for all } \rho \in P_4^{\mathbf{t}^{l\sigma}},$$

so we are engrafting twigs as explained in the last section.

Next we fix the language τ and choose our candidate for a τ-skeleton, the link between function symbols and the actual functions of M_Y.

Definition 6.3. (i) *Let* τ *be a vocabulary with no predicates consisting of an infinite set of free variables* X *and the set*

$$\mathcal{F} = \{F^{\mathbf{s}\sigma} \mid \mathbf{s} \in \mathfrak{T}^{\mathrm{red}},\ \sigma : |P_4^{\mathbf{s}}| \to |P_4^{\mathbf{s}}| \textit{ permutation}\} \quad \textit{of function symbols.}$$

(ii) *Let* M_Y *be the* τ*-structure with universe* $\mathfrak{T}^{\mathrm{red}}_Y$ *interpreting every function symbol* $F^{\mathbf{s}\sigma}$ *as the function* $F^{\mathbf{s}\sigma}_Y$*.*

Remark 6.4. (i) We point out that M_Y depends on Y while the family $\mathcal{F}$ of function symbols (in Definition 6.3) is independent of Y. This also reflects the purpose of these elements: Depending on Y, the τ-skeletons M_Y can be arbitrarily large. However, the set $\mathcal{F}$ describes the fixed family of function symbols of our universal language τ of these skeletons.

(ii) Figure 17 illustrates the application of $F^{\mathbf{s}\sigma}_Y$. Moreover, note that by the arguments of the last section concerning actions on disjoint areas the repeated application of our Construction Tool II does not depend on the order of these steps $[\eta_l, \mathbf{t}^{l\sigma}]$ but will in general not be reduced. This explains the final reduction before we get $F^{\mathbf{s}\sigma}_Y(\mathbf{t}^0_Y, \dots, \mathbf{t}^{n-1}_Y) \in \mathfrak{T}^{\mathrm{red}}_Y$.

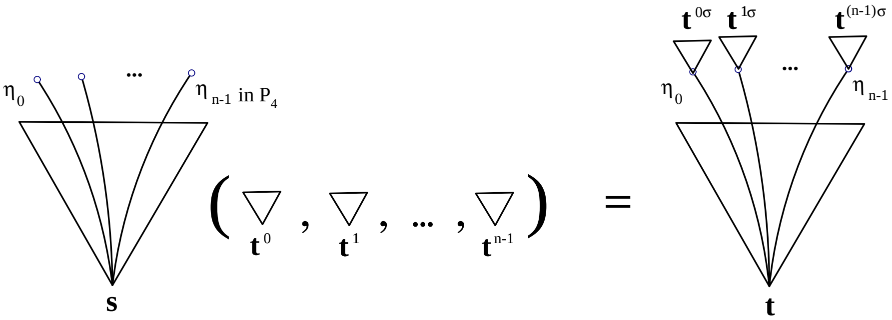


Fig. 17

(iii) Allowing the trivial case $Y = \emptyset$ we may identify $\mathfrak{T}_\emptyset = \mathfrak{T}$, $\mathfrak{T}^{\mathrm{red}}_\emptyset = \mathfrak{T}^{\mathrm{red}}$ and

$$F^{\mathbf{s}\sigma}_\emptyset(\mathbf{t}^0, \dots, \mathbf{t}^{n-1}) = \mathbf{t}$$

in Definition 6.2. This extension will be applied in the proof of Main Theorem 6.5.

As already noted, the function symbols $F^{\mathbf{s}\sigma}$ serve as source for our language τ. Finally, we have to do the necessary bookkeeping and identify the function symbols from $\{F_{\mathrm{id}}, 1, *\} \subseteq \tau_0$.

We begin with the particular function symbols F_{id}, $1, *$ and identify $* = F^{\mathbf{s}\sigma}$ with the following reduced tree $\mathbf{s} \in \mathfrak{T}^{\mathrm{red}}$ and the following permutation σ (see also Figure 18):

$$u^{\mathbf{s}} = \{\perp, \langle 0\rangle, \langle 1\rangle\},\ P_1^{\mathbf{s}} = \{\perp\},\ P_4^{\mathbf{s}} = \{\langle 0\rangle, \langle 1\rangle\},\ P_0^{\mathbf{s}} = P_2^{\mathbf{s}} = P_3^{\mathbf{s}} = P_5^{\mathbf{s}} = \emptyset,\ \sigma = \mathrm{id}\,.$$

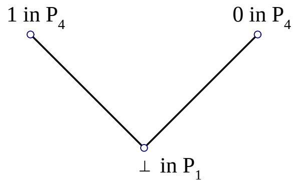


Fig. 18

The constant 1 is represented as $1 = F^{\mathbf{s}\sigma}$ by the reduced tree

$$u^{\mathbf{s}} = P_0^{\mathbf{s}} = \{\perp\}, \qquad P_1^{\mathbf{s}} = P_2^{\mathbf{s}} = P_3^{\mathbf{s}} = P_4^{\mathbf{s}} = P_5^{\mathbf{s}} = \emptyset, \qquad \text{with } \sigma = \emptyset.$$

Observe that this tree appears twice: as a function symbol in $\mathcal{F}$ and as a Y-colored tree in M_Y. This is the case for every constant in $\mathcal{F}$; it reflects the canonical identification of function symbols of arity 0 with certain elements in M_Y. Moreover, put $F_{\mathrm{id}} = F^{\mathbf{s}\sigma}$ for

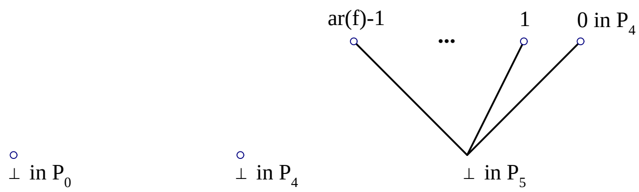
$$u^{\mathbf{s}} = P_4^{\mathbf{s}} = \{\perp\}, \qquad P_0^{\mathbf{s}} = P_1^{\mathbf{s}} = P_2^{\mathbf{s}} = P_3^{\mathbf{s}} = P_5^{\mathbf{s}} = \emptyset, \qquad \sigma = \mathrm{id}\,.$$

Using the mapping properties (Definition 6.2), it follows that F_{id} acts as identity on M_Y.

Every other function symbol $f \in \tau_0 \setminus \{F_{\mathrm{id}}, 1, *\}$ will be interpreted as $f = F^{\mathbf{s}\sigma}$ with

$$u^{\mathbf{s}} = \{\perp, \langle l\rangle \mid l < \mathrm{ar}(f)\}, \qquad P_4 = \{\langle l\rangle \mid l < \mathrm{ar}(f)\},$$
$$P_5 = \{\perp\}, \qquad P_0^{\mathbf{s}} = P_1^{\mathbf{s}} = P_2^{\mathbf{s}} = P_3^{\mathbf{s}} = \emptyset, \qquad G^{\mathbf{s}}(\perp) = f, \qquad \sigma = \mathrm{id}\,.$$

Thus we get the following functions acting on M_Y:



1 **$\mathbf{F_{id}}$** **f**

Fig. 19

With each $y \in Y$ we associate the reduced Y-colored tree $(\mathbf{t}_y, f_y) \in M_Y$ of the form

$$u^{\mathbf{t}_y} = P_4^{\mathbf{t}_y} = \{\perp\}, \qquad P_0^{\mathbf{t}_y} = P_1^{\mathbf{t}_y} = P_2^{\mathbf{t}_y} = P_3^{\mathbf{t}_y} = P_5^{\mathbf{t}_y} = \emptyset, \qquad f_y(\perp) = y.$$

We will write $y = (\mathbf{t}_y, f_y)$ and thus $Y \subseteq M_Y$ (without loss of generality).

The last remarks are the initial steps to establish the following

Main Theorem 6.5. *If Y is a non-empty set, then M_Y is a τ-skeleton of size $|M_Y| = |\tau_0| + |Y|$ and $Y \subseteq M_Y$ canonically. If Y is infinite, then M_Y is also faithful and nice.*

Proof. Obviously $|M_Y| \leq |\mathfrak{T}^{\text{red}}| \times |{}^{\omega>}Y| = (\aleph_0 + |\tau_0|) \times |Y| = |\tau_0| + |Y|$, and the canonical embedding $Y \hookrightarrow M_Y$ given above shows $|M_Y| \geq |Y| \geq \aleph_0$. Another injective mapping $\tau_0 \setminus \{F_{\text{id}}, 1, *\} \hookrightarrow M_Y$ is easily defined by replacing every argument in $f \in \tau_0$ by $1 \in M_Y$. Thus also $|M_Y| \geq |\tau_0|$, and $|M_Y| = |\tau_0| + |Y|$ follows. Next we have to verify the skeleton axioms given in Definition 2.1.

(i) We already checked $F_{\text{id}}(\mathbf{t}_Y) = \mathbf{t}_Y$ on M_Y. We must now show that $\mathbf{t}_Y * 1 = \mathbf{t}_Y$. When calculating $\mathbf{t}_Y * 1 = *(\mathbf{t}_Y, 1)$, we note that $1 \in M_Y$ is represented by $u^{\mathbf{s}} = P_0^{\mathbf{s}} = \{\perp\}$, and the tree $\mathbf{t}_Y * 1$ reduces by Case A to $\mathbf{t}_Y$; see Figure 20.

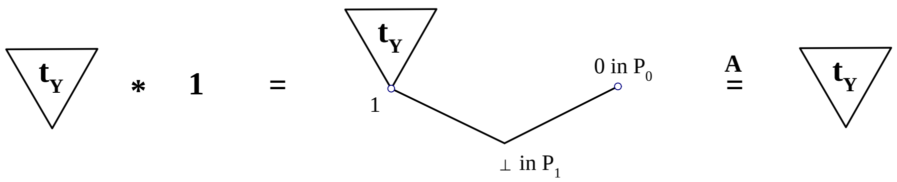


Fig. 20

(ii) For every n-ary function symbol $F^{\mathbf{s}\sigma} \in \mathcal{F}$ and any permutation $\pi : n \to n$ we have from Definition 6.2 the obvious identity

$$F_Y^{\mathbf{s}\sigma}(\mathbf{t}_Y^{0\pi}, \mathbf{t}_Y^{1\pi}, \ldots, \mathbf{t}_Y^{(n-1)\pi}) = F_Y^{\mathbf{s}\pi\sigma}(\mathbf{t}_Y^0, \mathbf{t}_Y^1, \ldots, \mathbf{t}_Y^{n-1}).$$

We leave it to the reader to check (iii) with the help of Figure 21. We first reformulate this axiom, so that the argument is more direct:

When proving the substitution law we are given function symbols $F^{\mathbf{r}\rho}, F^{\mathbf{s}_i\sigma_i} \in \mathcal{F}$ with obvious arities, and it remains to check the equality

$$F_Y^{\mathbf{r}\rho}(F_Y^{\mathbf{s}_0\sigma_0}(\mathbf{t}_Y^0, \ldots, \mathbf{t}_Y^{n_0-1}), F_Y^{\mathbf{s}_1\sigma_1}(\mathbf{t}_Y^{n_0}, \ldots, \mathbf{t}_Y^{n_0+n_1-1}), \ldots, F_Y^{\mathbf{s}_{k-1}\sigma_{k-1}}(\mathbf{t}_Y^{n-n_{k-1}}, \ldots, \mathbf{t}_Y^{n-1})) \\ = F_Y^{\text{red}\,\mathbf{t}\,\pi}(\mathbf{t}_Y^0, \ldots, \mathbf{t}_Y^{n-1}),$$

where $\mathbf{t} = F_\emptyset^{\mathbf{r}\rho}(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{k-1})$ and $\pi : n \to n$ is a suitable permutation. Figure 21 visualizes this equation, where we rely on unreduced trees and their unique reduction by small cancelation.

In fact, $\pi = \pi_1\pi_2$ is a product of two permutations. The first permutation $\pi_1 : n \to n$ comes from the reduction of the tree $\mathbf{t}$. If $P_4^{\mathbf{t}} = \langle \mu_0, \ldots, \mu_{n-1}\rangle$ and $P_4^{\text{red}\,\mathbf{t}} = \langle \mu'_0, \ldots, \mu'_{n-1}\rangle$, then

$$i\pi_1 = j \quad \text{for} \quad g^{\text{red}\,\mathbf{t},\,\mathbf{t}}(\mu'_i) = \mu_j \quad (0 \leq i, j < n). \tag{6.1}$$

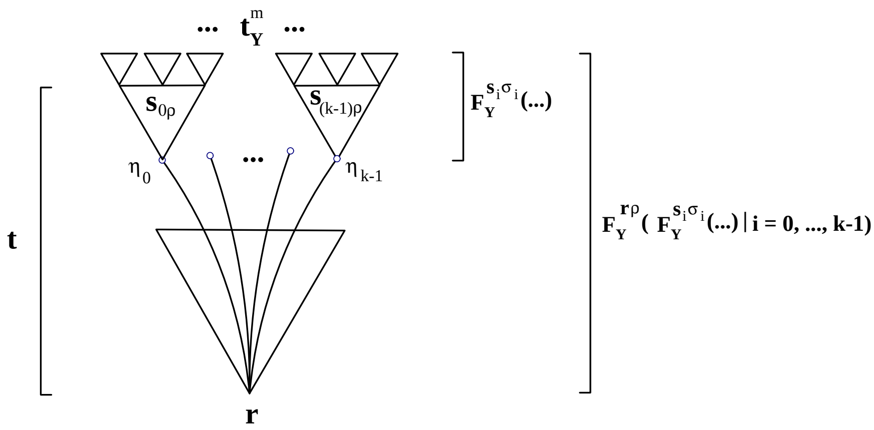


Fig. 21

The second permutation $\pi_2 : n \to n$ comes from engrafting the twigs and is defined by

$$m\pi_2 = \sum_{j=0}^{l\rho-1} n_j + i\sigma_{l\rho} \quad \text{for} \quad m = \sum_{j=0}^{l-1} n_{j\rho} + i \qquad (0 \le l < k,\ 0 \le i < n_{l\rho}).$$

(iv) For the $*$-scalar product law we have $F^{\mathbf{s}\sigma}(x_0, \dots, x_{m-1}, y, x_{m+1}, \dots, x_{n-1})$ and must specify a function symbol $F^{\mathrm{red}\,\mathbf{t}\,\pi} \in \mathcal{F}$ such that

$$F^{\mathbf{s}\sigma}(x_0, \dots, x_{m-1}, y, x_{m+1}, \dots, x_{n-1}) = y * F^{\mathrm{red}\,\mathbf{t}\,\pi}(x_0, \dots, x_{m-1}, x_{m+1}, \dots, x_{n-1}).$$

We determine $\mathbf{t}$ and π first:

For $P_4^{\mathbf{s}} = \langle \eta_0, \dots, \eta_{n-1} \rangle$ let $\nu = \eta_{m\sigma^{-1}}$ be the branch corresponding to the argument y in $F^{\mathbf{s}\sigma}$. Now we characterize the tree $\mathbf{t}$ by $\mathbf{s}$ and the defining relations

$$\perp \in P_2^{\mathbf{t}}, \qquad F^{\mathbf{t}}(\perp) = \langle 1 \rangle^\wedge \nu \in P_3^{\mathbf{t}}, \qquad \mathbf{t}^{\ge \langle 1 \rangle} = \mathbf{s}$$

with the help of Construction Tool I (see the previous section). Figure 22 shows what $\mathbf{t}$ looks like.

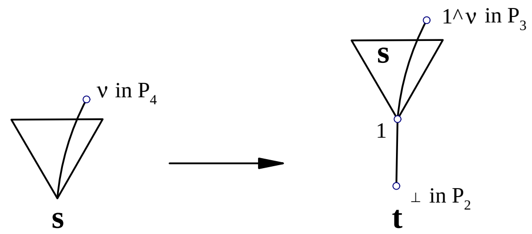


Fig. 22

The appropriate permutation $\pi : n-1 \to n-1$ is easy to visualize but not so easy to describe: we can express $\pi = \pi_1\pi_2$ as a product of two permutations. The first permutation $\pi_1 : n-1 \to n-1$ comes from the reduction of the tree $\mathbf{t}$ as in (6.1). The second permutation $\pi_2 : n-1 \to n-1$ is the result of the construction of the tree $\mathbf{t}$ from $\mathbf{s}$. Using the auxiliary functions $\rho_k : n-1 \to n$ $(0 \le k < n)$ with $i\rho_k = i$ for $i < k$ and $i\rho_k = i+1$ for $i \ge k$ we let $\pi_2 = \rho_{m\sigma^{-1}}\sigma\rho_m^{-1}$.

Now $F^{\mathrm{red}\,\mathbf{t}\pi}$ is determined and it remains to check

$$\mathbf{t}_Y^m * F_Y^{\mathrm{red}\,\mathbf{t}\pi}(\mathbf{t}_Y^0, \ldots, \mathbf{t}_Y^{m-1}, \mathbf{t}_Y^{m+1}, \ldots, \mathbf{t}_Y^{n-1}) = F_Y^{\mathbf{s}\sigma}(\mathbf{t}_Y^0, \ldots, \mathbf{t}_Y^{n-1}).$$

This equation is clear by Figure 23 using Definition 6.2 (as the left-hand side reduces with small cancelation by Case B to the right-hand side). Again we rely on the unique reduction of trees. Therefore also the $*$-scalar product law holds.

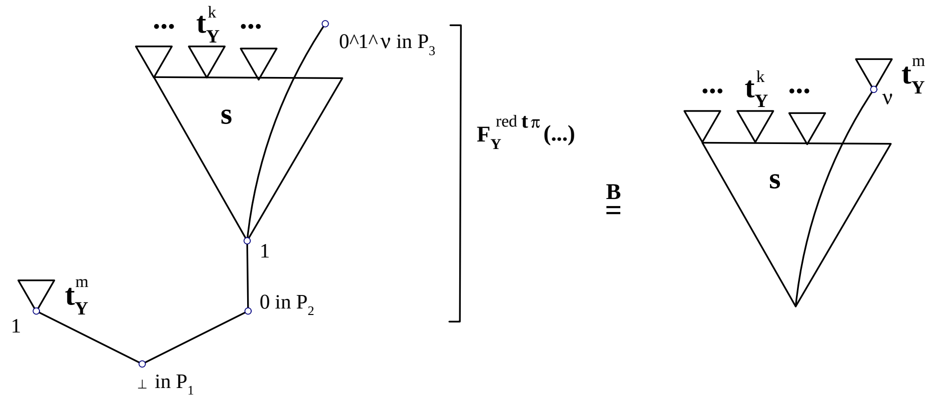


Fig. 23

Thus M_Y is a skeleton containing Y as a subset. Next we claim that

$$M_Y \text{ is faithful for } Y \text{ infinite.} \tag{6.2}$$

To show (6.2) we make essential use of the additional small cancelation steps by Cases C and D, while up to now only small cancelation by Cases A and B applied. And indeed, if we do not allow small cancelation by Cases C and D, then for $\mathbf{s}_Y = (\mathbf{s}, f^{\mathbf{s}})$, $\mathbf{t}_Y = (\mathbf{t}, f^{\mathbf{t}})$ with

$$u^{\mathbf{s}} = P_4^{\mathbf{s}} = \{\perp\}, \quad f^{\mathbf{s}}(\perp) = y \quad (\text{thus } \mathbf{s}_Y = y \in Y)$$

and

$$u^{\mathbf{t}} = \{\perp, \langle 1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}, \quad P_1^{\mathbf{t}} = \{\langle 1\rangle\}, \quad P_2^{\mathbf{t}} = \{\perp\}, \quad P_3^{\mathbf{t}} = \{\langle 1,1\rangle\}, \quad P_4^{\mathbf{t}} = \{\langle 1,0\rangle\},$$
$$F^{\mathbf{t}}(\perp) = \langle 1,1\rangle, \quad f^{\mathbf{t}}(\langle 1,0\rangle) = y = f^{\mathbf{s}}(\perp)$$

both $\mathbf{s}_Y \ne \mathbf{t}_Y$ are reduced in this restricted sense (see Figure 24).

But applying small cancelation by Case B to $\mathbf{r}_Y * \mathbf{t}_Y$ we see immediately that $\mathbf{r}_Y * \mathbf{t}_Y = \mathbf{r}_Y * \mathbf{s}_Y$ for all $\mathbf{r}_Y \in M_Y$ (see Figure 25).

The way out of this problem is to add the additional two cases C and D of small cancelations to ensure that $\mathbf{s}_Y, \mathbf{t}_Y$ above reduce to the same Y-colored tree.

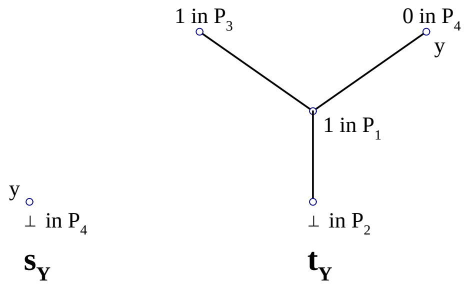


Fig. 24

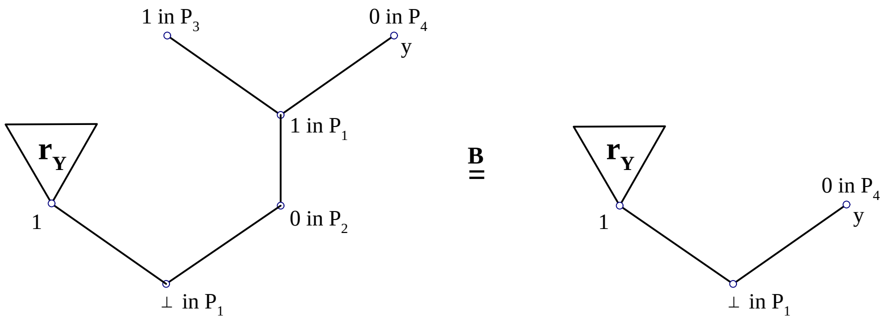


Fig. 25

Now we prove (6.2) and apply the $*$-scalar product to particular test elements from M_Y. Let $\mathbf{s}_Y = (\mathbf{s}, f^{\mathbf{s}})$, $\mathbf{t}_Y = (\mathbf{t}, f^{\mathbf{t}})$ be two elements from M_Y and choose some $y \in Y \subseteq M_Y$ which does not appear as a color of $\mathbf{s}_Y$, $\mathbf{t}_Y$, respectively. (Here we use the fact that Y is infinite!) We now compare $\mathbf{s}'_Y = y * \mathbf{s}_Y = (\mathbf{s}', f^{\mathbf{s}'}) \in \mathfrak{T}$ and $\mathbf{t}'_Y = y * \mathbf{t}_Y = (\mathbf{t}', f^{\mathbf{t}'}) \in \mathfrak{T}_Y$ and suppose that

$$\operatorname{red} \mathbf{s}'_Y = \operatorname{red} \mathbf{t}'_Y. \tag{6.3}$$

It remains to show that also $\mathbf{s}_Y = \mathbf{t}_Y$. The arguments depend on the small cancelation steps we can apply to $\mathbf{s}'_Y$ and $\mathbf{t}'_Y$. The two corresponding, possibly not yet reduced diagrams look as in Figure 26.

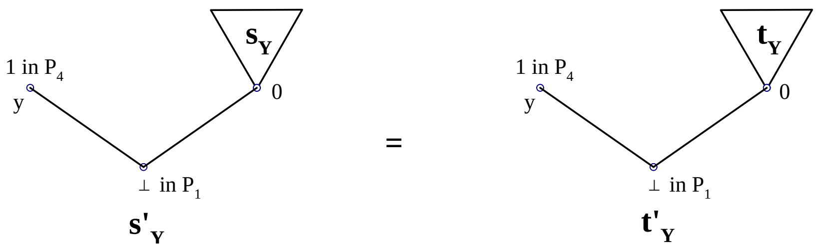


Fig. 26

Here we use y (a new color which does not appear in $\mathbf{s}_Y$ and $\mathbf{t}_Y$) to mark distinctively branches of the two trees $\mathbf{s}'_Y$, $\mathbf{t}'_Y$. This will be particularly important if the trees get mixed up by using small cancelation steps.

Thus, if we are lucky, and the two trees $\mathbf{s}'_Y$, $\mathbf{t}'_Y$ are both reduced, then automatically both right-hand components must coincide, so $\mathbf{s}_Y = \mathbf{t}_Y$.

In the remaining cases we must check all possible small cancelation steps of $\mathbf{s}'_Y$ and $\mathbf{t}'_Y$. By the last remark one of these trees must reduce further while obviously $\operatorname{can}(\mathbf{s}')$, $\operatorname{can}(\mathbf{t}')$ $\subseteq \{\perp\}$. Suppose that $\mathbf{s}'_Y$ can be reduced. Thus $\operatorname{can}(\mathbf{s}') = \{\perp\}$ and $\mathbf{s}'$ reduces either by small cancelation Case A or Case B.

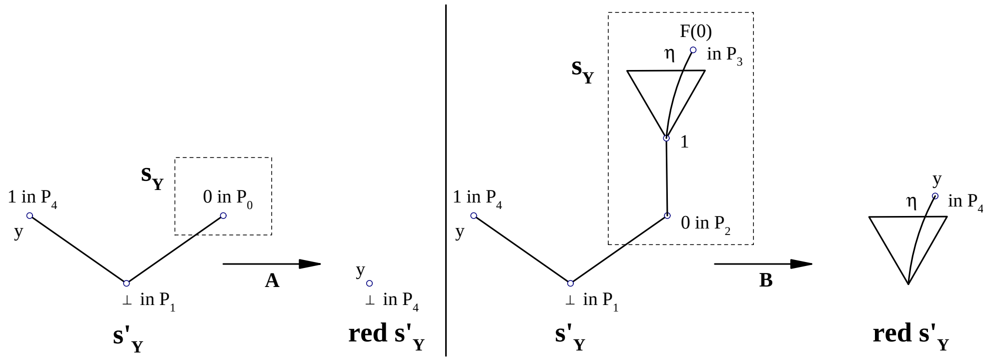


Fig. 27

Observe that in both cases the outcoming tree will be reduced, thus small cancelation stops after the first step.

Comparing now the reductions of $\mathbf{s}'_Y$ and $\mathbf{t}'_Y$ there are only two possibilities:

Case 1: If $\langle 0\rangle \in P_0^{\mathbf{s}'}$, then the right-hand component of $\mathbf{s}'$ is a maximal branch and the tree reduces with small cancelation by Case A to $y \in M_Y$. Now from (6.3) it follows that also $\mathbf{t}'$ must reduce. If $\mathbf{t}'$ reduces with small cancelation by Case A, then $\mathbf{s}_Y = \mathbf{t}_Y$ is immediate. If $\mathbf{t}'$ reduces with Case B, then (6.3) brings $\mathbf{t}$ into the form $u^{\mathbf{t}} = \{\perp, \langle 1\rangle\}$ with $P_2^{\mathbf{t}} = \{\perp\}$, $P_3^{\mathbf{t}} = \{\langle 1\rangle\}$, $F^{\mathbf{t}}(\perp) = \langle 1\rangle$, which reduces to $\mathbf{s}$ by Case C, but we assumed that $\mathbf{t}_Y$ is reduced, hence this case is impossible.

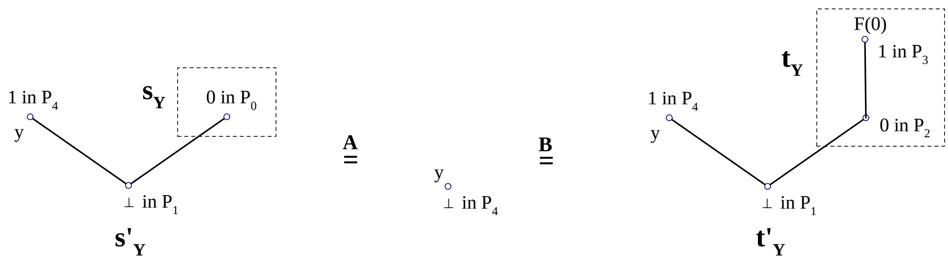


Fig. 28

Case 2: Now without loss of generality we can assume that neither $\mathbf{s}'$ nor $\mathbf{t}'$ reduces with small cancelation by Case A, while $\langle 0\rangle \in P_2^{\mathbf{s}'}$. Thus the right-hand component of $\mathbf{s}'$ extends further with $F^{\mathbf{s}'}(\langle 0\rangle) = \langle 0, 1\rangle^\wedge\eta$ and it reduces by Case B with $\eta \in P_4^{\mathrm{red}\,\mathbf{s}'}$. If $\mathbf{t}'$ also reduces by Case B, then $\mathbf{t}_Y = \mathbf{s}_Y$ is immediate. Suppose that $\mathbf{t}'$ is already reduced. Then comparing the two reduced trees gives

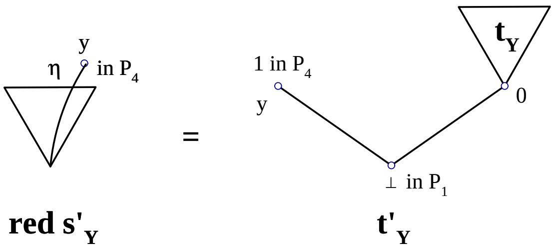


Fig. 29

and we use the marker y to identify the respective branches to reconstruct $\mathbf{s}$ as

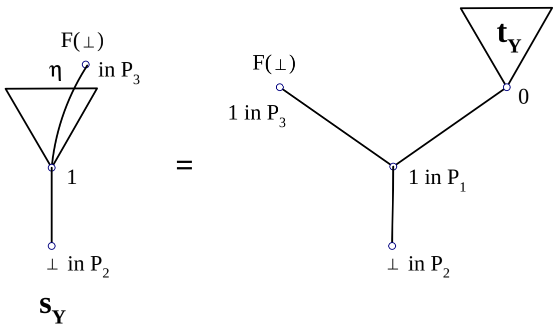


Fig. 30

But now $\mathbf{s}$ reduces with small cancelation by Case D to $\mathbf{t}$, which is a contradiction. Hence M_Y is faithful (as witnessed by the action of $*$ on $Y \subseteq M_Y$).

Finally, we must show that

$$M_Y \text{ is nice for } Y \text{ infinite.} \tag{6.4}$$

For the convenience of the reader we now repeat the essentials of the technical, but harmless Definition 2.10. Given integers $2 \le k_i \le n_i$ $(i < i_*)$ we let $k_* = \max\{k_i \mid i < i_*\}$ and

$$^{i}\bar{\mathbf{t}}_Y = \langle {}^{ij}\mathbf{t}_Y \mid j \in n_i \setminus k_i\rangle \in {}^{n_i - k_i}M_Y,$$

where

$$^{ij}\mathbf{t}_Y = ({}^{ij}\mathbf{t}, {}^{ij}f) \in M_Y \quad \text{with} \quad {}^{ij}f : P_4^{{}^{ij}\mathbf{t}} \to Y.$$

For any family

$$\sigma_i(x, {}^i\bar{\mathbf{t}}_Y) = \sigma_i^{M_Y}(\bar{x}^{k_i}, {}^i\bar{\mathbf{t}}_Y) = F_Y^{\mathbf{r}_i\rho_i}(\bar{x}^{k_i}, {}^i\bar{\mathbf{t}}_Y) \quad (i < i_*)$$

of strict, pairwise non-M_Y-equivalent M_Y-terms with n_i-ary function symbols $F^{\mathbf{r}_i\rho_i}$ we must find a suitable sequence $\bar{\mathbf{s}}_Y \in {}^{k_*}M_Y$ satisfying the conclusion of Definition 2.10.

Recall that $\Lambda_{k_i,k_*} = {}^{k_i}k_* \setminus \{\bar{n}^{k_i} \mid n < k_*\}$ and choose $\bar{\mathbf{s}}_Y = \langle y_i \mid i < k_*\rangle$ for distinct variables $y_i \in Y \setminus \bigcup_{i,j} \operatorname{Im}{}^{ij}f$. These elements exist because Y is infinite and will serve as distinguished markers. For (6.4) we claim that the elements $F_Y^{\mathbf{r}_i\rho_i}(\bar{\mathbf{s}}_{Y\eta}, {}^i\bar{\mathbf{t}}_Y)$ for $i < k_*$, $\eta \in \Lambda_{k_i,k_*}$ are pairwise distinct. Thus assume that

$$F_Y^{\mathbf{r}_i\rho_i}(\bar{\mathbf{s}}_{Y\eta}, {}^i\bar{\mathbf{t}}_Y) = F_Y^{\mathbf{r}_j\rho_j}(\bar{\mathbf{s}}_{Y\theta}, {}^j\bar{\mathbf{t}}_Y) \quad \text{for some pairs } (i,\eta), (j,\theta). \tag{6.5}$$

If $i \neq j$, then the assumption easily yields $F_Y^{\mathbf{r}_i\rho_i}(\bar{\mathbf{t}}_Y^{k_i}, {}^i\bar{\mathbf{t}}_Y) = F_Y^{\mathbf{r}_j\rho_j}(\bar{\mathbf{t}}_Y^{k_j}, {}^j\bar{\mathbf{t}}_Y)$ for all $\mathbf{t}_Y \in M_Y$, hence these terms are M_Y-equivalent, a contradiction. Thus $i = j$, and now comparing branch colors directly in (6.5) also, $\eta = \theta$. Hence (6.4) and Main Theorem 6.5 hold. ◻

Remark 6.6. (i) In Definition 6.3 we have deliberately relaxed the requirements on the pair (τ, T), in fact we kept essentially silent about the exact skeleton theory T, because the missing details did not play any role until now (and they are more delicate). Now we are ready to adjoin the precise list of equations to T which is revealed in detail in the proof of Main Theorem 6.5. Here it is important to recall that the predicted function symbols F' from Definition 2.1 are indeed fixed inside τ and independent from Y, thus no quantifier $\exists F'$ is used.
For future use (e.g. Proposition 7.1) we should remember that the constructed canonical τ-skeletons M_Y are actually canonical τ-skeletons *with respect to this canonical skeleton theory T.*

(ii) We now consider the generating sets τ_0 of τ and Y of M_Y as the trees allow a very suggestive interpretation. For this purpose let $\mathbf{t} = (u^{\mathbf{t}}, P_l^{\mathbf{t}}, F^{\mathbf{t}}, G^{\mathbf{t}} \mid l \leq 5) \in \mathfrak{T}^{\text{red}}$ be a reduced tree as in Definition 4.1 and $F^{\mathbf{t}\pi} \in \mathcal{F}$ an appropriate function symbol. As in a dictionary, translate the knots of the tree $\mathbf{t}$: every element in $P_0^{\mathbf{t}}$ interprets as an appearance of the constant 1, the elements of $P_4^{\mathbf{t}}$ represent free variables, while the elements of $P_3^{\mathbf{t}}$ correspond to bounded variables. Furthermore, elements in $P_1^{\mathbf{t}}$ represent $*$-scalar multiplication signs, while the elements of $P_5^{\mathbf{t}}$ correspond to function symbols in $\tau_0 \setminus \{F_{\text{id}}, 1, *\}$ by the map $G^{\mathbf{t}}$. Lastly, $P_2^{\mathbf{t}}$ is used to accommodate new function symbols F' generated by the $*$-scalar product law

$$F(x_0, x_1, \ldots, x_{m-1}, y, x_{m+1}, \ldots) = y * F'(x_0, x_1, \ldots, x_{m-1}, x_{m+1}, \ldots).$$

During this process the branch η of $\mathbf{t}$ associated to the free variable y is moved from $P_4^{\mathbf{t}}$ to $P_3^{\mathbf{t}}$, while y becomes bounded and the map $F^{\mathbf{t}}$ keeps track of this process. In terms of λ-calculus this is equivalent to saying that in

$$F'(x_0, x_1, \ldots, x_{m-1}, x_{m+1}, \ldots) = \lambda y F(x_0, x_1, \ldots, x_{m-1}, y, x_{m+1}, \ldots)$$

the variable y is bounded; see Barendregt [2]. Thus the function symbol $F^{\mathbf{t}\pi}$ can be interpreted as a τ_0-term (in distinct free variables from X) allowing some λ-calculus and permutation arguments and thus is generated by τ_0 in this sense. Similarly every element $\mathbf{t}_Y = (\mathbf{t}, f^{\mathbf{t}}) \in M_Y$ can be viewed as being generated by τ_0 and Y only.

(iii) In the given approach the basic set for the τ-skeleton is the family $\mathfrak{T}_Y^{\mathrm{red}}$ of reduced trees. We could have followed a different road: for example, constructing the rational numbers $\mathbb{Q}$ or free (non-commutative) groups, we either use uniquely determined reduced ('minimal') representatives of the obvious equivalence classes or the family of equivalence classes themselves as members of $\mathbb{Q}$ or of the free groups, respectively. Similarly, here we could have used the equivalence classes obtained by small cancelation of trees as well and created the skeletons M_Y this way.

(iv) Following Definition 2.8 we will call a τ-body B faithful if the $*$-multiplication is faithful, i.e. for all $a \neq b \in B$ there is some $c \in B$ such that $c * a \neq c * b$. Arguing as in Main Theorem 6.5 the canonical τ_R-body $\mathrm{Lin}_R\, M_Y$ is faithful for Y infinite as follows by the action of the $*$-scalar product on $Y \subseteq M_Y \subseteq \mathrm{Lin}_R\, M_Y$.

According to Theorem 6.5 the τ-skeleton M_Y is faithful for any infinite set Y. Thus with Lemma 2.9 we have a well-defined associative multiplication $\cdot$ on M_Y which we want to investigate next. By Definition 2.8 we identify $\cdot = F^{\mathbf{s}\sigma}$ with the following reduced tree $\mathbf{s} \in \mathfrak{T}^{\mathrm{red}}$ and the following permutation σ (see also Fig. 31):

$$u^{\mathbf{s}} = \{\perp, \langle 1\rangle, \langle 1,0\rangle, \langle 1,1\rangle, \langle 1,1,0\rangle, \langle 1,1,1\rangle\}, \quad P_1^{\mathbf{s}} = \{\langle 1\rangle, \langle 1,1\rangle\}, \quad P_2^{\mathbf{s}} = \{\perp\},$$
$$P_3^{\mathbf{s}} = \{\langle 1,1,1\rangle\}, \quad P_4^{\mathbf{s}} = \{\langle 1,0\rangle, \langle 1,1,0\rangle\}, \quad P_0^{\mathbf{s}} = P_5^{\mathbf{s}} = \emptyset,$$
$$F^{\mathbf{s}}(\perp) = \langle 1,1,1\rangle, \quad \sigma = \mathrm{id}\,.$$

Thanks to this identity $y_1 \cdot y_2 \neq y_2 \cdot y_1$ is obvious for all generators $y_1 \neq y_2$ from Y, which guarantees that we construct non-commutative R-algebras later on.

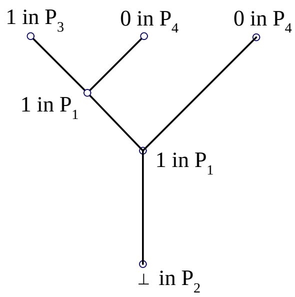


Fig. 31

Furthermore, $\cdot = F^{\mathbf{s}\sigma} \in \mathcal{F}$ allows a natural generalization of $\cdot$ to τ-skeletons M_Y over arbitrary (in particular finite) sets Y and we have the following

Lemma 6.7. *For any non-empty set Y the pair $(M_Y, \cdot)$ with the multiplication $\cdot$ from above is a non-commutative associative monoid with* 1.

Proof. The associativity of the multiplication $\cdot$ is demonstrated in Figure 32 as both $(\mathbf{r}_Y \cdot \mathbf{s}_Y) \cdot \mathbf{t}_Y$ and $\mathbf{r}_Y \cdot (\mathbf{s}_Y \cdot \mathbf{t}_Y)$ reduce by small cancelation B to the same Y-colored tree for any $\mathbf{r}_Y, \mathbf{s}_Y, \mathbf{t}_Y \in M_Y$.

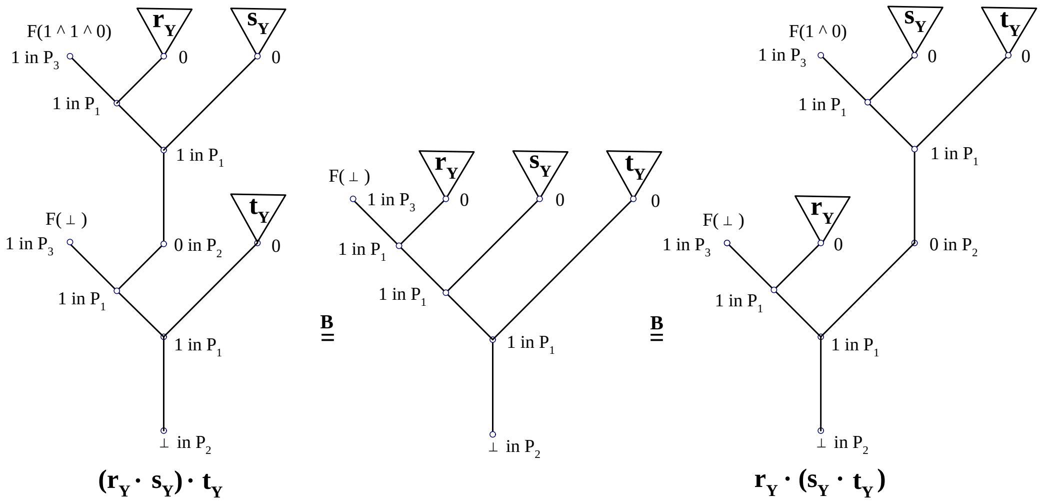


Fig. 32

Similarly $1 \cdot \mathbf{t}_Y = \mathbf{t}_Y = \mathbf{t}_Y \cdot 1$ holds for any $\mathbf{t}_Y \in M_Y$ using small cancelation by Cases A and D. And finally, the non-commutativity of the multiplication $\cdot$ follows by Figure 33 from $(1 * y) \cdot y \neq y \cdot (1 * y)$ for any generator $y \in Y$. $\square$

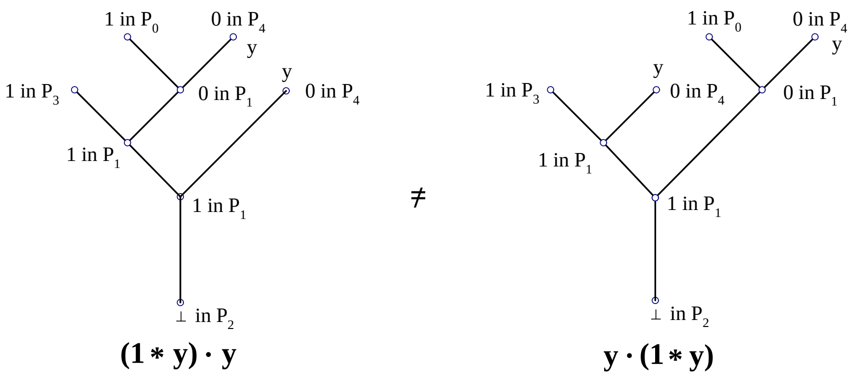


Fig. 33

An inspection of the last proof allows the following obvious generalization.

Observation 6.8. *If M is a τ-skeleton induced by the pair (τ, T) of the language τ from Definition* 6.3 *and the theory T from Theorem* 6.5*, then the pair $(M, \cdot)$ with the multiplication $\cdot$ from above is an associative monoid with* 1.

7. The algebraic structure of the free body $\mathbb{B}_Y$ from M_Y

Let R be a cotorsion-free commutative ring with 1 and $p \in R$ be the element in charge of the p-adic completion $\widehat{R}$; see Section 1. Moreover, saying that G is a pure submodule of H (or $G \subseteq_* H$) actually means that the R-module G is a p-pure submodule of the R-module H where p is the distinguished element from above.

In this section we want to investigate those τ_R-bodies which are induced by the pair (τ, T) of the language τ and the theory T introduced in Definition 6.3 and Main Theorem 6.5. We will use the letter $\mathbb{B}$ for these bodies to emphasize this special choice of (τ, T).

We recall from Section 3 that $\mathbb{B}$ is an R-module with a constant element 1, a $*$-scalar product and a family of multi-linear functions $\mathcal{F}$ satisfying the axioms from Definition 2.1. In particular, $m * 1 = m$ for all $m \in \mathbb{B}$, the substitution axiom holds for $\mathcal{F}$, and the following important law is satisfied in $\mathcal{F}$:

For any function symbol $F \in \mathcal{F}$ of arity n and any $0 \leq m < n$ some function symbol $F' \in \mathcal{F}$ of arity $n-1$ satisfies $\forall y : F(x_0, \ldots, x_{m-1}, y, x_{m+1}, \ldots) = y * F'(x_0, \ldots, x_{m-1}, x_{m+1}, \ldots)$, and the corresponding equations for functions $F^{\mathbb{B}}$, $F'^{\mathbb{B}}$ etc. on $\mathbb{B}$ hold for all $x_i, y \in \mathbb{B}$. Furthermore, by Observation 6.8 there exists a canonically defined multiplication $\cdot$ which turns $\mathbb{B}$ into an associative R-algebra $(\mathbb{B}, +, \cdot)$ with 1.

The axioms of τ_R-bodies are equations. Hence the class of τ_R-bodies constitutes a variety, and this class is closed under taking cartesian products, subobjects and quotients. Moreover, for each non-empty set Y there exists a *free* τ_R-body $\mathbb{B}_Y$ for the skeleton M_Y from the last section with $Y \subseteq \mathbb{B}_Y$. If $\mathbb{B}_{Y'}$ is another free body generated by Y', then any bijection between these sets Y, Y' induces a body isomorphism. Hence free τ_R-bodies are (up to isomorphism) uniquely determined by $|Y|$. To make this precise, we choose one of the (equivalent) definitions for freeness (for varieties). The τ_R-body $\mathbb{B}$ is free (as a body) if there is a subset $Y \subseteq \mathbb{B}$ such that any map $\varphi : Y \to \mathbb{B}'$ into another τ_R-body $\mathbb{B}'$ extends uniquely to a body homomorphism (also called) $\varphi : \mathbb{B} \to \mathbb{B}'$. The set Y is a family of *free generators* of $\mathbb{B}$, which we also call a (body) *basis* of $\mathbb{B}$.

We recall that a map $\varphi : \mathbb{B} \to \mathbb{B}'$ between two τ_R-bodies $\mathbb{B}, \mathbb{B}'$ is a τ_R-body homomorphism if the following holds:

- $\varphi : {}_R\mathbb{B} \to {}_R\mathbb{B}'$ is an R-module homomorphism.
- φ is compatible with all function symbols $F \in \mathcal{F}$: clearly the map φ induces a canonical map $\mathbb{B}^n \to \mathbb{B}'^n$ acting componentwise like φ, which we also denote by φ. If $F \in \mathcal{F}$ is a function symbol and $F^{\mathbb{B}} : \mathbb{B}^n \to \mathbb{B}$, $F^{\mathbb{B}'} : \mathbb{B}'^n \to \mathbb{B}'$ are the corresponding functions on $\mathbb{B}$ and $\mathbb{B}'$, respectively, then φ satisfies the equation $F^{\mathbb{B}}\varphi = \varphi F^{\mathbb{B}'}$.

The above remarks will follow immediately from the next

Proposition 7.1. *If Y is a non-empty set, then the τ_R-body $\mathbb{B}_Y = \mathrm{Lin}_R\, M_Y$ is a free τ_R-body with basis Y and $(\mathbb{B}_Y, +, \cdot)$ is a non-commutative R-algebra. If Y is infinite, then $(\mathbb{B}_Y, +, \cdot)$ is also faithful.*

Proof. The multiplication $\cdot$ on $\mathbb{B}_Y$ is obviously non-commutative by Lemma 6.7, while for the proof that the $*$-scalar product is faithful we refer to Remark 6.6(iv) [and Main

Theorem 6.5]. Thus it remains to prove freeness and we recall Definition 3.3 for $\mathrm{Lin}_R\, M_Y$ and $Y \subseteq M_Y \subseteq \mathbb{B}_Y$.

We must show that any map $\varphi : Y \to \mathbb{B}'$ into a τ_R-body $\mathbb{B}'$ extends uniquely to a body homomorphism. If $m \in \mathbb{B}_Y$, then we must define $m\varphi \in \mathbb{B}'$. Using multi-linearity it is enough to define $\mathbf{t}_Y\varphi$ for elements $\mathbf{t}_Y = (\mathbf{t}, f^{\mathbf{t}}) \in M_Y$, which are the reduced trees from Definition 6.1.

If $\mathbf{t}_Y$ is represented by the reduced tree $\mathbf{t}$ with $P_4^{\mathbf{t}} = \langle \eta_0, \dots, \eta_{n-1}\rangle$ and the coloring map $f^{\mathbf{t}} : P_4^{\mathbf{t}} \to Y$, then $\mathbf{t}_Y = F_Y^{\mathbf{t}\,\mathrm{id}}(f^{\mathbf{t}}(\eta_0), \dots, f^{\mathbf{t}}(\eta_{n-1}))$ with the help of Definition 6.2 and we let

$$\mathbf{t}_Y\varphi = (F^{\mathbf{t}\,\mathrm{id}})^{\mathbb{B}'}(f^{\mathbf{t}}(\eta_0)\varphi, \dots, f^{\mathbf{t}}(\eta_{n-1})\varphi).$$

It is now easy to show that the extended map φ is well-defined and a uniquely determined body homomorphism. □

We apply the last proposition to

Corollary 7.2. *Let $\mathbb{B}_Y$ be a free τ_R-body with (non-empty) basis Y and $Y' \subseteq \mathbb{B}_Y$ be a set of linearly independent elements of the underlying R-module ${}_R\mathbb{B}_Y$ such that*

$$|Y'| = |Y| \quad \text{and} \quad {}_R\langle Y\rangle = {}_R\langle Y'\rangle \text{ as } R\text{-submodules of } {}_R\mathbb{B}_Y.$$

Then Y' is a τ_R-body basis of $\mathbb{B}_Y$.

Proof. There is a bijection $\varphi : Y \to Y'$ which gives rise to an R-automorphism of ${}_R\langle Y\rangle$. Now from Proposition 7.1 both maps φ and φ^{-1} extend uniquely to a pair of mutually inverse body endomorphisms of $\mathbb{B}_Y$. Therefore Y' is an automorphic image of the body basis Y and thus itself a body basis of $\mathbb{B}_Y$. □

By Corollary 7.2 with $d \in Y$ and $Y' = (Y \setminus \{d\}) \cup \{d+e\}$ where $e \in {}_R\langle Y \setminus \{d\}\rangle$ it follows that Y' is a body basis. The next claim strengthens this a little.

Lemma 7.3. *Let $\mathbb{B}_Y$ be a free τ_R-body with (non-empty) basis Y, $d \in Y$ and $e \in \mathbb{B}_{Y\setminus\{d\}} \subseteq \mathbb{B}_Y$. Then $Y' = (Y \setminus \{d\}) \cup \{d + e\}$ is a τ_R-body basis of $\mathbb{B}_Y$.*

Proof. We have $\mathbb{B}_Y = \mathrm{Lin}_R\, M_Y$ and $\mathbb{B}_{Y\setminus\{d\}} = \mathrm{Lin}_R\, M_{Y\setminus\{d\}} \subseteq \mathbb{B}_Y$. Define the bijection $\varphi : Y \to Y'$ by $d\varphi = d + e$ and $y\varphi = y$ for $y \in Y \setminus \{d\}$ and similarly a second map $\psi : Y \to \mathbb{B}_Y$ by $d\psi = d - e$ and $y\psi = y$ for $y \in Y \setminus \{d\}$. By Proposition 7.1 both maps φ and ψ extend uniquely to body endomorphisms of $\mathbb{B}_Y$ with

$$\varphi{\restriction}\mathbb{B}_{Y\setminus\{d\}} = \psi{\restriction}\mathbb{B}_{Y\setminus\{d\}} = \mathrm{id}_{\mathbb{B}_{Y\setminus\{d\}}}.$$

Now obviously $(d - e)\varphi = (d + e)\psi = d$ and $\varphi\psi{\restriction}Y = \psi\varphi{\restriction}Y = \mathrm{id}_Y$, and consequently φ and ψ are mutually inverse body automorphisms of $\mathbb{B}_Y$. Therefore Y' is an automorphic image of the body basis Y and thus itself a body basis of $\mathbb{B}_Y$. □

8. The step lemma

We now prepare those skeletons and bodies implemented into the Step Lemmas.

Notation 8.1. (i) We will consider free bodies $\mathbb{B}' \subseteq \mathbb{B}$ with a basis Y of $\mathbb{B}$ such that $Y \cap \mathbb{B}'$ is a basis of $\mathbb{B}'$. If this is the case, then we will say that $\mathbb{B}$ is *free over* $\mathbb{B}'$.

(ii) Suppose that $\mathbb{B}$ is a body and $S \subseteq \mathbb{B}$ is a subset. The subbody generated by S is the image $\langle S\rangle_{\mathbb{B}} \subseteq \mathbb{B}$ under the substitution map $\varphi : \mathbb{B}_X \to \mathbb{B}$ with $x_s \mapsto s$ $(s \in S)$, where $X = \langle x_s \mid s \in S\rangle$ is a set of free variables.

(iii) We say that for free bodies $\mathbb{B}' := \mathbb{B}_{Y'}$, $\mathbb{B} := \mathbb{B}_Y$ with $\mathbb{B}' \subseteq \mathbb{B}$ an element $g \in \mathbb{B}$ is *free* over $\mathbb{B}'$ if the body $\langle \mathbb{B}', g\rangle_{\mathbb{B}} = \langle Y', g\rangle_{\mathbb{B}}$ is free with body basis $Y' \cup \{g\}$.

It follows that the free body $\mathbb{B}_Y$ is a free R-module, thus also cotorsion-free. Moreover, $\mathbb{B}_Y$ naturally is a pure R-submodule of its p-adic completion $\widehat{\mathbb{B}}_Y$ which is p-adically closed, thus by continuity we are allowed to build infinite p-adic sums in $\widehat{\mathbb{B}}_Y$. Also by continuity R-linear functions on $\mathbb{B}_Y$, in particular all functions coming from the function symbols in $\mathcal{F}$, extend uniquely to $\widehat{\mathbb{B}}_Y$, which thus becomes a $\tau_{\widehat{R}}$-body and $\mathbb{B}_Y$ is also a τ_R-subbody of $\widehat{\mathbb{B}}_Y$. Furthermore, for Y uncountable the $*$-multiplication on $\widehat{\mathbb{B}}_Y$ is obviously faithful again as witnessed by the action of $*$ on $Y \subseteq M_Y \subseteq \widehat{\mathbb{B}}_Y$.

We now fix our notation for the construction using the diamond principle which will be used throughout this section.

(i) Let $Y_\omega = \bigcup_{n<\omega} Y_n$ be a strictly increasing sequence of infinite sets Y_n of free variables and fix a sequence $\overline{y}$ of elements $y_k \in Y_{k+1} \setminus Y_k$ $(k < \omega)$.

(ii) Let $M_n := M_{Y_n}$ be the free skeleton over Y_n for $n \leq \omega$.

(iii) Let $\mathbb{B}_n := \mathrm{Lin}_R\, M_{Y_n}$ be the corresponding free τ_R-body and $G_n := {}_R\mathbb{B}_n$ be the underlying R-module for $n \leq \omega$.

(iv) For $\overline{y}$ from (i), $b \in \mathbb{B}_0$, $\pi = \sum_{k<\omega} p^k r_k \in \widehat{R}$, $r_k \in R$ and $n < \omega$ we define

$$\pi_n = \sum_{k\geq n} p^{k-n} r_k, \quad w_n = w_n(\overline{y}) = \sum_{k\geq n} p^{k-n} y_k \in \widehat{\mathbb{B}}_\omega, \quad v_n = \pi_n b + w_n \tag{8.1}$$

and set $v = v_0$ and $w = w_0$.

It is easy to check that for all $n < \omega$,

$$w_n - p w_{n+1} = y_n \in G_{n+1} \setminus G_n \quad \text{and} \quad v_n - p v_{n+1} = y_n + r_n b \in G_{n+1}. \tag{8.2}$$

Using these notations, we first prove a sequence of lemmata analyzing the algebraic structure of the bodies needed for the forthcoming generalized $E(R)$-algebras. We begin with a proposition related to the diamond case showing that freeness can be maintained during the construction.

Proposition 8.2. *Let $\overline{y}$ be as above and $Y := (Y_\omega \setminus \{y_i \mid i < \omega\}) \cup V$ with $V := \{v_i \mid i < \omega\}$. If σ denotes the substitution $y_i \mapsto v_i$ for $i < \omega$ and $\mathbb{B}_{\omega+1} := \mathbb{B}_\omega \sigma$, $G_{\omega+1} := {}_R\mathbb{B}_{\omega+1}$, then the following holds.*

(a) $G_\omega \subseteq_* G_{\omega+1} \subseteq_* \widehat{G}_\omega$.

(b) $G_{\omega+1} = {}_R(\langle G_\omega, v\rangle_{\mathbb{B}})_* \subseteq \widehat{G}_\omega$.

(c) *$G_{\omega+1}/G_\omega$ is p-divisible.*
(d) *The τ_R-body $\mathbb{B}_{\omega+1} \subseteq \widehat{\mathbb{B}}_\omega$ is freely generated by Y, thus $\mathbb{B}_{\omega+1} \cong \operatorname{Lin}_R M_Y$ canonically.*
(e) *$\mathbb{B}_{\omega+1}$ is free over $\mathbb{B}_n$ (as body) for any $n < \omega$.*

Proof. (a) We have the canonical τ_R-bodies $\mathbb{B}_\omega$ and $\widehat{\mathbb{B}}_\omega$. If we now substitute $y_i \mapsto v_i \in \widehat{\mathbb{B}}_\omega$ for $i < \omega$ and $y \mapsto y \in \widehat{\mathbb{B}}_\omega$ for $y \in Y_\omega \setminus \{y_i \mid i < \omega\}$, then (by Proposition 7.1) we obtain a body homomorphism $\sigma : \mathbb{B}_\omega \to \widehat{\mathbb{B}}_\omega$, and $\mathbb{B}_{\omega+1} = \mathbb{B}_\omega\sigma \subseteq \widehat{\mathbb{B}}_\omega$ is obviously a τ_R-subbody. Moreover, every basis element of the free R-module G_ω is a reduced tree $\mathbf{t}_{Y_\omega} = (\mathbf{t}, f^{\mathbf{t}})$ with $P_4^{\mathbf{t}} = \langle \eta_0, \ldots, \eta_{n-1}\rangle$, a coloring map $f^{\mathbf{t}} : P_4^{\mathbf{t}} \to Y_\omega$ as discussed in Section 6 (see Definition 6.1) and

$$\mathbf{t}_{Y_\omega}\sigma = F_{Y_\omega}^{\mathbf{t}\,\mathrm{id}}(f^{\mathbf{t}}(\eta_0)\sigma, \ldots, f^{\mathbf{t}}(\eta_{n-1})\sigma) \in \widehat{G}_\omega.$$

This representation of $\mathbf{t}_{Y_\omega}\sigma$ naturally corresponds to the element $(\mathbf{t}, f^{\mathbf{t}}\sigma)$ from the free τ_R-body $\operatorname{Lin}_R M_Y$. Now, on the other hand, we identify $Y \subseteq \widehat{\mathbb{B}}_\omega$. Using (8.1), by multilinearity and continuity the term $\mathbf{t}_{Y_\omega}\sigma$ can also be expressed as a p-adic sum of basis elements in G_ω, while clearly by (8.2) the elements of G_ω belong to $G_{\omega+1}$. Thus

$$G_\omega \subseteq G_{\omega+1} \subseteq \widehat{G}_\omega \quad \text{as } R\text{-modules.} \tag{8.3}$$

Next we note that the inclusions (8.3) are pure R-submodules. By definition of the p-adic topology clearly G_ω is a p-pure R-submodule of $\widehat{G}_\omega$, thus $G_\omega \subseteq G_{\omega+1}$ is p-pure. If $g \in G_{\omega+1}$ and $ph = g$ for some $h \in \widehat{G}_\omega$, then using (8.2) we can write $g = g' + pg''$ with $g' \in G_\omega$, $g'' \in G_{\omega+1}$, thus $g' = p(h - g'') \in G_\omega$ and by the p-purity of $G_\omega \subseteq \widehat{G}_\omega$ we have $h' = h - g'' \in G_\omega$. Now $g = g' + pg'' = p(h' + g'')$ with $h' + g'' \in G_{\omega+1}$, so $G_{\omega+1} \subseteq \widehat{G}_\omega$ is p-pure as well and (a) is shown.

(b) An argument similar to (a) shows the equality

$$G_{\omega+1} = {}_R\mathbb{B}_{\omega+1} = {}_R(\langle Y\rangle_{\mathbb{B}}) = {}_R(\langle Y_\omega, V\rangle_{\mathbb{B}}) = {}_R(\langle G_\omega, V\rangle_{\mathbb{B}}) = {}_R(\langle G_\omega, v\rangle_{\mathbb{B}})_*.$$

(c) By the p-adic completion it is obvious that $\widehat{G}_\omega/G_\omega$ is p-divisible (by density). From (a) and (b) it follows that $G_{\omega+1}/G_\omega \subseteq \widehat{G}_\omega/G_\omega$ is p-pure, hence p-divisible as well.

(d) For any $n < \omega$ let φ_n be the substitution $y_n \mapsto v_n$. We now claim that

$$Y' := (Y_\omega \setminus \{y_n\}) \cup \{v_n\} \quad \text{is a free body basis of } \mathbb{B}_\omega\varphi_n. \tag{8.4}$$

We will follow the proof of Lemma 7.3 and begin with the canonical τ_R-bodies $\mathbb{B}_\omega = \operatorname{Lin}_R M_{Y_\omega}$, $\mathbb{B}_{Y_\omega\setminus\{y_n\}} = \operatorname{Lin}_R M_{Y_\omega\setminus\{y_n\}} \subseteq \mathbb{B}_\omega$, $\widehat{\mathbb{B}}_\omega$ and $\widehat{\mathbb{B}}_{Y_\omega\setminus\{y_n\}} \subseteq \widehat{\mathbb{B}}_\omega$. Moreover, by (8.2) we have $y_n + r_n b + pv_{n+1} = v_n$ with $r_n b + pv_{n+1} \in \widehat{\mathbb{B}}_{Y_\omega\setminus\{y_n\}}$.

Define the bijection $\varphi_n : Y_\omega \to Y'$ by $y_n\varphi_n = y_n + r_n b + pv_{n+1} = v_n$ and $y\varphi_n = y$ for $y \in Y_\omega \setminus \{y_n\}$ and similarly we find another map $\psi_n : Y_\omega \to \widehat{\mathbb{B}}_\omega$ with $y_n\psi_n = y_n - r_n b - pv_{n+1}$ and $y\psi_n = y$ for $y \in Y_\omega \setminus \{y_n\}$. By Proposition 7.1 both maps φ_n and ψ_n extend uniquely to body homomorphisms $\mathbb{B}_\omega \to \widehat{\mathbb{B}}_\omega$ that by continuity extend further to body endomorphisms $\widehat{\mathbb{B}}_\omega \to \widehat{\mathbb{B}}_\omega$ with

$$\varphi_n\restriction\widehat{\mathbb{B}}_{Y_\omega\setminus\{y_n\}} = \psi_n\restriction\widehat{\mathbb{B}}_{Y_\omega\setminus\{y_n\}} = \mathrm{id}_{\widehat{\mathbb{B}}_{Y_\omega\setminus\{y_n\}}}.$$

Now obviously $(y_n - r_n b - pv_{n+1})\varphi_n = (y_n + r_n b + pv_{n+1})\psi_n = y_n$ and $\varphi_n\psi_n\restriction Y_\omega = \psi_n\varphi_n\restriction Y_\omega = \mathrm{id}_{Y_\omega}$, and consequently φ_n and ψ_n are mutually inverse body automorphisms of $\widehat{\mathbb{B}}_Y$. Thus $\mathbb{B}_\omega\varphi_n$ is a free τ_R-body with associated body basis $Y\varphi_n = Y'$ and (8.4) holds.

Combining Lemma 7.3, (8.2) and (8.4) shows immediately that for any $n < \omega$ the set $(Y_\omega \setminus \{y_i \mid i \le n\}) \cup \{v_i \mid i \le n\}$ is a body basis of $\mathbb{B}_\omega\varphi_n$. Using the fact that the body basis property is of finite character, it now easily follows that Y is a body basis of $\mathbb{B}_\omega\sigma$ as required.

(e) By Proposition 7.3, (8.2) and (c) the set

$$Y'_n := (Y_\omega \setminus \{y_i \mid n \le i < \omega\}) \cup \{v_i \mid n \le i < \omega\}$$

is a body basis of $\mathbb{B}_{\omega+1}$ for $n < \omega$, where $Y_n \subseteq Y'_n$. $\square$

Proposition 8.2 can be generalized to a non-free setting. Thus it is convenient to ease our notations.

Notation 8.3. (i) If $\mathbb{B}_Y$ is a τ_R-body freely generated by Y, then (as before) $\mathbb{B}_Y = \mathrm{Lin}_R\, M_Y$ and the elements m of its skeleton M_Y (which are trees) will be called *body monomials* (or body R-monomials if we consider rm with $r \in R$). All other elements of $\mathbb{B}_Y$ which are sums of R-monomials will be called *body polynomials* and allow a unique representation as reduced sum (i.e. as sum of minimal length) of body R-monomials.

(ii) If $\mathbb{B}_\omega \subseteq \mathbb{B} \subseteq \widehat{\mathbb{B}}_\omega$ and $v \in \widehat{\mathbb{B}}_\omega$, then we will write $\mathbb{B}[v] = \langle \mathbb{B}, v\rangle_{\mathbb{B}}$ for the subbody of $\widehat{\mathbb{B}}_\omega$ generated by $\mathbb{B}$ and v.

(iii) As mentioned in Notation 8.1, for $S \subseteq \mathbb{B}$ the subbody generated by this set is $\langle S\rangle_{\mathbb{B}} \subseteq \mathbb{B}$. It is induced by a free body $\mathbb{B}_X$ through the substitution map φ with $x_s \mapsto s$ ($s \in S$). If $m = (\mathbf{s}, f^{\mathbf{s}})$ is a body monomial of $\mathbb{B}_X$, then we have the coloring map $f^{\mathbf{s}} : P_4^{\mathbf{s}} = \langle \eta_0, \dots, \eta_{n-1}\rangle \to X$ and with a mild abuse of notation we will write $m = m(x_{s_0}, \dots, x_{s_{n-1}})$ to emphasize the colors $x_{s_i} = f^{\mathbf{s}}(\eta_i)$ contributing to the colored tree m. This expression also agrees with the representation

$$m = F_X^{\mathbf{s}\,\mathrm{id}}(f^{\mathbf{s}}(\eta_0), \dots, f^{\mathbf{s}}(\eta_{n-1})) = m(x_{s_0}, \dots, x_{s_{n-1}})$$

resulting from Definition 6.2 and the canonical identification $X \subseteq \mathbb{B}_X$. In this new notation we will write

$$m\varphi = (F^{\mathbf{s}\,\mathrm{id}})^{\mathbb{B}}(f^{\mathbf{s}}(\eta_0)\varphi, \dots, f^{\mathbf{s}}(\eta_{n-1})\varphi) = m(s_0, \dots, s_{n-1})$$

and call $m\varphi$ a *body monomial* in the generators s_i of the subbody $\langle S\rangle_{\mathbb{B}} \subseteq \mathbb{B}$. Replacing id by an arbitrary permutation σ it is possible to change the order of the generators s_i.

Notation 8.3(ii) should remind us of rings $R[x]$ with an element x adjoined. However, we must keep in mind that the two notions are fundamentally distinct. But nevertheless, it will help us as a guideline for arguments.

Now the next two lemmata will generalize our results from Proposition 8.2 on the p-purification of bodies. For this we still apply our notation for the diamond case.

Lemma 8.4. *Let $\mathbb{B}_\omega \subseteq \mathbb{B} \subseteq \widehat{\mathbb{B}}_\omega$ and $\mathbb{B}$ be a body. If $\mathbb{B}_* = \{b \in \widehat{\mathbb{B}}_\omega \mid \exists n < \omega$ with $p^n b \in \mathbb{B}\}$ denotes the p-purification of $\mathbb{B}$, then $\mathbb{B}_*$ is a subbody of $\widehat{\mathbb{B}}_\omega$.*

Proof. We must show that $\mathbb{B}_*$ is closed under the functions arriving from the function symbols in our body theory T_R (see Definition 3.1). Let $F \in \mathcal{F}$ be such an additional function symbol of arity $\mathrm{ar}(F) = i_*$. If $m_i \in \mathbb{B}_*$ for $i < i_*$, then there is $n < \omega$ such that $p^n m_i \in \mathbb{B}$ for all $i < i_*$. By the multi-linearity of F it follows that

$$p^{ni_*} F^{\widehat{\mathbb{B}}_\omega}(m_0, \dots, m_{i_*-1}) = F^{\widehat{\mathbb{B}}_\omega}(p^n m_0, \dots, p^n m_{i_*-1}) \in \mathbb{B}.$$

Thus also $F^{\widehat{\mathbb{B}}_\omega}(m_0, \dots, m_{i_*-1}) \in \mathbb{B}_*$. □

Lemma 8.5. *Let $\mathbb{B}_\omega \subseteq \mathbb{B} \subseteq_* \widehat{\mathbb{B}}_\omega$ and $\mathbb{B}$ be a body. Moreover, let $\overline{y}$ be the sequence from* (i) *above, $\pi \in \widehat{R}$ and $v = \pi b + w \in \widehat{\mathbb{B}}_\omega$ also be as above. Then*

$$\mathbb{B}[v]_* = \mathbb{B}[v_i \mid i < \omega] \subseteq_* \widehat{\mathbb{B}}.$$

Proof. From Lemma 8.4 it follows that $\mathbb{B}[v]_*$ is a body. If $c \in \mathbb{B}[v]_*$, there is $k < \omega$ with $p^k c \in \mathbb{B}[v]$. By (8.2) we can write $v = v' + p^k v_k$ for some $v' \in \mathbb{B}$. The element $p^k c \in \mathbb{B}[v]$ can be represented as a finite linear combination of body monomials $m(\overline{v}^j, b_0, \dots, b_{n-1})$ in v and some additional generators $b_i \in \mathbb{B}$ (see Notation 8.3). So we can substitute for v the above sum and by multi-linearity we can express this v-sum of monomials by a similar v'-sum living in $\mathbb{B}$ plus a new summand $p^k c'$ with $c' \in \mathbb{B}[v_i \mid i < \omega]$. Thus $p^k(c - c') \in \mathbb{B} \subseteq_* \widehat{\mathbb{B}}_\omega$, and $c - c' \in \mathbb{B}$ follows by purity. Hence also $c \in \mathbb{B}[v_i \mid i < \omega]$. This shows the desired equality. □

Next we introduce the notations of the Black Box construction which constitute an obvious generalization of the setting in the diamond case.

(1) Let Y be an infinite set of free variables that is well-ordered by some limit ordinal and let α be an ordinal number.
(2) Let $M := M_Y$ be the free skeleton over Y, $\mathbb{B}_Y := \mathrm{Lin}_R\, M_Y$ be the corresponding free τ_R-body and $G_Y := {}_R\mathbb{B}_Y$ be the underlying R-module.
(3) For every $\beta \le \alpha$ let $\overline{y}_\beta = \{y_n^\beta \mid n < \omega\} \subseteq Y$ be a strictly increasing sequence with respect to the well-ordering. For $b_\beta \in \mathbb{B}_Y$ define the divisibility chains $\pi_n^\beta, w_n^\beta, v_n^\beta$ as in (8.1).
(4) For $\beta_1 \ne \beta_2 \le \alpha$ let
$$\{y_n^{\beta_1} \mid n < \omega\} \cap \{y_n^{\beta_2} \mid n < \omega\} \quad \text{be finite.} \tag{8.5}$$
(5) For $g \in \widehat{\mathbb{B}}_Y$ a support $[g]_Y \subseteq Y$ is well-defined by
$$[g]_Y = \bigcap \{Y' \subseteq Y \mid g \in \widehat{\mathbb{B}}_{Y'} \subseteq \widehat{\mathbb{B}}_Y\}.$$

Now we can formulate and prove the $\aleph_1$-free counterpart of Proposition 8.2 for the Black Box construction.

Proposition 8.6. *Let R be a cotorsion-free commutative ring with* 1*, and let Y and $\overline{y}_\beta$ $(\beta < \alpha)$ be as above. Then the body $\mathbb{B} := \mathbb{B}_Y[v_\beta \mid \beta < \alpha] \subseteq \widehat{\mathbb{B}}_Y$ is freely generated by $Y \cup \{v_\beta \mid \beta < \alpha\}$ and the R-module $G := {}_R\mathbb{B}$ is free.*

Proof. We only sketch the arguments as they are very similar to the preceding proofs.

Let $X = \{x_\beta \mid \beta < \alpha\}$ be a set of additional free variables and let $\mathbb{B}_{Y\cup X}$ be the induced free body. We now have to show that the substitution $\varphi : \mathbb{B}_{Y\cup X} \to \mathbb{B}$ with $y \mapsto y$ $(y \in Y)$ and $x_\beta \mapsto v_\beta$ $(\beta < \alpha)$ is a body isomorphism, in particular it is injective. As this is a property of finite character, we can restrict ourselves to proving the finite case $\alpha < \omega$.

Thus let $\alpha < \omega$ be finite. Using (8.5) we can choose some $m < \omega$ with

$$\{y_n^{\beta_1} \mid m \le n < \omega\} \cap \{y_n^{\beta_2} \mid m \le n < \omega\} \cap \bigcup_{\beta<\alpha} [b_\beta]_Y = \emptyset \quad \text{for all } \beta_1 \ne \beta_2 < \alpha. \tag{8.6}$$

Set $\mathbb{B}' := \mathbb{B}_Y[v_\beta \mid \beta < \alpha]_*$. Then $\mathbb{B}' = \mathbb{B}_Y[v_n^\beta \mid \beta < \alpha, m \le n < \omega]$ (Lemma 8.5) and similarly to Proposition 8.2 by (8.6) the body $\mathbb{B}'$ is free having the body basis

$$Y' = (Y \setminus \{y_n^\beta \mid \beta < \alpha,\ m \le n < \omega\}) \cup \{v_n^\beta \mid \beta < \alpha,\ m \le n < \omega\}. \tag{8.7}$$

Now from (8.7) it follows that $(Y \setminus \{y_n^\beta \mid \beta < \alpha,\ m \le n < \omega\}) \cup \{v_m^\beta \mid \beta < \alpha\} \subseteq Y'$ and

$$(Y \setminus \{y_n^\beta \mid \beta < \alpha,\ m \le n < \omega\}) \cup \{p^m v_m^\beta \mid \beta < \alpha\} \tag{8.8}$$

are body bases of their induced bodies, respectively. Furthermore, combining (8.8) with (8.2), (8.6) and Lemma 7.3, also

$$(Y \setminus \{y_n^\beta \mid \beta < \alpha, m \le n < \omega\}) \cup \{v_\beta \mid \beta < \alpha\} \quad \text{is a body basis} \tag{8.9}$$

of its induced body. Since the body basis property is of finite character and (8.6) and in its wake (8.9) actually hold for every $m < \omega$ large enough, we finally conclude that $Y \cup \{v_\beta \mid \beta < \alpha\}$ is a body basis of its induced body $\mathbb{B}$. □

Similar to Proposition 8.2, it is not $\mathbb{B} := \mathbb{B}_Y[v_\beta \mid \beta < \alpha] \subseteq \widehat{\mathbb{B}}_Y$ but its purification $\mathbb{B}' := \mathbb{B}_* \subseteq \widehat{\mathbb{B}}_Y$ which is our main object of interest. We will start studying $\mathbb{B}'$ and its properties by introducing a refined support inspired from the proof of Proposition 8.6.

Notation 8.7. (i) Let B_X be a free body with body basis X. We recall ${}_R\mathbb{B}_X = \bigoplus_{t\in M_X} Rt$ from Definition 3.3, thus ${}_R\widehat{\mathbb{B}}_X \subseteq \prod_{t\in M_X} \widehat{R}t$ and every element $g \in \widehat{\mathbb{B}}_X$ has a unique representation $g = \sum_{t\in M_X} r_t t$ with coefficients $r_t \in \widehat{R}$. We will call $r_t t$ the *t-component* of g and r_t the *t-coefficient*.

(ii) In Proposition 8.6 we only use support arguments. Therefore the τ_R-body isomorphism $\varphi : \mathbb{B}_{Y\cup X} \to \mathbb{B}$ extends uniquely to a canonical $\tau_{\widehat{R}}$-body isomorphism $\varphi : \langle \mathbb{B}_{Y\cup X}\rangle_{\widehat{R}} \to \langle \mathbb{B}\rangle_{\widehat{R}}$ of the related $\tau_{\widehat{R}}$-bodies and we may again equate coefficients. The proofs of Lemma 8.9 and Step Lemma 8.10 will make extensive use of this fact.

Our aim now is to show that $\mathbb{B}' := \mathbb{B}_Y[v_\beta \mid \beta < \alpha]_* \subseteq \widehat{\mathbb{B}}_Y$ has a cotorsion-free R-module structure. For this we recall Definition 1.1 and state the following crucial

Corollary 8.8. *If R is a cotorsion-free commutative ring with* 1*, then any $\aleph_0$-free R-module is cotorsion-free.*

Proof. Let $\sigma : \widehat{R} \to M$ be an R-module homomorphism into the $\aleph_0$-free R-module M and choose a free and p-pure submodule $F \subseteq M$ with $1\sigma \in F$ and p-reduced quotient M/F. Now with $\widehat{R}/R$ also $(\widehat{R}\sigma + F)/F$ is p-divisible, while as a submodule of M/F it is also p-reduced. Hence $\widehat{R}\sigma \subseteq F$ which is free, and therefore cotorsion-free by assumption on R. Thus $\widehat{R}\sigma = 0$ follows and M is cotorsion-free. □

Now everything is prepared for the proof of

Lemma 8.9. *Let R be a cotorsion-free commutative ring with* 1*, and let Y and $\overline{y}_\beta$ $(\beta < \alpha)$ be as above. Then for $\mathbb{B} := \mathbb{B}_Y[v_\beta \mid \beta < \alpha] \subseteq \widehat{\mathbb{B}}_Y$ and $\mathbb{B}' := \mathbb{B}_* \subseteq \widehat{\mathbb{B}}_Y$ the R-module $G' := {}_R\mathbb{B}'$ is $\aleph_0$-free* (*and thus cotorsion-free*).

Proof. For every finite subset $S \subseteq G'$ there exist finite sets $\emptyset \neq Y' \subseteq Y$ and $I \subseteq \alpha$ with $S \subseteq \mathbb{B}_{Y'}[v_\beta \mid \beta \in I]_* \subseteq \mathbb{B}'$. Set

$$Y'' := Y' \cup \bigcup_{\beta \in I} [v_\beta]_Y \subseteq Y. \tag{8.10}$$

Then also $S \subseteq \mathbb{B}'' := \mathbb{B}_{Y''}[v_\beta \mid \beta \in I]_* \subseteq \mathbb{B}'$. Furthermore, by (8.7) the body $\mathbb{B}''$ is free and thus $F := {}_R\mathbb{B}''$ is a free and p-pure submodule of G'.

It remains to show that the quotient G'/F is p-reduced. For this let $h \in G'$ be such that $h + F$ is p-divisible. Thus there exist elements $g_n \in G'$ and $f_n \in F$ with

$$h = p^n g_n + f_n \quad (n < \omega). \tag{8.11}$$

We now take advantage of Notation 8.7 to compare coefficients of equation (8.11) inside $\widehat{\mathbb{B}}_Y$.

First assume that $h \notin \mathbb{B}_Y[v_\beta \mid \beta \in I]_*$. Then the representation of h needs some v_γ with $\gamma \notin I$. Now from (8.5) and (8.10) it follows that $[v_\gamma]_Y \setminus Y'' \neq \emptyset$ and there exists some non-trivial component in the reduced representation of h that does not contribute to any f_n $(n < \omega)$. By (8.11) the coefficient of this component has to be a non-zero p-divisible element of $\widehat{R}$, a contradiction. Thus $h \in \mathbb{B}_Y[v_\beta \mid \beta \in I]_*$.

Next assume that $h \in \mathbb{B}_Y[v_\beta \mid \beta \in I]_*$, but $h \notin \mathbb{B}'' = \mathbb{B}_{Y''}[v_\beta \mid \beta \in I]_*$. Then the representation of h needs some basis element from $Y \setminus Y'' \neq \emptyset$ and we proceed as before to derive a contradiction. Thus $h \in \mathbb{B}''$ and $h + F = 0$ follows. □

The remaining part of this section is devoted to proving the required Step Lemmas. We will continue using the notations basic for the Black Box construction and prove the appropriate version of the Step Lemma first. Recall that for any body $\mathbb{B}_Y \subseteq \mathbb{B} \subseteq \widehat{\mathbb{B}}_Y$ and any $g \in \mathbb{B}$ we have an induced R-endomorphism $*_g : {}_R\mathbb{B} \to {}_R\mathbb{B}$ $(x \mapsto x * g)$ on ${}_R\mathbb{B}$ by multi-linearity of the $*$-scalar product. Thus, as $*$ is faithful on $\mathbb{B}$ for uncountable sets Y (as seen by its action on $Y \subseteq \mathbb{B}$), we can view these endomorphisms as canonical embedding $\mathbb{B} \subseteq \operatorname{End}_R \mathbb{B}$. Finally, we must consider the remaining endomorphisms in $\operatorname{End}_R \mathbb{B} \setminus \mathbb{B}$. By the following crucial lemma it will be possible to get rid of them.

Step Lemma 8.10. *Let R be a cotorsion-free commutative ring with 1, Y and $\overline{y}_\beta$ $(\beta \leq \alpha)$ be as above, and set $\mathbb{B} = \mathbb{B}_Y[v_\beta \mid \beta < \alpha] \subseteq \widehat{\mathbb{B}}_Y$ and $\mathbb{B}' = \mathbb{B}_Y[v_\beta \mid \beta \leq \alpha] \subseteq \widehat{\mathbb{B}}_Y$. Moreover, let $\{y_n^\alpha \mid n < \omega\} \subseteq Y'$ be a subset of Y and $P := \mathbb{B}_{Y'} \subseteq \mathbb{B}_Y$ be the induced subbody. If $\varphi : P \to \mathbb{B}_* \subseteq \widehat{\mathbb{B}}_Y$ is an R-module homomorphism with $(p^N\varphi - *_g)\restriction P \not\equiv 0$ for all $N < \omega$ and $g \in \mathbb{B}_*$, then we can choose $\pi_\alpha \in \widehat{R}$ and $b_\alpha \in P \subseteq \mathbb{B}_Y$ such that $v_\alpha\varphi \notin \mathbb{B}'_* \subseteq \widehat{\mathbb{B}}_Y$ for $v_\alpha = \pi_\alpha b_\alpha + w_\alpha \in \widehat{P}$.*

Proof. We consider the R-module homomorphism $\varphi : P \to \mathbb{B}_*$, and claim that (for $\overline{y}_\alpha$ as above) there are $b_\alpha \in P$ and $\pi_\alpha \in \widehat{R}$ with $v_\alpha = \pi_\alpha b_\alpha + w_\alpha$ and $v_\alpha\varphi \notin \mathbb{B}'_* = \mathbb{B}[v_\alpha]_* \subseteq \widehat{\mathbb{B}}_Y$. In the opposite case we may assume that for all $b_\alpha \in P$ and $\pi_\alpha \in \widehat{R}$ we have $v_\alpha\varphi \in \mathbb{B}'_*$. Furthermore, to ease notation we will abbreviate from now on $\pi_\alpha, b_\alpha, w_\alpha, v_\alpha$ by π, b, w, v, respectively.

In particular (for $\pi = 0$) it follows that $w\varphi \in \mathbb{B}[w]_*$. By purity we find $N < \omega$ such that

$$p^N w\varphi = \sum_{i\in I} c_i m_i(\overline{w}^{n_i}) \in \mathbb{B}[w] \quad \text{is in reduced form} \tag{8.12}$$

with $c_i \in R$ and body monomials $m_i(\overline{w}^{n_i}) = m_i(\overline{w}^{n_i}, \overline{v}_i, \overline{y}_i)$ in w and finite sequences $\overline{v}_i$ and $\overline{y}_i$ of additional generators from $\{v_\beta \mid \beta < \alpha\}$ and Y, respectively. Let $n = \max\{n_i \mid i \in I\}$ be the maximal number of w's appearing in one of the monomials m_i. Similarly we treat any other $v = \pi b + w$ (but we will specify the actual elements b and π only later); and by assumption we also get $v\varphi \in \mathbb{B}[v]_*$. The same way we find $N' < \omega$ such that

$$p^{N'}(\pi b + w)\varphi = \sum_{i\in I'} d_i m'_i(\overline{v}^{n'_i}) \in \mathbb{B}' = \mathbb{B}[\pi b + w] \quad \text{is in reduced form} \tag{8.13}$$

with $d_i \in R$ and body monomials $m'_i(\overline{v}^{n'_i}) = m'_i(\overline{v}^{n'_i}, \overline{v}'_i, \overline{y}'_i)$, and we set $n' = \max\{n'_i \mid i \in I'\}$. Next we want to exclude

Case $n' > 1$: We may assume $N = N'$, and subtracting (8.12) from (8.13) we get

$$p^N \pi b\varphi = \sum_{i\in I'} d_i m'_i(\overline{v}^{n'_i}) - \sum_{i\in I} c_i m_i(\overline{w}^{n_i}) \in \langle \mathbb{B}[w]\rangle_{\widehat{R}}. \tag{8.14}$$

For the last inclusion we use $v = \pi b + w$ and multi-linearity of the body monomials m_i, m'_i to rewrite the first sum accordingly.

By Proposition 8.6 and Notation 8.7 the body $\mathbb{B}[w]$ is freely generated by $Y \cup \{w, v_\beta \mid \beta < \alpha\}$ and we can compare coefficients of components in (8.14). As $b \in P \subseteq \mathbb{B}_Y$ there will be no non-trivial component on the left-hand side containing the generator w. In the first sum on the right-hand side for every $n'_i = n' > 1$ we have a non-trivial $m'_i(\overline{w}^{n'})$-component that needs to be eliminated by the second sum, thus $n' \leq n$. By symmetry also $n \leq n'$, thus $n = n'$ and we must rule out the possibility $n = n' > 1$.

We first restrict (8.14) to those components with the maximal number $n = n'$ of occurrences of the generator w. We get the new equation

$$0 = \sum_{i\in I',\, n'_i = n} d_i m'_i(\overline{w}^n) - \sum_{i\in I,\, n_i = n} c_i m_i(\overline{w}^n). \tag{8.15}$$

We equate the coefficients of both sums on the right-hand side of (8.15) and get without loss of generality

$$\begin{aligned} &\{i \in I \mid n_i = n\} = \{i \in I' \mid n'_i = n\} =: J, \\ &c_i = d_i \quad \text{and} \quad m_i(\overline{w}^n) = m'_i(\overline{w}^n) \quad \text{for all } i \in J. \end{aligned} \tag{8.16}$$

Next we restrict (8.14) to all those components with exactly $n-1 \geq 1$ entries from w. Again, we use multi-linearity of the body monomials m_i, m'_i to decompose $v = \pi b + w$ and collect summands according to the appearance of w. We get the following equation:

$$\begin{aligned} 0 = &\sum_{i\in J} \sum_{j=0}^{n-1} d_i m'_i(\overline{w}^j, \pi b, \overline{w}^{n-1-j}) + \sum_{i\in I',\, n'_i=n-1} d_i m'_i(\overline{w}^{n-1}) \\ &- \sum_{i\in I,\, n_i=n-1} c_i m_i(\overline{w}^{n-1}). \end{aligned} \tag{8.17}$$

Using again multi-linearity of the body monomials m'_i and continuity (of linear maps in the p-adic topology), we can extract π, and applying (8.16) the first sum becomes

$$\pi \sum_{i\in J} \sum_{j=0}^{n-1} d_i m'_i(\overline{w}^j, b, \overline{w}^{n-1-j}) = \pi \sum_{i\in J} \sum_{j=0}^{n-1} c_i m_i(\overline{w}^j, b, \overline{w}^{n-1-j}).$$

We next want to fix the appropriate choice of $b \in P$ and $\pi \in \widehat{R}$. For this we consider the function

$$\psi : \mathbb{B}[w] \to \mathbb{B}[w], \quad x \mapsto \sum_{i\in J} \sum_{j=0}^{n-1} c_i m_i(\overline{w}^j, x, \overline{w}^{n-1-j}).$$

Observe that ψ depends only on the representation of $w\varphi$ in (8.12), but is independent of $b, \pi, v\varphi$ and (8.13). Moreover, $\psi(x)$ is defined by a $\mathbb{B}[w]$-term over the free τ_R-body $\mathbb{B}[w]$ which is linear and homogeneous in x. Thus by multi-linearity ψ is an R-module endomorphism of $\mathbb{B}[w]$, and from Theorems 6.5 and 3.5 we can write $\psi(x) = x * g$ for some $g \in \mathbb{B}[w]$. Next observe that by our reduced representation (8.12),

$$\psi(w) = n \sum_{i\in J} c_i m_i(\overline{w}^n) \neq 0 \quad \text{for } w \in \widehat{P}.$$

Thus ψ is not the zero endomorphism and in particular there is $b \in P \subseteq \mathbb{B}[w]$ such that $\psi(b) = b * g \neq 0$. Moreover, $b * g \in \mathbb{B}[w]$, and ${}_R\mathbb{B}[w]$ is a free R-module by Proposition 8.6, thus cotorsion-free, and $\widehat{R}(b * g) \nsubseteq \mathbb{B}[w]$. Hence we also find $\pi \in \widehat{R}$ such that $\pi(b * g) \notin \mathbb{B}[w]$. But from (8.17) we derive

$$\pi(b * g) = \pi\psi(b) = \sum_{i\in I,\, n_i=n-1} c_i m_i(\overline{w}^{n-1}) - \sum_{i\in I',\, n'_i=n-1} d_i m'_i(\overline{w}^{n-1}) \in \mathbb{B}[w],$$

which is a contradiction. It remains to consider

Case $n, n' \leq 1$**:** According to the previous case this restriction on the representations (8.12), (8.13) holds for every $b \in P$ and $\pi \in \widehat{R}$. The equations now become much simpler; we have for (8.12) the new expression

$$p^N w\varphi = \sum_{i \in I,\, n_i = 1} c_i m_i(w) + \sum_{i \in I,\, n_i = 0} c_i m_i \in \mathbb{B}[w] \quad \text{in reduced form.} \tag{8.18}$$

Here the second sum does not involve a generator w and therefore is an element of $\mathbb{B}$ while the first sum has only body monomials with exactly one occurrence of w. We now consider the function

$$\psi : \mathbb{B} \to \mathbb{B}, \quad x \mapsto \sum_{i \in I, n_i = 1} c_i m_i(x),$$

and a similar argument to that in Case $n' > 1$ applies. We can write $\psi(x) = x * g$ for some $g \in \mathbb{B}$. As ψ extends to $\widehat{\mathbb{B}}_Y$, by continuity also $\psi(w) = w * g$ and we can write (8.18) as

$$p^N w\varphi = w * g + c \quad \text{for some } c, g \in \mathbb{B}. \tag{8.19}$$

Now by assumption of the lemma, $(p^N \varphi - *_g) \not\equiv 0$ and we can choose $b \in P$ with $0 \neq b(p^N \varphi - *_g) = p^N b\varphi - b * g \in \mathbb{B}$. Furthermore, the R-module ${}_R\mathbb{B}_*$ is cotorsion-free by Lemma 8.9 and there is some $\pi \in \widehat{R}$ with

$$\pi(p^N b\varphi - b * g) \notin \mathbb{B}_*. \tag{8.20}$$

This will be our fixed choice of $b \in P$ and $\pi \in \widehat{R}$. Similarly to (8.19) equation (8.13) becomes

$$p^{N'}(\pi b + w)\varphi = (\pi b + w) * g' + c' \quad \text{for some } c', g' \in \mathbb{B} \text{ and } N' < \omega. \tag{8.21}$$

From (8.19) and (8.21) we derive the difference

$$\begin{aligned} p^{N+N'} \pi b\varphi &= p^{N+N'}(\pi b + w)\varphi - p^{N+N'} w\varphi \\ &= p^N((\pi b + w) * g' + c') - p^{N'}(w * g + c) \\ &= \pi(b * p^N g') + w * (p^N g' - p^{N'} g) + p^N c' - p^{N'} c, \end{aligned}$$

thus

$$w * (p^N g' - p^{N'} g) = \pi(p^{N+N'} b\varphi - b * p^N g') + p^{N'} c - p^N c' \in \langle \mathbb{B} \rangle_{\widehat{R}}.$$

Again we use Notation 8.7 to compare components and conclude $p^N g' = p^{N'} g$ from the left-hand side of the last equation. It follows that

$$p^{N'} \pi(p^N b\varphi - b * g) = \pi(p^{N+N'} b\varphi - b * p^N g') = p^N c' - p^{N'} c \in \mathbb{B}$$

and $\pi(p^N b\varphi - b * g) \in \mathbb{B}_*$, contradicting (8.20). $\square$

We formulate the Step Lemma for the diamond construction of the next section as a direct consequence of Step Lemma 8.10 applied to the special notations of the diamond construction.

Step Lemma 8.11. *Let R be a cotorsion-free commutative ring with* 1, $\mathbb{B}_n := \mathbb{B}_{Y_n}$ ($n \leq \omega + 1$) *be as in Proposition* 8.2 *and* $\varphi \in \operatorname{End}_R \mathbb{B}_\omega$ *be such that* $(p^N\varphi - *_g)\restriction\mathbb{B}_0 \not\equiv 0$ *for all* $N < \omega$ *and* $g \in \mathbb{B}_\omega$. *Then we can choose* $\pi \in \widehat{R}$ *and* $b \in \mathbb{B}_0$ *such that* $v\varphi \notin \mathbb{B}_{\omega+1} = \mathbb{B}_\omega[v]_* \subseteq \widehat{\mathbb{B}}_\omega$ *for* $v = \pi b + w$ *as in* (8.1).

We will apply the next simple proposition to both the diamond construction (Section 9) and the Black Box construction (Section 10) of generalized $E(R)$-algebras. Thus its setting must be adjusted to either case by

Definition 8.12. *Let* $\mathbb{B}$ *be a* τ_R*-body and let* $\varphi \in \operatorname{End}_R \mathbb{B}$ *be an* R*-module homomorphism. A family* $\mathcal{P}$ *of subbodies of* $\mathbb{B}$ *is* useful for φ *if the following conditions hold:*

(i) $\bigcup_{P\in\mathcal{P}} P = \mathbb{B}$.
(ii) $\mathcal{P}$ *is* (upwards) directed, *i.e. for any* $P, P' \in \mathcal{P}$ *there is* $P'' \in \mathcal{P}$ *with* $P \cup P' \subseteq P''$.
(iii) *The* $*$*-scalar product is faithful on* $\mathbb{B}$ *as witnessed by the action on every* $P \in \mathcal{P}$, *i.e. for every pair of distinct elements* $a, b \in \mathbb{B}$ *and* $P \in \mathcal{P}$ *there is* $c \in P$ *such that* $c * a \neq c * b$.
(iv) *For every* $P \in \mathcal{P}$ *there are* $N < \omega$ *and* $g_P \in \mathbb{B}$ *such that* $P(p^N\varphi - *_{g_P}) = 0$.

Proposition 8.13. *Let* $\mathbb{B}$ *be a* τ_R*-body such that* ${}_R\mathbb{B}$ *is torsion-free, let* $\varphi \in \operatorname{End}_R \mathbb{B}$ *and* $\mathcal{P}$ *be a useful family for* φ. *Then there are* (*uniform*) $N < \omega$ *and* $g \in \mathbb{B}$ *such that* $p^N\varphi = *_g$.

Proof. We apply Definition 8.12. If $P, P' \in \mathcal{P}$, then we can choose $P'' \in \mathcal{P}$ with $P \cup P' \subseteq P''$ and note by condition (iv) that there are $N, N', N'' < \omega$ and elements $g_P, g_{P'}, g_{P''} \in \mathbb{B}$ such that

$$P(p^N\varphi - *_{g_P}) = P'(p^{N'}\varphi - *_{g_{P'}}) = P''(p^{N''}\varphi - *_{g_{P''}}) = 0. \tag{8.22}$$

From (8.22) it follows that $x * p^{N''}g_P = p^{N+N''}x\varphi = x * p^N g_{P''}$ for all $x \in P \subseteq P''$. Thus $p^{N''}g_P = p^N g_{P''}$ by condition (iii) and we may assume $N = N''$ and $g_P = g_{P''}$. Applying the same argument for $x \in P' \subseteq P''$ shows that we can choose $N = N'' = N'$ and $g_P = g_{P''} = g_{P'}$ uniformly. □

9. The main construction using the diamond principle

We will next construct the κ-filtration $\bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$ of free τ_R-bodies for application using $\diamondsuit_\kappa E$ for some non-reflecting stationary subset $E \subseteq \kappa^o = \{\alpha < \kappa \mid \operatorname{cf}(\alpha) = \omega\}$.

9.1. *Construction of a κ-filtration of free bodies*

The desired body $\mathbb{B}$ will be constructed as a κ-filtration $\mathbb{B} = \bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$ of free τ_R-bodies $\mathbb{B}_\alpha$ of cardinality $< \kappa$. We choose this κ-filtration such that $|\mathbb{B}_\alpha| = |\alpha| + |R| = |\mathbb{B}_{\alpha+1} \setminus \mathbb{B}_\alpha|$ for all $\alpha < \kappa$, and let $\{\gamma_\alpha \mid \alpha \in E\}$ be the family of Jensen functions given by $\diamondsuit_\kappa E$.

Next we define the free bodies $\mathbb{B}_\alpha$ by transfinite induction and let $\mathbb{B}_0 := \mathbb{B}_Y$ be a free τ_R-body of rank $|Y| = |R|$. All subsequent bodies will be constructed according to the following rules:

(i) If $\alpha < \kappa$ is a limit ordinal, then $\mathbb{B}_\alpha = \bigcup_{\beta<\alpha} \mathbb{B}_\beta$ by continuity.

(ii) Suppose that $\alpha \in E$ and there exists some strictly increasing sequence of ordinals $\alpha_n \in \alpha \setminus E$ $(n < \omega)$ with $\sup_{n<\omega} \alpha_n = \alpha$ such that with $\mathbb{B}_n := \mathbb{B}_{\alpha_n}$ $(n < \omega)$ the hypotheses of Step Lemma 8.11 hold for $\bigcup_{n<\omega} \mathbb{B}_n$ and $\varphi := \gamma_\alpha$. Then we choose a suitable sequence $\overline{y}_\alpha$ as in (8.1) and identify $\mathbb{B}_{\alpha+1}$ with $\mathbb{B}_{\omega+1} = \mathbb{B}_\alpha[v_\alpha]_* \subseteq \widehat{\mathbb{B}}_\alpha$ from Step Lemma 8.11 (so γ_α does not extend to an R-module endomorphism of $\mathbb{B}_{\alpha+1}$).

(iii) Suppose $\alpha < \kappa$ is such that (ii) fails, and that $\mathbb{B}_\alpha := \mathbb{B}_{Y_\alpha}$ is a free τ_R-body with basis Y_α. Then we define $\mathbb{B}_{\alpha+1} := \mathbb{B}_{Y_\alpha \cup \{y_\alpha\}}$ as the free τ_R-body with some new free variable y_α.

Furthermore, the following conditions will hold for the constructed chain $\mathbb{B} = \bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$:

(iv) $\mathbb{B}_\alpha$ is a free τ_R-body of rank $|\mathbb{B}_\alpha| = |\alpha| + |R|$ for all $\alpha < \kappa$.

(v) $\mathbb{B}_\alpha$ is a free τ_R-body over $\mathbb{B}_\beta$ for all $\beta \in \alpha \setminus E$ (see Notation 8.1).

Next we must check by transfinite induction that conditions (iv) and (v) hold indeed. We distinguish the case of limit ordinals α and of successor ordinals $\alpha + 1$:

Let $\alpha + 1$ be a successor ordinal. If (iii) applies for $\mathbb{B}_\alpha$, then (iv) and (v) are immediate. If on the other hand (ii) applies, then Step Lemma 8.11 is designed to guarantee. Condition (iv) is the freeness of $\mathbb{B}_{\omega+1}$ in Proposition 8.2. Condition (v) needs that $\mathbb{B}_{\alpha+1}$ is a free body over $\mathbb{B}_\beta$. However, $\mathbb{B}_\beta \subseteq \mathbb{B}_{\alpha_n}$ for some large enough $n < \omega$. Hence (v) follows from the freeness of $\mathbb{B}_{\alpha_n}$ over $\mathbb{B}_\beta$ (inductively) and the freeness of $\mathbb{B}_{\alpha+1}$ over $\mathbb{B}_{\alpha_n}$ (Proposition 8.2).

Now let α be a limit ordinal. In this case we can choose a strictly increasing, continuous sequence $\alpha_\nu \in \alpha \setminus E$ $(\nu < \mathrm{cf}(\alpha))$ with $\sup_{\nu<\mathrm{cf}(\alpha)} \alpha_\nu = \alpha$. For $\mathrm{cf}(\alpha) = \omega$ this is easily done by choosing only successor ordinals α_ν, while for $\mathrm{cf}(\alpha) > \omega$ we use the fact that E is non-reflecting. Now $\mathbb{B}_\alpha = \bigcup_{\nu<\mathrm{cf}(\alpha)} \mathbb{B}_{\alpha_\nu}$ with the help of (iv) and (v) is a free τ_R-body and free over $\mathbb{B}_\beta$.

Conditions (iv) and (v) in turn guarantee that the construction of the chain $\bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$ follows exclusively the rules of (i) to (iii). Thus we can proceed and obtain $\mathbb{B} = \bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$, which is a κ-filtration of free τ_R-bodies. □

It remains to show that $_R\mathbb{B}$ has only the desired R-module endomorphisms.

9.2. *Proof of the main theorem with $\diamondsuit_\kappa E$*

Main Lemma 9.1. *Let $|R| < \kappa$ be a regular, uncountable cardinal with $\diamondsuit_\kappa E$ for a non-reflecting stationary subset $E \subseteq \kappa^o$. Moreover, let $\mathbb{B} = \bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$ be the κ-filtration of free τ_R-bodies $\mathbb{B}_\alpha$ just constructed and define $A := (\mathbb{B}, +, \cdot)$ as the R-algebra structure of the τ_R-body $\mathbb{B}$. Then A is a strongly κ-free, non-commutative R-algebra of cardinality $|A| = \kappa$ and for every $\varphi \in \mathrm{End}_R A$ there is a unique $g \in A$ such that $x\varphi = x * g$ for all $x \in A$.*

Proof. By conditions (iv) and (v) of the construction, A is a strongly κ-free R-module of cardinality κ (see Definition 1.2) and a non-commutative R-algebra (see Proposition 7.1).

For the last part of the claim let $\{\gamma_\alpha \mid \alpha \in E\}$ be the set of Jensen functions given by $\diamondsuit_\kappa E$. If $\varphi \in \mathrm{End}_R A$, then $C = \{\alpha < \kappa \mid \mathbb{B}_\alpha\varphi \subseteq \mathbb{B}_\alpha\}$ is a cub and $E_\varphi = \{\alpha \in E \mid \varphi\restriction\mathbb{B}_\alpha = \gamma_\alpha\} \cap C$ is a stationary subset of E.

In the first case we assume that there exist $\alpha < \kappa$ and $\alpha < \beta \in E_\varphi$ with $(p^N\varphi - *_g)\restriction\mathbb{B}_\alpha \not\equiv 0$ for all $N < \omega$ and $g \in \mathbb{B}_\beta$. Then Step Lemma 8.11 applies (any strictly increasing sequence of ordinals $\beta_n \in \beta \setminus E$ $(n < \omega)$ with $\sup_{n<\omega} \beta_n = \beta$ and $\alpha < \beta_0$ will do) and $v_\beta\varphi \notin \mathbb{B}_{\beta+1}$ by construction step (ii). But (v) implies that any $\mathbb{B}_\gamma$ $(\beta < \gamma)$ is a free body over $\mathbb{B}_{\beta+1}$, thus $\mathbb{B}_{\beta+1} \subseteq \mathbb{B}$ is the p-adic closure of $\mathbb{B}_\beta$ in $\mathbb{B}$ and therefore $v_\beta\varphi \in \mathbb{B} \cap \widehat{\mathbb{B}}_\beta \subseteq \mathbb{B}_{\beta+1}$, a contradiction.

Now, in the opposite case we may assume that for any $\alpha < \kappa$ there are $N < \omega$ and $g_\alpha \in \mathbb{B}$ such that $\mathbb{B}_\alpha(p^N\varphi - *_{g_\alpha}) = 0$. The family $\mathcal{P} := \{\mathbb{B}_\alpha \mid \alpha < \kappa\}$ constitutes an ascending chain of free τ_R-bodies and with Remark 6.6 and (v) the $*$-scalar product is faithful on $\mathbb{B} = \bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$ as observed by the action on every body basis Y_α of $\mathbb{B}_\alpha$ for $\alpha \in \kappa \setminus E$; thus $\mathcal{P}$ is directed and useful for φ (in the sense of Definition 8.12) and we can apply Proposition 8.13 to find a uniform $N < \omega$ and $g' \in \mathbb{B}$ with $p^N\varphi = *_{g'}$.

For the last step of the proof we apply similar arguments to those for Theorem 6.5: Choosing $\alpha < \kappa$ with $g' \in \mathbb{B}_\alpha$, some body basis Y_α of $\mathbb{B}_\alpha$ and a generator $y \in Y_\alpha$ that is not involved in the reduced representation of g' with respect to Y_α we can show that $p^N \mid y * g'$ implies $p^N \mid g'$ in $\mathbb{B}_\alpha$. Thus there is $g \in \mathbb{B}$ with $p^N g = g'$, and $\varphi = *_g$ follows. □

The final step of the proof of Theorem 1.4 is given in the introduction (Section 1).

10. The main construction in ZFC

10.1. The adjusted Black Box

In order to prove Theorem 1.5 about the existence of generalized $E(R)$-algebras in ZFC, we need a prediction principle which is based on the ordinary axioms of set theory, but nothing else. The obvious candidate is a version of the Black Box, adjusted to the present setting. Hence we only have to explain its basic definitions. Its proof will then be an easy modification of the arguments in Göbel and Trlifaj [20, The General Black Box 9.2.27, pp. 340–351].

In this section we will outline these definitions of the Black Box and discuss those minor modifications relevant in the present setting starting with the used trees.

For any non-empty set U we define the *canonical tree* $T_U = \{\tau : n \to U,\ n < \omega\}$ on U and let $\mathrm{Br}(T_U) = \{\tau : \omega \to U\}$. The ordering on T_U is extension of maps. The element $\tau \in T_U$ $(\tau : n \to U)$ has length $l(\tau) := n$ and the set $\{\tau\restriction n : n \le l(\tau)\}$ of initial segments of τ is called the *finite branch* induced by τ. Similarly, for any $f \in \mathrm{Br}(T_U)$ the set $\{f\restriction n : n < \omega\} \subseteq T_U$ is the *infinite branch* induced by f. We do not distinguish between $\tau \in T_U$ $(f \in \mathrm{Br}(T_U))$ and its induced branch. A non-empty subset of T_U is a *subtree* of T_U if it is closed under taking initial segments. If V is another non-empty set, then the *product* $U' \times V'$ of subtrees $U' \subseteq T_U$, $V' \subseteq T_V$ is naturally given by

$$\{\tau : n \to U \times V \mid \tau(m) = (\tau_1(m), \tau_2(m)) \text{ for } m < n,\\ \tau_1 \in U',\ \tau_2 \in V',\ l(\tau_1) = l(\tau_2) = n\}.$$

Let κ be an uncountable cardinal with $\kappa^{\aleph_0} = \kappa$. We now consider the product $T := T_{\aleph_1 \times \kappa}$ of the two trees $T_{\aleph_1}$ and T_κ. Following [20], the first subtree $T_{\aleph_1}$ is reserved for the *norm* on T:

$$\|\tau\| = \sup_{i<l(\tau)} \tau_1(i) \in \aleph_1 \quad \text{for any } \tau = (\tau_1, \tau_2) \in T.$$

This norm extends naturally to countable subsets $X \subseteq T$ by $\|X\| = \sup_{\tau \in X} \|\tau\| < \aleph_1$. In particular, $\|f\| < \aleph_1$ for any branch $f \in \mathrm{Br}(T)$ (remember $f \subseteq T$); and we call f a *stretched* branch if $\|f\restriction n\| < \|f\restriction(n+1)\|$ for all $n < \omega$.

Next we define the basic τ_R-body $\mathbb{B}$ for constructing the final generalized $E(R)$-algebras; compare [20, p. 324 ff.]:

Let $Y := \overline{T} = \{y_{\tau\mu} \mid \tau \in T,\ \mu < \omega\}$ be a family of variables and choose $\mathbb{B} := \mathbb{B}_Y$ as the induced free τ_R-body. We will also consider the p-adic completion $\widehat{\mathbb{B}}$ and define for $b \in \widehat{\mathbb{B}}$ the T-*support* $[[b]] \subseteq T$ as the set

$$[[b]] = \bigcap\{S \subseteq T \mid b \in \widehat{\mathbb{B}}_{\overline{S}} \subseteq \widehat{\mathbb{B}}\} \quad \text{with} \quad \overline{S} = \{y_{\tau\mu} \mid \tau \in S, \mu < \omega\} \subseteq Y$$

and a more refined Y-*support* $[b] \subseteq Y$,

$$[b] = \bigcap\{S \subseteq Y \mid b \in \widehat{\mathbb{B}}_S \subseteq \widehat{\mathbb{B}}\}.$$

Note here that $\mathbb{B}_S$ $(S \subseteq Y)$ and $\mathbb{B}_{\overline{S}}$ $(S \subseteq T)$ are defined for non-empty sets S as the canonically induced free subbodies of $\mathbb{B}_Y$, while in the special case of empty sets we define

$$M_\emptyset = \{\mathbf{t}_Y = (\mathbf{t}, f^{\mathbf{t}}) \in M_Y \mid P_4^{\mathbf{t}} = \emptyset\} \subseteq M_Y$$

as the canonical τ-subskeleton of all "color-free" Y-colored trees in M_Y and

$$\mathbb{B}_\emptyset := \mathbb{B}_{\overline{\emptyset}} := \mathrm{Lin}_R\, M_\emptyset \subseteq \mathbb{B}_Y.$$

Furthermore, for every $b \in \widehat{\mathbb{B}}$ the supports $[[b]] \subseteq T$, $[b] \subseteq \overline{[[b]]} \subseteq Y$ are at most countable with $b \in \widehat{\mathbb{B}}_{[b]} \subseteq \widehat{\mathbb{B}}_{\overline{[[b]]}}$. We state the following simple support lemma.

Lemma 10.1. *For the Y-support (and similarly for the T-support) the following holds.*

(a) $[rb] \subseteq [b]$ *for all* $r \in \widehat{R}$ *and* $b \in \widehat{\mathbb{B}}$.
(b) $[b_1 + b_2] \subseteq [b_1] \cup [b_2]$ *for all* $b_1, b_2 \in \widehat{\mathbb{B}}$. *Furthermore,* $[b_1 + b_2] = [b_1] \cup [b_2]$ *for* $[b_1] \cap [b_2] = \emptyset$.
(c) $[b_1 * b_2] \subseteq [b_1] \cup [b_2]$ *and* $[b_1 \cdot b_2] \subseteq [b_1] \cup [b_2]$ *for all* $b_1, b_2 \in \widehat{\mathbb{B}}$.

The support extends naturally to countable subsets X (and also to their generated subbodies $\langle X\rangle \subseteq \widehat{\mathbb{B}}$) by $[X] = \bigcup_{x\in X}[x]$ and $[[X]] = \bigcup_{x\in X}[[x]]$; moreover, $\|X\| := \|[[X]]\|$ also extends naturally. Following [20] we finally define traps of the Black Box.

Definition 10.2. *A pair $p = (f, \varphi)$ is called a* trap *if the following conditions hold:*

(i) $f : \omega \to \aleph_1 \times \kappa$ *is a stretched branch of the tree* $T_{\aleph_1 \times \kappa}$.
(ii) $\varphi : P \to \widehat{P}$ *is an R-module endomorphism with* $P := \mathbb{B}_S$ *for some countable* $S \subseteq Y$.

(iii) $[[P]] \subseteq T$ *is a subtree with* $f \subseteq [[P]]$.
(iv) $S = \overline{[[P]]}$ *and* $\|x\| < \|P\|$ *for all* $x \in P$.
(v) $\|p\| := \|f\| = \|P\| \in \aleph_1^o$, *where* $\aleph_1^o = \{\alpha < \aleph_1 \mid \mathrm{cf}(\alpha) = \omega\} = \aleph_1 \cap \mathrm{LORD}$.

We are now ready to state the Black Box (Theorem) which is only slightly different from [20, The General Black Box 9.2.27].

The Black Box 10.3. *Let κ be an uncountable cardinal with $\kappa^{\aleph_0} = \kappa$. Moreover, let the tree T and the τ_R-body $\mathbb{B} = \mathbb{B}_Y$ be as just defined. Then there exist an ordinal $\kappa^* < \kappa^+$ and a list of traps $p_\alpha = (f_\alpha, \varphi_\alpha)$ $(\alpha < \kappa^*)$ with the following properties for all $\alpha, \beta < \kappa^*$:*

(i) *If $\beta \leq \alpha$, then $\|p_\beta\| \leq \|p_\alpha\|$.*
(ii) *If $\alpha \neq \beta$, then $|f_\alpha \cap f_\beta| < \aleph_0$.*
(iii) *If $\beta + 2^{\aleph_0} \leq \alpha$, then $|f_\alpha \cap [[P_\beta]]| < \aleph_0$.*
(iv) *If $X \subseteq \widehat{\mathbb{B}}$ is a countable subset and $\varphi \in \mathrm{End}\,\widehat{\mathbb{B}}$, then there is an $\alpha < \kappa^*$ such that the trap p_α catches X and φ, i.e. the following holds:*
   (a) $X \subseteq P_\alpha := \mathrm{Dom}\,\varphi_\alpha$.
   (b) $\|X\| < \|p_\alpha\|$.
   (c) $\varphi \restriction P_\alpha = \varphi_\alpha$.

The proof is similar to the proof of the General Black Box 9.2.27 in [20]. □

We are now in a position to construct the τ_R-body $\overline{\mathbb{B}}$ for Theorem 1.5 using the Black Box 10.3.

10.2. Construction of an ascending chain of bodies

Given $Y = \overline{T} = \{y_{\tau\mu} \mid \tau \in T,\ \mu < \omega\}$ and $\mathbb{B} = \mathbb{B}_Y$ as in the Black Box 10.3, we want to define the desired τ_R-body $\mathbb{B} \subseteq \overline{\mathbb{B}} \subseteq \widehat{\mathbb{B}}$ as continuous ascending chain $\overline{\mathbb{B}} = \mathbb{B}_{\kappa^*} = \bigcup_{\alpha \leq \kappa^*} \mathbb{B}_\alpha$ by transfinite induction on $\alpha \leq \kappa^*$ and begin with $\mathbb{B}_0 := \mathbb{B}$. By continuity of this chain we let $\mathbb{B}_\alpha = \bigcup_{\beta<\alpha} \mathbb{B}_\beta$ for limit ordinals $\alpha \leq \kappa^*$, and if $\mathbb{B}_\alpha$ $(\alpha < \kappa^*)$ is given, then we will define $\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha[v_\alpha]_* \subseteq \widehat{\mathbb{B}}$. In fact, we will express $v_\alpha = \pi_\alpha b_\alpha + w_\alpha$ as in (8.1) for suitable elements

$$\pi_\alpha \in \widehat{R},\ b_\alpha \in P_\alpha \text{ and } w_\alpha = \sum_{k<\omega} p^k y_k^\alpha \text{ with } y_k^\alpha := y_{f_\alpha \restriction (k+1), e_\alpha(k)} \in Y \quad \text{for some map } e_\alpha : \omega \to \omega,$$

and we set $\overline{y}_\alpha = \{y_n^\alpha \mid n < \omega\}$. Thus $w_\alpha \in \widehat{P_\alpha}$ by Definition 10.2, and by continuity the R-module homomorphism φ_α is well-defined on $v_\alpha \in \widehat{P_\alpha}$. Furthermore, $b_\alpha \in P_\alpha \subseteq \mathbb{B}$, and property (ii) of the Black Box guarantees (8.5). Thus, equipping Y with some well-ordering respecting the norm on T we can use the notations of the Black Box from Section 8.

As a direct consequence of Lemmas 8.5 and 8.9 and without any further specific knowledge about the elements v_α we can summarize the following general algebraic properties.

Lemma 10.4. *For $\mathbb{B}_\alpha$ $(\alpha \leq \kappa^*)$ defined as above the following holds.*

(a) $\bigcup_{\alpha \leq \kappa^*} \mathbb{B}_\alpha$ *is a strictly ascending continuous chain of subbodies* $\mathbb{B} \subseteq \mathbb{B}_\alpha \subseteq \widehat{\mathbb{B}}$.

(b) $\mathbb{B}_\alpha = \mathbb{B}[v_\beta \mid \beta < \alpha]_* = \mathbb{B}[v_i^\beta \mid \beta < \alpha,\ i < \omega] \subseteq \widehat{\mathbb{B}}$ *for any* $\alpha \leq \kappa^*$.

(c) ${}_R\mathbb{B}_\alpha$ *is an* $\aleph_0$*-free* R*-module* (*and in particular cotorsion-free*) *for any* $\alpha \leq \kappa^*$.

We next want to specify the elements v_α $(\alpha < \kappa^*)$ we are going to use for $\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha[v_\alpha]_*$. To begin with we want to choose the map $e_\alpha : \omega \to \omega$ such that the construction of $\mathbb{B}_\alpha$ at earlier stages will not be demolished by the new choice of v_α. This is taken care of by a lemma which is entirely based on support arguments, so it can be taken over *verbatim* from earlier constructions of modules with prescribed endomorphism rings. For details we refer to Corner and Göbel [4, Lemma 3.9, p. 457] ("there are no useless ordinals").

Lemma 10.5. *For some $\alpha < \kappa^*$ let $\mathbb{B}_\beta$ $(\beta \leq \alpha)$ be as above. Then there exists a family of functions $e_{\alpha\gamma} : \omega \to \omega$ $(\gamma < 2^{\aleph_0})$ such that for $v_{\alpha\gamma} = \pi_{\alpha\gamma} b_{\alpha\gamma} + w_{\alpha\gamma}$ $(\gamma < 2^{\aleph_0})$ with arbitrary elements $\pi_{\alpha\gamma} \in \widehat{R}$, $b_{\alpha\gamma} \in P_\alpha$ and $w_{\alpha\gamma} = \sum_{k<\omega} p^k y_{f_\alpha \restriction (k+1), e_{\alpha\gamma}(k)}$ the following properties hold:*

(i) *The function $e_{\alpha\gamma}$ is injective for any $\gamma < 2^{\aleph_0}$.*

(ii) *If $\gamma \neq \gamma'$, then $|\mathrm{Im}\, e_{\alpha\gamma} \cap \mathrm{Im}\, e_{\alpha\gamma'}| < \aleph_0$.*

(iii) *If $\gamma \neq \gamma'$, then $|[x] \cap [w_{\alpha\gamma'}]| < \aleph_0$ for any $x \in \mathbb{B}_\alpha[v_{\alpha\gamma}]_* \subseteq \widehat{\mathbb{B}}$.*

(iv) *There exist $\gamma \neq \gamma'$ with $v_\beta \varphi_\beta \notin \mathbb{B}_\alpha[v_{\alpha\gamma}]_*, \mathbb{B}_\alpha[v_{\alpha\gamma'}]_*$ for all $\beta < \alpha$ with $v_\beta\varphi_\beta \notin \mathbb{B}_{\beta+1}$.*

Proof. We can choose the family of functions $e_{\alpha\gamma} : \omega \to \omega$ $(\gamma < 2^{\aleph_0})$ independent of the chain $\bigcup_{\beta \leq \alpha} \mathbb{B}_\beta$. For (i) and (ii) we only have to observe that $\mathrm{Br}(T_\omega)$ is interpretable as a family of almost disjoint subsets of T_ω with $|T_\omega| = \aleph_0$ and $|\mathrm{Br}(T_\omega)| = 2^{\aleph_0}$. An easy support argument similar to Section 8 shows (iii).

Finally, (iv) is immediate for ordinals $\beta + 2^{\aleph_0} \leq \alpha$ by claim (iii) of the Black Box while for the remaining ordinals β we can apply

$$\mathbb{B}_\alpha[v_{\alpha\gamma}]_* \cap \mathbb{B}_\alpha[v_{\alpha\gamma'}]_* = \mathbb{B}_\alpha \quad (\gamma \neq \gamma')$$

and a simple pigeonhole argument as in [4, Lemma 3.9, p. 457]. □

We now want to choose $v_\alpha = \pi_\alpha b_\alpha + w_\alpha$ $(\alpha < \kappa^*)$ such that Step Lemma 8.10 holds for v_α and φ_α $(\alpha \in I)$. At the same time we define the set $I = I_{\kappa^*} = \bigcup_{\alpha \leq \kappa^*} I_\alpha \subseteq \kappa^*$ recursively as a continuous ascending chain of subsets $I_\alpha \subseteq \alpha$ and a family of auxiliary elements $w'_\alpha \in \widehat{P}_\alpha$ $(\alpha < \kappa^*)$ for subsequent support arguments. By continuity we set $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$ for all limit ordinals $\alpha \leq \kappa^*$ and in the case of successor ordinals we distinguish the following two cases:

(I) If $P_\alpha(p^N \varphi_\alpha - *_g) = 0$ for some $N < \omega$ and $g \in \mathbb{B}_\alpha$, we set $\pi_{\alpha\gamma} = b_{\alpha\gamma} = 0$ for every map $e_{\alpha\gamma}$ $(\gamma < 2^{\aleph_0})$ from Lemma 10.5. Here we define $I_{\alpha+1} := I_\alpha$, $v_\alpha := v_{\alpha\gamma}$, $w'_\alpha := w_{\alpha\gamma'}$ and $\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha[v_\alpha]_*$ for ordinals $\gamma, \gamma' < 2^{\aleph_0}$ according to condition (iv) of Lemma 10.5.

(II) If otherwise $(p^N\varphi_\alpha - *_g)\restriction P_\alpha \not\equiv 0$ for all $N < \omega$ and $g \in \mathbb{B}_\alpha$, then the hypotheses of Step Lemma 8.10 hold and for every map $e_{\alpha\gamma}$ ($\gamma < 2^{\aleph_0}$) from Lemma 10.5 there are $\pi_{\alpha\gamma} \in \widehat{R}$ and $b_{\alpha\gamma} \in P_\alpha$ such that $v_{\alpha\gamma}\varphi_\alpha \notin \mathbb{B}_\alpha[v_{\alpha\gamma}]_*$ for $v_{\alpha\gamma} = \pi_{\alpha\gamma}b_{\alpha\gamma} + w_{\alpha\gamma}$. We now choose ordinals $\gamma, \gamma' < 2^{\aleph_0}$ according to condition (iv) of Lemma 10.5 and set $I_{\alpha+1} := I_\alpha \cup \{\alpha\}$, $v_\alpha := v_{\alpha\gamma}$, $w'_\alpha := w_{\alpha\gamma'}$ and $\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha[v_\alpha]_*$.

As an immediate consequence of this construction and Lemma 10.5 we add to Lemma 10.4 the following

Lemma 10.6. *For $\mathbb{B}_\alpha$ ($\alpha \le \kappa^*$) defined as above the following holds.*

(a) $v_\alpha\varphi_\alpha \in \mathbb{B}_{\alpha+1}$ *for any* $\alpha \notin I$.
(b) $v_\alpha\varphi_\alpha \notin \mathbb{B}_\beta$ *for any* $\alpha \in I$ *and* $\beta \le \kappa^*$.
(c) $[w'_\alpha] \subseteq P_\alpha \cap Y$ *is a countable set for any* $\alpha \le \kappa^*$.
(d) $|[x] \cap [w'_\alpha]| < \aleph_0$ *for any* $\alpha, \beta \le \kappa^*$ *and* $x \in \mathbb{B}_\beta$.

It remains to show that ${}_R\overline{\mathbb{B}}$ has only the desired R-module endomorphisms.

10.3. Proof of the Main Theorem with the General Black Box

Main Lemma 10.7. *Let $|R| \le \kappa$ be an uncountable cardinal with $\kappa^{\aleph_0} = \kappa$. Moreover, let $\overline{\mathbb{B}} = \bigcup_{\alpha\le\kappa^*} \mathbb{B}_\alpha$ be the ascending chain of τ_R-bodies $\mathbb{B}_\alpha$ just constructed and define $A := (\overline{\mathbb{B}}, +, \cdot)$ as the R-algebra structure of the τ_R-body $\overline{\mathbb{B}}$. Then A is an $\aleph_0$-free, non-commutative R-algebra of cardinality $|A| = \kappa$ and for every $\varphi \in \mathrm{End}_R A$ there is a unique $g \in A$ such that $x\varphi = x * g$ for all $x \in A$.*

Proof. Obviously A is an $\aleph_0$-free R-module with $\kappa = |\mathbb{B}| \le |A| \le |\widehat{\mathbb{B}}| = \kappa^{\aleph_0} = \kappa$ by Lemma 10.4, and a non-commutative R-algebra as $\mathbb{B} \subseteq A$ and by Proposition 7.1.

For the last claim let $p_\alpha = (f_\alpha, \varphi_\alpha)$ with $P_\alpha = \mathrm{Dom}\,\varphi_\alpha$ ($\alpha < \kappa^*$) be the list of traps given by the Black Box 10.3 and define for any $\varphi \in \mathrm{End}_R A$ the set $J = \{\alpha < \kappa^* \mid \varphi\restriction P_\alpha = \varphi_\alpha\}$.

If now construction case (II) applies for some $\alpha \in J$, then $\alpha \in I$ and $v_\alpha\varphi = v_\alpha\varphi_\alpha \notin \overline{\mathbb{B}}$ follows from Lemma 10.6, contradicting $\varphi \in \mathrm{End}_R A$. Thus for any $\alpha \in J$ there are $N < \omega$ and $g_\alpha \in \overline{\mathbb{B}}$ such that $P_\alpha(p^N\varphi - *_{g_\alpha}) = P_\alpha(p^N\varphi_\alpha - *_{g_\alpha}) = 0$. Furthermore, the family $\mathcal{P} := \{P_\alpha \mid \alpha \in J\}$ of free τ_R-bodies is upwards directed by condition (iv) of the Black Box and with Remark 6.6 and Lemma 10.6 the $*$-scalar product is faithful on $\overline{\mathbb{B}} = \bigcup_{\alpha\in J} P_\alpha$ as observed by the action on $[w'_\alpha] \subseteq P_\alpha$; thus $\mathcal{P}$ is useful for φ (in the sense of Definition 8.12) and we can apply Proposition 8.13 to find uniform $N < \omega$ and $g' \in \overline{\mathbb{B}}$ with $p^N\varphi = *_{g'}$, and similar to Lemma 9.1 the existence of some $g \in \overline{\mathbb{B}}$ with $\varphi = *_g$ follows. □

The final step of the proof of Theorem 1.5 is given in the introduction (Section 1).

10.4. Appendix: Rigid systems of generalized $E(R)$-algebras

We sketch the five steps for extending the existence of a non-commutative generalized $E(R)$-algebra (Theorem 1.5) to the existence of a fully rigid system of them.

(1) Let $\overline{\mathbb{B}}$ be the τ_R-body and generalized $E(R)$-algebra constructed in Main Lemma 10.7. Consider the set $\mathfrak{U} = \{\alpha < \kappa^* \mid \alpha$ satisfies case (I) in Section 10.2$\}$. We note the trivial fact that $|\mathfrak{U}| = \kappa$. If $U \subseteq \mathfrak{U}$, then let $U^\sharp := (\kappa^* \setminus \mathfrak{U}) \cup U \subseteq \kappa^*$ and define by
$$\overline{\mathbb{B}}_U := \mathbb{B}[v_\beta \mid \beta \in U^\sharp]_* \subseteq \overline{\mathbb{B}} = \overline{\mathbb{B}}_{\mathfrak{U}}$$
a family of $2^{|\mathfrak{U}|} = 2^\kappa$ τ_R-bodies.

(2) If $U, V \subseteq \mathfrak{U}$ and $\varphi \in \mathrm{Hom}_R(\overline{\mathbb{B}}_U, \overline{\mathbb{B}}_V)$, then $\overline{\mathbb{B}}_U\varphi \subseteq \overline{\mathbb{B}}_V$ and arguments similar to Main Lemma 10.7 apply, showing that $\varphi = *_g$ for some $g \in \overline{\mathbb{B}}$. From this we even conclude that $g \in \overline{\mathbb{B}}_V \subseteq \overline{\mathbb{B}}$ with an easy test argument: If $g \notin \overline{\mathbb{B}}_V$, then for any generator $y \in Y \setminus [g] \subseteq \overline{\mathbb{B}}_U$ we have $y\varphi = y * g \notin \overline{\mathbb{B}}_V$, contradicting $\varphi \in \mathrm{Hom}_R(\overline{\mathbb{B}}_U, \overline{\mathbb{B}}_V)$.

(3) If $U \subseteq V \subseteq \mathfrak{U}$ and $\varphi \in \mathrm{Hom}_R(\overline{\mathbb{B}}_U, \overline{\mathbb{B}}_V)$, then $\overline{\mathbb{B}}_U \subseteq \overline{\mathbb{B}}_V$ and by (2) there is $g \in \overline{\mathbb{B}}_V$ with $\varphi = *_g$. Conversely, $*_g \in \mathrm{Hom}_R(\overline{\mathbb{B}}_U, \overline{\mathbb{B}}_V)$ for any $g \in \overline{\mathbb{B}}_V$. Thus
$$\mathrm{Hom}_R(\overline{\mathbb{B}}_U, \overline{\mathbb{B}}_V) = \overline{\mathbb{B}}_V^* := \{*_g \mid g \in \overline{\mathbb{B}}_V\}.$$
In particular, $\mathrm{End}_R\,\overline{\mathbb{B}}_U = \overline{\mathbb{B}}_U^* \cong \overline{\mathbb{B}}_U$ for any $U \subseteq \mathfrak{U}$ and $\overline{\mathbb{B}}_U$ is always a non-commutative generalized $E(R)$-algebra.

(4) If $U, V \subseteq \mathfrak{U}$ and $U \not\subseteq V$, then we claim that
$$\mathrm{Hom}_R(\overline{\mathbb{B}}_U, \overline{\mathbb{B}}_V) = 0.$$
Indeed, if $0 \neq \varphi \in \mathrm{Hom}_R(\overline{\mathbb{B}}_U, \overline{\mathbb{B}}_V)$, then by (2) there is $0 \neq g \in \overline{\mathbb{B}}_V$ with $\varphi = *_g$. We now choose any $\alpha \in U \setminus V$ and the corresponding element $v_\alpha = \pi_\alpha b_\alpha + w_\alpha \in \overline{\mathbb{B}}_U$. Then $v_\alpha\varphi = v_\alpha * g \notin \overline{\mathbb{B}}_V$, contradicting $\varphi \in \mathrm{Hom}_R(\overline{\mathbb{B}}_U, \overline{\mathbb{B}}_V)$.

(5) If we choose a maximal family U_α $(\alpha < 2^\kappa)$ of pairwise incomparable subsets $U_\alpha \subseteq \mathfrak{U}$ and put $\overline{\mathbb{B}}_\alpha := \overline{\mathbb{B}}_{U_\alpha}$, then $\mathrm{Hom}_R(\overline{\mathbb{B}}_\alpha, \overline{\mathbb{B}}_\beta) = \overline{\mathbb{B}}_\alpha^* \delta_{\alpha\beta}$ for all $\alpha, \beta < 2^\kappa$.

Hence we derive the following

Corollary 10.8. *Let R be a cotorsion-free commutative ring R with* 1 *and $|R| \leq \kappa$ be an uncountable cardinal with $\kappa = \kappa^{\aleph_0}$. Then there are a set $\mathfrak{U}$ of cardinality $|\mathfrak{U}| = \kappa$ and an $\aleph_0$-free* (*thus cotorsion-free*), *non-commutative R-algebra $\overline{\mathbb{B}}$ of cardinality $|\overline{\mathbb{B}}| = \kappa$ with a fully rigid family of 2^κ subalgebras $\overline{\mathbb{B}}_U$* ($U \subseteq \mathfrak{U}$) *which are non-commutative generalized $E(R)$-algebras of cardinality κ, such that*
$$\mathrm{Hom}_R(\overline{\mathbb{B}}_U, \overline{\mathbb{B}}_V) = \begin{cases} \overline{\mathbb{B}}_V^* & \text{if } U \subseteq V \subseteq \mathfrak{U}, \\ 0 & \text{if } U \not\subseteq V,\ U, V \subseteq \mathfrak{U}. \end{cases}$$
Moreover, $\overline{\mathbb{B}}_U \subseteq \overline{\mathbb{B}}_V \subseteq \overline{\mathbb{B}}$ if $U \subseteq V \subseteq \mathfrak{U}$.

In particular, there is a rigid family $\overline{\mathbb{B}}_\alpha$ $(\alpha < 2^\kappa)$ of maximal size 2^κ of non-commutative generalized $E(R)$-algebras of cardinality κ such that
$$\mathrm{Hom}_R(\overline{\mathbb{B}}_\alpha, \overline{\mathbb{B}}_\beta) = \overline{\mathbb{B}}_\alpha^* \delta_{\alpha\beta}.$$

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