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Hypersurfaces in \mathbb{H}^{n+1} and conformally invariant equations: the generalized Christoffel and Nirenberg problems

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Abstract. Our first objective in this paper is to give a natural formulation of the Christoffel problem for hypersurfaces in \mathbb{H}^{n+1} , by means of the hyperbolic Gauss map and the notion of hyperbolic curvature radii for hypersurfaces. Our second objective is to provide an explicit equivalence of this Christoffel problem with the famous problem of prescribing scalar curvature on \mathbb{S}^n for conformal metrics, posed by Nirenberg and Kazdan–Warner. This construction lets us translate into the hyperbolic setting the known results for the scalar curvature problem, and also provides a hypersurface theory interpretation of such an intrinsic problem from conformal geometry. Our third objective is to place the above result in a more general framework. Specifically, we will show how the problem of prescribing the hyperbolic Gauss map and a given function of the hyperbolic curvature radii in \mathbb{H}^{n+1} is strongly related to some important problems on conformally invariant PDEs in terms of the Schouten tensor. This provides a bridge between the theory of conformal metrics on \mathbb{S}^n and the theory of hypersurfaces with prescribed hyperbolic Gauss map in \mathbb{H}^{n+1} . The fourth objective is to use the above correspondence to prove that for a wide family of Weingarten functionals $\mathcal{W}(\kappa_1, \dots, \kappa_n)$, the only compact immersed hypersurfaces in \mathbb{H}^{n+1} on which \mathcal{W} is constant are round spheres.

Keywords. Christoffel problem, Nirenberg problem, Kazdan–Warner conditions, Schouten tensor, hyperbolic Gauss map, Weingarten hypersurfaces

1. Introduction

Some of the most interesting problems in the theory of geometric PDEs come from the following classical question: *given a diffeomorphism $G : \mathbb{S}^n \rightarrow \mathbb{S}^n$ and a smooth function $F : \mathbb{S}^n \rightarrow \mathbb{R}$, can one find a (necessarily strictly convex) hypersurface $f : \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$ with Gauss map G and with F as a prescribed function of its principal curvatures?* Possibly the oldest particular case of this problem is the famous *Christoffel problem* [Chr] that prescribes F as the mean of the curvature radii of the hypersurface:

$$F = \frac{1}{n} \sum_{i=1}^n R_i, \quad R_i := \frac{1}{\kappa_i}, \quad (1.1)$$

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where $\kappa_1, \dots, \kappa_n$ are the principal curvatures of the hypersurface. The Christoffel problem was classically solved in [Fi1, Fi2].

It is very natural to ask for the extension of the Christoffel problem to space forms. But surprisingly, even though many interesting contributions on hypersurfaces with prescribed Weingarten curvatures in space forms have been made in the past, a satisfactory development of the Christoffel problem in \mathbb{S}^{n+1} or \mathbb{H}^{n+1} remains unknown. The reason for that seems to be, as Oliker [OI1] points out, that the classical Gauss map is not available on those spaces. Indeed, the unit normal takes its values in the unit tangent bundle of the space form, no longer identified with \mathbb{S}^n , which makes it unexpectedly subtle even how to formulate the Christoffel problem in \mathbb{S}^{n+1} or \mathbb{H}^{n+1} .

Our first goal in this paper is to show that the Christoffel problem can be naturally formulated in the context of hypersurfaces $M^n \subset \mathbb{H}^{n+1}$ in the hyperbolic space. For that we substitute the Euclidean Gauss map by the *hyperbolic Gauss map* $G : M^n \rightarrow \mathbb{S}^n$, which is widely accepted among specialists in hyperbolic geometry to be the right analogue of the classical Gauss map. In addition, the inverses of the principal curvatures of $M^n \subset \mathbb{H}^{n+1}$ do not serve anymore as curvature radii in this context. We will overcome this difficulty by introducing the *hyperbolic curvature radii* of $M^n \subset \mathbb{H}^{n+1}$, defined as $\mathcal{R}_i := 1/|1 - \kappa_i|$, and they will be shown to play the role in \mathbb{H}^{n+1} of the Euclidean curvature radii from several different perspectives.

The second objective of the paper is to provide a geometric back-and-forth procedure which shows that the Christoffel problem in \mathbb{H}^{n+1} is *essentially* equivalent to a famous problem from the theory of geometric PDEs, namely, the Nirenberg problem (or Kazdan–Warner problem) of prescribing scalar curvature in \mathbb{S}^n : *given a map $S : \mathbb{S}^n \rightarrow \mathbb{R}$, does there exist a metric $g = e^{2\rho} g_0$ conformal to the standard metric g_0 of \mathbb{S}^n , and whose scalar curvature function is given by S ?* Equivalently, this problem is to find functions S on \mathbb{S}^n for which the non-linear elliptic PDE

$$\begin{cases} -\Delta^{g_0} \rho + 1 = \frac{e^{2\rho}}{2} S(x) & \text{if } n = 2, \\ -\Delta^{g_0} u + \frac{n(n-2)}{4} u = \frac{n-2}{4(n-1)} S(x) u^{\frac{n+2}{n-2}} & \text{if } n > 2, u^{4/(n-2)} =: e^{2\rho}, \end{cases} \quad (1.2)$$

admits a solution globally defined on \mathbb{S}^n .

The Nirenberg–Kazdan–Warner problem (*Nirenberg problem* for short from now on) has gained an impressive number of contributions over the last 30 years. As a result, researchers on geometric PDEs have clarified to a large extent which smooth functions on \mathbb{S}^n arise as the scalar curvature functions of conformal metrics. We may cite [AM, BC, BE, Cha, CY1, CY2, CLi1, CLi2, CLn, ES, KW1, KW2, L1, L2, Mo] to name just a few of these works (see the survey [L6] for more details). However, a complete characterization of scalar curvature functions on \mathbb{S}^n is still unknown.

The back-and-forth construction that we develop here will let us translate all these results on the Nirenberg problem into results for the Christoffel problem in \mathbb{H}^{n+1} . And conversely, our construction also provides a hypersurface theory interpretation of an abstract problem of conformal geometry such as the Nirenberg problem. This is not an immediate fact, since the conformal flatness of solutions to the Nirenberg problem is rarely satisfied

by the induced metric of a hypersurface in a model space. Moreover, a remarkable consequence of (1.2) is that the Christoffel problem in \mathbb{H}^{n+1} cannot be reduced to a linear PDE, in contrast with the classical problem in \mathbb{R}^{n+1} .

The third objective of the paper is to analyze the scope of the previous construction. For that, we will consider the *generalized Christoffel problem* in \mathbb{H}^{n+1} , in which we prescribe the hyperbolic Gauss map and a given functional of the hyperbolic curvature radii of a compact surface $M^n \subset \mathbb{H}^{n+1}$, not just its mean. Problems of this nature for convex hypersurfaces in \mathbb{R}^{n+1} have been intensively studied: see [GMa, GLM, GMZ] and references therein. Again, we will show that this question in \mathbb{H}^{n+1} is tightly linked to an important problem from the theory of geometric PDEs that we describe next.

Recall first of all that on a Riemannian manifold (M^n, g) , $n > 2$, one has the following decomposition:

$$\text{Riem} = W_g + \text{Sch}_g \odot g,$$

where Riem is the Riemann curvature tensor, W_g is the Weyl tensor, \odot is the Kulkarni–Nomizu product, and

$$\text{Sch}_g := \frac{1}{n-2} \left(\text{Ric}_g - \frac{S(g)}{2(n-1)} g \right)$$

is the *Schouten tensor*. As the Weyl tensor is conformally invariant, the above decomposition reveals that the Schouten tensor encodes all the information on how curvature varies under a conformal change of metric. For this reason the Schouten tensor is the main object of study in conformal geometry. It is also remarkable that W_g vanishes identically in case (M^n, g) is locally conformally flat, which is the situation of the present paper. The *eigenvalues* of Sch_g are defined as the eigenvalues of the symmetric endomorphism $g^{-1}\text{Sch}_g$ obtained by raising an index to Sch_g .

We will show that the generalized Christoffel problem is equivalent to the problem of prescribing a functional of the eigenvalues of the Schouten tensor for conformal metrics $g = e^{2\rho} g_0$ on \mathbb{S}^n , under the regularity condition that $g - 2\text{Sch}_g$ is positive definite. An equivalent version of this regularity condition appeared in [Sc] in connection with the existence of hypersurfaces in \mathbb{H}^{n+1} with a given horospherical metric, but was not linked there to the Schouten tensor of g .

A remarkable consequence of this equivalence is that the theory of locally conformally flat Riemannian manifolds can be identified to a large extent with the local theory of hypersurfaces in \mathbb{H}^{n+1} with prescribed regular hyperbolic Gauss map. This fact yields a new way of applying methods from geometric PDEs to investigate hypersurfaces in \mathbb{H}^{n+1} , but is also of great interest in the opposite direction: on the one hand, the above equivalence motivates new problems for conformally invariant PDEs that are very interesting from the viewpoint of hypersurfaces in \mathbb{H}^{n+1} , although they do not appear so naturally in conformal geometry. And on the other hand, the hypersurface theory interpretation reveals non-trivial *superposition principles* via which one may obtain new solutions to a geometric PDE starting from a previously known one (in the spirit of Bäcklund transformations, for instance). An example of this use is given in Theorem 35.

Finally, the fourth objective of the paper is to give a more concrete application of the general correspondence between compact hypersurfaces in \mathbb{H}^{n+1} with regular hyperbolic Gauss map and conformal metrics on \mathbb{S}^n . Specifically, we will prove the existence of a wide family of smooth functions $\mathcal{W}(x_1, \dots, x_n)$ with the following property: *if $M^n \subset \mathbb{H}^{n+1}$ is an immersed compact hypersurface whose principal curvatures satisfy $\mathcal{W}(\kappa_1, \dots, \kappa_n) = 1$, then M^n is a totally umbilical round sphere.* This will follow from a deep theorem in [LL1] and the above correspondence.

The above result constitutes a relevant advance in what refers to sphere theorems for Weingarten hypersurfaces, since: (a) we are not assuming *a priori* that the hypersurfaces are embedded, and (b) the family of Weingarten functionals $\mathcal{W}(x_1, \dots, x_n)$ for which the result holds is extremely large, i.e. it is not just a specific Weingarten relation. We shall also prove a similar classification theorem for horospheres among Weingarten hypersurfaces in \mathbb{H}^{n+1} with one regular end.

The paper is organized as follows. Section 2 will review the hyperbolic Gauss map for hypersurfaces $M^n \subset \mathbb{H}^{n+1}$ and its relation to tangent horospheres. We will introduce *horospherical ovaloids* as compact hypersurfaces with regular hyperbolic Gauss map and analyze their properties, especially regarding the *horospherical metric* induced on the hypersurface via its associated space of tangent horospheres. Section 3 analyzes the possible formulations of the Christoffel problem in \mathbb{H}^{n+1} . We will show that this problem is naturally formulated in the class of horospherical ovaloids, and that this leads to the notion of *hyperbolic curvature radii* $\mathcal{R}_i := 1/|1 - \kappa_i|$, in terms of which the Christoffel problem in \mathbb{H}^{n+1} is satisfactorily formulated. These two preliminary sections constitute an important part of the paper, because they discuss in great detail why the problem we are considering here seems to be the most natural formulation of the Christoffel problem in \mathbb{H}^{n+1} .

In Section 4 we will prove that the Nirenberg problem on \mathbb{S}^n (modulo dilations) is equivalent to the Christoffel problem in \mathbb{H}^{n+1} (modulo parallel translations). This equivalence is made explicit by means of a representation formula for hypersurfaces in terms of the hyperbolic Gauss map and the *horospherical support function*, and shows how to translate into the Christoffel problem the known results for the Nirenberg one.

In Section 5 we generalize the above result, by showing that the generalized Christoffel problem in \mathbb{H}^{n+1} is equivalent to the problem of prescribing a given function of the eigenvalues of the Schouten tensor for conformal metrics on \mathbb{S}^n . Some applications of this relationship are explored, in particular regarding problems involving the elementary symmetric functions. In Section 6 we shall prove the above explained characterization of round spheres and horospheres in \mathbb{H}^{n+1} among a very general class of Weingarten hypersurfaces. At the very end of the paper, we will take the inverse approach and use hypersurface theory in \mathbb{H}^{n+1} to prove: (1) an inversion formula for the eigenvalues of Sch_g on an arbitrary locally conformally flat manifold (M^n, g) , and (2) a characterization of constant curvature metrics on \mathbb{S}^2 by the eigenvalues of the 2-dimensional analogue of the Schouten tensor.

From a conceptual viewpoint, this paper investigates hypersurfaces in \mathbb{H}^{n+1} by analyzing the local variation of their tangent horospheres, thus bifurcating from the usual perspective in which the variation of tangent hyperplanes is considered. This approach appears in other works [Ep1, Ep2, Ep3, Sc, FR, GMM, GM], but always from different

perspectives. In particular, topics such as the Nirenberg problem or the Schouten tensor have not been linked to hypersurface theory in \mathbb{H}^{n+1} before.

2. The hyperbolic Gauss map

In this preliminary section we study the hypersurfaces in \mathbb{H}^{n+1} with regular hyperbolic Gauss map in terms of their principal curvatures and their tangent horospheres.

Horospheres and the hyperbolic Gauss map

Let \mathbb{H}^{n+1} denote the $(n + 1)$ -dimensional hyperbolic space of constant curvature -1 , and let $\mathbb{S}_\infty^n = \partial_\infty \mathbb{H}^{n+1}$ denote its ideal boundary. In what follows, *horospheres* of \mathbb{H}^{n+1} play a central role. These hypersurfaces are easily visualized in the Poincaré ball model (\mathbb{B}^{n+1}, ds^2) for \mathbb{H}^{n+1} , where, as usual, $\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| < 1\}$. In this model, horospheres correspond to the n -spheres that are tangent at one point to the sphere at infinity \mathbb{S}_∞^n . In this way, two horospheres are always congruent, and they are at a constant distance if their respective points at infinity coincide. In addition, given a point $p \in \mathbb{S}_\infty^n$, the horospheres having p as their point at infinity provide a foliation of \mathbb{H}^{n+1} .

From a hypersurface theory viewpoint, horospheres are flat totally umbilical hypersurfaces in \mathbb{H}^{n+1} , and they are complete and embedded.

All of this suggests that horospheres can be naturally regarded in many ways as *hyperplanes* in the hyperbolic space \mathbb{H}^{n+1} , even though they are not totally geodesic.

Definition 1 ([Ep1, Ep2, Br]). *Let $\phi : M^n \rightarrow \mathbb{H}^{n+1}$ denote an immersed oriented hypersurface in \mathbb{H}^{n+1} with unit normal η . The hyperbolic Gauss map*

$$G : M^n \rightarrow \mathbb{S}_\infty^n \equiv \mathbb{S}^n$$

of ϕ is defined as follows: for every $p \in M^n$, $G(p) \in \mathbb{S}_\infty^n$ is the point at infinity of the unique horosphere in \mathbb{H}^{n+1} passing through $\phi(p)$ and whose inner unit normal at p coincides with $\eta(p)$.

Let us point out here that horospheres are globally convex, which allows us to talk about the *inner orientation* of a horosphere, meaning simply that the unit normal points to the convex side of the horosphere. With respect to this orientation, the second fundamental form of the horosphere is positive definite. Moreover, it turns out that inner oriented horospheres are the only hypersurfaces in \mathbb{H}^{n+1} with constant hyperbolic Gauss map.

There is an equivalent definition: the *hyperbolic Gauss map* $G : M^n \rightarrow \mathbb{S}_\infty^n \equiv \mathbb{S}^n$ of M^n sends each $p \in M^n$ to the point $G(p)$ at the ideal boundary \mathbb{S}_∞^n reached by the unique geodesic γ of \mathbb{H}^{n+1} that starts at $\phi(p)$ with initial speed $\eta(p)$.

The hyperbolic Gauss map is an analogue in the hyperbolic space of the classical Gauss map for hypersurfaces of \mathbb{R}^{n+1} , especially if we want tangent horospheres, as we do here, to play the role of tangent hyperplanes in the Euclidean theory. It must however

be remarked that the *a priori* chosen orientation for the hypersurface matters for the hyperbolic Gauss map. Indeed, if we change the orientation of M^n , then G turns into the *negative hyperbolic Gauss map* $G^- : M^n \rightarrow \mathbb{S}^n$, whose behaviour is totally different from that of G .

Regularity of the hyperbolic Gauss map

We shall work in the Minkowski model of \mathbb{H}^{n+1} . For that, consider the Minkowski space \mathbb{L}^{n+2} with canonical coordinates (x_0, \dots, x_{n+1}) and the Lorentzian metric

$$\langle \cdot, \cdot \rangle = -dx_0^2 + \sum_{i=1}^{n+1} dx_i^2.$$

The hyperbolic space is then realized in this model as the hyperquadric

$$\mathbb{H}^{n+1} = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = -1, x_0 > 0\}.$$

In the same way, the de Sitter $(n + 1)$ -space and the light cone are given, respectively, by

$$\mathbb{S}_1^{n+1} = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 1\}, \quad \mathbb{N}_+^{n+1} = \{x \in \mathbb{L}^{n+2} : \langle x, x \rangle = 0, x_0 > 0\}.$$

Let $\phi : M^n \rightarrow \mathbb{H}^{n+1}$ be an immersed oriented hypersurface, and let $\eta : M^n \rightarrow \mathbb{S}_1^{n+1}$ denote its unit normal. Then we can define a normal map associated to ϕ , taking values in the light cone, as

$$\psi = \phi + \eta : M^n \rightarrow \mathbb{N}_+^{n+1}. \tag{2.1}$$

The map ψ is strongly related to the hyperbolic Gauss map $G : M^n \rightarrow \mathbb{S}_\infty^n$ of ϕ . Indeed, the ideal boundary of \mathbb{N}_+^{n+1} coincides with \mathbb{S}_∞^n , and can be identified with the projective quotient space $\mathbb{N}_+^{n+1}/\mathbb{R}_+$. So, altogether, we have $G = [\psi] : M^n \rightarrow \mathbb{S}_\infty^n \cong \mathbb{N}_+^{n+1}/\mathbb{R}_+$.

Moreover, if we write $\psi = (\psi_0, \dots, \psi_{n+1})$, then we may interpret G as the map $G : M^n \rightarrow \mathbb{S}^n$ given by

$$G = \frac{1}{\psi_0}(\psi_1, \dots, \psi_{n+1}). \tag{2.2}$$

In this way, if we label $e^\rho := \psi_0$, we get the useful relation

$$\psi = e^\rho(1, G) : M^n \rightarrow \mathbb{N}_+^{n+1}. \tag{2.3}$$

Observe also that, by differentiating (2.3),

$$\langle d\psi, d\psi \rangle = e^{2\rho} \langle dG, dG \rangle_{\mathbb{S}^n}. \tag{2.4}$$

We thus introduce the following terminology, in analogy with the Euclidean setting.

Definition 2. *The smooth function $e^\rho : M^n \rightarrow \mathbb{R}$ will be called the horospherical support function, or just the support function, of the hypersurface $\phi : M^n \rightarrow \mathbb{H}^{n+1}$.*

Moreover, if $\{e_1, \dots, e_n\}$ denotes an orthonormal basis of principal directions of ϕ at p , and if $\kappa_1, \dots, \kappa_n$ are the associated principal curvatures, it is immediate that

$$\langle d\psi(e_i), d\psi(e_j) \rangle = (1 - \kappa_i)^2 \delta_{ij}. \quad (2.5)$$

Thus we have:

Lemma 3. *Let $\phi : M^n \rightarrow \mathbb{H}^{n+1}$ be an oriented hypersurface. The following conditions are equivalent at $p \in M^n$.*

- (i) *The hyperbolic Gauss map G is a local diffeomorphism.*
- (ii) *The associated light cone map ψ in (2.1) is regular.*
- (iii) *All principal curvatures of M^n are $\neq 1$.*

The regularity of the hyperbolic Gauss map gives rise to a notion of convexity specific to the hyperbolic setting, and weaker than the usual geodesic convexity notion:

Definition 4 ([Sc]). *Let $M^n \subset \mathbb{H}^{n+1}$ be an immersed oriented hypersurface, and let \mathcal{H}_p denote the horosphere in \mathbb{H}^{n+1} that is tangent to M^n at p , and whose interior unit normal at p agrees with the one of M^n . We will say that M^n is horospherically convex at p if there exists a neighborhood $V \subset M^n$ of p so that $V \setminus \{p\}$ does not intersect \mathcal{H}_p , and in addition the distance function of the hypersurface to the horosphere does not vanish up to the second order at p in any direction.*

This definition can be immediately characterized as follows.

Corollary 5. *An oriented hypersurface $M^n \subset \mathbb{H}^{n+1}$ is horospherically convex at $p \in M^n$ if and only if all the principal curvatures of M^n at p satisfy simultaneously $\kappa_i(p) < 1$ or $\kappa_i(p) > 1$.*

In particular, if M^n is horospherically convex at p , then any of the equivalent conditions in Lemma 3 holds.

Horospherical ovaloids

Definition 6. *A compact immersed hypersurface $\phi : M^n \rightarrow \mathbb{H}^{n+1}$ will be called a horospherical ovaloid of \mathbb{H}^{n+1} if it can be oriented so that it is horospherically convex at every point.*

Equivalently, a compact hypersurface is a horospherical ovaloid if and only if it can be oriented so that its hyperbolic Gauss map is a global diffeomorphism. This equivalence follows directly from Lemma 3 and Corollary 5 by a simple topological argument, bearing in mind that every compact hypersurface in \mathbb{H}^{n+1} has a point p at which $|\kappa_i(p)| > 1$ for every i . In particular, M^n is diffeomorphic to \mathbb{S}^n .

It is also immediate from the existence of the point with $|\kappa_i(p)| > 1$ that every horospherical ovaloid has a unique orientation such that $\kappa_i < 1$ everywhere for every $i = 1, \dots, n$. We call this orientation the *canonical orientation* of the horospherical

ovaloid. It follows that the hyperbolic Gauss map of a canonically oriented horospherical ovaloid is always a global diffeomorphism. This is not necessarily true for the other possible orientation. Let us also point out that if p is a point of a canonically oriented horospherical ovaloid $M^n \subset \mathbb{H}^{n+1}$, then M^n lies around p in the concave part of the unique horosphere that passes through p and whose interior unit normal at p coincides with the unit normal of M^n .

Recall that a compact hypersurface $M^n \subset \mathbb{H}^{n+1}$ is a (strictly convex) *ovaloid* if all its principal curvatures are non-zero and of the same sign. Thus, any ovaloid is a horospherical ovaloid, but the converse is not true.

Horospherical ovaloids in \mathbb{H}^{n+1} seem to be an unexplored topic that is of independent interest as a generalization of the usual geodesic ovaloids in \mathbb{H}^{n+1} . However, let us point out that a horospherical ovaloid is not necessarily embedded. For instance, take a regular curve $\alpha : [0, 1] \rightarrow \mathbb{H}^2$ with geodesic curvature smaller than 1 at every point, and such that $\alpha(0) = \alpha(1)$ and, moreover, $\alpha'(0) = -\alpha'(1)$. Then by considering \mathbb{H}^2 as a totally geodesic surface of \mathbb{H}^3 and after rotating α about the geodesic of \mathbb{H}^2 that meets α orthogonally at $\alpha(0)$, we get a surface of revolution in \mathbb{H}^3 that is a non-embedded horospherical ovaloid.

This lack of embeddedness shows that one cannot talk in general about the outer orientation of a horospherical ovaloid, and justifies the way we introduced the canonical orientation for them.

Another interesting feature of canonically oriented horospherical ovaloids is their good behaviour with regard to the parallel flow. As usual, the *parallel flow* of an oriented hypersurface $\phi : M^n \rightarrow \mathbb{H}^{n+1}$ is defined for every $t \in \mathbb{R}$ as $\phi_t : M^n \rightarrow \mathbb{H}^{n+1}$,

$$\phi_t(p) = \exp_{\phi(p)}(t\eta_p) : M^n \rightarrow \mathbb{H}^{n+1}, \quad (2.6)$$

where \exp denotes the exponential map of \mathbb{H}^{n+1} , and η_p is the unit normal of ϕ at p . It is then easy to check that if ϕ is a canonically oriented horospherical ovaloid, then the *forward* flow $\{\phi_t\}_t$, $t \geq 0$, is made up by regular canonically oriented horospherical ovaloids. This is no longer true in general for the *backward* flow (i.e. $t < 0$) due to the possible appearance of wave front singularities of the hypersurfaces.

The horospherical metric

It will be important for our purposes to introduce a natural metric on the space of horospheres in \mathbb{H}^{n+1} . This construction has appeared in other works previously, but we reproduce it here in order to put special emphasis on some aspects.

Let \mathcal{M} denote the space of horospheres in \mathbb{H}^{n+1} . Let us also fix an arbitrary point $p \in \mathbb{H}^{n+1}$, which we will regard without loss of generality as the origin in the Poincaré ball model. Then we can view each horosphere \mathcal{H} as a pair $(x, t) \in \mathbb{S}^n \times \mathbb{R}$, where x is the point at infinity of \mathcal{H} and t is the (signed) hyperbolic distance of \mathcal{H} to the point p . Here, t is negative if p is contained in the convex domain bounded by \mathcal{H} . Thus we may identify $\mathcal{M} \equiv \mathbb{S}^n \times \mathbb{R}$.

Let us now construct a natural metric on this space of horospheres. Points of the form $(x, 0)$ correspond to horospheres passing through the origin in the Poincaré ball model. It is then natural to endow each of these points with the canonical metric g_0 of \mathbb{S}^n evaluated at x .

But now, the horosphere (x, t) is a parallel hypersurface of $(x, 0)$, and the induced metric in \mathbb{H}^{n+1} of this parallel horosphere is a dilation of the one of $\mathcal{H} \equiv (x, 0)$, with factor e^{2t} . Thus, the natural metric to define at (x, t) is the dilated metric $e^{2t}g_0$ evaluated at x . Consequently, we may view the space of horospheres in \mathbb{H}^{n+1} as the product $\mathbb{S}^n \times \mathbb{R}$ endowed with the natural degenerate metric

$$\langle \cdot, \cdot \rangle_\infty := e^{2t}g_0.$$

Observe that the vertical rulings of $\mathbb{S}^n \times \mathbb{R}$ are null lines with respect to this degenerate metric.

Definition 7. Let $\phi : M^n \rightarrow \mathbb{H}^{n+1}$ denote an oriented hypersurface in \mathbb{H}^{n+1} , and let $\mathcal{H}_\phi : M^n \rightarrow \mathcal{M} \equiv \mathbb{S}^n \times \mathbb{R}$ be its tangent horosphere map at every point. We define the horospherical metric g_∞ of ϕ as

$$g_\infty := \mathcal{H}_\phi^*(\langle \cdot, \cdot \rangle_\infty),$$

i.e. as the pullback metric via \mathcal{H}_ϕ of the degenerate metric $\langle \cdot, \cdot \rangle_\infty$.

It turns out that the horospherical metric is everywhere regular if and only if the hyperbolic Gauss map of the hypersurface is a local diffeomorphism. This is a consequence of Lemma 3 and the following interpretation of the horospherical metric in the Minkowski model of \mathbb{H}^{n+1} , i.e. the model in which we will be working.

In the Minkowski model, horospheres of $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ are the intersections of affine degenerate hyperplanes of \mathbb{L}^{n+2} with \mathbb{H}^{n+1} . A simple calculation shows that horospheres are characterized by the fact that its associated light cone map is constant: $\phi + \eta = v \in \mathbb{N}_+^{n+1}$. Moreover, if we write $v = e^\rho(1, x)$, we see that $x \in \mathbb{S}^n$ is the point at infinity of the horosphere, and that parallel horospheres correspond to collinear vectors in \mathbb{N}_+^{n+1} . This shows that the space of horospheres in \mathbb{H}^{n+1} is naturally identified with the positive null cone \mathbb{N}_+^{n+1} . Thus, it is natural to endow this space with the canonical degenerate metric of the light cone, and it is quite obvious from the above construction that this light cone metric coincides with the degenerate metric $\langle \cdot, \cdot \rangle_\infty$ defined above.

Consequently, the horospherical metric on a hypersurface in \mathbb{H}^{n+1} is simply the pullback metric of its associated light cone map. Thus, it is regular if and only if the hyperbolic Gauss map is a local diffeomorphism.

All this construction is clearly reminiscent of the usual identification of the space of oriented vector hyperplanes in \mathbb{R}^{n+1} with the unit sphere \mathbb{S}^n . In this sense, just as the canonical \mathbb{S}^n metric is used in order to measure geometric quantities associated to the Euclidean Gauss map of a hypersurface in \mathbb{R}^{n+1} , we will use the horospherical metric for measuring geometrical quantities with respect to the hyperbolic Gauss map. Let us explain this in more detail, as was first done by Epstein [Ep3].

First, observe that the ideal boundary \mathbb{S}_∞^n of \mathbb{H}^{n+1} does not carry a geometrically useful metric (although it has a natural conformal structure), so we cannot endow G with a pullback metric from the ideal boundary. Moreover, note that to define the hyperbolic Gauss map G we need to know the exact point $p \in \mathbb{H}^{n+1}$ at which we are working (this does not happen for the Euclidean Gauss map). The additional knowledge of this point is then equivalent to the knowledge of the tangent horosphere to the hypersurface at the point. So, it is natural to use the horospherical metric for measuring lengths associated to the hyperbolic Gauss map. An alternative justification can be found in [Ep3] in connection with the parallel flow of hypersurfaces.

It is interesting to observe that the horospherical metric has played an important role in several different theories. For instance, it is equivalent to the Kulkarni–Pinkall metric [KP] (see [Sc]). It also happens that the area of a Bryant surface in \mathbb{H}^3 with respect to the horospherical metric is exactly the total curvature of the induced metric of the surface.

3. The Christoffel problem in \mathbb{H}^{n+1}

The formulation for the Christoffel problem in \mathbb{H}^{n+1} that seems most reasonable at first sight is: *given a diffeomorphism $G : \mathbb{S}^n \rightarrow \mathbb{S}^n$ and a function $F : \mathbb{S}^n \rightarrow \mathbb{R}$, does there exist a hypersurface $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ with hyperbolic Gauss map G and such that (1.1) holds for its principal curvatures $\kappa_1, \dots, \kappa_n$?* Disappointingly, this is not a natural problem in \mathbb{H}^{n+1} , because the two required hypotheses belong to different contexts. Specifically, a compact hypersurface $M^n \subset \mathbb{H}^{n+1}$ whose hyperbolic Gauss map is a global diffeomorphism is not necessarily convex (it is just horospherically convex at every point), and therefore the functional (1.1) may not be defined at some points of M^n . On the other hand, the convexity condition that is required on M^n for defining (1.1) is quite meaningless for the hyperbolic Gauss map. These limitations do not appear in the Euclidean setting, where the Gauss map is a global diffeomorphism exactly when the hypersurface is an ovaloid, which is the precise condition needed to define the functional (1.1).

Another *a priori* natural choice for replacement of curvature radii for the Christoffel problem in \mathbb{H}^{n+1} are the *contact radii*

$$q_i(p) := \coth^{-1}(\kappa_i(p)). \quad (3.1)$$

These quantities arise from the following fact: if α is a curve in \mathbb{H}^2 with geodesic curvature k_g at p , then the inverse of the curvature of the unique circle in \mathbb{H}^2 having an order two contact with α at p is given by $\coth^{-1}(k_g)$. However, we again see that the quantities (3.1) are not well defined if $\kappa_i \in [-1, 1]$, so we also have to discard them.

The above discussion shows that we must seek an alternative formulation of the Christoffel problem in \mathbb{H}^{n+1} . More specifically, we need to find a more suitable notion of curvature radii for hypersurfaces in \mathbb{H}^{n+1} that makes sense exactly when the hypersurface is a horospherical ovaloid.

The hyperbolic curvature radii

The key observation at this point is that the Euclidean Gauss map $N : M^n \rightarrow \mathbb{S}^n$ of a strictly convex hypersurface $M^n \subset \mathbb{R}^{n+1}$ is related in a very simple way to the Euclidean curvature radii R_i , as follows. Let

$$\alpha_i : (-\epsilon, \epsilon) \rightarrow M^n$$

denote a curve in $M^n \subset \mathbb{R}^{n+1}$ with $\alpha_i(0) = p$ and such that $\alpha'_i(0) = e_i$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of principal directions in $T_p M^n$. Let $L_0^t(\alpha_i)$ (resp. $L_0^t(N \circ \alpha_i)$) denote the length of $\alpha_i([0, t])$ (resp. of $N \circ \alpha_i([0, t])$), where N is the unit normal of M^n , is assumed to be a local diffeomorphism at p . Then

$$\lim_{t \rightarrow 0} \frac{L_0^t(\alpha_i)}{L_0^t(N \circ \alpha_i)} = \lim_{t \rightarrow 0} \frac{\int_0^t |\alpha'_i(u)| du}{\int_0^t |(N \circ \alpha_i)'(u)| du} = \frac{|\alpha'_i(0)|}{|(N \circ \alpha_i)'(0)|} = \frac{1}{|\kappa_i(p)|} = R_i(p). \tag{3.2}$$

This relation is relevant to the Christoffel problem, since it indicates that the curvature radii admit an interpretation in terms of the Gauss map.

The above construction can also be carried out in \mathbb{H}^{n+1} , with the Euclidean Gauss map substituted by the hyperbolic Gauss map. Specifically, let us consider an oriented hypersurface $M^n \subset \mathbb{H}^{n+1}$ that is horospherically convex at $p \in M^n$, and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of principal directions in $T_p M^n$. If we now take a regular curve $\alpha_i : (-\epsilon, \epsilon) \rightarrow M^n$ with $\alpha_i(0) = p$ and $\alpha'_i(0) = e_i$, we may define, in analogy with the Euclidean situation, the *hyperbolic curvature radii* of M^n at p as

$$\mathcal{R}_i(p) := \lim_{t \rightarrow 0} \frac{L_0^t(\alpha_i)}{L_0^t(G \circ \alpha_i)}.$$

Here the length of the hyperbolic Gauss map along α_i is obviously taken with respect to the horospherical metric g_∞ . In addition, the quotients make sense since G is a local diffeomorphism at p . Finally, by an argument analogous to (3.2) we deduce that $\mathcal{R}_i(p) = 1/|1 - \kappa_i(p)|$. So, we propose the following definition as the natural analogue in \mathbb{H}^{n+1} of the Euclidean curvature radii for geometrical problems involving the hyperbolic Gauss map.

Definition 8. *Let $M^n \subset \mathbb{H}^{n+1}$ be a hypersurface that is horospherically convex at $p \in M^n$. We define the hyperbolic curvature radii $\{\mathcal{R}_1, \dots, \mathcal{R}_n\}$ of M^n at p as*

$$\mathcal{R}_i(p) = \frac{1}{|1 - \kappa_i(p)|},$$

where $\{\kappa_1, \dots, \kappa_n\}$ are the principal curvatures of M^n at p .

Remark 9. We may observe that $1/\mathcal{R}_i$ is simply the length of $dG_p(e_i)$ with respect to the horospherical metric, where e_i is a principal unit vector at p . As the same property is true in the Euclidean setting, this indicates again that \mathcal{R}_i is the proper extension to \mathbb{H}^{n+1} of the Euclidean curvature radii.

The above arguments suggest the following formulation of the Christoffel problem in \mathbb{H}^{n+1} as the most natural one: *given a diffeomorphism $G : \mathbb{S}^n \rightarrow \mathbb{S}^n$ and a smooth function $C : \mathbb{S}^n \rightarrow \mathbb{R}$, does there exist an oriented hypersurface $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ (necessarily a horospherical ovaloid) with hyperbolic Gauss map G and with C as the mean of the hyperbolic curvature radii? It must be stressed here that the hyperbolic curvature radii \mathcal{R}_i make sense in the compact case exactly for the class of horospherical ovaloids, which is the condition we were looking for.*

It is convenient to work with a simplified (but equivalent) version of the above Christoffel problem. In order to present this simplification (see next page), we introduce the next remark, as well as the following subsection.

Remark 10. As G is a global diffeomorphism in the Christoffel problem, it can be used as a global parametrization of the horospherical ovaloid. In other words, we may assume that $G(x) = x$ on \mathbb{S}^n without losing generality.

Orientation and the parallel flow

It is interesting to observe the behaviour of the Christoffel problem under the parallel flow. Let $\{\phi_t\}_{t \in \mathbb{R}}$ denote the parallel flow of a solution $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ to the Christoffel problem for the function $C(x)$. Then the hyperbolic Gauss map $G(x) = x$ remains invariant under this flow, and the horospherical metric of ϕ_t is $g_{\infty,t} = e^{2t}g_\infty$. Moreover, the principal curvatures κ_i^t of ϕ_t at regular points are given by

$$\kappa_i^t(p) = \frac{\kappa_i(p) - \tanh(t)}{1 - \kappa_i(p) \tanh(t)}, \tag{3.3}$$

and so the mean $C_t(x)$ of the hyperbolic curvature radii of ϕ_t is, at its regular points,

$$C_t(x) = \frac{1}{2} - \frac{e^{-2t}}{2}(1 - 2C(x)). \tag{3.4}$$

Unfortunately, ϕ_t is not always regular. Indeed, the first fundamental form of ϕ_t is given by

$$I_t(e_i, e_j) = (\cosh(t) - \kappa_i \sinh(t))^2 \delta_{ij}, \tag{3.5}$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of principal directions of M^n , and it can be singular. Nonetheless, it is immediate from (3.5) that there exists $t_0 \in \mathbb{R}$ such that ϕ_t is regular (and hence solves the Christoffel problem for the function $C_t(x)$ in (3.4)) for $t \geq t_0$. In this way, the solutions to the Christoffel problem come in 1-parameter families determined by the parallel flow in the above way. Moreover, if $\kappa_i < 1$, i.e. the solution to the Christoffel problem is canonically oriented, then ϕ_t is regular for every $t \geq 0$.

Remark 11. In the Euclidean Christoffel problem it is usually assumed that the ovaloid $M^n \subset \mathbb{R}^{n+1}$ is canonically oriented, so that $R_i = 1/\kappa_i > 0$. In contrast, in the hyperbolic case it is at first sight restrictive to deal only with canonically oriented horospherical

ovaloids M^n in \mathbb{H}^{n+1} , as a change of orientation on M^n transforms the hyperbolic Gauss map G into the *negative* hyperbolic Gauss map G^- , which is totally different from G .

Nevertheless, if $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ is a negatively oriented horospherical ovaloid, there exists $t_0 > 0$ such that if $t \geq t_0$ the parallel hypersurface $\phi_t : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ is a canonically oriented horospherical ovaloid.

This property lets us work without loss of generality with the canonically oriented situation in the Christoffel problem in \mathbb{H}^{n+1} , as this problem is invariant under the parallel flow.

Formulation of the Christoffel problem in \mathbb{H}^{n+1}

Taking all of this into account, we formulate the Christoffel problem as follows:

The Christoffel problem in \mathbb{H}^{n+1} . Let $C : \mathbb{S}^n \rightarrow \mathbb{R}_+$ denote a positive smooth function. Find if there exists a canonically oriented horospherical ovaloid $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ such that its hyperbolic Gauss map and its mean of the hyperbolic curvature radii are, respectively,

$$G(x) = x \quad \text{and} \quad C(x) = \frac{1}{n} \sum_{i=1}^n \frac{1}{1 - \kappa_i} \quad (3.6)$$

for every $x \in \mathbb{S}^n$. Here $\kappa_1, \dots, \kappa_n$ are the principal curvatures of ϕ .

Definition 12. We will say that a smooth function $C : \mathbb{S}^n \rightarrow \mathbb{R}_+$ is a Christoffel function if it arises as the mean of the hyperbolic curvature radii of some canonically oriented horospherical ovaloid ϕ in \mathbb{H}^{n+1} with hyperbolic Gauss map $G(x) = x$.

Convention. From now on, and unless otherwise stated, by a horospherical ovaloid we will always mean a canonically oriented one. So, we will have

$$\mathcal{R}_i = \frac{1}{1 - \kappa_i}. \quad (3.7)$$

In [O12] one can find an alternative formulation of a Christoffel problem in \mathbb{H}^{n+1} , which is very different from the one here. However, the Christoffel-type problem proposed in that work is not invariant under isometries of \mathbb{H}^{n+1} (since it implicitly uses the Euclidean Gauss map), while the one formulated here does not have this flaw.

4. Solution of the Christoffel problem

This section is devoted to showing the equivalence of the Christoffel problem in \mathbb{H}^{n+1} and the Nirenberg problem (or Kazdan–Warner problem) on prescribing scalar curvature on \mathbb{S}^n , by means of an explicit back-and-forth procedure.

A representation formula

Firstly, we shall deduce a formula that represents locally a hypersurface in \mathbb{H}^{n+1} in terms of its hyperbolic Gauss map G and its support function e^ρ . As we shall work at a point around which G is a local diffeomorphism, we may use the hyperbolic Gauss map to parametrize the hypersurface, i.e. we may assume that the hypersurface is given by $\phi : U \subset \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ with $G(x) = x$, where U is an open subset of \mathbb{S}^n .

Theorem 13. *Let $\phi : U \subset \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ denote a locally horospherically convex hypersurface with hyperbolic Gauss map $G(x) = x$, and support function $e^\rho : U \rightarrow (0, +\infty)$. Then*

$$\phi = \frac{e^\rho}{2}(1 + e^{-2\rho}(1 + \|\nabla^{g_0} \rho\|_{g_0}^2))(1, x) + e^{-\rho}(0, -x + \nabla^{g_0} \rho). \tag{4.1}$$

Proof. We will give a constructive proof, in order to explain the origin of the expression (4.1). Let $g := e^{2\rho}g_0$, where g_0 is the canonical metric in \mathbb{S}^n . It follows from (2.4) that $\langle d\psi, d\psi \rangle = e^{2\rho} \langle dG, dG \rangle_{\mathbb{S}^n} = e^{2\rho}g_0 = g$. Our first objective is to prove the formula

$$\phi = \frac{1}{n} \Delta^g \psi + \frac{S(g) + n(n-1)}{2n(n-1)} \psi, \tag{4.2}$$

where $S(g)$ is the scalar curvature of g , and $\Delta^g \psi$ stands for the Laplacian of ψ with respect to g .

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of principal directions at p , with principal curvatures $\kappa_1, \dots, \kappa_n$. We set $v_i := (1/(1-\kappa_i))e_i$. It follows that $\langle d\psi(v_i), d\psi(v_j) \rangle = \delta_{ij}$. Let us now view $\psi : M^n \rightarrow \mathbb{N}_+^{n+1} \subset \mathbb{L}^{n+2}$ as a spacelike codimension-2 submanifold of \mathbb{L}^{n+2} . Its normal space at every point is spanned by $\{\phi, \eta\}$, and its second fundamental form $\alpha : \mathfrak{X}(M^n) \times \mathfrak{X}(M^n) \rightarrow \mathbb{L}^{n+2}$ is given by

$$\alpha(v_i, v_j) = \left(\frac{1}{1-\kappa_i} \phi + \frac{\kappa_i}{1-\kappa_i} \eta \right) \delta_{ij}. \tag{4.3}$$

If $K(x, y)$ denotes the sectional curvature of ψ , the Gauss equation in \mathbb{L}^{n+2} lets us infer from (4.3) that

$$K(v_i, v_j) = \langle \alpha(v_i, v_i), \alpha(v_j, v_j) \rangle - \|\alpha(v_i, v_j)\|^2 = 1 - \frac{1}{1-\kappa_i} - \frac{1}{1-\kappa_j}.$$

Hence

$$S(g) = n(n-1) - 2(n-1) \sum_{i=1}^n \frac{1}{1-\kappa_i}. \tag{4.4}$$

Also by (4.3) we infer that the mean curvature vector of ψ in \mathbb{L}^{n+2} is

$$\begin{aligned} \mathbf{H} &= \frac{1}{n} \sum_{i=1}^n \alpha(v_i, v_i) = \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{1-\kappa_i} \phi + \frac{\kappa_i}{1-\kappa_i} \eta \right) \\ &= \phi + \frac{1}{n} \left(\sum_{i=1}^n \frac{\kappa_i}{1-\kappa_i} \right) \psi = \phi + \left(-1 + \frac{1}{n} \sum_{i=1}^n \frac{1}{1-\kappa_i} \right) \psi. \end{aligned}$$

Now, if we recall the general relation $\Delta^g \psi = n\mathbf{H}$ that holds for any spacelike n -submanifold of \mathbb{L}^{n+2} , we find that

$$\phi = \frac{1}{n} \Delta^g \psi + \left(1 - \frac{1}{n} \sum_{i=1}^n \frac{1}{1 - \kappa_i} \right) \psi.$$

Finally, (4.4) shows that the above equation yields (4.2).

Let us now compute $\Delta^g \psi$. We shall work at a fixed point $x \in U \subset \mathbb{S}^n$. There exists an orthonormal basis $\{e_1, \dots, e_n\}$ of (\mathbb{S}^n, g_0) around x such that

$$\nabla_{e_i}^{g_0} e_j = 0$$

at x . From now on we will suppress the point x from our notation when possible, as we will always be working at that point.

Let $\{v_0, \dots, v_{n+1}\}$ denote the canonical basis of \mathbb{L}^{n+2} , and write $\psi = (\psi_0, \dots, \psi_{n+1})$ in canonical coordinates. Recalling now that $\psi = e^\rho(1, x)$, and after expressing the point $x \in \mathbb{S}^n \subset \mathbb{R}^{n+1} \equiv \{v \in \mathbb{L}^{n+2} : v_0 = 0\}$ as $x = \sum_{k=1}^{n+1} x_k v_k$, we compute

$$\begin{aligned} \Delta^{g_0} \psi &= \left(\Delta^{g_0}(e^\rho), \Delta^{g_0}(e^\rho)x + e^\rho \Delta^{g_0} x + 2e^\rho \sum_{k=1}^{n+1} g_0(\nabla^{g_0} x_k, \nabla^{g_0} \rho) v_k \right) \\ &= (\Delta^{g_0}(e^\rho), \Delta^{g_0}(e^\rho)x + e^\rho \Delta^{g_0} x + 2e^\rho \nabla^{g_0} \rho) \\ &= (e^{-\rho} \Delta^{g_0}(e^\rho)) \psi + (0, e^\rho \Delta^{g_0} x + 2e^\rho \nabla^{g_0} \rho). \end{aligned}$$

Now, since $\Delta^{g_0} x = -nx$ and $\Delta^{g_0} e^\rho = e^\rho(\Delta^{g_0} \rho + \|\nabla^{g_0} \rho\|_{g_0}^2)$, we have

$$\Delta^{g_0} \psi = (\Delta^{g_0} \rho + \|\nabla^{g_0} \rho\|_{g_0}^2) \psi + e^\rho(0, -nx + 2\nabla^{g_0} \rho). \tag{4.5}$$

Let us now introduce the following notation, for any $Z \in \mathfrak{X}(\mathbb{S}^n)$:

$$g_0(\nabla^{g_0} \psi, Z) := \sum_{k=0}^{n+1} g_0(\nabla^{g_0} \psi_k, Z) v_k = \sum_{k=0}^{n+1} Z(\psi_k) v_k = Z(\psi) \in \mathfrak{X}(\psi) \equiv \mathfrak{X}(\phi).$$

Then we have

$$\begin{aligned} g_0(\nabla^{g_0} \psi, \nabla^{g_0} \rho) &= (\nabla^{g_0} \rho)(\psi) = \sum_{i=1}^n e_i(\rho) e_i(\psi) \\ &= \sum_{i=1}^n e_i(\rho) (e^\rho(e_i(\rho))(1, x) + e^\rho(0, e_i)) \\ &= e^\rho \left(\sum_{i=1}^n (e_i(\rho))^2 \right) (1, x) + e^\rho \left(0, \sum_{i=1}^n e_i(\rho) e_i \right) \\ &= \|\nabla^{g_0} \rho\|_{g_0}^2 \psi + e^\rho(0, \nabla^{g_0} \rho). \end{aligned} \tag{4.6}$$

We now recall the usual relation between the Laplacians of two conformal metrics to deduce that, since $g = e^{2\rho}g_0$, we have

$$\Delta^g \psi = e^{-2\rho}(\Delta^{g_0} \psi + (n-2)g_0(\nabla^{g_0} \rho, \nabla^{g_0} \psi)). \quad (4.7)$$

Thus, by (4.5) and (4.6), we deduce from (4.7) that

$$\Delta^g \psi = e^{-2\rho}(\Delta^{g_0} \rho + (n-1)\|\nabla^{g_0} \rho\|_{g_0}^2)\psi + ne^{-\rho}(0, -x + \nabla^{g_0} \rho). \quad (4.8)$$

On the other hand, it is well known that if $g = e^{2\rho}g_0$ is a conformal metric on \mathbb{S}^n , then the scalar curvature $S(g)$ of g is related to ρ by means of the following elliptic PDE:

$$\Delta^{g_0} \rho + \frac{n-2}{2}\|\nabla^{g_0} \rho\|_{g_0}^2 - \frac{n}{2} + \frac{e^{2\rho}}{2(n-1)}S(g) = 0. \quad (4.9)$$

Thus,

$$e^{-2\rho}(\Delta^{g_0} \rho + (n-1)\|\nabla^{g_0} \rho\|_{g_0}^2) = -\frac{S(g)}{2(n-1)} + \frac{ne^{-2\rho}}{2}(1 + \|\nabla^{g_0} \rho\|_{g_0}^2). \quad (4.10)$$

If we substitute (4.10) into (4.8) we obtain

$$\Delta^g \psi = \left(-\frac{S(g)}{2(n-1)} + \frac{ne^{-2\rho}}{2}(1 + \|\nabla^{g_0} \rho\|_{g_0}^2)\right)\psi + ne^{-\rho}(0, -x + \nabla^{g_0} \rho). \quad (4.11)$$

Finally, plugging (4.11) into (4.2) we get (4.1), as we wished. \square

Remark 14. The parallel flow $\{\phi_t\}_{t \in \mathbb{R}}$ of ϕ in $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ is given by

$$\phi_t = \cosh(t)\phi + \sinh(t)(\psi - \phi) = e^{-t}\phi + \sinh(t)\psi.$$

Thus, using (4.1) we obtain the explicit formula

$$\phi_t = \frac{e^t}{2}e^\rho \left(1 + \frac{e^{-2\rho}}{e^{2t}}(1 + \|\nabla^{g_0} \rho\|_{g_0}^2)\right)(1, x) + \frac{e^{-\rho}}{e^t}(0, -x + \nabla^{g_0} \rho). \quad (4.12)$$

A solution for the Christoffel problem in \mathbb{H}^{n+1}

We are now in a position to prove one of our main results.

Theorem 15. *Let $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ be a solution of the Christoffel problem in \mathbb{H}^{n+1} for the smooth function $C : \mathbb{S}^n \rightarrow \mathbb{R}_+$. Then its horospherical metric g_∞ is a solution to the Nirenberg problem in \mathbb{S}^n for the scalar curvature function $S : \mathbb{S}^n \rightarrow \mathbb{R}$ given by*

$$S(x) = n(n-1)(1 - 2C(x)). \quad (4.13)$$

Conversely, let $g = e^{2\rho} g_0$ denote a solution to the Nirenberg problem for the scalar curvature function $S : \mathbb{S}^n \rightarrow \mathbb{R}$. Then there exists $\tau_0 > 0$ such that for every $\tau \geq \tau_0$ the map $\phi_\tau : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ given by

$$\phi_\tau = \frac{\tau}{2} e^\rho \left(1 + \frac{e^{-2\rho}}{\tau^2} (1 + \|\nabla^{g_0} \rho\|_{g_0}^2) \right) (1, x) + \frac{e^{-\rho}}{\tau} (0, -x + \nabla^{g_0} \rho) \tag{4.14}$$

is a solution to the Christoffel problem in \mathbb{H}^{n+1} for the smooth function

$$C_\tau(x) = \frac{1}{2} \left(1 - \frac{S(x)}{\tau^2 n(n-1)} \right). \tag{4.15}$$

Moreover, the horospherical metric of ϕ_τ is actually $g_\infty = \tau^2 g$.

Remark 16. Theorem 15 proves that the Christoffel problem in \mathbb{H}^{n+1} and the Nirenberg problem on \mathbb{S}^n are essentially equivalent problems. On the other hand, we cannot claim that they are completely equivalent. For instance, a scalar curvature function $S(x)$ on \mathbb{S}^n that arises as the horospherical metric of some horospherical ovaloid must satisfy $S(x) < n(n-1)$ by (4.13), while this estimate does not hold for general scalar curvature functions on \mathbb{S}^n . Alternatively, one can say that given a conformal metric $g = e^{2\rho} g_0$ on \mathbb{S}^n , the map $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ given by the representation formula (4.1) will be a solution to the Christoffel problem in \mathbb{H}^{n+1} if and only if it is free of singular points, which is not always the case. We will write down in Corollary 23 an explicit condition for g that is equivalent to the regularity of ϕ .

Remark 17. Let us explain in more detail the converse in Theorem 15. First, note that two conformal metrics g, \tilde{g} on \mathbb{S}^n differ just by a dilation (i.e. $\tilde{g} = e^{2t} g$ for some fixed $t \in \mathbb{R}$) if and only if their associated hypersurfaces $\phi, \tilde{\phi} : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ are parallel (specifically, $\tilde{\phi} = \phi_t$). Thus, given a conformal metric $g = e^{2\rho} g_0$ in \mathbb{S}^n , there exists some $\tau_0 > 0$ such that for $\tau \geq \tau_0$ the hypersurface $\phi_\tau : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ associated to $g_\tau := \tau^2 g$ is everywhere regular, and hence a solution to the Christoffel problem. This is the way that we prevented in Theorem 15 the appearance of singular points for ϕ .

Proof of Theorem 15. Let $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ denote a solution to the Christoffel problem in \mathbb{H}^{n+1} . Then its hyperbolic Gauss map is $G(x) = x$. So, by (2.4), the horospherical metric of ϕ is

$$g_\infty = \langle d\psi, d\psi \rangle = e^{2\rho} \langle dG, dG \rangle_{\mathbb{S}^n} = e^{2\rho} g_0,$$

where g_0 is the canonical metric of \mathbb{S}^n . Hence, the horospherical metric of ϕ is globally conformal to g_0 . Moreover, by (4.4) and (3.6) we see that (4.13) holds. This proves the first assertion.

Conversely, let $g = e^{2\rho} g_0$ denote a conformal metric on \mathbb{S}^n with scalar curvature function $S(x)$, and view $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ in the usual way. Consider in addition a positive constant $\tau > 0$ and the conformal metric $g_\tau := \tau^2 g$, whose scalar curvature function is obviously $S_\tau(x) = (1/\tau^2)S(x)$. Then we can construct the map

$$\psi_\tau = \tau e^\rho (1, x) : \mathbb{S}^n \rightarrow \mathbb{N}_+^{n+1} \subset \mathbb{L}^{n+2} \equiv \mathbb{L} \times \mathbb{R}^{n+1}, \tag{4.16}$$

which has the property that $\langle d\psi_\tau, d\psi_\tau \rangle = \tau^2 e^{2\rho} g_0 = g_\tau$. If we let

$$\xi := (0, -x + \nabla^{g_0} \rho),$$

we may write (4.14) as

$$\phi_\tau = \frac{1}{2} \left(1 + \frac{e^{-2\rho}}{\tau^2} (1 + \|\nabla^{g_0} \rho\|_{g_0}^2) \right) \psi_\tau + \frac{e^{-\rho}}{\tau} \xi. \tag{4.17}$$

Next, observe that $\langle \psi_\tau, \psi_\tau \rangle = 0$, and also that

$$\langle \psi_\tau, \xi \rangle = -\tau e^\rho + \tau e^\rho \langle x, \nabla^{g_0} \rho \rangle = -\tau e^\rho,$$

and in the same way $\langle \xi, \xi \rangle = 1 + \|\nabla^{g_0} \rho\|_{g_0}^2$. Therefore $\langle \phi_\tau, \psi_\tau \rangle = -1$ (thus the first coordinate of ϕ in \mathbb{L}^{n+2} is positive), and

$$\langle \phi_\tau, \phi_\tau \rangle = -1 - \frac{e^{-2\rho}}{\tau^2} (1 + \|\nabla^{g_0} \rho\|_{g_0}^2) + \frac{e^{-2\rho}}{\tau^2} (1 + \|\nabla^{g_0} \rho\|_{g_0}^2) = -1.$$

This proves that ϕ_τ takes its values in $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$. Finally, let $Z \in \mathfrak{X}(\mathbb{S}^n)$. Then, using the fact that $Z(x) = Z$, $\langle x, Z \rangle = 0$ and $\langle \nabla^{g_0} \rho, Z \rangle = Z(\rho)$, we see by (4.14) and (4.16) that

$$\begin{aligned} \langle \phi_\tau, Z(\psi_\tau) \rangle &= \tau \langle \phi_\tau, e^\rho Z(\rho)(1, x) + e^\rho (0, Z) \rangle \\ &= \langle (0, -x + \nabla^{g_0} \rho), Z(\rho)(1, x) + (0, Z) \rangle = -Z(\rho) + \langle \nabla^{g_0} \rho, Z \rangle = 0. \end{aligned}$$

Putting all of this together we have found that $\phi_\tau : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ is a hypersurface possibly with singular points, but such that at its regular points its associated light cone immersion is $\psi_\tau : \mathbb{S}^n \rightarrow \mathbb{N}_+^{n+1}$. This happens because $\langle \phi_\tau, \psi_\tau \rangle = -1$, $\langle \psi_\tau, \psi_\tau \rangle = 0$ and $\langle d(\phi_\tau), \psi_\tau \rangle = 0$, i.e. ψ_τ is normal to ϕ_τ . In particular, its hyperbolic Gauss map at regular points is $G(x) = x$, and it is horospherically convex at those points. Moreover, the horospherical metric of ϕ_τ at regular points is $g_\infty = \langle d\psi_\tau, d\psi_\tau \rangle = g_\tau$, which has the scalar curvature function $S_\tau(x) = \tau^{-2} S(x)$. Thus, by (4.4) the mean of the hyperbolic curvature radii of ϕ_τ at any regular point is given by

$$C_\tau = \frac{1}{n} \sum_{i=1}^n \mathcal{R}_i^\tau = \frac{1}{n} \sum_{i=1}^n \frac{1}{1 - \kappa_i^\tau} = \frac{1}{2} \left(1 - \frac{S(x)}{\tau^2 n(n-1)} \right),$$

which is exactly (4.15). Thus, in order to finish the proof we only need to ensure the existence of some $\tau_0 > 0$ such that $\psi_\tau : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ is everywhere regular whenever $\tau \geq \tau_0$. First of all, set $\phi := \phi_1$ and $\psi := \psi_1$. Then we may easily observe that

$$\phi_\tau = \frac{1}{\tau} \phi + \frac{\tau^2 - 1}{2\tau} \psi. \tag{4.18}$$

Let $U\mathbb{S}^n$ denote the unit tangent bundle of \mathbb{S}^n , i.e.

$$U\mathbb{S}^n = \{(p, v) \in \mathbb{S}^n \times \mathbb{S}^n : \langle p, v \rangle = 0\}.$$

By (4.18) we have $d(\phi_\tau)_p(v) = (1/\tau)d\phi_p(v) + ((\tau^2 - 1)/(2\tau))d\psi_p(v)$ for every $(p, v) \in U\mathbb{S}^n$. Thus, ϕ_τ is everywhere regular if and only if $d(\phi_\tau)_p(v) \neq 0$ for every $(p, v) \in U\mathbb{S}^n$, if and only if

$$d\phi_p(v) \neq \frac{1}{2}(1 - \tau^2)d\psi_p(v) \quad \text{for every } (p, v) \in U\mathbb{S}^n. \quad (4.19)$$

But now, as $U\mathbb{S}^n$ is compact, the sets

$$\Lambda := \{d\psi_p(v) \in \mathbb{L}^{n+2} : (p, v) \in U\mathbb{S}^n\}, \quad \Omega := \{d\phi_p(v) \in \mathbb{L}^{n+2} : (p, v) \in U\mathbb{S}^n\}$$

are compact in $\mathbb{R}^{n+2} \cong \mathbb{L}^{n+2}$. Moreover, as $d\psi_p(v) \neq 0$ always, we have $0 \notin \Lambda$, and thus we may infer the existence of some $r_0 > 0$ such that if $r \geq r_0$, then $\Omega \cap r\Lambda = \emptyset$. In particular, there exists some $\tau_0 > 0$ such that if $\tau \geq \tau_0$, the condition (4.19) holds. This proves that ϕ_τ is everywhere regular if $\tau \geq \tau_0$, and finishes the proof. \square

Applications

As a straightforward consequence of Theorem 15, we can rephrase in the context of the Christoffel problem in \mathbb{H}^{n+1} all the results on the prescribed scalar curvature problem in \mathbb{S}^n . It is our aim now to make explicit some of them.

The following result is a translation into our setting via Theorem 15 of the classical necessary conditions by Kazdan–Warner [KW1] and Bourguignon–Ezin [BE] on prescribing scalar curvature in \mathbb{S}^n .

Corollary 18 (Necessary conditions). *Let $C : \mathbb{S}^n \rightarrow \mathbb{R}_+$ be a Christoffel function. Then:*

- (i) $C(x) < 1/2$ for some $x \in \mathbb{S}^n$.
- (ii) If x_1, \dots, x_{n+1} denote the coordinate functions of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, then

$$\int_{\mathbb{S}^n} g_0(\nabla^{g_0} C, \nabla^{g_0} x_i) dv_g = 0, \quad i = 1, \dots, n+1,$$

where dv_g is the volume element of the horospherical metric of the horospherical ovaloid ϕ . In particular, C cannot be a monotonic function of a coordinate x_i .

- (iii) More generally,

$$\int_{\mathbb{S}^n} X(C) dv_g = 0$$

for any conformal vector field $X \in \mathfrak{X}(\mathbb{S}^n)$.

The following very interesting result on the moduli space of solutions to the Nirenberg problem was obtained by Y.Y. Li in [L1], as a strong generalization of a previous density result by Bourguignon–Ezin: *Smooth scalar curvature functions on \mathbb{S}^n are C^0 dense among functions on \mathbb{S}^n that are positive somewhere*. As a consequence of this result and Theorem 15 we have:

Corollary 19. *Let $F : \mathbb{S}^n \rightarrow \mathbb{R}$ denote a smooth function such that $F(x_0) < 1/2$ for some $x_0 \in \mathbb{S}^n$. Then for every $\epsilon > 0$ there exists a Christoffel function $C : \mathbb{S}^n \rightarrow \mathbb{R}_+$ and some $t \in \mathbb{R}$ such that*

$$\|F - C_t\|_{C^0(\mathbb{S}^n)} < \epsilon, \quad C_t(x) := \frac{1}{2} - \frac{e^{-2t}}{2}(1 - 2C(x)).$$

Moreover, the Christoffel function C can be chosen so that $F(x) = C_t(x)$ on the exterior of an arbitrarily small ball of \mathbb{S}^n centred at x_0 .

Concerning sufficient conditions for the Nirenberg problem, we cannot make justice in a few lines to the diversity of results that are known once some technical condition on the scalar curvature function is imposed (a good reference for that is the survey [L6]). Let us simply say here that each of these sufficient conditions can be rephrased in our context to yield sufficient conditions for a function to be a Christoffel function.

We shall nonetheless point out just a couple of sufficient conditions under symmetry assumptions on the Christoffel function. They follow from [Mo, ES, CLi2]. Let us say that a function $C \in C^\infty(\mathbb{S}^n)$ is a *generalized Christoffel function* if there exists $t_0 \in \mathbb{R}$ such that for every $t \geq t_0$ the map

$$C_t(x) := \frac{1}{2} - \frac{e^{-2t}}{2}(1 - 2C(x)) : \mathbb{S}^n \rightarrow \mathbb{R}$$

is a Christoffel function.

Corollary 20. *Let $C \in C^\infty(\mathbb{S}^n)$ be a smooth function with $C(x_0) < 1/2$ for some $x_0 \in \mathbb{S}^n$. Then C is a generalized Christoffel function if it satisfies one of the following conditions.*

- (i) $C(x) = C(-x)$ and there is some $\bar{x} \in \mathbb{S}^n$ such that $C(\bar{x}) = \min C$ and all derivatives of C up to order $n - 2$ vanish at \bar{x} .
- (ii) $C = C(r)$ is rotationally invariant on \mathbb{S}^n , $C'(r)$ changes sign in the region(s) where $C < 1/2$, and near any critical point r_0 the following flatness condition holds:

$$C(r) = C(r_0) + a|r - r_0|^\alpha + h(|r - r_0|),$$

where $a \neq 0$, $n - 2 < \alpha < n$, and $h'(s) = o(s^{\alpha-1})$.

5. Generalized Christoffel problems

A natural extension of the Christoffel problem in hyperbolic space is to prescribe for a horospherical ovaloid in \mathbb{H}^{n+1} the hyperbolic Gauss map together with a given functional of the hyperbolic curvature radii. Our aim in this section is to show that this problem is equivalent to the question of prescribing a given functional of the eigenvalues of the Schouten tensor for a conformal metric on \mathbb{S}^n .

Given a Riemannian metric g on a manifold M^n , $n > 2$, the *Schouten tensor* of g is the symmetric $(0, 2)$ -type tensor given by

$$\text{Sch}_g := \frac{1}{n-2} \left(\text{Ric}_g - \frac{S(g)}{2(n-1)}g \right), \tag{5.1}$$

where Ric_g and $S(g)$ stand for the Ricci and scalar curvatures of g . Its study is an important issue in conformal geometry, as it represents the non-conformally invariant part of the Riemann curvature tensor. For instance, if $\tilde{g} = e^{2u}g$ is a conformal metric, then

$$\text{Sch}_{\tilde{g}} = \text{Sch}_g - \nabla^{2, \cdot} u + du \otimes du - \frac{1}{2} \|\nabla^g u\|_g^2 g. \tag{5.2}$$

Let us now consider a conformal metric $g = e^{2\rho}g_0$ on \mathbb{S}^n , $n > 2$. By (5.1) it follows immediately that $\text{Sch}_{g_0} = (1/2)g_0$, which is independent of n . Hence (5.2) yields

$$\text{Sch}_g = -\nabla^{2, \cdot} \rho + d\rho \otimes d\rho - \frac{1}{2}(-1 + \|\nabla^{g_0} \rho\|_{g_0}^2)g_0. \tag{5.3}$$

The Schouten tensor (5.1) is not defined for 2-dimensional metrics. Nevertheless, as (5.3) makes sense also for $n = 2$, we may naturally define the Schouten tensor for conformal metrics on \mathbb{S}^2 in a unifying way:

Definition 21. *Let $g = e^{2\rho}g_0$ denote a conformal metric on \mathbb{S}^2 . Then its Schouten tensor is defined as the symmetric $(0, 2)$ -type tensor given by (5.3).*

Bearing this in mind, we have

Theorem 22. *Let $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ denote a horospherical ovaloid with hyperbolic Gauss map $G(x) = x$, and let $g = e^{2\rho}g_0$ denote its horospherical metric. Then the first and second fundamental forms of ϕ at $x \in \mathbb{S}^n$ are given, respectively, by*

$$I_\phi(e_i, e_j) = \frac{e^{-2\rho}}{4} (g(e_i, e_j) - 2\text{Sch}_g(e_i, e_j))^2 \tag{5.4}$$

and

$$II_\phi = I_\phi - \frac{1}{2}g + \text{Sch}_g. \tag{5.5}$$

Here $e_1, \dots, e_n \in T_x \mathbb{S}^n$ is an orthonormal frame with respect to g_0 such that $\nabla_{e_i}^{g_0} e_j = 0$ at x for every i, j .

Proof. Let $\{e_1, \dots, e_n\}$ be the basis at $x \in \mathbb{S}^n$ as in the statement. Then by Theorem 13 we may write (4.1) as

$$\phi = f(1, x) + e^{-\rho}(0, \xi), \tag{5.6}$$

where

$$f := \frac{e^\rho}{2} (1 + e^{-2\rho} (1 + \|\nabla^{g_0} \rho\|_{g_0}^2)), \quad \xi := (0, -x + \nabla^{g_0} \rho).$$

Since $\nabla_{e_i}^{g_0} e_j = 0$ implies $e_i(e_j) = -\delta_{ij}x$ at x , we get

$$e_i(\xi) = \left(0, -e_i + e_i\left(\sum_{j=1}^n e_j(\rho)e_j\right)\right) = \left(0, -e_i - e_i(\rho)x + \sum_{j=1}^n e_i(e_j(\rho))e_j\right). \quad (5.7)$$

So, by (5.6) and (5.7) we get

$$e_i(\phi) = e_i(f)(1, x) + (f - e^{-\rho})(0, e_i) - e^{-\rho}\left(0, \sum_{j=1}^n \{e_i(\rho)e_j(\rho) - e_i(e_j(\rho))\}e_j\right). \quad (5.8)$$

From (5.8) we see directly that $\langle(1, x), e_i(\phi)\rangle = 0$. In addition,

$$\langle e_i(\phi), (0, e_j)\rangle = (f - e^{-\rho})\delta_{ij} - e^{-\rho}(e_i(\rho)e_j(\rho) - e_i(e_j(\rho))). \quad (5.9)$$

Thus, by (5.8), (5.9) we have

$$\begin{aligned} \langle e_i(\phi), e_j(\phi)\rangle &= e^{-2\rho}(f - e^{-\rho})^2\delta_{ij} - 2e^{-2\rho}(f - e^{-\rho})(e_i(\rho)e_j(\rho) - e_i(e_j(\rho))) \\ &\quad + e^{-2\rho}\left(\sum_{k=1}^n (e_i(\rho)e_k(\rho) - e_i(e_k(\rho)))(e_j(\rho)e_k(\rho) - e_j(e_k(\rho)))\right). \end{aligned}$$

In order to simplify this expression, let A, B denote the $n \times n$ matrices

$$A = (a_{ij}) \quad \text{with} \quad a_{ij} := e^{-\rho}(e_i(e_j(\rho)) - e_i(\rho)e_j(\rho)), \quad B := (f - e^{-\rho})\text{Id}_n.$$

Then the above expressions immediately imply that $\langle e_i(\phi), e_j(\phi)\rangle$ is the (i, j) entry of the matrix $(A + B)^2$. On the other hand, we may observe that A is the matrix expression of $e^{-\rho}(\nabla^{2, g_0}\rho - d\rho \otimes d\rho)$ with respect to the basis $\{e_1, \dots, e_n\}$. Therefore, $A + B$ is the matrix expression of

$$\frac{e^\rho}{2}g_0 + e^{-\rho}\left(\nabla^{2, g_0}\rho - d\rho \otimes d\rho + \frac{1}{2}(-1 + \|\nabla^{g_0}\rho\|_{g_0}^2)g_0\right),$$

which by (5.3) is just $e^{-\rho}(g/2 - \text{Sch}_g)$. Thus,

$$\langle e_i(\phi), e_j(\phi)\rangle = \frac{e^{-2\rho}}{4}(g(e_i, e_j) - 2\text{Sch}_g(e_i, e_j))^2,$$

which yields (5.4).

In order to compute the second fundamental form, let us first note that

$$e_i(\psi) = e^\rho e_i(\rho)(1, x) + e^\rho(0, e_i),$$

and so by (5.9) and (5.3) we have

$$\langle d\phi, d\psi\rangle = g/2 - \text{Sch}_g. \quad (5.10)$$

Therefore, as $II_\phi = -\langle d\phi, d\eta\rangle = \langle d\phi, d\phi\rangle - \langle d\phi, d\psi\rangle$, we obtain (5.5) from (5.10). \square

As an immediate consequence of this result and Theorem 15, we have

Corollary 23. *A smooth function $C : \mathbb{S}^n \rightarrow \mathbb{R}_+$ is a Christoffel function if and only if $S(x) := n(n - 1)(1 - 2C(x))$ is the scalar curvature function of some conformal metric $g = e^{2\rho}g_0$ such that $g - 2\text{Sch}_g$ is everywhere positive definite on \mathbb{S}^n .*

In [Sc] Schlenker gave a necessary and sufficient condition for a conformally flat metric to be realized as the horospherical metric of some locally horospherically convex hypersurface in \mathbb{H}^{n+1} . Thus, Corollary 23 can be seen as an easy consequence of Theorem 15 and [Sc], although the Schouten tensor Sch_g is never mentioned there.

Let $g = e^{2\rho}g_0$ be a conformal metric on \mathbb{S}^n , and take $x \in \mathbb{S}^n$. Then we can consider the eigenvectors $v_1, \dots, v_n \in T_x\mathbb{S}^n$ and the eigenvalues $\lambda_1, \dots, \lambda_n$ of the Schouten tensor Sch_g with respect to g . Thus, $g(v_i, v_j) = \delta_{ij}$ and $\text{Sch}_g(v_i, v_j) = \lambda_i\delta_{ij}$. Theorem 22 then yields the following conclusion:

Corollary 24. *Let $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ be a horospherical ovaloid, and let $\{\mathcal{R}_1, \dots, \mathcal{R}_n\}$ denote its hyperbolic curvature radii at $x \in \mathbb{S}^n$. Then*

$$\mathcal{R}_i = 1/2 - \lambda_i, \quad i = 1, \dots, n, \tag{5.11}$$

where $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of the Schouten tensor of the horospherical metric g of ϕ at x . Moreover, the eigendirections of Sch_g at x coincide with the principal directions of ϕ at x .

Proof. Let $\{e'_1, \dots, e'_n\}$ denote an orthonormal basis of principal directions of ϕ at x , and define $v_i := (1/(1 - \kappa_i))e'_i$. Then $g(v_i, v_j) = \delta_{ij}$ and

$$\langle d\phi(v_i), d\psi(v_j) \rangle = \frac{1}{1 - \kappa_i} \delta_{ij} = \mathcal{R}_i \delta_{ij}.$$

On the other hand, by (5.10),

$$\langle d\phi(v_i), d\psi(v_j) \rangle = \frac{1}{2} \delta_{ij} - \text{Sch}_g(v_i, v_j).$$

Thus $\text{Sch}_g(v_i, v_j) = (1/2 - \mathcal{R}_i)\delta_{ij}$, which yields (5.11). The assertion on the eigendirections of Sch_g also follows directly. \square

Let us point out that from (5.11) and (3.7) we get

$$\lambda_i = \frac{1}{2} - \mathcal{R}_i = \frac{1}{2} - \frac{1}{1 - \kappa_i} = -\frac{1 + \kappa_i}{2(1 - \kappa_i)}. \tag{5.12}$$

An interpretation in \mathbb{H}^{n+1} of the Schouten tensor. Let $\phi : M^n \rightarrow \mathbb{H}^{n+1}$ denote an oriented hypersurface that is horospherically convex at p , and let us assume that $|\kappa_i(p)| < 1$ or $|\kappa_i(p)| > 1$ simultaneously for every $i = 1, \dots, n$. Then ϕ with its opposite orientation is still horospherically convex at p . Thus, its negative hyperbolic Gauss map G^- is a local diffeomorphism at p , and we can define the *negative curvature radii*

$$\mathcal{R}_i^*(p) = \frac{1}{|1 + \kappa_i(p)|}.$$

We can now consider the quotient

$$\mathcal{D}_i(p) := \frac{\mathcal{R}_i(p)}{\mathcal{R}_i^*(p)} = \frac{|1 + \kappa_i(p)|}{|1 - \kappa_i(p)|} > 0, \tag{5.13}$$

which measures the variation in the hyperbolic curvature radii of ϕ after a change of orientation. We call $\mathcal{D}_i(p)$ the *dilation ratios* of ϕ at p . This concept does not have a Euclidean counterpart, i.e. it is specific to hyperbolic geometry.

Now, let $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ denote a horospherical ovaloid, and assume furthermore that $\kappa_i < -1$ everywhere. This condition is equivalent (by compactness) to requiring that $G^- : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a global diffeomorphism, and it forces the convexity of ϕ . We will say that ϕ is then a *strongly H-convex* ovaloid in \mathbb{H}^{n+1} .

Let $\{\lambda_1, \dots, \lambda_n\}$ denote the eigenvalues of the Schouten tensor for the horospherical metric of ϕ . Then by (5.12) we have

$$\lambda_i = \frac{1}{2} \mathcal{D}_i. \tag{5.14}$$

Hence, the eigenvalues λ_i of the Schouten tensor are identified on a strongly *H-convex* ovaloid as half the dilation ratios \mathcal{D}_i of ϕ .

Alternatively, by (3.1), we can conclude that

$$e^{2\varrho_i} = \frac{|1 + \kappa_i|}{|1 - \kappa_i|} = \mathcal{D}_i = 2\lambda_i, \tag{5.15}$$

which provides an interpretation of the eigenvalues λ_i in terms of the classical *contact radii* $\varrho_i = \coth^{-1}(|\kappa_i|)$.

We will work with a slightly more general notion of contact radii, which is more suitable for non-convex horospherical ovaloids.

Definition 25. Let $M^n \subset \mathbb{H}^{n+1}$ denote a horospherical ovaloid with principal curvatures $\kappa_1, \dots, \kappa_n$. We define the signed contact radii $\delta_1, \dots, \delta_n$ of M^n at p as

$$\delta_i(p) := \frac{1 + \kappa_i(p)}{1 - \kappa_i(p)} \quad (\text{thus } |\delta_i(p)| = \mathcal{D}_i(p) \text{ if } \kappa_i(p) \neq -1). \tag{5.16}$$

The generalized Christoffel problem. Let $\Gamma : \mathbb{S}^n \rightarrow \mathbb{R}$ be a smooth function. Find if there exists a hypersurface $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ (necessarily a horospherical ovaloid) whose hyperbolic Gauss map is $G(x) = x$ for every $x \in \mathbb{S}^n$, and such that

$$\mathcal{F}(\mathcal{R}_1, \dots, \mathcal{R}_n) = \Gamma, \tag{5.17}$$

where $\mathcal{F}(\mathcal{R}_1, \dots, \mathcal{R}_n)$ is a prescribed functional of the hyperbolic curvature radii \mathcal{R}_i of the hypersurface ϕ .

Remark 26. By (3.7) it is obvious that prescribing a relation of the form (5.17) for the hyperbolic curvature radii $\mathcal{R}_i > 0$ can also be seen as prescribing a relation of the form $\mathcal{W}(\kappa_1, \dots, \kappa_n) = \Gamma$ for the principal curvatures $\kappa_1, \dots, \kappa_n$, with the restriction $\kappa_i \in (-\infty, 1)$. The novelty of the generalized Christoffel problem with respect to other previous works on hypersurfaces in space forms satisfying $\mathcal{W}(\kappa_1, \dots, \kappa_n) = \Gamma$ is that we are also prescribing here the hyperbolic Gauss map G .

In this generality, the problem is quite hopeless. Nonetheless, we can deduce from Corollaries 23 and 24 the following:

Corollary 27. *Let $\phi : \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ denote a solution to the generalized Christoffel problem, let g denote its horospherical metric, and define*

$$\mathcal{G}(t_1, \dots, t_n) := \mathcal{F}(1/2 - t_1, \dots, 1/2 - t_n). \quad (5.18)$$

Then the eigenvalues λ_i of Sch_g satisfy $\lambda_i < 1/2$ and $\mathcal{G}(\lambda_1, \dots, \lambda_n) = \Gamma(x)$.

Conversely, if $g = e^{2\rho} g_0$ is a conformal metric on \mathbb{S}^n such that the eigenvalues λ_i of its Schouten tensor satisfy $\lambda_i < 1/2$ and $\mathcal{G}(\lambda_1, \dots, \lambda_n) = \Gamma(x)$ for a certain functional \mathcal{G} , then there exists a solution to the generalized Christoffel problem for \mathcal{F} given by (5.18), and whose horospherical metric is g .

This result paves the way for studying the generalized Christoffel problem in special cases, as the behaviour of the eigenvalues of the Schouten tensor is a widely studied subject. We next discuss some interesting points of the correspondence given by Corollary 27.

The k -th symmetric problems on \mathbb{S}^n and \mathbb{H}^{n+1}

The problem of prescribing a certain functional of the eigenvalues of the Schouten tensor for a conformal metric on a Riemannian manifold has gained several contributions in the last few years: see [Cha, CGY1, CGY2, Vi, JLX] and specially the book [Gu] and references therein. In particular, special attention has been paid to the study of the σ_k -curvatures of a Riemannian metric g , which can be defined as follows.

First, let $S_k(x_1, \dots, x_n)$ be the k -th elementary symmetric function of (x_1, \dots, x_n) , i.e.

$$S_k(x_1, \dots, x_n) = \sum x_{i_1} \cdots x_{i_k}, \quad (5.19)$$

where the sum is taken over all strictly increasing sequences i_1, \dots, i_k of indices from the set $\{1, \dots, n\}$. Then, given a metric g on a manifold M^n , the σ_k -curvature of g is defined as $S_k(\lambda_1, \dots, \lambda_n)$, where λ_i are the eigenvalues of the Schouten tensor of g .

The problem of finding a conformal metric on \mathbb{S}^n with a prescribed σ_k -curvature has been specifically treated, as a generalization of the Nirenberg problem. By Corollary 27 and (5.14), this σ_k -Nirenberg problem is equivalent (up to dilations) to a natural problem for horospherical ovaloids in \mathbb{H}^{n+1} , namely, the generalized Christoffel problem in which the k -th elementary symmetric function of the signed dilation ratios δ_i in (5.16) is prescribed. Consequently, the local estimates in [Gu] or the Kazdan–Warner type obstructions in [Ha] can be translated into analogous results for the hyperbolic setting. We omit the details, as the process is clear from Corollary 24.

The σ_n -problem is of special interest in \mathbb{H}^{n+1} . Recall that the contact radii ϱ_i in (3.1) can be seen to some extent as analogues of the classical curvature radii in \mathbb{R}^{n+1} . The negative point was that they only make sense for strongly H -convex ovaloids in \mathbb{H}^{n+1} , which is quite restrictive. Nonetheless, by (5.15) we can see that

$$\Lambda := \frac{1}{n} \sum_{i=1}^n \varrho_i = \frac{\log(2^n \sigma_n)}{2n}. \quad (5.20)$$

Hence, by Corollary 27, the σ_n -Nirenberg problem on \mathbb{S}^n is equivalent (up to a regularity condition) to the *contact* Christoffel problem for strongly H -convex ovaloids in \mathbb{H}^{n+1} , in which the hyperbolic Gauss map together with the mean Λ of the contact radii ϱ_i are prescribed.

On the other hand, it is a natural question to consider *the k -th symmetric Christoffel problem in \mathbb{H}^{n+1}* , i.e. the generalized Christoffel problem in \mathbb{H}^{n+1} with the specific choice $\mathcal{F} = S_k$.

It is clear that if we take $k = 1$ we get the Christoffel problem in \mathbb{H}^{n+1} . The k -th symmetric Christoffel problem is a natural analogue in \mathbb{H}^{n+1} of the *Christoffel–Minkowski problem* for ovaloids in \mathbb{R}^{n+1} , which prescribes the Gauss map as well as the k -th symmetric function of the principal curvature radii. However, in this hyperbolic setting, the Minkowski problem (i.e. prescribing the hyperbolic Gauss map along with the Gauss–Kronecker curvature) does not appear as any of the k -th symmetric problems.

Now, by (5.11) we have

$$S_k(\mathcal{R}_1, \dots, \mathcal{R}_n) = \sum_{j=0}^k \binom{n-j}{k-j} (-1)^j 2^{j-k} S_j(\lambda_1, \dots, \lambda_n).$$

So, by this relation and Corollary 27, the above k -th symmetric problem in \mathbb{H}^{n+1} is equivalent (up to a regularity condition) to the Nirenberg-type problem in \mathbb{S}^n in which the following linear combination of the σ_k -curvatures is prescribed:

$$\sum_{j=0}^k c_j \sigma_j(g) = \Gamma, \quad c_j := \binom{n-j}{k-j} (-1)^j 2^{j-k}.$$

Thus, a natural situation in \mathbb{H}^{n+1} motivates the (seemingly unexplored) problem of prescribing a linear combination of the σ_k -curvatures for conformal metrics on \mathbb{S}^n .

Let us finally point out that there are many other interesting problems for hypersurfaces in \mathbb{H}^{n+1} that can be formulated in terms of geometric PDE problems by means of Corollaries 24 and 27: for instance, Dirichlet problems at infinity for horospherically convex hypersurfaces with prescribed Weingarten curvatures, or Christoffel–Minkowski type problems for horospherically convex hypersurfaces with one or two points at the ideal boundary of \mathbb{H}^{n+1} . So, an important point of the construction developed in this section is that it motivates new interesting problems for conformally invariant PDEs in terms of the Schouten tensor.

6. Weingarten hypersurfaces in \mathbb{H}^{n+1}

An oriented hypersurface in a model space \mathbb{R}^{n+1} , \mathbb{S}^{n+1} , or \mathbb{H}^{n+1} is a *Weingarten hypersurface* if there exists a non-trivial relation

$$\mathcal{W}(\kappa_1, \dots, \kappa_n) = c \in \mathbb{R} \tag{6.1}$$

between its principal curvatures $\kappa_1, \dots, \kappa_n$. In particular, totally umbilical hypersurfaces are Weingarten hypersurfaces. An important problem in hypersurface theory is to establish for which functionals \mathcal{W} the only examples that satisfy (6.1) together with a global condition (usually compactness) are totally umbilical.

Our next objective is to exhibit a wide family of functionals \mathcal{W} for which the only compact hypersurfaces satisfying (6.1) are totally umbilical round spheres. This result will be obtained as a consequence of Corollary 24 and a deep result by A. Li and Y.Y. Li [LL1, Corollary 1.6] on the Schouten tensor of conformal metrics on \mathbb{S}^n .

There are two remarkable facts regarding our result. To start, it seems to be the first example of a wide family of Weingarten functionals for which round spheres can be characterized as the only compact examples satisfying (6.1). Previous results in this line for specific Weingarten functionals in \mathbb{H}^{n+1} may be found, for instance, in [Al, Bi, CD, Ko, Ro] and references therein. The other remarkable feature of our result is that we are not assuming *a priori* that the hypersurface is embedded. This additional topological hypothesis appears in most of the previous geometric works on the characterization of compact Weingarten hypersurfaces in model spaces.

In order to state the result, some notation is needed. Let us first of all set $\Omega := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i < 1\}$, and consider the involutive diffeomorphism $\mathcal{T} : \Omega \rightarrow \Omega$ given by

$$\mathcal{T}(x_1, \dots, x_n) = \left(\frac{x_1 + 1}{x_1 - 1}, \dots, \frac{x_n + 1}{x_n - 1} \right). \quad (6.2)$$

Consider in addition

$$\text{an open convex symmetric cone } \Gamma \subset \mathbb{R}^n \text{ with vertex at the origin} \quad (6.3)$$

such that

$$\Gamma_n \subset \Gamma \subset \Gamma_1, \quad (6.4)$$

where $\Gamma_n := \{(x_1, \dots, x_n) : x_i > 0\}$, and $\Gamma_1 := \{(x_1, \dots, x_n) : \sum_{i=1}^n x_i > 0\}$. Here *symmetric* means symmetric with respect to the variables (x_1, \dots, x_n) .

Let us define next

$$\Gamma^* := \mathcal{T}(\Gamma \cap \Omega) \subset \Omega. \quad (6.5)$$

It is clear that $\Gamma_n^* \subset \Gamma^* \subset \Gamma_1^*$, where

$$\Gamma_n^* := \{(x_1, \dots, x_n) : x_i < -1\}, \quad \Gamma_1^* := \left\{ (x_1, \dots, x_n) : \frac{1}{n} \sum_{i=1}^n \frac{1}{1-x_i} < \frac{1}{2} \right\}.$$

Consider now a real function $\mathcal{W}(x_1, \dots, x_n)$ satisfying the following conditions:

$$\mathcal{W} \in C^1(\Gamma^*) \cap C^0(\overline{\Gamma^*}) \text{ is symmetric with respect to } x_1, \dots, x_n, \quad (6.6)$$

$$\mathcal{W} = 0 \text{ on } \partial\Gamma^* \text{ and } \partial\mathcal{W}/\partial x_i < 0 \text{ in } \Gamma^*, \quad (6.7)$$

$$s\mathcal{W}(x) = \mathcal{W}(\mathcal{T}(s(\mathcal{T}(x)))) \text{ for every } x \in \Gamma^*. \quad (6.8)$$

In this last condition, $s > 0$ is arbitrary except that the right hand side has to be well defined. Let us observe that if we set

$$f(x) := \mathcal{W}(\mathcal{T}(2x)) : \Gamma \cap \{(x_1, \dots, x_n) : x_i < 1/2\} \rightarrow \mathbb{R}, \tag{6.9}$$

then (6.8) simply means that $f(sx) = sf(x)$. This allows us to extend f by homogeneity to the whole cone Γ . This extension (still denoted by f) satisfies:

$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma}) \text{ is symmetric with respect to } x_1, \dots, x_n, \tag{6.10}$$

$$f = 0 \text{ on } \partial\Gamma \text{ and } \partial f / \partial x_i > 0 \text{ in } \Gamma, \tag{6.11}$$

$$f(sx) = sf(x) \text{ for every } x \in \Gamma, s > 0. \tag{6.12}$$

Under the conditions (6.3), (6.4), (6.10), (6.11), (6.12) for (Γ, f) , A. Li and Y.Y. Li [LL1] proved that if $g = e^{2\rho} g_0$ is a conformal metric on \mathbb{S}^n such that the eigenvalues of its Schouten tensor satisfy

$$f(\lambda_1, \dots, \lambda_n) = 1, \quad (\lambda_1, \dots, \lambda_n) \in \Gamma, \tag{6.13}$$

then g differs from g_0 by, at most, a dilation and a conformal isometry of \mathbb{S}^n .

As a consequence of this result in [LL1] and of our previous discussion, we can deduce the following:

Theorem 28. *Define (Γ^*, \mathcal{W}) by the conditions (6.3) to (6.8), and let $M^n \subset \mathbb{H}^{n+1}$, $n > 2$, denote an immersed oriented compact hypersurface in \mathbb{H}^{n+1} such that its principal curvatures κ_i satisfy*

$$\mathcal{W}(\kappa_1, \dots, \kappa_n) = 1. \tag{6.14}$$

Then M^n is a totally umbilical round sphere.

Proof. Since $\Gamma^* \subset \Omega$, we infer that $\kappa_i < 1$ always holds, and thus M^n is a canonically oriented horospherical ovaloid in \mathbb{H}^{n+1} . Therefore, M^n is diffeomorphic to \mathbb{S}^n and its horospherical metric g_∞ is conformal to the canonical metric of \mathbb{S}^n . Moreover, by (3.7) and (5.11) we can express the eigenvalues of the Schouten tensor of g_∞ as

$$2\lambda_i = \frac{\kappa_i + 1}{\kappa_i - 1}. \tag{6.15}$$

Let us recall at this point that the conditions imposed on (Γ^*, \mathcal{W}) ensure that (f, Γ) given by (6.3), (6.4), (6.5), (6.9) satisfies the conditions of [LL1, Corollary 1.6]. Moreover, we see from (6.8), (6.14), (6.15) that

$$f(\lambda_1, \dots, \lambda_n) = 1, \quad (\lambda_1, \dots, \lambda_n) \in \Gamma.$$

Consequently, g_∞ is a metric of constant (positive) curvature, and all eigenvalues λ_i are constant and equal. By (6.15) we see that M^n is a totally umbilical round sphere. \square

An important remark. Observe that if the Weingarten functional $\mathcal{W}(x_1, \dots, x_n)$, which is defined on Γ^* , admits an extension $\tilde{\mathcal{W}}$ to some larger domain $\tilde{\Gamma} \subset \mathbb{R}^n$, then the condition (6.14) on an immersed oriented hypersurface $M^n \subset \mathbb{H}^{n+1}$ simply means that

$$\tilde{\mathcal{W}}(\kappa_1, \dots, \kappa_n) = 1$$

and there exists some $p \in M^n$ with $(\kappa_1(p), \dots, \kappa_n(p)) \in \Gamma^*$. This happens because \mathcal{W} vanishes identically on $\partial\Gamma^*$. So, if M^n is compact, the condition $(\kappa_1(p), \dots, \kappa_n(p)) \in \Gamma^*$ is not restrictive because, by a change of orientation if necessary, there is some $p \in M^n$ with $\kappa_i(p) < -1$ for every i . That is, $(\kappa_1(p), \dots, \kappa_n(p)) \in \Gamma_n^* \subset \Gamma^*$.

Let us now consider a particular case of Theorem 28 that is specially important. Let $S_k(x_1, \dots, x_n)$ denote the elementary k -th symmetric function (5.19), and let $\Gamma_k \subset \mathbb{R}^n$ denote the connected component of

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : S_k(x_1, \dots, x_n) > 0\}$$

that contains the positive cone Γ_n . It is then known (see [LL1, LL2, L3]) that (Γ_k, f_k) satisfies conditions (6.3), (6.4), (6.10), (6.11), (6.12), where

$$f_k(x_1, \dots, x_n) := (S_k(x_1, \dots, x_n))^{1/k}.$$

Therefore, Theorem 28 and (6.15) yield

Corollary 29. *Let $M^n \subset \mathbb{H}^{n+1}$, $n > 2$, denote a horospherical ovaloid, and assume that a k -th symmetric elementary function of its signed contact radii is constant, i.e.*

$$S_k(\delta_1, \dots, \delta_n) = c > 0, \quad \delta_i := \frac{\kappa_i + 1}{\kappa_i - 1}. \quad (6.16)$$

Then M^n is a totally umbilical round sphere.

The particular case $k = n$ tells us (by (5.20)) that any strongly H -convex ovaloid in \mathbb{H}^{n+1} with constant mean of their contact radii, i.e. $\sum_{i=1}^n \varrho_i = c > 0$, is a round sphere.

The above corollary is relevant in the context of linear Weingarten hypersurfaces. Let us recall that for an oriented hypersurface $M^n \subset \mathbb{H}^{n+1}$ with principal curvatures $\kappa_1, \dots, \kappa_n$, the r -th mean curvature function is defined as

$$\binom{n}{r} H_r := S_r(\kappa_1, \dots, \kappa_n), \quad 1 \leq r \leq n.$$

In particular, $H = H_1$ is the mean curvature and $K = H_n$ is the Gauss–Kronecker curvature. With these notations, we recall the following

Definition 30. *An immersed oriented hypersurface $M^n \subset \mathbb{H}^{n+1}$ is a linear Weingarten hypersurface if there are constants $c_0, \dots, c_n \in \mathbb{R}$ such that*

$$\sum_{i=1}^n c_i H_i = c_0. \quad (6.17)$$

Now, observe that, for any oriented horospherically convex hypersurface, the equality (6.16) can be rewritten as (we use here r instead of k for clarity)

$$\sum_{i_1 < \dots < i_r} \frac{\kappa_{i_1} + 1}{\kappa_{i_1} - 1} \cdots \frac{\kappa_{i_r} + 1}{\kappa_{i_r} - 1} = c,$$

or equivalently,

$$\sum_{i_1 < \dots < i_r} (\kappa_{i_1} + 1) \cdots (\kappa_{i_r} + 1)(\kappa_{j_1} - 1) \cdots (\kappa_{j_{n-r}} - 1) = c(\kappa_1 - 1) \cdots (\kappa_n - 1),$$

where we have labelled $\{i_1, \dots, i_r, j_1, \dots, j_{n-r}\} = \{1, \dots, n\}$. This proves that the Weingarten relation given by (6.16) can be actually written as a linear Weingarten relation

$$\sum_{i=1}^n c_i H_i = c_0$$

for some constants c_0, \dots, c_n depending on c and r .

Thus, as a consequence of Corollary 29 we can conclude that *the only compact oriented immersed hypersurfaces in \mathbb{H}^{n+1} that satisfy (6.17) on Γ_r^* (given by (6.5) in terms of Γ_r) for the above constants c_j are totally umbilical round spheres*. This last conclusion was obtained in [FR] for $k = 1$.

Remark 31. A problem of interest in hypersurface theory is to classify the elements $(c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ for which the only compact linear Weingarten hypersurfaces satisfying (6.17) are round spheres. For that, one obviously needs some condition on the c_i 's in order to ensure that round spheres belong to this class. Again, this problem is tightly linked by Corollary 27 to the one of determining when a conformal metric $g = e^{2\rho} g_0$ on \mathbb{S}^n whose σ_k -curvatures satisfy a relation of the type

$$\sum_{k=1}^n c_k \sigma_k = c_0, \quad (c_0, \dots, c_n) \in \mathbb{R}^{n+1},$$

differs from g_0 at most by a dilation and a conformal isometry of \mathbb{S}^n .

Weingarten hypersurfaces with one end

It is possible to extend partially the above arguments to the case of non-compact Weingarten hypersurfaces. Again, the key is provided by some deep theorems by Y.Y. Li [L3, L4, L5] on the characterization of solutions globally defined on \mathbb{R}^n of some conformally invariant PDEs involving the Schouten tensor. Li's results generalize other previous works, like [CGS, CGY2, LL1, LL2]. For that we introduce the following notions.

Definition 32. *A manifold M^n is said to have one end if it is homeomorphic to a compact manifold K^n with one point removed, i.e. $M^n \simeq K^n \setminus \{p\}$.*

Definition 33. Let $M^n \subset \mathbb{H}^{n+1}$ denote an immersed oriented hypersurface with one end. We say that this end is regular (or that M^n has one regular end) provided the hyperbolic Gauss map G of $M^n \equiv K^n \setminus \{p\}$ extends continuously to K^n .

The most basic examples of non-compact hypersurfaces in \mathbb{H}^{n+1} with one regular end are horospheres (with the outer orientation). The following result shows that, for a large class of Weingarten functional equations $\mathcal{W}(\kappa_1, \dots, \kappa_n) = 0$, the only Weingarten hypersurfaces with one regular end are horospheres. We are not assuming here completeness or embeddedness of the hypersurface. Previous characterization results in \mathbb{H}^{n+1} for horospheres can be found in [FR, Cu].

Theorem 34. Let (Γ^*, \mathcal{W}) be given by (6.3) to (6.7) and such that $\mathcal{W} > 0$ on Γ^* . Let $M^n \subset \mathbb{H}^{n+1}$, $n > 2$, denote an immersed oriented hypersurface with one regular end, and whose principal curvatures $\kappa_1, \dots, \kappa_n$ satisfy

$$(\kappa_1, \dots, \kappa_n) \in \partial\Gamma^*, \quad \text{and so} \quad \mathcal{W}(\kappa_1, \dots, \kappa_n) = 0. \quad (6.18)$$

Then M^n is a horosphere (with its outer orientation).

Proof. Let us start by observing that if $G : M^n \rightarrow \mathbb{S}^n$ is the hyperbolic Gauss map of M^n , then G is a local diffeomorphism at every point of M^n (since from (6.18) we know that $\kappa_i < 1$). In addition, as the end of $M^n \equiv K^n \setminus \{p\}$ is regular, there exists a continuous map $\tilde{G} : K^n \rightarrow \mathbb{S}^n$ with $\tilde{G}|_{M^n} = G$.

Set $x_0 := \tilde{G}(p) \in \mathbb{S}^n$ and $N := G^{-1}(x_0) \subset M^n$. Obviously, N consists only of isolated points, since G is a local diffeomorphism. Thus $M^n \setminus N$ is a smooth manifold, and

$$G : M^n \setminus N \rightarrow \mathbb{S}^n \setminus \{x_0\}$$

is a covering map. Therefore $M^n \setminus N$ is simply connected, and a simple topological argument immediately implies that N has to be empty. Then we can infer that

$$G : M^n \rightarrow \mathbb{S}^n \setminus \{x_0\}$$

is a global diffeomorphism. In particular, M^n is homeomorphic to \mathbb{R}^n , and so we can view it as an immersion $\phi : \mathbb{R}^n \rightarrow \mathbb{H}^{n+1}$ with hyperbolic Gauss map $G = \pi^{-1}(x)$, where $\pi : \mathbb{S}^n \setminus \{\text{north pole}\} \rightarrow \mathbb{R}^n$ is the stereographic projection.

The rest of the proof is basically the same as the one of Theorem 28, but this time using the main results of [L3, L4, L5]. \square

Results on the Schouten tensor

We now use the interrelation between hypersurfaces in \mathbb{H}^{n+1} and conformal metrics on \mathbb{S}^n in the opposite direction, i.e. we will transfer results from hypersurface theory into conformal geometry.

First, we will prove a general duality for locally conformally flat metrics on a manifold M^n . It is an example of how a geometrically simple transformation (a change from positive to negative hyperbolic Gauss map in our situation) can yield a non-trivial superposition formula for a geometric PDE.

Theorem 35. *Let (M^n, g) , $n > 2$, denote a locally conformally flat Riemannian manifold, and let $\lambda_1, \dots, \lambda_n \in C^\infty(M^n)$ be the eigenvalues of its Schouten tensor. If none of the λ_i 's vanish on M^n , then there exists a new locally conformally flat Riemannian metric g^* on M^n whose Schouten tensor eigenvalues $\lambda_i^* \in C^\infty(M^n)$ are*

$$\lambda_i^* = 1/\lambda_i.$$

Proof. Let $p_0 \in M^n$ be an arbitrary point. Then there exists some relatively compact neighbourhood $U \subset M^n$ of p_0 and a conformal embedding $f : U \rightarrow \mathbb{S}^n$. Moreover, by [Ku], the map f on U is unique up to conformal transformations of \mathbb{S}^n .

Consider now the dilated metric $g_t := e^{2(\rho+t)}g_0$ on U , where $t > 0$ is a positive constant. As this is just a dilation of g with factor e^{2t} , the eigenvalues of the Schouten tensor of g_t are $\lambda_i^t = e^{-2t}\lambda_i \neq 0$. Therefore, as U is relatively compact, there exists $t_0 > 0$ such that $\lambda_i^t \neq 1/2$ everywhere on U if $t \geq t_0$.

Consider the map

$$\psi_t(x) := e^{\rho(x)+t}(1, x) : U \subset \mathbb{S}^n \rightarrow \mathbb{N}_+^{n+1},$$

which is an immersion with induced metric g_t . It follows from the proof of Theorem 15 that the map $\phi_t : U \subset \mathbb{S}^n \rightarrow \mathbb{H}^{n+1}$ given by

$$\phi_t = \frac{e^{\rho+t}}{2}(1 + e^{-2(\rho+t)}(1 + \|\nabla^{g_0}\rho\|_{g_0}^2))(1, x) + e^{-(\rho+t)}(0, -x + \nabla^{g_0}\rho) \quad (6.19)$$

has ψ_t as its associated light cone immersion for a specific orientation, i.e. $\langle d\phi_t, d\psi_t \rangle = 0$ and $\langle \phi_t, \psi_t \rangle = -1$. Moreover, by (5.4), the map ϕ_t is everywhere regular, and hence an immersed hypersurface in \mathbb{H}^{n+1} . We shall orient ϕ_t so that its unit normal η_t satisfies $\psi_t = \phi_t + \eta_t$.

Now, by (3.3) and (5.12) we find that the principal curvatures κ_i^t of ϕ_t satisfy

$$e^{-2t}\lambda_i = \frac{1}{2} - \frac{1}{1 - \kappa_i^t}, \quad (6.20)$$

and thus $\kappa_i^t \neq \pm 1$ at every point in U (recall that, by hypothesis, the eigenvalues λ_i never vanish).

This implies that the negative Gauss map of ϕ_t is a local diffeomorphism, since by Lemma 3 and after a change of orientation, this condition for

$$G_t^- := [\phi_t - \eta_t] : U \subset \mathbb{S}^n \rightarrow \mathbb{S}^n$$

is equivalent to $\kappa_i^t \neq -1$ for every $i \in \{1, \dots, n\}$. Moreover,

$$g_t^* := \langle d(\phi_t - \eta_t), d(\phi_t - \eta_t) \rangle \quad (6.21)$$

is a new regular metric on U , which is conformally flat since it is the induced metric of an immersion in the light cone. In other words, g_t^* is just the horospherical metric of ϕ_t after a change of orientation, which, as we already know, changes completely the geometry of the hyperbolic Gauss map.

Now, by (5.12), the eigenvalues $\lambda_i^{t,*}$ of the Schouten tensor of g_t^* satisfy the relation

$$\lambda_i^{t,*} = \frac{1}{2} - \frac{1}{1 - \kappa_i^t}.$$

From this expression and (6.20) we get

$$\lambda_i^{t,*} = \frac{e^{2t}}{4\lambda_i}.$$

It follows directly that the metric

$$g^* := e^{2t} g_t^* \tag{6.22}$$

defined on U has the Schouten tensor eigenvalues $\lambda_i^* = 1/(4\lambda_i)$.

Let us check finally that the definition of the conformally flat metric g^* on U does not depend on the choices of the positive constant $t > 0$ and of the conformal embedding f .

Let $t_1, t_2 \geq t_0 > 0$, and denote by

$$\psi_{t_1}(x) = e^{\rho(x)+t_1}(1, x), \quad \psi_{t_2}(x) = e^{\rho(x)+t_2}(1, x)$$

the associated light cone immersions. Then by Remark 14 we have

$$\phi_{t_2} = e^{-(t_2-t_1)}\phi_{t_1} + \sinh(t_2 - t_1)\psi_{t_1}.$$

Thus, $\phi_{t_2} - \eta_{t_2} = e^{t_1-t_2}(\phi_{t_1} - \eta_{t_1})$. But this equality together with (6.21) shows that $e^{2t_2}g_{t_2}^* = e^{2t_1}g_{t_1}^*$. This indicates that the definition of the metric g^* in (6.22) is independent of the value of the constant $t > 0$.

On the other hand, as we said, the conformal embedding f of U into \mathbb{S}^n is unique up to conformal transformations of \mathbb{S}^n . This implies that ψ_t, ϕ_t and thus $\phi_t - \eta_t$ are unique up to isometries of \mathbb{N}_+^{n+1} or \mathbb{H}^{n+1} . In particular, g^* is independent of the choice of the conformal embedding $f : U \rightarrow \mathbb{S}^n$.

These independence properties of the metric g^* clearly imply that if U_1, U_2 are two relatively compact neighbourhoods of M^n , and if g_1^*, g_2^* denote the respective associated metrics defined via (6.22), then $g_1^* = g_2^*$ on $U_1 \cap U_2$. In this way, the metric g^* is a locally conformally flat Riemannian metric globally defined on M^n . Moreover, its Schouten tensor eigenvalues at an arbitrary point are $1/(4\lambda_i)$. Finally, a dilation of this metric g^* by an appropriate constant gives the desired metric. \square

Let us point out regarding this result that the conformally flat metrics g, g^* are not in general conformal to each other on M^n .

The final result of this work is a 2-dimensional analogue of the results in [LL1] regarding conformal metrics on \mathbb{S}^n . It is remarkable that in this 2-dimensional case the hypotheses are much weaker.

Theorem 36. *Let $g = e^{2\rho} g_0$ denote a conformal metric on \mathbb{S}^2 , and let λ_1, λ_2 be the eigenvalues of its 2-dimensional Schouten tensor, i.e. of*

$$\text{Sch}_g := -\nabla^{2, g_0} \rho + d\rho \otimes d\rho - \frac{1}{2}(-1 + \|\nabla^{g_0} \rho\|_{g_0}^2) g_0.$$

Assume that $\mathcal{W}(\lambda_1, \lambda_2) = 0$ for a smooth function $\mathcal{W} : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\frac{\partial \mathcal{W}}{\partial \lambda_1} \frac{\partial \mathcal{W}}{\partial \lambda_2} > 0 \quad \text{whenever } \lambda_1 = \lambda_2.$$

Then $\lambda_1 \equiv \lambda_2$ and $g = \Phi^(g_0)$, where Φ is some linear fractional transformation.*

Proof. Reasoning as in Theorem 35, there exists a real number t and an immersion $\phi : \mathbb{S}^2 \rightarrow \mathbb{H}^3$ such that

- the eigenvalues $\bar{\lambda}_i$ of the new metric $e^{2t} g$ satisfy $\bar{\lambda}_i = e^{-2t} \lambda_i < 1/2$ on \mathbb{S}^2 ,
- the horospherical metric of ϕ is $e^{2t} g$,
- the principal curvatures k_i of ϕ satisfy $e^{-2t} \lambda_i = \frac{1}{2} - \frac{1}{1-k_i}$.

Therefore, ϕ is a Weingarten immersion with

$$\tilde{\mathcal{W}}(k_1, k_2) = \mathcal{W}\left(e^{2t} \left(\frac{1}{2} - \frac{1}{1-k_1}\right), e^{2t} \left(\frac{1}{2} - \frac{1}{1-k_2}\right)\right) = 0.$$

Now, using the fact that

$$\frac{\partial \tilde{\mathcal{W}}}{\partial k_1} \frac{\partial \tilde{\mathcal{W}}}{\partial k_2} = \frac{e^{4t}}{(1-k_1)^2(1-k_2)^2} \frac{\partial \mathcal{W}}{\partial \lambda_1} \frac{\partial \mathcal{W}}{\partial \lambda_2} > 0$$

whenever $k_1 = k_2$, we see that ϕ must be a totally umbilical immersion [HW, Che]. Let us remark here that [HW, Che] only consider surfaces in \mathbb{R}^3 , but their arguments carry over essentially unchanged to the space forms $\mathbb{S}^3, \mathbb{H}^3$, since they rely only on the Codazzi equation, which is the same in all those spaces.

Thus, the result follows easily. □

A closing remark. At the core of the results in the present paper there is an important phenomenon that we would like to emphasize.

It is well known that the metric geometry of \mathbb{H}^{n+1} is tightly linked to the conformal geometry of \mathbb{S}^n , in the following sense: any isometry of \mathbb{H}^{n+1} induces a conformal transformation of the ideal boundary $\partial_\infty \mathbb{H}^{n+1} \equiv \mathbb{S}^n$, and conversely, any conformal transformation of \mathbb{S}^n extends to an isometry of the unit ball $(\mathbb{B}^{n+1}, ds_{\mathbb{P}}^2) \equiv \mathbb{H}^{n+1}$ with respect to the Poincaré metric.

In this paper, we have shown that this relation is the base of a much more general situation. Indeed, we have proved that the conformal metrics $g = e^{2\rho} g_0$ on \mathbb{S}^n are in correspondence with the compact immersed hypersurfaces of \mathbb{H}^{n+1} with prescribed regular hyperbolic Gauss map. Moreover, the fundamental conformal invariants of g in \mathbb{S}^n , i.e.

the eigenvalues of the Schouten tensor Sch_g , are linked to the basic extrinsic quantities of the hypersurface in \mathbb{H}^{n+1} (the principal curvatures) by the simple relation

$$2\lambda_i = \frac{\kappa_i + 1}{\kappa_i - 1}.$$

It is our impression that this is a far reaching relation that goes beyond the specific results proved here, which seem to be just the most visible consequences of the above correspondence.

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