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Holomorphic functions and subelliptic heat kernels over Lie groups

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Abstract. A Hermitian form q on the dual space, $\mathfrak{g}^*$, of the Lie algebra, $\mathfrak{g}$, of a Lie group, G, determines a sub-Laplacian, Δ, on G. It will be shown that Hörmander's condition for hypoellipticity of the sub-Laplacian holds if and only if the associated Hermitian form, induced by q on the dual of the universal enveloping algebra, $\mathcal{U}'$, is non-degenerate. The subelliptic heat semigroup, $e^{t\Delta/4}$, is given by convolution by a C^∞ probability density ρ_t. When G is complex and $u : G \to \mathbb{C}$ is a holomorphic function, the collection of derivatives of u at the identity in G gives rise to an element, $\hat{u}(e) \in \mathcal{U}'$. We will show that if G is complex, connected, and simply connected then the "Taylor" map, $u \mapsto \hat{u}(e)$, defines a unitary map from the space of holomorphic functions in $L^2(G, \rho_t)$ onto a natural Hilbert space lying in $\mathcal{U}'$.

Keywords. Subelliptic, heat kernel, complex groups, universal enveloping algebra, Taylor map

Contents

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1. Introduction

Denote by G a real or complex Lie group, by $\mathfrak{g} = T_eG$ its Lie algebra, and by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Let q be a non-negative quadratic or Hermitian form on $\mathfrak{g}^*$ according to whether $\mathfrak{g}$ is real or complex. Let $K = \mathrm{Nul}(q) = \{\alpha \in \mathfrak{g}^* : q(\alpha, \alpha) = 0\}$ and

$$H = K^0 := \{\xi \in \mathfrak{g} : \alpha(\xi) = 0 \text{ for all } \alpha \in \mathrm{Nul}(q)\}$$

be the backward annihilator subspace of K in $\mathfrak{g}$. We say that q satisfies *Hörmander's condition* if H generates $\mathfrak{g}$ as a Lie algebra.

In Section 2 we are going to characterize those q for which Hörmander's condition holds in terms of the following natural seminorms on the dual space of the universal enveloping algebra of $\mathfrak{g}$. Denote by $q^{\otimes k}$ the extension of q to a non-negative quadratic/Hermitian form on $(\mathfrak{g}^*)^{\otimes k}$ where by convention, $(\mathfrak{g}^*)^{\otimes 0}$ is $\mathbb{R}$ or $\mathbb{C}$ according to whether G is real or complex and $q^{\otimes 0}(1) = 1$. If $T(\mathfrak{g})$ is the tensor algebra over $\mathfrak{g}$ then the algebraic dual space of $T(\mathfrak{g})$ is the direct product: $T(\mathfrak{g})' = \prod_{k=0}^{\infty} (\mathfrak{g}^*)^{\otimes k}$. For each $t > 0$ define

$$q_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} q^{\otimes k} \tag{1.1}$$

on $T(\mathfrak{g})'$, where we allow for the possibility that $q_t(\alpha)$ is infinite. On the subspace where q_t is finite it is the square of a seminorm. Because of the allowed degeneracy of q the seminorm may not be a norm. But we are going to restrict the domain of q_t further. Denote by J the two-sided ideal in $T(\mathfrak{g})$ generated by the elements $\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]$ where ξ and η run over $\mathfrak{g}$. We can identify the universal algebra $\mathcal{U}$ of $\mathfrak{g}$ with $T(\mathfrak{g})/J$ and then the algebraic dual space, $\mathcal{U}'$, may be identified with J^0, the annihilator of J in $T(\mathfrak{g})'$. Let

$$J_t^0 := \{\alpha \in J^0 : q_t(\alpha) < \infty\}.$$

We will show in Section 2 (see Theorem 2.7 and Corollary 2.14) that the following conditions on q are equivalent: 1) Hörmander's condition holds, 2) $T(\mathfrak{g}) = T(H) + J$ ($T(H)$ is the tensor algebra over H), and 3) for any $t > 0$, $q_t|_{J_t^0}$ is the square of a norm, i.e., $q_t|_{J_t^0}$ is the quadratic (Hermitian) form associated to a positive definite inner product on J_t^0.

For each $A \in \mathfrak{g}$, let $\tilde{A}$ denote the unique extension of A to a left invariant vector field on G. If G is real, for any basis $X_1, \ldots, X_M$ of $\mathfrak{g}$ with dual basis $\{X_j'\}$ the second order differential operator

$$\Delta_q = \sum_{j,k=1}^{M} q(X_j', X_k') \tilde{X}_j \tilde{X}_k \tag{1.2}$$

is easily seen to be independent of the choice of basis. Hörmander's theorem [20] states that Δ_q is hypoelliptic if and only if q satisfies Hörmander's condition.

Now suppose that G is a complex connected Lie group and q is a non-negative Hermitian form on $\mathfrak{g}^*$ satisfying Hörmander's condition. Let $\mathrm{Re}\, q$ denote the real part of the Hermitian inner product on $\mathfrak{g}^*$ with the complex structure forgotten. Since $\mathrm{Nul}(q) =$

$\mathrm{Nul}(\mathrm{Re}\, q)$, one easily shows that q satisfies Hörmander's condition iff $\mathrm{Re}\, q$ satisfies Hörmander's condition. We may form the associated hypoelliptic sub-Laplacian, $\Delta_{\mathrm{Re}\, q}$, as in (1.2) and in this case the heat semigroup, $\exp(\frac{t}{4}\Delta_{\mathrm{Re}\, q})$, is given by convolution by a C^∞ heat kernel ρ_t on G. Let $\mathcal{H} = \mathcal{H}(G)$ denote the space of holomorphic functions on G and for any function f in $\mathcal{H}$ and $x \in G$, let $\hat{f}(x) \in J^0 \cong \mathcal{U}'$ be the "Taylor coefficient" at the point x defined by

$$\langle \hat{f}(x), \beta \rangle = (\tilde{\beta} f)(x) \quad \text{for all } \beta \in T(\mathfrak{g}), \tag{1.3}$$

where $\tilde{\beta}$ is the left invariant differential operator on G associated to β (see Notation 2.4 below). Because of the results of Section 2, we know J_t^0 is a Hilbert space with respect to the norm $(q_t|_{J^0})^{1/2}$. The aim of this paper is to show that the Taylor map, $f \mapsto \hat{f}(e)$, is a unitary isomorphism of $\mathcal{H} \cap L^2(G, \rho_t)$ onto J_t^0 when G is simply connected.

This kind of unitary isomorphism of holomorphic function spaces with a Hilbert space of "Taylor coefficients" has a long history. A knowledgeable reader could "read out" of the 1932 paper [8] by the physicist V. A. Fock such an isomorphism in the classical case, where the complex group G is just $\mathbb{C}^M$, q is the usual Hermitian norm on $\mathbb{C}^M$, and ρ_t is a Gaussian density. However, the isomorphism in this classical setting was not actually made clear until the work of Segal [32, 33] and Bargmann [2]. (See also [17] for more history.) Inspired by related work of B. Hall [18], the first named author [4] proved such an isomorphism for a wide class of complex Lie groups G, for a strictly positive definite quadratic form q. This was subsequently extended to an arbitrary complex Lie group in [5] but again, for a strictly positive definite quadratic form q. A detailed exposition of this isomorphism along with a discussion of its extensive history may be found in the expository portion of the paper [17].

The Taylor map isomorphism has also been proven for some infinite-dimensional groups: in [11] and [10] M. Gordina found a precise analog of this unitary isomorphism for the infinite-dimensional complex Hilbert–Schmidt orthogonal group, and in [12] she proved the analog for the group of invertible operators in a factor of type II_1. Also M. Cecil, in [3], has shown that a unitary Taylor isomorphism holds for path groups over stratified nilpotent Lie groups. To our knowledge the present paper is the first work dealing with this isomorphism in the degenerate (i.e. subelliptic) case.

Section 4 establishes that $\hat{f}(e) \in J_t^0$ for every

$$f \in \mathcal{H} \cap L^2(G, \rho_t) =: \mathcal{H}L^2(G, \rho_t(x)\, dx)$$

and that the Taylor map,

$$f \in \mathcal{H}L^2(G, \rho_t) \mapsto \hat{f}(e) \in J_t^0,$$

is an isometry into J_t^0. The proof is similar to that for the non-degenerate case given in [5].

In Section 5 we will adapt the method first introduced in [4] to recover a holomorphic function from its Taylor coefficient $\hat{f}(e)$. Then, in Section 6, we will prove the surjectivity for an arbitrary simply connected group G. See Theorem 6.1 for a precise statement.

Our proof of surjectivity in this subelliptic setting depends on a delicate extension of the machinery of path dependent power series from the non-degenerate case, [5], to our subelliptic case (Sections 5 and 6). Although this extension is intrinsically interesting, we have tried to find other proofs for the subelliptic case that yield the surjectivity directly from the results already known in the non-degenerate case. In [6] we will report on these alternative approaches, which work primarily for nilpotent groups. We will also derive in [6] other interesting information for nilpotent groups besides surjectivity.

This paper continues a body of work in which the heat kernel on a Lie group G plays the role of a weight for the study of $L^2(G, w(x)\,dx)$. If G is complex then such a (rapidly decreasing) weight is required if this space is to contain non-constant holomorphic functions. In addition to a study of these holomorphic function spaces, $\mathcal{H} \cap L^2(G, w(x)\,dx)$, there are natural transforms into such spaces from function spaces over compact Lie groups. Heat kernel measures play a key role here also in place of Haar measure. For further background the reader may consult the recent surveys [16] and [19].

It may be useful to comment on the term "subelliptic" used in the title of this paper. Consider a second order differential operator $L = \sum \partial_i a_{i,j}(x)\partial_j$ with smooth coefficients in an open set $\Omega \subset \mathbb{R}^M$. The operator L is called *elliptic* if the matrix $(a_{i,j}(x))$ is everywhere positive definite (this is one of the standard usages of the term *elliptic*, see [22, 25]). The operator is called *subelliptic* if the matrix $(a_{i,j}(x))$ is everywhere positive semidefinite and there is a real $s \in (0, 1]$ such that L satisfies the *subelliptic* estimate

$$\forall u \in C_0^\infty(\Omega), \qquad \|u\|_{(2s)} \le C(\|u\| + \|Lu\|), \tag{1.4}$$

where $\|\cdot\|$ stands for the usual L^2-norm and $\|u\|_{(s)} = (\int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi)^{1/2}$ is the Sobolev norm of index s. See [25] and the references therein. Note that any elliptic operator satisfies (1.4) with $s = 1$, locally (see, e.g., Lemma 17.1.2 in [21]), and that any subelliptic operator is hypoelliptic (see Proposition 3.2 in [20]).

Now, if G is a real Lie group and $L = \sum_{i=1}^k \tilde{X}_i^2$ is the sum of the squares of left invariant vector fields on G, then L is (locally) subelliptic if and only if it satisfies *Hörmander's condition*, i.e., $\{X_1, \ldots, X_k\}$ generates the Lie algebra of G. See [20]. The term *subelliptic* heat kernel on G refers to the minimal solution of the Cauchy problem $\partial_t u = Lu$, $u_0 = \delta_e$, where $L = \sum_{i=1}^k \tilde{X}_i^2$ and is subelliptic.

2. Hörmander's condition and non-degeneracy of norms

Notation 2.1. We will denote by $\mathfrak{g}$ a real (respectively complex) finite-dimensional Lie algebra. We let q be a non-negative quadratic (respectively Hermitian) form on the dual space $\mathfrak{g}^*$. Thus

$$q(a) = (a, a)_q \tag{2.1}$$

for some, possibly degenerate, non-negative bilinear (respectively sesquilinear) form $(\ ,\ )_q$ on $\mathfrak{g}^*$. Let

$$K := \{a \in \mathfrak{g}^* : q(a) = 0\} \tag{2.2}$$

be the null space of q and let

$$H = K^0 := \{\xi \in \mathfrak{g} : \langle a, \xi \rangle = 0\ \forall a \in K\} \tag{2.3}$$

be the backward annihilator of K. Here, as elsewhere, we use $\langle\cdot,\cdot\rangle$ for the bilinear pairing between a vector space and its dual, while $(\cdot,\cdot)_q$ denotes the bilinear (or sesquilinear) form induced by q on $\mathfrak{g}^*$. We call H the *Hörmander space* associated to q.

The degenerate case is of primary interest to us. We will usually allow the kernel, K, of q to be non-trivial. The next elementary result gives an explicit characterization of q.

Lemma 2.2. *There is a unique inner product, $(\cdot,\cdot)_H$, on H such that for any orthonormal base $\{X_j\}_{j=1}^m$ ($m := \dim(H)$) of H we have*

$$(a,b)_q = \sum_{j=1}^m \langle a, X_j\rangle\overline{\langle b, X_j\rangle} \quad \textit{for all } a,b\in\mathfrak{g}^*.$$

In particular,

$$q(a) = (a,a)_q = \sum_{j=1}^m |\langle a, X_j\rangle|^2. \tag{2.4}$$

Proof. The form q descends to a strictly positive definite quadratic form, $\bar{q}$, on $\mathfrak{g}^*/K$ and the map

$$\mathfrak{g}^*/K \ni a + K \mapsto a|_H \in H^*$$

is a linear isomorphism of vector spaces. Using this isomorphism, $\bar{q}$ induces an inner product, $(\cdot,\cdot)_{H^*}$, on H^* and hence, by the Riesz theorem, an inner product, $(\cdot,\cdot)_H$, on H. Suppose that $\{X_j\}_{j=1}^m$ is any orthonormal basis of $(H,(\cdot,\cdot)_H)$ and $a,b\in\mathfrak{g}^*$. Then

$$(a,b)_q = (a+K, b+K)_{\bar{q}} = (a|_H, b|_H)_{H^*} = \sum_{j=1}^m \langle a, X_j\rangle\overline{\langle b, X_j\rangle}. \qquad \square$$

Notation 2.3. The form q induces a degenerate (real or Hermitian) quadratic form $q_k := q^{\otimes k}$ whose inner product, $(\cdot,\cdot)_{q_k}$, on $(\mathfrak{g}^*)^{\otimes k}$ is determined by

$$(a_1\otimes\cdots\otimes a_k, b_1\otimes\cdots\otimes b_k)_{q_k} = \prod_{j=1}^k (a_j,b_j)_q, \quad a_i,b_i\in\mathfrak{g}^*,\ i=1,\dots,k, \tag{2.5}$$

for $k\geq 1$. If $\alpha\in(\mathfrak{g}^*)^{\otimes k}$, we will write $q_k(\alpha)$ or $|\alpha|_{q_k}^2$ for $(\alpha,\alpha)_{q_k}$. By convention, $V^{\otimes 0}$ is $\mathbb{R}$ or $\mathbb{C}$ depending on whether V is a real or complex vector space respectively, and we define q_0 on $(\mathfrak{g}^*)^{\otimes 0}$ so that $q_0(1)=1$.

Notation 2.4 (Left invariant differential operators). Denote by $T(\mathfrak{g})$ the tensor algebra over $\mathfrak{g}$. An element of $T(\mathfrak{g})$ is a finite sum

$$\beta = \sum_{k=0}^N \beta_k, \quad \beta_k\in\mathfrak{g}^{\otimes k}. \tag{2.6}$$

We define a linear map $(\beta\mapsto\tilde{\beta})$ from $T(\mathfrak{g})$ to left invariant differential operators on G determined by: 1) $\tilde{1} = \mathrm{Id}$ and 2) for $\beta = A_1\otimes\cdots\otimes A_k\in\mathfrak{g}^{\otimes k}$, $\tilde{\beta} := \tilde{A}_1\dots\tilde{A}_k$.

The algebraic dual space $T(\mathfrak{g})'$ may be identified with the direct product $\prod_{k=0}^{\infty}(\mathfrak{g}^*)^{\otimes k}$ in the pairing

$$\langle \alpha, \beta \rangle = \sum_{k=0}^{\infty} \langle \alpha_k, \beta_k \rangle \tag{2.7}$$

where

$$\alpha = \sum_{k=0}^{\infty} \alpha_k, \qquad \alpha_k \in (\mathfrak{g}^*)^{\otimes k}. \tag{2.8}$$

Notation 2.5. Let J denote the two-sided ideal in $T(\mathfrak{g})$ generated by

$$\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] : \xi, \eta \in \mathfrak{g}\}.$$

The universal enveloping algebra of $\mathfrak{g}$ is the associative algebra $\mathcal{U} := T(\mathfrak{g})/J$ and the algebraic dual space $\mathcal{U}'$ can be identified with

$$J^0 := \{\alpha \in T(\mathfrak{g})' : \langle \alpha, J \rangle = \{0\}\}. \tag{2.9}$$

For $t > 0$ define

$$\|\alpha\|_t^2 := \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|_{q_k}^2 = q_t(\alpha) \tag{2.10}$$

when α is given by (2.8).

The function $\|\cdot\|_t$ defines a seminorm in the subspace of $T(\mathfrak{g})'$ on which $\|\alpha\|_t^2$ is finite. But we will, by restriction, always consider $\|\cdot\|_t$ to be a seminorm on

$$J_t^0 := \{\alpha \in J^0 : \|\alpha\|_t^2 < \infty\}. \tag{2.11}$$

It was shown in [5] that when $\mathfrak{g}$ is complex and q is non-degenerate then the Hilbert space J_t^0, in the norm $\|\cdot\|_t$, is naturally isomorphic to the Hilbert space of holomorphic functions in $L^2(G, \rho_t(x)\,dx)$ where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and convolution by $\rho_t(x)$ is the heat kernel operator for the left invariant Laplacian on G induced by q. The isomorphism is given by the Taylor map described in the Introduction above (cf. [5, Theorem 2.6]). In Sections 4–6 we will prove that the same result holds in the subelliptic case. But in the present section we will first characterize the circumstances under which the seminorm $\|\cdot\|_t$ on J_t^0 is actually a norm.

Definition 2.6. *We say that* Hörmander's condition holds *for q if the smallest Lie subalgebra,* Lie(H), *containing H is $\mathfrak{g}$.*

Theorem 2.7. *Let $t > 0$. The seminorm $\|\cdot\|_t$ on J_t^0 is a norm if and only if Hörmander's condition holds.*

Proof. The proof of this theorem is the contents of Lemmas 2.12 and 2.13 below whose proofs were motivated by the techniques developed in [15]. □

The Lie subalgebra containing H may be described explicitly as follows. Let H_n denote those elements of $\mathfrak{g}$ which may be written as linear combinations of elements of the form

$$A = \mathrm{ad}_{A_1} \dots \mathrm{ad}_{A_{k-1}} A_k = [A_1, [A_2, \dots [A_{k-1}, A_k]] \dots] \tag{2.12}$$

with $A_i \in H$ for $i \leq k$ and $k \leq n$. Here, for $k = 1$, we interpret $\mathrm{ad}_{A_1} \dots \mathrm{ad}_{A_{k-1}}$ to be the identity operator. In particular, $H_1 = H$.

Lemma 2.8. $\mathrm{Lie}(H) = H_n$ *for all sufficiently large* n.

Proof. It is clear that H_n is an increasing sequence of subspaces which are contained in $\mathrm{Lie}(H)$ and because $\mathfrak{g}$ is finite-dimensional, H_n must be independent of n for large n. So to finish the proof it suffices to show $\bigcup_n H_n$ is a Lie algebra and for this it suffices to show $[A, B] \in \bigcup_n H_n$ whenever A is as in (2.12) and

$$B = \mathrm{ad}_{B_1} \dots \mathrm{ad}_{B_{m-1}} B_m = [B_1, [B_2, \dots [B_{m-1}, B_m]] \dots] \tag{2.13}$$

for some $B_i \in H$. However, this is easily proved by induction on k. The case $k = 1$ is trivial. Now suppose that $[A, B] \in \bigcup_n H_n$ for any $k \leq k_0$. Let

$$A' := \mathrm{ad}_{A_2} \dots \mathrm{ad}_{A_{k_0}} A_{k_0+1} \quad \text{and} \quad A = \mathrm{ad}_{A_1} \dots \mathrm{ad}_{A_{k_0}} A_{k_0+1} = [A_1, A'].$$

Then, by the Jacobi identity,

$$[A, B] = \mathrm{ad}_A B = \mathrm{ad}_{[A_1, A']} B = \mathrm{ad}_{A_1} \mathrm{ad}_{A'} B - \mathrm{ad}_{A'} \mathrm{ad}_{A_1} B,$$

which is in $\bigcup_n H_n$ by the induction hypothesis and the fact that $\bigcup_n H_n$ is stable under ad_{A_1} with $A_1 \in H$. □

Notation 2.9. Let $r = \min\{n : H_n = \mathrm{Lie}(H)\}$.

The proof of Theorem 2.7 will depend on the following lemmas. Since the theorem has no content if q is non-degenerate we will assume throughout that q is degenerate.

Lemma 2.10. *Let* $\alpha \in (\mathfrak{g}^*)^{\otimes k}$ *for some* $k \geq 1$. *Then*

$$q_k(\alpha) > 0 \tag{2.14}$$

if and only if there exist vectors $\xi_1, \dots, \xi_k \in H$ *such that*

$$\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_k \rangle \neq 0. \tag{2.15}$$

Proof. From (2.4) and (2.5),

$$|\alpha|_{q_k}^2 = \sum_{j_1, \dots, j_k = 1}^{m} |\langle \alpha, X_{j_1} \otimes \cdots \otimes X_{j_k} \rangle|^2 \tag{2.16}$$

for any $\alpha \in (\mathfrak{g}^*)^{\otimes k}$. So $|\alpha|_{q_k}^2 > 0$ if and only if one of the terms on the right side of the last equality is not zero. □

Lemma 2.11. *If Hörmander's condition holds then there exists an* $r \in \mathbb{N}$ *and an algebra homomorphism,* $P : T(\mathfrak{g}) \to T(H)$*, such that:*

(i) *If* $\beta \in T(\mathfrak{g})$ *with maximum rank at most* n *then* $P\beta$ *has maximum rank at most* nr *in* $T(H)$.
(ii) $P|_{T(H)} = \mathrm{id}_{T(H)}$ *and in particular* P *is a projection operator.*
(iii) *For all* $\beta \in T(\mathfrak{g})$, $\beta - P\beta \in J$. *In particular,*

$$\mathrm{Nul}(P) \subset J,$$
$$T(\mathfrak{g}) = T(H) \oplus \mathrm{Nul}(P) = T(H) + J, \tag{2.17}$$
$$J = (J \cap T(H)) \oplus \mathrm{Nul}(P). \tag{2.18}$$

(iv) $P|_{\mathfrak{g}} : \mathfrak{g} \to \bigoplus_{k=1}^{r} H^{\otimes k} \subset T(H)$ *is a bounded linear operator.*

Proof. Given $\Gamma := (A_1, \ldots, A_n) \in \mathfrak{g}^n$, let

$$[\Gamma] := [A_1, [A_2, \ldots [A_{n-1}, A_n]] \ldots] = \mathrm{ad}_{A_1} \mathrm{ad}_{A_2} \ldots \mathrm{ad}_{A_{n-1}} A_n. \tag{2.19}$$

and let

$$\hat{\Gamma} := A_1 \wedge (A_2 \wedge \ldots (A_{n-1} \wedge A_n) \ldots) \in T(\mathfrak{g}), \tag{2.20}$$

where $u \wedge v = u \otimes v - v \otimes u$ for any two tensors u and v. A simple induction argument shows that $\hat{\Gamma} = [\Gamma] + j(\Gamma)$ with $j(\Gamma) \in J$. Indeed, if $n = 2$,

$$A_1 \wedge A_2 = [A_1, A_2] + j(A_1, A_2)$$

where $j(A_1, A_2) = A_1 \wedge A_2 - [A_1, A_2] \in J$. Similarly, if $A_0 \in \mathfrak{g}$, then

$$A_0 \wedge \hat{\Gamma} = A_0 \wedge [\Gamma] + A_0 \wedge j(\Gamma) = [A_0, [\Gamma]] + j(A_0, [\Gamma]) + A_0 \wedge j(\Gamma),$$

which completes the induction argument since J is an ideal. Clearly if $\Gamma \in H^n$ then $\hat{\Gamma} \in H^{\otimes n}$.

Choose a basis $X_1, \ldots, X_m, Y_1, \ldots, Y_\ell$ of $\mathfrak{g}$ (with $m + \ell = d = \dim \mathfrak{g}$) such that $X_1, \ldots, X_m$ is a basis for H. By Hörmander's condition each vector Y_k is a finite linear combination of commutators $[\Gamma]$ with $\Gamma \in H^n$ and $n \le r$. The corresponding linear combination, $\hat{Y}_k$, of such $\hat{\Gamma}$ lies in $\sum_{k=1}^{r} H^{\otimes k}$ while $\hat{Y}_k - Y_k$ lies in J. Define P on $\mathfrak{g}$ by

$$P\Big(\sum_{j=1}^{m} a_j X_j + \sum_{k=1}^{\ell} b_k Y_k\Big) = \sum_{j=1}^{m} a_j X_j + \sum_{k=1}^{\ell} b_k \hat{Y}_k \tag{2.21}$$

where a_j and b_k are in either $\mathbb{R}$ or $\mathbb{C}$ if $\mathfrak{g}$ is real or complex respectively. At this point $P : \mathfrak{g} \to \bigoplus_{k=1}^{r} H^{\otimes k} \subset T(H)$ is a linear operator such that: (a) $P(A) - A \in J$ for all $A \in \mathfrak{g}$, (b) $P(A) = A$ for all $A \in H$, and (c) P is bounded for any norm on $\mathfrak{g}$ because $\mathfrak{g}$ is finite-dimensional.

By the universal property of the tensor algebra, there is a unique extension of P to an algebra homomorphism $T(\mathfrak{g}) \to T(H)$, which we still denote by P, such that $P(1_{T(\mathfrak{g})}) = 1_{T(H)}$. Since, for $(A_1, \ldots, A_n) \in \mathfrak{g}^n$,

$$P(A_1 \otimes \cdots \otimes A_n) = PA_1 \otimes \cdots \otimes PA_n \in (A_1 + J) \otimes \cdots \otimes (A_n + J)$$

and J is an ideal, it follows that $P(A_1 \otimes \cdots \otimes A_n) - A_1 \otimes \cdots \otimes A_n \in J$. With this observation, the remaining stated properties of P are now easily verified. □

Lemma 2.12. *Assume that Hörmander's condition holds. If $\alpha \in J^0$ and $\|\alpha\|_t = 0$ for some $t > 0$ then $\alpha = 0$.*

Proof. If $\alpha \in J^0$ and $\|\alpha\|_t = 0$ for some $t > 0$ then, by Lemma 2.10 and the definition (2.10), $\alpha|_{T(H)} = 0$. By Lemma 2.11(iii), $\alpha = \alpha \circ P = \alpha|_{T(H)} \circ P = 0$. □

This proves half of Theorem 2.7. The next lemma proves the other half.

Lemma 2.13. *If Hörmander's condition fails then there is an element $\alpha \in J^0$ such that $\alpha \neq 0$ but $q_k(\alpha_k) = 0$ for $k = 0, 1, 2, \ldots,$ i.e. $\|\alpha\|_t = 0$ for all $t > 0$.*

Proof. Let r be as in Notation 2.9 so that $H_r = \mathrm{Lie}(H) \subsetneq \mathfrak{g}$. Then there exists an element $a \in \mathfrak{g}^*$ such that $a \neq 0$ while $a|_{H_r} \equiv 0$. Let $\tilde{a} \in T(\mathfrak{g})'$ be such that $\tilde{a}^j = 0$ if $j \neq 1$ and $\tilde{a}^1 = a$.

By the Poincaré–Birkhoff–Witt theorem, $T := T(\mathfrak{g})$ is the direct sum, $T = \mathcal{S} \oplus J$, where $\mathcal{S}$ is the space of symmetric tensors over $\mathfrak{g}$ (see e.g. [38, Lemma 3.3.3]). Let $P_{\mathcal{S}} : T \to \mathcal{S}$ be the projection onto $\mathcal{S}$ along J and let $\alpha := \tilde{a} \circ P_{\mathcal{S}}$. Then $\alpha \in J^0$. Since $\alpha^1 = a \neq 0$, we have $\alpha \neq 0$. So to finish the proof it suffices to show $q_k(\alpha) = 0$ for all k. Because of Lemma 2.10, this last assertion will be a consequence of

$$\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_k \rangle = 0 \quad \text{for all } \xi_1, \ldots, \xi_k \in H_r = \mathrm{Lie}(H) \text{ and } k = 1, 2, \ldots. \tag{2.22}$$

We will verify (2.22) by induction. The case $k = 1$ is trivial since $\alpha^1 = a = 0$ on H_r. Now suppose (2.22) holds up to some level $k \geq 1$ and let $\xi_i \in H_r$ for $i = 1, \ldots, k+1$. Using the fact that $\alpha \in J^0$, for any $i = 1, \ldots, k$ we have

$$\begin{aligned}\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{k+1} \rangle &- \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{i+1} \otimes \xi_i \otimes \cdots \otimes \xi_{k+1} \rangle \\ &= \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes [\xi_i, \xi_{i+1}] \otimes \xi_{i+2} \otimes \cdots \otimes \xi_{k+1} \rangle = 0,\end{aligned} \tag{2.23}$$

where the induction hypothesis along with the fact that $[\xi_i, \xi_{i+1}] \in H_r$ was used in the second equality. Since any permutation of $\{1, \ldots, k+1\}$ may be written as a product of transpositions of nearest neighbors, it follows from repeated use of (2.23) that

$$\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{k+1} \rangle = \langle \alpha, \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(k+1)} \rangle \tag{2.24}$$

for any permutation σ of $\{1, \ldots, k+1\}$. Averaging (2.24) over all permutations of $\{1, \ldots, k+1\}$ gives

$$\begin{aligned}\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{k+1} \rangle &= \left\langle \alpha, \frac{1}{(k+1)!} \sum_\sigma \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(k+1)} \right\rangle \\ &= \left\langle \tilde{a} \circ P_{\mathcal{S}}, \frac{1}{(k+1)!} \sum_\sigma \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(k+1)} \right\rangle \\ &= \left\langle \tilde{a}, \frac{1}{(k+1)!} \sum_\sigma \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(k+1)} \right\rangle = 0.\end{aligned}$$

□

Corollary 2.14. *Hörmander's condition holds if and only if*

$$T(\mathfrak{g}) = T(H) + J \qquad \text{(not necessarily a direct sum).} \tag{2.25}$$

Proof. We have already seen in Lemma 2.11 that (2.25) holds under Hörmander's condition. Conversely, if Hörmander's condition fails to hold, then by Lemma 2.13 there is a non-zero element $\alpha \in J^0$ (with $\alpha_0 = 0$) which annihilates $T(H)$. Thus α annihilates $T(H) + J$, which would be impossible if (2.25) were valid. □

Remark 2.15. Given two Hermitian forms, q^1 and q^2, satisfying Hörmander's condition, it is a natural question to ask how the norms q_t^1 and q_t^2 are related. We will not discuss this here, but the reader is referred to [6, 7] where this question is considered.

3. The subelliptic heat kernel

Section 2 gave an algebraic interpretation of Hörmander's condition in the tensor algebra (see Theorem 2.7). The rest of this paper is mostly analytic in nature and depends heavily on heat kernel estimates. This short section reviews the necessary material and gives pointers to the literature concerning subelliptic heat kernels.

Let G be a real connected Lie group equipped with its right Haar measure dx. Let q be a non-negative quadratic form on $\mathfrak{g}^*$ and let $(H, (\cdot,\cdot)_H)$ be the Hörmander space associated to q as defined in Section 2. Assume that $\mathrm{Lie}(H) = \mathfrak{g}$, i.e., the Hörmander condition is satisfied. Let $\{X_i : i = 1, \dots, m\}$ be an orthonormal basis of $(H, (\cdot,\cdot)_H)$. Set

$$\Delta = \Delta_q = \sum_{i=1}^{m} \widetilde{X}_i^2$$

where, as before, $\tilde{X}_i$ denotes the left invariant vector field on G which extends the vector $X_i \in \mathfrak{g} = T_e(G)$. Then Δ depends only on q. See (1.2).

It is straightforward to prove that any sum of squares, $L = \sum_{j=1}^m \tilde{X}_j^2$, of left invariant vector fields, $\tilde{X}_j$, is essentially self-adjoint on $C_c^\infty(G)$ in $L^2(G, dy)$ when dy is right invariant Haar measure. Indeed, it is sufficient to prove that $C_c^\infty(G)$ is a core for L^*. To this end one proves that L commutes with left convolution by any function $u \in C_c^\infty(G)$. This in turn implies that $C^\infty(G) \cap D(L^*)$ is a core for L^*. For any function f in this core the truncations $f_n(x) = h_n(x) f(x)$ are in $C_c^\infty(G)$ and converge to f in the L^* graph norm if the sequence $h_n \in C_c^\infty(G)$ converges to one on G in a strong enough sense, as e.g. in [5, Lemma 3.6]. A reader pursuing this route will find it necessary to prove the integration by parts identity $\int_G \sum_{j=1}^m (\tilde{X}_j f)^2\, dy = -(L^* f, f) < \infty$ for functions $f \in C^\infty(G) \cap D(L^*)$. This can be proved by inserting the sequence h_n in the left side before integrating by parts.

The exponential $e^{t\Delta/4}$ may therefore be defined by the spectral theorem. This semigroup commutes with left translations, and the associated quadratic form

$\int_G \sum_{j=1}^m (\tilde{X}_j f)^2\,dy$, $f \in D(\sqrt{-\Delta})$, is a Dirichlet form (see [9]). It follows that $e^{t\Delta/4}$ admits a transition kernel $\rho_t(x, dy)$ with $\rho(t, A) \geq 0$ for all Borel sets A and $\rho_t(x, G) \leq 1$ and such that

$$(e^{t\Delta/4} f)(x) = \int_G f(y)\rho_t(x, dy)$$

for all $f \in L^2(G, dx)$. We will see shortly that the measure $\rho_t(e, dy)$ admits a smooth positive density $x \mapsto \rho_t(x)$ with respect to the right invariant Haar measure on G. We call the measure $\rho_t(e, dx) = \rho_t(x)\,dx$ the *heat kernel measure* on G associated to the sub-Laplacian Δ. It plays a central role in this paper since one of the main objects of interest to us is the scale of Hilbert spaces of holomorphic functions on a complex Lie group that are in L^2 with respect to the heat kernel measure $\rho_t(e, dx) = \rho_t(x)\,dx$. In order to study these spaces, one needs information concerning the heat kernel ρ_t. In particular, the properties of ρ_t collected below play a key technical part in the proof of Theorem 4.2, outlined in Section 4.

The properties of the transition kernel $\rho_t(x, dy)$ are mostly derived through an understanding of the basic geometry associated to the operator Δ (i.e., the quadratic form q). More precisely, define the *intrinsic sub-Riemannian distance* d associated to Δ by setting

$$d(x, y) = \sup\Big\{ f(y) - f(x) : f \in C_c^\infty(G),\ \sum_{i=1}^m |\widetilde{X}_i f|^2 \leq 1 \Big\}. \tag{3.1}$$

It is well known that

$$d(x, y) = d_H(x, y)$$

where d_H is the horizontal distance obtained by minimizing the horizontal length of absolutely continuous curves as spelled out precisely in the next definition. See, e.g., [25] and [37, 35]. In what follows, θ will denote the *Maurer–Cartan form* on G, i.e., θ is the $\mathfrak{g}$-valued 1-form on G defined by $\theta(v) \equiv L_{g^{-1}*}v$ when $v \in T_g G$.

Definition 3.1. *Let $(H, (\cdot,\cdot)_H)$ be the Hörmander space associated to q as defined in Section 2 and set $|u|_H^2 = (u, u)_H$, $u \in H$.*

(i) *A path $g : [a, b] \to G$ is said to be* horizontal *if g is absolutely continuous and $\theta(g'(s)) \in H$ for a.e. s.*

(ii) *The* horizontal length *or H-*length *of a horizontal path $g : [a, b] \to G$ is defined to be*

$$\ell_H(g) = \int_a^b |\theta(g'(s))|_H\,ds. \tag{3.2}$$

If g is not horizontal we define $\ell_H(g) = \infty$.

(iii) *The* horizontal distance *between x and y is defined by*

$$d_H(x, y) = \inf\{\ell_H(g) : g(0) = x,\ g(1) = y\}. \tag{3.3}$$

Chow's theorem asserts (in a more general context, see, e.g., [27]) that Hörmander's condition implies that any two points in G can be joined by a horizontal path of finite H-length. Thus $d(x, y)$ is finite for all x, y. The Ball–Box theorem (see for example [27, Theorem 2.10] or [14, Section 0.5.A]) asserts that there exists $a > 0$ such that for any left invariant Riemannian distance function, d_{Riem}, $C_1 d_{\text{Riem}}(x, y) \leq d(x, y) \leq C_2 d_{\text{Riem}}(x, y)^a$ for all x, y such that $d(x, y) \leq 1$. Theorem 6.7 below implies the weaker result that $\{d(e, x) < r\}$ is an open neighborhood of e in the natural topology of G. By either of these results, it follows that d is continuous and induces the manifold topology of G.

Set

$$B(x, r) = \{y \in G : d(x, y) < r\}$$

and let $|B(x, r)|$ denote the right Haar measure of $B(x, r)$. One of the most basic results concerning the local analysis of the sub-Laplacian Δ is the following.

Theorem 3.2. *Referring to the above setting and notation, there are constants C_1, C_2 such that for any $x \in G$ and any $r \in (0, 1)$ we have:*

(i) $|B(x, 2r)| \leq C_1 |B(x, r)|$,
(ii) $\int_B |f(z) - f_B|^2 \, dz \leq C_2 r^2 \int_B \sum_{i=1}^m |\widetilde{X}_i f(z)|^2 \, dz$, $B = B(x, r)$, $f \in \mathrm{Lip}(\overline{B})$,

where $f_B := |B|^{-1} \int_B f(z) \, dz$ is the mean of f over B.

Proof. For the doubling property (i) we refer to [25, 28, 39]. In fact, there are constants $c_3, C_3 \in (0, \infty)$ and an integer $\nu = \nu_q$ such that

$$\forall r \in (0, 1), \quad c_3 r^\nu \leq |B(e, r)| \leq C_3 r^\nu. \tag{3.4}$$

The integer ν plays a role in the heat kernel estimates given below.

For the Poincaré inequality (ii), see [24, 25, 30, 31]. □

By the general results of [30, 36], Theorem 3.2 yields a powerful parabolic Harnack inequality and the heat kernel bounds stated in the following two theorems.

Theorem 3.3 (Parabolic Harnack inequality). *There exists a constant $C > 0$ such that, for any $T > 0$, if $(0, T) \times G \ni (t, x) \mapsto u(t, x)$ is any non-negative solution of $\partial u/\partial t = (1/4)\Delta u$ then*

$$u(s, x) \leq u(t, y) \exp\left(C\left[\frac{t}{s} + \frac{d(x, y)^2}{t - s}\right]\right) \tag{3.5}$$

for all $x \in G$ and $0 < s < t < T$.

Proof. See [30, Theorem 3.1] and the arguments in [31, Sec. 5.4.3]. See also [36, Theorem 3.5] and [39, Proposition IX.1.1]. □

One of the many consequences of Theorem 3.2 is that the transition kernel $\rho_t(x, dy)$ of the semigroup $e^{t\Delta/4}$ admits a continuous density, $\rho_t(x, dy) = h_t(x, y) \, dy$, with respect to the right invariant Haar measure on G. The function $(t, x, y) \mapsto h_t(x, y)$ is called the *heat kernel* associated to the sub-Laplacian Δ on G. Moreover, h_t is a fundamental solution of the heat equation on G, i.e., a solution of the initial value problem

$$\begin{cases} \partial h_t(x, \cdot)/\partial t = (1/4)\Delta h_t(x, \cdot), \\ h_t(x, y)\, dy \to \delta_x(dy) \quad \text{(weakly) as } t \to 0. \end{cases} \tag{3.6}$$

A further consequence of Theorem 3.2 is that uniqueness holds for the non-negative Cauchy problem associated with the heat equation (3.6). See [1].

By construction, the operator $e^{t\Delta/4}$ commutes with left translations whereas the Haar measure dy is right invariant. It follows that

$$h_t(x, y) = h_t(e, x^{-1}y)m(x)$$

where m denotes the modular function defined by $\int_G f(gx)\, dx = m(g) \int_G f(x)\, dx$ (the function m is a continuous multiplicative function). A reader may consult [29] for further details. In what follows we set

$$\rho_t(x) = h_t(e, x)$$

so that $\rho_t(x)$ is the density of the heat kernel measure

$$\rho_t(e,\, dx) = \rho_t(x)\, dx.$$

We will often refer to ρ_t, somewhat improperly, as the heat kernel.

Theorem 3.4. *Referring to the above setting and notation, the heat kernel $\rho_t(x)$ has the following properties:*

(i) (Regularity) $(t, x) \mapsto \rho_t(x)$ *is a smooth positive function on* $(0, \infty) \times G$.
(ii) (Conservation of heat) $\int_G \rho_t(x)\, dx = 1$.
(iii) (Gaussian upper bound) *For any $\kappa \in (0, 1)$, there exists $C_\kappa \in (0, \infty)$ such that for all $x \in G$ and all $t > 0$,*

$$\rho_t(x) \le C_\kappa (1 + 1/t)^{\nu/2} e^{C_\kappa t} e^{-\kappa d(e,x)^2/t}. \tag{3.7}$$

(iv) (Gaussian lower bound) *There are constants $C, c \in (0, \infty)$ such that, for all $x \in G$ and all $t > 0$,*

$$\rho_t(x) \ge c(1 + 1/t)^{\nu/2} e^{-Ct} e^{-Cd(e,x)^2/t}. \tag{3.8}$$

In the last two statements, ν is the integer introduced at (3.4).

Proof (outline). (i) That the heat kernel is smooth is a basic consequence of Hörmander's hypoellipticity theorem. That it is positive easily follows, for instance, from (3.5) although it can be obtained more directly.

(ii) This property (conservativeness) is again a consequence of Theorem 3.2 by way of a local Harnack inequality (see [34]). It also follows by the remark made above concerning uniqueness of solutions to the positive Cauchy problem (see [1]). Alternatively, one can use Grigor'yan's volume criterion (see [13] and [34]). Indeed, on any group, $r \mapsto |B(x,r)|$ grows at most exponentially fast.

(iii) This heat kernel upper bound is in [39, Theorem IX.1.2]. It also follows from the local parabolic Harnack inequality and the volume estimate (3.4). See [30].

(iv) This heat kernel lower bound is stated in [39, Theorem IX.1.2] for $0 < t < 1$. The global Harnack type inequality (3.5) easily gives the desired result for $t \geq 1$. □

Remark 3.5. Note that as κ tends to 1, the Gaussian factor $e^{-\kappa d(e,x)^2/t}$ in Theorem 3.4(iii) tends to its optimal value $e^{-d(e,x)^2/t}$ (recall that our heat semigroup is $e^{t\Delta/4}$). The fact that such an approximately optimal heat kernel upper bound holds is crucial for the analysis developed in this paper.

4. The Taylor map

Let G be a complex Lie group with Lie algebra $\mathfrak{g}$. Suppose we are given a non-negative Hermitian form q on the complex vector space $\mathfrak{g}^*$. As in Notation 2.1 and Lemma 2.2, let K be the kernel of q, $H = K^0$ be the backward annihilator of K in $\mathfrak{g}$, and let $\{X_j\}_{j=1}^m$ be an orthonormal basis for the complex inner product space H. Then the vectors $\{X_j, iX_j : j = 1, \ldots, m\}$, where $i = \sqrt{-1}$, form an orthonormal basis of H as a real vector space with inner product $\mathrm{Re}(\cdot,\cdot)_H$. The subspace $H \subset \mathfrak{g}$ generates the full Lie algebra $\mathfrak{g}$ over the complex numbers if and only if it generates $\mathfrak{g}$ as a real Lie algebra. Define

$$\Delta = \Delta_{\mathrm{Re}\,q} = \sum_{j=1}^{m} (\tilde{X}_j^2 + \widetilde{(iX_j)}^2) \tag{4.1}$$

where, as before, for $A \in \mathfrak{g}$, $\tilde{A}$ is the left invariant vector field on G such that $\tilde{A}(e) = A$. It is easy to see that the second order differential operator Δ is independent of the choice of orthonormal basis $X_1, \ldots, X_m$. By Hörmander's theorem Δ is subelliptic if and only if H generates $\mathfrak{g}$. Throughout this section we will assume that H does generate $\mathfrak{g}$. Let ρ_t in $C^\infty(G)$ be the heat kernel introduced in (3.6).

Notation 4.1. We denote by $\mathcal{H}$ the space of holomorphic functions on G and define

$$\mathcal{H}L^2(G, \rho_t(x)\,dx) = \mathcal{H} \cap L^2(G, \rho_t(x)\,dx). \tag{4.2}$$

For any finite-dimensional holomorphic representation, $\pi : G \to GL(n, \mathbb{C})$, polynomials in the matrix entries of π will lie in the space $\mathcal{H}L^2(G, \rho_t(x)\,dx)$ for any such subelliptic Laplacian. Such a representation π always exists when G is simply connected. For example by Ado's theorem [23, p. 199], $\mathfrak{g}$ has a faithful representation as a matrix subalgebra of $gl(n, \mathbb{C})$ for some n. Since G is simply connected, the Lie algebra representation integrates to a holomorphic representation, $\pi : G \to GL(n, \mathbb{C})$.

Recall that for each $\beta \in T(\mathfrak{g})$, $\tilde{\beta}$ is the corresponding left invariant partial differential operator on G as in Notation 2.4. If f is a holomorphic function defined in a neighborhood of the identity element of G then, as in (1.3), f defines a linear functional $\hat{f}(e)$ on $T(\mathfrak{g})$. Notice that $\hat{f}(e)$ is complex linear and that $\hat{f}(e) \in J^0$ where J^0 is the annihilator of $J \subset T(\mathfrak{g})$, defined in Notation 2.5. The complex linearity is a consequence of the fact that f is holomorphic. To see that $\hat{f}(e) \in J^0$, observe that $\tilde{\beta}_1 \tilde{h} \tilde{\beta}_2$ annihilates all functions if β_1 and β_2 are in $T(\mathfrak{g})$ and $h = A \otimes B - B \otimes A - [A, B]$ is a generator of J. Since J is the linear span of such elements, $\langle \hat{f}(e), \beta \rangle = (\tilde{\beta} f)(e) = 0$ for all $\beta \in J$.

Our main theorem in this section is the following.

Theorem 4.2. *Let G be a connected complex Lie group. Suppose that q is a non-negative Hermitian form on the dual space $\mathfrak{g}^*$ and assume that Hörmander's condition holds* (*cf. Definition* 2.6). *Let ρ_t denote the heat kernel associated to* (3.6). *Then the* ***Taylor map,***

$$f \mapsto \hat{f}(e), \tag{4.3}$$

is an isometry from $\mathcal{H}L^2(G, \rho_t(x)\,dx)$ into J_t^0.

Proof. The proof follows the pattern given in [5] for the case of non-degenerate q. We therefore just sketch the proof, emphasizing the issues that present a possible difference. The tensor $D^n f(x)$, of n^{th} order derivatives of f at x, is defined by

$$\langle (D^n f)(x), \xi_1 \otimes \cdots \otimes \xi_n \rangle = (\tilde{\xi}_1 \ldots \tilde{\xi}_n f)(x). \tag{4.4}$$

Let us first observe that the identity

$$(\Delta/4)|D^k f(x)|^2_{q_k} = |D^{k+1} f(x)|^2_{q_{k+1}} \quad \text{when} \;\; f \in \mathcal{H}(G) \tag{4.5}$$

holds in our degenerate case when the norms that appear are those induced on k-tensors by q. The proof is identical to that for the non-degenerate case (cf. [5, Remark 3.7]). Suppose now that $f \in \mathcal{H}L^2(G, \rho_t(x)\,dx)$ and define

$$F(s) = \int_G |f(x)|^2 \rho_s(x)\,dx, \qquad 0 \le s \le t. \tag{4.6}$$

We are going to proceed, at first, entirely informally and then discuss what needs to be done to justify the following computations. By definition ρ_s satisfies $\partial_s \rho_s(x) = (\Delta/4)\rho_s(x)$. Differentiate (4.6) and use (4.5) to find

$$\frac{dF(s)}{ds} = \frac{d}{ds} \int_G |f(x)|^2 \rho_s(x)\,dx \tag{4.7}$$

$$= \int_G |f(x)|^2 \frac{\partial}{\partial s} \rho_s(x)\,dx \tag{4.8}$$

$$= \int_G |f(x)|^2 (\Delta/4)\rho_s(x)\,dx$$

$$= \int_G \{(\Delta/4)|f(x)|^2\} \rho_s(x)\,dx \tag{4.9}$$

$$= \int_G \{|Df(x)|^2_q\} \rho_s(x)\,dx.$$

A similar derivation shows by induction that

$$F^{(k)}(s) = \int_G |D^k f(x)|^2_{q_k} \rho_s(x)\,dx, \quad k = 0, 1, 2, \ldots. \tag{4.10}$$

Were it possible to use these derivatives to expand F as a power series around $s = 0$, we would find from (4.10) and the expected relation $F^{(k)}(0) = \lim_{s\downarrow 0} F^{(k)}(s) = |D^k f(e)|^2$ that

$$F(s) = \sum_{k=0}^{\infty}(s^k/k!)F^{(k)}(0) = \sum_{k=0}^{\infty}(s^k/k!)|(D^k f)(e)|^2_{q_k} = |\hat{f}(e)|^2_{J^0_s}. \tag{4.11}$$

Therefore

$$\|f\|^2_{L^2(G,\rho_s(x)\,dx)} = \|\hat{f}(e)\|^2_{J^0_s}, \tag{4.12}$$

which, for $s = t$, is the isometry we wish to prove.

Among the previous steps the following clearly need justification:

(a) the interchange of d/ds with $\int_G$ in (4.8),
(b) the integration by parts in (4.9) (and in the similar derivation of (4.10)),
(c) the validity of the expansion in (4.11).

The only hypothesis available to us for these justifications is the assumption that $f \in \mathcal{H}L^2(G, \rho_t(x)\,dx)$. We do not have, for a general complex group, a method of approximating such rapidly growing holomorphic functions by more slowly growing holomorphic functions. Justification of the three items in (a)–(c) must therefore be done directly for the rapidly growing function f. The justification of these steps, developed in [5], consists in establishing expansion coefficient bounds, $\|\hat{f}(e)\|_s \le \|f\|_{L^2(\rho_s)}$ (cf. Proposition 3.3 in [5]), and pointwise bounds (cf. Corollary 3.10 in [5]),

$$|f(x)|^2 \le \|\hat{f}(e)\|^2_s e^{d(e,x)^2/s} \quad \text{for all } x \in G, \tag{4.13}$$

as well as similar pointwise bounds on the derivatives of f, $|D^k f(x)|_{q_k}$, where in [5], $d(e,x)$ refers to the Riemannian distance associated to q in the non-degenerate case. These estimates go over to the subelliptic case with no changes except that $d(e,x)$ should now be interpreted as the sub-Riemannian distance associated to q defined in either of (3.1) or (3.3). (The estimate in (4.13) and the analogous estimates for $|D^k f(x)|_{q_k}$ will be re-derived in Corollary 5.15 below.) One combines growth rates, such as (4.13), with known decay rates for subelliptic heat kernels (see Theorem 3.4) to justify the steps listed in (a)–(c). To prove (4.11), it is shown in [5, Section 4], using these rather detailed bounds on the derivatives $D^k f(x)$ and the consequent bounds on the derivatives $F^{(k)}(s)$ for $0 < s < t$, that F has a complex analytic extension to a complex neighborhood of $[0,t)$. The result of this procedure is to establish (4.12) for $s < t$. For $s = t$ one then uses a monotonicity argument on both sides of (4.12) as $s \uparrow t$ (cf. [5, Section 5 or Appendix 8]; one should replace the Li–Yau Harnack inequality used in [5] by the Harnack inequality stated in (3.5)). □

The following proposition complements Theorem 4.2 and makes use of the estimate (4.13). In words, it says that the inverse image of J^0_t under the Taylor map $f \mapsto \hat{f}(e)$ from $\mathcal{H}(G)$ into J^0 is contained in $\mathcal{H}L^2(G, \rho_t(x)\,dx)$.

Proposition 4.3. *Let $f \in \mathcal{H}(G)$ and assume that $\hat{f}(e) \in J_t^0$ (see (1.3)) for some $t > 0$. Then $f \in \mathcal{H}L^2(G, \rho_t(x)\,dx)$.*

Proof. As noted above, (4.13) and known heat kernel estimates (cf. (3.7)) show that if $\hat{f}(e) \in J_t^0$ then $f \in \mathcal{H}L^2(G, \rho_s)$ for $s < t$. By Theorem 4.2 we have

$$\|f\|_{L^2(G,\rho_s(x)\,dx)} = \|\hat{f}(e)\|_s \le \|\hat{f}(e)\|_t.$$

The desired conclusion follows because

$$\lim_{s\uparrow t} \|f\|_{L^2(G,\rho_s(x)\,dx)} = \|f\|_{L^2(G,\rho_t(x)\,dx)}.$$

See [5, Sect. 5 or Appendix 8]. The Li–Yau Harnack inequality used in [5] should be replaced by (3.5). □

5. Power series along a curve in a Lie group

If z is a point in $\mathbb{C}^n$ and $z^{\otimes k}$ is its k^{th} tensor power in $(\mathbb{C}^n)^{\otimes k}$ then the conventional power series representation of a holomorphic function f in a neighborhood of 0 may be written $f(z) = \langle \alpha, \Phi(z) \rangle$, where $\Phi(z) := \sum_{k=0}^{\infty} (k!)^{-1} z^{\otimes k}$ is an element of the (suitably completed) tensor algebra over $\mathbb{C}^n$ and α is in the dual space. In order to recover a holomorphic function f on a complex Lie group G from a knowledge of its Taylor coefficient $\alpha = \hat{f}(e)$ (cf. (1.3)), we will need to represent f locally and globally on G by an analogous kind of power series. Of course we do not have a global coordinate system as on $\mathbb{C}^n$. Consider a piecewise smooth curve $g : [0, 1] \to G$ beginning at the identity $e \in G$ and ending at a point $z \in G$. We are going to replace the tensor-valued function $\Phi(z)$ above by a path dependent tensor-valued function $\Psi(g)$ so that f is again given by $f(z) = \langle \alpha, \Psi(g) \rangle$, both locally and globally. When $G = \mathbb{C}^n$ and g is the straight-line path joining 0 to z our function $\Psi(g)$ reduces to $\Phi(z)$ and, in addition, $\langle \alpha, \Psi(g) \rangle = \langle \alpha, \Phi(z) \rangle$ for all paths g joining 0 to z.

In order to carry out the replacement we will first develop, in Section 5.1, the needed estimates in the space where $\Psi(g)$ will lie. In Section 5.2 we will describe the path dependent power series expansion associated to a local holomorphic function. And in Section 5.3 we will show that the seemingly path dependent series associated to a presumed Taylor coefficient α of limited size actually depends only on the homotopy class (with fixed endpoints) of the path. For the elliptic case this machinery has been developed in [4] and [5].

5.1. The Fréchet tensor algebra

Definition 5.1 (Fréchet tensor algebra). *Let V be a real or complex finite-dimensional vector space with an inner product $(\ ,\ )$ and associated norm $|\cdot|$. Let*

$$T_\infty(V) = \prod_{n=0}^{\infty} V^{\otimes n}$$

and for $A = \sum_{n=0}^{\infty} A_n \in T_\infty(V)$ *and* $B = \sum_{n=0}^{\infty} B_n \in T_\infty(V)$ *with* $A_n, B_n \in V^{\otimes n}$ *for all* n, *define*

$$AB := \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} A_k \otimes B_{n-k}\Big) \in T_\infty(V), \qquad \|A\|_t^2 := \sum_{n=0}^{\infty} \frac{n!}{t^n}|A_n|^2,$$

$$T_t(V) := \{A \in T_\infty(V) : \|A\|_t < \infty\}, \qquad T_+(V) = \lim_{t\downarrow 0} T_t(V) := \bigcap_{t>0} T_t(V).$$

Observe that

$$T(V) \subset T_+(V) \subset T_s(V) \subset T_t(V) \subset T_\infty(V) \quad \text{for } 0 < s < t < \infty.$$

The containment $T(V) \subset T_+(V)$ asserts that any finite rank tensor is in $T_t(V)$ for all $t > 0$, which is clear. $T_+(V)$ also contains some tensors of infinite rank. For example, if $A \in T_\infty(V)$ then $A \in T_+(V)$ if $|A_n| = O((n!)^{-\delta})$ for some $\delta > 1/2$. See Proposition 5.10 for more examples of elements of $T_+(V)$.

The following lemma is a technical improvement on [4, Lemma 2.18].

Lemma 5.2 ($T_+(V)$ is an algebra). *If* $s, t > 0$, $A \in T_t(V)$ *and* $B \in T_s(V)$, *then* $AB \in T_{s+t}(V)$ *and*

$$\|AB\|_{s+t} \le \|A\|_t \|B\|_s. \tag{5.1}$$

In particular, $T_+(V)$ *is an algebra.*

Proof. Write $A = \sum_{k=0}^{\infty} A_k$ and $B = \sum_{k=0}^{\infty} B_k$, where $A_k, B_k \in V^{\otimes k}$. Then

$$\begin{aligned}
|(AB)_n|^2 &= \Big|\sum_{k=0}^{n} A_k \otimes B_{n-k}\Big|^2 \le \Big(\sum_{k=0}^{n} |A_k|\,|B_{n-k}|\Big)^2 \\
&= \Bigg(\sum_{k=0}^{n} |A_k|\sqrt{\frac{k!}{t^k}}|B_{n-k}|\sqrt{\frac{(n-k)!}{s^{n-k}}} \cdot \sqrt{\frac{t^k s^{n-k}}{k!\cdot(n-k)!}}\Bigg)^2 \\
&\le \sum_{k=0}^{n}\Bigg(|A_k|^2\frac{k!}{t^k}|B_{n-k}|^2\frac{(n-k)!}{s^{n-k}}\Bigg)\sum_{k=0}^{n}\frac{t^k s^{n-k}}{k!\cdot(n-k)!} \\
&= \frac{(t+s)^n}{n!}\sum_{k=0}^{n}|A_k|^2\frac{k!}{t^k}\cdot|B_{n-k}|^2\frac{(n-k)!}{s^{n-k}},
\end{aligned}$$

and therefore

$$\begin{aligned}
\|AB\|_{s+t}^2 &\le \sum_{n=0}^{\infty}\frac{n!}{(t+s)^n}\cdot\frac{(t+s)^n}{n!}\sum_{k=0}^{n}|A_k|^2\frac{k!}{t^k}\cdot|B_{n-k}|^2\frac{(n-k)!}{s^{n-k}} \\
&= \sum_{0\le k\le n<\infty}|A_k|^2\frac{k!}{t^k}\cdot|B_{n-k}|^2\frac{(n-k)!}{s^{n-k}} = \|A\|_t^2\|B\|_s^2.
\end{aligned}$$

□

Remark 5.3. Lemma 5.2 is sharp, in the sense that

$$\sup_{A,B\in T_+(V)\setminus\{0\}} \frac{\|AB\|_r}{\|A\|_t\|B\|_s} = \infty \quad \text{if } r < s + t.$$

This can be seen by fixing $\xi \in V$ with $|\xi| = 1$ and then taking

$$A = \operatorname{Exp}(a\xi) := 1 + \sum_{n=1}^{\infty} \frac{a^n}{n!}\xi^{\otimes n}$$

and $B = \operatorname{Exp}(b\xi)$ while letting a and b range over $\mathbb{R}$.

Corollary 5.4. *Denote by L_β and R_β the left and right multiplication in $T_\infty(V)$. Thus $R_\beta\eta = \eta\beta$ and $L_\beta\eta = \beta\eta$ for all $\eta \in T_\infty(V)$. If $\beta \in T(V)$ then L_β and R_β are bounded operators from $T_s(V)$ into $T_t(V)$ whenever* $0 < s < t$.

Proof. By (5.1), $\|R_\beta\eta\|_t \le \|\eta\|_s\|\beta\|_{t-s}$, with the same inequality for L_β. Since $\|\beta\|_{t-s} < \infty$ the assertion follows. □

In the remainder of this section we will let G be a complex Lie group and let $\mathfrak{g} := \operatorname{Lie}(G)$ be its Lie algebra. Denote by q a non-negative Hermitian form on $\mathfrak{g}^*$. We will assume throughout this section that q satisfies Hörmander's condition (cf. Definition 2.6). Let $(H, (\cdot,\cdot)_H)$ be the inner product space described in (2.3) and Lemma 2.2 and let J_t^0 be as in (2.11). Lemma 5.2 will be often applied with $V = H$. In particular, if $\beta \in T(H)$ we define

$$\|\beta\|_t^2 = \sum_{n=0}^{\infty} \frac{n!}{t^n}|\beta_n|_H^2. \tag{5.2}$$

Remark 5.5. If $\alpha \in J_t^0$ and $\beta \in T(H)$, then, in view of (2.10) and (2.16),

$$|\langle \alpha, \beta\rangle| \le \|\alpha\|_t \cdot \|\beta\|_t,$$

and therefore $\alpha|_{T(H)}$ extends uniquely to an element of $T_t(H)^*$. We will continue to denote this extension by α. Moreover, using this identification, we have $\|\alpha\|_t = \|\alpha\|_{T_t(H)^*}$. But α is also a linear functional on $T(\mathfrak{g})$. No norms have been specified so far on $T(\mathfrak{g})$ and moreover there appears to be no natural norm on $T(\mathfrak{g})$ with respect to which α is continuous. Nevertheless, we will need to make use of $\langle \alpha, \beta\rangle$ for certain elements β which do not lie over H. To this end, in order to estimate the size of $\langle \alpha, \beta\rangle$, we will need to project these tensors into $T_\infty(H)$ along J. The key tool will be the projection constructed in Lemma 2.11.

Proposition 5.6. *Suppose that* $0 < s < \sigma$ *and that* $\alpha \in J_\sigma^0$. *If $\beta \in T(\mathfrak{g})$ then $\alpha \circ R_\beta$ and $\alpha \circ L_\beta$ are in J_s^0. Moreover, denoting by P the projection from $T(\mathfrak{g})$ onto $T(H)$ as constructed in Lemma* 2.11, *we have*

$$\|\alpha \circ R_\beta\|_s \le \|P\beta\|_{\sigma-s}\|\alpha\|_\sigma. \tag{5.3}$$

Proof. First observe that if $u \in J$ then $(\alpha \circ R_\beta)(u) = \alpha(u \otimes \beta) = 0$ if $\alpha \in J^0$. So $\alpha \circ R_\beta \in J^0$. Thus we need only focus on the size issue. Since J is an ideal, $\beta = P\beta \bmod J$, and α annihilates J, we may write, for any element $\eta \in T(H)$,

$$\begin{aligned}|\langle \alpha \circ R_\beta, \eta\rangle| &= |\langle \alpha, \eta \otimes \beta\rangle| = |\langle \alpha, \eta \otimes P\beta\rangle| \\ &\le \|\alpha\|_\sigma \|\eta \otimes P\beta\|_\sigma \le \|\alpha\|_\sigma \|\eta\|_s \|P\beta\|_{\sigma-s}.\end{aligned}$$

This proves (5.3). The proof for L_β is similar. □

Corollary 5.7. *Suppose that $\alpha \in J^0_\sigma$ for some $\sigma > 0$. Let $\psi := \sum_{n=0}^\infty \psi_n \in T_+(H)$. Define $\psi_{\le N} = \sum_{n=0}^N \psi_n$. Then $\alpha \circ L_\psi := \lim_{N\to\infty} \alpha \circ L_{\psi_{\le N}}$ exists in J^0_s for any $s \in (0, \sigma)$. Moreover,*

$$\|\alpha \circ L_\psi\|_s \le \|\psi\|_{\sigma-s}\|\alpha\|_\sigma. \tag{5.4}$$

Proof. By Proposition 5.6,

$$\|\alpha \circ L_{\psi_{\le N}}\|_s \le \|\psi_{\le N}\|_{\sigma-s}\|\alpha\|_\sigma, \tag{5.5}$$

$$\|\alpha \circ L_{\psi_{\le N}} - \alpha \circ L_{\psi_{\le K}}\|_s \le \|\psi_{\le N} - \psi_{\le K}\|_{\sigma-s}\|\alpha\|_\sigma. \tag{5.6}$$

Since $\psi_{\le N}$ converges to ψ in the sense of $T_{\sigma-s}(H)$, it follows from (5.6) that $\{\alpha \circ L_{\psi_{\le N}}\}_{N=1}^\infty$ is convergent in $T_s(H)$. Passing to the limit in (5.5) proves (5.4). □

5.2. A generalized power series

Definition 5.8. *A function, $g : [0, 1] \to G$, is a* piecewise C^k path *if:* (1) *g is continuous and* (2) *there exists a partition,*

$$D := \{0 = r_0 < r_1 < \cdots < r_l = 1\}, \tag{5.7}$$

of $[0, 1]$ *and functions, $g_i \in C^k([r_{i-1}, r_i], G)$, such that $g|_{[r_{i-1},r_i]} = g_i$ for $i = 1, \ldots, l$. We further say that a collection of paths $\{g_t\}_{t\in\mathbb{R}}$ are* piecewise C^2 paths depending differentiably on t, *if:* (1) *$(s, t) \mapsto g_t(s) \in G$ is continuous and* (2) *there exists a partition D as in* (5.7) *and functions $g_i \in C^2([r_{i-1}, r_i] \times \mathbb{R}, G)$ such that $g_t(s) = g_i(s, t)$ when $(s, t) \in [r_{i-1}, r_i] \times \mathbb{R}$. In particular, we are assuming that $\dot g_t(s) := \frac{d}{dt} g_t(s)$ exists for all $s \in [0, 1]$.*

For $0 \le r < s \le 1$, let

$$\Delta_n(r, s) := \{(s_1, \ldots, s_n) : r \le s_1 < \cdots < s_n \le s\}$$

and let $d\mathbf{s} = ds_1 \cdots ds_n$.

Notation 5.9. For $c \in L^1([0, 1], \mathfrak{g})$ and $0 \le r \le s \le 1$, define

$$\psi_{r,s}(c) = \sum_{n=0}^\infty \psi^n_{r,s}(c) \in T_\infty(\mathfrak{g}) \tag{5.8}$$

where $\psi^0_{r,s}(c) = 1$ and for $n \geq 1$,

$$\psi^n_{r,s}(c) = \int_{\Delta_n(r,s)} c(s_1) \otimes \cdots \otimes c(s_n)\, d\mathbf{s}. \tag{5.9}$$

Given a piecewise C^1 path, $g : [0, 1] \to G$, and $0 \leq r < s \leq 1$, let

$$\Psi^n_{r,s}(g) := \psi^n_{r,s}(c) \quad \text{and} \quad \Psi_{r,s}(g) := \psi_{r,s}(c) \tag{5.10}$$

where $c(s) := \theta(g'(s))$ and θ is the Maurer–Cartan form (cf. Section 3). For notational simplicity we will write $\psi_{0,s}(c)$ and $\Psi_{0,s}(g)$ simply as $\psi_s(c)$ and $\Psi_s(g)$ respectively. It is important to observe that if the path g is horizontal (cf. Definition 3.1), then $\Psi^n_{r,s}(g)$ lies in $H^{\otimes n}$.

The following proposition provides a key quantitative control on $\Psi_{r,s}$.

Proposition 5.10. *Suppose that $g : [0, 1] \to G$ is a piecewise C^1 horizontal path, and $0 \leq r < s \leq 1$. Then*

$$|\Psi^n_{r,s}(g)|_{H^{\otimes n}} \leq \frac{1}{n!}\left(\int_r^s |\theta(g'(\sigma))|\, d\sigma\right)^n = \frac{\ell_H(g|_{[r,s]})^n}{n!}. \tag{5.11}$$

For any $t > 0$,

$$\|\Psi_{r,s}(g)\|^2_{T_t(H)} \leq \exp\left(\frac{1}{t}\left[\int_r^s |\theta(g'(\sigma))|\, d\sigma\right]^2\right) = \exp\left\{\frac{1}{t}\ell_H(g|_{[r,s]})^2\right\}. \tag{5.12}$$

Moreover, if $0 < \sigma < t$ and $\beta \in T(H)$ then

$$\|\Psi_1(g) \otimes \beta\|^2_{T_t(H)} \leq \|\beta\|_{t-\sigma} e^{\ell_H(g)^2/\sigma}. \tag{5.13}$$

Proof. Letting $c(s) := \theta(g'(s))$, we may estimate (5.9) by

$$|\Psi^n_{r,s}(g)| = |\psi^n_{r,s}(c)| \leq \int_{\Delta_n(r,s)} |c(s_1)| \ldots |c(s_n)|\, d\mathbf{s} = \frac{1}{n!}\left(\int_r^s |c(\sigma)|\, d\sigma\right)^n, \tag{5.14}$$

which proves (5.11). The estimate (5.12) follows by squaring both sides of (5.11), multiplying the resulting estimate through by $n!/t^n$, and then summing on n. Inequality (5.13) now follows from (5.1). □

The following lemma summarizes some elementary properties of the various Ψ functions. We leave the proofs to the reader.

Lemma 5.11. *Let g be a piecewise C^1 horizontal path in G and let $\Psi(g)$ and $\Psi^n(g)$ be defined as in Notation* 5.9.

(i) *For $n \in \mathbb{N}$ and $0 \le r \le s \le 1$ with $r, s \notin D$ (cf.* (5.7)),

$$\frac{d}{ds}\Psi^n_{r,s}(g) = \Psi^{n-1}_{r,s}(g) \otimes c(s) \tag{5.15}$$

and

$$\frac{d}{dr}\Psi^n_{r,s}(g) = -c(r) \otimes \Psi^{n-1}_{r,s}(g). \tag{5.16}$$

(ii) $\Psi_{r,s}(g)$ *satisfies*

$$\frac{d}{ds}\Psi_{r,s}(g) = \Psi_{r,s}(g) \otimes c(s) \quad \text{with } \Psi_{r,r}(g) = 1 \tag{5.17}$$

and

$$\frac{d}{dr}\Psi_{r,s}(g) = -c(r) \otimes \Psi_{r,s}(g) \quad \text{with } \Psi_{s,s}(g) = 1 \tag{5.18}$$

where the derivatives exist in $T_t(H)$ for all $t > 0$.

Such tensor-valued functions of paths, $\Psi_{r,s}(g)$, were used in [4] to prove isomorphisms similar to those in this paper when G is the complexification of a reductive group. They have also been used as a tool in rough path analysis. See e.g. [26].

The following proposition explains the role of the path dependent Ψ function in the "power series" expansion of a local holomorphic function on G and motivates our reconstruction, in the next section, of a holomorphic function f_α from its "Taylor coefficient" $\alpha \in J^0$.

Notation 5.12. For any $\varepsilon > 0$, let

$$U^H_\varepsilon := \{x \in G : d(e,x) = d_H(e,x) < \varepsilon\}.$$

As we have already mentioned, U^H_ε is an open neighborhood of e.

Proposition 5.13. *Let $\varepsilon > 0$ and $a \in G$. If $f \in \mathcal{H}(aU^H_\varepsilon)$ and $g : [0,1] \to G$ is a piecewise C^1 path such that $g(0) = a$ and $\ell_H(g) < \varepsilon$, then*

$$f(g(1)) = \langle \hat{f}(a), \Psi_1(g)\rangle := \sum_{k=0}^{\infty} \langle \hat{f}(a), \Psi^k_1(g)\rangle \tag{5.19}$$

where the sum is absolutely convergent. More generally, if $\beta \in T(\mathfrak{g})$, then

$$(\tilde{\beta} f)(g(1)) = \langle \hat{f}(a), \Psi_1(g) \otimes \beta\rangle := \sum_{k=0}^{\infty} \langle \hat{f}(a), \Psi^k_1(g) \otimes \beta\rangle. \tag{5.20}$$

Proof. See [4, Proposition 5.1] where this same result is proved in the case $\varepsilon = \infty$ (i.e. $U^H_\varepsilon = G$) and $a = e$. The proof used there works here as well (when $a = e$) provided the parameter $z \in \mathbb{C}$ which appears in [4] is always required to satisfy $|z| < \varepsilon/\ell_H(g)$. The main point is that if we define $g_z(s) \in G$ as the solution to the ODE

$$\theta(g'_z(s)) = zc(s) \quad \text{with} \quad g_z(0) = e,$$

then $\ell_H(g_z) = |z|\ell_H(g) < \varepsilon$ provided that $|z| < \varepsilon/\ell_H(g)$. In particular, this implies that $g_z([0,1]) \subset U_\varepsilon^H$, and this is what is required to run the argument in [4, Proposition 5.1]. At the end of this argument we set $z = 1$, which is permissible since $\varepsilon/\ell_H(g) > 1$.

When $a \neq e$, apply the result with f replaced by $w(y) := f(ay)$ and $g(s)$ replaced by $a^{-1}g(s)$, in which case we learn that

$$f(g(1)) = w(a^{-1}g(1)) = \langle \hat{w}(e), \Psi_1(a^{-1}g)\rangle = \langle \hat{f}(a), \Psi_1(g)\rangle,$$

where the last equality holds because $\hat{f}(a) = \hat{w}(e)$ and $\theta((a^{-1}g)'(s)) = \theta(g'(s))$ so that $\Psi_1(a^{-1}g) = \Psi_1(g)$.

Applying (5.19) with f replaced by $\tilde{\beta} f$ implies

$$(\tilde{\beta} f)(g(1)) = \sum_{k=0}^{\infty} \langle \widehat{\tilde{\beta} f}(a), \Psi_1^k(g)\rangle,$$

which completes the proof since

$$\langle \widehat{\tilde{\beta} f}(a), \Psi_1^k(g)\rangle = (\widetilde{\Psi_1^k(g)}\tilde{\beta} f)(a) = ((\Psi_1^k(g) \otimes \beta)^{\sim} f)(a) = \langle \hat{f}(a), \Psi_1^k(g) \otimes \beta\rangle. \quad \square$$

Remark 5.14. It should be observed that the power series in (5.19) converges not because of some size restriction imposed on $\hat{f}(a)$, but because f is assumed to be holomorphic in a neighborhood of a (cf. [4, Proposition 5.1]). A size restriction, such as $\hat{f}(a) \in J_t^0$, yields strong bounds on the growth rate of the derivatives of f at a, much stronger than those that hold for a locally defined holomorphic function. The following corollary shows what kind of bounds on the derivatives of f are implied by such a strong condition. The inequalities (5.21) and (5.22) in the following corollary represent a generalization and an improvement over the corresponding inequality (3.25) in [5]. We thank M. Gordina for her proof of the improvement.

Corollary 5.15. *Let $a \in G$ and $f \in \mathcal{H}(aU_\varepsilon^H)$ be such that $\alpha := \hat{f}(a) \in J_t^0$. Suppose that $r, s > 0$ are such that $r + s \le t$. Then for every piecewise C^1 horizontal path $g : [0,1] \to aU_\varepsilon^H$ such that $g(0) = a$ and $\ell_H(g) < \varepsilon$,*

$$|D^k f(g(1))|_{q_k}^2 \le \frac{k!}{r^k}\|\alpha\|_t^2 e^{\ell_H(g)^2/s} \quad \text{for } k = 0, 1, 2, \dots. \tag{5.21}$$

Moreover, if $x \in aU_\varepsilon^H$, then

$$|D^k f(x)|_{q_k}^2 \le \frac{k!}{r^k}\|\alpha\|_t^2 e^{d_H(a,x)^2/s} \quad \text{for } k = 0, 1, 2, \dots. \tag{5.22}$$

Proof. From Proposition 5.13,

$$(\tilde{\beta} f)(g(1)) = \langle \alpha, \Psi_1(g) \otimes \beta\rangle \quad \text{for all } \beta \in H^{\otimes k}. \tag{5.23}$$

This identity along with the estimate (5.13) yields

$$|(\tilde{\beta} f)(g(1))|^2 \le \|\alpha\|_t^2 \|\Psi_1(g) \otimes \beta\|_t^2 \le \|\alpha\|_t^2 \frac{k!|\beta|^2}{r^k} e^{\ell_H(g)^2/s}. \tag{5.24}$$

Since

$$\begin{aligned}|D^k f(g(1))|_{q_k}^2 &= \sup\{|\langle D^k f(g(1)), \beta\rangle|^2 : \beta \in H^{\otimes k} \text{ with } |\beta| = 1\}\\ &= \sup\{|(\tilde{\beta} f)(g(1))|^2 : \beta \in H^{\otimes k} \text{ with } |\beta| = 1\}\\ &\leq \|\alpha\|_t^2 \frac{k!}{r^k} e^{\ell_H(g)^2/s},\end{aligned}$$

the estimate (5.21) is proved. If $x \in aU_\varepsilon^H$, by definition there exists a piecewise C^1 horizontal path, $g : [0,1] \to aU_\varepsilon^H$, such that $g(0) = a$, $g(1) = x$ and $\ell_H(g) < \varepsilon$. Therefore. from (5.21) we learn that

$$|D^k f(x)|_{q_k}^2 \leq \frac{k!}{r^k} \|\alpha\|_t^2 \inf\{e^{\ell_H(g)^2/s} : g(0) = a,\ g(1) = x\} = \frac{k!}{r^k} \|\alpha\|_t^2 e^{d_H(a,x)^2/s}. \quad \square$$

5.3. Dependence of power series on the endpoint

Theorem 5.16. *Let $s \mapsto g_t(s) \in G$ be a piecewise C^2* horizontal *path depending smoothly on a parameter t such that $g_t(0) = e \in G$ for all t. Suppose that $\alpha \in J_T^0$ for some $T > 0$. Then*

$$\frac{d}{dt}\bigg|_{t=0} \langle \alpha, \Psi_1(g_t)\rangle = \left\langle \alpha, \Psi_1(g_0) \otimes \theta\left(\frac{d}{dt}\bigg|_{t=0} g_t(1)\right)\right\rangle \tag{5.25}$$

where $\Psi_1(g_t)$ is defined in Notation 5.9.

Let us first give an informal but illustrative argument for (5.25). Let $c_t(s) := \theta(g_t'(s)) \in H$ and $h_t(s) := \theta(\dot{g}_t(s)) \in \mathfrak{g}$ where " $'$ " and " $\dot{}$ " are shorthand for $\partial/\partial s$ and $\partial/\partial t$ respectively. Then

$$\dot{c}_t(s) = \frac{d}{dt}\theta(g_t'(s)) = \frac{d}{ds}\theta(\dot{g}_t(s)) + d\theta(\dot{g}_t(s), g_t'(s)) = h_t'(s) + [c_t(s), h_t(s)]_{\mathfrak{g}}, \tag{5.26}$$

where we have used the structure equation $d\theta(v,w) + [\theta(v), \theta(w)] = 0$. Recall from Lemma 5.11 that $\Psi_s(g_t)$ and $\Psi_{s,1}(g_t)$ solve

$$\frac{d}{ds}\Psi_s(g_t) = \Psi_s(g_t) \otimes c_t(s) \quad \text{with } \Psi_0(g_t) = 1 \tag{5.27}$$

and

$$\frac{d}{ds}\Psi_{s,1}(g_t) = -c_t(s) \otimes \Psi_{s,1}(g_t) \quad \text{with } \Psi_{1,1}(g_t) = 1. \tag{5.28}$$

Differentiating (5.27) in t implies

$$\frac{d}{ds}\frac{d}{dt}\Psi_s(g_t) = \frac{d}{dt}\Psi_s(g_t) \otimes c_t(s) + \Psi_s(g_t) \otimes \dot{c}_t(s),$$

and using this identity along with (5.28) and (5.26) allows us to conclude

$$\begin{aligned}\frac{d}{ds}\left[\frac{d}{dt}\Psi_s(g_t)\cdot\Psi_{s,1}(g_t)\right] &= \Psi_s(g_t)\otimes\dot{c}_t(s)\otimes\Psi_{s,1}(g_t)\\ &= \Psi_s(g_t)\otimes(h_t'(s)+[c_t(s),h_t(s)]_{\mathfrak{g}})\otimes\Psi_{s,1}(g_t).\end{aligned}$$

Integrating this equation on s then gives

$$\frac{d}{dt}\Psi_1(g_t)=\int_0^1\Psi_s(g_t)\otimes(h_t'(s)+[c_t(s),h_t(s)]_{\mathfrak{g}})\otimes\Psi_{s,1}(g_t)\,ds. \tag{5.29}$$

An integration by parts along with (5.27) and (5.28) shows

$$\begin{aligned}\int_0^1 &\Psi_s(g_t)\otimes h_t'(s)\otimes\Psi_{s,1}(g_t)\,ds\\ &= \Psi_s(g_t)\otimes h_t(s)\otimes\Psi_{s,1}(g_t)|_{s=0}^{s=1}\\ &\quad -\int_0^1\Psi_s(g_t)\otimes c_t(s)\otimes h_t(s)\otimes\Psi_{s,1}(g_t)\,ds\\ &\quad +\int_0^1\Psi_s(g_t)\otimes h_t(s)\otimes c_t(s)\otimes\Psi_{s,1}(g_t)\,ds\\ &= \Psi_1(g_t)\otimes h_t(1)-\int_0^1\Psi_s(g_t)\otimes c_t(s)\wedge h_t(s)\otimes\Psi_{s,1}(g_t)\,ds.\end{aligned}$$

Using this identity in (5.29) gives

$$\frac{d}{dt}\Psi_1(g_t)=\Psi_1(g_t)\otimes h_t(1)+Z(t) \tag{5.30}$$

where

$$Z(t)=\int_0^1\Psi_s(g_t)\otimes([c_t(s),h_t(s)]_{\mathfrak{g}}-c_t(s)\wedge h_t(s))\otimes\Psi_{s,1}(g_t)\,ds.$$

By truncating Ψ_s and $\Psi_{s,1}$, we may write $Z(t)$ as a limit of elements in J and therefore argue that $\langle\alpha, Z_t\rangle = 0$ for $\alpha\in J_T^0$. Hence applying $\alpha\in J_T^0$ to (5.30) yields the desired result:

$$\frac{d}{dt}\langle\alpha,\Psi_1(g_t)\rangle=\langle\alpha,\Psi_1(g_t)\otimes h_t(1)\rangle=\left\langle\alpha,\Psi_1(g_t)\otimes\theta\left(\frac{d}{dt}g_t(1)\right)\right\rangle.$$

The remainder of this section will be devoted to making the above argument rigorous. The proof of Theorem 5.16 will be completed after Lemma 5.18 below.

Lemma 5.17. *Suppose that $h(\cdot) \in C([0,1],\mathfrak{g})$ is piecewise C^1 and that $c(\cdot) \in \mathfrak{g}$ is piecewise continuous. Let $v(s) = h'(s) + [c(s), h(s)]$. For an integer $N > 1$ define*

$$R_N = R_N(c,h) = \int_0^1 \sum_{m=0}^{N-1} \psi_s^m(c) \otimes [c(s), h(s)] \otimes \psi_{s,1}^{N-1-m}(c)\, ds. \tag{5.31}$$

There exists an element $Z_N = Z_N(c,h) \in J$ such that

$$\partial_v \Big(\sum_{n=0}^{N} \psi_1^n \Big)(c) = \sum_{n=0}^{N-1} \psi_1^n(c) \otimes h(1) + Z_N + R_N. \tag{5.32}$$

Proof. Let $\Delta_n := \Delta_n(0,1)$. Since $\psi_1^n(c)$ is a multi-linear form in c, it is easy to see that $\psi_1^n(c)$ is smooth in c and that

$$\begin{aligned} \partial_v \psi_1^n(c) &= \sum_{k=1}^{n} \int_{\Delta_n} c(s_1) \otimes \cdots \otimes c(s_{k-1}) \otimes v(s_k) \otimes c(s_{k+1}) \otimes \cdots \otimes c(s_n)\, d\mathbf{s} \\ &= \sum_{k=1}^{n} \int_0^1 \psi_s^{k-1}(c) \otimes v(s) \otimes \psi_{s,1}^{n-k}(c)\, ds. \end{aligned}$$

Thus the derivative of the n-linear functional $\psi_1^n(c)$ in the direction v may be written

$$(\partial_v \psi_1^n)(c) = \sum_{m+k=n-1} \int_0^1 \psi_s^m \otimes v(s) \otimes \psi_{s,1}^k\, ds \tag{5.33}$$

where, to simplify notation, we are writing $\psi_{r,s}^n$ for $\psi_{r,s}^n(c)$ and we have defined $\psi_{r,s}^k \equiv 1$ if $k = 0$ and $\psi_{r,s}^k \equiv 0$ if $k < 0$. Consider first the terms in (5.33) arising from the summand h' in v. An integration by parts, using (5.27) and (5.28), yields

$$\begin{aligned} \partial_{h'} \psi_1^n &= \sum_{m+k=n-1} \int_0^1 \psi_s^m \otimes h'(s) \otimes \psi_{s,1}^k\, ds \\ &= \sum_{m+k=n-1} \Big(\psi_s^m \otimes h(s) \otimes \psi_{s,1}^k |_{s=0}^1 \\ &\quad - \int_0^1 \{ \psi_s^{m-1} \otimes c(s) \otimes h(s) \otimes \psi_{s,1}^k - \psi_s^m \otimes h(s) \otimes c(s) \otimes \psi_{s,1}^{k-1} \}\, ds \Big). \end{aligned}$$

Since $h(0) = 0$ and $\psi_{1,1}^k = 0$ if $k \neq 0$, the boundary terms contain at most one non-zero term, $\psi_1^{n-1} \otimes h(1)$. Replace m by $m+1$ in the first integral and replace k by $k+1$ in the second integral. We may then write

$$\partial_{h'} \psi_1^n = \psi_1^{n-1} \otimes h(1) - \sum_{m+k=n-2} \int_0^1 \psi_s^m \otimes (c(s) \wedge h(s)) \otimes \psi_{s,1}^k\, ds.$$

Adding now the contribution to v from the term $[c(s), h(s)]$ we find, with the help of (5.33),

$$\partial_v \psi_1^n = \psi_1^{n-1} \otimes h(1) + \sum_{m+k=n-1} \int_0^1 \psi_s^m \otimes [c(s), h(s)] \otimes \psi_{s,1}^k \, ds - \sum_{m+k=n-2} \int_0^1 \psi_s^m \otimes (c(s) \wedge h(s)) \otimes \psi_{s,1}^k \, ds.$$

Summing this equation on n from 0 to N, keeping in mind that $\psi_1^0 = 1$ and $\psi_{r,s}^{-1} = 0$, we find

$$\begin{aligned}\partial_v \sum_{n=0}^{N} \psi_1^n = {} & \sum_{n=0}^{N-1} \psi_1^n \otimes h(1) \\ & + \sum_{n=1}^{N-1} \sum_{m+k=n-1} \int_0^1 \psi_s^m \otimes \{[c(s), h(s)] - c(s) \wedge h(s)\} \otimes \psi_{s,1}^k \, ds \\ & + \int_0^1 \sum_{m+k=N-1} \psi_s^m \otimes [c(s), h(s)] \otimes \psi_{s,1}^k \, ds.\end{aligned}$$

Since the middle line is in J the lemma is proved. □

Lemma 5.18. *Suppose that $\alpha \in J_T^0$ for some $T > 0$, $h(\cdot) \in C([0,1], \mathfrak{g})$ is piecewise C^1, $g(\cdot)$ is a piecewise C^2 horizontal path over $[0,1]$, and let $c(s) := \theta(g'(s))$. Let $R_N = R_N(c, h)$ be as in Lemma 5.17 and $\|h\|_\infty = \sup_{s\in[0,1]} |h(s)|$ where $|\cdot|$ is any given fixed norm on $\mathfrak{g}$, such that $|A|^2 = (A, A)_H$ for all $A \in H$. Then there exist constants, $\{C_N(T)\}_{N=1}^\infty$ such that $\lim_{N\to\infty} C_N(T)\lambda^N = 0$ for all $\lambda > 0$ and*

$$|\langle \alpha, R_N \rangle| \le \|\alpha\|_T \|h\|_\infty C_N(T) \ell_H(g)^n. \tag{5.34}$$

Proof. Let $u_m(s) = \psi_s^m(c)$ and $v_m(s) = \psi_{s,1}^{N-m-1}(c)$. Because g is horizontal, $u_m(s) \in H^{\otimes m}$ and $v_m(s) \in H^{\otimes(N-m-1)}$ for each $s \in [0,1]$. If $w(s) := [c(s), h(s)]$, then

$$\langle \alpha, R_N \rangle = \sum_{m=0}^{N-1} \int_0^1 \langle \alpha, u_m(s) \otimes w(s) \otimes v_m(s) \rangle \, ds \tag{5.35}$$

and we may find $K < \infty$ such that

$$\int_0^1 |w(s)|_{\mathfrak{g}} \, ds \le K \|h\|_\infty \int_0^1 |c(s)|_H \, ds = K \|h\|_\infty \ell_H(g) < \infty. \tag{5.36}$$

The integrability of w guarantees that the integrals in (5.35) and the integrals appearing in the argument below all exist. Although $u_m(s)$ and $v_m(s)$ lie in $T(H) \subset T(\mathfrak{g})$ the factor $w(s)$ may not lie in H. Since α is only continuous on tensor spaces over H we must replace the factor $w(s)$ before making estimates.

Let $P : T(\mathfrak{g}) \to T(H)$ be the projection operator constructed in Lemma 2.11 and let $L := P|_{\mathfrak{g}} : \mathfrak{g} \to \bigoplus_{k=1}^r H^{\otimes k}$. We may write, for all $A \in \mathfrak{g}$, $L(A) = \sum_{k=1}^r L_k(A)$ with each L_k being a linear map from $\mathfrak{g}$ into $H^{\otimes k}$. Since $\mathfrak{g}$ is finite-dimensional, there exists $K_1 < \infty$ such that $|L_k(A)|_{H^{\otimes k}} \le K_1 |A|_{\mathfrak{g}}$ for $k = 1, \dots, r$ and $A \in \mathfrak{g}$. With this notation, (5.35) may be written as

$$\begin{aligned}\langle \alpha, R_N \rangle &= \sum_{m=0}^{N-1} \int_0^1 \langle \alpha, u_m(s) \otimes L(w(s)) \otimes v_m(s) \rangle \, ds \\ &= \sum_{k=1}^{r} \sum_{m=0}^{N-1} \int_0^1 \langle \alpha, u_m(s) \otimes L_k(w(s)) \otimes v_m(s) \rangle \, ds.\end{aligned}$$

Using the estimate (5.11) and writing $|\alpha_j|$ for $|\alpha_j|_{q_j}$, we find

$$\begin{aligned}&\sum_{m=0}^{N-1} |\langle \alpha, u_m(s) \otimes L_k(w(s)) \otimes v_m(s) \rangle| \\ &\quad \le \sum_{m=0}^{N-1} |\alpha_{N-1+k}| \, |u_m(s)|_{H^{\otimes m}} |L_k(w(s))|_{H^{\otimes k}} |v_m(s)|_{H^{\otimes (N-m-1)}} \\ &\quad \le \sum_{m=0}^{N-1} |\alpha_{N-1+k}| \frac{\ell_H(g|_{[0,s]})^m}{m!} K_1 |w(s)|_{\mathfrak{g}} \frac{\ell_H(g|_{[s,1]})^{N-m-1}}{(N-m-1)!} \\ &\quad \le |\alpha_{N-1+k}| K_1 |w(s)|_{\mathfrak{g}} \frac{\ell_H(g)^{N-1}}{(N-1)!}\end{aligned}$$

where the binomial formula was used to obtain the last inequality. After integrating on s, summing on k, and using (5.36) in the previous estimate, we find

$$|\langle \alpha, R_N \rangle| \le K_1 K \frac{\ell_H(g)^N}{(N-1)!} \|h\|_\infty \sum_{k=1}^{r} |\alpha_{N-1+k}|. \tag{5.37}$$

By the definition (2.10) we see that $|\alpha_j|_{q_j} \le (j!/T^j)^{1/2} \|\alpha\|_T$, which combined with (5.37) gives

$$|\langle \alpha, R_N \rangle| \le K_1 K \frac{\ell_H(g)^N}{(N-1)!} \|h\|_\infty \|\alpha\|_T \sum_{k=1}^{r} \sqrt{\frac{(N-1+k)!}{T^{N-1+k}}},$$

which proves the lemma with

$$C_N(T) := K_1 K \frac{1}{(N-1)!} \sum_{k=1}^{r} \sqrt{\frac{(N-1+k)!}{T^{N-1+k}}}. \qquad \square$$

We are now in a position to complete the proof of Theorem 5.16.

Proof of Theorem 5.16. As at the beginning of this section, let $c_t(s) := \theta(g_t'(s)) \in H$ and $h_t(s) := \theta(\dot{g}_t(s)) \in \mathfrak{g}$ and recall from (5.26) that

$$\dot{c}_t(s) = h_t'(s) + [c_t(s), h_t(s)]_{\mathfrak{g}}.$$

Let $f(t) := \langle \alpha, \Psi_1(g_t) \rangle$ and

$$f_N(t) = \left\langle \alpha, \sum_{n=0}^{N} \Psi_1^n(g_t) \right\rangle = \left\langle \alpha, \sum_{n=0}^{N} \psi_1^n(c_t) \right\rangle$$

so that $f(t) = \lim_{N\to\infty} f_N(t)$. By Lemma 5.17 with $v(s) := h_t'(s) + [c_t(s), h_t(s)]_{\mathfrak{g}}$, $f_N(t)$ is differentiable and

$$\frac{df_N(t)}{dt} = \left\langle \alpha, \sum_{n=0}^{N} \psi_1^n(c_t) \otimes h_t(1) \right\rangle + \langle \alpha, R_N(c_t, \dot{c}_t) \rangle. \tag{5.38}$$

Because $\ell_H(g_t)$ and $\|\dot{c}_t(\cdot)\|_\infty$ are bounded for t near zero, Lemma 5.18 may be used to conclude that the remainder term $\langle \alpha, R_N(c_t, \dot{c}_t) \rangle$ goes to zero as $N \to \infty$ uniformly in a neighborhood of $t = 0$. Moreover, it is easily verified that

$$\lim_{N\to\infty} \left\langle \alpha, \sum_{n=0}^{N} \psi_1^n(c_t) \otimes h_t(1) \right\rangle = \langle \alpha, \psi_1(c_t) \otimes h_t(1) \rangle$$

with the above limit being uniform in t near zero. Hence we may conclude that $f(t)$ is differentiable near zero and that $\dot{f}(t) = \langle \alpha, \Psi_1(c_t) \otimes h_t(1) \rangle$. □

6. Reconstruction of f from its Taylor coefficients

The purpose of this section is to complete the proof of the following theorem, which is the main theorem of this paper.

Theorem 6.1. *Let G be a connected, simply connected complex Lie group. Suppose that q is a non-negative Hermitian form on the dual space $\mathfrak{g}^*$ and assume that Hörmander's condition holds (cf. Definition* 2.6*). Let ρ_t denote the heat kernel associated to* (3.1)*. Then the Taylor map $f \mapsto \hat{f}(e)$ is a unitary map from $\mathcal{H}L^2(G, \rho_t(x)\,dx)$ onto J_t^0.*

Proof. Since we have already proved the isometry property of the Taylor map in Theorem 4.2, it suffices to prove the map is surjective. In light of Proposition 4.3, it suffices to show that for each $\alpha \in J_t^0$ there exists $f \in \mathcal{H}(G)$ such that $\hat{f}(e) = \alpha$. But this is the content of Theorem 6.13 below. □

The remainder of this section is devoted to the proof of Theorem 6.13.

6.1. Holomorphic horizontal coordinates and paths

In this section, let G be a complex Lie group and $\mathfrak{g} := \mathrm{Lie}(G)$ be its complex Lie algebra.

Notation 6.2. For $g, h \in G$, let $[g, h] := g^{-1}h^{-1}gh$.

Lemma 6.3. *For* $\Gamma := (A_1, \ldots, A_n) \in \mathfrak{g}^n$ *and* $\varepsilon \in \mathbb{C} \setminus \{0\}$ *define*

$$v_\Gamma(\varepsilon) = \frac{d}{dt}\bigg|_{t=0} \big[e^{\frac{t}{\varepsilon^{n-1}} A_1}, [e^{\varepsilon A_2}, [e^{\varepsilon A_3}, \ldots [e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}]]] \ldots\big]. \tag{6.1}$$

Then

$$\lim_{\varepsilon \to 0} v_\Gamma(\varepsilon) = [\Gamma] \tag{6.2}$$

where $[\Gamma]$ *is defined as in* (2.19).

Proof. If $X \in \mathfrak{g}$ and $b \in G$ then

$$\frac{d}{dt}\bigg|_{t=0} [e^{tX}, b] = \frac{d}{dt}\bigg|_{t=0} e^{-tX} e^{t(\mathrm{Ad}_{b^{-1}})X} = (\mathrm{Ad}_{b^{-1}} - I)X. \tag{6.3}$$

Let

$$b = [e^{\varepsilon A_2}, [e^{\varepsilon A_3}, \ldots [e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}]] \ldots]$$

and let $\Gamma' = (A_2, \ldots, A_n)$. We assert that, for ε near 0, $b = e^{B(\varepsilon)}$ where

$$B(\varepsilon) = \varepsilon^{n-1}[\Gamma'] + \varepsilon^n C(\varepsilon) \tag{6.4}$$

and $C(\varepsilon)$ is an analytic $\mathfrak{g}$-valued function of ε for ε near 0. This may be proven by induction on n with the help of the Baker–Campbell–Hausdorff formula [38, Theorem 2.15.4] as follows. An application of [38, Theorem 2.15.4] shows

$$[e^X, e^Y] = e^{-X}e^{-Y}e^X e^Y = e^{[X,Y]+R_2(X,Y)} \tag{6.5}$$

where $R_2(X, Y)$ is an analytic function of X and Y defined in a neighborhood of 0 in $\mathfrak{g} \times \mathfrak{g}$ and which satisfies

$$|R_2(X, Y)| \le C_2(|X| + |Y|)|X|\,|Y|.$$

We further assert that

$$[e^{B_2}, [e^{B_3}, \ldots [e^{B_{n-1}}, e^{B_n}]] \ldots] = e^{[B_2,[B_3,\ldots[B_{n-1},B_n]]\ldots]+R_{n-1}(B_2,\ldots,B_{n-1},B_n)} \tag{6.6}$$

where $R_{n-1}(B_2, \ldots, B_{n-1}, B_n)$ is an analytic function of $(B_2, \ldots, B_{n-1}, B_n)$ in a neighborhood of $0 \in \mathfrak{g}^{n-1}$ which satisfies

$$|R_{n-1}(B_2, \ldots, B_{n-1}, B_n)| \le C_{n-1}\Big(\sum_{i=2}^{n} |B_i|\Big)|B_2| \ldots |B_n|. \tag{6.7}$$

Indeed, assuming (6.6) and (6.7) hold, it follows by using (6.5) that

$$[e^{B_1}[e^{B_2}, \ldots [e^{B_{n-1}}, e^{B_n}]] \ldots] = e^{[B_1,[B_2,\ldots[B_{n-1},B_n]]\ldots]+R_n(B_1,B_2,\ldots,B_{n-1},B_n)}$$

where

$$R_n(B_1, B_2, \dots, B_{n-1}, B_n) = [B_1, R_{n-1}(B_2, \dots, B_{n-1}, B_n)] + R_2(B_1, [B_2, [B_3, \dots [B_{n-1}, B_n]] \dots] + R_{n-1}(B_2, \dots, B_{n-1}, B_n)).$$

The function R_n is analytic for $(B_1, \dots, B_n)$ in a neighborhood of $0 \in \mathfrak{g}^n$ and is easily seen to satisfy

$$|R_n(B_1, \dots, B_n)| \le C_n \Big(\sum_{i=1}^{n} |B_i|\Big) |B_1| \dots |B_n|.$$

Formula (6.6), with $B_i = \varepsilon A_i$, implies $B(\varepsilon)$ is an analytic function of ε in a neighborhood of 0 such that

$$B(\varepsilon) = \varepsilon^{n-1}[\Gamma'] + O(\varepsilon^n).$$

Taking $b = e^{B(\varepsilon)}$ and $X = \varepsilon^{-(n-1)} A_1$ in (6.3) implies

$$\begin{aligned} v_\Gamma(\varepsilon) &= (e^{-\operatorname{ad}_{B(\varepsilon)}} - I)\varepsilon^{-(n-1)} A_1 = \sum_{k=1}^{\infty} \frac{1}{k!} (-\operatorname{ad}_{B(\varepsilon)})^k \varepsilon^{-(n-1)} A_1 \\ &= -\operatorname{ad}_{[\Gamma']} A_1 + O(\varepsilon) = [\Gamma] + O(\varepsilon). \end{aligned}$$ □

Formula (6.4) can be viewed as a version of [28, Lemma 2.2.1]. Such a commutator identity frequently plays a role in subelliptic estimates and goes back at least to Hörmander [20].

Notation 6.4. For $\Gamma := (A_1, \dots, A_n) \in \mathfrak{g}^n$ and $\varepsilon > 0$, let $\phi_{\Gamma,\varepsilon} : \mathbb{C} \to G$ be defined by

$$\phi_{\Gamma,\varepsilon}(z) := \big[e^{\frac{z}{\varepsilon^{n-1}} A_1}, [e^{\varepsilon A_2}, [e^{\varepsilon A_3}, \dots [e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}]]] \dots\big]. \tag{6.8}$$

The function $\phi(z) := \phi_{\Gamma,\varepsilon}(z) \in G$ is a holomorphic function of z whose derivative at $z = 0$ in the direction w is given by

$$\begin{aligned} \phi_*(w_0) = w\phi_*(1_0) &= w \frac{d}{dt}\Big|_{t=0} \phi(t) \\ &= w \frac{d}{dt}\Big|_{t=0} \big[e^{\frac{t}{\varepsilon^{n-1}} A_1}, [e^{\varepsilon A_2}, [e^{\varepsilon A_3}, \dots [e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}]]] \dots\big] \\ &= w \cdot v_\Gamma(\varepsilon) \to [\Gamma] w \quad \text{as } \varepsilon \to 0, \end{aligned}$$

where w_z refers to the complex number w as an element of $T_z(\mathbb{C})$. For each $\Gamma := (A_1, \dots, A_n) \in \mathfrak{g}^n$, $\varepsilon \neq 0$, and $z \in \mathbb{C}$ we are going to define a piecewise C^∞ horizontal path, $\sigma_{\Gamma,\varepsilon,z}$, which depends holomorphically on z as follows. First observe that

$$\big[e^{z\varepsilon^{-(n-1)} A_1}, [e^{\varepsilon A_2}, [e^{\varepsilon A_3}, \dots [e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}]]] \dots\big]$$

is the product of $N_n := 3 \cdot 2^{n-1} - 2$ exponentials of the form $e^{B_1} \dots e^{B_{N_n}}$ with each B_i being an element from the set

$$S(\Gamma, \varepsilon, z) := \{\pm\varepsilon A_1, \dots, \pm\varepsilon A_{n-1}\} \cup \left\{\pm \frac{z}{\varepsilon^{n-1}} A_n\right\}. \tag{6.9}$$

Hence if $k = 1, \dots, N_n$ and $s \in [(k-1)/N_n, k/N_n]$, let

$$\sigma_{\Gamma,\varepsilon,z}(s) := e^{B_1} \dots e^{B_{k-1}} e^{(N_n s - k + 1)B_k}. \tag{6.10}$$

The following proposition summarizes what we have done.

Proposition 6.5. *Assume that* $\Gamma \in H^n$. *The path* $\sigma_{\Gamma,\varepsilon,z}$ *in* (6.10) *is a piecewise* C^∞ *horizontal path from* e *to* $\phi_{\Gamma,\varepsilon}(z)$ *which depends holomorphically on* $z \in \mathbb{C}$. *Moreover, for* $s \in ((k-1)/N_n, k/N_n)$, $\theta(\sigma'_{\Gamma,\varepsilon,z}(s)) = N_n B_k \in S(\Gamma, \varepsilon, z)$ (*see* (6.9)) *and hence* $\theta(\sigma'_{\Gamma,\varepsilon,z}(s))$ *is either constant in* z *or depends on* z *linearly in each of the intervals* $\{((k-1)/N_n, k/N_n) : k = 1, \dots, N_n\}$.

Let $\mathcal{X} := \{X_j\}_{j=1}^m$ be an orthonormal basis for the Hörmander subspace H. For $l = m+1, \dots, M := \dim \mathfrak{g}$, let $n_l \in \mathbb{N}$ and $\Gamma_l \in \mathcal{X}^{n_l}$ be chosen so that

$$\{X_j\}_{j=1}^m \cup \{[\Gamma_l] : l = m+1, \dots, M\}$$

is a basis for $\mathfrak{g}$. We may apply Lemma 6.3 to find (and fix once and for all) an $\varepsilon \in \mathbb{C} \setminus \{0\}$ sufficiently close to zero such that

$$\{X_j\}_{j=1}^m \cup \{[X_l := v_{\Gamma_l}(\varepsilon)] : l = m+1, \dots, M\} \tag{6.11}$$

is still a basis for $\mathfrak{g}$. For $z \in \mathbb{C}$, let

$$\phi_j(z) := \begin{cases} e^{zX_j} & \text{if } 1 \le j \le m, \\ \phi_{\Gamma_j,\varepsilon}(z) & \text{if } m+1 \le j \le M, \end{cases} \tag{6.12}$$

where $\phi_{\Gamma_j,\varepsilon}$ has been defined in (6.8).

Notation 6.6 (Horizontal charts and paths). For $z = (z_1, \dots, z_M) \in \mathbb{C}^M$, let

$$\varphi(z) := \phi_1(z_1)\phi_2(z_2)\dots\phi_M(z_M) \in G$$

and let $\sigma_z(s) \in G$ be the horizontal path defined, for $s \in [(j-1)/M, j/M]$, by

$$\sigma_z(s) = \phi_1(z_1)\phi_2(z_2)\dots\phi_{j-1}(z_{j-1})\sigma_{\Gamma_j,\varepsilon,z_j}(Ms - j + 1).$$

Theorem 6.7. *The function* $\varphi : \mathbb{C}^M \to G$ *is a local bi-holomorphism from an open neighborhood,* Ω, *of* $0 \in \mathbb{C}^M$ *to an open neighborhood,* U, *of* $e \in G$. *The path* $\sigma_z(s)$ *is a piecewise* C^∞ *horizontal path in* G *from* e *to* $\varphi(z)$ *which depends holomorphically on* $z \in \mathbb{C}^M$. *More precisely, there is a partition of* $[0, 1]$,

$$D = \{0 = s_0 < s_1 < \dots < s_N = 1\},$$

such that for $s \in (s_{l-1}, s_l)$, $l \in \{1, \dots, N\}$, *either* $\theta(\sigma'_z(s)) = X'_l$ *or* $\theta(\sigma'_z(s)) = z_{j_l} X'_l$ *for some* $j_l \in \{1, \dots, M\}$, *where each* $X'_l \in H$ *is a real multiple of one of the elements from the orthonormal basis* $\mathcal{X} \subset H$.

Proof. Since $\varphi_*([e_j]_0) = X_j$ for $j = 1, \dots, M$ where X_j are defined in (6.11), the first assertion is a consequence of the inverse function theorem. The remaining assertions have already been proved prior to the statement of the theorem. □

6.2. *Local existence of f_α*

Notation 6.8. We will want to consider α and its "translates" in various of the spaces J_t^0. Noting that $J_\sigma^0 \subset J_s^0$ if $0 < s < \sigma$, we define

$$J_+^0 := \bigcup_{t>0} J_t^0. \tag{6.13}$$

A consequence of Corollary 5.7 is that if $\alpha \in J_+^0$ and $\psi \in T_+(H)$ then $\alpha \circ L_\psi \in J_+^0$.

Theorem 6.9 (Local existence). *Let $\Omega \subset \mathbb{C}^M$ and $U \subset G$ be as in Theorem 6.7. For each $\alpha \in J_+^0$ and $x \in G$ there exists $f = f_\alpha \in \mathcal{H}(xU)$ such that $\hat{f}(x) = \alpha$. This function has the additional property that*

$$\hat{f}(x\varphi(z)) = \alpha \circ L_{\Psi_1(\sigma_z)} \quad \textit{for all } z \in \Omega. \tag{6.14}$$

In particular, $\hat{f}(y) \in J_+^0$ for all $y \in xU$.

Proof. The proof will consist of showing that the function $f : xU \to \mathbb{C}$ defined by

$$f(x\varphi(z)) = f(x\sigma_z(1)) := \langle \alpha, \Psi_1(\sigma_z)\rangle =: u(z) \quad \text{for all } z \in \Omega$$

is the desired function. By Proposition 5.10,

$$|\langle \alpha, \Psi_1^n(\sigma_z)\rangle| \le |\alpha_n|_{q_n} |\Psi_1^n(\sigma_z)|_{H^{\otimes n}} \le |\alpha_n|_{q_n} \frac{1}{n!} K^n$$

where $K := \sup_{z\in\Omega} \ell_H(\sigma_z)$, and therefore

$$\sum_{n=0}^{\infty} |\alpha_n|_{q_n} K^n \frac{1}{n!} \le \left(\sum_{n=0}^{\infty} |\alpha_n|_{q_n}^2 \frac{t^n}{n!}\right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{K^{2n}}{n!t^n}\right)^{1/2} = \|\alpha\|_t e^{K^2/2t}.$$

Therefore the sum $\sum_{n=0}^{\infty} \langle \alpha_n, \Psi_1^n(\sigma_z)\rangle$ defining $u(z)$ is uniformly and absolutely convergent. Moreover, it is easy to verify that each summand $u_n(z) := \langle \alpha_n, \Psi_1^n(\sigma_z)\rangle$ is a holomorphic polynomial in z of degree n and thus $u(z)$ is holomorphic as well.

Using Theorem 5.16, we learn that

$$\begin{aligned} u_* w_z &= \frac{d}{dt}\bigg|_{t=0} u(z+tw) = \frac{d}{dt}\bigg|_{t=0} \langle \alpha, \Psi_1(\sigma_{z+tw})\rangle \\ &= \left\langle \alpha, \Psi_1(\sigma_z) \otimes \theta\left(\frac{d}{dt}\bigg|_{t=0} \sigma_{z+tw}(1)\right)\right\rangle = \left\langle \alpha, \Psi_1(\sigma_z) \otimes \theta\left(\frac{d}{dt}\bigg|_{t=0} \varphi(z+tw)\right)\right\rangle \\ &= \langle \alpha, \Psi_1(\sigma_z) \otimes \theta(\varphi_* w_z)\rangle, \end{aligned}$$

while on the other hand

$$u_* w_z = \frac{d}{dt}\bigg|_{t=0} f(x\varphi(z+tw)) = \langle Df(x\varphi(z)), \theta(\varphi_* w_z)\rangle.$$

Comparing these two equations shows that

$$\langle Df(x\varphi(z)), \theta(\varphi_* w_z)\rangle = \langle \alpha, \Psi_1(\sigma_z) \otimes \theta(\varphi_* w_z)\rangle$$

for all $w \in \mathbb{C}^M$ and $z \in \Omega$, which implies

$$\langle Df(x\varphi(z)), A\rangle = \langle \alpha, \Psi_1(\sigma_z) \otimes A\rangle \quad \text{for all } z \in \Omega \text{ and } A \in \mathfrak{g}. \tag{6.15}$$

By Proposition 5.6, $\alpha_A := \alpha \circ R_A$ is in J_+^0. With this notation, (6.15) reads

$$(\tilde{A} f)(x\varphi(z)) = \langle \alpha_A, \Psi_1(\sigma_z)\rangle \quad \text{for all } A \in \mathfrak{g}.$$

Applying the above results with f replaced by $\tilde{A} f$ and α replaced by α_A, we learn that

$$(\tilde{B}\tilde{A} f)(x\varphi(z)) = \langle (\alpha_A)_B, \Psi_1(\sigma_z)\rangle = \langle \alpha_A, \Psi_1(\sigma_z) \otimes B\rangle = \langle \alpha, \Psi_1(\sigma_z) \otimes B \otimes A\rangle.$$

Moreover, a simple induction argument now shows that

$$(\tilde{A}_1 \ldots \tilde{A}_n f)(x\varphi(z)) = \langle \alpha, \Psi_1(\sigma_z) \otimes A_1 \otimes \cdots \otimes A_n\rangle \quad \text{for all } A_i \in \mathfrak{g},$$

which is equivalent to (6.14). In light of Corollary 5.7, the proof is complete. □

6.3. Global construction of f_α

In what follows, we will fix an inner product on $\mathfrak{g}$. Such a choice induces a unique left invariant Riemannian metric on G. Fix $\delta > 0$ such that the Riemannian ball $U = U_\delta^{\mathrm{Riem}}$ of radius δ and centered at e is geodesically convex, and such that there exists an open neighborhood, Ω, of 0 in $\mathbb{C}^M$ for which the results of Theorems 6.7 are valid. In particular, for every $\alpha \in J_+^0$ and $x \in G$ there exists $f \in \mathcal{H}(xU)$ such that $\hat{f}(x) = \alpha$ by Theorem 6.9. The following two simple observations will be used repeatedly below: (1) A point $x \in G$ is in yU iff $y \in xU$. (2) If S is a non-empty finite subset of G such that

$$\operatorname{diam}(S) := \sup\{d_{\mathrm{Riem}}(x, y) : x, y \in S\} < \delta,$$

then $\bigcap_{a\in S}(aU)$ is a non-empty, (pathwise) connected open subset of G containing S. The latter holds because $\bigcap_{a\in S}(aU)$ is a non-empty, geodesically convex, open subset of G containing S.

Theorem 6.10 (Analytic continuation). *Suppose that $g \in C([0,1], G)$ is a path such that $g(0) = e$. Then to each $\alpha \in J_+^0$, there exists a unique family of functions,*

$$\{f_t \in \mathcal{H}(g(t)U) : 0 \le t \le 1\}, \tag{6.16}$$

satisfying:

(i) $\hat{f}_0(e) = \alpha$,

(ii) *if* $0 \le a \le b \le 1$ *with* $\operatorname{diam}(g([a,b])) < \delta$, *then* $f_s = f_t$ *on* $g(s)U \cap g(t)U$ *for all* $s, t \in [a, b]$.

Moreover, $\hat{f}_t(x) \in J_+^0$ *for all* $x \in g(t)U$ *and all* $t \in [0, 1]$.

Proof. Uniqueness. Suppose that $\{k_t \in \mathcal{H}(g(t)U) : 0 \le t \le 1\}$ is another collection of holomorphic functions with the same properties as $\{f_t : t \in [0,1]\}$ and let

$$T_0 := \sup\{T \in [0,1] : f_t = k_t \text{ for } 0 \le t \le T\}.$$

Since holomorphic functions are determined by their Taylor coefficients and $\hat{f}_0 = \alpha = \hat{k}_0$, it follows that $f_0 = k_0$ on U. Moreover, if $T > 0$ is chosen so that $g([0, T]) \subset U$, then for $0 \le t \le T$, we have $f_t = f_0 = k_0 = k_t$ on $g(t)U \cap U$, which is a non-empty open subset of $g(t)U$. Since $g(t)U$ is a connected open set it follows that $f_t = k_t$ on all of $g(t)U$. Hence we have shown $T_0 > 0$.

Choose $0 < a < T_0 \le b \le 1$ such that $\operatorname{diam}(g([a, b])) < \delta$ and $b > T_0$ if $T_0 < 1$. Then for $t \in [a, b]$, $f_t = f_a$ and $k_t = k_a$ on $g(a)U \cap g(t)U$ and $f_a = k_a$ on $g(a)U$. Therefore $f_t = k_t$ on $g(a)U \cap g(t)U$, which implies $f_t = k_t$ on the connected open set $g(t)U$. If $T_0 < 1$, we would conclude that $T_0 \ge b > T_0$, which is absurd. Hence $T_0 = b = 1$ and we conclude $f_1 = k_1$ and by the definition of T_0 that $f_t = k_t$ for $0 \le t < 1$.

Existence. From Theorem 6.9, there exists $f_0 \in \mathcal{H}(U)$ such that $\hat{f}_0(e) = \alpha$ and $\hat{f}_0(x) \in J_+^0$ for all $x \in J_+^0$. If $T > 0$ is chosen so that $\operatorname{diam}(g([0, T])) < \delta$, another application of Theorem 6.9 shows there exists $f_t \in \mathcal{H}(g(t)U)$ such that $\hat{f}_t(g(t)) = \hat{f}_0(g(t))$ for all $t \in [0, T]$. Since f_t and f_0 have the same derivatives at $g(t)$, it follows that $f_t = f_0$ in a neighborhood of $g(t)$ and therefore on the connected open set $g(t)U \cap U$. Hence if $s, t \in [0, T]$, then $f_s = f_0$ on $g(s)U \cap U$, and $f_t = f_0$ on $g(t)U \cap U$, which implies $f_s = f_t$ on the non-empty open set $g(s)U \cap g(t)U \cap U$. So again $f_s = f_t$ on the connected open set $g(s)U \cap g(t)U$.

Let T_0 be the supremum of all $T \in [0, 1]$ such that there exists a (unique) family of functions, $f_t \in \mathcal{H}(g(t)U)$ for $0 \le t \le T$, with the properties listed in the statement of the theorem (with “1” replaced by T everywhere) including the assertion that $\hat{f}_t(x) \in J_+^0$ for all $x \in g(t)U$ and all $t \in [0, T]$. The previous paragraph shows that $T_0 > 0$.

Suppose, for the sake of contradiction, that $T_0 < 1$. Choose $0 \le T_- < T_0 < T_+ \le 1$ such that $\operatorname{diam}(g([T_-, T_+])) < \delta$. Applying Theorem 6.9 as above, we may find $f_t \in \mathcal{H}(g(t)U)$ such that $\hat{f}_t(g(t)) = \hat{f}_{T_-}(g(t))$ for all $t \in [T_-, T_+]$. Let us now suppose that $0 \le a \le b \le T_+$ with $\operatorname{diam}(g([a, b])) < \delta$ and that $a \le s \le t \le b$. If $t \le T_-$ then $f_s = f_t$ on $g(s)U \cap g(t)U$ by definition of T_0. If $s, t \in [T_-, T_+]$, then arguing as above, we see that $f_s = f_{T_-}$ on $g(s)U \cap g(T_-)U$, and $f_t = f_{T_-}$ on $g(t)U \cap g(T_-)U$, and therefore $f_s = f_t$ on $g(s)U \cap g(t)U \cap g(T_-)U$, which implies $f_s = f_t$ on $g(s)U \cap g(t)U$. Finally, if $a \le s \le T_- \le t \le b$, then $f_s = f_{T_-}$ on $g(s)U \cap g(T_-)U$, $f_t = f_{T_-}$ on $g(t)U \cap g(T_-)U$ and so again $f_s = f_t$ on $g(s)U \cap g(t)U$. But this shows $T_0 \ge T_+ > T_0$, which is the desired contradiction and hence $T_0 = 1$.

So far we have constructed a family of functions, $\{f_t : 0 \le t < 1\}$, with the desired properties. It only remains to extend this family to all $t \in [0, 1]$ by defining $f_1 \in \mathcal{H}(g(1)U)$ so that $\hat{f}_1(g(1)) = \hat{f}_T(g(1))$, where $T \in (0, 1)$ is chosen so that $\operatorname{diam}(g([T, 1])) < \delta$. Arguing as above, the reader may verify that the family $\{f_t : 0 \le t \le 1\}$ so constructed satisfies the conclusions of the theorem. □

Notation 6.11. When $g \in C([0, 1], G)$ is a path such that $g(0) = e$ and $\alpha \in J_+^0$, write f_t^g for $f_t \in \mathcal{H}(g(t)U)$ as described in Theorem 6.10.

Theorem 6.12 (Monodromy theorem). *Let $\alpha \in J_+^0$ and $g, h \in C([0, 1], G)$ be such that $g(0) = h(0) = e$, $g(1) = x = h(1)$, and $d_{\mathrm{Riem}}(g(t), h(t)) < \delta/2$ for all t. Then $f_1^g = f_1^h$ on xU.*

Proof. Let $v_t := f_t^g$, $w_t := f_t^h$, and

$$T_0 := \sup\{T \in [0,1] : v_t = w_t \text{ on } g(t)U \cap h(t)U \text{ for all } 0 \le t \le T\}. \tag{6.17}$$

Since $\hat{v}_0(e) = \hat{w}_0(e)$ we know that $v_0 = w_0$. Suppose that $T > 0$ is such that $\operatorname{diam}(g([0,T]) \cup h([0,T])) < \delta$ and that $t \in [0,T]$. Then $v_t = v_0$ on $g(t)U \cap U$, $w_t = w_0$ on $h(t)U \cap U$ and hence $v_t = w_t = v_0$ on $g(t)U \cap h(t)U \cap U$ and thus $v_t = w_t$ on $g(t)U \cap h(t)U$. This shows that $T_0 > 0$.

Choose $0 \le a < T_0 \le b \le 1$ such that $\operatorname{diam}(g([a,b])) < \delta/2$, $\operatorname{diam}(h([a,b])) < \delta/2$, and $b > T_0$ if $T_0 < 1$. Because $v_t = v_a$ on $g(t)U \cap g(a)U$ and $w_t = w_a$ on $h(t)U \cap h(a)U$, and $v_a = w_a$ on $g(a)U \cap h(a)U$, it follows that $v_t = w_t$ on $\mathcal{O}_t := g(t)U \cap g(a)U \cap h(t)U \cap h(a)U$. Since, for $t \in [a,b]$,

$$d_{\mathrm{Riem}}(h(t), g(a)) \le d_{\mathrm{Riem}}(h(t), h(a)) + d_{\mathrm{Riem}}(h(a), g(a)) < \delta/2 + \delta/2 = \delta,$$

$g(a) \in \mathcal{O}_t$ so that $\mathcal{O}_t$ is a non-empty open set contained in the connected open set $g(t)U \cap h(t)U$. So again we conclude $v_t = w_t$ on $g(t)U \cap h(t)U$ for all $t \in [a,b]$. Hence if $T_0 < 1$, we have shown $T_0 \ge b > T_0$, which is a contradiction. Thus $T_0 = b = 1$ and we have shown $v_1 = w_1$ on $g(1)U \cap h(1)U = xU$. □

Theorem 6.13. *Suppose that G is a simply connected complex Lie group. Then for each* $\alpha \in J_+^0$, *there exists a unique function* $f_\alpha \in \mathcal{H}(G)$ *such that* $\hat{f}_\alpha(e) = \alpha$.

Proof. For any $x \in G$, we may choose a path $g \in C([0,1], G)$ joining e to x, i.e. such that $g(0) = e$ and $g(1) = x$. We then define $f_\alpha(x) := f_1^g(x)$. If $h \in C([0,1], G)$ is another such path joining e to x, there is a homotopy, $g_t \in C([0,1], G)$, of paths joining e to x, which interpolates between g and h. By the Monodromy Theorem 6.12, one easily sees that $f_1^{g_t}$ is independent of t, and in particular $f_1^g = f_1^{g_0} = f_1^{g_1} = f_1^h$. This shows the function f_α is well defined.

Let $V := U_{\delta/2}^{\mathrm{Riem}}$, $y \in xV$, $h \in C([0,1], xV)$ be a path joining x to y, and

$$(h * g)(t) = \begin{cases} g(2t) & \text{if } t \in [0,1/2], \\ h(2t-1) & \text{if } t \in [1/2,1]. \end{cases}$$

Since $\operatorname{diam}[(h*g)([1/2,1])] = \operatorname{diam}[h([0,1])] < \delta$, we know by property (ii) of Theorem 6.10 that

$$f_1^{h*g} = f_{1/2}^{h*g} = f_1^g \quad \text{on } xU \cap yU \tag{6.18}$$

where we have used the (easily proved) fact that $f_t^{h*g} = f_{2t}^g$ for $t \in [0,1/2]$. Evaluating (6.18) at y shows that $f_\alpha(y) = f_1^g(y)$. Since $y \in xV$ was arbitrary, we have $f_\alpha = f_1^g$ on xV and hence f_α is holomorphic on xV. Since $x \in G$ was arbitrary, we have shown that f_α is holomorphic on all of G.

Taking $x = e$ and $g(t) \equiv e$ for all $t \in [0,1]$ in the above argument shows that $f_\alpha = f_1^g = f_0^g$ on $V = eV$. Therefore, by construction (see Theorem 6.10), $\hat{f}_\alpha(e) = \widehat{f_0^g}(e) = \alpha$. □

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