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Gennadiy Averkov · Gabriele Bianchi

Confirmation of Matheron's conjecture on the covariogram of a planar convex body

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Abstract. The covariogram g_K of a convex body K in $\mathbb{E}^d$ is the function which associates to each $x \in \mathbb{E}^d$ the volume of the intersection of K with $K + x$. In 1986 G. Matheron conjectured that for $d = 2$ the covariogram g_K determines K within the class of all planar convex bodies, up to translations and reflections in a point. This problem is equivalent to some problems in stochastic geometry and probability as well as to a particular case of the phase retrieval problem in Fourier analysis. It is also relevant for the inverse problem of determining the atomic structure of a quasicrystal from its X-ray diffraction image. In this paper we confirm Matheron's conjecture completely.

Keywords. Autocorrelation, covariogram, cut-and-project scheme, geometric tomography, image analysis, phase retrieval, quasicrystal, set covariance

1. Introduction

Let C be a compact set in the Euclidean space $\mathbb{E}^d$, $d \geq 2$. The *covariogram* g_C of C is the function on $\mathbb{E}^d$ defined by

$$g_C(x) := V_d(C \cap (C + x)), \qquad x \in \mathbb{E}^d, \tag{1.1}$$

where V_d stands for the d-dimensional Lebesgue measure. This function, which was introduced by G. Matheron in his book [Mat75, Section 4.3] on random sets, is also called *set covariance*. The covariogram g_C coincides with the *autocorrelation* of the characteristic function $\mathbf{1}_C$ of C, i.e.

$$g_C = \mathbf{1}_C * \mathbf{1}_{-C}. \tag{1.2}$$

The covariogram g_C is clearly unchanged with respect to translations and reflections of C, where, throughout the paper, *reflection* means reflection in a point. A *convex body* in $\mathbb{E}^d$ is a convex compact set with nonempty interior. In 1986 Matheron [Mat86, p. 20] asked the following question and conjectured a positive answer for the case $d = 2$.

G. Averkov: Faculty of Mathematics, University of Magdeburg, Universitätsplatz 2, D-39106 Magdeburg, Germany; e-mail: gennadiy.averkov@googlemail.com

G. Bianchi: Dipartimento di Matematica, Università di Firenze, Viale Morgagni 67a, 50134 Firenze, Italy; e-mail: gabriele.bianchi@unifi.it

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Covariogram Problem. Does the covariogram determine a convex body in $\mathbb{E}^d$, among all convex bodies, up to translations and reflections?

We are able to confirm Matheron's conjecture completely.

Theorem 1.1. *Every planar convex body is determined within all planar convex bodies by its covariogram, up to translations and reflections.*

The covariogram problem is equivalent to any of the following problems (for details see Remark 5.5).

P1. Determine a convex body K by the knowledge, for each unit vector u in $\mathbb{E}^d$, of the distribution of the lengths of the chords of K parallel to u.
P2. Determine a convex body K by the distribution of $X - Y$, where X and Y are independent random variables uniformly distributed over K.
P3. Determine the characteristic function $\mathbf{1}_K$ of a convex body K from the modulus of its Fourier transform $\widehat{\mathbf{1}_K}$.

In view of Theorem 1.1, for Problems P1 and P2 the determination holds within the class of planar convex bodies and for Problem P3 within the class of characteristic functions of planar convex bodies. In each of the three problems the determination is unique up to translations and reflections of the body.

In Problem P1 a random chord parallel to u is obtained by taking the intersection of K with a random invariant line L_u parallel to u, conditioned on $K \cap L_u \neq \emptyset$. Matheron [Mat75, p. 86] explained the relation between P1 and the covariogram of a set; see also Nagel [Nag93]. Blaschke [San04, §4.2] asked whether the distribution of the lengths of *all* chords (that is, not separated direction by direction) of a planar convex body determines that body, up to isometries in $\mathbb{E}^2$. Mallows and Clark [MC70] constructed polygonal examples that show that the answer is negative in general. Gardner, Gronchi, and Zong [GGZ05] observed that the distribution of the lengths of the chords of K parallel to u coincides, up to a multiplicative factor, with the *rearrangement* of the X-ray of K in direction u, and rephrased P1 in these terms. Chord-length distributions are of wide interest beyond mathematics, as Mazzolo, Roesslinger, and Gille [MRG03] describe. See also Schmitt [Sch93a] and Cabo and Baddeley [CB03].

Problem P2 was asked by Adler and Pyke [AP91] in 1991; see also [AP97].

Problem P3 is a special case of the *phase retrieval problem*, where $\mathbf{1}_K$ is replaced by a function with compact support. The phase retrieval problem has applications in *X-ray crystallography, optics, electron microscopy* and other areas, references to which may be found in [BSV02].

Recently, Baake and Grimm [BG07] have observed that the covariogram problem is relevant for the inverse problem of finding the atomic structure of a *quasicrystal* from its *X-ray diffraction image*. It turns out that quasicrystals can often be described by means of the so-called *cut-and-project scheme*; see [BM04]. In this scheme a quasiperiodic discrete subset S of $\mathbb{E}^d$, which models the atomic structure of a quasicrystal, is described as the canonical projection of $Z \cap (\mathbb{E}^d \times W)$ onto $\mathbb{E}^d$, where W (called the *window*) is a subset of $\mathbb{E}^n$, $n \in \mathbb{N}$, and Z is a lattice in $\mathbb{E}^d \times \mathbb{E}^n$. For many quasicrystals, the lattice Z can be recovered from the diffraction image of S. Thus, in order to determine S, it is necessary to know W. The covariogram problem enters at this point, since g_W can be obtained from the diffraction image of S. Note that the set W is in many cases a convex body.

In [GZ98, Theorem 6.2 and Question 6.3] the covariogram problem was transformed to a question for the so-called *radial mean bodies*.

A planar convex body K can be determined by its covariogram in a class $\mathcal{C}$ of sets which is much larger than that of convex bodies. This is a consequence of Theorem 1.1 and of a result of Benassi, Bianchi, and D'Ercole [BBD]. In [BBD] a suitable class $\mathcal{C}$ is defined and it is proved that a body $C \in \mathcal{C}$ whose covariogram is equal to that of a convex body is necessarily convex. The class $\mathcal{C}$ contains all planar bodies whose interior has at most two components, and it also contains all planar bodies whose boundary consists of a finite number of closed disjoint simple polygonal curves (each with finitely many edges). On the other hand, in Theorem 1.1 the assumption that K is convex is crucial, since there exist examples of non-convex sets which are neither translations nor reflections of each other and have equal covariograms; see [GGZ05], Rataj [Rat04], and [BBD].

The first partial solution of Matheron's conjecture was given by Nagel [Nag93] in 1993, who confirmed it for all convex polygons. Schmitt [Sch93a], in the same year, gave a constructive proof of the determination of each set in a suitable class of polygons by its covariogram. This class contains each convex polygon without parallel edges and also some non-convex polygons. In 2002 Bianchi, Segala and Volčič [BSV02] gave a positive answer to the covariogram problem for all planar convex bodies whose boundary has strictly positive continuous curvature. Bianchi [Bia05] proved a common generalization of this and Nagel's result. In [AB07] the authors of the present paper studied how much of the covariogram data is needed for the uniqueness of the determination, and also extended the class of bodies for which the conjecture was confirmed.

The covariogram problem in the general setting has a negative answer, as Bianchi [Bia05] proved by finding counterexamples in $\mathbb{E}^d$ for every $d \geq 4$. For other results in dimensions higher than two we refer to Goodey, Schneider, and Weil [GSW97, p. 87] and [Bia08a]. In [Bia08a] it is proved that a convex three-dimensional polytope is determined by its covariogram. This proof requires the following generalization of the covariogram problem. The *cross covariogram* of two convex bodies K and L in $\mathbb{E}^2$ is the function defined for each $x \in \mathbb{E}^2$ by $g_{K,L}(x) := V_2(K \cap (L + x))$. Bianchi [Bia08b] proves that if K and L are convex polygons, then $g_{K,L}$ determines *both* K *and* L, with exclusion of a completely described family of exceptions. The family of exceptions is composed of pairs of parallelograms.

In view of results from [Bia05], to prove Theorem 1.1 it suffices to derive the following statement.

Proposition 1.2. *Let K and L be planar strictly convex and* C^1 *regular bodies with equal covariograms. Then L possesses a non-degenerate boundary arc whose translation or reflection lies in the boundary of K.*

Theorem 1.1 follows directly from Proposition 1.2 and the following two statements.

Theorem 1.3 (Bianchi [Bia05]). *Let K and L be planar convex bodies with equal covariograms. Assume that one of them is not strictly convex or not* C^1 *regular. Then K and L are translations or reflections of each other.*

Proposition 1.4 (Bianchi [Bia05]). *Let K and L be planar convex bodies with equal covariograms and a common non-degenerate boundary arc. Then K and L coincide up to translations and reflections.*

The heart of the proof is contained in Section 5, which also provides, at the beginning, an explanation of the main ideas. Many natural questions for the covariogram problem are still open. We mention here some of them.

1. Which four-dimensional convex polytopes are determined by their covariogram?
2. All known examples of convex bodies that are not determined by their covariogram are Cartesian products. Do there exist other examples?
3. Is the answer to the covariogram problem positive for all three-dimensional convex bodies whose boundary has continuous and strictly positive principal curvatures?

2. Preliminaries

The closure, boundary, interior, linear hull, affine hull, and convex hull of a set, and the support of a function, are abbreviated, in the standard way, by cl, bd, int, lin, aff, conv, and supp, respectively. We denote by o, $\langle \cdot, \cdot \rangle$, $|\cdot|$, and $\mathbb{S}^{d-1}$ the origin, scalar product, Euclidean norm, and Euclidean unit sphere in $\mathbb{E}^d$, respectively. In analytic expressions elements of $\mathbb{E}^d$ are identified with real column vectors of length d. Thus, $\langle x, y\rangle = x^\top y$, where $(\cdot)^\top$ denotes matrix transposition. Throughout the paper we use the matrix $\mathcal{R} := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ of 90-degree rotation in the counterclockwise orientation. We do not distinguish between 2×2 matrices over $\mathbb{R}$ and linear operators in $\mathbb{E}^2$. For vectors $x, y \in \mathbb{E}^2$ we put

$$\det(x, y) := \det \begin{bmatrix} | & | \\ x & y \\ | & | \end{bmatrix} = -x^\top \mathcal{R} y, \tag{2.1}$$

where $\left[\begin{smallmatrix} | & | \\ x & y \\ | & | \end{smallmatrix}\right]$ stands for the matrix whose first column is x and the second is y.

Regarding standard notations and notions from the theory of convex sets we mostly follow the monograph [Sch93b]. The *difference body* of a convex body K is the set $DK := K + (-K) = \{x - y : x, y \in K\}$. It is not hard to see that $\operatorname{supp} g_K = DK$. A boundary point p of a convex body K is said to be C^1 *regular* if there exists precisely one hyperplane supporting K at p. Furthermore, a convex body K is said to be C^1 *regular* if all boundary points of K are C^1 regular. We say that a convex polygon $P \subseteq \mathbb{E}^2$ is *inscribed* in a convex body $K \subseteq \mathbb{E}^2$ if all vertices of P lie in bd K. Given $q_1, q_2 \in \operatorname{bd} K$, the chord $[q_1, q_2]$ is said to be an *affine diameter* of K if for some $u \in \mathbb{E}^2 \setminus \{o\}$ the vectors u and $-u$ are outward normals of K at q_1 and q_2, respectively. It is well known that $[q_1, q_2]$ is an affine diameter of K if and only if $q_1 - q_2 \in \operatorname{bd} DK$. If K is a planar convex body and p, q are two distinct boundary points of K, then $[p, q]_K$ stands for the counterclockwise boundary arc of K starting at p and terminating at q.

3. Gradient of covariogram and inscribed parallelograms

Let K be a strictly convex and C^1 regular convex body in $\mathbb{E}^2$. Consider an arbitrary $x \in \operatorname{int}\operatorname{supp} g_K \setminus \{o\}$. Then there exist points $p_i(K,x),\ i \in \{1,\dots,4\}$, in counterclockwise order on K, such that $x = p_1(K,x) - p_2(K,x) = p_4(K,x) - p_3(K,x)$; see Fig. 3.1, also regarding notations introduced below. Then the set

$$P(K,x) := \operatorname{conv}\{p_1(K,x),\dots,p_4(K,x)\} \tag{3.1}$$

is a parallelogram inscribed in K, whose edges are translates of $[o,x]$ and $[o,D(K,x)]$ with

$$D(K,x) := p_1(K,x) - p_4(K,x),$$

By $u_i(K,x)$ we denote the outward unit normal of K at $p_i(K,x)$.

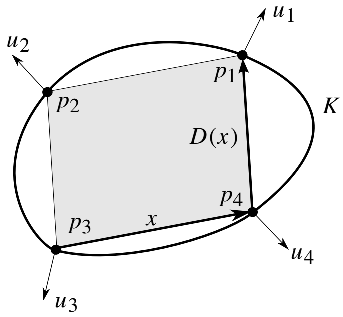

Fig. 3.1

For the sake of brevity, dealing with functionals $f(K,x)$ depending on K and x, we shall also use the notations $f(x)$ or f instead of $f(K,x)$, provided the choice of K and/or x is clear from the context.

Theorem 3.1. *Let K be a strictly convex and C^1 regular body in $\mathbb{E}^2$ and let $x \in \operatorname{int}\operatorname{supp} g_K \setminus \{o\}$. Then the following statements hold.*

I. *The covariogram g_K is continuously differentiable at x. Moreover,*

$$\nabla g_K(x) = \mathcal{R}(D(x)). \tag{3.2}$$

II. *The functions $P,\ D,\ p_i,\ u_i$ with $i \in \{1,\dots,4\}$ are continuous at x.*

III. *For every strictly convex C^1 regular body L with $g_L = g_K$, the parallelogram $P(L,x)$ is a translate of $P(K,x)$.*

Proof. Part I is known; see [Mat86, p. 3] and [MRS93, p. 282]. Let us prove part II. If $x \in \operatorname{int}\operatorname{supp} g_K \setminus \{o\}$, then $\operatorname{bd} K$ and $\operatorname{bd} K + x$ intersect precisely at $p_1(x)$ and $p_4(x)$. Furthermore, the intersection is transversal. By the Implicit Function Theorem, this implies that $p_1(x)$ and $p_4(x)$ depend continuously on x. Since K is C^1 regular, the outward unit normal $u(p)$ of K at a boundary point p of K depends continuously on p. Therefore, for $i \in \{1, \ldots, 4\}$ the function $u_i(x) = u(p_i(x))$ depends continuously on x. Part III follows directly from (3.2). □

In what follows, notations involving an integer subscript i ranging in a certain interval are extended periodically to all $i \in \mathbb{Z}$. For example, we set $p_i(K, x) := p_j(K, x)$, where $i \in \mathbb{Z}$, $j \in \{1, \ldots, 4\}$ and $i = j \pmod 4$.

Throughout the paper, the parallelograms $P(K, x)$ will provide a convenient geometric representation of some information contained in the covariogram, in view of (3.2). A priori, for two planar convex bodies K and L with $g_K = g_L$, the translation that carries $P(K, x)$ to $P(L, x)$ may depend on x. One crucial step of the proof is to show that the above translation is in fact independent of x. See the beginning of Section 5 for a brief sketch of the above mentioned argument.

4. Second derivatives, Monge–Ampère equation, and central symmetry

If the covariogram of K is twice differentiable at x, we introduce the Hessian matrix

$$G(K, x) := \left[\frac{\partial^2 g_K(x)}{\partial x_i \partial x_j} \right]_{i,j=1}^{2}.$$

The relations given in Theorem 4.1 below are reformulations of the relations presented in [Mat86, pp. 12–18]. Part I of Theorem 4.1 is extended to every dimension in [MRS93]. We omit the proof of part I and present a short proof of parts II and III.

Theorem 4.1. *Let K be a strictly convex and C^1 regular body in $\mathbb{E}^2$. Then $g_K(x)$ is twice continuously differentiable at every $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$. Furthermore, for every $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$, the following statements hold true.*

I. *The Hessian $G(x)$ can be represented by*

$$G = \frac{u_2 u_1^\top}{\det(u_2, u_1)} - \frac{u_3 u_4^\top}{\det(u_3, u_4)} = \frac{u_1 u_2^\top}{\det(u_2, u_1)} - \frac{u_4 u_3^\top}{\det(u_3, u_4)}. \tag{4.1}$$

II. *The determinant of $G(x)$ depends continuously on x and satisfies*

$$\det G = -\frac{\det(u_2, u_3)\det(u_4, u_1)}{\det(u_3, u_4)\det(u_1, u_2)} < 0, \tag{4.2}$$

$$1 + \det G = \frac{\det(u_2, u_4)\det(u_1, u_3)}{\det(u_3, u_4)\det(u_1, u_2)}. \tag{4.3}$$

III. *The vectors u_1, u_3 and the matrix G are related by*

$$u_1^\top G^{-1} u_3 = 0. \tag{4.4}$$

Proof. The fact that g_K is twice continuously differentiable on $\operatorname{int}\operatorname{supp} g_K \setminus \{o\}$ is proved in [MRS93, pp. 283–284]. For the proof of part I see [Mat86, pp. 12–18] and [MRS93, pp. 283–284].

Part II: From (4.1) we get

$$G\mathcal{R}u_2 = -\frac{u_4 u_3^\top \mathcal{R}u_2}{\det(u_3, u_4)} \overset{(2.1)}{=} \frac{\det(u_3, u_2)}{\det(u_3, u_4)} u_4,$$
$$G\mathcal{R}u_3 = \frac{u_1 u_2^\top \mathcal{R}u_3}{\det(u_2, u_1)} \overset{(2.1)}{=} -\frac{\det(u_2, u_3)}{\det(u_2, u_1)} u_1 = \frac{\det(u_3, u_2)}{\det(u_2, u_1)} u_1.$$

The above two equalities imply

$$G\mathcal{R}\begin{bmatrix} | & | \\ u_2 & u_3 \\ | & | \end{bmatrix} = \det(u_3, u_2)\begin{bmatrix} | & | \\ u_4 & u_1 \\ | & | \end{bmatrix}\begin{bmatrix} \frac{1}{\det(u_3,u_4)} & 0 \\ 0 & \frac{1}{\det(u_2,u_1)} \end{bmatrix}.$$

Taking determinants of both sides we obtain

$$\det G \cdot \det(u_2, u_3) = \det(u_2, u_3)^2 \cdot \det(u_4, u_1) \cdot \frac{1}{\det(u_3, u_4)\det(u_2, u_1)}, \tag{4.5}$$

Let us notice that

$$\det(u_i, u_{i+1}) > 0 \tag{4.6}$$

for every $i \in \{1, \dots, 4\}$. For instance, $\det(u_1, u_2)$ is positive because, if w denotes the unit outward normal to the edge $[p_4, p_1]$ of $P(K, x)$, then w, u_1, u_2, and $-w$ are in this counterclockwise order on $\mathbb{S}^1$, by the strict convexity of K. By (4.6), we may divide (4.5) by $\det(u_2, u_3)$, arriving at the equality in (4.2). The inequality in (4.2) follows from (4.6).

Equality (4.3) follows directly from (4.2) and the algebraic identity

$$\det(v_1, v_3)\det(v_2, v_4) = \det(v_2, v_3)\det(v_1, v_4) + \det(v_4, v_3)\det(v_2, v_1), \tag{4.7}$$

which holds for all $v_1, \dots, v_4 \in \mathbb{E}^2$ and can be found, in a much more general form, in [BLVS⁺99, p. 127]. Notice that (4.7) can be easily proved by fixing v_2 and v_4 arbitrarily, considering both sides as bilinear functions of v_1 and v_3 and checking the equality for the case when v_1 and v_3 belong to the standard orthonormal basis. The continuity of $\det G(x)$ is a consequence of (4.2) and Theorem 3.1.

Part III: We multiply (4.1) by $u_1^\top \mathcal{R}$ from the left and by $\mathcal{R}u_3$ from the right getting $u_1^\top \mathcal{R}G\mathcal{R}u_3 = 0$. Expressing the entries of $\mathcal{R}G\mathcal{R}$ by the entries of G one can see that $\mathcal{R}G\mathcal{R} = -\det G \cdot G^{-1}$. Hence, taking into account that $\det G \neq 0$, we arrive at (4.4). □

Theorem 4.2. *Let K be a strictly convex and C^1 regular body in $\mathbb{E}^2$. The following conditions are equivalent.*

(i) *The body K is centrally symmetric.*
(ii) *At least one diagonal of each parallelogram inscribed in K is an affine diameter of K.*

(iii) *The covariogram* g_K *is a solution of the Monge–Ampère differential equation* $\det G(x) = -1$ *for* $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$.

Proof. The implication (i)⇒(ii) is trivial. The equivalence of (ii) and (iii) follows from (4.3). It remains to prove that (ii) implies (i).

Let us first prove that (ii) implies that both diagonals of each parallelogram inscribed in K are affine diameters. Assume the contrary. Then, for some $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$, exactly one diagonal of $P(x)$, say $[p_1(x), p_3(x)]$, is an affine diameter. Let $q(t)$, $t \in [0, 1]$, be a continuous parametrization of a small arc of $\operatorname{bd} K$ with $q(0) = p_4(x)$. If we define $x(t) := q(t) - p_3(x)$ then $p_3(x(t)) = p_3(x)$ and $p_4(x(t)) = q(t)$. We claim that there exists a sufficiently small $t > 0$ such that no diagonal of $P(x(t))$ is an affine diameter of K. In fact, $[p_2(x(t)), p_4(x(t))]$ is not an affine diameter, because it is close to $[p_2(x), p_4(x)]$, which is not an affine diameter. On the other hand, assume that there exists $\varepsilon > 0$ such that for each $t \in [0, \varepsilon]$ the diagonal $[p_1(x(t)), p_3(x(t))]$ is an affine diameter. Since $p_1(x)$ is the only point of $\operatorname{bd} K$ with unit outer normal opposite to the one in $p_3(x)$, and $p_3(x) = p_3(x(t))$, we have $[p_1(x(t)), p_3(x(t))] = [p_1(x), p_3(x)]$. Thus, the reflection of $\{q(t) : t \in [0, \varepsilon]\}$ in the midpoint of $[p_1(x), p_3(x)]$ is contained in $\operatorname{bd} K$, and this implies that $[p_2(x), p_4(x)]$ is an affine diameter too, a contradiction. This proves the claim.

It remains to prove that if both diagonals of each parallelogram inscribed in K are affine diameters, then K is centrally symmetric. Let $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$, let $q(t)$, $t \in [0, 1]$, be a continuous parametrization of the arc $[p_1(x), p_3(x)]_K$, and let $x(t)$ be as above. Arguing as above one can prove that, for each t, we have $[p_1(x(t)), p_3(x(t))] = [p_1(x), p_3(x)]$. Therefore, for each t, $q(t)$ and its reflection in the midpoint c of $[p_1(x), p_3(x)]$ belong to $\operatorname{bd} K$. Thus K is centrally symmetric with respect to c. □

5. Determination of an arc of the boundary

The crucial point of the proof of Proposition 1.2 is the statement that outer normals of K are determined by g_K, up to the ambiguities arising from reflections of the body. More precisely, we need to prove the following.

Proposition 5.1. *Let* K *be a strictly convex and* C^1 *regular body in* $\mathbb{E}^2$. *Then, for every* $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$ *with* $\det G(x) \neq -1$, *the set* $\{u_1(x), -u_3(x)\}$ *is uniquely determined by* g_K.

Let us sketch the proof of Proposition 5.1. Due to the assumptions of Proposition 5.1 and (4.3) we have $u_1(x) \neq -u_3(x)$. We prove that there is y, with $y \neq x$, such that $P(x)$ and $P(y)$ have the opposite vertices p_1 and p_3 in common, i.e. $p_1(x) = p_1(y)$ and $p_3(x) = p_3(y)$. This clearly implies $u_1(x) = u_1(y)$ and $u_3(x) = u_3(y)$. Thus, $u_1(x)$ and $u_3(x)$ satisfy the system given by the two equations obtained by evaluating (4.4) at both x and y. Using the geometric interpretation of the action of G contained in Lemma 5.2, in Lemma 5.3 we express the vectors $u_1(x)$ and $u_3(x)$ in terms of the eigenvectors of $G(x)G(y)^{-1}$. In order to make this expression of $u_1(x)$ and $u_3(x)$ dependent only on the covariogram, it remains to prove that the property that $P(x)$ and $P(y)$ share a diagonal is preserved across bodies with equal covariograms. The latter is done in Proposition 5.4.

Let us now sketch how Proposition 1.2 follows from Proposition 5.1. Let K and L be strictly convex C^1 regular bodies with $g_K = g_L$, and let us choose $x_0 \in \operatorname{int}\operatorname{supp} g_K \setminus \{o\}$ such that $\det G(x_0) \neq -1$. We will prove the following claim:

If x belongs to a suitable neighborhood U of x_0, and if $P(K, x)$ and $P(K, x_0)$ share their vertex p_3 (i.e. $p_3(K, x) = p_3(K, x_0)$), then also $P(L, x)$ and $P(L, x_0)$ share their vertex p_3.

Indeed, Proposition 5.1 together with some continuity argument allows us to prove that when x is close to x_0 and $u_3(K, x) = u_3(K, x_0)$ then we have $u_3(L, x) = u_3(L, x_0)$. In view of the strict convexity of K and L, this implies the claim above.

Let now $x(t)$, for $t \in [0, 1]$, be a parametrization of a curve contained in U with the property that for each $t \in [0, 1]$ the parallelograms $P(K, x_0)$ and $P(K, x(t))$ share their vertex p_3. The previous claim implies that the arc of $\operatorname{bd} K$ spanned by the vertex $p_4(K, x(t))$ when t varies in $[0, 1]$ is a translate of the arc of $\operatorname{bd} L$ spanned by the vertex $p_4(L, x(t))$. Therefore, up to translations, $\operatorname{bd} K$ and $\operatorname{bd} L$ have an arc in common.

Lemma 5.2. *Let K be a strictly convex and C^1 regular body in $\mathbb{E}^2$ and let $x \in \operatorname{int}\operatorname{supp} g_K \setminus \{o\}$. Let $h \in \mathbb{E}^2 \setminus \{o\}$ be such that the vectors $u_1, u_2, -h/|h|, u_3, u_4, h/|h|$ and u_1 are in counterclockwise order on $\mathbb{S}^1$. Consider the convex quadrilateral $Q(x, h)$ with consecutive vertices $q_1(x, h), \dots, q_4(x, h)$ such that $q_1(x, h) = h$, $q_3(x, h) = o$ and, for each $i \in \{1, \dots, 4\}$, the vector $u_i(x)$ is an outward normal of the side $[q_i(x, h), q_{i+1}(x, h)]$; see Fig. 5.1. Then $q_4(x, h) - q_2(x, h) = -\mathcal{R}G(x)h$.*

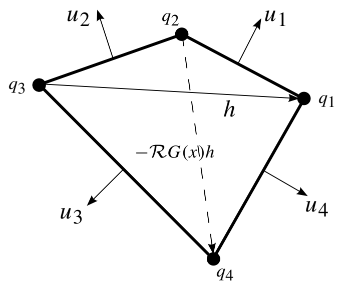

Fig. 5.1. The quadrilateral $Q(x, h)$.

Proof. Given two linearly independent vectors $v_1, v_2 \in \mathbb{E}^2$, we denote by $\Pi_{v_1}^{v_2}$ the operator of projection onto $\operatorname{lin} v_1$ along the vector v_2, that is, $\Pi_{v_1}^{v_2} y := \alpha_1 v_1$ for $y = \alpha_1 v_1 + \alpha_2 v_2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. By Cramer's rule, $\alpha_1 = \det(y, v_2)/\det(v_1, v_2)$, and hence

$$\Pi_{v_1}^{v_2} y = \frac{\det(v_2, y)}{\det(v_2, v_1)} v_1 \overset{(2.1)}{=} -\frac{v_1 v_2^\top \mathcal{R} y}{\det(v_2, v_1)},$$

which implies

$$\Pi_{v_1}^{v_2} = -\frac{v_1 v_2^\top \mathcal{R}}{\det(v_2, v_1)}.$$

Thus, (4.1) is equivalent to

$$G = \Pi_{u_1}^{u_2}\mathcal{R} - \Pi_{u_4}^{u_3}\mathcal{R}. \tag{5.1}$$

It can be easily verified that $\Pi_{\mathcal{R}v_1}^{\mathcal{R}v_2} = -\mathcal{R}\Pi_{v_1}^{v_2}\mathcal{R}$. Therefore, (5.1) implies

$$-\mathcal{R}G = \Pi_{\mathcal{R}u_1}^{\mathcal{R}u_2} - \Pi_{\mathcal{R}u_4}^{\mathcal{R}u_3}. \tag{5.2}$$

We have

$$\begin{aligned} q_4(x,h) - q_2(x,h) &= (q_1(x,h) - q_2(x,h)) + (q_4(x,h) - q_1(x,h)) \\ &= \Pi_{\mathcal{R}u_1}^{\mathcal{R}u_2} h - \Pi_{\mathcal{R}u_4}^{\mathcal{R}u_3} h \overset{(5.2)}{=} -\mathcal{R}Gh. \end{aligned}$$ □

Lemma 5.3. *Let K be a strictly convex and C^1 regular body in $\mathbb{E}^2$. Let $x,\ y$ be distinct vectors from* $\operatorname{int}\operatorname{supp} g_K \setminus \{o\}$ *such that $p_i := p_i(x) = p_i(y)$ for $i \in \{1,3\}$ and the segment $[p_1, p_3]$ is not an affine diameter of K. Then the matrix $G(x)G(y)^{-1}$ has two distinct real eigenvalues. Furthermore, if $v_1, v_3 \in \mathbb{S}^1$ are distinct eigenvectors of $G(x)G(y)^{-1}$ satisfying $\langle x, v_1\rangle \ge 0,\ \langle x, v_3\rangle \ge 0$, then $\{u_1(x), -u_3(x)\} = \{v_1, v_3\}$.*

Proof. The assumptions $p_i(x) = p_i(y)$ for $i \in \{1,3\}$ imply that $u_i := u_i(x) = u_i(y)$ for $i \in \{1,3\}$. By (4.4) applied at x and y, we get

$$u_1^\top G(x)^{-1} u_3 = 0, \tag{5.3}$$

$$u_1^\top G(y)^{-1} u_3 = 0. \tag{5.4}$$

From (5.3) and (5.4) we see that u_1 is orthogonal to both $G(x)^{-1}u_3$ and $G(y)^{-1}u_3$. Then $G(x)^{-1}u_3$ and $G(y)^{-1}u_3$ are parallel, which implies that $G(x)G(y)^{-1}u_3$ is parallel to u_3. Thus, u_3 is an eigenvector of the matrix $G(x)G(y)^{-1}$. Analogous arguments show that also u_1 is an eigenvector of $G(x)G(y)^{-1}$. We show that it is not possible that all vectors from $\mathbb{E}^2 \setminus \{o\}$ are eigenvectors of $G(x)G(y)^{-1}$. We introduce the centrally symmetric hexagon $H := \operatorname{conv}(P(x) \cup P(y))$. After possibly interchanging the roles of x and y, we assume that the points $p_1(x),\ p_2(y),\ p_2(x),\ p_3(x)$ are in counterclockwise order on $\operatorname{bd} K$; see Fig. 5.2.

Let h be an outward normal of the side $[p_1(x), p_4(x)]$ of H. Let $Q(x,h)$ and $Q(y,h)$ be quadrilaterals constructed as in the statement of Lemma 5.2. By the choice of H we have $u_1(x) = u_1(y)$, $u_3(x) = u_3(y)$, while $u_2(x)$ follows $u_2(y)$, and $u_4(x)$ follows $u_4(y)$, in counterclockwise order on $\mathbb{S}^1$. Consequently, $[o, h]$ is a common diagonal of $Q(x,h)$ and $Q(y,h)$, while the vertices $q_2(y,h)$ and $q_4(y,h)$ of $Q(y,h)$ lie in the relative interiors of the sides $[q_1(x,h), q_2(x,h)]$ and $[q_3(x,h), q_4(x,h)]$, respectively, of $Q(x,h)$; see Fig. 5.3. The latter implies that the diagonals $[q_2(x,h), q_4(x,h)]$ and $[q_2(y,h), q_4(y,h)]$ of the quadrilaterals $Q(x,h)$ and $Q(y,h)$, respectively, are not parallel. Hence, by Lemma 5.2, $G(x)h$ and $G(y)h$ are not parallel, which implies that $G(y)h$ is not an eigenvector of $G(x)G(y)^{-1}$. Consequently, $\operatorname{lin} u_1(x)$ and $\operatorname{lin} u_3(x)$ are two distinct eigenspaces of $G(x)G(y)^{-1}$, and we arrive at the assertion. □

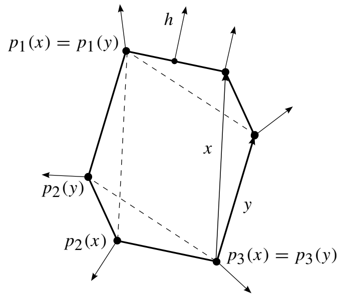

Fig. 5.2. The hexagon H and the normals of K at the vertices of H

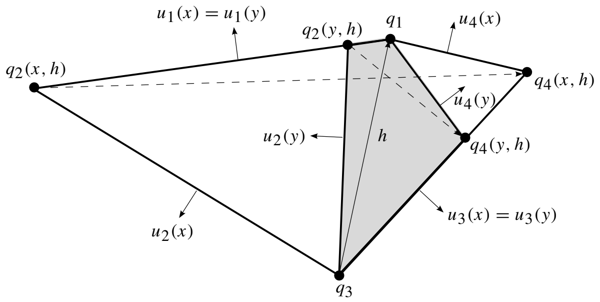

Fig. 5.3. The boundaries of $Q(x, h)$ and $Q(y, h)$ are plotted in bold; $Q(y, h)$ is shaded.

Proposition 5.4. *Let K be a strictly convex and C^1 regular body in $\mathbb{E}^2$. Let H be a centrally symmetric convex hexagon with consecutive vertices $h_1, \ldots, h_6$ in counterclockwise order. For $i \in \{1, 2, 3\}$ we introduce the vectors $x_i := h_{2i+1} - h_{2i-1}$; see Fig.* 5.4. *Then a translate of H is inscribed in K if and only if*

$$D(x_i) = h_{2i+2} - h_{2i+1} \tag{5.5}$$

for every $i \in \{1, 2, 3\}$ and

$$\prod_{i=1}^{3} (1 + \det G(x_i)) \geq 0. \tag{5.6}$$

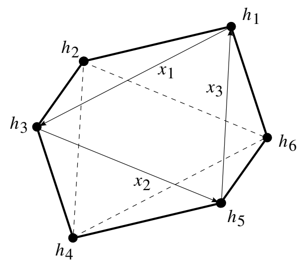

Fig. 5.4. The hexagon H and the vectors x_1, x_2 and x_3.

Proof. Let us show the necessity. Since conditions (5.5) and (5.6) are invariant with respect to translations of H, we can assume that H itself is inscribed in K. From the definition of x_i and p_j it follows that

$$\begin{aligned} p_1(x_i) &= h_{2i+2}, & p_2(x_i) &= h_{2i-2}, \\ p_3(x_i) &= h_{2i-1}, & p_4(x_i) &= h_{2i+1}, \end{aligned} \tag{5.7}$$

where $i \in \mathbb{Z}$. Thus, (5.5) follows directly from (5.7) and the definition of the function D. Let us obtain (5.6). By (4.3) we have

$$s := \prod_{i=1}^{3}(1 + \det G(x_i)) = \frac{s_1 s_2}{s_3},$$

where

$$\begin{aligned} s_1 &:= \prod_{i=1}^{3} \det(u_1(x_i), u_3(x_i)), \\ s_2 &:= \prod_{i=1}^{3} \det(u_2(x_i), u_4(x_i)), \\ s_3 &:= \prod_{i=1}^{3} \det(u_1(x_i), u_2(x_i)) \det(u_3(x_i), u_4(x_i)). \end{aligned}$$

The determinants $\det(u_1(x_i), u_2(x_i))$ and $\det(u_3(x_i), u_4(x_i))$ are strictly positive; see (4.6). Consequently, $s_3 > 0$. From (5.7) we get the equalities $p_1(x_{i+1}) = p_2(x_i)$ and $p_3(x_{i+1}) = p_4(x_i)$ and hence also the equalities

$$\det(u_1(x_{i+1}), u_3(x_{i+1})) = \det(u_2(x_i), u_4(x_i))$$

for $i \in \{1, 2, 3\}$. Hence we see that $s_1 = s_2$ and therefore $s \geq 0$.

Now let us show the sufficiency by contradiction. Assume that for some K and H satisfying the assumptions of the proposition, conditions (5.5) and (5.6) are fulfilled but no translate of H is inscribed in K. By (5.5) we see that for every $i \in \mathbb{Z}$ the parallelogram $P(x_i)$ is a translate of $\operatorname{conv}\{h_{2i-2}, h_{2i-1}, h_{2i+1}, h_{2i+2}\}$, and moreover

$$\begin{aligned} p_1(x_i) &= a_i + h_{2i+2}, \quad & p_2(x_i) &= a_i + h_{2i-2}, \\ p_3(x_i) &= a_i + h_{2i-1}, \quad & p_4(x_i) &= a_i + h_{2i+1}, \end{aligned} \tag{5.8}$$

with appropriate $a_i \in \mathbb{E}^2$. If for some $i, j \in \{1, 2, 3\}$ with $i \neq j$ the parallelograms $P(x_i)$, $P(x_j)$ share a diagonal, it follows that H is a translate of $\operatorname{conv}(P(x_i) \cup P(x_j))$, a contradiction. Now we consider the case when no two distinct parallelograms $P(x_i)$ and $P(x_j)$ share a diagonal. Let $i \in \mathbb{Z}$. In view of (5.8), we get

$$h_{2i-2} - h_{2i+1} = p_1(x_{i+1}) - p_3(x_{i+1}) = p_2(x_i) - p_4(x_i).$$

Thus the diagonals $[p_1(x_{i+1}), p_3(x_{i+1})]$ and $[p_2(x_i), p_4(x_i)]$ of $P(x_{i+1})$ and $P(x_i)$, respectively, are translates of each other. By the assumption, these diagonals are distinct. Thus, $[p_1(x_{i+1}), p_3(x_{i+1})]$ and $[p_2(x_i), p_4(x_i)]$ are distinct chords of K which are translates of $[h_{2i+1}, h_{2i-2}]$; see Fig. 5.5.

The strict convexity of K implies that

$$\operatorname{sign} \det(u_1(x_{i+1}), u_3(x_{i+1})) = -\operatorname{sign} \det(u_2(x_i), u_4(x_i)) \neq 0$$

for $1 \leq i \leq 3$. The latter yields $\operatorname{sign} s_1 = -\operatorname{sign} s_2 \neq 0$. But since $s_3 > 0$ we obtain $s < 0$, a contradiction to (5.6). □

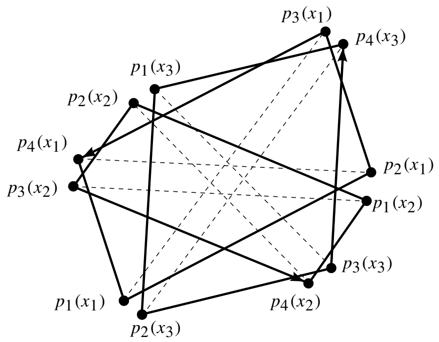

Fig. 5.5. The parallelograms $P(x_1)$, $P(x_2)$, $P(x_3)$ and their diagonals.

Proof of Proposition 5.1. First we show that there exists $y \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$ with $y \neq x$ such that $p_i(x) = p_i(y)$ for $i \in \{1, 3\}$. Let c be the center of $P(x)$ and K_c be the reflection of K with respect to c; see Fig. 5.6. Assume first that $1 + \det G(x) > 0$. Then, in view of (4.3) and (4.6),

$$\operatorname{sign} \det(u_1(x), u_3(x)) = \operatorname{sign} \det(u_2(x), u_4(x)) \neq 0.$$

Therefore $\operatorname{bd} K$ and $\operatorname{bd} K_c$ intersect transversally at $p_i(x)$ for every $i \in \{1, \dots, 4\}$. Moreover, either a small subarc of $[p_1(x), p_2(x)]_K$ with endpoint $p_1(x)$ is contained in K_c and a small subarc of $[p_1(x), p_2(x)]_K$ with endpoint $p_2(x)$ is contained in $\mathbb{E}^2 \setminus \operatorname{int} K_c$ or vice versa (that is, a small subarc of $[p_1(x), p_2(x)]_K$ with endpoint $p_1(x)$ is contained in $\mathbb{E}^2 \setminus \operatorname{int} K_c$ and a small subarc of $[p_1(x), p_2(x)]_K$ with endpoint $p_2(x)$ is contained in K_c). Consequently, the arcs $[p_1(x), p_2(x)]_K$ and $[p_1(x), p_2(x)]_{K_c}$ intersect at some point q distinct from $p_1(x)$ and $p_2(x)$. We define $y := p_1(x) - q$. By construction, $p_1(x)$, q, $p_3(x)$, and $2c - q$ are consecutive vertices of $P(y)$; see Fig. 5.6. Therefore y satisfies the desired conditions.

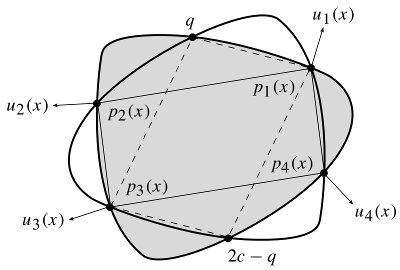

Fig. 5.6. The bodies K (shaded) and K_c and the parallelograms $P(x)$ (continuous line) and $P(y)$.

In the case $1 + \det G(x) < 0$ we can use similar arguments showing that the arcs $[p_4(x), p_1(x)]_K$ and $[p_4(x), p_1(x)]_{K_c}$ intersect at some point q distinct from $p_1(x)$ and $p_4(x)$. Thus, for that case we can define $y := q - p_3(x)$.

Now let L be a strictly convex and C^1 regular planar convex body with the same covariogram as K. By Proposition 5.4, a translate of $H := \mathrm{conv}(P(K,x) \cup P(K,y))$ is inscribed in L. Without loss of generality we assume H itself is inscribed in L, that is, $P(K,x) = P(L,x)$ and $P(K,y) = P(L,y)$. Notice that the inequality $1+\det G(x) \neq 0$ implies that $[p_1, p_3]$ is not an affine diameter of K or L. Then, by Lemma 5.3, we have $\{u_1(K,x), -u_3(K,x)\} = \{u_1(L,x), -u_3(L,x)\}$, and we are done. □

Proof of Proposition 1.2. By Theorem 4.2, if K is centrally symmetric, then so is L. In this case K and L are translates of $\frac{1}{2}\,\mathrm{supp}\, g_K = \frac{1}{2}\,\mathrm{supp}\, g_L$, and the proof is concluded.

Now assume that K is not centrally symmetric. Then, by Theorem 4.2, there exists $x_0 \in \mathrm{int}\,\mathrm{supp}\, g_K \setminus \{o\}$ such that $\det G(x_0) \neq -1$. This implies $u_1(K,x_0) \neq -u_3(K,x_0)$. Let N_1 and N_3 be disjoint open neighborhoods of $u_1(K,x_0)$ and $-u_3(K,x_0)$, respectively. In view of Theorem 3.1(III) and Proposition 5.1, replacing L by an appropriate translation or reflection, we can assume that $P(K,x_0) = P(L,x_0)$ and $u_i(K,x_0) = u_i(L,x_0)$ for $i \in \{1,3\}$. Let $q(t)$, $0 \leq t \leq 1$, be a continuous, counterclockwise parametrization of a small boundary arc of K such that $q(0) = p_4(K,x_0)$ and, for $x(t) := q(t) - p_3(K,x_0)$, one has $\det G(x(t)) \neq -1$, $u_1(K,x(t)) \in N_1$ and $-u_3(K,x(t)) \in N_3$ for every $0 \leq t \leq 1$.

We show by contradiction that for every $0 \leq t \leq 1$ the equalities

$$u_i(K,x(t)) = u_i(L,x(t)), \qquad i \in \{1,3\}, \tag{5.9}$$

hold. Assume the contrary. Then, by Proposition 5.1, there exists t_1 with $0 < t_1 \leq 1$ such that $u_1(K,x(t_1)) = -u_3(L,x(t_1))$ and $-u_3(K,x(t_1)) = u_1(L,x(t_1))$. In particular, we have $u_1(L,x(t_1)) \in N_3$. Since $u_1(L,x(0)) = u_1(K,x(0)) \in N_1$ and since N_1 and N_3 are disjoint, there exists t_2 with $0 < t_2 < t_1$ such that $u_1(L,x(t_2))$ lies outside $N := N_1 \cup N_3$. Hence $\{u_1(L,x(t_2)), -u_3(L,x(t_2))\} \not\subseteq N$. But, by construction, we have $\{u_1(K,x(t_2)), -u_3(K,x(t_2))\} \subseteq N$, a contradiction to Proposition 5.1.

The definition of $x(t)$ implies $p_3(K,x(t)) = p_3(K,x_0)$ for each $t \in [0,1]$, and therefore also $u_3(K,x(t)) = u_3(K,x_0)$. Hence, in view of (5.9), we get $u_3(L,x(t)) = u_3(L,x_0)$. Consequently, by the strict convexity of L, we also have $p_3(L,x(t)) = p_3(L,x_0)$. The latter implies

$$[p_4(K,x(0)), p_4(K,x(1))]_K = [p_4(L,x(0)), p_4(L,x(1))]_L,$$

and concludes the proof. □

Remark 5.5. Let us show the equivalence of the covariogram problem and problems P1–P3 in the introduction. The equivalence of the covariogram problem and P3 follows by applying the Fourier transform to (1.2) and using the relation $\widehat{\mathbf{1}_{-K}} = \overline{\widehat{\mathbf{1}_K}}$. Regarding P2, by (1.2) we see that the distribution of $X - Y$ coincides with $g_K(x)/V_d(K)^2$. Since $g_K(o) = V_d(K)$, knowing the covariogram is equivalent to knowing the distribution of $X - Y$. Finally, let us discuss the equivalence of the covariogram problem and P1. Let X_u be the length of the chord $L_u \cap K$, where L_u is an invariant line parallel to u conditioned on $L_u \cap K \neq \emptyset$. It was noticed in [Mat75, p. 86] that, for $r > 0$, the probability of the event $\{X_u \geq r\}$ is equal to $-\left(\partial g_K/\partial u\right)(ru)/V_{d-1}(K|u^{\perp})$, where $K|u^{\perp}$ denotes the orthogonal projection of K on $u^\perp$. Integrating the above expression with respect to r we determine $f(ru) := g_K(ru)/V_{d-1}(K|u^\perp)$. Consequently, the distribution of X_u given for each direction u determines $f(ru)/f(0u) = g_K(ru)/V_d(K)$, for every $r > 0$ and every unit vector u. The latter is equivalent to the determination of $g_K(x)/V_d(K)$ for every $x \in \mathbb{E}^d$. By (1.2) we have $\int_{\mathbb{R}^d} g_K(x)\,dx = V_d(K)^2$. Hence the integration of $g_K/V_d(K)$ over $\mathbb{R}^d$ yields $V_d(K)$, and we determine g_K. Conversely, since $(\partial^+ g_K/\partial u)(o) = V_{d-1}(K|u^\perp)$ (where ∂^+ stands for the right derivative), g_K determines the distribution of X_u.

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