



Rowan Killip · Terence Tao · Monica Visan

The cubic nonlinear Schrödinger equation in two dimensions with radial data

Received October 20, 2008

Abstract. We establish global well-posedness and scattering for solutions to the mass-critical nonlinear Schrödinger equation $iu_t + \Delta u = \pm|u|^2u$ for large spherically symmetric $L_x^2(\mathbb{R}^2)$ initial data; in the focusing case we require, of course, that the mass is strictly less than that of the ground state. As a consequence, we deduce that in the focusing case, any spherically symmetric blowup solution must concentrate at least the mass of the ground state at the blowup time.

We also establish some partial results towards the analogous claims in other dimensions and without the assumption of spherical symmetry.

Contents

1. Introduction	1203
2. Notation and linear estimates	1217
3. Almost periodic solutions	1223
4. Three enemies	1228
5. The self-similar solution	1234
6. An in/out decomposition	1239
7. Additional regularity	1241
8. The double high-to-low frequency cascade	1246
9. Death of a soliton	1248
10. A concentration result	1250

1. Introduction

We primarily consider the Cauchy problem for the cubic nonlinear Schrödinger equation (NLS)

$$iu_t + \Delta u = \mu|u|^2u \tag{1}$$

in two space dimensions with L_x^2 initial data. Here $\mu = \pm 1$, with $\mu = +1$ known as the *defocusing* case and $\mu = -1$ as the *focusing* case. (Other nonzero values of μ can be reduced to these two cases by rescaling the values of u .)

R. Killip, T. Tao: Department of Mathematics, University of California, Los Angeles, 405 Hilgard Ave, Los Angeles, CA 90095, USA; e-mail: killip@math.ucla.edu, tao@math.ucla.edu

M. Visan: Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA; e-mail: mvisan@math.uchicago.edu

Mathematics Subject Classification (2000): 35Q55

The cubic nonlinearity is the most common nonlinearity in applications. It arises as a simplified model for studying Bose–Einstein condensates [22, 27, 51], Kerr media in nonlinear optics [33, 61], and even freak waves in the ocean [21, 29].

From a mathematical point of view, the cubic NLS in two dimensions is remarkable for being mass-critical.

1.1. *The mass-critical nonlinear Schrödinger equation*

In arbitrary dimensions, $d \geq 1$, the mass-critical (or *pseudoconformal*) nonlinear Schrödinger equation is given by

$$iu_t + \Delta u = F(u) \quad \text{with } F(u) := \mu|u|^{4/d}u. \tag{2}$$

The name is a testament to the fact that there is a scaling symmetry (see Definition 1.6) that leaves both the equation and the mass invariant. Mass is a term used in physics to represent the square of the L_x^2 -norm:

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t, x)|^2 dx.$$

For (2), this is a conserved quantity; see Theorem 1.4.

Our main result is to construct global strong $L_x^2(\mathbb{R}^d)$ solutions to (2) for spherically symmetric initial data in two space dimensions, $d = 2$. Many of our arguments continue to hold in greater generality, namely, for arbitrary dimensions $d \geq 1$ and without the assumption of spherical symmetry. We will discuss this more general problem whenever it does not disrupt the flow of our argument.

Let us first make the notion of a solution more precise:

Definition 1.1. *A function $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ on a nonempty time interval $I \subset \mathbb{R}$ (possibly infinite or semi-infinite) is a strong $L_x^2(\mathbb{R}^d)$ solution (or solution for short) to (2) if it lies in the class $C_t^0 L_x^2(K \times \mathbb{R}^d) \cap L_{t,x}^{2(d+2)/d}(K \times \mathbb{R}^d)$ for all compact $K \subset I$, and we have the Duhamel formula*

$$u(t_1) = e^{i(t_1-t_0)\Delta}u(t_0) - i \int_{t_0}^{t_1} e^{i(t_1-t)\Delta}F(u(t)) dt \tag{3}$$

for all $t_0, t_1 \in I$. Note that by Lemma 2.5, the condition $u \in L_{t,x}^{2(d+2)/d}$ implies that the second term in the Duhamel formula above exists as a weak integral in L_x^2 . Here, $e^{it\Delta}$ is the propagator for the free Schrödinger equation defined via the Fourier transform

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$$

by

$$\widehat{e^{it\Delta} f}(\xi) = e^{-it|\xi|^2} \hat{f}(\xi).$$

We refer to the interval I as the lifespan of u . We say that u is a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. We say that u is a global solution if $I = \mathbb{R}$.

Definition 1.2. We say that a solution u to (2) blows up forward in time if there exists a time $t_0 \in I$ such that

$$\int_{t_0}^{\sup I} \int_{\mathbb{R}^d} |u(t, x)|^{2(d+2)/d} dx dt = \infty$$

and that u blows up backward in time if there exists a time $t_0 \in I$ such that

$$\int_{\inf I}^{t_0} \int_{\mathbb{R}^d} |u(t, x)|^{2(d+2)/d} dx dt = \infty.$$

Remark. The condition that u is in $L_{t,x}^{2(d+2)/d}$ locally in time is natural. This is the space that appears in the original Strichartz inequality, [59]. As a consequence all solutions to the linear problem lie in this space. Existence of solutions to (2) in this space is guaranteed by the local theory discussed below; it is also necessary in order to ensure uniqueness of solutions in this local theory. Solutions to (2) in this class have been intensively studied (see for example [1, 5, 9, 10, 11, 37, 45, 67, 70, 71, 72]).

Next, we recall some basic facts from the local theory.

Definition 1.3. Let $u^{(n)} : I^{(n)} \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a sequence of solutions to (2), let $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ be another solution, and let K be a compact time interval. We say that $u^{(n)}$ converges uniformly to u on K if we have $K \subset I$ and $K \subset I^{(n)}$ for all sufficiently large n , and furthermore, $u^{(n)}$ converges strongly to u in $C_t^0 L_x^2(K \times \mathbb{R}^d) \cap L_{t,x}^{2(d+2)/d}(K \times \mathbb{R}^d)$ as $n \rightarrow \infty$. We say that $u^{(n)}$ converges locally uniformly to u if $u^{(n)}$ converges uniformly to u on every compact interval $K \subset I$.

The local theory for (2) was worked out by Cazenave and Weissler [10]. They constructed local-in-time solutions for arbitrary initial data in $L_x^2(\mathbb{R}^d)$; however, due to the critical nature of the equation, the resulting time of existence depends on the profile of the initial data and not merely on its L_x^2 norm. Cazenave and Weissler also constructed global solutions for small initial data. We summarize their results in the theorem below.

Theorem 1.4 (Local well-posedness, [10, 11]). *Given $u_0 \in L_x^2(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}$, there exists a unique maximal-lifespan solution u to (2) with $u(t_0) = u_0$. We will write I for the maximal lifespan. This solution also has the following properties:*

- (Local existence) I is an open neighbourhood of t_0 .
- (Mass conservation) The solution u has a conserved mass

$$M(u) = M(u(t)) := \int_{\mathbb{R}^d} |u(t, x)|^2 dx. \tag{4}$$

- (Blowup criterion) If $\sup(I)$ is finite, then u blows up forward in time; if $\inf(I)$ is finite, then u blows up backward in time.
- (Continuous dependence) If $u_0^{(n)}$ is a sequence converging to u_0 in $L_x^2(\mathbb{R}^d)$ and $u^{(n)} : I^{(n)} \times \mathbb{R}^d \rightarrow \mathbb{C}$ are the associated maximal-lifespan solutions, then $u^{(n)}$ converges locally uniformly to u .

- (Scattering) If $\sup(I) = +\infty$ and u does not blow up forward in time, then u scatters forward in time, that is, there exists a unique $u_+ \in L_x^2(\mathbb{R}^d)$ such that

$$\lim_{t \rightarrow +\infty} \|u(t) - e^{it\Delta} u_+\|_{L_x^2(\mathbb{R}^d)} = 0.$$

Similarly, if $\inf(I) = -\infty$ and u does not blow up backward in time, then u scatters backward in time, that is, there is a unique $u_- \in L_x^2(\mathbb{R}^d)$ so that

$$\lim_{t \rightarrow -\infty} \|u(t) - e^{it\Delta} u_-\|_{L_x^2(\mathbb{R}^d)} = 0.$$

- (Spherical symmetry) If u_0 is spherically symmetric, then u remains spherically symmetric for all time.
- (Small data global existence) If $M(u_0)$ is sufficiently small depending on d , then u is a global solution which does not blow up either forward or backward in time. Indeed, in this case

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t, x)|^{2(d+2)/d} dx dt \lesssim M(u).$$

A variant of the local well-posedness theorem above is the following

Lemma 1.5 (Stability, [69]). Fix μ and d . For every $A > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ is an approximate solution to (2) in the sense that

$$\|iu_t + \Delta u - F(u)\|_{L_{t,x}^{2(d+2)/(d+4)}(I \times \mathbb{R}^d)} \leq \delta$$

and also obeys

$$\|u\|_{L_{t,x}^{2(d+2)/d}(I \times \mathbb{R}^d)} \leq A,$$

and $t_0 \in I$ and $v_0 \in L_x^2(\mathbb{R}^d)$ are such that

$$\|u(t_0) - v_0\|_{L_x^2(\mathbb{R}^d)} \leq \delta,$$

then there exists a solution $v : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ to (2) with $v(t_0) = v_0$ such that

$$\|u - v\|_{L_{t,x}^{2(d+2)/d}(I \times \mathbb{R}^d)} \leq \varepsilon.$$

In particular, by the Strichartz inequality,

$$\|u - v\|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim \delta + \varepsilon(A + \varepsilon)^{4/d}.$$

Remark. This generalizes the continuous dependence statement of Theorem 1.4. It also implies the existence and uniqueness of maximal-lifespan solutions in Theorem 1.4. Analogous stability results for the energy-critical NLS (in $\dot{H}_x^1(\mathbb{R}^d)$ instead of $L_x^2(\mathbb{R}^d)$, of course) have appeared in [16, 35, 52, 68].

The class of solutions to (2) enjoys a large number of important mass-preserving symmetries. We first discuss the symmetries which fix the initial surface $t = 0$. We employ the notations from [70].

Definition 1.6 (Symmetry group). For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}^d$, frequency $\xi_0 \in \mathbb{R}^d$, and scaling parameter $\lambda > 0$, we define the unitary transformation $g_{\theta, x_0, \xi_0, \lambda} : L_x^2(\mathbb{R}^d) \rightarrow L_x^2(\mathbb{R}^d)$ by the formula

$$[g_{\theta, \xi_0, x_0, \lambda} f](x) := \frac{1}{\lambda^{d/2}} e^{i\theta} e^{ix \cdot \xi_0} f\left(\frac{x - x_0}{\lambda}\right).$$

We let G be the collection of such transformations. If $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$, we define $T_{g_{\theta, \xi_0, x_0, \lambda}} u : \lambda^2 I \times \mathbb{R}^d \rightarrow \mathbb{C}$ where $\lambda^2 I := \{\lambda^2 t : t \in I\}$ by the formula

$$[T_{g_{\theta, \xi_0, x_0, \lambda}} u](t, x) := \frac{1}{\lambda^{d/2}} e^{i\theta} e^{ix \cdot \xi_0} e^{-it|\xi_0|^2} u\left(\frac{t}{\lambda^2}, \frac{x - x_0 - 2\xi_0 t}{\lambda}\right),$$

or equivalently,

$$[T_{g_{\theta, \xi_0, x_0, \lambda}} u](t) = g_{\theta - t|\xi_0|^2, \xi_0, x_0 + 2\xi_0 t, \lambda} \left(u\left(\frac{t}{\lambda^2}\right) \right).$$

Note that if u is a solution to (2), then $T_g u$ is a solution to (2) with initial data gu_0 .

We also let $G_{\text{rad}} \subset G$ denote the collection of transformations in G which preserve spherical symmetry, or more explicitly,

$$G_{\text{rad}} := \{g_{\theta, 0, 0, \lambda} : \theta \in \mathbb{R}/2\pi\mathbb{Z}, \lambda > 0\}.$$

Remark. One easily verifies that G is a group with G_{rad} as a subgroup and that the map $g \mapsto T_g$ is a homomorphism. The map $u \mapsto T_g u$ maps solutions to (2) to solutions with the same mass and $L_{t,x}^{2(d+2)/d}$ norm as u , that is, $M(T_g u) = M(u)$ and

$$\|T_g u\|_{L_{t,x}^{2(d+2)/d}(\lambda^2 I \times \mathbb{R}^d)} = \|u\|_{L_{t,x}^{2(d+2)/d}(I \times \mathbb{R}^d)}.$$

Furthermore, u is a maximal-lifespan solution if and only if $T_g u$ is a maximal-lifespan solution.

Lemma 1.7 (Further symmetries of solutions). Let u be a solution to (2) with lifespan I .

- (Time reversal) The function

$$\tilde{u}(t, x) := \overline{u(-t, x)} \tag{5}$$

is a solution to (2) with lifespan $-I := \{-t : t \in I\}$ and mass $M(\tilde{u}) = M(u)$. If u is a maximal-lifespan solution, then so is \tilde{u} .

- (Time translation) For any $t_0 \in \mathbb{R}$, the function

$$u_{t_0}(t, x) := u(t + t_0, x) \tag{6}$$

is a solution to (2) with lifespan $I - t_0 := \{t - t_0 : t \in I\}$ and mass $M(u_{t_0}) = M(u)$. If u is a maximal-lifespan solution, then so is u_{t_0} .

- (Pseudoconformal transformation) If $0 \notin I$, then the function

$$v(t, x) := |t|^{-d/2} e^{i|x|^2/4t} u(-1/t, x/t) \tag{7}$$

is a solution to (2) with lifespan $-1/I := \{-1/t : t \in I\}$ and mass $M(v) = M(u)$.

Furthermore, u, \tilde{u}, u_{t_0} , and v have the same $L_{t,x}^{2(d+2)/d}$ norm on their respective lifespans, and if u is spherically symmetric then so are \tilde{u}, u_{t_0} , and v .

Proof. Direct computation. □

1.2. The scattering conjecture

From Theorem 1.4 we see that all maximal-lifespan solutions of sufficiently small mass are automatically global and do not blow up either forward or backward in time. But in the focusing case $\mu = -1$ it is well known that this assertion can fail for solutions of large mass. In particular, if we define the *ground state* to be the unique¹ positive radial Schwartz solution $Q : \mathbb{R}^d \rightarrow \mathbb{R}^+$ to the elliptic equation

$$\Delta Q + Q^{1+4/d} = Q, \tag{8}$$

then $u(t, x) := e^{it} Q(x)$ is a solution to (2). This shows that a solution of mass $M(Q)$ can blow up both forward and backward in time in the sense of Definition 1.2; moreover, by applying the pseudoconformal transformation (7), one obtains a Schwartz solution of mass $M(Q)$ that blows up in finite time.

It is however widely believed that this ground state example is the minimal mass obstruction to global well-posedness and scattering in the focusing case, and that no such obstruction exists in the defocusing case. More precisely, we have

Conjecture 1.8 (Global existence and scattering). *Let $d \geq 1$ and $\mu = \pm 1$. In the defocusing case $\mu = +1$, all maximal-lifespan solutions to (2) are global and do not blow up either forward or backward in time. In the focusing case $\mu = -1$, all maximal-lifespan solutions u to (2) with $M(u) < M(Q)$ are global and do not blow up either forward or backward in time.*

Remarks. 1. While this conjecture is phrased for $L_x^2(\mathbb{R}^d)$ solutions, it is equivalent to a scattering claim for smooth solutions; see [1, 8, 37, 67]. In [67] (and also in the earlier work [3]), it is also shown that the global existence and the scattering claims are equivalent in the $L_x^2(\mathbb{R}^d)$ category.

2. Let us reiterate that blowup refers to infinite spacetime norm. As noted in Theorem 1.4, finiteness of the $L_{t,x}^4$ Strichartz norm implies scattering. By Lemma 1.5, it also implies (quantitative) continuous dependence upon initial data and stability under external forcing. To this one may add stability of well-posedness under perturbations of the equation; see [69].

¹ The existence and uniqueness of Q was established in [2] and [38] respectively.

3. By Theorem 1.4, the conjecture is valid when the solution u has sufficiently small mass; the interesting questions are when the mass is very large in the defocusing case $\mu = +1$ or close to the mass $M(Q)$ of the ground state in the focusing case $\mu = -1$.

Conjecture 1.8 has been the focus of much intensive study and several partial results for various choices of d, μ , and sometimes with the additional assumption of spherical symmetry. The most compelling evidence in favour of this conjecture stems from results obtained under the assumption that u_0 has additional regularity. For the defocusing equation, it is easy to prove global well-posedness for initial data in H_x^1 ; this follows from the usual subcritical argument combined with the conservation of mass and energy; see, for example, [11]. Recall that the energy is given by

$$E(u(t)) := \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u(t, x)|^2 + \mu \frac{d}{2(d+2)} |u(t, x)|^{2(d+2)/d} \right) dx. \tag{9}$$

Note that for general L_x^2 initial data, the energy need not be finite.

The focusing equation with data in H_x^1 was treated by Weinstein [77]. A key ingredient was his proof of the sharp Gagliardo–Nirenberg inequality:

Theorem 1.9 (Sharp Gagliardo–Nirenberg, [77]).

$$\int_{\mathbb{R}^d} |f(x)|^{2(d+2)/d} dx \leq \frac{d+2}{d} \left(\frac{\|f\|_{L^2}^2}{\|Q\|_{L^2}^2} \right)^{2/d} \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx. \tag{10}$$

As noticed by Weinstein, this inequality implies that the energy (9) is positive once $M(u_0) < M(Q)$; indeed, it gives an upper bound on the \dot{H}_x^1 norm of the solution at all times of existence. Combining this with a contraction mapping argument and the conservation of mass and energy, Weinstein proved global well-posedness for the focusing equation with initial data in H_x^1 and mass smaller than that of the ground state.

Note that the iterative procedure used to obtain a global solution both for the defocusing and the focusing equations with initial data in H_x^1 does not yield finite spacetime norms; in particular, scattering does not follow even for more regular initial data.

There has been much work [5, 14, 15, 17, 18, 19, 20, 26, 73, 76] devoted to lowering the regularity of the initial data from H_x^1 toward $L_x^2(\mathbb{R}^d)$ and thus, toward establishing the conjecture. For instance, in the defocusing case $\mu = +1, d = 2$, the best known result is in [14], where global well-posedness is established for $u_0 \in H_x^s(\mathbb{R}^2)$ whenever $s > 2/5$. In the focusing case $\mu = -1, d = 2$, with $M(u_0) < M(Q)$, the best result is in [17], which achieved the same claim for $s > 1/2$. In [3] it was shown that such global well-posedness results in the $H_x^s(\mathbb{R}^d)$ class lead (via the pseudoconformal transform (7)) to global existence and scattering results in the weighted space $L_x^2(\mathbb{R}^d; \langle x \rangle^{2s} dx)$; thus, for instance, when $\mu = +1, d = 2$, one has global existence and scattering whenever $u_0 \in L_x^2(\mathbb{R}^d; \langle x \rangle^{2s} dx)$ for some $s > 2/5$. (Scattering results of this type were first obtained in [72].)

For $d = 1$ or $d = 2$, Conjecture 1.8 would imply (by standard persistence of regularity theory) that for any $s \geq 0$ and for u as in the conjecture, the $H_x^s(\mathbb{R}^d)$ norm of the solution

$u(t)$ at an arbitrary time t would be bounded by the $H_x^s(\mathbb{R}^d)$ norm of $u(t_0)$ for any fixed t_0 , times a quantity depending only on $M(u)$ and the dimension d , that is,

$$\|u(t)\|_{H_x^s} \leq C(M(u), d) \|u(t_0)\|_{H_x^s}.$$

In particular, these norms would be bounded uniformly in t . In this direction, some polynomial upper bounds on the growth in time of $H_x^s(\mathbb{R}^d)$ norms in the case $d = 2$ were established in [4, 13, 54, 55].

In a slightly different direction, it was shown by Nakanishi [48] that one has global existence and scattering for H_x^1 initial data in the defocusing case $\mu = +1$, $d = 2$, whenever the mass-critical nonlinearity $|u|^2u$ is increased in strength to the mass-supercritical (but energy-subcritical) nonlinearity $|u|^{2+\varepsilon}u$ for some $\varepsilon > 0$.

In the case of spherically symmetric solutions, the conjecture was recently settled in the high-dimensional defocusing case $\mu = +1$, $d \geq 3$ in [71]. The focusing case and the non-spherically-symmetric case remain open in all higher dimensions. Both [71] and the current paper build on techniques developed in order to treat the energy-critical NLS; see [6, 16, 35, 52, 64, 74, 75]. We will better explain our debt to this work when we outline our argument. For the energy-critical problem, the analogue of Conjecture 1.8 is mostly settled, with the only currently outstanding problem being the focusing case $\mu = -1$ with non-spherically-symmetric data.

1.3. Main result

Our main result settles the scattering conjecture in the spherically symmetric case in two dimensions:

Theorem 1.10. *Let $d = 2$. Then Conjecture 1.8 is true (for either choice of sign μ) whenever u is spherically symmetric.*

In particular, in the defocusing case one now has global well-posedness and scattering in the class $L_x^2(\mathbb{R}^2)_{\text{rad}}$ of spherically symmetric $L_x^2(\mathbb{R}^2)$ functions for arbitrarily large mass, while in the focusing case one has a similar claim under the additional assumption $M(u) < M(Q)$.

Neither Theorem 1.10 nor Conjecture 1.8 address the focusing problem for masses greater than or equal to that of the ground state. In this case, blowup solutions exist and attention has been focused on describing their properties. Finite-time blowup solutions with finite energy and mass equal to that of the ground state have been completely characterized by Merle [39]; they are precisely the ground state solution up to symmetries of the equation.

Several works have shown that finite-time blowup solutions must concentrate a positive amount of mass around the blowup time T^* . For finite energy data, Merle and Tsutsumi [44] (for radial data) and Nawa [49] and Weinstein [78] (for general data) proved the following: there exists $x(t) \in \mathbb{R}^d$ such that

$$\liminf_{t \nearrow T^*} \int_{|x-x(t)| \leq R} |u(t, x)|^2 dx \geq M(Q)$$

for any $R > 0$. For merely L^2 initial data, Bourgain [5] proved that some small amount of mass must concentrate in parabolic windows (at least along a subsequence):

$$\limsup_{t \nearrow T^*} \sup_{x_0 \in \mathbb{R}^2} \int_{|x-x_0| \leq (T^*-t)^{1/2}} |u(t, x)|^2 dx \geq c,$$

where c is a small constant depending on the mass of u . This result was extended to other dimensions in [1, 37]. Note that by a construction of Perelman [50], the parabolic window cannot be made smaller by more than the square root of a double-logarithmic factor. Our contribution to this line of investigation is Corollary 1.12 below.

In a series of papers [41, 42, 43], Merle and Raphael gave a more or less complete description of blowup behaviour at masses slightly above that of the ground state (and finite energy). In particular, they show that the blowup rate observed by Perelman is generic.

Keraani [37, Theorem 1.12] also describes blowup behaviour for masses close to the critical mass. (See also [45, Theorem 3] for a closely related result.) In light of Theorem 1.10 (and using the pseudoconformal transformation), his result implies

Corollary 1.11 (Small blowup solutions concentrate mass $M(Q)$). *Let u be a spherically symmetric solution to (1) with $M(u) < 2M(Q)$ that blows up forward in time. If the blowup time T^* is finite, then*

$$\liminf_{t \rightarrow T^*} \int_{|x| \leq R(t)} |u(t, x)|^2 dx \geq M(Q)$$

for any function $R(t)$ obeying $(T^* - t)^{-1/2}R(t) \rightarrow \infty$ as $t \rightarrow T^*$. If $T^* = \infty$, then

$$\liminf_{t \rightarrow \infty} \int_{|x| \leq R(t)} |u(t, x)|^2 dx \geq M(Q)$$

for any function $R(t)$ obeying $t^{-1/2}R(t) \rightarrow \infty$ as $t \rightarrow \infty$. A similar statement holds in the negative time direction.

As discussed in [45, 37], the main obstacle to treating general masses is the spectre of quadratic oscillation. We will (partially) circumvent this problem using some ideas from [5]. The price we must pay is to replace \liminf by \limsup , just as in [5].

Corollary 1.12 (Blowup solutions concentrate the mass of the ground state). *Let u be a spherically symmetric solution to (1) that blows up at time $0 < T^* \leq \infty$. If $T^* < \infty$, then there exists a sequence $t_n \nearrow T^*$ such that for any sequence $R_n \in (0, \infty)$ obeying $(T^* - t_n)^{-1/2}R_n \rightarrow \infty$,*

$$\limsup_{n \rightarrow \infty} \int_{|x| \leq R_n} |u(t_n, x)|^2 dx \geq M(Q). \tag{11}$$

If $T^* = \infty$, then there exists a sequence $t_n \rightarrow \infty$ such that for any sequence $R_n \in (0, \infty)$ with $t_n^{-1/2}R_n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \int_{|x| \leq R_n} |u(t_n, x)|^2 dx \geq M(Q). \tag{12}$$

The analogous statement holds in the negative time direction.

Remark. The argument used to deduce Corollary 1.12 works in any dimension and without the assumption of spherical symmetry; it just relies on an affirmative answer to Conjecture 1.8. However, as this remains open in such generality, we present the argument in context of Theorem 1.10.

The arguments in this paper rely heavily on the spherical symmetry in many places, for example, in order to localise the solution at the spatial origin $x = 0$ and frequency origin $\xi = 0$, and also to provide some strong spatial decay as $|x| \rightarrow \infty$. We do not see how to remove this assumption, but it seems that if one wishes to do so, one should first look at the higher-dimensional defocusing case $d \geq 3$, $\mu = +1$, in which the numerology is more favourable (in particular, the dispersive inequality becomes more powerful) and one has additional tools such as Morawetz inequalities available.

1.4. Reduction to almost periodic solutions

Bourgain's seminal work [6] on the energy-critical NLS first realized the important role played by solutions that are simultaneously localised in both frequency and space. The sufficiency of treating such solutions stems from his 'induction on energy' argument. These ideas were pursued further in the work that followed, for example, [16, 52, 64, 74, 75].

A new and much more efficient approach to the energy-critical NLS (albeit nonquantitative) was introduced by Kenig and Merle [35], which employed a (concentration) compactness technique in place of the earlier laborious (quantitative) localisation arguments. Earlier steps in this direction include [1, 5, 36, 37, 45].

In the Schrödinger context, the role of concentration compactness is to provide a linear profile decomposition. The main technical ingredients from the quantitative argument (for example, refined Strichartz estimates) appear in the proof of such a decomposition theorem. The two-dimensional mass-critical version of the linear profile decomposition theorem reads as follows:

Theorem 1.13 (Linear profiles, [45]). *Let u_n , $n = 1, 2, \dots$, be a bounded sequence in $L_x^2(\mathbb{R}^2)$. Then (after passing to a subsequence if necessary) there exists a sequence of functions $\phi^j \in L_x^2(\mathbb{R}^2)$, group elements $g_n^j \in G$, and $t_n^j \in \mathbb{R}$ such that we have the decomposition*

$$u_n = \sum_{j=1}^J g_n^j e^{it_n^j \Delta} \phi^j + w_n^J \quad (13)$$

for all $J = 1, 2, \dots$; here, $w_n^J \in L_x^2(\mathbb{R}^2)$ is such that its linear evolution has asymptotically vanishing scattering size:

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|e^{it\Delta} w_n^J\|_{L_{t,x}^4} = 0. \quad (14)$$

Moreover, for any $j \neq j'$,

$$\frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \lambda_n^j \lambda_n^{j'} |\xi_n^j - \xi_n^{j'}|^2 + \frac{|x_n^j - x_n^{j'}|^2}{\lambda_n^j \lambda_n^{j'}} + \frac{|t_n^j (\lambda_n^j)^2 - t_n^{j'} (\lambda_n^{j'})^2|}{\lambda_n^j \lambda_n^{j'}} \rightarrow \infty. \tag{15}$$

Furthermore, for any $J \geq 1$ we have the mass decoupling property

$$\lim_{n \rightarrow \infty} \left[M(u_n) - \sum_{j=1}^J M(\phi^j) - M(w_n^J) \right] = 0. \tag{16}$$

Lastly, if u_n are assumed to be spherically symmetric and $d \geq 2$, one can choose ϕ_j and w_n^j to be spherically symmetric and $g_n^j \in G_{\text{rad}}$.

Remark. Note that spherical symmetry is only a significant additional hypothesis when $d \geq 2$. When $d = 1$, any nonsymmetric solution u can be used to generate a spherically symmetric solution of approximately twice the mass by considering the near-solution $u(t, x - x_0) + u(t, x + x_0)$ for some very large x_0 and using perturbation theory such as Lemma 1.5 to convert this to an actual solution; we leave the details to the reader. Because of this, we do not expect the spherically symmetric case to be significantly different from the nonsymmetric case in one dimension.

The concentration compactness approach reduces matters to the study of almost periodic solutions (modulo symmetries). This reduction does not depend upon the dimension, nor does it require the assumption of spherical symmetry. Indeed, in his recent lectures [34], Kenig stressed that these ideas should be applicable to virtually any dispersive equation with a good local theory; how (and indeed whether) this helps in settling global well-posedness questions is very much equation dependent.

Definition 1.14. Let $d \geq 1$ and $\mu = \pm 1$. A solution u with lifespan I is said to be almost periodic modulo G if there exist (possibly discontinuous) functions $N : I \rightarrow \mathbb{R}^+$, $\xi : I \rightarrow \mathbb{R}^d$, $x : I \rightarrow \mathbb{R}^d$ and a function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\int_{|x-x(t)| \geq C(\eta)/N(t)} |u(t, x)|^2 dx \leq \eta$$

and

$$\int_{|\xi-\xi(t)| \geq C(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi \leq \eta$$

for all $t \in I$ and $\eta > 0$. We refer to the function N as the frequency scale function for the solution u , ξ as the frequency centre function, x as the spatial centre function, and C as the compactness modulus function. Furthermore, if we can select $x(t) = \xi(t) = 0$, then we say that u is almost periodic modulo G_{rad} .

Remarks. 1. The parameter $N(t)$ measures the frequency scale of the solution at time t , and $1/N(t)$ measures the spatial scale; see [70, 71] for further discussion. Note that we

have the freedom to modify $N(t)$ by any bounded function of t , provided that we also modify the compactness modulus function C accordingly. In particular, one could restrict $N(t)$ to be a power of 2 if one wished, although we will not do so here. Alternatively, the fact that the solution trajectory $t \mapsto u(t)$ is continuous in $L_x^2(\mathbb{R}^d)$ can be used to show that the functions N, ξ, x may be chosen to depend continuously on t .

2. One can view $\xi(t)$ and $x(t)$ as roughly measuring the (normalised) momentum and centre-of-mass, respectively, at time t , although as u is only assumed to lie in $L_x^2(\mathbb{R}^d)$, these latter quantities are not quite rigorously defined.

3. By the Ascoli–Arzelà Theorem, a family of functions is precompact in $L_x^2(\mathbb{R}^d)$ if and only if it is norm-bounded and there exists a compactness modulus function C such that

$$\int_{|x| \geq C(\eta)} |f(x)|^2 dx + \int_{|\xi| \geq C(\eta)} |\hat{f}(\xi)|^2 d\xi \leq \eta$$

for all functions f in the family. Thus, an equivalent formulation of Definition 1.14 is as follows: u is almost periodic modulo G (respectively modulo G_{rad}) if and only if there exists a compact subset K of $L_x^2(\mathbb{R}^d)$ such that the orbit $\{u(t) : t \in I\}$ is contained inside $GK := \{gf : g \in G, f \in K\}$ (respectively $G_{\text{rad}}K$). This may help explain the terminology ‘almost periodic’.

In [70] the following result was established (see also [1, 37]), showing that any failure of Conjecture 1.8 must be ‘caused’ by a very special type of solution.

Theorem 1.15 (Reduction to almost periodic solutions). *Fix μ and d and suppose that Conjecture 1.8 failed for this choice of μ and d . Then there exists a maximal-lifespan solution u which is almost periodic modulo G and which blows up both forward and backward in time, and in the focusing case we also have $M(u) < M(Q)$. If furthermore $d \geq 2$ and Conjecture 1.8 failed for spherically symmetric data, then we can also ensure that u is spherically symmetric and almost periodic modulo G_{rad} .*

Proof. See [70, Theorem 1.13] and [70, Theorem 7.2]. □

Remark. Theorem 1.15 was the first step in the resolution of Conjecture 1.8 in the case $d \geq 3, \mu = +1$ with spherical symmetry in [71], just as the analogous statement for the energy-critical NLS was the first step in [35].

1.5. Outline of the proof

Under the assumption that Conjecture 1.8 fails, Theorem 1.15 constructs an almost periodic solution u with some frequency scale function $N(t)$, but provides little information about the behaviour of $N(t)$ over the lifespan I of the solution. In this paper we refine Theorem 1.15 by showing that the failure of Conjecture 1.8 implies the existence of one of three types of almost periodic solutions u for which $N(t)$ and I have very particular properties. The argument is independent of the dimension, the sign of μ , and makes no assumption of spherical symmetry. Indeed, the argument is predominantly combinatorial and, as with the reduction to almost periodic solutions, should be applicable whenever there is a satisfactory local theory.

Theorem 1.16 (Three special scenarios for blowup). *Fix μ and d and suppose that Conjecture 1.8 fails for this choice of μ and d . Then there exists a maximal-lifespan solution u which is almost periodic modulo G , blows up both forward and backward in time, and in the focusing case also obeys $M(u) < M(Q)$. If furthermore $d \geq 2$ and Conjecture 1.8 fails for spherically symmetric data, then u may be chosen to be spherically symmetric and almost periodic modulo G_{rad} .*

With or without spherical symmetry, we can also ensure that the lifespan I and the frequency scale function $N : I \rightarrow \mathbb{R}^+$ match one of the following three scenarios:

I. (Soliton-like solution) *We have $I = \mathbb{R}$ and*

$$N(t) = 1 \quad \text{for all } t \in \mathbb{R} \tag{17}$$

(thus the solution stays in a bounded space/frequency range for all time).

II. (Double high-to-low frequency cascade) *We have $I = \mathbb{R}$,*

$$\liminf_{t \rightarrow -\infty} N(t) = \liminf_{t \rightarrow +\infty} N(t) = 0, \tag{18}$$

and

$$\sup_{t \in \mathbb{R}} N(t) < \infty \quad \text{for all } t \in I. \tag{19}$$

III. (Self-similar solution) *We have $I = (0, +\infty)$ and*

$$N(t) = t^{-1/2} \quad \text{for all } t \in I. \tag{20}$$

Remark. Elementary scaling arguments show that one may assume that $N(t)$ is either bounded from above or from below at least on half of its maximal lifespan. This observation was used in [71, Theorem 3.3] and also in [35, 40]. However, the proof of Theorem 1.10 seems to require the finer control given by Theorem 1.16 on the nature of the blowup as one approaches either endpoint of the interval I .

The proof of Theorem 1.16 can be found in Section 4; it uses the basic properties of almost periodic solutions developed in Section 3.

Thus, by the end of Section 4 we have isolated our three main enemies (i.e. Scenarios I, II, and III); the remainder of the paper is devoted to defeating them. Our arguments to achieve this are very much specific to the equation and depend heavily on the radial assumption. In all three scenarios, the key step is to prove that u has additional regularity, indeed, more than one derivative in L_x^2 . In this regard, our approach differs somewhat from the strategy in [6, 16, 52, 64, 71, 74, 75], which primarily sought to use spacetime estimates (especially those arising from Morawetz inequalities) to exclude blowup scenarios; control of regularity (that is, decay at very high or very low frequencies) only appeared later in the argument. But in two dimensions, the Morawetz inequalities are unfavourable and we have to rely on the virial inequality instead (see Proposition 9.2), which does not provide good long-term spacetime control. Thus, we need to establish regularity even in the absence of such spacetime estimates. In this regard, our approach is more similar to papers such as [35, 49, 63, 66]. One reward for making do with the

virial identity is that it opens up the possibility of treating the focusing problem, as was conclusively demonstrated by Kenig and Merle [35].

At first glance, additional regularity may seem unreasonable since u is *a priori* only known to have finite mass and dispersive equations such as (2) do not exhibit global smoothing properties. However, the solutions appearing in Theorem 1.16 are very special in nature, for example, they are almost periodic modulo symmetries. The possibility of proving such regularity becomes still more plausible when one recalls how Theorem 1.15 is proved. The solution u is selected to have minimal mass among all blowup solutions; this means that there can be no waste. Adding high-frequency ripples that do not directly contribute to blowup would constitute an example of such waste. (The same rationale explains why u must concentrate in a single bubble both in space and in frequency.) A further manifestation of this ‘eco-friendly’ (that is, no waste) property is the absence of a scattered wave at the endpoints of the lifespan I ; more formally, we have the following Duhamel formula, which plays an important role in proving additional regularity.

Lemma 1.17 ([70, Section 6]). *Let u be an almost periodic solution to (2) on its maximal-lifespan I . Then, for all $t \in I$,*

$$u(t) = \lim_{T \nearrow \sup I} i \int_t^T e^{i(t-t')\Delta} F(u(t')) dt' = - \lim_{T \searrow \inf I} i \int_T^t e^{i(t-t')\Delta} F(u(t')) dt', \quad (21)$$

as weak limits in L_x^2 .

Additional regularity for the self-similar solution is proved in Section 5. The argument relies on iterating various versions of the Strichartz inequality (including a recent refinement of that inequality in the spherically symmetric case due to Shao [53]), taking full advantage of the self-similarity to control the motion of mass between frequencies. In the latter part of Section 5 we disprove the existence of the self-similar solution by noting that H_x^1 solutions are global (see the discussion after Conjecture 1.8), while the self-similar solution is not.

For the remaining two cases, higher regularity is proved in Section 7. The idea is to exploit the global existence, together with the almost periodicity modulo scaling, by applying the Duhamel formula (21) both in the past and in the future. If done naively, neither of these Duhamel integrals will be absolutely convergent. However, in Section 6 we describe the decomposition into incoming waves (which we propagate backwards in time) and outgoing waves (which we propagate forward in time). We can then take advantage of the radial symmetry, which concentrates the solution near the origin. In this way, we obtain convergent integrals and regularity will then be obtained by a simple iteration argument.

In Section 8, we use the additional regularity together with the conservation of energy to preclude the double high-to-low frequency cascade. In Section 9, we disprove the existence of soliton-like solutions using a truncated virial identity in much the same manner as [35].

Corollary 1.12 is proved in Section 10.

2. Notation and linear estimates

This section contains the basic linear estimates we use repeatedly in the paper.

2.1. Some notation

We use $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leq CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$. We write $X \sim Y$ whenever $X \lesssim Y \lesssim X$. The fact that these constants depend upon the dimension d will be suppressed. If C depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_u Y$ denotes the assertion that $X \leq C_u Y$ for some C_u depending on u ; similarly for $X \sim_u Y, X = O_u(Y)$, etc.

We use the ‘Japanese bracket’ convention $\langle x \rangle := (1 + |x|^2)^{1/2}$.

We use $L_t^q L_x^r$ to denote the Banach space with norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u(t, x)|^r dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R} \times \mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I \times \mathbb{R}^d$. When $q = r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$.

2.2. Basic harmonic analysis

Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 11/10\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For each number $N > 0$, we define the Fourier multipliers

$$\begin{aligned} \widehat{P_{\leq N} f}(\xi) &:= \varphi(\xi/N) \widehat{f}(\xi), & \widehat{P_{>N} f}(\xi) &:= (1 - \varphi(\xi/N)) \widehat{f}(\xi), \\ \widehat{P_N f}(\xi) &:= \psi(\xi/N) \widehat{f}(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N)) \widehat{f}(\xi). \end{aligned}$$

We similarly define $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M < \cdot \leq N} := P_{\leq N} - P_{\leq M} = \sum_{M < N' \leq N} P_{N'}$$

whenever $M < N$. We will normally use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N to not be a power of 2. Note that P_N is not truly a projection; to get around this, we will occasionally need to use fattened Littlewood–Paley operators:

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}. \tag{22}$$

These obey $P_N \tilde{P}_N = \tilde{P}_N P_N = P_N$.

As with all Fourier multipliers, the Littlewood–Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many times, including

Lemma 2.1 (Bernstein estimates). *For $1 \leq p \leq q \leq \infty$,*

$$\begin{aligned} \|P_{\leq N} f\|_{L_x^q(\mathbb{R}^d)} &\lesssim N^{d/p-d/q} \|P_{\leq N} f\|_{L_x^p(\mathbb{R}^d)}, \\ \|P_N f\|_{L_x^q(\mathbb{R}^d)} &\lesssim N^{d/p-d/q} \|P_N f\|_{L_x^p(\mathbb{R}^d)}. \end{aligned}$$

One of the ways in which we exploit the spherical symmetry assumption in this paper is by using the stronger spatial decay as $|x| \rightarrow \infty$ that spherically symmetric functions enjoy. One expression of this is the following ‘endpoint’ radial Sobolev embedding estimate.

Lemma 2.2 (Radial Sobolev embedding). *For spherically symmetric $f : \mathbb{R}^d \rightarrow \mathbb{C}$,*

$$\| |x|^{(d-1)/2} f_N \|_{L_x^\infty} \lesssim N^{1/2} \|f_N\|_{L_x^2} \tag{23}$$

for any frequency $N \in 2^{\mathbb{Z}}$.

Proof. The claim follows from a more general inequality obeyed by radial functions,

$$\| |x|^{(d-1)/2} f \|_{L_x^\infty}^2 \lesssim \|f\|_{L_x^2} \|\nabla f\|_{L_x^2}. \tag{24}$$

This in turn, can be deduced as follows. By the Fundamental Theorem of Calculus and Hölder’s inequality,

$$\begin{aligned} |g(r)|^2 &= \left| \int_r^\infty g(\rho)g'(\rho) d\rho \right| \leq r^{1-d} \left| \int_r^\infty g(\rho)g'(\rho)\rho^{d-1} d\rho \right| \\ &\leq r^{1-d} \|g\|_{L^2(\rho^{d-1} d\rho)} \|g'\|_{L^2(\rho^{d-1} d\rho)} \end{aligned}$$

for any Schwartz function $g : [0, \infty) \rightarrow \mathbb{C}$. To obtain (24), we apply this to $g(|x|) = f(x)$. □

In a series of papers culminating in [12], Christ and Kiselev proved that under certain hypotheses, truncations of bounded integral operators are themselves bounded. We will only use the simplest form of their results (the proof is based on a Whitney decomposition of the half space $t < s$):

Lemma 2.3 (Christ–Kiselev, [12]). *For each $s, t \in \mathbb{R}$, let $T(t, s) : L_x^r \rightarrow L_x^{\tilde{r}}$ denote a linear operator. If*

$$\left\| \int_{\mathbb{R}} T(t, s)F(s) ds \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^p L_x^{\tilde{r}}}$$

for some $q > p$, then

$$\left\| \int_{-\infty}^t T(t, s)F(s) ds \right\|_{L_t^q L_x^r} \lesssim_{p,q} \|F\|_{L_t^p L_x^{\tilde{r}}}.$$

This lemma is handy because the Duhamel formula (3) comes with the constraint $s < t$, but it is much easier to prove estimates after discarding it.

2.3. Strichartz estimates

Naturally, everything that we do for the nonlinear Schrödinger equation builds on basic properties of the linear propagator $e^{it\Delta}$.

From the explicit formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{-i|x-y|^2/4t} f(y) dy,$$

we deduce the standard dispersive inequality

$$\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L^1(\mathbb{R}^d)} \quad \text{for all } t \neq 0. \tag{25}$$

Finer bounds on the (frequency localised) linear propagator can be derived using stationary phase:

Lemma 2.4 (Kernel estimates). *For any $m \geq 0$, the kernel of the linear propagator obeys the following estimates:*

$$|(P_N e^{it\Delta})(x, y)| \lesssim_m \begin{cases} |t|^{-d/2}, & |x - y| \sim N|t|, \\ \frac{N^d}{|N^2 t|^m \langle N|x - y| \rangle^m}, & \text{otherwise,} \end{cases} \tag{26}$$

for $|t| \geq N^{-2}$, and

$$|(P_N e^{it\Delta})(x, y)| \lesssim_m N^d \langle N|x - y| \rangle^{-m} \tag{27}$$

for $|t| \leq N^{-2}$.

We also record the following standard Strichartz estimate:

Lemma 2.5 (Strichartz). *Let I be an interval, let $t_0 \in I$, and let $u_0 \in L_x^2(\mathbb{R}^d)$ and $f \in L_{t,x}^{2(d+2)/(d+4)}(I \times \mathbb{R}^d)$. Then the function u defined by*

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^t e^{i(t-t')\Delta} f(t') dt'$$

obeys the estimate

$$\|u\|_{C_t^0 L_x^2(I \times \mathbb{R}^d)} + \|u\|_{L_{t,x}^{2(d+2)/d}(I \times \mathbb{R}^d)} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^d)} + \|f\|_{L_{t,x}^{2(d+2)/(d+4)}(I \times \mathbb{R}^d)}.$$

Moreover, if $d = 2$ and both u and f are spherically symmetric,

$$\|u\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^2)} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^2)} + \|f\|_{L_{t,x}^{4/3}(I \times \mathbb{R}^2)}.$$

Proof. See, for example, [25, 59] for the first inequality and [56, 62] for the endpoint case in dimension $d = 2$. In [46] it is shown that the $L_t^2 L_x^\infty$ endpoint estimate fails without the assumption of spherical symmetry. \square

Remark. There are of course more Strichartz estimates available than the ones listed here, but these (together with their adjoints) are the only ones we will need. By the Duhamel formula (3), one can of course write $u(t_0)$ in place of u_0 and $iu_t + \Delta u$ in place of f .

We will also need three variants of the Strichartz inequality. We will only record the two-dimensional case of these. The first is a bilinear variant, which will be useful for controlling interactions between widely separated frequencies:

Lemma 2.6 (Bilinear Strichartz). *For any spacetime slab $I \times \mathbb{R}^2$, any $t_0 \in I$, and any $M, N > 0$, we have*

$$\begin{aligned} \|(P_{\geq N}u)(P_{\leq M}v)\|_{L^2_{t,x}(I \times \mathbb{R}^2)} &\lesssim (M/N)^{1/2}(\|P_{\geq N}u(t_0)\|_{L^2} + \|(i\partial_t + \Delta)P_{\geq N}u\|_{L^{4/3}_{t,x}(I \times \mathbb{R}^2)}) \\ &\quad \times (\|P_{\leq M}v(t_0)\|_{L^2} + \|(i\partial_t + \Delta)P_{\leq M}v\|_{L^{4/3}_{t,x}(I \times \mathbb{R}^2)}) \end{aligned}$$

for all functions u, v on I .

Proof. See [75, Lemma 2.5], which builds on earlier versions in [7, 16]. More general estimates hold for other dimensions and other exponents, but we will not need those here. □

Next, we observe a weighted endpoint Strichartz estimate, which exploits the spherical symmetry heavily in order to obtain spatial decay. It is very useful in regions of space far from the origin $x = 0$.

Lemma 2.7 (Weighted endpoint Strichartz). *Let I be an interval, let $t_0 \in I$, and let $u_0 \in L^2_x(\mathbb{R}^2)$ and $f \in L^{4/3}_{t,x}(I \times \mathbb{R}^2)$ be spherically symmetric. Then the function u defined by*

$$u(t) := e^{i(t-t_0)\Delta}u_0 - i \int_{t_0}^t e^{i(t-t')\Delta} f(t') dt'$$

obeys the estimate

$$\||x|^{1/2}u\|_{L^4_t L^\infty_x(I \times \mathbb{R}^2)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^2)} + \|f\|_{L^{4/3}_{t,x}(I \times \mathbb{R}^2)}.$$

Proof. As in the usual proof of Strichartz inequality, the main ingredients are a dispersive estimate and fractional integration.

Using Lemma 2.5 and the Christ–Kiselev lemma, we see that it suffices to prove the free estimate

$$\||x|^{1/2}e^{it\Delta}u_0\|_{L^4_t L^\infty_x(\mathbb{R} \times \mathbb{R}^2)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^2)} \tag{28}$$

for spherically symmetric u_0 . By the TT^* method, this is equivalent to

$$\left\| \int_{\mathbb{R}} |x|^{1/2} e^{i(t-t')\Delta} |x|^{1/2} f(t') dt' \right\|_{L^4_t L^\infty_x(\mathbb{R} \times \mathbb{R}^2)} \lesssim \|f\|_{L^{4/3}_t L^1_x(\mathbb{R} \times \mathbb{R}^2)}.$$

By the Hardy–Littlewood–Sobolev theorem of fractional integration, it thus suffices to prove the following weighted dispersive inequality:

$$\||x|^{1/2}e^{i(t-t')\Delta}|x|^{1/2}f(t')\|_{L^\infty_x(\mathbb{R}^2)} \lesssim |t-t'|^{-1/2}\|f(t')\|_{L^1_x(\mathbb{R}^2)}.$$

In order to achieve this, we note that for radial functions the propagator has integral kernel

$$e^{it\Delta}(x, y) = \frac{1}{4\pi it} \int_0^{2\pi} \exp\left\{ \frac{i|x|^2 + i|y|^2 - 2i|x||y|\cos(\theta)}{4t} \right\} \frac{d\theta}{2\pi}.$$

Thus, a standard application of stationary phase (or the behaviour of the J_0 Bessel function) reveals that $|e^{it\Delta}(x, y)| \lesssim (|x||y||t|)^{-1/2}$. In particular,

$$\| |x|^{1/2} e^{it\Delta} |x|^{1/2} \|_{L_x^1 \rightarrow L_x^\infty} \lesssim |t|^{-1/2},$$

and the claim follows. □

We will rely crucially on a slightly different type of improvement to the Strichartz inequality in the spherically symmetric case due to Shao [53], which improves the space-time decay of the solution after localising in frequency:

Lemma 2.8 (Shao’s Strichartz estimate, [53, Corollary 6.2]). *For $f \in L_{\text{rad}}^2(\mathbb{R}^2)$ we have*

$$\| P_N e^{it\Delta} f \|_{L_{t,x}^q(\mathbb{R} \times \mathbb{R}^2)} \lesssim_q N^{1-4/q} \| f \|_{L_x^2(\mathbb{R}^2)}, \tag{29}$$

provided $q > 10/3$.

The key point for us is that q can go below 4, which is the exponent given by Lemma 2.5. The Knapp counterexample (a wave-packet whose momentum is concentrated in a single direction) shows that such an improvement is not possible without the radial assumption. One can combine this estimate with the usual Strichartz estimates by the standard Christ–Kiselev trick to obtain similar estimates for solutions to an inhomogeneous Schrödinger equation, but we will not need those estimates here.

Our last lemma is essentially Bourgain’s concentration argument from [5, §2–3]. It may be interpreted as an inverse Strichartz inequality. It roughly says that the Strichartz norm cannot be large without there being a bubble of concentration in spacetime. While results of this genre constitute an important precursor to the concentration compactness technique, our only application of it will be in the proof of Corollary 1.12; see Section 10.

Lemma 2.9. *Given $\phi \in L_x^2(\mathbb{R}^2)$ and $\eta > 0$, there exists $C = C(M(\phi), \eta)$ such that if*

$$\int_I \int_{\mathbb{R}^2} |e^{it\Delta} \phi|^4 dx dt \geq \eta$$

for some interval $I \subseteq \mathbb{R}$, then there exist $x_0 \in \mathbb{R}^2$ and $J \subseteq I$ such that

$$\int_{|x-x_0| \leq C|J|^{1/2}} |e^{it\Delta} \phi|^2 dx \geq C^{-1} \quad \text{for all } t \in J. \tag{30}$$

(Notice that C does not depend on I or J .) If ϕ is spherically symmetric, then we may take $x_0 = 0$.

Proof. As noted, a proof can be found in [5]. For the convenience of the reader, we will give a short proof in the radial case using the tools described in this section.

Making a Littlewood–Paley decomposition of u and then applying Hölder followed by the Strichartz and bilinear Strichartz inequalities, we find

$$\begin{aligned} \eta &\lesssim \sum_{N \geq M} \|e^{it\Delta} \phi_N e^{it\Delta} \phi_M\|_{L^2_{t,x}(I \times \mathbb{R}^2)} \|e^{it\Delta} \phi_N\|_{L^4_{t,x}(I \times \mathbb{R}^2)} \|e^{it\Delta} \phi_M\|_{L^4_{t,x}(I \times \mathbb{R}^2)} \\ &\lesssim \sum_{N \geq M} (M/N)^{1/2} \|\phi_N\|_{L^2_x}^2 \|\phi_M\|_{L^2_x} \|e^{it\Delta} \phi_M\|_{L^4_{t,x}(I \times \mathbb{R}^2)} \lesssim \|\phi\|_{L^2_x}^3 \sup_M \|e^{it\Delta} \phi_M\|_{L^4_{t,x}(I \times \mathbb{R}^2)}. \end{aligned}$$

Therefore,

$$\sup_M \|e^{it\Delta} \phi_M\|_{L^4_{t,x}(I \times \mathbb{R}^2)} \gtrsim \eta \|\phi\|_{L^2_x}^{-3}.$$

On the other hand, by Bernstein’s inequality (Lemma 2.1),

$$\int_I \int_{\mathbb{R}^2} |e^{it\Delta} \phi_M|^4 dx dt \lesssim |I| M^2 \|\phi\|_{L^2_x}^4.$$

Combining these two, we see that there is $M \gtrsim_{M(\phi)} \eta^{1/2} |I|^{-1/2}$ such that

$$\|e^{it\Delta} \phi_M\|_{L^4_{t,x}(I \times \mathbb{R}^2)} \gtrsim_{M(\phi)} \eta. \tag{31}$$

By Lemma 2.8 we obtain the upper bound

$$\|e^{it\Delta} \phi_M\|_{L^{7/2}_{t,x}(I \times \mathbb{R}^2)} \lesssim M^{-1/7} \|\phi\|_{L^2_x},$$

which combined with (31) and Hölder’s inequality yields

$$\|e^{it\Delta} \phi_M\|_{L^\infty_{t,x}(I \times \mathbb{R}^2)} \gtrsim_{M(\phi)} \eta^8 M.$$

Thus, there exist $t_0 \in I$ and $x_0 \in \mathbb{R}^2$ such that

$$|[e^{it_0\Delta} \phi_M](x_0)| \gtrsim_{M(\phi)} \eta^8 M.$$

Using the fact that the kernel of P_M is concentrated in the ball of radius M^{-1} and that it has L^2_x norm comparable to M , we may conclude that

$$\int_{|x-x_0| \lesssim M^{-1}} |\tilde{P}_M e^{it_0\Delta} \phi(x)|^2 dx \gtrsim_{M(\phi), \eta} 1.$$

Here $\tilde{P}_M = P_{M/2} + P_M + P_{2M}$ as in (22). By enlarging the ball a little and using (27) we obtain

$$\int_{|x-x_0| \lesssim M^{-1}} |e^{it\Delta} \phi(x)|^2 dx \gtrsim_{M(\phi), \eta} 1$$

for all $|t - t_0| \leq M^{-2}$. Let $J := \{t \in I : |t - t_0| \leq M^{-2}\}$. To obtain (30), we simply note that because of our lower bound on M , the length of J obeys $|J| \gtrsim_{M(\phi), \eta} M^{-2}$.

When ϕ is spherically symmetric, we must have $|x_0| \lesssim_{M(\phi)} |I|^{1/2}$; otherwise, we may find a large number of disjoint balls each containing η amount of mass, which contradicts the finiteness of the L^2_x norm of ϕ . Thus, it is possible to set $x_0 = 0$ by simply enlarging C . \square

3. Almost periodic solutions

In this section, we record basic facts about the frequency scale function $N(t)$ that will be needed in the proof of Theorem 1.16 in the next section. Much of the theory here is implicit in [37, 70]. Throughout this section, we fix the dimension d and the sign μ . The results do not rely on spherical symmetry.

Lemma 3.1 (Quasi-uniqueness of N). *Let u be a nonzero solution to (2) with lifespan I that is almost periodic modulo G with frequency scale function $N : I \rightarrow \mathbb{R}^+$ and compactness modulus function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and also almost periodic modulo G with frequency scale function $N' : I \rightarrow \mathbb{R}^+$ and compactness modulus function $C' : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. Then*

$$N(t) \sim_{u,C,C'} N'(t) \quad \text{for all } t \in I.$$

Proof. By symmetry, it suffices to establish the bound $N'(t) \lesssim_{u,C,C'} N(t)$. We write $x'(t)$ for the spatial centre associated to N' and C' . Similarly, $\xi(t)$ is the frequency centre associated to N and C .

Fix t and let $\eta > 0$ to be chosen later. By Definition 1.14 we have

$$\int_{|x-x'(t)| \geq C'(\eta)/N'(t)} |u(t, x)|^2 dx \leq \eta$$

and

$$\int_{|\xi-\xi(t)| \geq C(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi \leq \eta.$$

We split $u(t, x) = u_1(t, x) + u_2(t, x)$, where $u_1(t, x) := u(t, x)1_{|x-x'(t)| \geq C'(\eta)/N'(t)}$ and $u_2(t, x) := u(t, x)1_{|x-x'(t)| < C'(\eta)/N'(t)}$. Then by Plancherel’s theorem we have

$$\int_{\mathbb{R}^d} |\hat{u}_1(t, \xi)|^2 d\xi \lesssim \eta, \tag{32}$$

while from Cauchy–Schwarz we have

$$\sup_{\xi \in \mathbb{R}^d} |\hat{u}_2(t, \xi)|^2 \lesssim_{\eta,C'} M(u)N'(t)^{-d}.$$

Integrating the last inequality over the ball $|\xi - \xi(t)| \leq C(\eta)N(t)$ and using (32), we conclude that

$$\int_{\mathbb{R}^d} |\hat{u}(t, \xi)|^2 d\xi \lesssim \eta + O_{\eta,C,C'}(M(u)N(t)^d N'(t)^{-d}).$$

Thus, by Plancherel and mass conservation,

$$M(u) \lesssim \eta + O_{\eta,C,C'}(M(u)N(t)^d N'(t)^{-d}).$$

Choosing η to be a small multiple of $M(u)$ (which is nonzero by hypothesis), we obtain the claim. □

Lemma 3.2 (Quasi-continuous dependence of N on u). *Let $u^{(n)}$ be a sequence of solutions to (2) with lifespans $I^{(n)}$, which are almost periodic modulo scaling with frequency scale functions $N^{(n)} : I^{(n)} \rightarrow \mathbb{R}^+$ and compactness modulus function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, independent of n . Suppose that $u^{(n)}$ converge locally uniformly to a nonzero solution u to (2) with lifespan I . Then u is almost periodic modulo scaling with a frequency scale function $N : I \rightarrow \mathbb{R}^+$ and compactness modulus function C . Furthermore,*

$$N(t) \sim_{u,C} \liminf_{n \rightarrow \infty} N^{(n)}(t) \sim_{u,C} \limsup_{n \rightarrow \infty} N^{(n)}(t) \quad \text{for all } t \in I. \tag{33}$$

Finally, if all $u^{(n)}$ are spherically symmetric, then u is also.

Proof. We first show that

$$0 < \liminf_{n \rightarrow \infty} N^{(n)}(t) \leq \limsup_{n \rightarrow \infty} N^{(n)}(t) < \infty \quad \text{for all } t \in I. \tag{34}$$

Indeed, if one of these inequalities failed for some t , then (by passing to a subsequence if necessary) $N^{(n)}(t)$ would converge to zero or to infinity as $n \rightarrow \infty$. Thus, by Definition 1.14, $u^{(n)}(t)$ would converge weakly to zero, and hence, by the local uniform convergence, would converge strongly to zero. But this contradicts the hypothesis that u is not identically zero. This establishes (34).

From (34), we see that for each $t \in I$ the sequence $N^{(n)}(t)$ has at least one limit point $N(t)$. Thus, using the local uniform convergence we easily verify that u is almost periodic modulo scaling with frequency scale function N and compactness modulus function C . It is also clear that if all $u^{(n)}$ are spherically symmetric, then u is also.

It remains to establish (33), which we prove by contradiction. Suppose it fails. Then given any $A = A_u$, there exists a $t \in I$ for which $N^{(n)}(t)$ has at least two limit points which are separated by a ratio of at least A , and so u has two frequency scale functions with compactness modulus function C which are separated by this ratio. But this contradicts Lemma 3.1 for A large enough depending on u . Hence (33) holds. \square

The following claim is easily verified.

Lemma 3.3 (Symmetries of almost periodic solutions). *Let u be a solution to (2) with lifespan I which is almost periodic modulo G (respectively G_{rad}) with frequency scale function $N : I \rightarrow \mathbb{R}^+$ and compactness modulus function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. Let the notation be as in Lemma 1.7. Then*

- *The time reversal \tilde{u} is almost periodic modulo G (respectively G_{rad}) with frequency scale function $\tilde{N}(t) := N(-t)$ and compactness modulus function C .*
- *For any $t_0 \in \mathbb{R}$, the time translation u_{t_0} is almost periodic modulo G (respectively G_{rad}) with frequency scale function $N_{t_0}(t) := N(t + t_0)$ and compactness modulus function C .*
- *For any $\lambda > 0$, the rescaled solution $T_{g_{0,0,\lambda}} u$ is almost periodic modulo G (respectively G_{rad}) with frequency scale function $N^{[\lambda]}(t) := N(t/\lambda^2)/\lambda$ and compactness modulus function C .*

There is an analogous claim for the pseudoconformal transformation, but it is more complicated and we will not need it here. The difference is that the compactness modulus function C changes; the closer 0 is to I , the worse it becomes.

In the nonradial case one can also easily track the changes in the functions $\xi(t)$ and $x(t)$, as well as the actions of more general elements of G , but we will not need to do so here.

Next, we state a useful compactness result. First, a definition:

Definition 3.4. *Let u be a solution to (2), which is almost periodic modulo G with frequency scale function N , position centre function x , and frequency centre function ξ . We say that u is normalised if the lifespan I contains zero and*

$$N(0) = 1, \quad x(0) = \xi(0) = 0.$$

More generally, we can define the normalisation of a solution u at a time t_0 in its lifespan I to be

$$u^{[t_0]} := T_{g_0, -\xi(t_0)/N(t_0), -x(t_0)N(t_0), N(t_0)}(u_{t_0}). \tag{35}$$

Observe that $u^{[t_0]}$ is a normalised solution which is almost periodic modulo G and has lifespan

$$I^{[t_0]} := \{s \in \mathbb{R} : t_0 + s/N(t_0)^2 \in I\}$$

(so, in particular, $0 \in I^{[t_0]}$). It has frequency scale function

$$N^{[t_0]}(s) := N(t_0 + s/N(t_0)^2)/N(t_0)$$

and the same compactness modulus function as u . Furthermore, if u is maximal-lifespan then so is $u^{[t_0]}$, and if u is spherically symmetric and almost periodic modulo G_{rad} then so is $u^{[t_0]}$.

Lemma 3.5 (Compactness of almost periodic solutions). *Let $u^{(n)}$ be a sequence of normalised maximal-lifespan solutions to (2) with lifespans $I^{(n)} \ni 0$, which are almost periodic modulo G with frequency scale functions $N^{(n)} : I^{(n)} \rightarrow \mathbb{R}^+$ and a uniform compactness modulus function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. Assume that we also have a uniform mass bound*

$$0 < \inf_n M(u^{(n)}) \leq \sup_n M(u^{(n)}) < \infty. \tag{36}$$

Then, after passing to a subsequence if necessary, there exists a nonzero maximal-lifespan solution u to (2) with lifespan $I \ni 0$ that is almost periodic modulo G , such that $u^{(n)}$ converge locally uniformly to u . Moreover, if all $u^{(n)}$ are spherically symmetric and almost periodic modulo G_{rad} , then u is also.

Proof. By hypothesis and Definition 1.14, we see that for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{|x| \geq R} |u^{(n)}(0, x)|^2 dx \leq \varepsilon \quad \text{and} \quad \int_{|\xi| \geq R} |\widehat{u^{(n)}}(0, \xi)|^2 d\xi \leq \varepsilon$$

for all n . From this, (36), and the Ascoli–Arzelà theorem, we see that the sequence $u^{(n)}(0)$ is precompact in the strong topology of $L_x^2(\mathbb{R}^d)$. Thus, by passing to a subsequence if necessary, we can find $u_0 \in L_x^2(\mathbb{R}^d)$ such that $u^{(n)}(0)$ converge strongly to u_0 in $L_x^2(\mathbb{R}^d)$. From (36) we see that u_0 is not identically zero.

Now let u be the maximal Cauchy development of u_0 from time 0, with lifespan I . By Theorem 1.4, $u^{(n)}$ converge locally uniformly to u . The remaining claims now follow from Lemma 3.2. \square

To illustrate how the above theory is used, let us now record some simple consequences that will be useful in what follows.

Corollary 3.6 (Local constancy of N). *Let u be a nonzero maximal-lifespan solution to (2) with lifespan I that is almost periodic modulo G with frequency scale function $N : I \rightarrow \mathbb{R}^+$. Then there exists a small number δ , depending on u , such that for every $t_0 \in I$ we have*

$$[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I \quad (37)$$

and

$$N(t) \sim_u N(t_0) \quad \text{whenever} \quad |t - t_0| \leq \delta N(t_0)^{-2}. \quad (38)$$

Proof. Let us first establish (37). We argue by contradiction. Assume (37) fails. Then there exist sequences $t_n \in I$ and $\delta_n \rightarrow 0$ such that $t_n + \delta_n N(t_n)^{-2} \notin I$ for all n . Define the normalisations $u^{[t_n]}$ of u from time t_n by (35). Then $u^{[t_n]}$ are maximal-lifespan normalised solutions whose lifespans $I^{[t_n]}$ contain 0 but not δ_n ; they are also almost periodic modulo G with frequency scale functions

$$N^{[t_n]}(s) := N(t_n + s N(t_n)^{-2}) / N(t_n) \quad (39)$$

and the same compactness modulus function C as u . Applying Lemma 3.5 (and passing to a subsequence if necessary), we conclude that $u^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution v with some lifespan $J \ni 0$. By Theorem 1.4, J is open and so contains δ_n for all sufficiently large n . This contradicts the local uniform convergence as, by hypothesis, δ_n does not belong to $I^{[t_n]}$. Hence (37) holds.

We now show (38). Again, we argue by contradiction, shrinking δ if necessary. Assume (38) fails no matter how small one selects δ . Then one can find sequences $t_n, t'_n \in I$ such that $s_n := (t'_n - t_n)N(t_n)^2 \rightarrow 0$ but $N(t'_n)/N(t_n)$ converge to either zero or infinity. If we define $u^{[t_n]}$ and $N^{[t_n]}$ as before and apply Lemma 3.5 (passing to a subsequence if necessary), we see once again that $u^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution v with some open lifespan $J \ni 0$. But then $N^{[t_n]}(s_n)$ converge to either zero or infinity and thus, by Definition 1.14, $u^{[t_n]}(s_n)$ are converging weakly to zero. On the other hand, since s_n converge to zero and $u^{[t_n]}$ are locally uniformly convergent to $v \in C_{t,\text{loc}}^0 L_x^2(J \times \mathbb{R}^d)$, we may conclude that $u^{[t_n]}(s_n)$ converge strongly to $v(0)$ in $L_x^2(\mathbb{R}^2)$. Thus $v(0) = 0$ and $M(u^{[t_n]})$ converge to $M(v) = 0$. But since $M(u^{(n)}) = M(u)$, we see that u vanishes identically, a contradiction. Thus (38) holds. \square

Corollary 3.7 (Blowup criterion). *Let u be a nonzero maximal-lifespan solution to (2) with lifespan I that is almost periodic modulo G with frequency scale function $N : I \rightarrow \mathbb{R}^+$. If T is any finite endpoint of I , then $N(t) \gtrsim_u |T - t|^{-1/2}$; in particular, $\lim_{t \rightarrow T} N(t) = \infty$.*

Proof. This is immediate from (37). □

Lemma 3.8 (Local quasi-boundedness of N). *Let u be a nonzero solution to (2) with lifespan I that is almost periodic modulo G with frequency scale function $N : I \rightarrow \mathbb{R}^+$. If K is any compact subset of I , then*

$$0 < \inf_{t \in K} N(t) \leq \sup_{t \in K} N(t) < \infty.$$

Proof. We only prove the first inequality; the argument for the last is similar.

We argue by contradiction. Suppose that the first inequality fails. Then there exists a sequence $t_n \in K$ such that $\lim_{n \rightarrow \infty} N(t_n) = 0$ and hence, by Definition 1.14, $u(t_n)$ converge weakly to zero. Since K is compact, we can assume t_n converge to a limit $t_0 \in K$. As $u \in C_t^0 L_x^2(K \times \mathbb{R}^d)$, we see that $u(t_n)$ converge strongly to $u(t_0)$. Thus $u(t_0)$ must be zero, contradicting the hypothesis. □

Finally, we establish a spacetime bound.

Lemma 3.9 (Spacetime bound). *Let u be a nonzero solution to (2) with lifespan I , which is almost periodic modulo G with frequency scale function $N : I \rightarrow \mathbb{R}^+$. If J is any subinterval of I , then*

$$\int_J N(t)^2 dt \lesssim_u \int_J \int_{\mathbb{R}^d} |u(t, x)|^{2(d+2)/d} dx dt \lesssim_u 1 + \int_J N(t)^2 dt. \tag{40}$$

Proof. We first prove

$$\int_J \int_{\mathbb{R}^d} |u(t, x)|^{2(d+2)/d} dx dt \lesssim_u 1 + \int_J N(t)^2 dt. \tag{41}$$

Let $0 < \eta < 1$ to be chosen momentarily and partition J into subintervals I_j so that

$$\int_{I_j} N(t)^2 dt \leq \eta; \tag{42}$$

this requires at most $\eta^{-1} \times \text{RHS}(41)$ intervals. For each j , we may choose $t_j \in I_j$ so that

$$N(t_j)^2 |I_j| \leq 2\eta. \tag{43}$$

By Strichartz' inequality, we have the following estimates on the spacetime slab $I_j \times \mathbb{R}^d$:

$$\begin{aligned} \|u\|_{L_{t,x}^{2(d+2)/d}} &\lesssim \|e^{i(t-t_j)\Delta} u(t_j)\|_{L_{t,x}^{2(d+2)/d}} + \|u\|_{L_{t,x}^{(d+4)/d}}^{2(d+2)/d} \\ &\lesssim \|u_{\geq N_0}(t_j)\|_{L_x^2} + \|e^{i(t-t_j)\Delta} u_{\leq N_0}(t_j)\|_{L_{t,x}^{2(d+2)/d}} + \|u\|_{L_{t,x}^{(d+4)/d}}^{2(d+2)/d} \\ &\lesssim \|u_{\geq N_0}(t_j)\|_{L_x^2} + |I_j|^{d/2(d+2)} N_0^{d/(d+2)} \|u(t_j)\|_{L_x^2} + \|u\|_{L_{t,x}^{(d+4)/d}}^{2(d+2)/d}; \end{aligned}$$

the last step here used Hölder’s and Bernstein’s inequalities. Choosing N_0 as a large multiple of $N(t_j)$ and using Definition 1.14, one can make the first term as small as one wishes. Subsequently, choosing η sufficiently small depending on $M(u)$ and invoking (43), one may also render the second term arbitrarily small. Thus, by the usual bootstrap argument we obtain

$$\int_{I_j} \int_{\mathbb{R}^d} |u(t, x)|^{2(d+2)/d} dx dt \leq 1.$$

Using the bound on the number of intervals I_j , this leads to (41).

Now we prove

$$\int_J \int_{\mathbb{R}^d} |u(t, x)|^{2(d+2)/d} dx dt \gtrsim_u \int_J N(t)^2 dt. \tag{44}$$

Using Definition 1.14 and choosing η sufficiently small depending on $M(u)$, we can guarantee that

$$\int_{|x-x(t)| \leq C(\eta)N(t)^{-1}} |u(t, x)|^2 dx \gtrsim_u 1 \quad \text{for all } t \in J; \tag{45}$$

here $x(t)$ is the spatial centre given to us by compactness. On the other hand, a simple application of Hölder’s inequality yields

$$\int_{\mathbb{R}^d} |u(t, x)|^{2(d+2)/d} dx \gtrsim_u \left(\int_{|x-x(t)| \leq C(\eta)N(t)^{-1}} |u(t, x)|^2 \right)^{(d+2)/d} N(t)^2.$$

Thus, using (45) and integrating over J we derive (44). □

Corollary 3.10 (Maximal-lifespan almost periodic solutions blow up). *Let u be a maximal-lifespan solution to (2) which is almost periodic modulo G . Then u blows up both forward and backward in time.*

Proof. At a finite endpoint, Corollary 3.7 shows that the integral $\int_I N(t)^2 dt$ diverges in any neighbourhood of that endpoint. Thus by Lemma 3.9 the spacetime norm also diverges, which is the definition of blowup.

In the case of an infinite endpoint, we choose $t_0 \in I$ and note that by (37), $N(t) \gtrsim_u (t - t_0)^{-1/2}$. The claim now follows as before. □

4. Three enemies

We now prove Theorem 1.16. The proof is unaffected by spherical symmetry, the dimension d , or the sign of μ ; the essential ingredients are the scaling symmetry and the local constancy of $N(t)$.

Fix d and μ . Invoking Theorem 1.15, we can find a maximal-lifespan solution v of some lifespan J which is almost periodic modulo G and blows up both forward and backward in time; also, in the focusing case $\mu = -1$, we have $M(v) < M(Q)$.

Let $N_v(t)$ be the frequency scale function associated to v as in Definition 1.14, and let $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the compactness modulus function. The solution v partially satisfies the conclusions of Theorem 1.16, but we are not necessarily in one of the three scenarios listed there. To extract a solution u with such additional properties, we will have to perform some further manipulations primarily based on the scaling and time-translation symmetries.

For any $T \geq 0$, define the quantity

$$\text{osc}(T) := \inf_{t_0 \in J} \frac{\sup_{t \in J : |t-t_0| \leq T/N_v(t_0)^2} N_v(t)}{\inf_{t \in J : |t-t_0| \leq T/N_v(t_0)^2} N_v(t)}. \tag{46}$$

Roughly speaking, this measures the least possible oscillation one can find in N_v on time intervals of normalised duration T . This quantity is clearly nondecreasing in T . If $\text{osc}(T)$ is bounded, we will be able to extract a soliton-like solution; this is

Case I: $\lim_{T \rightarrow \infty} \text{osc}(T) < \infty$. In this case, we have arbitrarily long periods of stability for N_v . More precisely, we can find a finite number $A = A_v$, a sequence t_n of times in J , and a sequence $T_n \rightarrow \infty$ such that

$$\frac{\sup_{t \in J : |t-t_n| \leq T_n/N_v(t_n)^2} N_v(t)}{\inf_{t \in J : |t-t_n| \leq T_n/N_v(t_n)^2} N_v(t)} < A$$

for all n . Note that this, together with Corollary 3.6, implies that

$$[t_n - T_n/N_v(t_n)^2, t_n + T_n/N_v(t_n)^2] \subset J$$

and

$$N_v(t) \sim_v N_v(t_n)$$

for all t in this interval.

Now define the normalisations $v^{[t_n]}$ of v at times t_n as in (35). Then $v^{[t_n]}$ is a maximal-lifespan normalised solution with lifespan

$$J_n := \left\{ s \in \mathbb{R} : t_n + \frac{1}{N_v(t_n)^2} s \in J \right\} \supset [-T_n, T_n]$$

and mass $M(v)$. It is almost periodic modulo scaling with frequency scale function

$$N_{v^{[t_n]}}(s) := \frac{1}{N_v(t_n)} N_v\left(t_n + \frac{1}{N_v(t_n)^2} s\right)$$

and compactness modulus function C . In particular, we see that

$$N_{v^{[t_n]}}(s) \sim_v 1 \quad \text{for all } s \in [-T_n, T_n]. \tag{47}$$

We now apply Lemma 3.5 and conclude (passing to a subsequence if necessary) that $v^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution u with mass $M(v)$ defined on an open interval I containing 0 and which is almost periodic modulo symmetries. As

$T_n \rightarrow \infty$, Lemma 3.2 and (47) imply that the frequency scale function $N : I \rightarrow \mathbb{R}^+$ of u satisfies

$$0 < \inf_{t \in I} N(t) \leq \sup_{t \in I} N(t) < \infty.$$

In particular, by Corollary 3.7, I has no finite endpoints and hence $I = \mathbb{R}$. By modifying C by a bounded amount we may now normalise $N \equiv 1$. We have thus constructed a soliton-like solution in the sense of Theorem 1.16.

When $\text{osc}(T)$ is unbounded, we must seek a solution belonging to one of the remaining two scenarios. To distinguish between them, we introduce the quantity

$$a(t_0) := \frac{\inf_{t \in J : t \leq t_0} N_v(t) + \inf_{t \in J : t \geq t_0} N_v(t)}{N_v(t_0)}$$

for every $t_0 \in J$. This measures the extent to which $N_v(t)$ decays to zero on both sides of t_0 . Clearly, this quantity takes values in the interval $[0, 2]$.

Case II: $\lim_{T \rightarrow \infty} \text{osc}(T) = \infty$ and $\inf_{t_0 \in J} a(t_0) = 0$. In this case, there are no long periods of stability but there are times about which there are arbitrarily large cascades from high to low frequencies in both future and past directions. This will allow us to extract a solution with a double high-to-low frequency cascade as defined in Theorem 1.16.

As $\inf_{t_0 \in J} a(t_0) = 0$, there exists a sequence of times $t_n \in J$ such that $a(t_n) \rightarrow 0$ as $n \rightarrow \infty$. By the definition of a , we can also find times $t_n^- < t_n < t_n^+$ with $t_n^-, t_n^+ \in J$ such that

$$\frac{N_v(t_n^-)}{N_v(t_n)} \rightarrow 0 \quad \text{and} \quad \frac{N_v(t_n^+)}{N_v(t_n)} \rightarrow 0.$$

Choose $t_n^- < t'_n < t_n^+$ so that

$$N_v(t'_n) \sim \sup_{t_n^- \leq t \leq t_n^+} N_v(t);$$

then

$$\frac{N_v(t_n^-)}{N_v(t'_n)} \rightarrow 0 \quad \text{and} \quad \frac{N_v(t_n^+)}{N_v(t'_n)} \rightarrow 0.$$

We define the rescaled and translated times $s_n^- < 0 < s_n^+$ by

$$s_n^\pm := N_v(t'_n)^2 (t_n^\pm - t'_n)$$

and the normalisations $v^{[t'_n]}$ at times t'_n by (35). These are normalised maximal-lifespan solutions with lifespans containing $[s_n^-, s_n^+]$, which are almost periodic modulo G with frequency scale functions

$$N_{v^{[t'_n]}}(s) := \frac{1}{N_v(t'_n)} N_v\left(t'_n + \frac{1}{N_v(t'_n)^2} s\right). \tag{48}$$

By the way we chose t'_n , we see that

$$N_{v^{[t'_n]}}(s) \lesssim 1 \quad \text{for all } s_n^- \leq s \leq s_n^+. \tag{49}$$

Moreover,

$$N_{v^{[t'_n]}}(s_n^\pm) \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{50}$$

for either choice of sign.

We now apply Lemma 3.5 and conclude (passing to a subsequence if necessary) that $v^{[t'_n]}$ converge locally uniformly to a maximal-lifespan solution u of mass $M(v)$ defined on an open interval I containing 0, which is almost periodic modulo symmetries.

Let N be a frequency scale function for u . From Lemma 3.8 we see that $N(t)$ is bounded from below on any compact set $K \subset I$. From this and Lemma 3.2 (and Lemma 3.1), we see that $N_{v^{[t'_n]}}(t)$ is also bounded from below, uniformly in $t \in K$, for all sufficiently large n (depending on K). As a consequence of this and (50), we see that s_n^- and s_n^+ cannot have any limit points in K ; thus $K \subset [s_n^-, s_n^+]$ for all sufficiently large n . Therefore, s_n^\pm converge to the endpoints of I . Combining this with Lemma 3.2 and (49), we conclude that

$$\sup_{t \in I} N(t) < \infty. \tag{51}$$

Corollary 3.7 now implies that I has no finite endpoints, that is, $I = \mathbb{R}$.

In order to prove that u is a double high-to-low frequency cascade, we merely need to show that

$$\liminf_{t \rightarrow +\infty} N(t) = \liminf_{t \rightarrow -\infty} N(t) = 0. \tag{52}$$

By time reversal symmetry, it suffices to establish that $\liminf_{t \rightarrow +\infty} N(t) = 0$. Suppose that this is not the case. Then using (51) we may deduce $N(t) \sim_u 1$ for all $t \geq 0$. We conclude from Lemma 3.2 that for every $m \geq 1$, there exists an n_m such that

$$N_{v^{[t'_{n_m}]}}(t) \sim_u 1 \quad \text{for all } 0 \leq t \leq m.$$

But by (46) and (48) this implies that

$$\text{osc}(\varepsilon m) \lesssim_u 1$$

for all m and some $\varepsilon = \varepsilon(u) > 0$ independent of m . Note that ε is chosen as a lower bound on the quantities $N(t''_{n_m})^2/N(t'_{n_m})^2$ where $t''_{n_m} = t'_{n_m} + (m/2)N(t'_{n_m})^{-2}$. This contradicts the hypothesis $\lim_{T \rightarrow \infty} \text{osc}(T) = \infty$ and so settles Case II.

Case III: $\lim_{T \rightarrow \infty} \text{osc}(T) = \infty$ and $\inf_{t_0 \in J} a(t_0) > 0$. In this case, there are no long periods of stability and no double cascades from high to low frequencies; we will be able to extract a self-similar solution in the sense of Theorem 1.16.

Let $\varepsilon = \varepsilon(v) > 0$ be such that $\inf_{t_0 \in J} a(t_0) \geq 2\varepsilon$. We call a time t_0 *future-focusing* if

$$N_v(t) \geq \varepsilon N_v(t_0) \quad \text{for all } t \in J \text{ with } t \geq t_0 \tag{53}$$

and *past-focusing* if

$$N_v(t) \geq \varepsilon N_v(t_0) \quad \text{for all } t \in J \text{ with } t \leq t_0. \tag{54}$$

From the choice of ε we see that every time $t_0 \in J$ is either future-focusing or past-focusing, or possibly both.

We will now show that either all sufficiently late times are future-focusing, or all sufficiently early times are past-focusing. If this were false, there would be a future-focusing time t_0 and a sequence of past-focusing times t_n that converges to the righthand endpoint of J . For sufficiently large n , we have $t_n \geq t_0$. By (53) and (54) we then see that

$$N_v(t_n) \sim_v N_v(t_0)$$

for all such n . For any $t_0 < t < t_n$, we know that t is either past-focusing or future-focusing; thus we have either $N_v(t_0) \geq \varepsilon N_v(t)$ or $N_v(t_n) \geq \varepsilon N_v(t)$. Also, since t_0 is future-focusing, $N_v(t) \geq \varepsilon N_v(t_0)$. We conclude that

$$N_v(t) \sim_v N_v(t_0)$$

for all $t_0 < t < t_n$; since t_n converges to $\sup(J)$, this claim in fact holds for all $t_0 < t < \sup(J)$. In particular, from Corollary 3.7 we see that v does not blow up forward in finite time, that is, $\sup(J) = \infty$. The function N_v is now bounded above and below on the interval $(t_0, +\infty)$. This implies that $\lim_{T \rightarrow \infty} \text{osc}(T) < \infty$, a contradiction. This proves the assertion at the beginning of the paragraph.

We may now assume that future-focusing occurs for all sufficiently late times; more precisely, we can find a $t_0 \in J$ such that all times $t \geq t_0$ are future-focusing. The case when all sufficiently early times are past-focusing reduces to this via time-reversal symmetry.

We will now recursively construct a new sequence of times t_n . More precisely, we will explain how to choose t_{n+1} from t_n .

Since $\lim_{T \rightarrow \infty} \text{osc}(T) = \infty$, we have $\text{osc}(B) \geq 2/\varepsilon$ for some sufficiently large $B = B(v) > 0$. Given $J \ni t_n > t_0$ set $A = 2B\varepsilon^{-2}$ and $t'_n = t_n + \frac{1}{2}AN_v(t_n)^{-2}$. As $t'_n > t_0$, it is future-focusing and so $N_v(t'_n) \geq \varepsilon N_v(t_n)$. From this, we see that

$$\{t : |t - t'_n| \leq BN_v(t'_n)^{-2}\} \subseteq [t_n, t_n + AN_v(t_n)^{-2}]$$

and thus, by the definition of B and the fact that all $t \geq t_n$ are future-focusing,

$$\sup_{t \in J \cap [t_n, t_n + AN_v(t_n)^{-2}]} N_v(t) \geq 2N_v(t_n). \tag{55}$$

Using this and Corollary 3.6, we see that for every $t_n \in J$ with $t_n \geq t_0$ there exists a time $t_{n+1} \in J$ obeying

$$t_n < t_{n+1} \leq t_n + AN(t_n)^{-2} \tag{56}$$

such that

$$2N_v(t_n) \leq N_v(t_{n+1}) \lesssim_v N_v(t_n) \tag{57}$$

and

$$N_v(t) \sim_v N_v(t_n) \quad \text{for all } t_n \leq t \leq t_{n+1}. \tag{58}$$

From (56) we have

$$N_v(t_n) \geq 2^n N_v(t_0)$$

for all $n \geq 0$, which by (57) implies

$$t_{n+1} \leq t_n + O_v(2^{-2n} N_v(t_0)^{-2}).$$

Thus t_n converge to a limit and $N_v(t_n)$ to infinity. In view of Lemma 3.8, this implies that $\sup(J)$ is finite and $\lim_{n \rightarrow \infty} t_n = \sup(J)$.

Let $n \geq 0$. By (57),

$$N_v(t_{n+m}) \geq 2^m N_v(t_n)$$

for all $m \geq 0$, and so using (56) we obtain

$$0 < t_{n+m+1} - t_{n+m} \lesssim_v 2^{-2m} N_v(t_n)^{-2}.$$

Summing this series in m , we conclude that

$$\sup(J) - t_n \lesssim_v N_v(t_n)^{-2}.$$

Combining this with Corollary 3.7, we obtain

$$\sup(J) - t_n \sim_v N_v(t_n)^{-2}.$$

In particular, we have

$$\sup(J) - t_{n+1} \sim_v \sup(J) - t_n \sim_v N_v(t_n)^{-2}.$$

Applying (57) and (58) shows

$$\sup(J) - t \sim_v N_v(t)^{-2} \quad \text{for all } t_n \leq t \leq t_{n+1}.$$

Since t_n converges to $\sup(J)$, we conclude that

$$\sup(J) - t \sim_v N_v(t)^{-2} \quad \text{for all } t_0 \leq t < \sup(J).$$

As we have the freedom to modify $N(t)$ by a bounded function (modifying C appropriately), we may normalise

$$N_v(t) = (\sup(J) - t)^{-1/2} \quad \text{for all } t_0 \leq t < \sup(J).$$

It is now not difficult to extract our sought-after self-similar solution by suitably rescaling the interval $(t_0, \sup J)$ as follows.

Consider the normalisations $v^{[t_n]}$ of v at times t_n (cf. (35)). These are maximal-lifespan normalised solutions of mass $M(v)$, whose lifespans include the interval

$$\left(-\frac{\sup(J) - t_0}{\sup(J) - t_n}, 1 \right),$$

and which are almost periodic modulo scaling with compactness modulus function C and frequency scale functions

$$N_{v^{[t_n]}}(s) = (1 - s)^{1/2} \tag{59}$$

for all $-\frac{\sup(J)-t_0}{\sup(J)-t_n} < s < 1$. We now apply Lemma 3.5 and conclude (passing to a subsequence if necessary) that $v^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution u of mass $M(v)$ defined on an open interval I containing $(-\infty, 1)$, which is almost periodic modulo symmetries.

By Lemma 3.2 and (59), we see that u has a frequency scale function N obeying

$$N(s) \sim_v (1-s)^{-1/2} \quad \text{for all } s \in (-\infty, 1).$$

By modifying N (and C) by a bounded factor, we may normalise

$$N(s) = (1-s)^{-1/2}.$$

From this, Corollary 3.6, and Corollary 3.7 we see that we must have $I = (-\infty, 1)$. Applying a time translation (by -1) followed by a time reversal, we obtain our sought-after self-similar solution.

Remark. Theorem 1.16 may not be the final word on the matter. It may be that by a further passage to rescaled limits, one could improve the control in the case of a double high-to-low frequency cascade. We will not pursue this matter here.

5. The self-similar solution

In this section we preclude self-similar solutions. As mentioned in the Introduction, the key ingredient is additional regularity.

Theorem 5.1 (Regularity in the self-similar case). *Let $d = 2$ and let u be a spherically symmetric solution to (1) that is almost periodic modulo G_{rad} and self-similar in the sense of Theorem 1.16. Then $u(t) \in H_x^s(\mathbb{R}^2)$ for all $t \in (0, \infty)$ and all $s \geq 0$.*

In order to preclude self-similar solutions we only need Theorem 5.1 for $s = 1$, as we now show.

Corollary 5.2 (Absence of self-similar solutions). *For $d = 2$ there are no nonzero spherically symmetric solutions to (1) that are self-similar in the sense of Theorem 1.16.*

Proof. By Theorem 5.1, any such solution would obey $u(t) \in H_x^1(\mathbb{R}^2)$ for all $t \in (0, \infty)$. Then, by the H_x^1 global well-posedness theory described after Conjecture 1.8, there exists a global solution with initial data $u(t_0)$ at any time $t_0 \in (0, \infty)$; recall that we assume $M(u) < M(Q)$ in the focusing case. On the other hand, self-similar solutions blow up at time $t = 0$. These two facts (combined with the uniqueness statement in Theorem 1.4) yield a contradiction. \square

The remainder of this section is devoted to proving Theorem 5.1.

Let u be as in Theorem 5.1. For any $A > 0$, we define

$$\begin{aligned} \mathcal{M}(A) &:= \sup_{T>0} \|u_{>AT^{-1/2}}(T)\|_{L_x^2(\mathbb{R}^2)}, \\ \mathcal{S}(A) &:= \sup_{T>0} \|u_{>AT^{-1/2}}\|_{L_{t,x}^4([T,2T]\times\mathbb{R}^2)}, \\ \mathcal{N}(A) &:= \sup_{T>0} \|P_{>AT^{-1/2}}(F(u))\|_{L_{t,x}^{4/3}([T,2T]\times\mathbb{R}^2)}. \end{aligned} \tag{60}$$

The notation chosen indicates the quantity being measured, namely, the mass, the symmetric Strichartz norm, and the nonlinearity in the adjoint Strichartz norm, respectively. As u is self-similar, $N(t)$ is comparable to $T^{-1/2}$ for t in the interval $[T, 2T]$. Thus, the Littlewood–Paley projections are adapted to the natural frequency scale on each interval.

To prove Theorem 5.1 it suffices to show that for every $s > 0$ we have

$$\mathcal{M}(A) \lesssim_{s,u} A^{-s} \tag{61}$$

whenever A is sufficiently large depending on u and s . To establish this, we need a variety of estimates linking \mathcal{M} , \mathcal{S} , and \mathcal{N} . From mass conservation, Lemma 3.9, self-similarity, and Hölder’s inequality, we see that

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u 1 \quad \text{for all } A > 0. \tag{62}$$

From the Strichartz inequality (Lemma 2.5), we also see that

$$\mathcal{S}(A) \lesssim \mathcal{M}(A) + \mathcal{N}(A) \quad \text{for all } A > 0. \tag{63}$$

A similar application of Strichartz using (62) shows

$$\|u\|_{L_t^2 L_x^\infty([T,2T]\times\mathbb{R}^2)} \lesssim_u 1 \quad \text{for all } T > 0. \tag{64}$$

Next, we use the bilinear Strichartz inequality to obtain a further connection between these quantities.

Lemma 5.3 (Nonlinear estimate). *For all $A > 100$ we have*

$$\mathcal{N}(A) \lesssim_u \mathcal{S}(A/8)\mathcal{M}(\sqrt{A}) + A^{-1/4}[\mathcal{M}(A/8) + \mathcal{N}(A/8)].$$

Proof. Fix $A > 100$. It suffices to show that

$$\|P_{>AT^{-1/2}}(F(u))\|_{L_{t,x}^{4/3}([T,2T]\times\mathbb{R}^2)} \lesssim_u \mathcal{S}(A/8)\mathcal{M}(\sqrt{A}) + A^{-1/4}[\mathcal{M}(A/8) + \mathcal{N}(A/8)] \tag{65}$$

for arbitrary $T > 0$. To do this, we decompose our solution as

$$u = u_{>(A/8)T^{-1/2}} + u_{\sqrt{A}T^{-1/2} < \cdot \leq (A/8)T^{-1/2}} + u_{\leq \sqrt{A}T^{-1/2}}.$$

Any term in the resulting expansion of $P_{>AT^{-1/2}}(F(u))$ that does not contain at least one factor of $u_{>(A/8)T^{-1/2}}$ vanishes.

Consider a term which contains at least one factor of $u_{\sqrt{A}T^{-1/2} < \cdot \leq (A/8)T^{-1/2}}$. By Hölder’s inequality, the contribution of such a term to (65) can be bounded by

$$\|u_{>(A/8)T^{-1/2}}\|_{L^4_{t,x}([T,2T] \times \mathbb{R}^2)} \|u_{>\sqrt{A}T^{-1/2}}\|_{L^\infty L^2_x([T,2T] \times \mathbb{R}^2)} \|u\|_{L^2_t L^\infty_x([T,2T] \times \mathbb{R}^2)}.$$

By (60), the first factor is at most $\mathcal{S}(A/8)$ and the second is at most $\mathcal{M}(\sqrt{A})$, while the last is $O_u(1)$ by (64).

Finally, we consider the terms with one factor of $u_{>(A/8)T^{-1/2}}$ and two factors of $u_{\leq \sqrt{A}T^{-1/2}}$. By Hölder’s inequality and (62), the contribution of this term to (65) can be bounded by

$$\|u_{>(A/8)T^{-1/2}} u_{\leq \sqrt{A}T^{-1/2}}\|_{L^2_{t,x}([T,2T] \times \mathbb{R}^2)}.$$

Applying the bilinear Strichartz inequality (Lemma 2.6), we bound this by

$$A^{-1/4} (\|u_{>(A/8)T^{-1/2}}(T)\|_{L^2_x(\mathbb{R}^2)} + \|P_{>(A/8)T^{-1/2}} F(u)\|_{L^{4/3}_{t,x}([T,2T] \times \mathbb{R}^2)}) \\ \times (\|u_{\leq \sqrt{A}T^{-1/2}}(T)\|_{L^2_x(\mathbb{R}^2)} + \|u_{\leq \sqrt{A}T^{-1/2}} F(u)\|_{L^{4/3}_{t,x}([T,2T] \times \mathbb{R}^2)}),$$

which is acceptable by virtue of (60) and (62). □

We have some decay as $A \rightarrow \infty$:

Lemma 5.4 (Qualitative decay). *We have*

$$\lim_{A \rightarrow \infty} \mathcal{M}(A) = \lim_{A \rightarrow \infty} \mathcal{S}(A) = \lim_{A \rightarrow \infty} \mathcal{N}(A) = 0. \tag{66}$$

Proof. The vanishing of the first limit follows from Definition 1.14, self-similarity, and (60). The vanishing of the third limit then follows from that of the first and Lemma 5.3. Lastly, we can deduce $\lim_{A \rightarrow \infty} \mathcal{S}(A) = 0$ using (63). □

We have now gathered enough tools to prove some regularity, albeit in the symmetric Strichartz space. As such, the next result is the crux of this section.

Proposition 5.5 (Quantitative decay estimate). *Let $0 < \eta < 1$. Then, if A is sufficiently large depending on u and η ,*

$$\mathcal{S}(A) \leq \eta \mathcal{S}(A/16) + A^{-1/10}. \tag{67}$$

In particular, $\mathcal{S}(A) \lesssim_u A^{-1/10}$ for all $A > 0$.

Proof. Fix $\eta \in (0, 1)$. It suffices to show

$$\|u_{>AT^{-1/2}}\|_{L^4_{t,x}([T,2T] \times \mathbb{R}^2)} \lesssim_u \eta \mathcal{S}(A/16) + A^{-1/8} \quad \text{for all } T > 0, \tag{68}$$

since then (67) follows by redefining η and requiring A to be larger, both depending upon u .

Fix $T > 0$. By writing the Duhamel formula (3) beginning at $T/2$ and then using Lemma 2.5, we obtain

$$\begin{aligned} \|u_{>AT^{-1/2}}\|_{L^4_{t,x}([T,2T] \times \mathbb{R}^2)} &\lesssim \|P_{>AT^{-1/2}} e^{i(t-T/2)\Delta} u(T/2)\|_{L^4_{t,x}([T,2T] \times \mathbb{R}^2)} \\ &\quad + \|P_{>AT^{-1/2}} F(u)\|_{L^{4/3}_{t,x}([T/2,2T] \times \mathbb{R}^2)}. \end{aligned}$$

First, we consider the second term. By (60), we have

$$\|P_{>AT^{-1/2}} F(u)\|_{L^{4/3}_{t,x}([T/2,2T] \times \mathbb{R}^2)} \lesssim \mathcal{N}(A/2).$$

Using Lemma 5.3 combined with Lemma 5.4 (choosing A sufficiently large depending on u and η) and (62), we derive

$$\|P_{>AT^{-1/2}} F(u)\|_{L^{4/3}_{t,x}([T/2,2T] \times \mathbb{R}^2)} \lesssim_u \text{RHS}(68).$$

Thus, the second term is acceptable.

We now consider the first term. It suffices to show

$$\|P_{>AT^{-1/2}} e^{i(t-T/2)\Delta} u(T/2)\|_{L^4_{t,x}([T,2T] \times \mathbb{R}^2)} \lesssim_u A^{-1/8}, \tag{69}$$

which we will deduce by first proving two estimates at a single frequency scale, interpolating between them, and then summing.

From Lemma 2.8 and mass conservation, we have

$$\|P_{BT^{-1/2}} e^{i(t-T/2)\Delta} u(T/2)\|_{L^q_{t,x}([T,2T] \times \mathbb{R}^2)} \lesssim_{u,q} (BT^{-1/2})^{1-4/q} \tag{70}$$

for all $10/3 < q < 4$ and $B > 0$. This is our first estimate.

Using the Duhamel formula (3), we write

$$P_{BT^{-1/2}} e^{i(t-T/2)\Delta} u(T/2) = P_{BT^{-1/2}} e^{i(t-\varepsilon)\Delta} u(\varepsilon) - i \int_{\varepsilon}^{T/2} P_{BT^{-1/2}} e^{i(t-t')\Delta} F(u(t')) dt'$$

for any $\varepsilon > 0$. By self-similarity, the former term converges strongly to zero in L^2_x as $\varepsilon \rightarrow 0$. Convergence to zero in L^∞_x then follows from Lemma 2.1. Thus, using the dispersive estimate (25), we estimate

$$\begin{aligned} \|P_{BT^{-1/2}} e^{i(t-T/2)\Delta} u(T/2)\|_{L^\infty_{t,x}([T,2T] \times \mathbb{R}^2)} &\lesssim \left\| \int_0^{T/2} \frac{1}{t-t'} \|F(u(t'))\|_{L^1_x} dt' \right\|_{L^\infty_t([T,2T])} \\ &\lesssim \frac{1}{T} \|F(u)\|_{L^1_{t,x}((0,T/2] \times \mathbb{R}^2)} \lesssim \frac{1}{T} \sum_{0 < \tau \leq T/4} \|F(u)\|_{L^1_{t,x}([\tau,2\tau] \times \mathbb{R}^2)} \\ &\lesssim \frac{1}{T} \sum_{0 < \tau \leq T/4} \tau^{1/2} \|u\|_{L^4_{t,x}([\tau,2\tau] \times \mathbb{R}^2)}^2 \|u\|_{L^\infty_t L^2_x([\tau,2\tau] \times \mathbb{R}^2)} \lesssim_u T^{-1/2}. \end{aligned}$$

The last step used the fact that $\|u\|_{4,4} \sim 1$ on dyadic intervals, as follows from self-similarity plus Lemma 3.9.

Interpolating between the estimate just proved and (70) with $q = 7/2$, we obtain

$$\|P_{BT^{-1/2}}e^{i(t-T/2)\Delta}u(T/2)\|_{L^4_{t,x}([T,2T]\times\mathbb{R}^2)} \lesssim_u B^{-1/8}.$$

Summing this over dyadic $B \geq A$ yields (69) and hence (68).

We now explain why (67) implies $\mathcal{S}(A) \lesssim_u A^{-1/10}$. Choose $\eta = 1/2$. Then there exists A_0 , depending on u , so that (67) holds for $A \geq A_0$. By (62), we need only bound $\mathcal{S}(A)$ for $A \geq A_0$.

Choose $k \geq 1$ so that $2^{-4k}A \leq A_0 \leq 2^{-4(k-1)}A$. By iterating (67) k times and then using (62), we obtain

$$\begin{aligned} \mathcal{S}(A) &\leq [1 + \eta 2^{4/10} + \dots + (\eta 2^{4/10})^{k-1}]A^{-1/10} + \eta^k \mathcal{S}(2^{-4k}A) \\ &\lesssim_u A^{-1/10} + \eta^k \lesssim_u A^{-1/10}. \end{aligned}$$

Note that the last inequality uses the way we chose η and k . □

Corollary 5.6. *For any $A > 0$ we have*

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u A^{-1/10}.$$

Proof. The bound on \mathcal{S} was derived in Proposition 5.5. The bound on \mathcal{N} follows from this, Lemma 5.3, and (62).

We now turn to the bound on \mathcal{M} . By Lemma 1.17,

$$\|P_{>AT^{-1/2}}u(T)\|_2 \lesssim \sum_{k=0}^{\infty} \left\| \int_{2^k T}^{2^{k+1}T} e^{i(T-t')\Delta} P_{>AT^{-1/2}} F(u(t')) dt' \right\|_2, \tag{71}$$

where weak convergence has become strong convergence because of the frequency projection and the fact that $N(t) = t^{-1/2} \rightarrow 0$ as $t \rightarrow \infty$. Intuitively, the reason for using (21) forward in time is that the solution becomes smoother as $N(t) \rightarrow 0$.

Combining (71) with Lemma 2.5 and (60), we get

$$\mathcal{M}(A) = \sup_{T>0} \|P_{>AT^{-1/2}}u(T)\|_2 \lesssim \sum_{k=0}^{\infty} \mathcal{N}(2^{k/2}A). \tag{72}$$

The desired bound on \mathcal{M} now follows from that on \mathcal{N} . □

We have succeeded in proving that self-similar solutions have almost 1/10 derivatives in L^2_x . Upgrading this to arbitrarily many derivatives now follows by a simple inductive argument.

Proof of Theorem 5.1. Combining Lemma 5.3 with Corollary 5.6 shows

$$\mathcal{N}(A) \lesssim_u A^{-1/20}[\mathcal{S}(A/8) + \mathcal{M}(A/8) + \mathcal{N}(A/8)].$$

Together with (63) and (72), this allows us to deduce

$$\mathcal{S}(A) + \mathcal{M}(A) + \mathcal{N}(A) \lesssim_u A^{-\sigma} \Rightarrow \mathcal{S}(A) + \mathcal{M}(A) + \mathcal{N}(A) \lesssim_u A^{-\sigma-1/20}$$

for any $\sigma > 0$. Iterating this statement shows that $u(t) \in H^s_x(\mathbb{R}^2)$ for all $s > 0$. □

6. An in/out decomposition

Spherical symmetry forces the amplitude of a wave to diminish as it moves away from the origin. This section provides a tool that we will use to exploit this idea in the next section. To set the stage however, let us briefly try to convey the guiding principle. Given t , we will split $u(t)$ into incoming and outgoing waves. Writing the Duhamel formula (21) with integration over the past, the source terms producing these incoming waves will lie at large radii. By contrast, it is favourable to represent outgoing waves using Duhamel with integration over the future.

For spherically symmetric functions, we have the Fourier–Bessel formula

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^2} J_0(|x| |\xi|) f(x) dx,$$

where J_0 is the Bessel function of the first kind

$$J_0(r) := \int_0^{2\pi} e^{ir \cos(\theta)} \frac{d\theta}{2\pi}$$

(see e.g. [58, Ch. IV]). Similarly we have the Fourier–Bessel inversion formula

$$f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^2} J_0(|x| |\xi|) \hat{f}(\xi) d\xi,$$

which expresses f as a linear combination of standing waves. These standing waves can then be decomposed into their incoming and outgoing components:

$$J_0(|\xi| |x|) = \frac{1}{2} H_0^{(1)}(|\xi| |x|) + \frac{1}{2} H_0^{(2)}(|\xi| |x|), \tag{73}$$

Here, $H_0^{(1)}$ denotes the Hankel function of the first kind and order zero; for real arguments, the Hankel function of the second kind is its complex conjugate.

With these considerations in mind, it is natural to make the following

Definition 6.1. Let P^+ denote the projection onto outgoing spherical waves,

$$\begin{aligned} [P^+ f](x) &= \frac{1}{2} (2\pi)^{-d/2} \int_{\mathbb{R}^2} H_0^{(1)}(|\xi| |x|) \hat{f}(\xi) d\xi \\ &= \frac{1}{2} (2\pi)^{-d} \int_{\mathbb{R}^2 \times \mathbb{R}^2} H_0^{(1)}(|\xi| |x|) J_0(|\xi| |y|) f(y) d\xi dy, \end{aligned} \tag{74}$$

and let P^- denote the projection onto incoming spherical waves, whose kernel is the complex conjugate of that of P^+ . We will write P_N^\pm for the product $P^\pm P_N$, where P_N is a Littlewood–Paley projection.

Proposition 6.2 (Properties of P^\pm).

- The P^\pm are bounded on $L^2(\mathbb{R}^2)$.
- $P^+ + P^-$ represents the projection from L^2 onto L^2_{rad} .

- For $|x| \gtrsim N^{-1}$ and $t \gtrsim N^{-2}$, the integral kernel obeys

$$|[P_N^\pm e^{\mp it\Delta}](x, y)| \lesssim \begin{cases} (|x| |y| |t|)^{-1/2}, & |y| - |x| \sim Nt, \\ \frac{N^2}{(N|x|)^{1/2} (N|y|)^{1/2}} \langle N^2 t + N|x| - N|y| \rangle^{-m}, & \text{otherwise,} \end{cases}$$

for all $m \geq 0$.

- For $|x| \gtrsim N^{-1}$ and $|t| \lesssim N^{-2}$, the integral kernel obeys

$$|[P_N^\pm e^{\mp it\Delta}](x, y)| \lesssim \frac{N^2}{(N|x|)^{1/2} (N|y|)^{1/2}} \langle N|x| - N|y| \rangle^{-m}$$

for all $m \geq 0$.

Proof. By performing the ξ integral in (74), we obtain

$$[P^\pm f](x) = \frac{f(x)}{2} \pm \frac{i}{2\pi^2} \int_{\mathbb{R}^2} \frac{f(y) dy}{|x|^2 - |y|^2} \tag{75}$$

for radial Schwartz functions f , while $P^\pm f \equiv 0$ if f is orthogonal to radial functions, that is, has zero mean on all circles centred at the origin. Switching to polar coordinates and changing variables reveals that the integral in (75) is simply the Hilbert transform in disguise. This shows that P^+ is bounded on $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$, settling claim (i). Observation (ii) also follows from this discussion.

The third claim is an exercise in stationary phase. We will only provide the details for $P_N^+ e^{-it\Delta}$; the other kernel is its complex conjugate. By (74) we have the following formula for the kernel:

$$[P_N^+ e^{-it\Delta}](x, y) = \frac{1}{2} (2\pi)^{-d} \int_{\mathbb{R}^2} H_0^{(1)}(|\xi| |x|) e^{it|\xi|^2} J_0(|\xi| |y|) \psi(\xi/N) d\xi, \tag{76}$$

where ψ is the multiplier from the Littlewood–Paley projection. To proceed, we use the following information about Bessel/Hankel functions.

$$J_0(r) = \frac{a(r)e^{ir}}{\langle r \rangle^{1/2}} + \frac{\bar{a}(r)e^{-ir}}{\langle r \rangle^{1/2}}, \tag{77}$$

where $a(r)$ obeys the symbol estimates

$$\left| \frac{\partial^m a(r)}{\partial r^m} \right| \lesssim \langle r \rangle^{-m} \quad \text{for all } m \geq 0. \tag{78}$$

The Hankel function $H_0^{(1)}(r)$ has a logarithmic singularity at $r = 0$; however, for $r \gtrsim 1$,

$$H_0^{(1)}(r) = \frac{b(r)e^{ir}}{r^{1/2}} \tag{79}$$

for a smooth function $b(r)$ obeying (78). As we assume $|x| \gtrsim N^{-1}$, the logarithmic singularity does not enter into our considerations.

Substituting (77) and (79) into (76), we see that a stationary phase point can only occur in the term containing $\bar{a}(r)$ and even then only if $|y| - |x| \sim Nt$. In this case, stationary phase yields the first estimate. In all other cases, integration by parts yields the second estimate.

Part (iv) is also a consequence of (76) and stationary phase techniques. Since t is so small, $e^{i|\xi|^{2t}}$ shows no appreciable oscillation and can be incorporated into $\psi(\xi/N)$. For $||y| - |x|| \leq N^{-1}$, the result follows from the naive L^1 estimate. For larger $|x| - |y|$, one integrates by parts m times. \square

7. Additional regularity

This section is devoted to a proof of

Theorem 7.1 (Regularity in the global case). *Let $d = 2$ and let u be a global spherically symmetric solution to (1) that is almost periodic modulo G_{rad} . Suppose also that $N(t) \lesssim 1$ for all $t \in \mathbb{R}$. Then $u \in L_t^\infty H_x^s(\mathbb{R} \times \mathbb{R}^2)$ for all $s \geq 0$.*

That $u(t)$ is smooth will follow from a careful study of the Duhamel formulae (21). Different phenomena are involved at times near t and those far from t . Near t , the important fact is that there is little mass at high frequencies, as is implied by the definition of almost periodicity and the boundedness of the frequency scale function $N(t)$. Far from t , the driving force behind regularity is the spherical symmetry of the solution. Of course this symmetry is only valuable at large radii. Consequently, we are only able to exploit it by using the in/out decomposition described in Section 6.

Let us now begin the proof. For the remainder of the section, u will denote a solution to (1) that obeys the hypotheses of Theorem 7.1.

We first record some basic local estimates. From mass conservation we have

$$\|u\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} \lesssim_u 1, \tag{80}$$

while from Definition 1.14 and the fact that $N(t)$ is bounded we have

$$\lim_{N \rightarrow \infty} \|u_{\geq N}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} = 0.$$

From Lemma 3.9 and $N(t) \lesssim 1$, we have

$$\|u\|_{L_{t,x}^4(J \times \mathbb{R}^2)} \lesssim_u \langle |J| \rangle^{1/4} \tag{81}$$

for all intervals $J \subset \mathbb{R}$. By Hölder, this implies

$$\|F(u)\|_{L_{t,x}^{4/3}(J \times \mathbb{R}^2)} \lesssim_u \langle |J| \rangle^{3/4}$$

and then, by the (endpoint) Strichartz inequality (Lemma 2.5),

$$\|u\|_{L_t^2 L_x^\infty(J \times \mathbb{R}^2)} \lesssim_u \langle |J| \rangle^{1/2}. \tag{82}$$

More precisely, one first treats the case $|J| = O(1)$ using (81) and then larger intervals by subdivision. Similarly, from the weighted Strichartz inequality (Lemma 2.7),

$$\| |x|^{1/2} u \|_{L_t^4 L_x^\infty(J \times \mathbb{R}^2)} \lesssim_u \langle |J| \rangle^{1/4}. \tag{83}$$

Now, for any dyadic number N , define

$$\mathcal{M}(N) := \|u_{\geq N}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)}. \tag{84}$$

From the discussion above, we see that $\mathcal{M}(N) \lesssim_u 1$ and

$$\lim_{N \rightarrow \infty} \mathcal{M}(N) = 0. \tag{85}$$

To prove Theorem 7.1, it suffices to show that $\mathcal{M}(N) \lesssim_{u,s} N^{-s}$ for any $s > 0$ and all N sufficiently large depending on u and s . This will immediately follow from iterating the following proposition with a suitably small choice of η (depending on u and s):

Proposition 7.2 (Regularity). *Let u be as in Theorem 7.1 and let $\eta > 0$ be a small number. Then*

$$\mathcal{M}(N) \lesssim_u \eta \mathcal{M}(N/8)$$

whenever N is sufficiently large depending on u and η .

The rest of this section is devoted to proving Proposition 7.2. Our task is to show that

$$\|u_{\geq N}(t_0)\|_{L_x^2(\mathbb{R}^2)} \lesssim_u \eta \mathcal{M}(N/8)$$

for all times t_0 and all N sufficiently large (depending on u and η). By time translation symmetry, we may assume $t_0 = 0$. As suggested above, one of the keys to obtaining additional regularity is Lemma 1.17. Specifically, we have

$$\begin{aligned} u_{\geq N}(0) &= (P^+ + P^-)u_{\geq N}(0) \\ &= \lim_{T \rightarrow \infty} i \int_0^T P^+ e^{-it\Delta} P_{\geq N} F(u(t)) dt - i \lim_{T \rightarrow \infty} \int_{-T}^0 P^- e^{-it\Delta} P_{\geq N} F(u(t)) dt, \end{aligned} \tag{86}$$

where the limit is to be interpreted as a weak limit in L^2 . However, this representation is not useful for $|x|$ small because the kernels of P^\pm have a logarithmic singularity at $x = 0$. To deal with this, we will use a different representation for $|x| \leq N^{-1}$, namely

$$u_{\geq N}(0) = \lim_{T \rightarrow \infty} i \int_0^T e^{-it\Delta} P_{\geq N} F(u(t)) dt, \tag{87}$$

also as a weak limit. To deal with the poor nature of these limits, we note that

$$f_T \rightarrow f \text{ weakly along a subsequence} \Rightarrow \|f\| \leq \limsup_{T \rightarrow \infty} \|f_T\|, \tag{88}$$

or equivalently, that the unit ball is weakly closed.

Despite the fact that different representations will be used depending on the size of $|x|$, some estimates can be dealt with in a uniform manner. The first such example is a bound on integrals over short times.

Lemma 7.3 (Local estimate). *For any $\eta > 0$, there exists $\delta = \delta(u, \eta) > 0$ such that*

$$\left\| \int_0^\delta e^{-it\Delta} P_{\geq N} F(u(t)) dt \right\|_{L_x^2} \lesssim_u \eta \mathcal{M}(N/8),$$

provided N is sufficiently large depending on u and η . An analogous estimate holds for integration over $[-\delta, 0]$ and after pre-multiplication by P^\pm (they are bounded operators on L_x^2).

Proof. By Lemma 2.5, it suffices to prove

$$\|P_{\geq N} F(u)\|_{L_t^2 L_x^1(J \times \mathbb{R}^2)} \lesssim_u \eta \mathcal{M}(N/8)$$

for any interval J of length $|J| \leq \delta$ and all sufficiently large N depending on u and η .

From (85), there exists $N_0 = N_0(u, \eta)$ such that

$$\|u_{\geq N_0}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} \leq \eta. \tag{89}$$

Let $N > 8N_0$. We decompose

$$u = u_{\geq N/8} + u_{N_0 \leq \cdot < N/8} + u_{< N_0}.$$

Any term in the resulting decomposition of $P_{\geq N} F(u)$ which does not involve at least one $u_{\geq N/8}$ will vanish.

Consider first a term with two factors of the form $u_{< N_0}$. Using Hölder’s inequality, (80), (84), and Lemma 2.1, we estimate

$$\begin{aligned} & \|u_{\geq N/8} u_{< N_0}^2\|_{L_t^2 L_x^1(J \times \mathbb{R}^2)} \\ & \lesssim |J|^{1/2} \|u_{\geq N/8}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} \|u_{< N_0}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} \|u_{< N_0}\|_{L_t^\infty L_x^\infty(\mathbb{R} \times \mathbb{R}^2)} \\ & \lesssim_u |J|^{1/2} \mathcal{M}(N/8) N_0. \end{aligned}$$

Choosing δ sufficiently small depending on η and N_0 , we see that this term is acceptable.

It remains only to consider those components of $P_{\geq N} F(u)$ which involve $u_{\geq N/8}$ and at least one other term which is not $u_{< N_0}$. We estimate such terms using Hölder’s inequality, (82), (84), and (89) in the following fashion:

$$\begin{aligned} \|u_{\geq N/8} u_{\geq N_0} u\|_{L_t^2 L_x^1(J \times \mathbb{R}^2)} & \lesssim \|u_{\geq N_0}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} \|u_{\geq N/8}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} \|u\|_{L_t^2 L_x^\infty(J \times \mathbb{R}^2)} \\ & \lesssim_u \eta \mathcal{M}(N/8) \langle |J| \rangle^{1/2}. \end{aligned}$$

This completes the proof of the lemma. □

We now turn our attention to $|t| \geq \delta$. In this case we make the decomposition

$$P_{\geq N} = \sum_{M \geq N} P_M \tilde{P}_M,$$

where $\tilde{P}_M := P_{M/2} + P_M + P_{2M}$. In this way, (87) becomes

$$\begin{aligned} u_{\geq N}(0, x) &= i \int_0^\delta e^{-it\Delta} P_{\geq N} F(u(t)) dt \\ &+ \lim_{T \rightarrow \infty} \sum_{M \geq N} i \int_\delta^T \int_{\mathbb{R}^2} [P_M e^{-it\Delta}](x, y) [\tilde{P}_M F(u(t))](y) dy dt, \tag{90} \end{aligned}$$

which we will use when $|x| \leq N^{-1}$. The analogous reformulation of (86), namely

$$\begin{aligned}
 u_{\geq N}(0, x) &= i \int_0^\delta P^+ e^{-it\Delta} P_{\geq N} F(u(t)) dt - i \int_{-\delta}^0 P^- e^{-it\Delta} P_{\geq N} F(u(t)) dt \\
 &\quad + \lim_{T \rightarrow \infty} \sum_{M \geq N} i \int_\delta^T \int_{\mathbb{R}^2} [P_M^+ e^{-it\Delta}](x, y) [\tilde{P}_M F(u(t))](y) dy dt \\
 &\quad - \lim_{T \rightarrow \infty} \sum_{M \geq N} i \int_{-T}^{-\delta} \int_{\mathbb{R}^2} [P_M^- e^{-it\Delta}](x, y) [\tilde{P}_M F(u(t))](y) dy dt, \quad (91)
 \end{aligned}$$

will be used when $|x|$ is large.

To estimate the integrals where $|t| \geq \delta$, we break the region of (t, y) integration into two pieces, namely, where $|y| \gtrsim M|t|$ and $|y| \ll M|t|$. The former is the more significant region; it contains the points where the integral kernels $P_M e^{it\Delta}(x, y)$ and $P_M^\pm e^{it\Delta}(x, y)$ are large (see Lemma 2.4 and Proposition 6.2). More precisely, when $|x| \leq N^{-1}$, we use (90); in this case $|y - x| \sim M|t|$ implies $|y| \gtrsim M|t|$ for $|t| \geq \delta \geq N^{-2}$. (This last condition can be subsumed under our hypothesis N sufficiently large depending on u and η .) When $|x| \geq N^{-1}$, we use (91); in this case $|y| - |x| \sim M|t|$ implies $|y| \gtrsim M|t|$.

The next lemma bounds the integrals over the significant region $|y| \gtrsim M|t|$. Let χ_k denote the characteristic function of the set

$$\{(t, y) : 2^k \delta \leq |t| \leq 2^{k+1} \delta, |y| \gtrsim M|t|\}.$$

Lemma 7.4 (Main contribution). *Let $\eta > 0$ be a small number and let δ be as in Lemma 7.3. Then*

$$\sum_{M \geq N} \sum_{k=0}^\infty \left\| \int_{2^k \delta}^{2^{k+1} \delta} \int_{\mathbb{R}^2} [P_M e^{it\Delta}](x, y) \chi_k(t, y) [\tilde{P}_M F(u(t))](y) dy dt \right\|_{L_x^2} \lesssim_u \eta \mathcal{M}(N/8)$$

for all N sufficiently large depending on u and η . An analogous estimate holds with P_M replaced by P_M^+ or P_M^- ; moreover, the time integrals may be taken over $[-2^{k+1} \delta, -2^k \delta]$.

Proof. By the adjoint of the weighted Strichartz estimate (28) and then Hölder’s inequality and (84),

$$\begin{aligned}
 &\left\| \int_{2^k \delta}^{2^{k+1} \delta} \int_{\mathbb{R}^2} [P_M e^{it\Delta}](x, y) \chi_k(t, y) [\tilde{P}_M F(u(t))](y) dy dt \right\|_{L_x^2} \\
 &\lesssim (M2^k \delta)^{-1/2} \|\chi_k \tilde{P}_M F(u)\|_{L_t^{4/3} L_x^1} \\
 &\lesssim (M2^k \delta)^{-1/2} \|u_{\geq M/8}\|_{L_t^\infty L_x^2} \|u\|_{L_t^\infty L_x^2} \{ \|\chi_k u_{\geq M}\|_{L_t^{4/3} L_x^\infty} + \|\chi_k u_{< M}\|_{L_t^{4/3} L_x^\infty} \} \\
 &\lesssim_u (M2^k \delta)^{-1/2} \mathcal{M}(N/8) \{ M^{-1/2} \| |y|^{1/2} u_{\geq M} \|_{L_t^4 L_x^\infty([2^k \delta, 2^{k+1} \delta] \times \mathbb{R}^2)} \\
 &\quad + M^{-1/2} (2^k \delta)^{1/4} \| |y|^{1/2} u_{< M} \|_{L_{t,x}^\infty} \}.
 \end{aligned}$$

By (83),

$$\| |y|^{1/2} u_{\geq M} \|_{L_t^4 L_x^\infty([2^k \delta, 2^{k+1} \delta] \times \mathbb{R}^2)} \lesssim_u \langle 2^k \delta \rangle^{1/4},$$

while by Lemma 2.2 (with $d = 2$) and the conservation of mass,

$$\| |y|^{1/2} u_{<M} \|_{L_{t,x}^\infty} \lesssim_u M^{1/2}.$$

Thus,

$$\begin{aligned} & \left\| \int_{2^k \delta}^{2^{k+1} \delta} \int_{\mathbb{R}^2} [P_M e^{it\Delta}](x, y) \chi_k(t, y) [\tilde{P}_M F(u(t))](y) dy dt \right\|_{L_x^2} \\ & \lesssim_u (M 2^k \delta)^{-1/2} \mathcal{M}(N/8) (M^{-1/2} \langle 2^k \delta \rangle^{1/4} + (2^k \delta)^{1/4}). \end{aligned}$$

Summing first over $k \geq 0$ and then over $M \geq N$, we obtain

$$\begin{aligned} & \sum_{M \geq N} \sum_{k=0}^\infty \left\| \int_{2^k \delta}^{2^{k+1} \delta} \int_{\mathbb{R}^2} [P_M e^{it\Delta}](x, y) \chi_k(t, y) [\tilde{P}_M F(u(t))](y) dy dt \right\|_{L_x^2} \\ & \lesssim_u (N^{-1} \delta^{-1/2} + N^{-1/2} \delta^{-1/4}) \mathcal{M}(N/8). \end{aligned}$$

Choosing N sufficiently large depending on δ and η (and hence only on u and η), we obtain the desired bound.

The last claim follows from the L_x^2 -boundedness of P^\pm and the time reversal symmetry of the argument just presented. \square

We now turn to the region of (t, y) integration where $|y| \ll M|t|$. First, we describe the bounds that we will use for the kernels of the propagators. For $|x| \leq N^{-1}$, $|y| \ll M|t|$, and $|t| \geq \delta \gg N^{-2}$,

$$|P_M e^{-it\Delta}(x, y)| \lesssim \frac{M^2}{(M^2|t|)^{50}}; \tag{92}$$

this follows from Lemma 2.4 since under these constraints, $|y - x| \ll M|t|$. For $|x| \geq N^{-1}$ and y and t as above,

$$|P_M^\pm e^{-it\Delta}(x, y)| \lesssim \frac{M^2}{(M^2|t|)^{50} \langle M|x| - M|y| \rangle^{50}} \tag{93}$$

by Proposition 6.2. Note that we have used $|y| - |x| \ll M|t|$ and

$$\langle M^2|t| + M|x| - M|y| \rangle^{-100} \lesssim (M^2|t|)^{-50} \langle M|x| - M|y| \rangle^{-50}$$

in order to simplify the bound.

Let $\tilde{\chi}_k$ denote the characteristic function of the set

$$\{(t, y) : 2^k \delta \leq |t| \leq 2^{k+1} \delta, |y| \ll M|t|\}.$$

Lemma 7.5 (The tail). *Let $\eta > 0$ be a small number and let δ be as in Lemma 7.3. Then*

$$\begin{aligned} & \sum_{M \geq N} \sum_{k=0}^\infty \left\| \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{M^2}{(M^2|t|)^{50} \langle M|x| - M|y| \rangle^{50}} \tilde{\chi}_k(t, y) [\tilde{P}_M F(u(t))](y) dy dt \right\|_{L_x^2} \\ & \lesssim_u \eta \mathcal{M}(N/8) \end{aligned}$$

for all N sufficiently large depending on u and η .

Proof. Using Young’s inequality, Hölder’s inequality, (82), and (84), we have

$$\begin{aligned} & \left\| \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{M^2}{(M^2|t|)^{50} \langle M|x| - M|y| \rangle^{50}} \tilde{\chi}_k(t, y) [\tilde{P}_M F(u(t))](y) dy dt \right\|_{L_x^2} \\ & \lesssim M^2 (M^2 2^k \delta)^{-50} \|\tilde{\chi}_k \tilde{P}_M F(u)\|_{L_t^1 L_x^2} \\ & \lesssim M^2 (M^2 2^k \delta)^{-50} \|u_{\geq M/8}\|_{L_t^\infty L_x^2} \|u\|_{L_t^2 L_x^\infty([2^k \delta, 2^{k+1} \delta] \times \mathbb{R}^2)}^2 \\ & \lesssim_u M^2 (M^2 2^k \delta)^{-50} \mathcal{M}(N/8) \langle 2^k \delta \rangle. \end{aligned}$$

Summing over $k \geq 0$ and $M \geq N$, we get

$$\sum_{M \geq N} \sum_{k=0}^\infty \left\| \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{M^2}{(M^2|t|)^{50}} \tilde{\chi}_k(t, y) [\tilde{P}_M F(u(t))](y) dy dt \right\|_{L_x^2} \lesssim_u N^{-98} \delta^{-50} \mathcal{M}(N/8).$$

The claim follows by choosing N sufficiently large depending on δ and η (and hence only on u and η). □

We have now gathered enough information to complete the

Proof of Proposition 7.2. Naturally, we may bound $\|u_{\geq N}\|_{L^2}$ by separately bounding the L^2 norm on the ball $\{|x| \leq N^{-1}\}$ and on its complement. On the ball, we use (90), while outside the ball we use (91). Invoking (88) and the triangle inequality, we reduce the proof to bounding certain integrals. The integrals over short times were estimated in Lemma 7.3. For $|t| \geq \delta$, we further partition the region of integration into two main pieces. The first piece, where $|y| \gtrsim M|t|$, was dealt with in Lemma 7.4. To estimate the remaining piece, $|y| \ll M|t|$, one combines (92), (93), and Lemma 7.5. □

8. The double high-to-low frequency cascade

In this section, we use the additional regularity provided by Theorem 7.1 to preclude double high-to-low frequency cascade solutions.

Proposition 8.1 (Absence of double cascades). *Let $d = 2$. There are no nonzero global spherically symmetric solutions to (1) that are double high-to-low frequency cascades in the sense of Theorem 1.16.*

Proof. Suppose to the contrary that there is such a solution u . By Theorem 7.1, u lies in $C_t^0 H_x^1(\mathbb{R} \times \mathbb{R}^2)$. Hence the energy

$$E(u) = E(u(t)) := \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla u(t, x)|^2 + \mu \frac{1}{4} |u(t, x)|^4 \right) dx$$

is finite and conserved (see e.g. [11]). As we have $M(u) < M(Q)$ in the focusing case, the sharp Gagliardo–Nirenberg inequality (reproduced here as Theorem 1.9) gives

$$\|\nabla u(t)\|_{L_x^2(\mathbb{R}^2)}^2 \sim_u E(u) \sim_u 1 \quad \text{for all } t \in \mathbb{R}. \tag{94}$$

We will now reach a contradiction by proving that $\|\nabla u(t)\|_2 \rightarrow 0$ along any sequence where $N(t) \rightarrow 0$. The existence of two such time sequences is guaranteed by the fact that u is a double high-to-low frequency cascade.

Let $\eta > 0$ be arbitrary. By Definition 1.14, we can find $C = C(\eta, u) > 0$ such that

$$\int_{|\xi| \geq CN(t)} |\hat{u}(t, \xi)|^2 d\xi \leq \eta^2 \quad \text{for all } t.$$

Meanwhile, by Theorem 7.1, $u \in C_t^0 H_x^s(\mathbb{R} \times \mathbb{R}^2)$ for some $s > 1$. Thus,

$$\int_{|\xi| \geq CN(t)} |\xi|^{2s} |\hat{u}(t, \xi)|^2 d\xi \lesssim_u 1 \quad \text{for all } t \text{ and some } s > 1.$$

By Hölder’s inequality (or interpolation), we thus obtain

$$\int_{|\xi| \geq CN(t)} |\xi|^2 |\hat{u}(t, \xi)|^2 d\xi \lesssim_u \eta^{2(s-1)/s}.$$

On the other hand, from mass conservation and Plancherel’s theorem we have

$$\int_{|\xi| \leq CN(t)} |\xi|^2 |\hat{u}(t, \xi)|^2 d\xi \lesssim_u C^2 N(t)^2.$$

Summing these last two bounds and using Plancherel’s theorem again, we obtain

$$\|\nabla u(t)\|_{L_x^2(\mathbb{R}^2)} \lesssim_u \eta^{(s-1)/s} + CN(t) \quad \text{for all } t.$$

As $\eta > 0$ is arbitrary and there exists a sequence of times $t_n \rightarrow \infty$ such that $N(t_n) \rightarrow 0$ (u is a double high-to-low frequency cascade), we conclude $\|\nabla u(t_n)\|_2 \rightarrow 0$. This contradicts (94). \square

Remark. The role of the dimensional and spherical symmetry hypotheses in Proposition 8.1 is to guarantee regularity, more precisely that $u \in C_t^0 H_x^s$ for some $s > 1$. With a few modifications, the argument presented shows that the hypothesis of spherical symmetry can be replaced by this regularity assumption, as we now explain. For such regular solutions u , we may define the total momentum $\int_{\mathbb{R}^2} \text{Im}(\bar{u}\nabla u)$, which is conserved. By a Galilean transformation, we can set this momentum equal to zero; thus $\int_{\mathbb{R}^2} \xi |\hat{u}(t, \xi)|^2 d\xi = 0$. From this, mass conservation, and the uniform H_x^s bound for some $s > 1$, one can show that $\xi(t) \rightarrow 0$ whenever $N(t) \rightarrow 0$. On the other hand, a modification of the above argument gives

$$1 \sim_u \|\nabla u(t)\|_2 \lesssim \eta^{(s-1)/s} + C(N(t) + |\xi(t)|),$$

which is absurd.

Moreover, if regularity is assumed, the argument presented also shows that even a single-sided cascade is impossible. By a single-sided cascade we mean a solution with $N(t)$ bounded on a semi-infinite interval, say $[T, \infty)$, with $\liminf_{t \rightarrow \infty} N(t) = 0$. In particular, this argument provides an alternative way of deducing Corollary 5.2 from Theorem 5.1.

9. Death of a soliton

In this section, we use the additional regularity proved in Theorem 7.1 to rule out the third and final enemy, the soliton-like solution. Our approach here is similar to that in [35].

The key ingredient in disproving the existence of a soliton is a monotonicity formula related to the virial identity. Like all formulae of Morawetz type, it expresses the fact that as time passes, the wave moves from being incoming to being outgoing. In these crude terms, one expects that if $\vec{a}(x)$ is a vector field with $(x/|x|) \cdot \vec{a}(x)$ an increasing function of $|x|$, then

$$M_a(t) := 2 \operatorname{Im} \int_{\mathbb{R}^2} \bar{u}(t, x) \vec{a}(x) \cdot \nabla u(x, t) \, dx \tag{95}$$

should be an increasing function of time. Note that $2 \operatorname{Im}(\bar{u} \nabla u)$ is the momentum density (or mass current). We choose $\vec{a}(x) := x \psi(|x|/R)$, where ψ is a smooth function obeying

$$\psi(r) = \begin{cases} 1, & r \leq 1, \\ 0, & r \geq 2, \end{cases}$$

and R denotes a radius to be chosen momentarily. For solutions u to (1) belonging to $C_t^0 H_x^1$, $M_a(t)$ is a well-defined function. Indeed,

$$|M_a(t)| \lesssim R \|u(t)\|_2 \|\nabla u(t)\|_2 \lesssim_u R. \tag{96}$$

Although \vec{a} vanishes for large radii, a soliton-like solution (in the sense of Theorem 1.16) is space localised and so we may still expect monotonicity provided we choose $R \gg 1$. Substituting our choice of \vec{a} into (95) leads to the following

Lemma 9.1.

$$\begin{aligned} \partial_t M_a(t) &= 8E(u(t)) \\ &\quad - \int_{\mathbb{R}^2} \left[\frac{3}{R|x|} \psi' \left(\frac{|x|}{R} \right) + \frac{5}{R^2} \psi'' \left(\frac{|x|}{R} \right) + \frac{|x|}{R^3} \psi''' \left(\frac{|x|}{R} \right) \right] |u(t, x)|^2 \, dx \end{aligned} \tag{97}$$

$$+ 4 \int_{\mathbb{R}^2} \left[\psi \left(\frac{|x|}{R} \right) - 1 + \frac{|x|}{R} \psi' \left(\frac{|x|}{R} \right) \right] |\nabla u(t, x)|^2 \, dx \tag{98}$$

$$+ \mu \int_{\mathbb{R}^2} \left[2\psi \left(\frac{|x|}{R} \right) - 2 + \frac{|x|}{R} \psi' \left(\frac{|x|}{R} \right) \right] |u(t, x)|^4 \, dx, \tag{99}$$

where $E(u)$ is the energy of u as defined in (9).

Proposition 9.2 (Absence of solitons). *Let $d = 2$. There are no nonzero global spherically symmetric solutions to (1) that are soliton-like in the sense of Theorem 1.16.*

Proof. Assume to the contrary that there is such a solution u . Then, by Theorem 7.1, $u \in C_t^0 H_x^s$ for any $s > 1$. In particular,

$$|M_a(t)| \lesssim_u R. \tag{100}$$

Recall that in the focusing case, $M(u) < M(Q)$. As a consequence, the sharp Gagliardo–Nirenberg inequality (reproduced here as Theorem 1.9) implies that the energy is a positive quantity in the focusing case as well as in the defocusing case. Indeed,

$$E(u) \gtrsim_u \int_{\mathbb{R}^2} |\nabla u(t, x)|^2 dx > 0.$$

We will show that (97) through (99) constitute only a small fraction of $E(u)$. Combining this fact with Lemma 9.1, we conclude that $\partial_t M_a(t) \gtrsim E(u) > 0$, which contradicts (100).

We first turn our attention to (97). This is trivially bounded by

$$|(97)| \lesssim_u R^{-2}. \tag{101}$$

We now study (98) and (99). Let $\eta > 0$ be a small number to be chosen later. By Definition 1.14 and the fact that $N(t) = 1$ for all $t \in \mathbb{R}$, there exists $R = R(\eta, u)$ sufficiently large such that

$$\int_{|x| \geq R/4} |u(t, x)|^2 dx \leq \eta \tag{102}$$

for all $t \in \mathbb{R}$. Let χ denote a smooth cutoff to the region $|x| \geq R/2$; in particular, $\nabla \chi$ is bounded by R^{-1} and supported where $|x| \sim R$. As $u \in C_t^0 H_x^s$ for some $s > 1$, using interpolation and (102), we estimate

$$\begin{aligned} |(98)| &\lesssim \|\chi \nabla u(t)\|_2^2 \lesssim \|\nabla(\chi u(t))\|_2^2 + \|u(t) \nabla \chi\|_2^2 \lesssim \|\chi u(t)\|_2^{2(s-1)/s} \|u(t)\|_{H_x^s}^{2/s} + \eta \\ &\lesssim_u \eta^{(s-1)/s} + \eta. \end{aligned} \tag{103}$$

Finally, we are left to consider (99). Using the same χ as above together with the Gagliardo–Nirenberg inequality and (102), we find

$$|(99)| \lesssim \|\chi u(t)\|_4^4 \lesssim \|\chi u(t)\|_2^2 \|\nabla(\chi u(t))\|_2^2 \lesssim_u \eta. \tag{104}$$

Combining (101), (103), and (104) and choosing η sufficiently small depending on u and R sufficiently large depending on u and η , we obtain

$$|(97)| + |(98)| + |(99)| \leq \frac{1}{100} E(u).$$

This completes the proof of the proposition for the reasons explained in the third paragraph. \square

Remark. The argument just presented can also be used to preclude both the (spherically symmetric) self-similar and double cascade solutions, once it is known that they are sufficiently regular. The idea is to study the behaviour of $M_a(t)$ on intervals $[1, T]$ for T very large; however, in this case we must choose $R \gg_u T^{1/2}$ rather than $R \gg_u 1$, which was the case above. A contradiction arises from $\partial_t M_a(t) > 0$ but $|M_a(t)| \lesssim_u T^{1/2}$ on $[1, T]$.

We did not delay the ruling out of these other two types of solutions until now because the arguments presented earlier are much simpler (and also more robust) than that just sketched.

10. A concentration result

In this section we prove Corollary 1.12. The proof is a consequence of Theorem 1.10 and relies on the linear profile decomposition Theorem 1.13.

It suffices to prove (11) since (12) can be deduced from this using the pseudoconformal transformation. To this end, let u be a spherically symmetric solution to (1) that blows up in finite time $0 < T^* < \infty$. Fix a sequence of times $t_n \nearrow T^*$.

After passing to a subsequence if necessary, we have the decomposition

$$u_n(0) := u(t_n) = \sum_{j=1}^J g_n^j e^{it_n^j \Delta} \phi^j + w_n^J$$

as in Lemma 1.13. Let ψ_n^j be the maximal-lifespan solution to (1) with initial data $\psi_n^j(0) = g_n^j e^{it_n^j \Delta} \phi^j$.

From (16) we obtain the mass decoupling

$$\sum_{j=1}^{\infty} M(\phi^j) \leq M(u). \tag{105}$$

Thus (reordering the indices j if necessary) we may assume that there exist $\varepsilon > 0$ and $J_0 \geq 1$ such that

$$M(\psi_n^j) = M(\phi^j) \leq M(Q) - \varepsilon \quad \text{for all } j \geq J_0 \tag{106}$$

and $M(\psi_n^j) = M(\phi^j) \geq M(Q)$ for $1 \leq j < J_0$. Combining the small data theory (see Theorem 1.4) with Theorem 1.10 and using (105), we conclude

$$\sum_{j \geq J_0} \|\psi_n^j\|_{L_{t,x}^4(\mathbb{R} \times \mathbb{R}^2)}^4 \lesssim \sum_{j \geq J_0} M(\phi^j) \lesssim_u 1. \tag{107}$$

Our next job is to find a profile responsible for the finite-time blowup of u . By (107), we should look amongst ψ_n^j with $1 \leq j < J_0$.

Lemma 10.1 (One bad profile). *There exists $1 \leq j_0 < J_0$ so that*

$$M(\phi^{j_0}) \geq M(Q) \quad \text{and} \quad \limsup_{n \rightarrow \infty} \|\psi_n^{j_0}\|_{L^4_{t,x}([0, T^* - t_n] \times \mathbb{R}^2)} = \infty. \tag{108}$$

Moreover, for each $\eta > 0$ there exists $1 \leq j_1 < J_0$ such that for infinitely many n , the $L^4_{t,x}$ norm of $\psi_n^{j_1}$ reaches η first, that is, there exists $0 < \tau_n < T^* - t_n$ such that

$$\|\psi_n^j\|_{L^4_{t,x}([0, \tau_n] \times \mathbb{R}^2)} \leq \|\psi_n^{j_1}\|_{L^4_{t,x}([0, \tau_n] \times \mathbb{R}^2)} = \eta \tag{109}$$

for all $1 \leq j < J_0$.

Proof. Suppose that (108) does not hold for any choice of j_0 . Then, by (107),

$$\sum_{j \geq 1} \|\psi_n^j\|_{L^4_{t,x}([0, T^* - t_n] \times \mathbb{R}^2)}^4 \lesssim_u 1 \tag{110}$$

for n sufficiently large. We will reach a contradiction by deducing that for n large, $\|u\|_{L^4_{t,x}([t_n, T^*] \times \mathbb{R}^2)} < \infty$, which is inconsistent with the assumption that u blows up at time T^* .

We will obtain our bound on u by combining perturbation theory with bounds on an approximation to $u(t)$, namely,

$$u_n^J(t) := \sum_{j=1}^J \psi_n^j(t) + e^{it\Delta} w_n^J.$$

We first note that for each $J \geq 1$,

$$M(u_n^J(0) - u_n(0)) = 0. \tag{111}$$

Next we establish finite $L^4_{t,x}$ bounds for u_n^J on $[0, T^* - t_n]$. Using (15), it is easy to check that

$$\lim_{n \rightarrow \infty} \|\psi_n^j \psi_n^{j'}\|_{L^2_{t,x}([0, T^* - t_n] \times \mathbb{R}^2)} = 0; \tag{112}$$

for details see [36, Lemma 2.7]. Combining this with (110), we estimate

$$\lim_{J \rightarrow \infty} \lim_{n \rightarrow \infty} \left\| \sum_{j=1}^J \psi_n^j \right\|_{L^4_{t,x}([0, T^* - t_n] \times \mathbb{R}^2)}^4 = \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{j=1}^J \|\psi_n^j\|_{L^4_{t,x}([0, T^* - t_n] \times \mathbb{R}^2)}^4 \lesssim_u 1,$$

and hence by the triangle inequality and Strichartz,

$$\lim_{J \rightarrow \infty} \lim_{n \rightarrow \infty} \|u_n^J\|_{L^4_{t,x}([0, T^* - t_n] \times \mathbb{R}^2)}^4 \lesssim_u 1 + \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|e^{it\Delta} w_n^J\|_{L^4_{t,x}([0, T^* - t_n] \times \mathbb{R}^2)}^4 \lesssim_u 1. \tag{113}$$

We now show that u_n^J asymptotically solves (1) in the sense that

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|(i\partial_t + \Delta)u_n^J - F(u_n^J)\|_{L^{4/3}_{t,x}([0, T^* - t_n] \times \mathbb{R}^2)} = 0. \tag{114}$$

By the definition of u_n^J , we have

$$(i\partial_t + \Delta)u_n^J = \sum_{j=1}^J F(\psi_n^j)$$

and so, by the triangle inequality, it suffices to show that

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|F(u_n^J - e^{it\Delta}w_n^J) - F(u_n^J)\|_{L_{t,x}^{4/3}([0, T^* - t_n] \times \mathbb{R}^2)} = 0$$

and

$$\lim_{n \rightarrow \infty} \left\| F\left(\sum_{j=1}^J \psi_n^j\right) - \sum_{j=1}^J F(\psi_n^j) \right\|_{L_{t,x}^{4/3}([0, T^* - t_n] \times \mathbb{R}^2)} = 0$$

for each J . That the first limit is zero follows from Hölder’s inequality, (14), and (113). For the second limit, we use the elementary inequality

$$\left| F\left(\sum_{j=1}^J z_j\right) - \sum_{j=1}^J F(z_j) \right| \lesssim_J \sum_{j \neq j'} |z_j| |z_{j'}|^2$$

together with Hölder’s inequality, (112), and (113).

Next we use the stability result, Lemma 1.5. For any small $\delta > 0$, one may choose J sufficiently large (depending on δ) and then n sufficiently large (depending on J, δ) so that

$$\|(i\partial_t + \Delta)u_n^J - F(u_n^J)\|_{L_{t,x}^{4/3}([0, T^* - t_n] \times \mathbb{R}^2)} \leq \delta.$$

Invoking (111) and (113) and applying Lemma 1.5 (for δ chosen small enough), we conclude that for n sufficiently large,

$$\|u\|_{L_{t,x}^4([t_n, T^*] \times \mathbb{R}^2)} < \infty.$$

This contradicts the assumption that u blows up at T^* . Thus, there exists at least one j_0 satisfying (108). In particular, $J_0 \geq 2$.

We now turn our attention to (109). By virtue of (108), there are infinitely many n such that

$$\|\psi_n^{j_0}\|_{L_{t,x}^4([0, T^* - t_n] \times \mathbb{R}^2)} > \eta.$$

For such n there exists $1 \leq j(n) < J_0$ that wins the $L_{t,x}^4$ race in the sense of (109). Choose j_1 to be any value that the sequence $j(n)$ achieves infinitely many times. \square

For the remainder of this section, we assume that (109) holds; this amounts to restricting n to a subsequence. Choosing $\eta > 0$ sufficiently small and using (109) and the Strichartz inequality, we conclude

$$\|e^{it\Delta}\psi_n^{j_1}(0)\|_{L_{t,x}^4([0, \tau_n] \times \mathbb{R}^2)} \gtrsim \|\psi_n^{j_1}\|_{L_{t,x}^4([0, \tau_n] \times \mathbb{R}^2)} - \|\psi_n^{j_1}\|_{L_{t,x}^4([0, \tau_n] \times \mathbb{R}^2)}^3 \gtrsim \eta.$$

Combining this with Lemma 2.9 (accounting for scaling), we extract a bubble of concentration that is not too close to τ_n . More precisely, there exists $\tau'_n \in (0, \tau_n)$ such that

$$\int_{|x| \lesssim |\tau_n - \tau'_n|^{1/2}} |e^{i\tau'_n \Delta} \psi_n^{j_1}(0)|^2 dx \gtrsim_{\phi^{j_1, \eta}} 1.$$

Let $t'_n := t_n + \tau'_n$. Using the fact that $\tau_n \in (0, T^* - t_n)$, we derive

$$\int_{|x| \lesssim |T^* - t'_n|^{1/2}} |e^{it'_n \Delta} \psi_n^{j_1}(0)|^2 dx \gtrsim_{\phi^{j_1, \eta}} 1. \tag{115}$$

As the next lemma shows, we can extract the whole mass of $\psi_n^{j_1}(0)$ by enlarging the diameter of the bubble.

Lemma 10.2 (Compactness of profiles). *Suppose*

$$\int_{|x-x_k| \leq r_k} |e^{it_k \Delta} \psi|^2 dx \geq \varepsilon$$

for some $\varepsilon > 0$ and sequences $t_k \in \mathbb{R}$, $x_k \in \mathbb{R}^2$, and $r_k > 0$. Then for any sequence $a_k \rightarrow \infty$,

$$\int_{|x| \leq a_k r_k} |e^{it_k \Delta} \psi|^2 dx \rightarrow M(\psi). \tag{116}$$

Proof. Without loss of generality, we may assume that $t_k \rightarrow t_\infty \in [-\infty, \infty]$. We first consider the case of a finite limit. In this case, $e^{it_k \Delta} \psi \rightarrow e^{it_\infty \Delta} \psi$ in L^2 and so

$$\liminf_{k \rightarrow \infty} \int_{|x-x_k| \leq r_k} |e^{it_\infty \Delta} \psi|^2 dx \geq \varepsilon.$$

This implies that $\liminf r_k > 0$ (also $\limsup |x_k|/r_k < \infty$) and so

$$\lim_{k \rightarrow \infty} \int_{|x| \leq a_k r_k} |e^{it_\infty \Delta} \psi|^2 dx = M(\psi).$$

Using $e^{it_k \Delta} \psi \rightarrow e^{it_\infty \Delta} \psi$ once again finishes the argument.

In the case that $t_k \rightarrow \pm\infty$, we use the Fraunhofer formula: for $t \rightarrow \pm\infty$,

$$\| [e^{it \Delta} \psi](x) - (2it)^{-1} e^{i|x|^2/4t} \hat{\psi}(x/2t) \|_{L_x^2} \rightarrow 0. \tag{117}$$

It allows us to conclude

$$\liminf_{k \rightarrow \infty} \int_{|y-y_k| \leq r_k/|2t_k|} |\hat{\psi}(y)|^2 dy \geq \varepsilon,$$

where $y_k = x_k/(2t_k)$; this shows that $\liminf r_k/|t_k| > 0$. Thus

$$\lim_{k \rightarrow \infty} \int_{|y| \leq a_k r_k/|2t_k|} |\hat{\psi}(y)|^2 dy = M(\psi),$$

which leads to (116) via another application of (117). □

Returning to (115) and applying Lemma 10.2 (accounting for scaling again), we obtain

$$\lim_{n \rightarrow \infty} \int_{|x| \leq R_n} |e^{i\tau'_n \Delta} \psi_n^{j_1}(0)|^2 dx = M(\psi_n^{j_1}) = M(\phi^{j_1}) \geq M(Q) \tag{118}$$

for any sequence R_n obeying $(T^* - t'_n)^{-1/2} R_n \rightarrow \infty$. It remains to show that a similar bubble can be found in $u(t'_n)$. This will be effected using perturbation theory (and asymptotic orthogonality) in much the same way as it was used to prove (108). This time we approximate $u_n(t)$ by u_n^J on $[0, \tau_n]$. Again $u_n^J(0) = u_n(0)$. To bound the $L^4_{t,x}$ norm of u_n^J on $[0, \tau_n]$ we argue as for (113) using (107) and (109) in place of (110). Repeating the argument that proved (114) we obtain

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|(i\partial_t + \Delta)u_n^J - F(u_n^J)\|_{L^{4/3}_{t,x}([0, \tau_n] \times \mathbb{R}^2)} = 0.$$

Therefore, by Lemma 1.5, we conclude

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|u(t'_n) - u_n^J(\tau'_n)\|_{L^2_x} = 0. \tag{119}$$

To continue, we note that by (15),

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} |\langle u_n^J(\tau'_n), \psi_n^{j_1}(\tau'_n) \rangle_{L^2_x}| = M(\psi_n^{j_1}) = M(\phi^{j_1}) \geq M(Q). \tag{120}$$

Here, we also used the fact that $w_n^J \rightharpoonup 0$ weakly in $G_{\text{rad}} \setminus L^2_x$ by virtue of (14). Putting together (119) and (120), we obtain

$$\limsup_{n \rightarrow \infty} |\langle u(t'_n), \psi_n^{j_1}(\tau'_n) \rangle_{L^2_x}| \geq M(Q).$$

By (109) and the Strichartz inequality, this implies

$$\limsup_{n \rightarrow \infty} |\langle u(t'_n), e^{i\tau'_n \Delta} \psi_n^{j_1}(0) \rangle_{L^2_x}| \geq M(Q) - \eta$$

provided $\eta > 0$ is sufficiently small. Combining this with (118), we conclude

$$\limsup_{n \rightarrow \infty} \int_{|x| \leq R_n} |u(t'_n, x)|^2 dx \geq M(Q) - \eta.$$

Sending η to zero completes the proof of Corollary 1.12.

Acknowledgments. We are grateful to Shuanglin Shao for access to preliminary drafts of his work.

R. K. was supported by NSF grant DMS-0401277 and a Sloan Foundation Fellowship. He is also grateful to the Institute for Advanced Study (Princeton) for its hospitality. T. T. was supported by a grant from the MacArthur Foundation. M. V. was supported under NSF grant DMS-0111298 and as a Liftoff Fellow of the Clay Mathematics Institute.

Any opinions, findings and conclusions or recommendations expressed are those of the authors and do not reflect the views of the National Science Foundation.

References

- [1] Bégout, P., Vargas, A.: Mass concentration phenomena for the L^2 -critical nonlinear Schrödinger equation. *Trans. Amer. Math. Soc.* **359**, 5257–5282 (2007) Zbl pre05178492 MR 2327030
- [2] Berestycki, H., Lions, P.-L.: Existence d'ondes solitaires dans des problèmes nonlinéaires du type Klein–Gordon. *C. R. Acad. Sci. Paris Sér. A-B* **288**, A395–A398 (1979) Zbl 0397.35024 MR 0552061
- [3] Blue, P., Colliander, J.: Global well-posedness in Sobolev space implies global existence for weighted L^2 initial data for L^2 -critical NLS. *Comm. Pure Appl. Anal.* **5**, 691–708 (2006) Zbl 1137.35066 MR 2246003
- [4] Bourgain, J.: On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE. *Int. Math. Res. Notices* **1996**, no. 6, 277–304 Zbl 0934.35166 MR 1386079
- [5] Bourgain, J.: Refinements of Strichartz' inequality and applications to 2d-NLS with critical nonlinearity. *Int. Math. Res. Notices* **1998**, no. 5, 253–284 Zbl 0917.35126 MR 1616917
- [6] Bourgain, J.: Global well-posedness of defocusing 3D critical NLS in the radial case. *J. Amer. Math. Soc.* **12**, 145–171 (1999) Zbl 0958.35126 MR 1626257
- [7] Bourgain, J.: Global solutions of nonlinear Schrödinger equations. *Amer. Math. Soc. Colloq. Publ.* 46, Amer. Math. Soc., Providence, RI (1999) Zbl 0933.35178 MR 1691575
- [8] Carles, R.: Critical nonlinear Schrödinger equations with and without harmonic potential. *Math. Models Methods Appl. Sci.* **12**, 1513–1523 (2002) Zbl 1029.35208 MR 1933935
- [9] Carles, R., Keraani, S.: On the role of quadratic oscillations in nonlinear Schrödinger equation II. The L^2 -critical case. *Trans. Amer. Math. Soc.* **359**, 33–62 (2007) Zbl 1115.35119 MR 2247881
- [10] Cazenave, T., Weissler, F. B.: Some remarks on the nonlinear Schrödinger equation in the subcritical case. In: *New Methods and Results in Nonlinear Field Equations* (Bielefeld, 1987), *Lecture Notes in Phys.* 347, Springer, Berlin, 59–69 (1989) Zbl 0699.35217 MR 1030060
- [11] Cazenave, T.: *Semilinear Schrödinger Equations*. *Courant Lecture Notes in Math.* 10, Amer. Math. Soc. (2003) Zbl 1055.35003 MR 2002047
- [12] Christ, M., Kiselev, A.: Maximal functions associated to filtrations. *J. Funct. Anal.* **179**, 409–425 (2001) Zbl 0974.47025 MR 1809116
- [13] Colliander, J., Delort, J., Kenig, C., Staffilani, G.: Bilinear estimates and applications to 2D NLS. *Trans. Amer. Math. Soc.* **353**, 3307–3325 (2001) Zbl 0970.35142 MR 1828607
- [14] Colliander, J., Grillakis, M., Tzirakis, N.: Improved interaction Morawetz inequalities for the cubic nonlinear Schrödinger equation on \mathbb{R}^2 . *Int. Math. Res. Notices* **2007**, no. 23, art. ID rnm 090, 30 pp. Zbl 1142.35085 MR 2377216
- [15] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation. *Math. Res. Lett.* **9**, 659–682 (2002) Zbl 1152.35491 MR 1906069
- [16] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in \mathbb{R}^3 . *Ann. of Math.* **167**, 767–865 (2008) Zbl pre05578705 MR 2415387
- [17] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Resonant decompositions and the I -method for the cubic nonlinear Schrödinger equation on \mathbb{R}^2 . *Discrete Contin. Dynam. Systems* **21**, 665–686 (2008) Zbl 1147.35095 MR 2399431
- [18] Colliander, J., Raynor, S., Sulem, C., Wright, J. D.: Ground state mass concentration in the L^2 -critical nonlinear Schrödinger equation below H^1 . *Math. Res. Lett.* **12**, 357–375 (2005) Zbl 1084.35088 MR 2150890

- [19] De Silva, D., Pavlović, N., Staffilani, G., Tzirakis, N.: Global well-posedness for the L^2 -critical nonlinear Schrödinger equation in higher dimensions. *Comm. Pure Appl. Anal.* **6**, 1023–1041 (2007) Zbl 1152.35106 MR 2341818
- [20] De Silva, D., Pavlović, N., Staffilani, G., Tzirakis, N.: Global well-posedness and polynomial bounds for the defocusing L^2 -critical nonlinear Schrödinger equation in \mathbb{R} . *Comm. Partial Differential Equations* **33**, 1395–1429 (2008) Zbl 1155.35310 MR 2450163
- [21] Dysthe, K. B.: Modelling a “rogue wave”—speculations or a realistic possibility? In: *Rogue Waves 2000*, Éditions Ifremer, Plouzané, 255–264 (2001)
- [22] Erdős, L., Schlein, B., Yau, H.-T.: Derivation of the Gross–Pitaevskii hierarchy for the dynamics of Bose–Einstein condensate. *Comm. Pure Appl. Math.* **59**, 1659–1741 (2006) Zbl 1122.82018 MR 2257859
- [23] Glassey, R. T.: On the blowing up of solution to the Cauchy problem for nonlinear Schrödinger operators. *J. Math. Phys.* **8**, 1794–1797 (1977) Zbl 0372.35009 MR 0460850
- [24] Ginibre, J., Velo, G.: Scattering theory in the energy space for a class of nonlinear Schrödinger equations. *J. Math. Pures Appl.* **64**, 363–401 (1985) Zbl 0535.35069 MR 0839728
- [25] Ginibre, J., Velo, G.: Smoothing properties and retarded estimates for some dispersive evolution equations. *Comm. Math. Phys.* **144**, 163–188 (1992) Zbl 0762.35008 MR 1151250
- [26] Grillakis, M., Fang, Y.: On the global existence of rough solutions of the cubic defocusing Schrödinger equation in \mathbb{R}^{2+1} . *J. Hyperbolic Differential Equations* **4**, 233–257 (2007) Zbl 1122.35132 MR 2329384
- [27] Gross, E. P.: Hydrodynamics of a superfluid condensate. *J. Math. Phys.* **4**, 195–207 (1963)
- [28] Hayashi, N., Tsutsumi, Y.: Remarks on the scattering problem for nonlinear Schrödinger equations. In: *Differential Equations and Mathematical Physics* (Birmingham, AL, 1986), Lecture Notes in Math. 1285, Springer, Berlin, 162–168 (1987) Zbl 0633.35059 MR 0921265
- [29] Henderson, K. L., Peregrine, D. H., Dold, J. W.: Unsteady water wave modulations: fully nonlinear solutions and comparison with the nonlinear Schrödinger equation. *Wave Motion* **29**, 341–361 (1999) Zbl 1074.76513 MR 1686798
- [30] Kato, T.: On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* **46**, 113–129 (1987) Zbl 0632.35038 MR 0877998
- [31] Kato, T.: On nonlinear Schrödinger equations, II. H^s -solutions and unconditional well-posedness. *J. Anal. Math.* **67**, 281–306 (1995) Zbl 0848.35124 MR 1383498
- [32] Keel, M., Tao, T.: Endpoint Strichartz estimates. *Amer. J. Math.* **120**, 955–980 (1998) Zbl 0922.35028 MR 1646048
- [33] Kelley, P. L.: Self-focusing of optical beams. *Phys. Rev. Lett.* **15**, 1005–1008 (1965)
- [34] Kenig, C.: Global well-posedness and scattering for the energy critical focusing nonlinear Schrödinger and wave equations. Lectures given at “Analyse des équations aux dérivées partielles”, Evian-les-Bains (2007)
- [35] Kenig, C., Merle, F.: Global well-posedness, scattering, and blowup for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Invent. Math.* **166**, 645–675 (2006) Zbl 1115.35125 MR 2257393
- [36] Keraani, S.: On the defect of compactness for the Strichartz estimates of the Schrödinger equations. *J. Differential Equations* **175**, 353–392 (2001) Zbl 1038.35119 MR 1855973
- [37] Keraani, S.: On the blow-up phenomenon of the critical nonlinear Schrödinger equation. *J. Funct. Anal.* **235**, 171–192 (2006) Zbl 1099.35132 MR 2216444
- [38] Kwong, M. K.: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in \mathbb{R}^n . *Arch. Ration. Mech. Anal.* **105**, 243–266 (1989) Zbl 0676.35032 MR 0969899
- [39] Merle, F.: Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equation with critical power. *Duke Math. J.* **69**, 427–453 (1993) Zbl 0808.35141 MR 1203233

- [40] Merle, F.: Existence of blow-up solutions in the energy space for the critical generalized KdV equation. *J. Amer. Math. Soc.* **14**, 555–578 (2001) Zbl 0970.35128 MR 1824989
- [41] Merle, F., Raphael, P.: Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation. *Geom. Funct. Anal.* **13**, 591–642 (2003) Zbl 1061.35135 MR 1995801
- [42] Merle, F., Raphael, P.: On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation. *Invent. Math.* **156**, 565–672 (2004) Zbl 1067.35110 MR 2061329
- [43] Merle, F., Raphael, P.: The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Ann. of Math.* **161**, 157–222 (2005) Zbl pre02204253 MR 2150386
- [44] Merle, F., Tsutsumi, Y.: L^2 concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power nonlinearity. *J. Differential Equations* **84**, 205–214 (1990) Zbl 0722.35047 MR 1047566
- [45] Merle, F., Vega, L.: Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D. *Int. Math. Res. Notices* **1998**, no. 8, 399–425 Zbl 0913.35126 MR 1628235
- [46] Montgomery-Smith, S. J.: Time decay for the bounded mean oscillation of solutions of the Schrödinger and wave equations. *Duke Math. J.* **91**, 393–408 (1998) Zbl 0955.35012 MR 1600602
- [47] Morawetz, C.: Time decay for the nonlinear Klein–Gordon equation. *Proc. Roy. Soc. A* **306**, 291–296 (1968) Zbl 0157.41502 MR 0234136
- [48] Nakanishi, K.: Energy scattering for non-linear Klein–Gordon and Schrödinger equations in spatial dimensions 1 and 2. *J. Funct. Anal.* **169**, 201–225 (1999) Zbl 0942.35159 MR 1726753
- [49] Nawa, H.: Asymptotic and limiting profiles of blowup solutions of the nonlinear Schrödinger equation with critical power. *Comm. Pure Appl. Math.* **52**, 193–270 (1999) Zbl 0964.37014 MR 1653454
- [50] Perelman, G.: On the formation of singularities in solutions of the critical nonlinear Schrödinger equation. *Ann. H. Poincaré* **2**, 605–673 (2001) Zbl 1007.35087 MR 1852922
- [51] Pitaevskii, L. P.: Vortex lines in an imperfect Bose gas. *Sov. Phys. JETP* **13**, 451–454 (1961)
- [52] Ryckman, E., Visan, M.: Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in \mathbb{R}^{1+4} . *Amer. J. Math.* **129**, 1–60 (2007) Zbl 1160.35067 MR 2288737
- [53] Shao, S.: Sharp linear and bilinear restriction estimates for the paraboloid in the cylindrically symmetric case. Preprint
- [54] Staffilani, G.: Quadratic forms for a 2D semilinear Schrödinger equation. *Duke Math. J.* **86**, 79–107 (1997) Zbl 0872.35104 MR 1427846
- [55] Staffilani, G.: On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations. *Duke Math. J.* **86**, 109–142 (1997) Zbl 0874.35114 MR 1427847
- [56] Stefanov, A.: Strichartz estimates for the Schrödinger equation with radial data. *Proc. Amer. Math. Soc.* **129**, 1395–1401 (2001) Zbl 1009.35016 MR 1814165
- [57] Stein, E. M.: *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton Univ. Press (1993) Zbl 0821.42001 MR 1232192
- [58] Stein, E. M., Weiss, G.: Fractional integrals on n -dimensional Euclidean space. *J. Math. Mech.* **7**, 503–514 (1958) Zbl 0082.27201 MR 0098285
- [59] Strichartz, R. S.: Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.* **44**, 705–774 (1977) Zbl 0372.35001 MR 0512086
- [60] Sulem, C., Sulem, P.-L.: *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*. Springer, New York (1999) Zbl 0928.35157

- [61] Talanov, V. I.: Self modelling wave beams in a nonlinear dielectric. *Radiophysics and Quantum Electronics* **9**, 260–261 (1967)
- [62] Tao, T.: Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation. *Comm. Partial Differential Equations* **25**, 1471–1485 (2000) Zbl 0966.35027 MR 1765155
- [63] Tao, T.: On the asymptotic behavior of large radial data for a focusing non-linear Schrödinger equation. *Dynam. Partial Differential Equations* **1**, 1–48 (2004) Zbl 1082.35144 MR 2091393
- [64] Tao, T.: Global well-posedness and scattering for the higher-dimensional energy-critical non-linear Schrödinger equation for radial data. *New York J. Math.* **11**, 57–80 (2005) Zbl 1119.35092 MR 2154347
- [65] Tao, T.: *Nonlinear Dispersive Equations. Local and Global Analysis.* CBMS Reg. Conf. Ser. Math. 106, Amer. Math. Soc., Providence, RI (2006) Zbl 1106.35001 MR 2233925
- [66] Tao, T.: A (concentration-)compact attractor for high-dimensional non-linear Schrödinger equations. *Dynam. Partial Differential Equations* **4**, 1–53 (2007) Zbl 1142.35088 MR 2304091
- [67] Tao, T.: A pseudoconformal compactification of the nonlinear Schrödinger equation and applications. *math.AP/0606254*
- [68] Tao, T., Visan, M.: Stability of energy-critical nonlinear Schrödinger equations in high dimensions. *Electron. J. Differential Equations* **2005**, no. 118, 28 pp. Zbl pre05002687 MR 2174550
- [69] Tao, T., Visan, M., Zhang, X.: The nonlinear Schrödinger equation with combined power-type nonlinearities. *Comm. Partial Differential Equations* **32**, 1281–1343 (2007) Zbl pre05204438 MR 2354495
- [70] Tao, T., Visan, M., Zhang, X.: Minimal-mass blowup solutions of the mass-critical NLS. *Forum Math.* **20**, 881–919 (2008) Zbl 1154.35085 MR 2445122
- [71] Tao, T., Visan, M., Zhang, X.: Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions. *Duke Math. J.* **140**, 165–202 (2007) Zbl pre05208403 MR 2355070
- [72] Tsutsumi, Y.: Scattering problem for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* **43**, 321–347 (1985) Zbl 0612.35104 MR 0824843
- [73] Tzirakis, N.: The Cauchy problem for the semilinear quintic Schrödinger equation in one dimension. *Differential Integral Equations* **18**, 947–960 (2005) MR 2150447
- [74] Visan, M.: The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. *Duke Math. J.* **138**, 281–374 (2007) Zbl 1131.35081 MR 2318286
- [75] Visan, M.: The defocusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. Ph.D. Thesis, UCLA (2006)
- [76] Visan, M., Zhang, X.: On the blowup for the L^2 -critical focusing nonlinear Schrödinger equation in higher dimensions below the energy class. *SIAM J. Math. Anal.* **39**, 34–56 (2007) Zbl 1135.35079 MR 2318374
- [77] Weinstein, M.: Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.* **87**, 567–576 (1983) Zbl 0527.35023 MR 0691044
- [78] Weinstein, M.: The nonlinear Schrödinger equation—singularity formation, stability and dispersion. In: *The Connection between Infinite-Dimensional and Finite-Dimensional Dynamical Systems* (Boulder, CO, 1987), *Contemp. Math.* 99, Amer. Math. Soc., Providence, RI, 213–232 (1989) Zbl 0703.35159 MR 1034501