J. Eur. Math. Soc. 11, 1285–1305

Alessio Porretta · Laurent Véron

Separable p-harmonic functions in a cone and related quasilinear equations on manifolds

Received November 17, 2007 and in revised form February 4, 2008

Abstract. Considering a class of quasilinear elliptic equations on a Riemannian manifold, we give a new proof of Tolksdorf's result on the construction of separable p-harmonic functions in a cone.

Keywords. p-harmonic functions, conical singularities, Ricci curvature, ergodic constant

1. Introduction

Let (r, σ) be the spherical coordinates in $\mathbb{R}^N$. If u is a harmonic function in $\mathbb{R}^N \setminus \{0\}$ written in the separable form

$$u(x) = r^{-\beta}\omega(\sigma) \tag{1.1}$$

it is straightforward to check that ω is an eigenfunction of the Laplace–Beltrami operator $-\Delta_{S^{N-1}}$ on the unit sphere $S^{N-1} \subset \mathbb{R}^N$ and β is a root of

$$X^2 - (N-2)X - \lambda = 0, \tag{1.2}$$

where $\lambda \geq 0$ is the corresponding eigenvalue. The function ω is called a *spherical harmonic* and its properties are well known, since such functions are the restrictions to the sphere of homogeneous harmonic polynomials. More generally, if $C_S \subset \mathbb{R}^N$ is the cone with vertex 0 and opening $S \subsetneq S^{N-1}$, there exist positive harmonic functions u in C_S of the form (1.1) which vanish on $\partial C_S \setminus \{0\}$ if and only if β is a root of (1.2), where, in that case, $\lambda := \lambda_S$ is the first eigenvalue of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(S)$. These separable harmonic functions play a fundamental role in the description of isolated interior or boundary singularities of solutions of second order linear elliptic equations. If the Laplace equation is replaced by the p-Laplace equation

$$-\Delta_p u := -\operatorname{div}(|Du|^{p-2}Du) = 0 \tag{1.3}$$

A. Porretta: Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Roma, Italy; e-mail: porretta@mat.uniroma2.it

L. Véron: Université François Rabelais, Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des Sciences–Parc de Grandmont, F37200 Tours, France; e-mail: veronl@univ-tours.fr

Mathematics Subject Classification (2000): 35J60, 58J32

($p > 1$), the same question of existence of separable p-harmonic functions, i.e. solutions of (1.3) of the form (1.1), was considered by Krol' [16], Tolksdorf [21], and Kichenassamy and Véron [15]. If u in (1.1) is p-harmonic, then the function ω must be a solution of the spherical p-harmonic equation,

$$-\mathrm{div}((\beta^2\omega^2+|\nabla'\omega|^2)^{p/2-1}\nabla'\omega) = \beta(\beta(p-1)+p-N)(\beta^2\omega^2+|\nabla'\omega|^2)^{p/2-1}\omega \tag{1.4}$$

on S^{N-1}, where ∇' and div are respectively the covariant derivative identified with the "tangential gradient" and the divergence operator acting on vector fields on S^{N-1}. Two special cases arise when either $p = 2$ or $N = 2$: if $p = 2$, (1.4) is just an eigenvalue problem

$$-\Delta'\omega = \beta(\beta + 2 - N)\omega, \tag{1.5}$$

where Δ' is the Laplace–Beltrami operator on S^{N-1}. When $N = 2$, equation (1.4) becomes

$$-((\beta^2\omega^2 + |\omega_\theta|^2)^{p/2-1}\omega_\theta)_\theta = \beta(\beta(p-1) + p - 2)(\beta^2\omega^2 + |\omega_\theta|^2)^{p/2-1}\omega, \tag{1.6}$$

where $\theta \in [0, \pi]$. Introducing the new unknown $\phi := \omega_\theta/\omega$ (1.6) transforms into a separable equation,

$$-((\beta^2 + \phi^2)^{p/2-1}\phi)_\theta = ((p-1)\phi^2 + \beta(\beta(p-1) + p - 2))(\beta^2 + \phi^2)^{p/2-1}. \tag{1.7}$$

This equation was completely integrated by Krol' [16] in the case $\beta < 0$, and Kichenassamy and Véron [15] in the case $\beta > 0$. It turns out that for any integer $k > 0$ there exist two couples $(\tilde{\beta}_k, \tilde{\phi}_k)$ and (β_k, ϕ_k) where $\tilde{\beta}_k < 0$, $\beta_k > 0$, and $\tilde{\phi}_k$ and ϕ_k are π/k-anti-periodic solutions of the corresponding equation (1.7). Furthermore, $\tilde{\phi}_k$ and ϕ_k are uniquely determined, up to a homothety.

An important step in analyzing the local behaviour of p-harmonic functions was realized by Tolksdorf [21] who proved that for any smooth domain $S \subset S^{N-1}$ there exists a couple (β, ϕ) where $\beta < 0$ and $\phi \in C^1(\bar{S})$ is positive in S, vanishes on ∂S and solves (1.4) in S. Furthermore, $\beta := \tilde{\beta}_S$ is unique and ϕ is determined up to a multiplicative constant. Tolksdorf's result is obtained by constructing a p-harmonic function u in the cone C_S generated by S with a compactly supported boundary data and by proving, thanks to a kind of Harnack inequality up to the boundary, the "equivalence principle", that the asymptotic behaviour of u is self-similar. Later on the existence of a couple (β, ϕ), with $\beta := \beta_S > 0$ and ϕ, as above, a positive solution of (1.4) in S vanishing on ∂S is proved by the same method in [23], therefore we shall refer to the two cases $\beta > 0$ and $\beta < 0$ as Tolksdorf's results. The structure of these spherical p-harmonic functions is studied in [6]. These regular ($\beta < 0$) and singular ($\beta > 0$) separable p-harmonic functions play a fundamental role in describing the behaviour of solutions of quasilinear equations near a regular or singular boundary point [16], [17], [4], [7].

In this article, we give a new proof of Tolksdorf's results, entirely different from his. Actually, performing a change of variable, we embed our problem into a wider class of quasilinear equations. Indeed, if $\omega \in W_0^{1,p}(S)$ is a positive solution of (1.4) in $S \subset S^{N-1}$, which vanishes on ∂S, then the function v defined by

$$v = -\frac{1}{\beta} \ln \omega$$

solves

$$\begin{cases} -\operatorname{div}((1+|\nabla' v|^2)^{p/2-1}\nabla' v) + \beta(p-1)(1+|\nabla' v|^2)^{p/2-1}|\nabla' v|^2 \\ \qquad\qquad = -(\beta(p-1)+p-N)(1+|\nabla' v|^2)^{p/2-1} \quad \text{in } S, \\ \lim\limits_{\sigma\to\partial S} v(\sigma) = \infty. \end{cases} \tag{1.8}$$

Notice that this equation is never degenerate and v is C^2 (actually C^∞) in S and satisfies the equation and the boundary condition in the classical sense. Our construction of solutions of (1.4) relies on a careful study of the quasilinear problem (1.8), and on the interpretation of the constant on the right hand side of (1.8) as the analogue of an "ergodic constant". Furthermore, having an intrinsic independent interest, this study will be performed on any compact smooth subdomain of a Riemannian manifold, without referring to the p-Laplace equation (1.3). Our main result is the following:

Theorem A. *Let (M, g) be a d-dimensional Riemannian manifold, and let ∇ and* div_g *be respectively the covariant derivative and the divergence operator on M. Then for any compact smooth subdomain $S \subset M$ and any $\beta > 0$ there exists a unique positive constant λ_β such that the problem*

$$\begin{cases} -\operatorname{div}_g((1+|\nabla v|^2)^{p/2-1}\nabla v) + \beta(p-1)(1+|\nabla v|^2)^{p/2-1}|\nabla v|^2 \\ \qquad\qquad = -\lambda_\beta(1+|\nabla v|^2)^{p/2-1} \quad \textit{in } S, \\ \lim\limits_{x\to\partial S} v(x) = \infty, \end{cases} \tag{1.9}$$

admits a solution $v \in C^2(S)$. Furthermore, v is unique up to an additive constant.

By formal analogy to the case $p = 2$, the result of Theorem A is the typical statement of an ergodic problem, although no real link with probability theory seems to exist in the quasilinear case. Therefore, we shall call λ_β the *ergodic constant* for the equation obtained after dividing by $(1+|\nabla v|^2)^{p/2-1}$ (see (2.1)); we shall prove its uniqueness for a given β. Observe also that (1.9) may be reformulated if we set $\omega = e^{-\beta v}$: then ω is a solution of

$$\begin{cases} -\operatorname{div}_g((\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\nabla\omega) = \beta\lambda_\beta(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\omega \quad \text{in } S, \\ \omega = 0 \quad \text{on } \partial S, \end{cases} \tag{1.10}$$

When $p = 2$, problem (1.10) reduces to an eigenvalue problem since $\beta\lambda_\beta = \lambda_1(S)$, the principal eigenvalue of the Laplace–Beltrami operator in S. In that case the connection between (1.9) and (1.10) dates back to the stochastic interpretation of principal eigenvalues (see e.g. [14], [18]). In the nonlinear framework with $p \neq 2$, by proving that the mapping $\beta \mapsto \lambda_\beta$ is continuous, decreasing and tends to ∞ as $\beta \to 0^+$, we conclude that the equation $\lambda_\beta = \beta(p-1)+p-d-1$ has a unique positive solution. As a consequence we generalize Tolksdorf's result as follows.

Theorem B. *Under the assumptions of Theorem A, for any compact smooth subdomain S of M there exists a unique $\beta := \beta_S > 0$ such that the problem*

$$\begin{cases} -\operatorname{div}_g((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) \\ \qquad = \beta(\beta(p-1)+p-d-1)(\beta^2\omega^2+|\nabla\omega|^2)^{p/2-1}\omega \quad \text{in } S, \\ \omega = 0 \quad \text{on } \partial S, \end{cases} \tag{1.11}$$

admits a positive solution $\omega \in C^1(\bar{S}) \cap C^2(S)$. Furthermore, ω is unique up to a homothety.

Of course, we find similarly that for $\beta < 0$ there exists a unique $\beta := \tilde{\beta}_S < 0$ such that $\lambda_\beta = \beta(p-1) + p - d - 1$. Tolksdorf's results then follow as a particular case by taking $(M, g) = (S^{N-1}, g_0)$, where S^{N-1} is equipped with the standard metric g_0 induced by the Euclidean structure in $\mathbb{R}^N$. Because the spherical domain S is assumed to be smooth, this method does not give a construction for signed spherical p-harmonic functions: if one wants to construct such functions, the natural way is to consider a tesselation of S^{N-1} obtained via the action of a finite group of isometries generated by reflections in hyperplanes, to construct, in a fundamental domain S, a positive spherical p-harmonic function vanishing on ∂S, and to extend it by reflections through the boundary. However, the difficulty comes from the fact that S is necessarily Lipschitz (except if S is a hemisphere). This non-smooth case will be considered in a forthcoming article. Notice that a large class of explicit spherical p-harmonic functions are obtained in [6, Sec. 4] as products of $N-1$ functions depending only on one spherical coordinate.

2. The singular case

In the following, we consider a general geometric setting and we recall some elements of Riemannian geometry (see e.g. [9], [13]). Let (M, g) be a complete d-dimensional Riemannian manifold with metric tensor $g = (g_{ij})$, inverse $g^{-1} = (g^{ij})$ and determinant $|g|$. If X and Y are two tangent vector fields to M, we denote by

$$X.Y = \sum_{ij} g_{ij}(x) X^i Y^j$$

their scalar product in the tangent space $T_x M$. Let x_j, $j = 1, \ldots, d$, be a local system of coordinates: if $u \in C^1(M)$, the gradient of u, denoted by ∇u, is the vector field with components $(\nabla u)^i = \sum_k g^{ik} u_{x_k}$. Therefore

$$\nabla u.\nabla u = |\nabla u|^2 = \sum_{ij} g^{ij}(x) u_{x_i} u_{x_j}.$$

If $X = (X^i)$ is a C^1 vector field on M, the divergence of X is defined by

$$\operatorname{div}_g X = \frac{1}{\sqrt{|g|}} \sum_k (\sqrt{|g|} X^k)_{x_k}.$$

Recalling that, in local coordinates, the Christoffel symbols are

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l}\left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right)g^{lk},$$

the second covariant derivatives of a C^2 function u are

$$\nabla_{ij}u = u_{x_i x_j} - \sum_{k}\Gamma_{ij}^{k}u_{x_k},$$

while the Hessian is the 2-tensor $D^2u = (\nabla_{ij}u)$. Finally, $\Delta_g u = \mathrm{trace}(D^2u) = \mathrm{div}_g\,\nabla u$ is the Laplace–Beltrami operator on M, locally expressed by

$$\Delta_g u = \frac{1}{\sqrt{|g|}}\sum_{ij}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}\,g^{ij}\frac{\partial u}{\partial x_j}\right) = \sum_{ij}\frac{\partial}{\partial x_i}\left(g^{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{ijk}\Gamma_{ik}^{k}\,g^{ij}\frac{\partial u}{\partial x_j}.$$

We denote by Ricc_g the Ricci curvature tensor of the metric g. In particular, if $(M, g) = (S^{N-1}, g_0)$, then $\mathrm{Ricc}_{g_0} = (N-1)g_0$.

In all the following, $p > 1$ is a real number. We now prove Theorem A, which we restate here for the reader's convenience.

Theorem 2.1. *Let $S \subset M$ be a smooth bounded open domain of M. Then for any $\beta > 0$ there exists a unique $\lambda_\beta > 0$ such that there exists a function $v \in C^2(S)$ satisfying*

$$\begin{cases} -\Delta_g v - (p-2)\dfrac{D^2v\nabla v.\nabla v}{1+|\nabla v|^2} + \beta(p-1)|\nabla v|^2 = -\lambda_\beta & \text{in } S, \\ \lim\limits_{x\to\partial S} v(x) = \infty. \end{cases} \tag{2.1}$$

Furthermore, v is unique up to an additive constant.

Proof. We start by considering the problem

$$\begin{cases} -\Delta_g v_\epsilon - (p-2)\dfrac{D^2v_\epsilon\nabla v_\epsilon.\nabla v_\epsilon}{1+|\nabla v_\epsilon|^2} + \beta(p-1)|\nabla v_\epsilon|^2 + \epsilon v_\epsilon = 0 & \text{in } S, \\ \lim\limits_{x\to\partial S} v_\epsilon(x) = \infty, \end{cases} \tag{2.2}$$

where $\epsilon > 0$, and then we study the limit when $\epsilon \to 0$.

Step 1: Construction of supersolutions and subsolutions. Since ∂S is C^2, the distance function $\rho(x) = \mathrm{dist}(x, \partial S)$, where the distance is the geodesic distance, is a positive C^2 function is some relative neighbourhood $\mathcal{N}_\delta = \{x \in M : |\dot\rho(x)| < \delta\}$ of ∂S; here $\dot\rho(x)$ is the signed distance, equal to $\pm\rho(x)$ according as $x \in S$ or $x \in M \setminus S$. Then $|\nabla\dot\rho(x)| = 1$ in $\mathcal{N}_\delta$. We extend $\dot\rho$ outside $\mathcal{N}_\delta$ to a $C^2(M)$ function $\tilde\rho$. Next we consider the function

$$\bar u(x) = -\frac{1}{\beta}\ln(\tilde\rho(x)) - M_0\tilde\rho(x) + \frac{M_1}{\epsilon} \qquad \forall x \in S, \tag{2.3}$$

where the $M_j > 0$ are to be chosen later. Using the fact that

$$|\nabla \bar{u}(x)|^2 = \frac{1 + 2\beta M_0 \rho(x) + O(\rho^2(x))}{\beta^2 \rho^2(x)} \quad \text{as } \rho(x) \to 0,$$

and $D^2\bar{u}\nabla\bar{u}.\nabla\bar{u} = \frac{1}{2}\nabla(|\nabla\bar{u}|^2).\nabla\bar{u}$, after some lengthy but standard computations one obtains

$$\begin{aligned} -\Delta_g \bar{u} - (p-2)\frac{D^2\bar{u}\nabla\bar{u}.\nabla\bar{u}}{1+|\nabla\bar{u}|^2} + \beta(p-1)|\nabla\bar{u}|^2 + \epsilon\bar{u} \\ = \frac{1}{\tilde{\rho}}\left(\frac{\Delta_g\tilde{\rho}}{\beta} - \frac{\varepsilon}{\beta}\tilde{\rho}\ln(\tilde{\rho}) + 2(p-1)M_0|\nabla\tilde{\rho}|^2\right) + \psi_\beta(x) + M_1, \end{aligned} \tag{2.4}$$

where ψ_β is a function depending on β (and on M_0), but which is bounded on S, uniformly when β remains in a compact subset of $(0, \infty)$. Since $|\nabla\tilde{\rho}| = 1$ near the boundary, it is possible to choose M_0 and M_1 such that $\bar{u}$ defined by (2.3) is a supersolution for (2.2). Moreover, M_0 and M_1 can be chosen independent of β whenever it varies in a compact subset of $(0, \infty)$.

One finds similarly that the function

$$\underline{u}(x) = -\frac{1}{\beta}\ln(\tilde{\rho}(x)) + M_0\tilde{\rho}(x) - \frac{M_1}{\epsilon} \quad \forall x \in S \tag{2.5}$$

is a subsolution of (2.2), with M_0 and M_1 chosen as for $\bar{u}$. Moreover, for $0 < h < \delta$, we can approximate $\bar{u}$ and $\underline{u}$ respectively from above and from below by

$$\bar{u}_h(x) = -\frac{1}{\beta}\ln(\tilde{\rho}(x) - h) - M_0(\tilde{\rho}(x) - h) + \frac{M_{1,h}}{\epsilon}, \tag{2.6}$$

$$\underline{u}_h(x) = -\frac{1}{\beta}\ln(\tilde{\rho}(x) + h) + M_0(\tilde{\rho}(x) + h) - \frac{M_{1,h}}{\epsilon}, \tag{2.7}$$

which are, respectively, a supersolution in $\{x \in S : \rho(x) > h\}$ and a subsolution in S. Together with the comparison principle, these supersolutions and subsolutions will be used to derive estimates on the solutions of (2.2).

Step 2: Basic estimates. In this part, by using the classical Bernstein method ([3]), we derive the fundamental gradient estimate for the solutions $u \in C^2(S)$ of

$$-\Delta_g u - (p-2)\frac{D^2u\nabla u.\nabla u}{1+|\nabla u|^2} + \beta(p-1)|\nabla u|^2 + \epsilon u = 0 \quad \text{in } S. \tag{2.8}$$

We recall the Weitzenböck formula (see e.g. [2]):

$$\frac{1}{2}\Delta_g|\nabla u|^2 = |D^2u|^2 + \nabla(\Delta_g u).\nabla u + \mathrm{Ricc}_g(\nabla u, \nabla u), \tag{2.9}$$

and the Cauchy–Schwarz inequality for D^2u,

$$|D^2u|^2 \geq \frac{1}{d}|\Delta_g u|^2.$$

Let $m = \inf\{\mathrm{Ricc}_g(\nabla u, \nabla u) : |\nabla u| = 1\}$. Then

$$\frac{1}{2}\Delta_g |\nabla u|^2 \geq \frac{1}{d}|\Delta_g u|^2 + m|\nabla u|^2 + \nabla(\Delta_g u).\nabla u. \tag{2.10}$$

If we set $z = |\nabla u|^2$, we can rewrite (2.8) as

$$\Delta_g u = -\frac{p-2}{2}\frac{\nabla z.\nabla u}{1+|\nabla u|^2} + \beta(p-1)z + \epsilon u \quad \text{ in } S, \tag{2.11}$$

and we obtain

$$\begin{aligned}\nabla(\Delta_g u).\nabla u = & -\frac{p-2}{2}\frac{D^2 z\nabla u.\nabla u}{1+|\nabla u|^2} - \frac{p-2}{4}\frac{|\nabla z|^2}{1+|\nabla u|^2} + \frac{p-2}{2}\frac{(\nabla z.\nabla u)^2}{(1+|\nabla u|^2)^2} \\ & + \beta(p-1)\nabla z.\nabla u + \epsilon z.\end{aligned}$$

Since, from (2.11),

$$|\Delta_g u|^2 \geq c_0 z^2 - c_1\left((\epsilon u^-)^2 + \frac{(\nabla z.\nabla u)^2}{(1+|\nabla u|^2)^2}\right),$$

we derive from (2.10)

$$\begin{aligned}\Delta_g z + (p-2)\frac{D^2 z\nabla u.\nabla u}{1+|\nabla u|^2} \geq & \frac{2c_0 z^2}{d} - \frac{2c_1}{d}\left((\epsilon u^-)^2 + \frac{(\nabla z.\nabla u)^2}{(1+|\nabla u|^2)^2}\right) + 2(m+\epsilon)z \\ & - \frac{p-2}{2}\frac{|\nabla z|^2}{1+|\nabla u|^2} + (p-2)\frac{(\nabla z.\nabla u)^2}{(1+|\nabla u|^2)^2} \\ & + 2\beta(p-1)\nabla z.\nabla u,\end{aligned}$$

which yields, by Young's inequality, the fact that $z = |\nabla u|^2$, and $\frac{2c_0}{d}z^2 + 2(m+\epsilon)z \geq \frac{c_0}{d}z^2 - c_2$,

$$-\Delta_g z - (p-2)\frac{D^2 z\nabla u.\nabla u}{1+|\nabla u|^2} + C_0 z^2 \leq C_1\frac{|\nabla z|^2}{1+z} + C_2 \tag{2.12}$$

for some positive constants C_j ($j = 0, 1, 2$), possibly depending on β, with the constant C_2 also depending on $\|\epsilon u^-\|_\infty$. Next we introduce the operator $\mathcal{A}$ defined by

$$\mathcal{A}(z) = -\Delta_g z - (p-2)\frac{D^2 z\nabla u.\nabla u}{1+|\nabla u|^2}. \tag{2.13}$$

Working in local coordinates, one can see that $\mathcal{A}$ can be written as

$$\mathcal{A}(z) = -\sum_{i,j} a_{ij} z_{x_i x_j} + \sum_i b_i z_{x_i}, \tag{2.14}$$

where the b_i are bounded and the a_{ij} are uniformly elliptic and bounded, in particular

$$\min(p-1, 1)g^{ij}\xi_i\xi_j \leq a_{ij}\xi_i\xi_j \leq \max(1, p-1)g^{ij}\xi_i\xi_j.$$

Therefore from (2.12), z is a positive subsolution of an equation of the type

$$\mathcal{A}(z) + h(z) + g(z)|\nabla z|^2 = f, \tag{2.15}$$

where $g(z) = -C_1(1+z)^{-1}$, $h(z) = C_0 z^2$ and $f = C_2$. Since g and h are increasing functions of the nonnegative variable z, it follows that the comparison principle holds between supersolutions and subsolutions of

$$-\Delta_g z - (p-2)\frac{D^2 z \nabla u.\nabla u}{1+|\nabla u|^2} + C_0 z^2 - C_1\frac{|\nabla z|^2}{1+z} = C_2. \tag{2.16}$$

Standard computations show that, if λ and μ are positive constants large enough, the function

$$\bar{z}(x) = \frac{\lambda}{\tilde{\rho}^2(x)} + \mu$$

is a supersolution of (2.16), which in addition blows up on ∂S. We conclude that any bounded subsolution of (2.16) satisfies $z(x) \le \bar{z}(x)$, and therefore so does any subsolution by replacing S by $\{x \in S : \rho(x) > h\}$ and $\tilde{\rho}(x)$ by $\tilde{\rho}(x) - h$.

Finally, we have proved that any $u \in C^2(S)$ which is a solution of (2.8) satisfies

$$|\nabla u(x)| \le \frac{L_0}{\tilde{\rho}(x)} + L_1 \quad \forall x \in S, \tag{2.17}$$

for some constants L_0, L_1 depending on $\|\varepsilon\, u^-\|_\infty$. Moreover, L_0 and L_1 can be chosen uniformly bounded with respect to β, provided β remains in a compact subset of $(0, \infty)$.

To conclude with the estimates on solutions of (2.8), it is classical from the theory of quasilinear elliptic equations (see e.g. [12]) that local Lipschitz estimates imply local $C^{2,\alpha}$ estimates since the equation is smooth and uniformly elliptic.

Step 3: Existence for the approximate equation. As in [18], we consider, for $n \in \mathbb{N}$, the solution $v_{n,\epsilon} := v$ of

$$\begin{cases} -\Delta_g v - (p-2)\dfrac{D^2 v \nabla v.\nabla v}{1+|\nabla v|^2} + \beta(p-1)|\nabla v|^2 + \epsilon v = 0 & \text{in } S, \\ v(x) = n \quad \text{on } \partial S. \end{cases} \tag{2.18}$$

By previous steps, the following estimates hold in S:

$$0 \le v_{n,\epsilon}(x) \le -\frac{1}{\beta}\ln\tilde{\rho}(x) - M_0\tilde{\rho}(x) + \frac{M_1}{\epsilon}, \tag{2.19}$$

$$|\nabla v_{n,\epsilon}(x)| \le \frac{L_0}{\tilde{\rho}(x)} + L_1. \tag{2.20}$$

Moreover, the sequence $\{v_{n,\epsilon}\}$ is bounded in $C^{2,\alpha}_{\mathrm{loc}}(S)$, which ensures the local compactness of the gradients. Since $n \mapsto v_{n,\epsilon}$ is increasing, the limit $v_\epsilon = \lim_{n\to\infty} v_{n,\epsilon}$ exists and v_ϵ is a solution of (2.2) which satisfies (2.19) and (2.20).

Step 4: The ergodic limit. From Step 1, by comparison with $\bar{u}_h$ and $\underline{u}_h$ defined in (2.6)–(2.7) (and letting $h \to 0$), we know that in S,

$$-\frac{1}{\beta}\ln\tilde{\rho}(x) + M_0\tilde{\rho}(x) - \frac{M_1}{\epsilon} \le v_\epsilon(x) \le -\frac{1}{\beta}\ln\tilde{\rho}(x) - M_0\tilde{\rho}(x) + \frac{M_1}{\epsilon}. \qquad (2.21)$$

Therefore ϵv_ϵ is locally bounded in S. Since ∇v_ϵ is also locally bounded in S, $\epsilon_n v_{\epsilon_n}$ converges (for some sequence $\{\epsilon_n\}$) to some constant $\lambda_0 \ge 0$ in the C_{loc}-topology of S. We fix $x_0 \in S$ and set $w_\epsilon(x) := v_\epsilon(x) - v_\epsilon(x_0)$. Because w_ϵ is locally bounded in $C^1_{\text{loc}}(S)$ and w_ϵ satisfies

$$-\Delta_g w_\epsilon - (p-2)\frac{D^2 w_\epsilon \nabla w_\epsilon . \nabla w_\epsilon}{1+|\nabla w_\epsilon|^2} + \beta(p-1)|\nabla w_\epsilon|^2 + \epsilon w_\epsilon = -\epsilon v_\epsilon(x_0) \quad \text{in } S \qquad (2.22)$$

the regularity theory for elliptic equations implies that w_ϵ is locally bounded in $C^{2,\alpha}(S)$. Passing to a subsequence, the limit $w_0 = \lim_{n\to\infty} w_{\epsilon_n}$ exists, and w_0 is a solution of

$$-\Delta_g w_0 - (p-2)\frac{D^2 w_0 \nabla w_0 . \nabla w_0}{1+|\nabla w_0|^2} + \beta(p-1)|\nabla w_0|^2 = -\lambda_0 \quad \text{in } S. \qquad (2.23)$$

The only point which remains to be proved is that w_0 blows up at the boundary. We set

$$\underline{\psi}(x) = -\frac{1}{\beta}\ln\tilde{\rho}(x) + M_0\tilde{\rho}(x),$$

and we have, with the same computations as for (2.4),

$$\begin{aligned}-\Delta_g\underline{\psi} - (p-2)\frac{D^2\underline{\psi}\nabla\underline{\psi}.\nabla\underline{\psi}}{1+|\nabla\underline{\psi}|^2} + \beta(p-1)|\nabla\underline{\psi}|^2 + \epsilon\underline{\psi}\\ = \frac{1}{\tilde{\rho}}\left(\frac{\Delta_g\tilde{\rho}}{\beta} - \frac{\varepsilon}{\beta}\tilde{\rho}\ln(\tilde{\rho}) - 2(p-1)M_0|\nabla\tilde{\rho}|^2\right) + \psi_\beta(x),\end{aligned}$$

where ψ_β is a bounded function (depending on β, M_0). Noticing that $|\nabla\tilde{\rho}| = 1$ in a neighbourhood of ∂S, and that $\epsilon v_\epsilon(x_0)$ is uniformly bounded, we can choose M_0, ρ_0 such that the function $\underline{\psi}$ is a subsolution of (2.22) in $\{x \in S : 0 < \rho(x) < \rho_0\}$. Since, whenever $\rho(x) = \rho_0$, we have $w_\epsilon(x) \ge -c_0$ for some $c_0 > 0$ (due to the gradient estimate for v_ϵ), and since $\underline{\psi} - c$ is still a subsolution for any positive constant c, we derive

$$w_\epsilon(x) \ge -\frac{1}{\beta}\ln\tilde{\rho}(x) + M_0\tilde{\rho}(x) - c \quad \forall x \text{ with } \rho(x) \le \rho_0. \qquad (2.24)$$

Letting ϵ tend to 0 implies that $\lim_{x\to\partial S} w_0(x) = \infty$.

Step 5: Uniqueness of the ergodic limit. We claim that there exists a unique constant $\lambda_0 > 0$ such that there exists a solution $v_0 \in C^2(S)$ of

$$\begin{cases} -\Delta_g v_0 - (p-2)\dfrac{D^2 v_0 \nabla v_0 . \nabla v_0}{1+|\nabla v_0|^2} + \beta(p-1)|\nabla v_0|^2 = -\lambda_0 & \text{in } S, \\ \lim\limits_{x\to\partial S} v_0(x) = \infty. \end{cases} \qquad (2.25)$$

For this purpose, the following will be useful:

Lemma 2.2. *A function* $v_0 \in C^2(S)$ *is a solution of* (2.25) *if and only if the function* $\omega_0 = e^{-\beta v_0} \in C^2(S) \cap C(\bar{S})$ *is a solution of*

$$\begin{cases} -\operatorname{div}_g((\beta^2\omega_0^2 + |\nabla\omega_0|^2)^{p/2-1}\nabla\omega_0) = \beta\lambda_0(\beta^2\omega_0^2 + |\nabla\omega_0|^2)^{p/2-1}\omega_0 & \textit{in } S, \\ \omega_0 = 0 \quad \textit{on } \partial S. \end{cases} \tag{2.26}$$

Moreover, $\omega_0 \in C^{1,\gamma}(\bar{S})$ *for some* $\gamma > 0$, *and* $\partial_\nu\omega_0 < 0$ *on* ∂S.

Proof. Let $v_0 \in C^2(S)$ be a solution of (2.25). As in the previous steps, the functions

$$\underline{\phi}(x) = -\frac{1}{\beta}\ln\tilde{\rho}(x) + M_0\tilde{\rho}(x) - M^* \quad \text{and} \quad \bar{\phi}(x) = -\frac{1}{\beta}\ln\tilde{\rho}(x) - M_0\tilde{\rho}(x) + M^*$$

are respectively a subsolution and a supersolution of (2.25) in $\{x : \rho(x) < \delta\}$ for some $\delta > 0$ small enough (where M^* depends on the value of v_0 on the set $\{x \in S : \rho(x) = \delta\}$). Then we obtain, by comparison,

$$\left| v_0(x) + \frac{\ln\tilde{\rho}(x)}{\beta} \right| \le M^*. \tag{2.27}$$

By the gradient estimates of Step 2,

$$|\nabla v_0(x)| \le L_0/\tilde{\rho}(x) + L_1. \tag{2.28}$$

Now set $\omega_0 = e^{-\beta v_0}$. Then $\omega_0 \in W^{1,\infty}(S) \cap C(\bar{S})$ solves the problem (2.26) in the weak sense. By the regularity theory for degenerate equations of p-Laplacian type (see the Appendix, Theorem A.1 and related references), we can deduce that $\omega_0 \in C^{1,\gamma}(\bar{S})$. Moreover, since (2.27) implies

$$e^{-\beta M^*} \le \omega_0/\rho(x) \le e^{\beta M^*} \tag{2.29}$$

we deduce that $\partial_\nu\omega_0 \le -e^{-\beta M^*} < 0$ on ∂S. As a consequence, since $\omega_0 \in C^1(\bar{S})$ and is positive in S, we deduce that problem (2.26) is uniformly elliptic, so that the classical regularity theory applies to give $\omega_0 \in C^{2,\alpha}(S)$.

Of course, the converse is also true: given a solution ω_0 of (2.26), clearly $v_0 = -\beta^{-1}\ln\omega_0$ is a solution of (2.25). □

Assume now that there exist two ergodic constants, λ_1 and λ_2, associated with two solutions v_1, v_2, and let $\omega_i = e^{-\beta v_i}$ be the corresponding solutions of (2.26). Notice that multiplying (2.26) by ω_0 and integrating on S, we get actually $\lambda_0 > 0$. Thus $\lambda_i > 0$ and, say, $\lambda_2 > \lambda_1$.

Since $\omega_1/\omega_2 \in L^\infty(S)$ (from estimate (2.29)), we define

$$\theta = \sup_S \omega_1/\omega_2.$$

Because equation (2.26) is homogeneous we can assume that $\theta = 1$ and either there exists $x_0 \in S$ such that $\omega_1(x_0) = \omega_2(x_0)$, $\nabla\omega_1(x_0) = \nabla\omega_2(x_0)$ and $\omega_1(x) \le \omega_2(x)$ for $x \in \bar{S}$, or $\omega_1(x) < \omega_2(x)$ for $x \in S$ and there exists $x_0 \in \partial S$ such that $\partial_\nu\omega_1(x_0) = \partial_\nu\omega_2(x_0)$.

In the first case, it turns out that the function $z = v_1 - v_2$ is nonnegative in S, achieves a minimum at $x_0 \in S$ and satisfies

$$-\Delta_g z(x_0) - (p-2)\frac{D^2 z(x_0)\nabla v_1(x_0).\nabla v_1(x_0)}{1 + |\nabla v_1(x_0)|^2} = \lambda_2 - \lambda_1 > 0,$$

which is impossible because of ellipticity. In the second case, $\partial_\nu(\omega_1 - \omega_2)(x_0) = 0$, whereas $\omega_1 - \omega_2$ is negative in S and $(\omega_1 - \omega_2)(x_0) = 0$. Since the problem (2.26) is uniformly elliptic (recall that the functions ω_i satisfy $\beta^2 w_i^2 + |\nabla \omega_i|^2 > 0$ on $\overline{S}$) this contradicts the Hopf maximum principle. Therefore $\omega_1 = \omega_2$, which implies $\lambda_1 = \lambda_2$ by the equation. Thus the ergodic constant is unique.

In a similar way one can prove that ω_0 is unique up to a multiplicative constant, and so v_0 is unique up to an additive constant (as a consequence, the whole sequence w_ϵ, constructed in Step 4, converges to w_0 as $\epsilon \to 0$). However, the uniqueness of v_0 can be proved with a more general argument, concerning directly problem (2.25), which is a variant as well as a generalization of previous uniqueness results for explosive solutions. Since it can have its own interest, we present it here.

First of all, we recall that any C^2 solution v_0 of (2.25) satisfies (2.27) and (2.28). Moreover, by Lemma 2.2 we have $\omega_0 = e^{-\beta v_0} \in C^1(\bar{S})$ and $\partial_\nu \omega_0 < 0$ on ∂S, hence, using the fact that $\nabla v_0 = -(e^{\beta v_0}/\beta)\nabla \omega_0$ and the estimate (2.27), we conclude that there exists a constant $\sigma > 0$ such that, in a neighbourhood of ∂S,

$$|\nabla v_0| \geq \sigma/\rho(x). \tag{2.30}$$

In addition, it is possible to deduce from (2.27)–(2.28) that there exists a constant $C_0 > 0$ such that

$$|D^2 v_0| \leq C_0/\tilde{\rho}(x)^2 \quad \forall x \in S. \tag{2.31}$$

Indeed, take $x_0 \in S$ and let $\rho_0 = \rho(x_0)/2$, where we recall that $\rho(x_0) = \mathrm{dist}(x_0, \partial S)$. Then consider (in a local neighbourhood of x_0) the rescaled function

$$u_0(\xi) = v_0(x_0 + \rho_0\,\xi) + \frac{\ln \rho_0}{\beta}$$

for $\xi \in B(0, 1)$. Note that $\rho(x_0 + \rho_0\,\xi) \in (\rho_0, 3\rho_0)$ so that (2.28) and (2.30) imply $\sigma/3 \leq |Du_0| \leq L_0 + L_1\,\rho_0$. Since v_0 is a solution of (2.25), a simple scaling in local coordinates shows that u_0 is a solution of

$$-\Delta_g u_0 - (p-2)\frac{D^2 u_0 \nabla u_0.\nabla u_0}{\rho_0^2 + |\nabla u_0|^2} + \beta(p-1)|\nabla u_0|^2 = -\lambda_0\,\rho_0^2 \quad \text{for } \xi \in B(0,1)$$

with a slight abuse of notation since now, in local coordinates, the derivatives are taken with respect to the variable ξ. Since the second order operator is uniformly elliptic, by the classical regularity theory (see e.g. [12, Theorem 13.6] to deduce the Hölder estimates for Du_0 and then apply the Schauder estimates, Chapter 6) we have

$$|D^2 u_0(\xi)| \leq C \quad \forall \xi \in B(0, 1/2)$$

where C is a constant depending on $\sup_{B(0,1)}(|u_0| + |Du_0|)$. Using the estimates (2.27)–(2.28) we can bound this last quantity only depending on M^*, L_0, L_1, hence we conclude that $|D^2u_0(0)| \le C$, which gives (2.31).

Now, take two solutions v_1, v_2 of (2.25) corresponding to λ_1, λ_2 with, say, $\lambda_1 \le \lambda_2$. We adapt an argument in [18]: consider the function $\hat{v} = \theta v_2$ for $\theta < 1$, and compute

$$-\Delta_g \hat{v} - (p-2)\frac{D^2\hat{v}\nabla\hat{v}.\nabla\hat{v}}{1+|\nabla\hat{v}|^2} + \beta(p-1)|\nabla\hat{v}|^2 = -\theta\lambda_2$$
$$+ (1-\theta^2)\theta(p-2)\frac{D^2v_2\nabla v_2.\nabla v_2}{(1+|\nabla v_2|^2)(1+\theta^2|\nabla v_2|^2)} - (1-\theta)\theta\beta(p-1)|\nabla v_2|^2.$$

Using (2.28), (2.31) and (2.30), and recalling that $\lambda_2 \ge \lambda_1$, we deduce that $\hat{v}$ satisfies, for some constant $C > 0$,

$$-\Delta_g \hat{v} - (p-2)\frac{D^2\hat{v}\nabla\hat{v}.\nabla\hat{v}}{1+|\nabla\hat{v}|^2} + \beta(p-1)|\nabla\hat{v}|^2 \le -\lambda_1 + (1-\theta)[\lambda_1 + C - \beta(p-1)\theta|\nabla v_2|^2].$$

Thanks to (2.30), we conclude that there exists $\delta > 0$, independent of θ, such that $\hat{v}$ satisfies

$$-\Delta_g \hat{v} - (p-2)\frac{D^2\hat{v}\nabla\hat{v}.\nabla\hat{v}}{1+|\nabla\hat{v}|^2} + \beta(p-1)|\nabla\hat{v}|^2 \le -\lambda_1$$

in $\{x \in S : \rho(x) < \delta\}$. However, from the estimate (2.27) which holds for v_1 and v_2 we see that $v_1 - \hat{v} \to +\infty$ as $\rho(x) \to 0$, hence $v_1 - \hat{v}$ has a minimum in $\{x \in S : \rho(x) < \delta\}$ and, by the standard maximum principle, it is reached when $\rho(x) = \delta$. Letting $\theta \to 1$, we conclude that

$$\min\{(v_1 - v_2)(x) : \rho(x) \le \delta\} = \min\{(v_1 - v_2)(x) : \rho(x) = \delta\}.$$

On the other hand, looking at the equations of v_1, v_2 in $\{x \in S : \rho(x) > \delta\}$, we also know (again by the maximum principle) that

$$\min\{(v_1 - v_2)(x) : \rho(x) \ge \delta\} = \min\{(v_1 - v_2)(x) : \rho(x) = \delta\},$$

hence $v_1 - v_2$ should have a global minimum reached at a point $x_0 \in S$ such that $\rho(x_0) = \delta$. Since x_0 lies inside the domain, and the function $z = v_1 - v_2$ satisfies a smooth elliptic equation around x_0, using the strong maximum principle we conclude that $v_1 - v_2$ is constant. This proves the uniqueness, up to a constant, of the solution of (2.25) (note that this argument shows at the same time that $\lambda_1 = \lambda_2$, i.e. the uniqueness of the ergodic constant which we already proved before). □

Remark 2.3. The argument in the last step of the previous proof also provides a general uniqueness result for explosive solutions of

$$\begin{cases} -\Delta_g v - (p-2)\dfrac{D^2v\nabla v.\nabla v}{1+|\nabla v|^2} + \beta(p-1)|\nabla v|^2 + \epsilon v = f & \text{in } S, \\ \lim\limits_{x\to\partial S} v(x) = \infty. \end{cases} \tag{2.32}$$

Precisely, if f is a Lipschitz function, and $\epsilon > 0$, the problem (2.32) has a unique solution $v \in C^2(S)$. To our knowledge, such a result is new even in the euclidean setting $M = \mathbb{R}^N$.

We now proceed to study how the ergodic constant λ_β depends on β, which will lead to the proof of Theorem B.

Proposition 2.4. *Under the assumptions of Theorem* 2.1*, the mapping* $\beta \mapsto \lambda_\beta$ *is continuous and decreasing from* $(0, \infty)$ *to* $(0, \infty)$*, and*

$$\lim_{\beta \to 0} \lambda_\beta = \infty. \tag{2.33}$$

Proof. Step 1: monotonicity. Let $0 < \beta_1 < \beta_2$ and let $v_{\epsilon,1}$ and $v_{\epsilon,2}$ be the corresponding solutions of (2.2) with β respectively replaced by β_1 and β_2. Since the $v_{\epsilon,i}$ are limits of solutions with finite boundary value, we have $v_{\epsilon,1} > v_{\epsilon,2}$ by the comparison principle. Therefore

$$\lambda_{\beta_1} := \lim_{\epsilon \to 0} \epsilon v_{\epsilon,1} \geq \lambda_{\beta_2} := \lim_{\epsilon \to 0} \epsilon v_{\epsilon,2}.$$

Next, if we assume that there exist β_i $(i = 1, 2)$ such that $0 < \beta_1 < \beta_2$ and $\lambda_{\beta_1} = \lambda_{\beta_2} = \lambda$ and if ω_1 and ω_2 are the corresponding solutions of (2.26) with $\beta = \beta_i$ and $\lambda = \lambda_{\beta_1} = \lambda_{\beta_2}$, then (2.27) implies

$$m^{-1}\rho(x) \leq \omega_i \leq m\rho(x) \quad \forall x \in S,$$

for some $m > 0$. Set $\tilde{\omega} = \omega_1^{\beta_2/\beta_1}$. Then

$$\begin{aligned} &-\operatorname{div}_g((\beta_2^2\tilde{\omega}^2 + |\nabla\tilde{\omega}|^2)^{p/2-1}\nabla\tilde{\omega}) - \beta_2\lambda(\beta_2^2\tilde{\omega}^2 + |\nabla\tilde{\omega}|^2)^{p/2-1}\tilde{\omega} \\ &= (p-1)\left(1 - \frac{\beta_2}{\beta_1}\right)\left(\frac{\beta_2}{\beta_1}\right)^{p-1} \omega_1^{(p-1)(\beta_2/\beta_1 - 1)}(\beta_1^2\omega_1^2 + |\nabla\omega_1|^2)^{(p-2)/2}\frac{|\nabla\omega_1|^2}{\omega_1}. \end{aligned} \tag{2.34}$$

Therefore $\tilde{\omega}$ is a strict subsolution. By homogeneity, and since $\partial_\nu\tilde{\omega}$ vanishes on ∂S, we can assume that $\tilde{\omega} \leq \omega_2$, that there exists $x_0 \in S$ such that $\tilde{\omega}(x_0) = \omega_2(x_0)$, and that the coincidence set of $\tilde{\omega}$ and ω_2 is a subset of S. Let

$$z = -\frac{1}{\beta_2}(\ln\omega_2 - \ln\tilde{\omega}) = v_2 - \tilde{v}.$$

Then $z \leq 0$, it is not identically zero, $z(x_0) = 0$ and $z(x) \to -\infty$ as $\rho(x) \to \partial S$. Since (2.34) implies that $\tilde{v}$ is a strict supersolution of the equation satisfied by v_2, we conclude that, at $x = x_0$,

$$\begin{aligned} &-\Delta_g z - (p-2)\frac{D^2z\nabla v_2.\nabla v_2}{1 + |\nabla v_2|^2} \\ &\quad + (p-2)\left[\frac{D^2\tilde{v}\nabla\tilde{v}.\nabla\tilde{v}}{1 + |\nabla\tilde{v}|^2} - \frac{D^2\tilde{v}\nabla v_2.\nabla v_2}{1 + |\nabla v_2|^2}\right] + \beta_2(p-1)[|\nabla v_2|^2 - |\nabla\tilde{v}|^2] \leq 0. \end{aligned}$$

Since $\tilde{v}$, v_2 are C^2 in S, the strong maximum principle yields a contradiction. Therefore $\beta \mapsto \lambda_\beta$ is decreasing.

Step 2: continuity. Let $\{\beta_n\}$ be a positive sequence such that $\beta_n \to \beta_0$ and v_{β_n} be the corresponding solution of

$$\begin{cases} -\Delta_g v_{\beta_n} - (p-2)\dfrac{D^2 v_{\beta_n} \nabla v_{\beta_n}.\nabla v_{\beta_n}}{1+|\nabla v_{\beta_n}|^2} + \beta_n(p-1)|\nabla v_{\beta_n}|^2 = -\lambda_{\beta_n} \quad \text{in } S, \\ \lim\limits_{x\to\partial S} v_{\beta_n}(x) = \infty, \end{cases} \tag{2.35}$$

and let v_{ϵ,β_n} be the corresponding solutions of (2.2) with $\beta = \beta_n$. Since $\epsilon v_{\epsilon,\beta_n}$ remains locally bounded in S when β_n remains in a compact subset of $(0,\infty)$ and converges to λ_{β_n} locally uniformly as $\epsilon \to 0$, the set $\{\lambda_{\beta_n}\}$ is bounded. Up to a subsequence (not relabeled) we can assume that $\lambda_{\beta_n} \to \bar\lambda$ as $n \to \infty$. Thanks to (2.27) and (2.28),

$$\left| v_{\beta_n} + \frac{\ln \rho(x)}{\beta_n} \right| \le C_0 \quad \text{and} \quad |\nabla v_{\beta_n}| \le \frac{C_1}{\rho(x)} \tag{2.36}$$

for some constants C_0, C_1, hence the sequence $\{v_{\beta_n}\}$ remains locally bounded in $W^{1,\infty}_{\mathrm{loc}}(S)$ and therefore in $C^{2,\alpha}_{\mathrm{loc}}(S)$. Up to a subsequence $v_{\beta_n} \to \bar v$ in $C^2_{\mathrm{loc}}(S)$, and $\bar v$ is a solution of

$$\begin{cases} -\Delta_g \bar v - (p-2)\dfrac{D^2 \bar v \nabla \bar v.\nabla \bar v}{1+|\nabla \bar v|^2} + \beta_0(p-1)|\nabla \bar v|^2 = -\bar\lambda \quad \text{in } S, \\ \lim\limits_{x\to\partial S} \bar v(x) = \infty. \end{cases}$$

By uniqueness of the ergodic limit, $\bar\lambda = \lambda_{\beta_0}$, and $\lambda_{\beta_n} \to \lambda_{\beta_0}$ for the whole sequence.

Step 3: (2.33) holds. Let ω be a positive solution of

$$\begin{cases} -\operatorname{div}_g((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega) = \beta\lambda_\beta(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega \quad \text{in } S, \\ \omega = 0 \quad \text{on } \partial S. \end{cases} \tag{2.37}$$

We normalize ω by

$$\int_S |\nabla\omega|^p \, dv_g = 1.$$

Therefore, if μ_S is the first eigenvalue of $-\operatorname{div}_g(|\nabla.|^{p-2}\nabla.)$ in $W^{1,p}_0(S)$, then

$$\int |\omega|^p \, dv_g \le \frac{1}{\mu_S}.$$

Multiplying (2.37) by ω and integrating over S yields

$$\int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{p/2} \, dv_g = \beta(\lambda_\beta + \beta)\int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega^2 \, dv_g. \tag{2.38}$$

Clearly

$$\int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{p/2} \, dv_g \ge \int_S |\nabla\omega|^p \, dv_g = 1.$$

If $p \geq 2$, then

$$\int_S (\beta^2 \omega^2 + |\nabla \omega|^2)^{p/2-1} \omega^2 \, dv_g \leq c \int_S (\omega^p + \omega^2 |\nabla \omega|^{p-2}) \, dv_g$$
$$\leq c\left(1 + \frac{2}{p}\right) \int_S \omega^p \, dv_g + c\left(1 - \frac{2}{p}\right) \int_S |\nabla \omega|^p) \, dv_g \leq C_{p,S},$$

with $c = \max(1, 2^{p/2-2})$. This implies

$$\beta(\lambda_\beta + \beta) \geq \frac{1}{C_{p,S}} \;\Rightarrow\; \lambda_\beta \geq \frac{1}{C_{p,S}\beta} - \beta. \tag{2.39}$$

If $1 < p < 2$, then

$$\int_S \frac{\omega^2 \, dv_g}{(\beta^2 \omega^2 + |\nabla \omega|^2)^{1-p/2}} \leq \beta^{p-2} \int_S |\omega|^p \, dv_g \leq \frac{\beta^{p-2}}{\mu_S}.$$

Therefore

$$\beta^{p-1}(\lambda_\beta + \beta) \geq \mu_S \;\Rightarrow\; \lambda_\beta \geq \frac{\mu_S}{\beta^{p-1}} - \beta. \tag{2.40}$$

Clearly (2.39) and (2.40) imply (2.33). □

Remark. Using the uniform ellipticity and the maximum principle, one can possibly improve inequalities (2.39) and (2.40) to $\lambda_\beta \geq C/\beta$. However, this improved inequality plays no better role than the former ones in what follows.

We now have all the ingredients for the proof of Theorem B.

Proof of Theorem B. If we set $\omega = e^{-\beta v}$ where v is a solution of (2.1), then ω is defined up to a multiplicative constant and satisfies (2.37). By Lemma 2.2, $\omega \in C^1(\bar{S}) \cap C^2(S)$. Therefore the desired conclusion follows if we can prove that there exists a unique $\beta := \beta_S > 0$ such that

$$\lambda_\beta = \beta(p-1) + p - d - 1. \tag{2.41}$$

But the mapping $\beta \mapsto \lambda_\beta - \beta(p-1)$ is continuous and decreasing on $(0, \infty)$. Clearly

$$\lim_{\beta \to \infty} (\lambda_\beta - \beta(p-1)) = -\infty, \quad \lim_{\beta \to 0} (\lambda_\beta - \beta(p-1)) = \infty,$$

by Proposition 2.4. The result follows by continuity. □

3. The regular case and Tolksdorf's result

If $\beta < 0$, the equation satisfied by a separable p-harmonic function u of the form (1.1) is unchanged. However, if we set $\tilde{\beta} = -\beta$, then (1.4) turns into

$$-\operatorname{div}((\tilde{\beta}^2\omega+|\nabla'\omega|^2)^{p/2-1}\nabla'\omega) = \tilde{\beta}(\tilde{\beta}(p-1)+N-p)(\tilde{\beta}^2\omega+|\nabla'\omega|^2)^{p/2-1}\omega. \tag{3.1}$$

Furthermore, if a solution ω of (3.1) in $S \subset S^{N-1}$ exists and vanishes on ∂S, then we have $\tilde{\beta}(p-1) + N - p > 0$ by multiplying by ω and integration over S. If we set

$$v = -\frac{\ln \omega}{\tilde{\beta}},$$

then v satisfies

$$\begin{cases} -\operatorname{div}((1+|\nabla' v|^2)^{p/2-1}\nabla' v) + \beta(p-1)(1+|\nabla' v|^2)^{p/2-1}|\nabla' v|^2 \\ \qquad\qquad = -(\tilde{\beta}(p-1)+N-p)(1+|\nabla' v|^2)^{p/2-1} \quad \text{in } S, \\ \lim\limits_{\sigma\to\partial S} v(\sigma) = \infty. \end{cases}$$

In the general setting of a Riemannian manifold, Theorem 2.1 and Proposition 2.4 are valid with β replaced by $\tilde{\beta}$. The proof of Theorem B holds except that (2.41) is replaced by

$$\lambda_{\tilde{\beta}} = \tilde{\beta}(p-1) + d + 1 - p. \tag{3.2}$$

Because the function $\tilde{\beta} \mapsto \lambda_{\tilde{\beta}} - \tilde{\beta}(p-1)$ is unchanged, the proof of Theorem B applies and shows that there exists a unique $\tilde{\beta} := \tilde{\beta}_S > 0$ such that (3.2) holds. Consequently, we have proved the following result which contains Tolksdorf's initial result if $(M, g) = (S^{N-1}, g_0)$.

Corollary 3.1. *Under the assumptions of Theorem* 2.1 *there exists a unique $\tilde{\beta} := \tilde{\beta}_S > 0$ such that the problem*

$$\begin{cases} -\operatorname{div}_g((\tilde{\beta}^2\omega^2+|\nabla\omega|^2)^{p/2-1}\nabla\omega) \\ \qquad\qquad = \tilde{\beta}(\tilde{\beta}(p-1)+d+1-p)(\tilde{\beta}^2\omega^2+|\nabla\omega|^2)^{p/2-1}\omega \quad \text{in } S, \\ \omega = 0 \quad \text{on } \partial S, \end{cases}$$

admits a positive solution $\omega \in C^1(\bar{S}) \cap C^2(S)$. Furthermore, ω is unique up to homothety.

A. Appendix

We prove here the $C^{1,\gamma}$ regularity up to the boundary, stated in Lemma 2.2, for solutions of degenerate equations in divergence form

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = B(x,u,\nabla u) \quad \text{in } S, \\ u = 0 \quad \text{on } \partial S. \end{cases} \tag{A.1}$$

We will assume that $a(x, s, \xi)$ satisfies the following conditions: there exist constants $\lambda, \Lambda, \beta > 0$, and $\alpha \in (0, 1]$, $p > 1$ and a continuous function $\mu : S \times \mathbb{R} \to \mathbb{R}$ such that, for all $s, t \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$, and a.e. $x \in \Omega$,

$$\frac{\partial a^i}{\partial \xi_j}(x, s, \xi)\, \eta_i \eta_j \geq \lambda(\mu(x, s)^2 + |\xi|^2)^{(p-2)/2} |\eta|^2, \tag{A.2}$$

$$\left| \frac{\partial a^i}{\partial \xi_j}(x, s, \xi) \right| \leq \Lambda(\mu(x, s)^2 + |\xi|^2)^{(p-2)/2}, \tag{A.3}$$

$$|a(x, s, \xi) - a(y, t, \xi)| \leq \beta(1 + |\xi|^{p-2} + |\xi|^{p-1})\, [|x - y|^\alpha + |s - t|^\alpha], \tag{A.4}$$

$$|B(x, s, \xi)| \leq \beta(1 + |\xi|^p). \tag{A.5}$$

The model we have in mind is clearly

$$a(x, u, \nabla u) = (\mu(x, u)^2 + |\nabla u|^2)^{(p-2)/2} \nabla u$$

where $p > 1$, and the function $\mu(x, s)$ is Lipschitz (or possibly Hölder) continuous. In many cases, as in the proof of Lemma 2.2, the a priori information that u is Lipschitz (or Hölder) continuous could allow us to consider only the case $\mu = \mu(x)$.

The $C^{1,\gamma}$ estimates, or similar regularity results, are by now classical since the works of E. DiBenedetto [11] and P. Tolksdorf [22] for the p-Laplace equation: as far as the global regularity, up to the boundary, is concerned, we refer to the works of G. Lieberman (e.g. [19]) or to [10] (see also [1], [20]). Despite a large amount of literature available, it seems that no exact reference applies to our model, so that, for the sake of completeness, we feel like giving a proof of this result, at least detailing the possible slight modifications in order that previous results can be generalized. To this purpose, we observe that while the case $p \geq 2$ is contained, if not in previous statements, at least in previous arguments (specifically, we refer to [19]), this does not seem sure for the case $p < 2$ because of our growth assumption (A.4) (roughly speaking, the (x, s)-derivatives may grow like $|\xi|^{p-2}$). Finally, we note that the next result would still hold for a nonhomogeneous boundary condition ($u = \varphi$ on ∂S) provided φ belongs to $C^{1,\alpha}(\partial S)$.

Theorem A.1. *Let S be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^N$, and assume that* (A.2)–(A.5) *hold true. If u is a bounded weak solution of* (A.1)*, then there exists $\gamma \in (0, 1)$ such that $u \in C^{1,\gamma}(\overline{S})$ and moreover*

$$\|u\|_{C^{1,\gamma}(\overline{S})} \leq C(\Lambda/\lambda, \alpha, \|u\|_\infty, p, N, S).$$

Proof. Because our specific interest is in the boundary estimate, we only prove the regularity of u around a point $x_0 \in \partial S$ (the inner regularity is treated in the same manner). Up to straightening the boundary, we can assume that locally $\partial S = \{x : x_N = 0\}$ and $S = \{x : x_N > 0\}$.

We follow the standard approach via a perturbation argument. We set $B_R = \{x : |x - x_0| < R\}$, $B_R^+ = B_R \cap S$, and consider a solution v of

$$\begin{cases} -\operatorname{div}(a(x_0, u(x_0), \nabla v)) = 0 & \text{in } B_R^+, \\ v = u & \text{on } \partial B_R^+. \end{cases} \tag{A.6}$$

Problem (A.6) has a unique solution $v \in W^{1,p}(B_R^+)$. Due to assumptions (A.2)–(A.3), the estimates concerning v are well-established ([11], [22], [19]). In particular, from Lemma 5 in [19] we have, for some $\sigma > 0$,

$$\operatorname*{osc}_{B_r^+} \nabla v \leq C\left(\frac{r}{R}\right)^{\sigma}\left(R^{-N}\int_{B_R^+}|\nabla v|^p\,dx\right)^{1/p} \quad \forall r < R/2 \tag{A.7}$$

where C, here and below, depends only on the constants appearing in the hypotheses and possibly on $\|u\|_\infty$, in particular through the quantity $\sup\{|\mu(x,s)| : x \in \bar{S}, |s| \leq \|u\|_\infty\}$. Moreover, since $a(x,s,\xi)\cdot\xi \geq c(|\xi|^p - |\mu|^p)$, one easily deduces from (A.6), using $v-u$ as test function and Young's inequality, that

$$\int_{B_R^+}|\nabla v|^p\,dx \leq C\left(R^N + \int_{B_R^+}|\nabla u|^p\,dx\right). \tag{A.8}$$

Finally, the maximum principle gives $\inf_{\partial B_R^+} u \leq v \leq \sup_{\partial B_R^+} u$, which yields

$$\operatorname*{osc}_{B_R^+} v \leq \operatorname*{osc}_{B_R^+} u. \tag{A.9}$$

Now take $u-v$ as test function both in (A.1) (restricted to B_R^+) and in (A.6) to obtain

$$\int_{B_R^+} a(x,u,\nabla u)\cdot\nabla(u-v)\,dx - \int_{B_R^+} a(x_0,u(x_0),\nabla v)\cdot\nabla(u-v)\,dx = \int_{B_R^+} B(x,u,\nabla u)(u-v)\,dx.$$

Define $D_v := \{x \in B_R^+ : |\nabla u| < |\nabla v|\}$ and $D_u := \{x \in B_R^+ : |\nabla v| \leq |\nabla u|\}$; hence we have

$$\begin{aligned}
&\int_{D_v}[a(x,u,\nabla u) - a(x,u,\nabla v)]\cdot\nabla(u-v)\,dx\\
&\qquad + \int_{D_u}[a(x_0,u(x_0),\nabla u) - a(x_0,u(x_0),\nabla v)]\cdot\nabla(u-v)\,dx\\
&\quad = \int_{D_v}[a(x_0,u(x_0),\nabla v) - a(x,u,\nabla v)]\cdot\nabla(u-v)\,dx\\
&\qquad + \int_{D_u}[a(x_0,u(x_0),\nabla u) - a(x,u,\nabla u)]\cdot\nabla(u-v)\,dx\\
&\qquad + \int_{B_R^+} B(x,u,\nabla u)(u-v)\,dx.
\end{aligned} \tag{A.10}$$

Using (A.4) and the definition of D_v, we have

$$\begin{aligned}\int_{D_v} &[a(x_0, u(x_0), \nabla v) - a(x, u, \nabla v)] \cdot \nabla(u - v)\, dx \\ &\le 2\beta \int_{D_v} (1 + |\nabla v|^{p-2} + |\nabla v|^{p-1})|\nabla v|\,[|x - x_0|^\alpha + |u(x) - u(x_0)|^\alpha]\, dx \\ &\le C[R^\alpha + (\operatorname{osc}_{B_R^+} u)^\alpha] \int_{D_v} (1 + |\nabla v|^p)\, dx.\end{aligned}$$

Similarly we estimate the second term on the right hand side of (A.10), and using also (A.5) we deduce

$$\begin{aligned}\int_{D_v} &[a(x, u, \nabla u) - a(x, u, \nabla v)] \cdot \nabla(u - v)\, dx \\ &\qquad + \int_{D_u} [a(x_0, u(x_0), \nabla u) - a(x_0, u(x_0), \nabla v)] \cdot \nabla(u - v)\, dx \\ &\le C[R^\alpha + (\operatorname{osc}_{B_R^+} u)^\alpha + \operatorname{osc}_{B_R^+} u] \int_{B_R^+} (1 + |\nabla v|^p + |\nabla u|^p)\, dx,\end{aligned}$$

where we have used the fact that $\operatorname{osc}_{B_R^+}(u - v) \le 2 \operatorname{osc}_{B_R^+} u$ thanks to (A.9).

Now, in both terms on the left hand side we use (A.2), which implies, for every (x, s, ξ),

$$[a(x, s, \xi) - a(x, s, \eta)] \cdot (\xi - \eta) \ge c(\lambda)(\mu(x, s)^2 + |\xi|^2 + |\eta|^2)^{(p-2)/2} |\xi - \eta|^2. \quad \text{(A.11)}$$

If $p < 2$ we get (recall that the generic constant C may depend on $\|u\|_\infty$)

$$\begin{aligned}\int_{D_v} &[a(x, u, \nabla u) - a(x, u, \nabla v)] \cdot \nabla(u - v)\, dx \\ &\qquad + \int_{D_u} [a(x_0, u(x_0), \nabla u) - a(x_0, u(x_0), \nabla v)] \cdot \nabla(u - v)\, dx \\ &\ge C \int_{D_v \cup D_u} [1 + |\nabla u|^2 + |\nabla v|^2]^{(p-2)/2} |\nabla(u - v)|^2\, dx,\end{aligned}$$

hence using Hölder's inequality we end up with

$$\int_{B_R^+} |\nabla(u - v)|^p\, dx \le C[R^\alpha + (\operatorname{osc}_{B_R^+} u)^\alpha + \operatorname{osc}_{B_R^+} u]^q \int_{B_R^+} (1 + |\nabla v|^p + |\nabla u|^p)\, dx$$

with $q = p/2$. If $p \ge 2$ we simply get rid of the term μ^2 in (A.11) and obtain the same inequality with $q = 1$. Therefore, using also (A.8), we conclude that for any $p > 1$,

$$\int_{B_R^+} |\nabla(u - v)|^p\, dx \le C[R^\alpha + (\operatorname{osc}_{B_R^+} u)^\alpha + \operatorname{osc}_{B_R^+} u]^q \int_{B_R^+} (1 + |\nabla u|^p)\, dx \quad \text{(A.12)}$$

with $q = \min(1, p/2)$.

Starting from the inequality (A.12) it is possible to deduce the Hölder regularity of ∇u following well-known arguments. In particular, if u is Lipschitz continuous (as in our application in Lemma 2.2) the conclusion is straightforward, since (A.12) implies

$$\int_{B_R^+} |\nabla(u - v)|^p \, dx \le C R^{N+\alpha q}$$

and (A.7)–(A.8) yield $\operatorname{osc}_{B_r^+} \nabla v \le C(r/R)^\sigma$.

Then, defining $(F)_r = |B_r^+|^{-1} \int_{B_r^+} F(y)\, dy$ for $F = \nabla u$ or ∇v, we deduce

$$\begin{aligned}\int_{B_r^+} |\nabla u - (\nabla u)_r|^p \, dx &\le C\left[\int_{B_r^+} |\nabla u - \nabla v|^p + \int_{B_r^+} |\nabla v - (\nabla v)_r|^p\right] \\ &\le C[R^{N+\alpha q} + r^N \, (r/R)^{\sigma p}],\end{aligned}$$

and if we choose $R = r^\theta$ for some suitable $\theta < 1$ the conclusion follows from the results of Campanato [8].

In the general case, i.e. when a Lipschitz estimate on u is not available, one needs further work to estimate the right hand side of (A.12). For this purpose, starting from (A.12), we can follow the arguments of G. Lieberman ([19, Section 3]) and still get the conclusion. ◻

References

[1] Acerbi, E., Fusco, N.: Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$. J. Math. Anal. Appl. **140**, 115–135 (1989) Zbl 0686.49004 MR 0997847

[2] Berger, M., Gauduchon, P., Mazet, E.: Le spectre d'une variété riemannienne. Lecture Notes in Math. 194, Springer, Berlin (1971) Zbl 0223.53034 MR 0282313

[3] Bernstein, S.: Sur la généralisation du problème de Dirichlet II. Math. Ann. **69**, 82–136 (1910) JFM 41.0427.02 MR 1511579

[4] Bidaut-Véron, M.-F., Borghol, R., Véron, L.: Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations. Calc. Var. Partial Differential Equations **27**, 159–177 (2006) Zbl 1143.35034 MR 2251991

[5] Bidaut-Véron, M.-F., Jazar, M., Véron, L.: Separable solutions of some quasilinear equations with source reaction. J. Differential Equations **244**, 274–308 (2008) Zbl 1136.35041 MR 2376199

[6] Borghol, R., Véron, L.: Boundary singularities of N-harmonic functions. Comm. Partial Differential Equations **32**, 1001–1015 (2007) Zbl pre05174545 MR 2334840

[7] Borghol, R., Véron, L.: Boundary singularities of solutions of N-harmonic equations with absorption. J. Funct. Anal. **241**, 611–637 (2006) Zbl pre05116349 MR 2271931

[8] Campanato, S.: Proprietà di Hölderianità di alcune classi di funzioni. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **17**, 175–188 (1963) Zbl 0121.29201 MR 0156188

[9] do Carmo, M. P.: Riemannian Geometry. Birkhäuser Boston, Boston, MA (1992) Zbl 0752.53001 MR 1138207

[10] Chen Y. Z., DiBenedetto, E.: Boundary estimates for solutions of nonlinear degenerate parabolic systems. J. Reine Angew. Math. **395**, 102–131 (1989) Zbl 0661.35052 MR 0983061

[11] DiBenedetto, E.: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. Nonlinear Anal. **7**, 827–850 (1983) Zbl 0539.35027 MR 0709038

[12] Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order. 2nd ed., Springer, Berlin (1983) Zbl 0562.35001 MR 0737190

[13] Greene, R. E., Wu, H.: Function Theory on Manifolds which Possess a Pole. Lecture Notes in Math. 699, Springer, Berlin (1979) Zbl 0414.53043 MR 0521983

[14] Holland, C. J.: A new energy characterization of the smallest eigenvalue of the Schrödinger equation. Comm. Pure Appl. Math. **30**, 755–765 (1977) Zbl 0358.35060 MR 0450768

[15] Kichenassamy, S., Véron, L.: Singular solutions of the p-Laplace equation. Math. Ann. **275**, 599–615 (1986) Zbl 0592.35031 MR 0886428

[16] Krol', I. N.: The behaviour of the solutions of a certain quasilinear equation near zero cusps of the boundary. Proc. Steklov Inst. Math. **125**, 140–146 (1973) Zbl 0286.35033 MR 0344671

[17] Krol', I. N., Maz'ya, V. G.: The absence of the continuity and Hölder continuity of the solutions of quasilinear elliptic equations near a nonregular boundary. Trudy Moskov. Mat. Obshch. **26**, 75–94 (1972) (in Russian) Zbl 0246.35061 MR 0377265

[18] Lasry, J.-M., Lions, P.-L.: Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem. Math. Ann. **283**, 583–630 (1989) Zbl 0688.49026 MR 0990591

[19] Lieberman, G.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**, 1203–1219 (1988) Zbl 0675.35042 MR 0969499

[20] Manfredi, J. J.: Regularity for minima of functionals with p-growth. J. Differential Equations **76**, 203–212 (1988) Zbl 0674.35008 MR 0969420

[21] Tolksdorf, P.: On the Dirichlet problem for quasilinear equations in domains with conical boundary points. Comm. Partial Differential Equations **8**, 773–817 (1983) Zbl 0515.35024 MR 0700735

[22] Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. J. Differential Equations **51**, 126–150 (1984) Zbl 0488.35017 MR 0727034

[23] Véron, L.: Some existence and uniqueness results for solutions of some quasilinear elliptic equations on compact Riemannian manifolds. In: Colloq. Math. Soc. János Bolyai 62, North-Holland, 317–352 (1991) Zbl 0822.58052 MR 1468764