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Geometry of the theta divisor of a compactified jacobian

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Abstract. The object of this paper is the theta divisor of the compactified jacobian of a nodal curve. We determine its irreducible components and give it a geometric interpretation. A characterization of hyperelliptic irreducible stable curves is appended as an application.

Keywords. Nodal curve, line bundle, compactified Picard scheme, theta divisor, Abel map, hyperelliptic stable curve

1. Introduction

Let X be a connected, projective curve of arithmetic genus g and $\operatorname{Pic}^d X$ its degree-d Picard variety, parametrizing line bundles of degree d. If X is smooth, $\operatorname{Pic}^d X$ is isomorphic to an abelian variety and it is endowed with a principal polarization: the theta divisor. If $d = g - 1$ the theta divisor can be intrinsically defined as the locus of $L \in \operatorname{Pic}^{g-1} X$ such that $h^0(X, L) \neq 0$.

If X is singular, $\operatorname{Pic}^d X$ may fail to be projective, so one often needs to replace it with some projective analogue, a so-called "compactified jacobian", or "compactified Picard variety". We shall always assume that X is reduced, possibly reducible, and has at most nodes as singularities.

Although there exist several different constructions of compactified jacobians in the literature, recent work of V. Alexeev shows that in case $d = g - 1$, there exists a "canonical" one. More precisely, in [Al04] the compactifications of T. Oda and C. S. Seshadri [OS79], of C. Simpson [Si94], and of [C94] are shown to be isomorphic if $d = g-1$, to be endowed with an ample Cartier divisor, the theta divisor $\Theta(X)$, and to behave consistently with the degeneration theory of principally polarized abelian varieties.

Some first results on the theta divisor of the (non-compactified) generalized jacobian of any nodal curve were obtained by A. Beauville [B77]. Years later, A. Soucaris [S94] and E. Esteves [E97] independently constructed the theta divisor (as a Cartier, ample divisor) on the compactified jacobian of an irreducible curve. The case of a reducible, nodal curve was handled in [Al04]. As a result, today we know that, in degree $g - 1$, the

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compactified Picard variety of any nodal curve has a polarization, the theta divisor, such that the pair (*compactified jacobian, theta divisor*) is a semiabelic stable pair in the sense of [Al02]. Furthermore, the above holds in the relative setting, i.e. for families of nodal curves.

These recent developements revive interest in the theory of Brill–Noether varieties for singular curves, of which the theta divisor is one of the principal objects.

The purpose of this paper is to investigate the geometry and the modular meaning of $\Theta(X)$ more closely. Our first result (Theorem 3.1.2) describes its irreducible components, establishing that every irreducible component of the compactified jacobian contains a unique irreducible component of the theta divisor, unless X has some separating node (see 4.2.1); in particular, we characterize singular curves whose theta divisor is irreducible (in 4.2.2). In more technical terms, we prove that for every fixed "stable" multidegree (cf. Definition 1.3.1) the theta divisor has a unique irreducible component. This result is sharp in the sense that irreducibility fails for non-stable multidegrees (see Examples 3.1.4). The idea and the strategy of the proof are described in 1.3.8.

We prove the irreducibility Theorem 3.1.2 using the Abel map, namely, the rational map from X^{g-1} to $\operatorname{Pic}^{g-1} X$, sending $(p_1, \dots, p_{g-1})$ to $[\mathcal{O}_X(\sum p_i)]$. As a by-product, the theta divisor is shown to be the closure of the image of the Abel map, for every stable multidegree. This fact, albeit trivial for smooth curves, fails if the multidegree is not stable (see Proposition 1.3.7 for a non-semistable multidegree, and Example 3.1.4 for a strictly semistable one).

In the second part of the paper we concentrate on the geometric interpretation of $\Theta(X)$ and precisely describe the objects it parametrizes. In Theorem 4.2.6 we exhibit a stratification by means of the theta divisors of the partial normalizations of X. We wish to observe that very similar stratifications have been proved to exist for several other compactified spaces, associated to singular curves (see 4.1.5, or Theorem 7.9 in [C05], for example). It is thus quite natural to ask whether all compactified moduli spaces associated to a singular curve admit an analogous stratification, or whether some general rules governing such a phenomenon exist. These questions are open at the moment.

Our stratification of $\Theta(X)$ yields a description in terms of effective line bundles on the partial normalizations of X, or (which turns out to be the same) in terms of line bundles on semistable curves stably equivalent to X.

In the final part, we apply our techniques to generalize to singular curves the characterization of smooth hyperelliptic curves via the singular locus of their theta divisor; recall that $\Theta(C)_{\text{sing}} = W^1_{g-1}(C)$ for every smooth curve C of genus $g \geq 3$. Furthermore, C is hyperelliptic if $\dim W^1_{g-1}(C) = g-3$, and non-hyperelliptic if $\dim W^1_{g-1}(C) = g-4$; we prove that the same holds if X is an irreducible singular curve (Theorem 5.2.4), but fails if X is reducible (see 5.2.5). On the other hand, the relation between $\Theta(X)_{\text{sing}}$ and $W^1_{g-1}(X)$ (and more generally $W^r_{g-1}(X)$), i.e. a Riemann Singularity Theorem for singular curves, is not known and it would be very interesting to establish it.

The paper consists of five sections. The first contains preliminaries and basic definitions; the second mostly consists of technical results. In the third section we prove the irreducibility theorem and study the dimension of the image of the Abel map (Proposition 3.2.1). In the fourth section we describe the compactification of the theta divisor inside the compactified jacobian. The fifth section contains the application to singular hyperelliptic curves.

1.1. Notation and conventions

1.1.1. We work over an algebraically closed field k. By a “curve” we mean a reduced, projective curve over k.

Throughout the paper, X will be a connected nodal curve of arithmetic genus g, having γ irreducible components and δ nodes. We let $\nu : Y \to X$ be the normalization of X, so that $Y = \coprod_{i=1}^{\gamma} C_i$ with C_i smooth of genus g_i, and $X = \bigcup \overline{C_i}$ with $\overline{C_i} = \nu(C_i)$. Recall that $g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$. Observe that this formula holds regardless of whether X is connected or not.

We denote by X_{sing} the set of nodes of X. For any set of nodes of X, $S \subset X_{\text{sing}}$, set $\#S = \delta_S$ and $S = \{n_1, \ldots, n_{\delta_S}\}$. The normalization of X at exactly the nodes in S will be denoted $\nu_S : Y_S \to X$ and γ_S will be the number of connected components of Y_S; thus $Y_S = \coprod_{i=1}^{\gamma_S} Y_i$ with Y_i a connected curve of arithmetic genus g_{Y_i}. We have

$$g = \sum_{i=1}^{\gamma_S} g_{Y_i} + \delta_S - \gamma_S + 1 \tag{1}$$

and, denoting $g_{Y_S} = p_a(Y_S)$,

$$g_{Y_S} = g - \delta_S = \sum_{i=1}^{\gamma_S} g_{Y_i} - \gamma_S + 1. \tag{2}$$

For every $j = 1, \ldots, \delta_S$ (or for every $n \in S$) we set

$$\nu_S^{-1}(n_j) = \{q_1^j, q_2^j\} \quad (\text{or } \ \nu_S^{-1}(n) = \{q_1, q_2\}). \tag{3}$$

1.1.2. The dual graph of a nodal curve Y, denoted Γ_Y, has vertices the irreducible components of Y and edges the nodes of Y. A node lying in a unique irreducible component C_i is a loop of Γ_Y based at the vertex C_i; a node lying in $C_i \cap C_j$ is an edge joining the vertices C_i and C_j.

1.1.3. The degree-d Picard variety $\operatorname{Pic}^d X$ has a decomposition into connected/irreducible components: $\operatorname{Pic}^d X = \coprod_{\underline{d} \in \mathbb{Z}^{\gamma} : |\underline{d}| = d} \operatorname{Pic}^{\underline{d}} X$, where $\operatorname{Pic}^{\underline{d}} X$ is the variety of isomorphism classes of line bundles of multidegree $\underline{d}$.

Let $\nu_S : Y_S \to X$ be as in 1.1.1. Consider the pull-back map

$$\operatorname{Pic} X \xrightarrow{\nu_S^*} \operatorname{Pic} Y_S \cong \prod_{i=1}^{\gamma_S} \operatorname{Pic} Y_i \to 0.$$

We shall usually identify $\operatorname{Pic} Y_S \cong \prod \operatorname{Pic} Y_i$ without mentioning it.

Let $M \in \operatorname{Pic} Y_S$. Then the fiber over M will be denoted

$$F_M(X) := \{L \in \operatorname{Pic} X : \nu_S^* L = M\} \cong (k^*)^{\delta_S - \gamma_S + 1}. \tag{4}$$

1.1.4. We shall now describe the isomorphism $F_M(X) \cong (k^*)^{\delta_S - \gamma_S + 1}$ explicitly to fix some conventions. Let us simplify the notation by omitting the subscript S (so, $\delta = \delta_S$, $Y = Y_S$, etc.). Assume first that Y is connected.

Let $\underline{c} = (c_1, \dots, c_\delta) \in (k^*)^\delta$; $\underline{c}$ determines a unique $L \in \operatorname{Pic} X$ such that $\nu^* L = M$ as follows. For every $j = 1, \dots, \delta$ consider the two fibers of M over q_1^j and q_2^j (recall that $\nu(q_1^j) = \nu(q_2^j) = n_j$), and fix an isomorphism between them. We define a line bundle $L = L^{(\underline{c})}$ on X which pulls back to M, by gluing $M_{q_1^j}$ to $M_{q_2^j}$ via the isomorphism

$$M_{q_1^j} \xrightarrow{\cdot c_j} M_{q_2^j}$$

given by multiplication by c_j. Conversely, every $L \in F_M(X)$ is of type $L^{(\underline{c})}$.

Now let Y have γ connected components; note that, since X is connected, we always have $\gamma - 1 \leq \delta$. There exist some subsets $T \subset S$ with $\#T = \gamma - 1$ such that if we remove from Γ_X every node that is not in T, the remaining graph is a connected tree (a so-called *spanning tree* of Γ_X).

Let us fix one such T and order the nodes in S so that the last $\gamma - 1$ are in T, i.e. $S = \{n_1, \dots, n_\delta\} = \{n_1, \dots, n_{\delta-\gamma+1}\} \cup T$. Now factor ν as

$$\nu : Y \xrightarrow{\nu_T} Y' \xrightarrow{\nu'} X$$

so that ν' is the partial normalization of X at $S \smallsetminus T$ and ν_T the normalization at the nodes of Y' preimages of the nodes in T. For example, if $S = X_{\text{sing}}$ (i.e. if Y is smooth) then Y' is a curve of compact type. The pull-back map ν_T^* induces an isomorphism $\operatorname{Pic} Y' \cong \operatorname{Pic} Y$, i.e. different gluing data determine isomorphic line bundles on Y'.

Now, to construct the fiber of $\operatorname{Pic} X \to \operatorname{Pic} Y'$ over M' we proceed as in the previous part.

Summarizing, to every $\underline{c} \in (k^*)^{\delta-\gamma+1}$ we associate a unique $L^{(\underline{c})} \in \operatorname{Pic} Y$; since the gluing data over the nodes in T is irrelevant, we shall fix $c_j = 1$ if $j \geq \delta - \gamma$ and use that as gluing constant over T.

Finally, observe that a section $s \in H^0(Y, M)$ descends to a section $\overline{s} \in H^0(X, L^{(\underline{c})})$ if and only if for every $j = 1, \dots, \delta$ we have

$$s(q_2^j) = c_j s(q_1^j). \tag{5}$$

1.2. Brill–Noether varieties and Abel maps

1.2.1. We recall some basic facts about Brill–Noether varieties for smooth curves, following the notation of [ACGH] to which we refer for details.

Let C be a smooth connected curve of genus $g \geq 0$, and let d and r be non-negative integers. The set $W_d^r(C) := \{L \in \operatorname{Pic}^d C : h^0(C, L) \geq r + 1\}$ has an algebraic structure and is called a *Brill–Noether variety*. It is closely related to the Abel map in degree d of C, that is, the map

$$\alpha_C^d : C^d \to \operatorname{Pic}^d C, \qquad (p_1, \dots, p_d) \mapsto \mathcal{O}_C\Big(\sum_{i=1}^{d} p_i\Big). \tag{6}$$

Then $\operatorname{Im} \alpha_C^d \subseteq W_d^0(C)$ for all $d \geq 0$ (see 1.2.3 for when equality occurs). Note that $W_d^r(C)$ may fail to be irreducible, so when talking about its dimension we will mean the maximum dimension of its components. The following is well known ([ACGH, Lemma 3.3, Ch. IV]).

Fact 1.2.2. *If $r \geq d - g$ then every irreducible component of $W_d^r(C)$ has dimension at least*

$$\rho(g, r, d) := g - (r+1)(r - d + g).$$

If $r \leq d - g$ then $W_d^r(C) = W_d^{d-g}(C)$.

There is also a simple upper bound

$$\dim W_d^r(C) \leq \min\{d - r, g\}. \tag{7}$$

Indeed, if $d - r \leq g$, it suffices to look at the Abel map of degree d to obtain $\dim W_d^r(C) \leq d - r$ (cf. [ACGH, Prop. 3.4, Ch. IV]). If $d - r \geq g$ then, by Riemann–Roch, $\dim W_d^r(C) = g$.

Remark 1.2.3. Denote by $r(d)$ the dimension of a general (non-empty) complete linear system of degree d. i.e. if $d \leq g$ set $r(d) = 0$, if $d \geq g$ set $r(d) = d - g$. Note that $W_d^{r(d)}(C) = \operatorname{Im} \alpha_C^d$. Now, $\min\{d - r(d), g\} = \min\{d, g\}$ and

$$\dim W_d^r(C) \begin{cases} = \min\{d, g\} & \text{if } r \leq r(d), \\ < \min\{d, g\} & \text{if } r > r(d). \end{cases}$$

To see that, assume first that $r \leq r(d)$. Then $W_d^r(C) = W_d^{r(d)}(C)$ by Riemann–Roch, so we may assume that $r = r(d)$. Now computing gives $\rho(d, g, r(d)) = \min\{d, g\}$, so by Fact 1.2.2 and (7) we get $\dim W_d^r(C) = \min\{d, g\}$. The case $r > r(d)$ follows from (7) and the fact that $\min\{d - r, g\} < \min\{d - r(d), g\}$.

1.2.4. For a nodal curve X of genus g having γ irreducible components, for any $\underline{d} \in \mathbb{Z}^\gamma$ and $r \geq 0$, we set $W_{\underline{d}}^r(X) = \{L \in \operatorname{Pic}^{\underline{d}} X : h^0(X, L) \geq r + 1\}$ and for any $d \in \mathbb{Z}$, $W_d^r(X) := \coprod_{|\underline{d}|=d} W_{\underline{d}}^r(X)$. In case $r = 0$ the superscript $r = 0$ is usually omitted. In particular

$$W_{g-1}(X) := \{L \in \operatorname{Pic}^{g-1} X : h^0(X, L) \geq 1\} = \coprod_{|\underline{d}|=g-1} W_{\underline{d}}(X).$$

With the notation of 1.1.3, if $\nu_S : Y_S \to X$ is a partial normalization and $M \in \operatorname{Pic} Y_S$, the fiber of $W_d^r(X)$ over M will be denoted (recall (4))

$$W_M^r(X) := \{L \in F_M(X) : h^0(X, L) \geq r + 1\} \tag{8}$$

and $W_M(X) := \{L \in F_M(X) : h^0(X, L) \geq 1\}$.

Remark 1.2.5. The above definitions make sense also for non-connected curves. Consider a disconnected curve, $Y = \coprod_{i=1}^{\gamma} C_i$, where C_i is smooth and connected (or more generally C_i irreducible) of genus g_i. For any $\underline{d} \in \mathbb{Z}^{\gamma}$, the variety $W_{\underline{d}}(Y)$ is easily described in terms of the C_i:

$$W_{\underline{d}}(Y) = \begin{cases} \prod_{i=1}^{\gamma} \operatorname{Pic}^{d_i} C_i & \text{if } \exists i : d_i \geq g_i, \\ \bigcup_{j=1}^{\gamma} (W_{d_j}(C_j) \times \prod_{i \neq j, i=1,\dots,\gamma} \operatorname{Pic}^{d_i} C_i) & \text{if } \forall i : d_i \leq g_i - 1. \end{cases}$$

We shall need the following very simple

Lemma 1.2.6. *Let $S \subset X_{\text{sing}}$, $\nu_S : Y_S \to X$ the normalization of X at S and $p \in X \smallsetminus S$. Let $M \in \operatorname{Pic} Y_S$ and assume that M has no base point in $\nu_S^{-1}(S \cup p)$. Then there exists $L \in W_M(X)$ such that L has no base point in p. In particular, if M has no base point over S then $W_M(X)$ is non-empty.*

Proof. To say that M has no base point in $\nu_S^{-1}(S \cup p)$ is to say that there exists $s \in H^0(Y_S, M)$ such that $s(q) \neq 0$ for every $q \in \nu_S^{-1}(S \cup p)$. We can use s to construct a line bundle $L \in W_M(X)$ by identifying the two fibers over pairs of corresponding branches. More precisely, with the notation of 1.1.4(5) for every $n_j \in S$ let q_1^j, q_2^j be the branches over n_j. Then set $c_j := s(q_2^j)/s(q_1^j)$ and define $L = L^{(\underline{c})}$. It is clear that s descends to a nonzero section $\overline{s}$ of L and that $\overline{s}(p) \neq 0$. □

1.2.7. *Abel maps.* We now introduce the Abel maps of a singular curve. Recall (see 1.1.1) that $X = \overline{C}_1 \cup \cdots \cup \overline{C}_{\gamma}$ denotes the decomposition of X into irreducible components. For every $\underline{d} = (d_1, \dots, d_{\gamma})$ such that $d_i \geq 0$ we set $X^{\underline{d}} = \overline{C}_1^{d_1} \times \cdots \times \overline{C}_{\gamma}^{d_{\gamma}}$. Now denote $\dot{X} = X \smallsetminus X_{\text{sing}}$, the smooth locus of X. The normalization map $Y = \bigcup C_i \xrightarrow{\nu} X = \bigcup \overline{C}_i$ induces an isomorphism of $\dot{X}$ with $Y \smallsetminus \nu^{-1}(X_{\text{sing}})$. We shall identify $\dot{X} = Y \smallsetminus \nu^{-1}(X_{\text{sing}})$ and denote $\dot{C}_i := \overline{C}_i \cap \dot{X}$. Finally, set

$$\dot{X}^{\underline{d}} := \dot{C}_1^{d_1} \times \cdots \times \dot{C}_{\gamma}^{d_{\gamma}} \subset X^{\underline{d}}$$

so that $\dot{X}^{\underline{d}}$ is a smooth irreducible variety of dimension $|\underline{d}|$, open and dense in $X^{\underline{d}}$. Set $d = |d|$. Then we have a regular map

$$\alpha_X^{\underline{d}} : \dot{X}^{\underline{d}} \to \operatorname{Pic}^{\underline{d}} X, \qquad (p_1, \dots, p_d) \mapsto \mathcal{O}_X\Big(\sum_{i=1}^{d} p_i\Big), \tag{9}$$

which we call the *Abel map of multidegree $\underline{d}$*. We denote

$$A_{\underline{d}}(X) := \overline{\alpha_X^{\underline{d}}(\dot{X}^{\underline{d}})} \subset \operatorname{Pic}^{\underline{d}} X.$$

Lemma 1.2.8. *Let X be a (connected, nodal) curve of genus $g \geq 0$. For every $d \geq 1$ and every multidegree $\underline{d}$ on X such that $\underline{d} \geq 0$ and $|\underline{d}| = d$ we have*

(i) *$A_{\underline{d}}(X)$ is irreducible and* $\dim A_{\underline{d}}(X) \leq \min\{d, g\}$*;*

(ii) *$A_{\underline{d}}(X) \subset W_{\underline{d}}(X)$.*

Proof. Obvious. □

We shall see that strict inequality in (i) does occur (cf. Proposition 3.2.1).

1.3. Stability and semistability

As we said in the introduction, there exist various modular descriptions for a compactified Picard variety, and they are equivalent if $d = g - 1$. We shall give the complete description later, in 4.1.1. For now it is enough to recall that, for every nodal curve X, the compactified Picard variety in degree $g-1$, $\overline{P_X^{g-1}}$, is a union of (finitely many) irreducible g-dimensional components each of which contains as an open subset a copy of the generalized jacobian of X. To study the irreducible components of the theta divisor of $\overline{P_X^{g-1}}$ there is no need to consider its boundary points. This explains why we chose to postpone the complete description of $\overline{P_X^{g-1}}$; see 4.1.1.

So, now only the open smooth locus of $\overline{P_X^{g-1}}$ will be described, using line bundles of "stable" multidegree on the normalization of X at its separating nodes.

There exist two different, equivalent definitions of semistability and stability (1.3.1 and 1.3.2 below); the simultaneous use of the two is a good tool to overcome technical difficulties of combinatorial type.

1.3.1. *Stability: Definition 1.* Let Y be a nodal curve of arithmetic genus g having γ irreducible components. Let $\underline{d} \in \mathbb{Z}^\gamma$ be such that $|\underline{d}| = g - 1$.

(a) We call $\underline{d}$ *semistable* if for every subcurve (equivalently, every connected subcurve) $Z \subset Y$ of arithmetic genus g_Z we have

$$d_Z \geq g_Z - 1 \tag{10}$$

where $d_Z := |\underline{d}_Z|$. The set of semistable multidegrees on Y is denoted

$$\Sigma^{ss}(Y) := \{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = g - 1, \ \underline{d} \text{ is semistable}\}.$$

(b) Assume Y is connected. If Y is irreducible, or if strict inequality holds in (10) for every (connected) subcurve $Z \subsetneq Y$, then $\underline{d}$ is called *stable*. If Y is not connected, we say that $\underline{d}$ is stable if its restriction to every connected component of Y is stable. We define

$$\Sigma(Y) := \{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = g - 1, \ \underline{d} \text{ is stable}\} \subset \Sigma^{ss}(Y).$$

We shall also use the following equivalent definition, originating from [B77].

1.3.2. *Stability: Definition 2.* Fix Y and $\underline{d}$ as in 1.3.1.

(A) $\underline{d}$ is *semistable* if the dual graph Γ_Y of Y (cf. 1.1.2) can be oriented in such a way that, denoting by b_i the number of edges pointing to the vertex corresponding to the irreducible component C_i of Y, we have

$$d_i = g_i - 1 + b_i$$

where g_i is the geometric genus of C_i (so that $g_i = p_a(C_i) - \#(C_i)_{\text{sing}}$).

(B) Assume Y is connected. Then $\underline{d}$ is stable if Γ_Y admits an orientation satisfying (A) and such that there exists no proper subcurve $Z \subsetneq Y$ such that the edges between Γ_Z and Γ_{Z^C} go all in the same direction, where $Z^C := \overline{Y \smallsetminus Z}$.

The equivalence of definitions 1.3.2 and 1.3.1 is Proposition 3.6 in [Al04]. The version given in (A) is due to A. Beauville, who used it in [B77] to define and study the theta divisor of a generalized jacobian. (In [B77, Lemma (2.1)] the dual graph is without loops by definition, whereas we need to include loops. This explains the difference between our definition and that of [B77].)

Version 1.3.1 actually extends to all degrees (other than degree $g-1$); it originates from D. Gieseker's construction of $\overline{M}_g$ and is crucial in [C94] (where (10) is generalized by the so-called "Basic Inequality"). V. Alexeev proved that the Basic Inequality yields the modular description of the compactified jacobians constructed by Oda–Seshadri and by C. Simpson using different approaches (see [Al04, 1.7(5)]). More details about this definition and its connection with Geometric Invariant Theory will be given in Section 4.

Remark 1.3.3. (i) Applying inequality (10) to all subcurves, we find that $\underline{d}$ is semistable if and only if for every connected $Z \subset Y$,

$$p_a(Z) - 1 \le d_Z \le p_a(Z) - 1 + \#Z \cap Z^C. \tag{11}$$

If X is connected, $\underline{d}$ is stable if and only if strict inequalities hold in (11) for all Z.

(ii) If $\underline{d} \in \Sigma^{ss}(X)$ and $V \subset X$ is a subcurve such that $d_V = g_V - 1$, then $\underline{d}_V$ is semistable on V.

(iii) If $\underline{d}$ is stable, then $\underline{d} \ge 0$.

Remark 1.3.4. The following convention turns out to be useful. Given a graph Γ (e.g. $\Gamma = \Gamma_Y$), every edge n determines two half-edges, denoted q_1^n and q_2^n (corresponding to the two branches of the node n of Y). If Γ is oriented we call q_1^n the starting half-edge of n and q_2^n the ending one.

$\Sigma^{ss}(X)$ is never empty (by [C05, Prop. 4.12]). On the other hand, we have

Lemma 1.3.5. $\Sigma(X) = \emptyset$ *if and only if* X *has a separating node.*

Proof. If X has a separating node, n, then $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \{n\}$. Let $\underline{d} \in \Sigma^{ss}(X)$. Using (11) we have $p_a(X_i) - 1 \le d_{X_i} \le p_a(X_i)$, so that strict inequalities cannot simultaneously occur. Hence $\underline{d}$ is not stable.

Conversely, assume that X has no separating node. We shall use Definition 1.3.2, and prove that the dual graph of X, $\Gamma = \Gamma_X$, admits a "stable orientation" (i.e. an orientation satisfying (B)). We use induction on the number δ of nodes that lie in two different irreducible components (the only nodes that matter), i.e. induction on the number of edges that are not loops. If $\delta = 1$ there is nothing to prove (the edge is necessarily separating); if $\delta = 2$ then Γ has two vertices so the statement is clear.

Let $\delta \ge 2$, pick an edge n and let $\Gamma' = \Gamma - n$; thus Γ' is connected. If Γ' has no separating edge, by induction Γ' admits a stable orientation, hence so does Γ, of course. Denote by $n_1, \ldots, n_t$ the separating edges of Γ'. The graph

$$\Gamma' - \{n_1, \ldots, n_t\} = \Gamma - \{n_0, n_1, \ldots, n_t\},$$

where $n = n_0$, has $t + 1$ connected components, $\overline{\Gamma_0}, \dots, \overline{\Gamma_t}$, each of which is free from separating edges.

We claim that the image $\Gamma_i \subset \Gamma$ of each $\overline{\Gamma_i}$ contains exactly two of the edges $n_0, n_1, \dots, n_t$.

Indeed, if (say) Γ_1 contains only one n_i with $i \geq 1$, call it n_1 and let Γ_2 be the other Γ_i containing n_1. Then n_0 connects Γ_1 to Γ_2 (for otherwise n_1 would be a separating node of Γ, which is not possible). Hence Γ_1 contains n_0 and n_1.

If Γ_1 contains two n_i with $i \geq 1$, say n_1 and n_2, let Γ_2 and Γ_3 be such that $n_i \in \Gamma_1 \cap \Gamma_{i+1}$, $i = 1, 2$. Then n_0 connects Γ_2 and Γ_3, thus $n_0 \notin \Gamma_1$. Therefore Γ_1 contains only n_1 and n_2.

If Γ_1 contains three n_i, $i \geq 1$, say n_1, n_2 and n_3, let Γ_2, Γ_3 and Γ_4 be such that $n_i \in \Gamma_1 \cap \Gamma_{i+1}$. Now n_0 is contained in at most two Γ_i, so say $n_0 \notin \Gamma_4$; but then n_3 is a separating node of Γ, which is a contradiction. Therefore, up to reordering the Γ_i, we can assume that

$$n_i \in \Gamma_i \cap \Gamma_{i-1}, \qquad i = 1, \dots, t, t + 1 = 0.$$

We now define an orientation on Γ by combining the stable orientation on each Γ_i with each edge n_i oriented from Γ_{i-1} to Γ_i. It suffices to prove that this is a stable orientation on Γ.

Indeed, let $Z \subset X$ and $\Gamma_Z \subset \Gamma$ the corresponding graph. If for some i we have $\emptyset \neq \Gamma_Z \cap \Gamma_i \subsetneq \Gamma_i$, then inside Γ_i there are edges both starting from and ending in Γ_Z. So the same holds in Γ and we are done. Hence we can assume that for every i either $\Gamma_i \subset \Gamma_Z$ or $\Gamma_Z \cap \Gamma_i = \emptyset$. Therefore

$$\Gamma_Z \cap \Gamma_{Z^c} \subset \{n_0, n_1, \dots, n_t\}.$$

We can thus reduce ourselves to considering the graph obtained by contracting every Γ_i to a point. This is of course a cyclic graph with $t+1$ vertices and $t+1$ edges $\{n_0, n_1, \dots, n_t\}$, oriented cyclically. This is a stable orientation, so we are done. $\square$

Example 1.3.6. Let X be a nodal connected curve of genus g, $X_{\text{sep}} \subset X_{\text{sing}}$ the set of its separating nodes and $\widetilde{X} \to X$ the normalization of X at X_{sep}. Assume $\#X_{\text{sep}} = c - 1 \geq 1$ so that $\widetilde{X}$ has c connected components $X_1, \dots, X_c$ and X_i is free from separating nodes for every $i = 1, \dots, c$. Thus $\Sigma(X_i) \neq \emptyset$ and

$$\Sigma(\widetilde{X}) = \Sigma(X_1) \times \cdots \times \Sigma(X_c).$$

Indeed, set $g_i := p_a(X_i)$. Then $p_a(\widetilde{X}) - 1 = (g - c + 1) - 1 = \sum_{i=1}^{c}(g_i - 1)$, and $\underline{d} \in \Sigma(\widetilde{X})$ if and only the restriction of $\underline{d}$ to X_i is stable on X_i.

Proposition 1.3.7 (Beauville). *Let X be a (connected, nodal) curve of genus $g \geq 1$, and let $\underline{d} \in \mathbb{Z}^\gamma$ be such that $|\underline{d}| = g - 1$.*

(i) *$\underline{d}$ is semistable iff there exists $L \in \operatorname{Pic}^{\underline{d}} X$ such that $h^0(X, L) = 0$.*
(ii) *If $\underline{d}$ is semistable then every irreducible component of $W_{\underline{d}}(X)$ has dimension $g - 1$.*
(iii) *If $\underline{d}$ is not semistable then $W_{\underline{d}}(X) = \operatorname{Pic}^{\underline{d}} X$.*

See Lemma (2.1) and Proposition (2.2) in [B77].

1.3.8. Our first theorem (Theorem 3.1.2) states that, if $\underline{d}$ is stable, then $W_{\underline{d}}(X)$ is irreducible and equal to $A_{\underline{d}}(X)$. The proof's strategy is the following. We know, by the above Proposition 1.3.7, that every irreducible component of $W_{\underline{d}}(X)$ has dimension $g-1$; we also know that $A_{\underline{d}}(X)$ is irreducible. We shall prove that if W is an irreducible component of $W_{\underline{d}}(X)$, not dominated by the image of the Abel map, then $\dim W \leq g-2$, and hence W must be empty.

To do that we consider the normalization $\nu : Y \to X$ and the pull-back map $\nu^* : \operatorname{Pic} X \to \operatorname{Pic} Y$. The dimension of W is then studied by fibering W using ν^*, and bounding the dimensions of the image and the fibers.

An important point is to show that, on the one hand, the divisors on Y supported over the nodes of X impose independent conditions on the general line bundle $M \in \operatorname{Pic}^{\underline{d}} Y$; see Lemma 2.3.3. On the other hand, if $M \in \operatorname{Pic} Y$ has this property (i.e. divisors supported in $\nu^{-1}(X_{\text{sing}})$ impose independent conditions on it), then the dimension of the locus of $L \in W_M(X)$ which do not lie in the image of the Abel map is small, hence the dimension of the fiber of W over M is small; see Proposition 2.3.5 and Corollary 2.3.7.

2. Technical groundwork

2.1. Basic estimates

Recall the set-up of 1.1.1.

Proposition 2.1.1. *Fix $\nu_S : Y_S \to X$ and let $M \in \operatorname{Pic} Y_S$.*

(i) *For every $L \in \operatorname{Pic} X$ such that $\nu_S^* L = M$ we have*

$$h^0(Y_S, M) - \delta_S \leq h^0(X, L) \leq h^0(Y_S, M). \tag{12}$$

(ii) *Let $h^0(Y_S, M) \geq \delta_S$. Assume that for some $h : \{1, \dots, \delta_S\} \to \{1, 2\}$,*

$$h^0\Big(Y_S, M\Big(-\sum_{j=1}^{\delta_S} q^j_{h(j)}\Big)\Big) = h^0(M) - \delta_S. \tag{13}$$

Then $W_M(X)$ is of pure dimension

$$\dim W_M(X) = \begin{cases} \delta_S - \gamma_S & \text{if } h^0(M) = \delta_S, \\ \delta_S - \gamma_S + 1 & \text{if } h^0(M) \geq \delta_S + 1. \end{cases}$$

Moreover, the general element $L \in W_M(X)$ satisfies

$$h^0(X, L) = \max\{h^0(Y_S, M) - \delta_S, 1\}. \tag{14}$$

Proof. Throughout the proof we shall simplify the notation by omitting the index S, i.e. set $Y = Y_S$, $\delta = \delta_S$, $\nu = \nu_S$ and $\gamma = \gamma_S$.

Let $L \in F_M(X)$. Then we have the exact sequence

$$0 \to L \to \nu_* M \to \sum_{n \in S} k_n \to 0 \tag{15}$$

and the associated long cohomology sequence

$$0 \to H^0(X, L) \xrightarrow{\alpha} H^0(Y, M) \xrightarrow{\beta} k^\delta \to H^1(X, L) \to H^1(Y, M) \to 0 \qquad (16)$$

from which we immediately get the upper bound on $h^0(X, L)$ stated in (12).

Fix $M \in \operatorname{Pic} Y$ and recall the description of the fiber of ν^* over M given in 1.1.4. Thus every $L \in F_M(X)$ is of the form $L = L^{(\underline{c})}$ for some $\underline{c} \in (k^*)^{\delta-\gamma+1}$. For convenience, we use the same set-up of 1.1.4, in particular we set $c_j = 1$ for $\delta - \gamma + 2 \le j \le \delta$.

To compute $H^0(X, L)$, set $l = h^0(Y, M)$ and pick a basis $s_1, \dots, s_l$ for $H^0(Y, M)$. Let $s \in H^0(Y, M)$, so $s = \sum_{i=1}^{l} x_i s_i$ where $x_i \in k$. Now s descends to a section of L (i.e. s lies in the image of α in (16)) if and only if

$$\sum_{i=1}^{l} x_i (s_i(q_2^j) - c_j s_i(q_1^j)) = 0 \quad \forall j = 1, \dots, \delta. \qquad (17)$$

The above is a linear system of δ homogeneous equations in the l unknowns $x_1, \dots, x_l$. The space of its solutions, $\Lambda(\underline{c})$, is identified with $H^0(X, L^{(\underline{c})})$. Now, $\Lambda(\underline{c})$ is a linear subspace of $H^0(Y, M)$ of dimension at least $l - \delta$. Hence $h^0(X, L) = \dim \Lambda(\underline{c}) \ge l - \delta$, proving (12).

For (ii) assume $l = h^0(Y, M) \ge \delta$; denote by $A(\underline{c})$ the $\delta \times l$ matrix of the system (17). By what we said

$$h^0(X, L^{(\underline{c})}) = \dim \Lambda(\underline{c}) = l - \operatorname{rank} A(\underline{c}) \qquad (18)$$

and

$$W_M(X) \cong \{\underline{c} : \Lambda(\underline{c}) \neq 0\} = \{\underline{c} : \operatorname{rank} A(\underline{c}) < l\}. \qquad (19)$$

We shall prove that $A(\underline{c})$ has rank δ unless $\underline{c}$ lies in a proper closed subset of $(k^*)^\delta$. For that, we apply the assumption (13) to choose the basis for $H^0(Y, M)$ as follows. First, up to renaming each pair of branches we can assume that $h(j) = 1$ for every j. By (13) we can pick δ linearly independent $s_1, \dots, s_\delta \in H^0(M)$ such that

$$s_i(q_1^j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } j \neq i, \ (j = 1, \dots, \delta). \end{cases}$$

If $l > \delta$ we choose the remaining basis elements however we like. Set $b_i^j := s_i(q_2^j) \in k$. Then the matrix $A(\underline{c})$ contains a $\delta \times \delta$ minor $B(\underline{c})$, (given by the first δ columns), whose diagonal is $(c_1 - b_1^1, \dots, c_\delta - b_\delta^\delta)$, and such that the c_j do not appear anywhere else in $B(\underline{c})$. Therefore the determinant of $B(\underline{c})$ is a non-zero polynomial in the c_j. This proves that the locus where the matrix has maximal rank (equal to δ) is open, non-empty.

Suppose $\delta = l$. Then $B(\underline{c}) = A(\underline{c})$. By (19), $W_M(X)$ is naturally identified with the locus of points of $F_M(X)$ where $\det A(\underline{c})$ vanishes. We conclude that $W_M(X)$ has pure dimension $\dim W_M(X) = \delta - \gamma$, proving (ii).

Moreover, for a general $L^{(\underline{c})} \in W_M(X)$, the rank of $A(\underline{c})$ is equal to $\delta - 1$. Indeed, by (19), $W_M(X)$ is identified to the hypersurface, W, of k^δ where $\det A(\underline{c})$ vanishes. Denote by $A_j^i(\underline{c})$ the minor of $A(\underline{c})$ obtained by removing the i-th row and the j-th column, and set $U_j^i = \{\underline{c} \in k^\delta : \det A_j^i(\underline{c}) \neq 0\}$. We must prove that $W \cap U_j^i \neq \emptyset$ for some

$1 \leq i, j \leq \delta$. Suppose c_1 appears in $\det A(\underline{c})$. On the other hand, c_1 does not appear in $\det A_1^1(\underline{c})$, as $A_1^1(\underline{c})$ does not contain c_1. Hence $W \cap U_1^1 \neq \emptyset$.

Therefore by (18) we get $h^0(X, L) = 1$, proving (14).

If $l > \delta$, then $W_M(X) = F_M(X)$ by (12). Furthermore, by (18),

$$h^0(X, L^{(\underline{c})}) = l - \operatorname{rank} A(\underline{c}) \geq l - \delta.$$

By looking at the matrix $A(\underline{c})$, we see that $h^0(X, L^{(\underline{c})}) = l - \delta$ on the non-empty open subset where $\det B(\underline{c})$ does not vanish; this proves (14). ◻

Lemma 2.1.2. *Let $\nu : Y \to X$ be the normalization of X and let $\underline{d} \in \Sigma^{ss}(X)$. For a general $M \in \operatorname{Pic}^{\underline{d}} Y$ we have*

(i) $h^0(Y, M) = \delta$;
(ii) *M satisfies condition* (13) *with respect to a suitable $h : \{1, \dots, \delta\} \to \{1, 2\}$;*
(iii) $\dim W_M(X) = \delta - \gamma$;
(iv) *the general L in $W_M(X)$ satisfies $h^0(X, L) = 1$.*

Proof. Using the notation of 1.1.1, $Y = \coprod C_i$ with C_i smooth of genus g_i, and $X = \bigcup \overline{C_i}$. The fact that $\underline{d}$ is semistable implies that $d_i \geq p_a(\overline{C_i}) - 1 \geq g_i - 1$ for every $i = 1, \dots, \gamma$. Therefore for M general in $\operatorname{Pic}^{\underline{d}} Y$,

$$h^0(Y, M) = \sum_i (d_i - g_i + 1) = g - 1 - \sum_i g_i + \gamma = \delta.$$

Let us prove (ii). We use definition 1.3.2 (A) of a semistable multidegree; Γ_X of X can be oriented so that, if b_i denotes the number of edges pointing at C_i, then for all $i = 1, \dots, \gamma$,

$$d_i = g_i - 1 + b_i. \tag{20}$$

Any such orientation gives us a choice of branches over each node. Namely, for every $n_j \in X_{\text{sing}}$ we denote by q_2^j the branch corresponding to the ending half-edge of n_j. We claim that (13) holds with respect to the map $h(j) = 2$ for every j. Indeed

$$h^0\Big(Y, M\Big(-\sum_{j=1}^{\delta} q_2^j\Big)\Big) = \sum_{i=1}^{\gamma} h^0\Big(C_i, M\Big(-\sum_{j=1}^{\delta} q_2^j\Big)_{|C_i}\Big).$$

Now by (20),

$$\deg_{C_i} M\Big(-\sum_{j=1}^{\delta} q_2^j\Big) = d_i - b_i = g_i - 1, \tag{21}$$

hence (M being general) $h^0(C_i, M(-\sum_{j=1}^{\delta} q_2^j)_{|C_i}) = 0$ for every i. We conclude that, by part (i),

$$h^0\Big(Y, M\Big(-\sum_{j=1}^{\delta} q_2^j\Big)\Big) = 0 = h^0(Y, M) - \delta$$

so that (13) is satisfied. Now, applying 2.1.1(ii), we get $\dim W_M(X) = \delta - \gamma$ and $h^0(X, L) = 1$ for a general $L \in W_M(X)$. So (iii) and (iv) are proved. ◻

Corollary 2.1.3. *Let $\underline{d} \in \Sigma^{ss}(X)$ and let L be a general line bundle in* $\mathrm{Pic}^{\underline{d}} X$*. For every subcurve $Z \subseteq X$ we have $h^0(Z, L_Z) = d_Z - g_Z + 1$.*

Proof. It suffices to assume Z is connected (by (2)). Consider the normalization $\nu : Y = \bigcup C_i \to X$ of X and $\nu^* L = M = (L_1, \dots, L_\gamma)$ with $L_i \in \mathrm{Pic}^{d_i} C_i$. Then L_i is general in $\mathrm{Pic}^{d_i} C_i$ (as L is general in $\mathrm{Pic}^{\underline{d}} X$); since $d_i \geq g_i - 1$ (as $\underline{d}$ is semistable) we get $h^0(C_i, L_i) = d_i - g_i + 1$. Now, denote by $Z^\nu \to Z$ the normalization of Z, order the irreducible components of X so that the first γ_Z are the irreducible components of Z, and set $S = Z_{\mathrm{sing}}$, so that $g_Z = \sum_{i=1}^{\gamma_Z} g_i + \delta_S - \gamma_Z + 1$. Let M_{Z^ν} be the restriction of M to Z^ν. Then

$$h^0(Z^\nu, M_{Z^\nu}) = \sum_{i=1}^{\gamma_Z} h^0(C_i, L_i) = \sum_{i=1}^{\gamma_Z} (d_i - g_i + 1) = d_Z - g_Z + \delta_S + 1.$$

Now, since $\underline{d}$ is semistable, $d_Z \geq g_Z - 1$ hence $h^0(Z^\nu, M_{Z^\nu}) \geq \delta_S$. Moreover, recall that by 2.1.2(ii), M satisfies condition (13); it is straightforward to check that the analogue holds for M_{Z^ν}, i.e. for a suitable choice of branches,

$$h^0\Big(Z^\nu, M_{Z^\nu}\Big(-\sum_{j=1}^{\delta_S} q^j_{h(j)}\Big)\Big) = h^0(M_{Z^\nu}) - \delta_S = 0.$$

This enables us to apply 2.1.1(14) to $Z^\nu \to Z$, thus getting

$$h^0(Z, L_Z) = h^0(Z^\nu, M_{Z^\nu}) - \delta_S = d_Z - g_Z + \delta_S + 1 - \delta_S = d_Z - g_Z + 1. \qquad \square$$

2.2. *Basic cases*

Recall the notation of 1.1.1, in particular (3). The following simple fact will be used several times.

Remark 2.2.1. *Let $\nu_S : Y_S \to X$ be the normalization of X at one node (i.e. $S = \{n\}$). Let $M \in \mathrm{Pic}\, Y_S$ be such that $h^0(M) \geq 2$. If $h^0(M(-q_1 - q_2)) = h^0(M) - 2$, every $L \in F_M(X)$ satisfies $h^0(X, L) = h^0(Y_S, M) - 1$.*

To prove it, pick $L \in F_M(X)$ and consider the cohomology sequence

$$0 \to H^0(X, L) \xrightarrow{\alpha} H^0(Y_S, M) \xrightarrow{\beta} k \to H^1(X, L) \to H^1(Y_S, M) \to 0 \qquad (22)$$

(associated to (15)). It suffices to show that β is non-zero. The assumption $h^0(M(-q_1 - q_2)) = h^0(M) - 2$ implies that $h^0(M(-q_h)) = h^0(M) - 1$ for $h = 1, 2$; hence M has a section s vanishing at q_1 but not at q_2; but then $\beta(s) \neq 0$.

2.2.2. Let $S \subset X_{\mathrm{sing}}$ and consider the partial normalization $Y_S \to X$. Fix a finite set S' of points of X (usually $S' \subseteq S$). For any $M \in \mathrm{Pic}\, Y_S$ set

$$W_M(X, S') := \{L \in W_M(X) : \forall s \in H^0(X, L)\ \exists n \in S' : s(n) = 0\} \qquad (23)$$

or equivalently (since S' is finite)

$$W_M(X, S') := \{L \in W_M(X) : \exists n \in S' : s(n) = 0 \, \forall s \in H^0(X, L)\}. \tag{24}$$

If $S = X_{\text{sing}}$ then $W_M(X, S)$ is equal to the set of points in $W_M(X)$ which do not lie in $\alpha_X^d(\dot{X}^{\underline{d}})$, where $\underline{d} = \underline{\deg} M$.

Lemma 2.2.3. *Fix $\nu_S : Y_S \to X$ and let $M \in \mathrm{Pic}^d Y_S$ be such that $h^0(Y_S, M) = 1$.*

(1) *If there exists $n_j \in S$ such that $h^0(Y_S, M(-q_1^j)) \neq h^0(Y_S, M(-q_2^j))$, then $W_M(X) = \emptyset$.*
(2) *If $h^0(Y_S, M(-q_1^j)) = h^0(Y_S, M(-q_2^j))$ for every j, there are two cases.*
   (a) *If $h^0(Y_S, M(-q_h^j)) = 0$ for every j and h, then Y_S is connected and there exists an $L_M \in F_M(X)$ such that $W_M(X) = \{L_M\}$ and $h^0(L_M) = 1$. Moreover, $W_M(X, S) = \emptyset$ (hence $L_M \in \alpha_X^d(\dot{X}^d)$).*
   (b) *If there exists j for which $h^0(Y_S, M(-q_1^j)) = h^0(Y_S, M(-q_2^j)) = 1$, then $W_M(X, S) = W_M(X)$. Moreover, if $h^0(Y_S, M(-q_h^j)) = 1$ for every j then $W_M(X) = F_M(X)$; otherwise $W_M(X) = \{L_M\}$.*

Proof. Let $s \in H^0(M)$ be a non-zero section. In case (1) we are assuming that (up to switching the branches over n_j) $s(q_1^j) = 0$ while $s(q_2^j) \neq 0$, so obviously s does not descend to a section of any $L \in F_M(X)$.

For case (2a) suppose, by contradiction, that $Y_S = \coprod_{i=1}^{\gamma} Z_i$ is not connected. Then $h^0(Y, M) = \bigoplus h^0(Z_i, M_{Z_i}) = 1$ so that there is only one component, say Z_1, such that $h^0(Z_1, M_{Z_1}) \neq 0$. Pick $q = q_h^j \in Z_2$. Then (as $h^0(Z_2, M_{Z_2}) = 0$) every $s \in H^0(M)$ vanishes at q so that $h^0(M(-q)) = h^0(M) = 1$, contradicting the hypothesis. So Y is connected. Now any nonzero $s \in H^0(Y, M)$ satisfies $s(q_h^j) \neq 0$ for $j = 1, \ldots, \delta$ and $h = 1, 2$. Let $c_j := s(q_2^j)/s(q_1^j) \in k^*$ and $\underline{c} = (c_1, \ldots, c_\delta)$; then $\underline{c}$ does not depend on the choice of s, as $h^0(M) = 1$. Using the construction of 1.1.4 set $L_M = L^{(\underline{c})}$; we get $W_M(X) = \{L_M\}$ and obviously s descends to a section of L_M that does not vanish at any n_j. So, $W_M(X, S)$ is empty, and by construction, $h^0(X, L_M) = 1$.

In case (2b), it is clear that for every $L \in W_M(X)$ and $s \in H^0(L)$ we have $s(n_j) = 0$, hence $W_M(X, S) = W_M(X)$. The last sentence is proved similarly. □

Lemma 2.2.4. *Let $\nu_S : Y_S \to X$ be the normalization of X at one node (i.e. $S = \{n\}$). Let $M \in \mathrm{Pic}^d Y_S$ be such that $h^0(Y_S, M) \geq 2$. Then $W_M(X) = F_M(X)$ and the following cases occur.*

(1) *If $h^0(M(-q_1 - q_2)) = h^0(M) - 2$ then $W_M(X, S) = \emptyset$ and $h^0(L) = h^0(M) - 1$ for every $L \in F_M(X)$.*
(2) *If $h^0(M(-q_1 - q_2)) = h^0(M(-q_h)) = h^0(M) - 1$ for $h = 1, 2$ then Y_S is connected and $W_M(X, S) = W_M(X) \smallsetminus \{L_M\}$ for a uniquely determined $L_M \in W_M(X)$ (hence $L_M \in \alpha_X^d(\dot{X}^d)$). Moreover, $h^0(L_M) = h^0(M)$ while for every $L \in W_M(X) - \{L_M\}$ we have $h^0(L) = h^0(M) - 1$.*

(3) *If* $h^0(M(-q_1)) = h^0(M) - 1$ *and* $h^0(M(-q_2)) = h^0(M)$ *then* $F_M(X) = W_M(X, S)$. *Moreover,* $h^0(L) = h^0(M) - 1$ *for every* $L \in F_M(X)$.
(4) *If* $h^0(M(-q_1)) = h^0(M(-q_2)) = h^0(M)$ *then* $F_M(X) = W_M(X, S)$. *Moreover,* $h^0(L) = h^0(M)$ *for every* $L \in F_M(X)$.

Proof. Pick $L \in F_M(X)$ and consider the cohomology sequence (22). It implies that $\alpha(H^0(X, L))$ has codimension at most 1, i.e. that $h^0(L) \geq h^0(Y, M) - 1 \geq 1$ so that $W_M(X) = F_M(X)$. We shall omit the subscript S during the proof.

In case (1), $H^0(Y, M(-q_1 - q_2))$ has codimension 2, hence $\alpha(H^0(X, L))$ cannot be contained in it. Therefore $H^0(X, L)$ contains sections that do not vanish at n. The rest has been proved in Remark 2.2.1.

For the remaining cases, note that every section of $H^0(M(-q_1 - q_2))$ descends to a section of every $L \in F_M(X)$.

In case (2), to show that Y is connected, suppose by contradiction that $Y = Y_1 \amalg Y_2$. Then (say) $q_1 \in Y_1$ and $q_2 \in Y_2$ and $h^0(M) = h^0(Y_1, M_1) + h^0(Y_2, M_2)$ (denoting $M_i = M_{Y_i}$). Furthermore,

$$h^0(M_1) + h^0(M_2) - 1 = h^0(M) - 1 = h^0(M(-q_1)) = h^0(M_1(-q_1)) + h^0(M_2),$$

hence $h^0(M_1(-q_1)) = h^0(M_1) - 1$. Similarly, $h^0(M_2(-q_2)) = h^0(M_2) - 1$. But then $h^0(M(-q_1-q_2)) = h^0(M_1(-q_1))+h^0(M_2(-q_2)) = h^0(M)-2$, which is a contradiction.

Now, there exists $s \in H^0(M)$ such that $s(q_h) \neq 0$ for $h = 1, 2$. Thus

$$H^0(M) = H^0(M(-q_1 - q_2)) \oplus ks. \tag{25}$$

Set $c = s(q_2)/s(q_1)$ and let $L_M = L^{(c)}$ (as in 1.1.4). The s descends to a section $\overline{s} \in H^0(L_M)$ such that $\overline{s}(n) \neq 0$. Hence $L_M \notin W_M(X, S)$ and $h^0(L_M) = h^0(M)$. Now, L_M is uniquely determined: indeed, if $s' \in H^0(M)$ is another section such that $s'(q_h) \neq 0$ for $h = 1, 2$, then by (25), $s' = at + bs$ for $t \in H^0(M(-q_1 - q_2))$ and $a, b \in k$ with $b \neq 0$. Thus $c = s'(q_2)/s'(q_1)$. This proves that for every $L \in W_M(X)$ such that $L \notin W_M(X, S)$ we have $L = L_M$.

In case (3), $H^0(M(-q_1 - q_2)) = H^0(M(-q_1))$ and these are the only sections that can be pull backs of sections of any $L \in F_M(X)$. Case (4) is obvious. □

Corollary 2.2.5. $W_{(0,\dots,0)}(X) = \{\mathcal{O}_X\}$ *for every connected, nodal curve* X.

2.3. Divisors imposing independent conditions

Let $Y_S \to X$ be some partial normalization of X and let $M \in \operatorname{Pic} Y_S$. The goal of this subsection is to bound the dimension of the locus of $L \in W_M(X)$ which are not contained in the image of the Abel map (i.e. with the notation of 2.2.2 the dimension of $W_M(X, S)$). The easy cases, $h^0(Y_S, M) = 1$ or $\#S = 1$, are dealt with by Lemmas 2.2.3 and 2.2.4. To treat the general case we introduce the following.

Definition 2.3.1. *Let* Y *be a nodal curve (possibly not connected). Let* $M \in \operatorname{Pic} Y$ *and let* E *be a Cartier divisor on* Y.

(A) *We say that* E *is* admissible *for* M *if for every subcurve* $V \subseteq Y$ *we have* $0 \leq \deg_V E \leq h^0(V, M_V)$ *(in particular,* E *is effective).*

(B) *We say that E* imposes independent conditions on M *if E is admissible for M and if* $h^0(V, M(-E)_V) = h^0(V, M_V) - \deg_V E$ *for every subcurve* $V \subseteq Y$.
(C) *For $R \subset Y \smallsetminus Y_{\text{sing}}$, we denote by $\mathcal{A}(M, R)$ the set of all admissible divisors for M with support contained in R.*

Remark 2.3.2. If R in part (C) is finite, then the set $\mathcal{A}(M, R)$ is also finite.

If C is a smooth irreducible curve, Definition 2.3.1 coincides with the classical one. Fix a finite subset $R \subset C$; then every admissible divisor E such that $\operatorname{Supp} E \subset R$ imposes independent conditions on the general $L \in \operatorname{Pic}^d C$. More generally:

Lemma 2.3.3. *Let $\nu : Y \to X$ be the normalization of X and $R \subset Y$ a finite subset. Let $\underline{d} \in \Sigma^{\text{ss}}(X)$ and $M \in \operatorname{Pic}^{\underline{d}} Y$ a general point. Then every divisor $E \in \mathcal{A}(M, R)$ imposes independent conditions on M.*

Proof. By Remark 2.3.2, it suffices to prove that a fixed E imposes independent conditions on the general $M \in \operatorname{Pic}^{\underline{d}} Y$.

Set as usual $Y = \coprod_{i=1}^{\gamma} C_i$. Given M and E as in the statement, define $M_i := M_{C_i}$, $E_i := E_{C_i}$ and $e_i = \deg_{C_i} E$. Now, for any line bundle N on Y and any subcurve $V \subset Y$ we have $H^0(V, N) = \bigoplus_{C_i \subset V} H^0(C_i, N_{C_i})$. Therefore it suffices to prove that $h^0(C_i, M(-E)_{C_i}) = h^0(C_i, M_i) - e_i$ for every $i = 1 \dots, \gamma$. Since M is general in $\operatorname{Pic}^{\underline{d}} Y = \prod \operatorname{Pic}^{d_i} C_i$, every M_i is general in $\operatorname{Pic}^{d_i} C_i$. The fact that $\underline{d}$ is semistable implies that $d_i \geq p_a(C_i) - 1 \geq g_i - 1$ (g_i is the genus of C_i), hence $h^0(C_i, M_i) = d_i - g_i + 1$. Now by (A) of 2.3.1 we have $e_i \leq d_i - g_i + 1$, hence

$$\deg_{C_i} M(-E) = d_i - e_i \geq g_i - 1. \tag{26}$$

At this point, observe that $M_i(-E_i)$ is a general point in $\operatorname{Pic}^{d_i - e_i} C_i$ (E_i is fixed and M_i is general in $\operatorname{Pic}^{d_i} C_i$) and hence (by (26))

$$h^0(C_i, M_i(-E_i)) = d_i - e_i - g_i + 1 = h^0(C_i, M_i) - e_i$$

as claimed. □

Example 2.3.4. Let $\nu : Y \to X$ the normalization of X and $\underline{d} \in \Sigma^{\text{ss}}(X)$. Then there exists a choice of branches $h : \{1, \dots, \delta\} \to \{1, 2\}$ such that the divisor $E = \sum_{j=1}^{\delta} q_{h(j)}^j$ is admissible for every $M \in \operatorname{Pic}^{\underline{d}} Y$. In fact, the construction of such an admissible divisor E has appeared in the proof of 2.1.2. Recall that $\deg_{C_i} M(-E) = g_i - 1$ for every $i = 1, \dots, \gamma$ (see (21)).

For the next result we need some notation. Recall that $\nu_S : Y_S \to X$ denotes the normalization of X at S. Let $Z \subset X$ be a subcurve. We denote by $Z_S := \nu_S^{-1}(Z)$ the corresponding subcurve in Y_S, so that Z_S is the normalization of Z at $S \cap Z_{\text{sing}}$. Obviously every subcurve of Y_S is of the form Z_S for a unique $Z \subset X$. We shall often simplify the notation by setting $H^0(Z_S, M) := H^0(Z_S, M_{Z_S})$.

Proposition 2.3.5. *Fix $\nu_S : Y_S \to X$ as above. Let $M \in \operatorname{Pic} Y_S$ be such that $h^0(Y_S, M) \geq \delta_S$, and assume that for every $Z_S \subsetneq Y_S$,*

$$h^0(Z_S, M_{Z_S}) \geq 1 + \#S \cap Z_{\mathrm{sing}}. \tag{27}$$

If every $E \in \mathcal{A}(M, \nu_S^{-1}(S))$ imposes independent conditions on M, then

$$\dim W_M(X, S) \leq \begin{cases} \delta_S - \gamma_S - 1 & \text{if } h^0(M) = \delta_S, \\ \delta_S - \gamma_S & \text{if } h^0(M) \geq \delta_S + 1. \end{cases}$$

Proof. We set $l = h^0(Y_S, M)$. By hypothesis, for every $q \in \nu^{-1}(S)$,

$$h^0(Y_S, M(-q)) = l - 1, \tag{28}$$

indeed by (27) every such q is admissible for M. Let $n \in S$ and set $\nu^{-1}(n) = \{q_1, q_2\}$. Suppose $l = 1$; then $\delta_S = 1$. By (28) applied to q_1 and q_2, we are in case (2a) of Lemma 2.2.3. Hence $W_M(X, S) = \emptyset$ and we are done.

From now on, we assume $l \geq 2$. Let $E = q_1 + q_2$. Then E is admissible, i.e. $\deg_{Z_S} E \leq h^0(Z_S, M_{Z_S})$ for every subcurve $Z_S \subset Y_S$. Indeed, for every Z_S, we have $h^0(Z_S, M_{Z_S}) \geq 1$ by (27). On the other hand, $\deg_{Z_S} E \leq 2$ and equality holds iff Z_S contains both q_1 and q_2, i.e. if and only if Z is singular at n. In this case, $h^0(Z_S, M_{Z_S}) \geq 2$ by (27). Therefore, by hypothesis, for every Z_S,

$$h^0(Z_S, M(-q_1 - q_2)) = h^0(Z_S, M_{Z_S}) - 2. \tag{29}$$

Assume $\delta_S = 1$. By (29) we are in case (1) of Lemma 2.2.4. Thus $W_M(X, S)$ is empty and we are done. We continue by induction on δ_S.

For every $j = 1, \dots, \delta$ set $S_j := S \smallsetminus \{n_j\}$. For any $\{j_1, j_2\} \subset \{1, \dots, \delta_S\}$,

$$W_M(X, S) = \bigcup_{j=1}^{\delta} W_M(X, S_j) = W_M(X, S_{j_1}) \cup W_M(X, S_{j_2}), \tag{30}$$

therefore it suffices to bound the dimension of $W_M(X, S_j)$ for a chosen pair of values of $j = 1, \dots, \delta$. We pick one of them and simplify the notation by setting $n = n_j$ and $T = S_j = S \smallsetminus \{n\}$. We factor ν_S as

$$\nu_S : Y_S \xrightarrow{\nu_1} Y_T \xrightarrow{\nu_T} X$$

where ν_T is the normalization of X at T and ν_1 the normalization at the remaining node n. We abuse notation by using the same names for points in Y_S, Y_T and X whenever the maps are local isomorphisms (e.g. n denotes a node in Y_T and in X). The following is the basic diagram to keep in mind:

$$\begin{array}{ccccc} \operatorname{Pic} X & \xrightarrow{\nu_T^*} & \operatorname{Pic} Y_T & \xrightarrow{\nu_1^*} & \operatorname{Pic} Y_S \\ W_M(X, T) & \to & W_M(Y_T) & \to & M \\ W_N(X, T) & \to & N & \mapsto & M \end{array} \tag{31}$$

where $N \in F_M(Y_T)$; since $l \geq 2$, $F_M(Y_T) = W_M(Y_T)$. By (29) and 2.2.1,

$$h^0(Y_T, N) = l - 1. \tag{32}$$

Case 1: *The node n lies in two different irreducible components of X.* By Lemma 2.3.6(i) (applied with $R = \nu_S^{-1}(S \smallsetminus n)$) every admissible divisor E_T on Y_T such that $\operatorname{Supp} E_T \subset \nu_T^{-1}(T)$ imposes independent conditions on N. Therefore we can use induction ($\#T = \#S - 1$) and obtain

$$\dim W_N(X, T) \leq \begin{cases} \delta_S - 1 - \gamma_T - 1 & \text{if } h^0(Y_T, N) = \delta_S - 1, \\ \delta_S - 1 - \gamma_T & \text{if } h^0(Y_T, N) \geq \delta_S, \end{cases}$$

i.e. using (32),

$$\dim W_N(X, T) \leq \begin{cases} \delta_S - \gamma_T - 2 & \text{if } l - 1 = \delta_S - 1, \\ \delta_S - \gamma_T - 1 & \text{if } l - 1 \geq \delta_S. \end{cases}$$

If n is not a separating node for X, then $F_M(Y_T) = W_M(Y_T) \cong k^*$ and $\gamma_S = \gamma_T$. Therefore, from diagram (31), $\dim W_M(X, T) \leq \dim W_N(X, T) + 1$. So, using the equality $\delta_S - \gamma_T - 1 = \delta_S - \gamma_S$, we are done.

If n is separating, then $\gamma_S = \gamma_T + 1$. On the other hand, $\dim F_M(Y_T) = 0$, hence $\dim W_M(X, T) \leq \dim W_N(X, T)$. Again, we are done.

Case 2: *The node n lies in only one irreducible component of X.* Denote by $\overline{C} \subset X$ the component containing n, and by $C \subset Y_S$ the component containing both q_1 and q_2. We are in the situation of Lemma 2.3.6(ii). Therefore there exists a finite set $P \subset F_M(Y_T)$ such that for every $N \in \operatorname{Pic} Y_T \smallsetminus P$, every admissible E supported on $\nu_T^{-1}(T)$ imposes independent conditions on N. We can use induction on every $N \in W_M(Y_T)$ such that $N \notin P$. We obtain

$$\dim W_N(X, T) \leq \begin{cases} \delta_S - 1 - \gamma_T - 1 & \text{if } h^0(Y_T, N) = \delta_S - 1, \\ \delta_S - 1 - \gamma_T & \text{if } h^0(Y_T, N) \geq \delta_S. \end{cases}$$

Consider diagram (31) and note that now $\dim W_M(Y_T) = \dim F_M(Y_T) = 1$. Hence, away from the fibers over P, the dimension of every irreducible component of $W_M(X, T)$ is at most

$$\dim W_M(Y_T) + \dim W_N(X, T) \leq \begin{cases} 1 + \delta - \gamma - 2 & \text{if } l = \delta, \\ 1 + \delta - \gamma - 1 & \text{if } l \geq \delta + 1, \end{cases}$$

(using (32)) as wanted.

It remains to bound the dimension of the fibers over every $N \in P$. Now, set $n = n_1$ and $T = \{n_2, \dots, n_{\delta_S}\}$.

If $l \geq \delta_S + 1$, i.e. if $h^0(Y_T, N) \geq \delta_T + 1$, then

$$\dim W_N(X) = \dim F_N(X) = \delta_T - \gamma_T + 1 = \delta_S - \gamma_S.$$

The fiber of $W_M(X, T) \to W_M(Y_T)$ over N is obviously contained in $W_N(X)$, hence it has dimension at most $\delta_S - \gamma_S$ and we are done.

Assume $\delta_S = l$. If

$$h^0(Y_T, N(-q_1^2 - \cdots - q_1^{\delta_S})) = 0, \tag{33}$$

then, by 2.1.1(ii), $W_N(X)$ has pure dimension equal to $\delta_T - \gamma_S = \delta_S - \gamma_S - 1$. Hence the dimension of the fiber of $W_M(X, T)$ over N is at most $\delta_S - \gamma_S - 1$ and we are done.

We shall complete the proof by showing that (33) holds for some choice of branches. Assume $h^0(Y_T, N(-q_1^2 - \cdots - q_1^{\delta_S})) \geq 1$.

Observe that $E := \sum_{j=2}^{\delta_S} q_1^j + q_2^{\delta_S}$ is admissible for M. Indeed, we have $\deg_{Z_S} E \leq 1 + \#T \cap Z_{\text{sing}}$ for every $Z_S \subset Y_S$; hence, by (27),

$$\deg_{Z_S} E \leq 1 + \#T \cap Z_{\text{sing}} \leq 1 + \#S \cap Z_{\text{sing}} \leq h^0(Z_S, M).$$

As E is admissible, we have

$$h^0\Big(Y_S, M\Big(-\sum_{j=2}^{\delta_S} q_1^j - q_2^{\delta_S}\Big)\Big) = 0, \tag{34}$$

also, by Lemma 2.2.4,

$$h^0(Y_T, N(-q_1^2 - \cdots - q_1^{\delta_S - 1} - q_2^{\delta_S})) \leq 1 \quad \text{and} \quad h^0(Y_T, N(-q_1^2 - \cdots - q_1^{\delta_S})) = 1.$$

If $h^0(Y_T, N(-q_1^2 - \cdots - q_1^{\delta_S - 1} - q_2^{\delta_S})) = 1$ then, of course,

$$h^0\Big(N\Big(-\sum_{j=2}^{\delta_S} q_1^j - q_2^{\delta_S}\Big)\Big) = 1, \tag{35}$$

which is impossible, by (34). Therefore $h^0(Y_T, N(-q_1^2 - \cdots - q_1^{\delta_S - 1} - q_2^{\delta_S})) = 0$, i.e. (33) holds for some choice of branches. The proof is complete. $\square$

In the proof of Proposition 2.3.5 we used the following

Lemma 2.3.6. *Let $\nu_1 : Y_S \to Y_T$ be the partial normalization of Y_T at a unique node n. Let $M \in \operatorname{Pic} Y_S$ be such that for every subcurve $Z_S \subset Y_S$,*

$$h^0(Z_S, M) \begin{cases} \geq 2 & \text{if } \nu_1^{-1}(n) \subset Z_S, \\ \geq 1 & \text{otherwise.} \end{cases}$$

Let R be a finite set of smooth points of Y_S. Assume that every divisor in $\mathcal{A}(M, \nu_1^{-1}(n) \cup R)$ imposes independent conditions on M.

(i) *If n lies in two irreducible components of Y_T, then for any $N \in F_M(Y_T)$, every divisor in $\mathcal{A}(N, \nu_1(R))$ imposes independent conditions on N.*
(ii) *If n lies in only one irreducible component of Y_T, there exists a finite subset $P \subset F_M(Y_T)$ such that for every $N \in F_M(Y_T) \smallsetminus P$, every divisor in $\mathcal{A}(N, \nu_1(R))$ imposes independent conditions on N.*

Proof. Let $\nu^{-1}(n) = \{q_1, q_2\}$. Then formula (29) holds (with the same proof). For every $Z_S \subset Y_S$, denote $Z_T := \nu_1(Z_S)$. By (29) and 2.2.1 we have

$$\{q_1, q_2\} \subset Z_S \;\Rightarrow\; h^0(Z_T, N_{Z_T}) = h^0(Z_S, M_{Z_S}) - 1, \tag{36}$$

$$\{q_1, q_2\} \not\subset Z_S \;\Rightarrow\; h^0(Z_T, N_{Z_T}) = h^0(Z_S, M_{Z_S}), \tag{37}$$

because in the latter case $Z_S \cong Z_T$ via ν_1. Thus for any $N \in F_M(Y_T)$, the number $h^0(Z_T, N_{Z_T})$ depends only on M, and not on the choice of N. Therefore the set $\mathcal{A}(N, \nu_1(R))$ depends only on M.

Pick $E_T \in \mathcal{A}(N, \nu_1(R))$. Denote $E_S := \nu_1^*(E_T)$, and observe that ν_1 is an isomorphism locally at every point in Supp E_S. Hence

$$\deg_{Z_S} E_S = \deg_{Z_T} E_T \le h^0(Z_T, N) \le h^0(Z_S, M). \tag{38}$$

Therefore E_S imposes independent conditions on M, i.e.

$$h^0(Z_S, M(-E_S)) = h^0(Z_S, M) - \deg_{Z_S} E_S. \tag{39}$$

If $\{q_1, q_2\} \not\subset Z_S$, ν_1 induces an isomorphism $Z_S \cong Z_T$, hence by (38) and (39) we get $h^0(Z_T, N(-E_T)) = h^0(Z_S, M(-E_S)) = h^0(Z_T, N) - \deg_{Z_T} E_T$, as wanted. So we need only consider the case $\{q_1, q_2\} \subset Z_S$.

For (i), let $q_1 \in C_1$ and $q_2 \in C_2$. Set $e_i := \deg_{C_i} E$ and $l_i := h^0(C_i, M_{C_i}) = h^0(C_i, N_{C_i})$. Consider the usual sequence

$$0 \to H^0(Z_T, N(-E_T)) \to H^0(Z_S, M(-E_S)) \xrightarrow{\beta} k \to \cdots . \tag{40}$$

If E_T is such that $e_i \le l_i - 1$ for $i = 1, 2$ then $E_S + q_1 + q_2$ imposes independent conditions on M. We get $h^0(Z_S, M(-E_S - q_1 - q_2)) = h^0(Z_S, M(-E_S)) - 2$, hence $h^0(Z_T, N(-E_T)) = h^0(Z_S, M(-E_S)) - 1$. By (38) and (39) we get

$$h^0(Z_T, N(-E_T)) = h^0(Z_S, M) - \deg_{Z_T} E_S - 1 = h^0(Z_T, N) - \deg_{Z_T} E_T,$$

as wanted. Now, E_T is admissible, hence $l_i \ge e_i$; so only two cases remain.

Case 1: $e_1 = l_1$ and $e_2 = l_2 - 1$. Then $H^0(C_1, M(-E_S)) = 0$, $h^0(C_2, M(-E_S)) = 1$ and $h^0(C_2, M(-E_S - q_2)) = 0$. Then all sections in $H^0(Z_S, M(-E_S))$ vanish at q_1 while there exist sections that do not vanish at q_2. Hence β is surjective and we are done.

Case 2: $l_i = e_i$ for $i = 1, 2$. Let $Z_T := \nu_1(C_1 \cup C_2) \subset Y_T$. By (36),

$$e_1 + e_2 = \deg_{Z_T} E_T \le h^0(Z_T, N) = h^0(Z_S, M) - 1 \le l_1 + l_2 - 1,$$

which is possible only if at least one e_i is less than l_i. So Case 2 does not occur and (i) is proved.

For (ii), denote by Call $C \subset Y_S$ the component of Y_S containing both q_1 and q_2, and $D := \nu_1(C)$. Set $e_D = \deg_D E_T = \deg_C E_S$; and (by (36))

$$l_D := h^0(C, M) = h^0(D, N) + 1 \tag{41}$$

so that $e_D \leq l_D - 1$. If $e_D \leq l_D - 2$ then $E_S + q_1 + q_2$ is admissible for M. Hence for every $Z_S \subset Y_S$ we have $h^0(Z_S, M(-E_S - q_1 - q_2)) = h^0(Z_S, M(-E)) - 2$ so that (using Remark 2.2.1)

$$h^0(Z_T, N(-E_T)) = h^0(Z_S, M(-E_S)) - 1 = h^0(Z_T, N) - \deg_{Z_T} E_T. \tag{42}$$

We are left with the case $e_D = l_D - 1$. Then $h^0(C, M(-E_S)) = 1$ and part (2a) of Lemma 2.2.3 applies. We conclude that there exists a unique line bundle in Pic D which pulls back to $M(-E_S)_C$ and has $h^0 = 1$. This in turn determines a (unique) line bundle N_D on D which pulls back to M_C, and finally a unique line bundle on Y_T which pulls back to M and restricts to N_D on D. This last line bundle on Y_T is uniquely determined by E_T; denote it by N^{E_T}. Set $P := \{N \in F_M(Y_T) : N = N^{E_T} \text{ for some } E_T\}$. We just showed that for any $N \in F_M(Y_T) \smallsetminus P$, every $E_T \in \mathcal{A}(N, \nu_1(R))$ imposes independent conditions on N. The finiteness of the set P follows at once from the finiteness of the set of E_T's. $\square$

Corollary 2.3.7. *Let $Y \to X$ be the normalization of X and $S = X_{\text{sing}}$. If $\underline{d} \in \Sigma(X)$ and $M \in \operatorname{Pic}^{\underline{d}} Y$ is a general point then* $\dim W_M(X, S) \leq \delta - \gamma - 1$.

Proof. If M is general, $h^0(Y, M) = \delta$ by 2.1.2. Moreover, as $\underline{d}$ is stable, (27) holds. Indeed, for every $Z \subset X$, $Z^\nu = Z_S$ is the normalization of Z and we have $d_Z \geq p_a(Z) = p_a(Z^\nu) + \#Z_{\text{sing}}$; hence $h^0(Z^\nu, M_{Z^\nu}) \geq \#Z_{\text{sing}} + 1$. Finally, by Lemma 2.3.3, M satisfies the assumption of Proposition 2.3.5. $\square$

3. Irreducibility and dimension

3.1. Irreducible components

We are ready to prove that $W_{\underline{d}}(X)$ is irreducible for every stable multidegree $\underline{d}$. This implies that, if X is free from separating nodes, the theta divisor $\Theta(X) \subset \overline{P_X^{g-1}}$ has one irreducible component for every irreducible component of $\overline{P_X^{g-1}}$. If X has some separating node this is false (see 3.1.4 and 4.2.7). The stability assumption on $\underline{d}$ is also essential, as one can see from counterexample 3.1.4.

If $|\underline{d}| \geq 1$ we shall use the Abel map $\alpha_X^{\underline{d}}$. If $|\underline{d}| \leq 0$, i.e. if $g = 0, 1$ the Abel map is not defined so we need to treat this case separately, which will be done in the following

Lemma 3.1.1. *Let X have genus $g \leq 0, 1$; let $\underline{d} \in \Sigma(X)$. Then*

$$W_{\underline{d}}(X) = \begin{cases} \emptyset & \text{if } \underline{d} \neq (0, \dots, 0), \\ \{\mathcal{O}_X\} & \text{if } \underline{d} = (0, \dots, 0) \text{ (hence } g = 1). \end{cases}$$

Proof. By hypothesis, for each $\underline{d} \in \Sigma(X)$ we have $|\underline{d}| = -1, 0$ depending on whether $g = 0, 1$. Recall that $X = \bigcup \overline{C_i}$ denotes the decomposition of X into irreducible components. Let $L \in \mathrm{Pic}^{\underline{d}} X$ and suppose that there exists a non-zero section $s \in H^0(X, L)$. Set

$$Z^- := \bigcup_{i\,:\,d_i<0} \overline{C_i}, \quad Z^0 := \bigcup_{i\,:\,d_i=0} \overline{C_i}, \quad Z^+ := \bigcup_{i\,:\,d_i>0} \overline{C_i}.$$

Note that $Z^- = \emptyset \Leftrightarrow \underline{d} = (0, \dots, 0)$. By contradiction, assume $Z^- \neq \emptyset$. Then s vanishes along a non-empty subcurve $Z \subset X$ which contains Z^-. Let Z^C be the complementary curve of Z, so that s does not vanish along any subcurve of Z^C. Since for every $n \in Z \cap Z^C$ we have $s(n) = 0$, the degree of s restricted to Z^C satisfies

$$d_{Z^C} \geq \#Z \cap Z^C. \tag{43}$$

On the other hand, $g \leq 1$ implies $p_a(Z^C) \leq 1$, hence the stability of $\underline{d}$ yields

$$d_{Z^C} \leq p_a(Z^C) + \#Z \cap Z^C < \#Z \cap Z^C$$

(cf. 1.3.3), a contradiction with (43). Therefore $Z^- = \emptyset$; we see that if $W_{\underline{d}}(X) \neq \emptyset$ then $\underline{d} = (0, \dots, 0)$; in particular, $g = 1$. Now we conclude by Corollary 2.2.5. $\square$

Recall that for $\underline{d}$ such that $\underline{d} \geq 0$ and $|\underline{d}| \geq 1$ we denote by $A_{\underline{d}}(X) \subset \mathrm{Pic}^{\underline{d}} X$ the closure of the image of the Abel map $\alpha_X^{\underline{d}}$ (see 1.2.7). If $\underline{d} \in \Sigma(X)$ is such that $|\underline{d}| = -1, 0$, we denote $A_{\underline{d}}(X) := W_{\underline{d}}(X)$.

Theorem 3.1.2. *Let X be a connected, nodal curve of arithmetic genus g. Let $\underline{d}$ be a stable multidegree on X such that $|\underline{d}| = g - 1$. Then*

(i) *$W_{\underline{d}}(X) = A_{\underline{d}}(X)$, hence $W_{\underline{d}}(X)$ is irreducible of dimension $g - 1$;*
(ii) *the general $L \in W_{\underline{d}}(X)$ satisfies $h^0(X, L) = 1$.*

Proof. If $g = 0, 1$ the theorem follows from Lemma 3.1.1; so we assume $g \geq 2$. (ii) follows from (i) and from 2.1.2(iv).

Let W be an irreducible component of $W_{\underline{d}}(X)$. By 1.3.7 we know that $\dim W = g - 1$. We shall prove the theorem by showing that if $A_{\underline{d}}(X)$ is not dense in W, i.e. if $W \neq \overline{W \cap \mathrm{Im}\, \alpha_X^{\underline{d}}}$, then $\dim W \leq g - 2$, and hence W must be empty.

Up to removing a proper closed subset of W, we can and will assume that $W \cap \mathrm{Im}\, \alpha_X^{\underline{d}} = \emptyset$. Consider the normalization $\nu : Y \to X$ of X with $Y = \coprod_{i=1}^{\gamma} C_i$ and let g_i be the genus of C_i. Recall that $g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$.

We let ρ denote the restriction to W of the pull-back map ν^*, so that

$$\mathrm{Pic}^{\underline{d}} X \supset W \xrightarrow{\rho} \rho(W) \subset \mathrm{Pic}^{\underline{d}} Y = \prod_{i=1}^{\gamma} \mathrm{Pic}^{d_i} C_i. \tag{44}$$

We shall bound the dimension of W by analyzing ρ.

To say that $L \in \operatorname{Pic}^{\underline{d}} X$ does not lie in the image of $\alpha_X^d : \dot{X}^d \to \operatorname{Pic} X$ is to say that L does not admit any section whose zero locus is contained in $\dot{X}$. In other words, setting $S = X_{\text{sing}}$, we have $L \in W_M(X, S)$ (cf. 2.2.2). Therefore for every M in $\rho(W)$ we have

$$\rho^{-1}(M) \subset W_M(X, S) \subset W_M(X).$$

From now on, M is a general point in $\rho(W)$. The proof is divided into four cases.

Case I: $\dim \rho(W) \leq \sum_{i=1}^{\gamma} g_i - 2$. It suffices to use the inequalities $\dim \rho^{-1}(M) \leq \dim F_M(X) = \delta - \gamma + 1$. Then

$$\dim W \leq \dim \rho(W) + \dim F_M(X) \leq \sum_{i=1}^{\gamma} g_i - 2 + \delta - \gamma + 1 = g - 2.$$

Case II: $\dim \rho(W) = \sum_{i=1}^{\gamma} g_i$. Now ρ is dominant, so that M is general in $\operatorname{Pic}^{\underline{d}} Y = \prod_{i=1}^{\gamma} \operatorname{Pic}^{d_i} C_i$. Then we can apply Corollary 2.3.7, which yields $\dim W_M(X, S) \leq \delta - \gamma - 1$, and hence

$$\dim W \leq \dim \rho(W) + \dim W_M(X, S) \leq \sum_{i=1}^{\gamma} g_i + \delta - \gamma - 1 = g - 2.$$

Remark 3.1.3. From now on we shall assume $\dim \rho(W) = \sum_{i=1}^{\gamma} g_i - 1$.

Denote by $\pi_i : \prod_{i=1}^{\gamma} \operatorname{Pic}^{d_i} C_i \to \operatorname{Pic}^{d_i} C_i$ the projection and $\rho_i := \pi_i \circ \rho$,

$$\rho_i : W \to \rho(W) \to \rho_i(W) \subset \operatorname{Pic}^{d_i} C_i.$$

As $\dim \prod_{i=1}^{\gamma} \operatorname{Pic}^{d_i} C_i = \sum_{i=1}^{\gamma} g_i$ and $\dim \rho(W) = \sum_{i=1}^{\gamma} g_i - 1$, we get

$$\dim \rho_i(W) \geq g_i - 1, \quad \forall i,$$

and there can be at most one index i for which $\dim \rho_i(W) = g_i - 1$.

Case III: $\dim \rho(W) = \sum_{i=1}^{\gamma} g_i - 1$ *and* $h^0(Y, M) \geq \delta + 1$. We claim that we can apply 2.3.5 to the general $M \in \rho(W)$. This would yield $\dim W_M(X, S) \leq \delta - \gamma$ so that we could conclude as follows:

$$\dim W \leq \dim \rho(W) + \dim W_M(X, S) \leq g_Y - 1 + \delta - \gamma = g - 2.$$

To prove that the hypotheses of 2.3.5 hold, observe that (27) follows from the fact that $\underline{d}$ is stable (see the proof of 2.3.7). To prove the remaining assumption we argue by contradiction. Assume that for some admissible divisor E with $\operatorname{Supp} E \subset \nu^{-1}(S)$ and $e := \deg E$ we have

$$h^0(Y, M(-E)) \geq h^0(Y, M) - e + 1$$

for M general in $\rho(W)$. As Y is the disjoint union of the C_i, we get

$$h^0(Y, M(-E)) = \sum_{i=1}^{\gamma} h^0(C_i, M_i(-E_i)) \geq \sum_{i=1}^{\gamma} (h^0(C_i, M_i) - e_i) + 1$$

where $E_i = E_{|C_i}$, $e_i := \deg E_i$ and $M_i = M_{|C_i}$. Therefore there exists at least one index, say $i = 1$, such that

$$h^0(C_1, M_1(-E)) \geq h^0(C_1, M_1) - e_1 + 1. \tag{45}$$

The fact that E is admissible implies that $e_1 \leq h^0(C_1, M_1)$. Now, as $d_1 \geq g_1$, there are two possiblities:

(a) $h^0(C_1, M_1) = d_1 - g_1 + 1$;
(b) $h^0(C_1, M_1) \geq d_1 - g_1 + 2$.

If (a) occurs, $\rho_1 : W \to \operatorname{Pic}^{d_1} C_1$ is dominant. In fact, by the assumption $h^0(M) \geq \delta + 1$, there exists an index $i \neq 1$ (say $i = 2$) such that $h^0(C_2, M_2) \geq d_2 - g_2 + 2$, i.e. M_2 is a special line bundle on C_2. Therefore $\rho_2(W)$ cannot be dense in $\operatorname{Pic}^{d_2} C_2$. By 3.1.3, $\rho_1(W)$ is dense in $\operatorname{Pic}^{d_1} C_1$. Therefore we can apply Lemma 2.3.3 (with $Y = X = C_1$ and $\underline{d} = d_1$) to deduce that E_1 imposes independent conditions on M_1, a contradiction with (45).

In case (b) we can assume $e_1 = h^0(C_1, M_1) = d_1 - g_1 + 2$. So M_1 is not a general point in $\operatorname{Pic}^{d_1} C_1$; by 3.1.3, $\dim \rho_1(W) = g_1 - 1$. Now (45) is $h^0(C_1, M_1(-E_1)) \geq 1$. Consider the map

$$u_{E_1} : W^0_{d_1 - e_1}(C_1) \to \operatorname{Pic}^{d_1} C_1, \qquad N \mapsto N(+E_1). \tag{46}$$

By what we said, $\operatorname{Im} u_{E_1}$ dominates $\rho_1(W)$, hence the variety $W^0_{d_1-e_1}(C_1)$ has dimension at least $g_1 - 1$. This is impossible, since (by (7))

$$\dim W^0_{d_1-e_1}(C_1) \leq \min\{d_1 - e_1, g_1\} \leq \min\{d_1 - (d_1 - g_1 + 2), g_1\} = g_1 - 2.$$

Case IV: $\dim \rho(W) = \sum_{i=1}^{\gamma} g_i - 1$ *and* $h^0(Y, M) = \delta$. If Proposition 2.3.5 applies, we can argue as for Case II and we are done. Observe that in order for 2.3.5 to apply, it suffices to show that for every $i = 1, \dots, \gamma$, every divisor $E_i \in \mathcal{A}(M_i, \nu^{-1}(S) \cap C_i)$ imposes independent conditions on M_i. Indeed, this implies that every $E \in \mathcal{A}(M, \nu^{-1}(S))$ imposes independent conditions on M. By 3.1.3 there are two possibilities.

(a) $\rho_i(W)$ is dense in $\operatorname{Pic}^{d_i} C_i$ for every i.
(b) There exists a unique index, say $i = 1$, such that $\dim \rho_1(W) = g_1 - 1$, whereas for $i \geq 2$, $\rho_i(W)$ is dense.

In case (a), M_i is general in $\operatorname{Pic}^{d_i} C_i$, hence by 2.3.3 and by what we observed above we can use 2.3.5 and we are done.

In case (b), we may assume that 2.3.5 cannot be applied. Let $E := \sum_{j=1}^{\delta} q^j_{h(j)}$ be an admissible divisor for M of the same type constructed in 2.3.4 (with the same notation). Recall from 2.3.4 that $\deg_{C_i} M(-E) = g_i - 1$ for all i.

If E imposes independent conditions, i.e. if $h^0(Y, M(-E)) = h^0(M) - \delta = 0$, we can apply 2.1.1(ii) to obtain $\dim W_M(X) = \gamma - \delta$. This is enough to conclude:

$$\dim W \leq \dim \rho(W) + \dim W_M(X) = \sum_{i=1}^{\gamma} g_i - 1 + \delta - \gamma = g - 2. \tag{47}$$

So, assume that $h^0(Y, M(-E)) \geq 1$. We have $h^0(C_i, M_i(-E_i)) = 0$ if $i \geq 2$, whereas $h^0(C_1, M_1(-E_1)) \geq 1$. As we said, $\deg M_1(-E_1) = g_1 - 1$; we claim that

$$h^0(C_1, M_1(-E_1)) = 1. \tag{48}$$

To prove it we argue as for Case III(b). Consider the map analogous to (46):

$$u^1_{E_1} : W^1_{g_1-1}(C_1) \to \operatorname{Pic}^{d_1} C_1$$

mapping N to $N(E_1)$. Now $\dim W^1_{g_1-1}(C_1) \leq g_1 - 3$ (well known); therefore, $u^1_{E_1}$ cannot dominate $\rho_1(W)$, whose dimension is $g_1 - 1$. So (48) is proved.

It is trivial to check that we can assume, for a suitable $q \in \operatorname{Supp} E_1$, that $E_1 = E'_1 + q$ with E'_1 imposing independent conditions on M_1, i.e.

$$h^0(C_1, M_1(-E'_1)) = 1$$

so that q is a base point of $M_1(-E'_1)$. Therefore

$$h^0(Y, M(-E)) = 1$$

and there exists a point $q \in E_1$ such that, setting $E' = E - q_1$, the divisor E' imposes independent conditions on M. Now let n be the node of X of which the point q_1 is a branch, and let $S' = S \smallsetminus n$; thus E' is supported on $\nu^{-1}(S')$. Let $\nu_n : X' \to X$ be the normalization of X at n, so that we can factor ν as

$$Y \xrightarrow{\nu'} X' \xrightarrow{\nu_n} X$$

and ν' is the normalization of X'. Of course, X' has $\delta' = \delta - 1$ nodes and $h^0(Y, M) = \delta' + 1$. As E' imposes independent conditions on M, we can apply 2.1.1 with respect to $\nu' : Y \to X'$. This gives us that $W_M(X') = F_M(X')$ and, for a general $L' \in W_M(X')$,

$$h^0(X', L') = h^0(Y, M) - \delta' = 1. \tag{49}$$

Consider the following diagram:

$$\begin{array}{ccccc} \operatorname{Pic} X & \xrightarrow{\nu_n^*} & \operatorname{Pic} X' & \xrightarrow{(\nu')^*} & \operatorname{Pic} Y \\ W_M(X) & \to & F_M(X') & \to & M. \end{array} \tag{50}$$

Observe that n is not a separating node of X (otherwise, by 1.3.5, $\Sigma(X)$ is empty and there is nothing to prove). Hence ν_n^* is a k^*-fibration and

$$\dim F_M(X') = \delta' - \gamma + 1 = \delta - \gamma.$$

We now claim that the fiber $W_{L'}(X)$ of $W_M(X)$ over the general point $L' \in F_M(X')$ has dimension ≤ 0. By (49) we are in the situation of Lemma 2.2.3, which tells us that the only case when $\dim W_{L'}(X) = 1$ is when L' has a base point in each of the two branches of n. Now this does not happen. Indeed, if $i \geq 2$, M_i is general and hence has no base point over X_{sing}; on the other hand, M_1 varies in a codimension 1 subset of $\operatorname{Pic}^{d_1} C_1$, hence it has at most one base point over X_{sing}; therefore we can apply Lemma 1.2.6.

Concluding: $\dim W_M(X) \leq \delta - \gamma$. Arguing as in (47) we are done. □

Example 3.1.4. The conclusion of Theorem 3.1.2 fails if we only assume $\underline{d}$ to be semistable. The simplest instance of $\underline{d} \in \Sigma^{ss}(X)$ with $W_{\underline{d}}(X)$ reducible is that of a curve of compact type, $X = C_1 \cup C_2$, where C_i is smooth of genus g_i, $\#C_1 \cap C_2 = 1$ and $\underline{d} = (g_1 - 1, g_2)$ (note that $\underline{d}$ is strictly semistable by 1.3.5). Then

$$W_{\underline{d}}(X) = (W_{g_1-1}(C_1) \times \operatorname{Pic}^{g_2} C_2) \cup (\operatorname{Pic}^{g_1-1} C_1 \times \Theta_{q_2}(C_2))$$

where $q_2 \in C_2$ is the point over the node and $\Theta_{q_2}(C_2) := \{L \in \operatorname{Pic}^{g_2} C_2 : h^0(C_2, L(-q_2)) \neq 0\}$. The interested reader will easily construct similar, more interesting, examples on curves not of compact type.

3.2. Dimension of the image of the Abel map

Proposition 3.2.1. *Let X be a curve of genus $g \geq 2$. Let $\underline{d} \in \mathbb{Z}^\gamma$ be a non-negative multidegree such that $|\underline{d}| = g - 1$. If $\underline{d}$ is semistable, then*

(a) *the general $L \in A_{\underline{d}}(X)$ satisfies $h^0(X, L) = 1$;*
(b) $\dim A_{\underline{d}}(X) = g - 1$.

Conversely, if $\underline{d}$ is not semistable, then

(A) *for every $L \in A_{\underline{d}}(X)$ we have $h^0(X, L) \geq 2$;*
(B) $\dim A_{\underline{d}}(X) \leq g - 2$.

Proof. If $\underline{d}$ is stable, by Theorem 3.1.2 we know that $A_{\underline{d}}(X) = W_{\underline{d}}(X)$, $\dim A_{\underline{d}}(X) = g - 1$ (by 1.3.7) and that the general point $L \in A_{\underline{d}}(X)$ has $h^0(X, L) = 1$. So, for the first half of the statement, we need to consider the case where X is reducible and $\underline{d}$ semistable but not stable. Thus, there exists a decomposition $X = V \cup Z$, where V and Z are subcurves of respective arithmetic genus g_V and g_Z, such that V is connected,

$$d_V = g_V - 1 \quad \text{and} \quad d_Z = g_Z + \delta_S - 1, \tag{51}$$

where $S := V \cap Z$ and $\delta_S := \#S$.

Observe that, since $\underline{d} \geq 0$, we get $g_V \geq 1$. By (1) we have

$$g = g_V + g_Z + \delta_S - 1. \tag{52}$$

Let L be a general point in $A_{\underline{d}}(X)$; we can assume that L is a line bundle on X of type $L = \mathcal{O}_X(D)$ where D is an effective divisor of multidegree $\underline{d}$ supported on the smooth locus of X. Consider the restrictions L_V and L_Z of L to V and Z; we have $h^0(V, L_V) \geq 1$. On the other hand, $h^0(Z, L_Z) \geq d_Z - g_Z + 1 = \delta_S$ (by Riemann–Roch and (51)); moreover, equality holds for a general $L_Z \in \operatorname{Pic}^{\underline{d}_Z} Z$, by Corollary 2.1.3. Denote the partial normalization of X at S by

$$\nu_S : Y_S = V \amalg Z \to X$$

and note that $\mathrm{Pic}^{\underline{d}}\, Y_S = \mathrm{Pic}^{\underline{d}_V}\, V \times \mathrm{Pic}^{\underline{d}_Z}\, Z$. Set $M = \nu_S^* L = (L_V, L_Z)$. Then for L general

$$h^0(Y_S, M) = h^0(V, L_V) + h^0(Z, L_Z) = \delta_S + 1, \tag{53}$$

hence by Proposition 2.1.1 (14), which we can apply by Lemma 2.1.2(ii), we obtain $h^0(X, L) = h^0(Y_S, M) - \delta_S = 1$.

Now we compute $\dim A_{\underline{d}}(X)$ using induction on the number of irreducible components of X. The case of X irreducible has already been settled. Assume X is reducible; by what we said above, the pull-back map ν_S^* restricted to $A_{\underline{d}}(X)$ gives a dominant rational map (denoted by ρ)

$$A_{\underline{d}}(X) \xrightarrow{\rho} A_{\underline{d}_V}(V) \times \mathrm{Pic}^{\underline{d}_Z}\, Z.$$

Now recall that $|\underline{d}_V| = g_V - 1 \geq 0$ by (51) and $\underline{d}_V$ is semistable on V because $\underline{d}$ is semistable on X (cf. 1.3.3). Furthermore, V has fewer components than X, hence we can use induction to conclude that $\dim A_{\underline{d}_V}(V) = d_V = g_V - 1$.

If M is a general point in the image of the above map ρ, then by (53) and 2.1.1(ii), we see that $W_M(X) = F_M(X)$. We claim that $W_M(X) \subset A_{\underline{d}}(X)$. Indeed, recall that $M = \nu_S^* \mathcal{O}_X(D)$ with $\mathrm{Supp}\, D \subset \dot{X}$, hence there exists an $L \in W_M(X)$ (namely, $L = \mathcal{O}_X(D)$) admitting a section that does not vanish at any node of X. Therefore the same holds for every line bundle in a dense open subset of $W_M(X)$ (which is irreducible, being equal to $F_M(X)$). This shows that $W_M(X) \subset A_{\underline{d}}(X)$. Therefore $\rho^{-1}(M) = F_M(X)$ and

$$\dim A_{\underline{d}}(X) = g_V - 1 + g_Z + \delta_S - \gamma_S + 1 = g - 1.$$

Conversely, assume that $\underline{d}$ is not semistable. Then there exists a decomposition $X = V \cup Z$, where (as before) V and Z are subcurves of genus g_V and g_Z such that

$$d_V \leq g_V - 2 \quad \text{and} \quad d_Z \geq g_Z + \delta_S \tag{54}$$

where $S := V \cap Z$ and $\delta_S := \#S$. Notice that $g_V \geq 2$ (as $\underline{d} \geq 0$).

We use the same notation as before. Let L be a general point in $A_{\underline{d}}(X)$, so L is of type $L = \mathcal{O}_X(D)$ with $D \geq 0$ supported on $\dot{X}$. We have $h^0(V, L_V) \geq 1$ and $h^0(Z, L_Z) \geq d_Z - g_Z + 1 \geq \delta_S + 1$.

Consider $\nu_S : Y_S = V \amalg Z \to X$ and set $M = \nu_S^* L = (L_V, L_Z)$. We have

$$h^0(Y_S, M) = h^0(V, L_V) + h^0(Z, L_Z) \geq \delta_S + 2, \tag{55}$$

hence by 2.1.1 (12), $h^0(X, L) \geq 2$, proving part (A). To compute $\dim A_{\underline{d}}(X)$ consider again the rational map

$$A_{\underline{d}}(X) \xrightarrow{\rho} A_{\underline{d}_V}(V) \times \mathrm{Pic}^{\underline{d}_Z}\, Z.$$

Since $\dim A_{\underline{d}_V}(V) \leq d_V$ (by Lemma 1.2.8) we get

$$\dim A_{\underline{d}}(X) \leq d_V + g_Z + \dim W_M(X) \leq g_V - 2 + g_Z + \dim W_M(X),$$

using (54) for the last inequality. Thus

$$\dim A_{\underline{d}}(X) \leq g_V - 2 + g_Z + \delta_S - \gamma_S + 1 = g - 2.$$

This proves (B) and we are done. $\square$

From the proof, it is clear that the farther $\underline{d}$ is from being semistable, the smaller the dimension of $A_{\underline{d}}(X)$ is. The following fact will be useful later on.

Corollary 3.2.2. *Let $R \subset X$ be a finite set of non-singular points of X and $\underline{d} \in \Sigma^{ss}(X)$. Then the general $L \in A_{\underline{d}}(X)$ has no base point in R.*

Proof. It obviously suffices to assume $\#R = 1$, so let $R = \{q\}$. If L is general in $A_{\underline{d}}(X)$ we can assume that $L \in \operatorname{Im} \alpha_X^{\underline{d}}$. Set $L' = L(-q)$ and $\underline{d}' := \underline{\deg}\, L'$. If q is a base point of L, then $L' \in \operatorname{Im} \alpha_X^{\underline{d}'}$. Therefore, if the general $L \in A_{\underline{d}}(X)$ has a base point in q, the map

$$\operatorname{Im} \alpha_X^{\underline{d}'} \to A_{\underline{d}}(X), \qquad L' \mapsto L'(q), \tag{56}$$

must be dominant. But this is not possible, as $\dim A_{\underline{d}}(X) = g - 1$ by 3.2.1, whereas obviously $\dim \operatorname{Im} \alpha_X^{\underline{d}'} \leq |\underline{d}'| = g - 2$. □

4. The compactified theta divisor

4.0.1. Let X be a connected nodal curve, $S \subset X_{\text{sing}}$, $\delta_S := \#S$ and $\nu_S : Y_S \to X$ the normalization of X at S. Let

$$\widehat{X}_S = Y_S \cup \bigcup_{i=1}^{\delta_S} E_i \tag{57}$$

be the connected, nodal curve obtained by "blowing up" X at S, so that $E_i \cong \mathbb{P}^1$ for all i and E_i is called an *exceptional component* of $\widehat{X}_S \to X$ (where this map is the contraction of all the exceptional components of $\widehat{X}_S$). We shall usually denote by $\widehat{M}$ a line bundle on $\widehat{X}_S$ and by $M \in \operatorname{Pic} Y_S$ its restriction to Y_S.

4.1. The compactified Picard variety

4.1.1. In what follows we shall recall what the points of $\overline{P_X^{g-1}}$ parametrize, and give a stratified description of it (in 4.1.5); our notation is that of [C05]. There is more than one place where details and proofs can be found, even though some terminology may be different from ours. We refer to [Al04] for a unifying account and other references.

To begin with, using the notation of 4.0.1, the compactified Picard variety, or compactified jacobian, $\overline{P_X^{g-1}}$, in degree $g - 1$, parametrizes equivalence classes of stable line bundles of degree $g - 1$ on the curves $\widehat{X}_S$ as S varies among all subsets of X_{sing}.

Let us define stable line bundles and the equivalence relation among them. For every $S \subset X_{\text{sing}}$ consider the blow-up of X at S, $\widehat{X}_S = Y_S \cup \bigcup_{i=1}^{\delta_S} E_i$ (cf. (57)). A stable line bundle $\widehat{M} \in \operatorname{Pic}^d \widehat{X}_S$ is such that, if we set $M := \widehat{M}_{Y_S}$, properties (1) and (2) below hold:

(1) $\underline{\deg}\, M \in \Sigma(Y_S)$;
(2) $\deg_{E_i} \widehat{M} = 1$ for $i = 1, \ldots, \delta_S$.

We call $\widehat{M} \in \operatorname{Pic}^d \widehat{X}_S$ *semistable* if it satisfies (2) as well as (1′), where

(1′) $\underline{\deg}\, M \in \Sigma^{ss}(Y_S)$.

In other words, a line bundle on $\widehat{X}_S$ is semistable (resp. stable) if its restriction to the complement of all the exceptional components of $\widehat{X}_S \to X$ has semistable (resp. stable) multidegree. Two stable line bundles $\widehat{M}$ and $\widehat{M}'$ on $\widehat{X}_S$ are defined to be equivalent iff their restrictions, M and M', to Y_S coincide.

4.1.2. Thus, the points in $\overline{P_X^{g-1}}$ are in one-to-one correspondence with equivalence classes of stable line bundles. Any such class is uniquely determined by S and by $M \in \operatorname{Pic} Y_S$ (provided that $\Sigma(Y_S)$ is not empty), therefore points of $\overline{P_X^{g-1}}$ will be denoted by pairs $[M, S]$, where $S \subset X_{\text{sing}}$ and $M \in \operatorname{Pic}^{\underline{d}} Y_S$ with $\underline{d} \in \Sigma(Y_S)$.

4.1.3. Although $\overline{P_X^{g-1}}$ is constructed as a GIT-quotient, our terminology "stable/semistable line bundles" does not precisely reflect the GIT stability/semistability. More precisely, denote by $q_X : H_X \to \overline{P_X^{g-1}}$ the GIT quotient defining $\overline{P_X^{g-1}}$ (so that H_X is a closed subset in the GIT-semistable locus of some Hilbert scheme). Note that H_X contains strictly GIT-semistable points, unless X is irreducible. Our stable line bundles correspond to GIT-semistable points in H_X having closed orbit.

4.1.4. For technical reasons we need to consider semistable multidegrees that are not stable. Let $\underline{d} \in \Sigma^{ss}(Y_S)$ be a semistable multidegree of Y_S; a node n of Y_S is called *destabilizing* for $\underline{d}$ if there exists a connected subcurve $Z \subset Y_S$ such that $n \in Z \cap Z^C$ and $d_Z = p_a(Z) - 1$ ($Z^C = \overline{Y \smallsetminus Z}$). We set

$$S(\underline{d}) := \{n \in (Y_S)_{\text{sing}} : n \text{ is destabilizing for } \underline{d}\}. \tag{58}$$

Observe that

$$S(\underline{d}) = \emptyset \;\Leftrightarrow\; \underline{d} \in \Sigma(Y_S). \tag{59}$$

We denote by $Y_S(\underline{d})$ the normalization of Y_S at $S(\underline{d})$, so that we have

$$Y_S(\underline{d}) = Y_{S \cup S(\underline{d})} \stackrel{\nu_{\underline{d}}}{\to} Y_S \stackrel{\nu_S}{\to} X \tag{60}$$

where $\nu_{\underline{d}}$ is the normalization map.

Assume that $\underline{d}$ is strictly semistable, i.e. $S(\underline{d})$ is not empty. Then the dual graph of Y_S has an orientation such that for every subcurve $Z \subset Y_S$ such that $d_Z = p_a(Z) - 1$, all the edges between Γ_Z and Γ_{Z^C} go from Γ_Z to Γ_{Z^C} (by 1.3.2). Therefore, if we consider $Y_S(\underline{d})$ and use the convention of 1.3.4, for every destabilizing node $n \in Z \cap Z^C$, we have $q_1^n \in Z$ and $q_2^n \in Z^C$ (abusing notation by denoting $Z = \nu_{\underline{d}}^{-1}(Z)$ and $Z^C = \nu_{\underline{d}}^{-1}(Z^C)$). We now introduce a divisor on $Y_S(\underline{d})$:

$$T(\underline{d}) := \sum_{n \in S(\underline{d})} q_2^n \quad \text{and} \quad \underline{t}(\underline{d}) := \underline{\deg}\, T(\underline{d}). \tag{61}$$

By construction, $\underline{d} - \underline{t}(\underline{d})$ is a stable multidegree for $Y_S(\underline{d})$. Set

$$\underline{d}^{\text{st}} := \underline{d} - \underline{t}(\underline{d}) \in \Sigma(Y_S(\underline{d})). \tag{62}$$

The following statement summarizes various known facts about $\overline{P_X^{g-1}}$. The only novelty is that we use line bundles on the partial normalizations of X, rather than torsion free sheaves on X (as in [AK80], [OS79], [Si94]) or line bundles on the blow-ups of X (as in [C94]).

Fact 4.1.5. $\overline{P_X^{g-1}}$ *is a connected, reduced, projective scheme of pure dimension* g. *It has a stratification*

$$\overline{P_X^{g-1}} = \coprod_{\substack{\emptyset\subseteq S\subseteq X_{\text{sing}}\\ \underline{d}\in\Sigma(Y_S)}} P_S^{\underline{d}}$$

such that the following properties hold:

(i) *For every* $S \subset X_{\text{sing}}$ *and every* $\underline{d} \in \Sigma(Y_S)$ *there is a canonical isomorphism* (*notation in* 4.1.2)

$$\operatorname{Pic}^{\underline{d}} Y_S \xrightarrow{\epsilon_S^{\underline{d}}} P_S^{\underline{d}}, \qquad M \mapsto [M, S]. \tag{63}$$

In particular, if $P_S^{\underline{d}} \neq \emptyset$, *then* $\dim P_S^{\underline{d}} = g - \delta_S + \gamma_S - 1$.

(ii) *More generally, for every* $S \subset X_{\text{sing}}$ *and every* $\underline{d} \in \Sigma^{\text{ss}}(Y_S)$ *there is a canonical surjective morphism* $\epsilon_S^{\underline{d}} : \operatorname{Pic}^{\underline{d}} Y_S \to P_{S(\underline{d})}^{\underline{d}^{\text{st}}}$ (*notation in* 4.1.4) *which factors as follows:*

$$\begin{array}{llll} \epsilon_S^{\underline{d}} : \operatorname{Pic}^{\underline{d}} Y_S & \xrightarrow{\tau} & \operatorname{Pic}^{\underline{d}^{\text{st}}} Y_S(\underline{d}) & \xrightarrow{\epsilon_{S(\underline{d})}^{\underline{d}^{\text{st}}}} P_{S(\underline{d})}^{\underline{d}^{\text{st}}} \\ \qquad L & \mapsto & \nu_{\underline{d}}^* L \otimes \mathcal{O}_{Y_{S(\underline{d})}}(-\sum_{n\in S(\underline{d})} q_2^n) & \end{array} \tag{64}$$

where τ *is surjective with fibers* $(k^*)^b$, $b = \delta_{S(\underline{d})} - \gamma_{S(\underline{d})} + 1$, *and* $\epsilon_{S(\underline{d})}^{\underline{d}^{\text{st}}}$ *is an isomorphism.*

(iii) *If* $P_{S'}^{\underline{d}'} \subset \overline{P_S^{\underline{d}}}$ *then* $S \subset S'$ *and* $\underline{d} \geq \underline{d}'$ (*i.e.* $d_i \geq d_i'$ *for all* $i = 1, \dots, \gamma$). *In such a case,* $\#((S' \smallsetminus S) \cap \overline{C_i}) = d_i - d_i'$ (*recall that* $X = \bigcup_{i=1}^{\gamma} \overline{C_i}$).

(iv) *Denote by* P_X^{g-1} *the smooth locus of* $\overline{P_X^{g-1}}$. *Then*

$$P_X^{g-1} = \coprod_{\underline{d}\in\Sigma(\widetilde{X})} P_{X_{\text{sep}}}^{\underline{d}} \cong \coprod_{\underline{d}\in\Sigma(\widetilde{X})} \operatorname{Pic}^{\underline{d}} \widetilde{X}$$

where $\widetilde{X} \to X$ *is the normalization at the separating nodes* (*cf.* 1.3.6) *and the isomorphism is the canonical one described in part* (i).

Given the normalization of X at all of its separating nodes, $\widetilde{X} \to X$, recall from 1.3.6 that $\widetilde{X} = \coprod_{i=1}^{c} X_i$ denotes the decomposition of $\widetilde{X}$ into connected components.

Corollary 4.1.6. $\overline{P_X^{g-1}}$ *is irreducible if and only if for every* $i = 1, \dots, c$ *either* X_i *is irreducible, or every irreducible component* C *of* X_i *meets* $\overline{X_i \smallsetminus C}$ *in exactly two points.*

Proof. Assume first $X = \widetilde{X}$. Then $\overline{P_X^{g-1}}$ is irreducible if and only if $\#\Sigma(X) = 1$. If X is irreducible, then $\Sigma(X) = \{g-1\}$ so $\overline{P_X^{g-1}}$ is irreducible. If every irreducible component $\overline{C}_i$ of X meets its complementary curve in two points, calling $\overline{g}_i$ the arithmetic genus of $\overline{C}_i$, we have $g - 1 = \sum_{i=1}^{\gamma} \overline{g}_i$. Therefore $\Sigma(X) = \{(\overline{g}_1, \dots, \overline{g}_\gamma)\}$, hence $\overline{P_X^{g-1}}$ is irreducible.

Conversely, assume that X is reducible and has an irreducible component, $\overline{C}_i$, such that $\delta_i := \#\overline{X \setminus C_i} \geq 3$. Then X may be obtained as the special fiber of a family of nodal curves X_t having exactly two irreducible components intersecting in δ_i points. Then $\#\Sigma(X_t) = \delta_i - 1 \geq 2$ (cf. 4.2.8), hence $\overline{P^{g-1}_{X_t}}$ has at least two irreducible components. Since $\overline{P^{g-1}_{X_t}}$ specializes to $\overline{P^{g-1}_X}$ we find that $\overline{P^{g-1}_X}$ has at least two irreducible components. So, if X has no separating node we are done.

In general, denote $\widetilde{b} := \#\Sigma(\widetilde{X})$. Then $\overline{P^{g-1}_X}$ is irreducible if and only if $\widetilde{b} = 1$; by 1.3.6 this is equivalent to $\#\Sigma(X_i) = 1$ for every $i = 1, \dots, c$. Then the result follows by applying the first part to each X_i. □

Remark 4.1.7. In combinatorial terms, consider the graph $\widetilde{\Gamma}_X$ obtained from Γ_X by removing every loop and every separating edge. Then $\overline{P^{g-1}_X}$ is irreducible if and only if every vertex of $\widetilde{\Gamma}_X$ has valency (or degree) equal to either 0 or 2.

4.2. Stratifying the theta divisor

We shall now define the theta divisor of $\overline{P^{g-1}_X}$ using the stratification given above. A natural thing to do is to consider the irreducible strata, $P^{\underline{d}}_S$, of dimension g of $\overline{P^{g-1}_X}$, consider $W_{\underline{d}}(X)$ in such strata and then take their closure. Recalling Lemma 1.3.5, the g-dimensional strata are easily listed. First, denote by $X_{\mathrm{sep}} \subset X_{\mathrm{sing}}$ the set of separating nodes of X and let $\widetilde{X} \to X$ be the normalization of X at X_{sep} (as in 4.1.5(iv)). Thus $\widetilde{X}$ is a nodal curve having $c = \#X_{\mathrm{sep}} + 1$ connected components. Finally, set $\widetilde{b} = \#\Sigma(\widetilde{X})$. We have

Lemma-Definition 4.2.1. *Let X be a connected nodal curve. Using $\epsilon^{\underline{d}}_S$ of* 4.1.5(i) *as an identification, we define the theta divisor $\Theta(X)$ of $\overline{P^{g-1}_X}$ as*

$$\Theta(X) := \overline{\bigcup_{\underline{d} \in \Sigma(\widetilde{X})} W_{\underline{d}}(\widetilde{X})} \subset \overline{P^{g-1}_X}.$$

$\Theta(X)$ has $c\widetilde{b}$ irreducible components, all of dimension $g - 1$.

Proof. If X is free from separating nodes (i.e. $c = 1$) the statement follows trivially from Theorem 3.1.2. Otherwise, let $\widetilde{X} = X_1 \amalg \cdots \amalg X_c$ be the decomposition into connected components. Then $g = \sum_{i=1}^{c} p_a(X_i)$ and

$$W_{\underline{d}}(\widetilde{X}) = \bigcup_{i=1}^{c} \Big(W_{\underline{d}_i}(X_i) \times \prod_{\substack{j \neq i \\ j=1,\dots,c}} \mathrm{Pic}^{\underline{d}_j} X_j \Big)$$

where $\underline{d}_i$ denotes the restriction of $\underline{d}$ to X_i. Since X_i is connected and $\underline{d}_i$ is stable, $W_{\underline{d}_i}(X_i)$ is irreducible of dimension $p_a(X_i) - 1$, hence we are done (cf. 1.3.6). □

Corollary 4.2.2. *$\Theta(X)$ is irreducible if and only if either X is irreducible, or every irreducible component of X meets its complementary curve in exactly two points.*

Proof. By 4.2.1, $\Theta(X)$ is irreducible if and only if $c = 1$ (i.e. X is free from separating nodes) and $\widetilde{b} = 1$.

Assume $\Theta(X)$ is irreducible; then X has no separating nodes and $\widetilde{b} = \#\Sigma(X) = 1$. Hence $\overline{P_X^{g-1}}$ is irreducible, by 4.1.5. Applying Corollary 4.1.6 we are done.

Conversely, if X is irreducible, then $\Theta(X)$ is irreducible by Theorem 3.1.2. If X is reducible and satisfies the hypothesis, obviously $c = 1$. Moreover, arguing as in the proof of Corollary 4.1.6 we conclude that X has only one stable multidegree: $\underline{d} = (\overline{g}_1, \dots, \overline{g}_\gamma)$, hence $\Theta(X)$ is irreducible. □

Remark 4.2.3. In combinatorial terms, let Γ_X^* be the graph obtained from Γ_X by removing every loop. Then $\Theta(X)$ is irreducible if and only if either Γ_X^* is a point, or every vertex of Γ_X^* has valency (i.e. degree) 2.

Remark 4.2.4. Definition 4.2.1 coincides with the one given in [E97] or (which is the same) in [Al04], by Theorem 4.2.6 below. In particular, $\Theta(X)$ is Cartier and ample.

For the following simple lemma we use the notation in 4.0.1.

Lemma 4.2.5. *Let $S \subset X_{\text{sing}}$ and $M \in \operatorname{Pic} Y_S$. Pick $\widehat{M} \in \operatorname{Pic} \widehat{X}_S$ such that $\widehat{M}_{|Y_S} = M$ and $\widehat{M}_E = \mathcal{O}_E(1)$ for every exceptional component E of $\widehat{X}_S$. Then $h^0(\widehat{X}_S, \widehat{M}) = h^0(Y_S, M)$.*

Proof. (Cf. [P07, 2.1] for an analogous statement.) For any pair of points $p_1, p_2 \in \mathbb{P}^1$ choose a trivialization of $\mathcal{O}_{\mathbb{P}^1}(1)$ locally at such points; now for any pair $a_1, a_2 \in k$ there exists a unique section $s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that $s(p_1) = a_i$ for $i = 1, 2$. So, every section $s_Y \in H^0(Y, M)$ extends to a unique section of $H^0(\widehat{X}_S, \widehat{M})$ determined by s_Y and by the gluing data defining $\widehat{M}$. Conversely, of course any section in $H^0(\widehat{X}_S, \widehat{M})$ restricts to a section of M. □

Theorem 4.2.6. *Let X be a connected nodal curve. The stratification of $\overline{P_X^{g-1}}$ given by 4.1.5 induces the following canonical stratification:*

$$\Theta(X) = \coprod_{\substack{\emptyset \subseteq S \subseteq X_{\text{sing}} \\ \underline{d} \in \Sigma(Y_S)}} \Theta_S^{\underline{d}} \text{ with canonical isomorphisms } \Theta_S^{\underline{d}} \cong W_{\underline{d}}(Y_S). \tag{65}$$

Equivalently, $\Theta(X) = \{[M, S] \in \overline{P_X^{g-1}} : h^0(\widehat{X}_S, \widehat{M}) \neq 0\}$.

Proof. The equivalence of the two descriptions follows immediately from 4.1.5 and Lemma 4.2.5. Furthermore, it is clear that

$$\Theta(X) \subset \{[M, S] \in \overline{P_X^{g-1}} : h^0(\widehat{X}_S, \widehat{M}) \neq 0\}$$

(by upper semicontinuity of h^0). So we need to prove the other inclusion.

Part 1: *Proof assuming X is free from separating nodes.* In this case, by definition,

$$\Theta(X) = \overline{\bigcup_{\underline{d} \in \Sigma(X)} W_{\underline{d}}(X)}.$$

We shall use Abel maps (see 1.2.7): recall that $\alpha^{\underline{d}}_{Y_S}$ is the $\underline{d}$-th Abel map of Y_S and the closure of its image in $\operatorname{Pic}^{\underline{d}} Y_S$ is denoted by $A_{\underline{d}}(Y_S)$.

Step 1. Assume $\#S = 1$ and let $\underline{d} \in \Sigma^{ss}(Y_S)$. Then there exists $\underline{e} \in \Sigma(X)$ such that (using Theorem 3.1.2 *for the equality below)*

$$\epsilon^{\underline{d}}_S(A_{\underline{d}}(Y_S)) \subset \overline{\epsilon^{\underline{e}}_{\emptyset}(W_{\underline{e}}(X))} = \overline{\epsilon^{\underline{e}}_{\emptyset}(A_{\underline{e}}(X))}.$$

In particular, if $[M, S] \in \overline{P^{g-1}_X}$ (so that $\underline{\deg}\, M \in \Sigma(Y_S)$) satisfies $\#S = 1$ and $h^0(\widehat{X}_S, \widehat{M}) \neq 0$, then $[M, S] \in \Theta(X)$.

Let $M \in \operatorname{Pic}^{\underline{d}}(Y_S)$ with $M \in A_{\underline{d}}(Y_S)$ and $\underline{\deg}\, M = \underline{d} \in \Sigma^{ss}(Y_S)$. As X is free from separating nodes, Y_S is connected.

Observe that, by 4.1.5(iv), $\overline{P^{g-1}_X}$ is the closure of its open subset

$$P^{g-1}_X = \coprod_{\underline{e} \in \Sigma(X)} P^{\underline{e}}_{\emptyset} \cong \coprod_{\underline{e} \in \Sigma(X)} \operatorname{Pic}^{\underline{e}} X.$$

Therefore there exists an $\underline{e} \in \Sigma(X)$ such that $\epsilon^{\underline{d}}_S(M) \in \overline{P^{\underline{e}}_{\emptyset}} = \overline{\operatorname{Pic}^{\underline{e}} X}$.

Since $\#S = 1$, $|\underline{d}| = p_a(Y_S) - 1 = g - 2 = |\underline{e}| - 1$. Furthermore, $\underline{d} \leq \underline{e}$ (by 4.1.5(iii)). Therefore there exists a unique index $i \in \{1, \dots, \gamma\}$, say $i = 1$, such that $d_1 = e_1 - 1$ and $d_i = e_i$ for $i \geq 2$.

Set $S = \{n\}$, let $\nu_S : Y_S \to X$ be the normalization at n, and let C_1 be the first component of Y_S. Since $d_1 = e_1 - 1$, by 4.1.5(iii) C_1 contains one of the two branches of n; call it q_1. Let now $p_t \in C_1$ be a moving point specializing to q_1.

We can assume that M is a general point in $A_{\underline{d}}(Y_S)$ (which is irreducible of dimension $p_a(Y_S) - 1$), in particular that M is in the image of the Abel map, that $h^0(Y_S, M) = 1$, and that M has no base point lying over n (by 3.2.1 and 3.2.2). Therefore there exists $L \in \operatorname{Pic} X$ such that $\nu_S^* L = M$ and $L \in \operatorname{Im} \alpha^{\underline{d}}_X$ (by 2.2.3(2a)). Set $L_t := L(p_t)$; then

$$\underline{\deg}\, L_t = \underline{d} + (1, 0, \dots, 0) = \underline{e} \in \Sigma(X)$$

and $L_t \in \operatorname{Im} \alpha^{\underline{e}}_X$, in particular $h^0(X, L_t) \neq 0$. As p_t specializes to q_1, it follows that $\epsilon^{\underline{e}}_{\emptyset}(L_t)$ specializes to $\epsilon^{\underline{d}}_S(M)$, so we are done with Step 1.

Step 2. For every S such that $\#S \geq 2$ and $\underline{d} \in \Sigma^{ss}(Y_S)$, there exist $S' \subset S$ such that $\#S' = \#S - 1$, and a $\underline{d}' \in \Sigma^{ss}(Y_{S'})$ such that

$$\epsilon^{\underline{d}}_S(A_{\underline{d}}(Y_S)) \subset \overline{\epsilon^{\underline{d}'}_{S'}(A_{\underline{d}'}(Y_{S'}))}.$$

Let $\underline{d}$ be a semistable multidegree for Y_S. Consider the dual graph Γ_{Y_S} and an orientation on it inducing $\underline{d}$. Note that Γ_{Y_S} is the subgraph of Γ_X obtained by removing the edges corresponding to S. It is clear that if we add to Γ_{Y_S} any edge n of Γ (so that $n \in S$), oriented however we like, we obtain a new oriented graph Γ' such that $\Gamma_{Y_S} \subset \Gamma' \subset \Gamma_X$. Set $S' = S \smallsetminus \{n\}$, thus Γ' is the dual graph of the curve $Y_{S'}$ obtained by normalizing X at S'. Thus we have a map $Y_S \to Y_{S'}$ which is the normalization of $Y_{S'}$ at n.

The given orientation on Γ' corresponds to a semistable multidegree $\underline{d}'$ such that $|\underline{d}'| = |\underline{d}| + 1$ and $\underline{d}' \geq \underline{d}$.

From now on we can argue as for Step 1, with $Y_{S'}$ playing the role of X. More precisely, if Y_S is connected, then the argument is exactly the same: start from a general $M \in A_{\underline{d}}(Y_S)$ and construct a family of line bundles $L_t = L(p_t) \in A_{\underline{d}'}(Y_{S'})$ such that p_t is a smooth point of $Y_{S'}$ specializing to n, and $L \in A_{\underline{d}}(Y_{S'})$ such that L pulls back to M. Then $\epsilon_{S'}^{\underline{d}'}(L_t)$ specializes to $\epsilon_S^{\underline{d}}(M)$.

If Y_S is not connected, then the general $M \in A_{\underline{d}}(Y_S)$ has $h^0(M) \geq 2$, and it has no base point over n (by 3.2.2). We now apply 2.2.4 to obtain $L \in \operatorname{Im} \alpha^{\underline{d}}_{Y_{S'}}$ which pulls back to M. The rest of the argument is the same as before.

This concludes the proof of Step 2.

Step 3. End of proof of Part 1. To prove the theorem, we pick $[M, S] \in \overline{P_X^{g-1}}$ such that $M \in W_{\underline{d}}(Y_S)$; since $\underline{d}$ is stable, we have $W_{\underline{d}}(Y_S) = A_{\underline{d}}(Y_S)$ by 3.1.2 (applied to every connected component of Y_S).

Using Step 2 we can decrease the cardinality of S at the cost of passing from a stable multidegree to a semistable one (which is why the assumption for Step 1 is that $\underline{d}$ is semistable, rather than stable). Iterating Step 2 finitely many times, we reduce the proof of the theorem to Step 1. So the theorem is proved for X free from separating nodes.

Part 2: *Proof assuming X_{sep} is not empty.* Recall that $\widetilde{X} \to X$ is the normalization of X at X_{sep} and $\widetilde{X} = \bigcup_{i=1}^{c} X_i$ denotes the decomposition of $\widetilde{X}$ into connected components; set $g_i = p_a(X_i)$. By Fact 4.1.5 we have a canonical isomorphism

$$\overline{P_X^{g-1}} \cong \prod_{i=1}^{c} \overline{P_{X_i}^{g_i-1}} \tag{66}$$

and, by Definition 4.2.1, another canonical isomorphism

$$\Theta(X) \cong \bigcup_{j=1}^{c} \Big(\Theta(X_j) \times \prod_{\substack{i \neq j \\ 1 \leq i \leq c}} \overline{P_{X_i}^{g_i-1}} \Big). \tag{67}$$

Let $[M, S] \in \overline{P_X^{g-1}}$ be such that $h^0(Y_S, M) \neq 0$. Now $S \supset X_{\text{sep}}$, hence we can factor

$$\nu_S : Y_S \xrightarrow{\widetilde{\nu_S}} \widetilde{X} \to X$$

and denote $Y_i = \widetilde{\nu_S}^{-1}(X_i)$, so that Y_S is the disjoint union of $Y_1, \ldots, Y_c$. Note that Y_i is the normalization of X_i at a certain set of nodes, S_i, of X_i. Therefore, under the isomorphism (66), the point $[M, S]$ corresponds to the point $([M_1, S_1], \ldots, [M_c, S_c]) \in \prod_{i=1}^{c} \overline{P_{X_i}^{g_i-1}}$ where $M_i = M_{Y_i}$.

Furthermore, $h^0(Y_S, M) = \sum_{i=1}^{c} h^0(Y_i, M_i)$, hence there exists an index, say $i = 1$, such that $h^0(Y_1, M_1) \neq 0$. Now, X_1 is free from separating nodes, therefore by the first part of the proof we obtain $[M_1, S_1] \in \Theta(X_1)$. By (67), this implies $[M, S] \in \Theta(X)$, finishing the proof. $\square$

Example 4.2.7. Let $X = C_1 \cup C_2$ with $\# C_1 \cap C_2 = 1$; then $\Sigma(X)$ is empty, while $\Sigma(Y) = \{(g_1 - 1, g_2 - 1)\}$ (Y is the normalization of X). The points of $\overline{P_X^{g-1}}$ correspond to line bundles of multidegree $(g_1 - 1, g_2 - 1)$ on Y or to equivalence classes of line bundles on the curve $\widehat{X}$ obtained by blowing up the unique node of X. More precisely, if we order the components of $\widehat{X}$ so that $\widehat{X} = C_1 \cup E \cup C_2$ (where $E \cong \mathbb{P}^1$), then $\overline{P_X^{g-1}}$ bijectively parametrizes line bundles of multidegree $(g_1 - 1, 1, g_2 - 1)$ on $\widehat{X}$. There is a canonical isomorphism

$$\overline{P_X^{g-1}} \cong \operatorname{Pic}^{g_1-1} C_1 \times \operatorname{Pic}^{g_2-1} C_2.$$

Now, $\Theta(X)$ is canonically isomorphic to $W_{(g_1-1,g_2-1)}(Y)$, which we can easily describe by means of 1.2.5. We obtain three different cases.

Case 1: $g_i \neq 0$, $i = 1, 2$. Then $\Theta(X)$ has two irreducible components:

$$\Theta(X) = (W_{g_1-1}(C_1) \times \operatorname{Pic}^{g_2-1} C_2) \cup (\operatorname{Pic}^{g_1-1} C_1 \times W_{g_2-1}(C_2)). \tag{68}$$

Case 2: $g_1 = 0$ and $g_2 \neq 0$. Then the first component in (68) is empty and we get $\Theta(X) \cong W_{g_2-1}(C_2) \cong \Theta(C_2)$.

Case 3: $g_1 = g_2 = 0$. Then $\Theta(X)$ is empty.

Example 4.2.8. Let $X = C_1 \cup C_2$ with $\# C_1 \cap C_2 = \delta \geq 2$; assume C_i is smooth (this assumption can easily be removed) of genus g_i. Then $g - 1 = g_1 + g_2 + \delta - 2$. We have $\Sigma(X) = \{(g_1, g_2+\delta-2), (g_1+1, g_2+\delta-1), \ldots, (g_1+\delta-2, g_2)\}$, so that $\overline{P_X^{g-1}}$ has $\delta-1$ irreducible components of dimension g. There is a canonical isomorphism (cf. 4.1.5(iv))

$$P_X^{g-1} = \coprod_{i=0}^{\delta-2} P_{\emptyset}^{(g_1+i, g_2+\delta-i-2)} \cong \coprod_{i=0}^{\delta-2} \operatorname{Pic}^{(g_1+i, g_2+\delta-i-2)} X.$$

For every set $S \subset X_{\text{sing}}$ such that $\# S = k$ with $1 \leq k \leq \delta - 2$, we have

$$\Sigma(Y_S) = \{(g_1, g_2 + \delta - k - 2), \ldots, (g_1 + \delta - k - 2, g_2)\};$$

so that $\overline{P_X^{g-1}}$ has a total of $(\delta - k - 1)\binom{\delta}{k}$ strata of codimension k, each of which is isomorphic to $\operatorname{Pic}^{\underline{d}} Y_S$. If $k = \delta - 1$ then for any choice of $\delta - 1$ nodes, the curve obtained by blowing up X at such nodes has a separating node, hence $\Sigma(Y_S)$ is empty. Finally, the last stratum corresponds to $S = X_{\text{sing}}$ and $\underline{d} = (g_1 - 1, g_2 - 1)$, and it has codimension $\delta - 1$. We have

$$P_{X_{\text{sing}}}^{(g_1-1, g_2-1)} \cong \operatorname{Pic}^{g_1-1} C_1 \times \operatorname{Pic}^{g_2-1} C_2.$$

Now, $\Theta(X)$ contains $\delta - 1$ irreducible strata of dimension $g - 1$, one for every component of $\overline{P_X^{g-1}}$. Indeed, for every $\underline{d} \in \Sigma(X)$ we have $\Theta_{\emptyset}^{\underline{d}} \cong W_{\underline{d}}(X)$, which is irreducible of dimension $g - 1$, by Theorem 3.1.2.

For every set $S \subset X_{\text{sing}}$ such that $\# S = k$ with $1 \leq k \leq \delta - 2$, Y_S is connected and free from separating nodes, so that for every $\underline{d} \in \Sigma(Y_S)$ we get an irreducible stratum of dimension $g - k - 1$ isomorphic to $W_{\underline{d}}(Y_S)$. If $k = \delta - 1$ there are no strata (as before). If $k = \delta$ we get a stratum isomorphic to the theta divisor computed in Example 4.2.7 (cf. (68)).

5. Characterizing hyperelliptic stable curves

We conclude the paper with a characterization of hyperelliptic irreducible curves, Theorem 5.2.4, extending a well known one for smooth curves. The irreducibility assumption is truly needed, as shown in counterexample 5.2.5.

5.1. Irreducible curves

If we restrict our interest to irreducible singular curves, not only does the description of the compactified jacobian simplifies substantially, but also the same description is valid for all degrees.

5.1.1. Let X be an irreducible curve. Then the definitions of stable and semistable multidegrees (given for $d = g - 1$) coincide and are trivial. Thus, for every normalization $Y_S \to X$ at a set S of δ_S nodes, we have $\Sigma(Y_S) = \Sigma^{ss}(Y_S) = \{p_a(Y_S) - 1\} = \{g - 1 - \delta_S\}$. So, that definition generalizes to all d, as follows. With the notation of 4.0.1, a line bundle $\widehat{M} \in \operatorname{Pic}^d \widehat{X}_S$ is stable if (1) and (2) hold, where (1) $\deg_{Y_S} \widehat{M} = d - \delta_S$, (2) $\deg_{E_i} \widehat{M} = 1$ for all $i = 1, \ldots, \delta_S$.

The equivalence relation is the same as for $d = g - 1$: two stable line bundles on $\widehat{X}_S$ are equivalent iff their pull-backs to Y_S coincide. An equivalence class is thus uniquely determined by S and by the restriction, M, of $\widehat{M}$ to Y_S; we shall maintain the notation of 4.0.1 and 4.1.2.

Exactly as in the case $d = g - 1$, we have the following. *The variety $\overline{P^d_X}$ is reduced and irreducible. It bijectively parametrizes the equivalence classes of stable line bundles on the curves $\widehat{X}_S$ associated to X as S varies among all subsets of X_{sing}.*

Moreover, as in 4.1.5, $\overline{P^d_X}$ has a canonical stratification into disjoint strata, called P_S, indexed by the subsets S of X_{sing}. Every P_S has a canonical isomorphism (usually viewed as an identification) $\epsilon_S : \operatorname{Pic}^{d-\delta_S} Y_S \xrightarrow{\cong} P_S \subset \overline{P^d_X}$. We have

$$\overline{P^d_X} = \coprod_{S \subset X_{\text{sing}}} P_S \cong \coprod_{S \subset X_{\text{sing}}} \operatorname{Pic}^{d-\delta_S} Y_S. \tag{69}$$

5.1.2. Given a family of irreducible curves, $f : \mathcal{X} \to B$, up to a finite base change there exists the compactified Picard scheme $\pi_d : \overline{P^d_f} \to B$ which contains the relative degree-d Picard scheme of f, denoted Pic^d_f, as an open subset (see [C05] for details). The fiber of π_d over a point $b \in B$ is $\overline{P^d_{X_b}}$.

In the next lemma we use the notation of 1.2.4, in particular (8).

Lemma 5.1.3. *Let $\nu : Y_S \to X$ be the normalization of X at a non-separating node n of X, and set $\nu^{-1}(n) = \{q_1, q_2\}$. Let $M \in W^r_d(Y_S)$. Then*

(1) $W^r_M(X) = \emptyset$ *iff* $h^0(Y_S, M) = r + 1$ *and one of the two cases below occurs: either*
   (a) $h^0(Y_S, M - q_1 - q_2)) = h^0(Y_S, M) - 2$, *or*
   (b) *up to interchanging q_1 with q_2,*

$$h^0(Y_S, M) = h^0(Y_S, M - q_1) \neq h^0(Y_S, M - q_2).$$

(2) $\dim W^r_M(X) = 0$ *iff* $h^0(Y_S, M) = r + 1$ *and*

$$h^0(Y_S, M - q_1 - q_2) = h^0(Y_S, M - q_h) = r, \qquad h = 1, 2.$$

In this case $W^r_M(X) = \{L_M\}$ *with* $h^0(X, L_M) = r + 1$.

(3) $\dim W^r_M(X) = 1$ *iff one of the two cases below occurs:*
   (a) $h^0(Y_S, M) = h^0(Y_S, M(-q_h))$ *for* $h = 1, 2$,
   (b) $h^0(Y, M) \geq r + 2$,

Proof. It is a straightforward consequence of Lemma 2.2.4. □

5.1.4. We recall a construction due to E. Arbarello and M. Cornalba (cf. [AC81, Section 2]). Let $h : T \to U$ be family of connected smooth projective curves and assume that h has a section. Then for every pair (d, r) of integers, there exists a U-scheme $\rho : W^r_{d,h} \to U$ such that for every $u \in U$, the fiber of ρ over u is the Brill–Noether variety $W^r_d(h^{-1}(u))$ of the corresponding fiber of h. Moreover there is a natural injective morphism of U-schemes, $W^r_{d,h} \hookrightarrow \mathrm{Pic}^d_h$, viewed here as an inclusion.

Now let $f : \mathcal{X} \to B$ be a one-parameter family of smooth curves specializing to an irreducible curve X, let $b_0 \in B$ be the point over which the fiber is X, and assume that the restriction of f to $U = B \smallsetminus b_0$ is smooth. Up to making a finite étale base change, we may asume that f has a section (this will not affect our conclusion). Denote by h the restriction of f to U and introduce the scheme $W^r_{d,h} \to U$ described above. Consider the Picard scheme $\mathrm{Pic}^d_f \to B$, which is integral, separated and of finite type. Let $\overline{W^r_{d,h}} \subset \mathrm{Pic}^d_f$ denote the closure of $W^r_{d,h}$ in Pic^d_f. Thus $\overline{W^r_{d,h}}$ is a scheme over B; we denote by $W^r_{d,X} := \overline{W^r_{d,h}} \cap \mathrm{Pic}^d X$ the fiber over b_0. Then, by upper semicontinuity of h^0, we have $W^r_{d,X} \subset W^r_d(X)$. Therefore, *if X is the specialization of a family of smooth curves X_b such that every irreducible component of $W^r_d(X_b)$ has dimension at least c (for some number c), then* $\dim W^r_d(X) \geq c$ (i.e. $W^r_d(X)$ has a component of dimension at least c). In particular: *If $r \geq d - g$, then* $\dim W^r_d(X) \geq \rho(g, r, d) = g - (r + 1)(r - d + g)$.

5.2. Hyperelliptic stable curves

Some of the subsequent results are probably known to experts, but an exhaustive reference has not been found.

Let $H_g \subset M_g$ be the locus of smooth hyperelliptic curves and $\overline{H_g}$ its closure in $\overline{M}_g$. We call a singular curve X *hyperelliptic* if it is contained in $\overline{H_g}$ (cf. [HM]).

Some parts of the following proposition can be found in, or easily derived from, [CH] and [HM]. We here need a unified statement.

Proposition 5.2.1. *Let X be an irreducible nodal curve of genus $g \geq 3$ with δ nodes and $\nu : Y \to X$ its normalization. For every node n_j set $\nu^{-1}(n_j) = \{q^j_1, q^j_2\}$. The following are equivalent.*

(i) *There exists a line bundle $H_X \in \mathrm{Pic}^2 X$ such that $h^0(X, H_X) = 2$.*
(ii) $[X] \in \overline{H_g} \subset \overline{M}_g$ *(i.e. X is hyperelliptic).*

(iii) *There exists a family of smooth hyperelliptic curves X_t specializing to X and such that the hyperelliptic class of X_t specializes to a line bundle, H_X, on X.*

(iv) *There exists a g^1_2, Λ, on Y such that $q_1^j + q_2^j$ is a divisor in Λ for every $j = 1, \dots, \delta$ (in particular, $h^0(Y, q_1^j + q_2^j) \geq 2$).*

If the above hold, for every $j = 1, \dots, \delta$ we have $\nu^* H_X = \mathcal{O}_Y(q_1^j + q_2^j)$ and $\Lambda \subset \mathbb{P}(H^0(Y, q_1^j + q_2^j)^*)$. Furthermore, $W^1_2(X) = \{H_X\}$; H_X will be called the hyperelliptic class *of X.*

Remark 5.2.2. The implications (iii)⇔(ii) and (iii)⇒(i) also hold if X is reducible.

Proof. The implications (iii)⇒(i) and (iii)⇒(ii) are obvious.

(i)⇒(iv). Let $\nu_1 : Y_1 \to X$ be the normalization of exactly one node n_1 of X. Let $M = \nu^* H_X$. Then ($g_Y \geq 2$) $h^0(Y_1, M) = 2 = h^0(X, H_X)$. Furthermore, M is base-point-free, hence we are in case (2) of Lemma 5.1.3. We obtain $M = \mathcal{O}_{Y_1}(q_1^1 + q_2^1)$ and H_X is uniquely determined (with the notation of 5.1.3(2), $H_X = L_M$). Finally, $\Lambda_1 := \mathbb{P}(H^0(Y_1, M)^*)$; set $H_1 = M$.

If Y_1 is smooth we are done, otherwise we iterate this procedure as follows. Let $\nu_2 : Y_2 \to Y_1$ be the normalization of one node, n_2, of Y_1. Set

$$\nu_{1,2} : Y_2 \xrightarrow{\nu_2} Y_1 \xrightarrow{\nu_1} X$$

and abuse the notation by using the same symbols for points in X, Y_1 and Y_2 whenever the normalization maps are local isomorphisms. Then $\nu_{1,2}^* H_X = \nu_2^* H_1 = \nu_2^* \mathcal{O}_{Y_1}(q_1^1 + q_2^1) = \mathcal{O}_{Y_2}(q_1^1 + q_2^1)$. Set $H_2 = \nu_2^* H_1 = \mathcal{O}_{Y_2}(q_1^1 + q_2^1)$. Note that the pull-back of the linear series Λ_1 to Y_2 is a g^1_2 containing $q_1^1 + q_2^1$; denote it $\Lambda_2 = \nu_2^* \Lambda_1$. Now we distinguish two cases.

Case 1: $\delta \leq g - 1$, i.e. $Y \neq \mathbb{P}^1$. In this case we certainly have $p_a(Y_2) \geq 1$, hence $h^0(Y_2, H_2) = 2$; thus we can argue as in the previous part to obtain $H_2 = \mathcal{O}_{Y_2}(q_1^2 + q_2^2)$ and $\Lambda_2 = \mathbb{P}(H^0(Y_2, q_1^j + q_2^j)^*)$ for $j = 1, 2$. This procedure can be repeated so we are done.

Case 2: $Y = \mathbb{P}^1$. We can argue as for Case 1 only for $\delta - 1$ steps, arriving at

$$\nu : Y = \mathbb{P}^1 \xrightarrow{\nu_\delta} Y_{\delta-1} \to X$$

where $Y_{\delta-1}$ has only one node and all the above morphisms are birational. Furthermore, for every $j = 1, \dots, \delta - 1$ the pull-back to $Y_{\delta-1}$ of H_X is $\mathcal{O}_{Y_{\delta-1}}(q_1^j + q_2^j)$ and $\Lambda_{\delta-1} = \mathbb{P}(H^0(Y_{\delta-1}, q_1^j + q_2^j)^*)$.

Now let $\Lambda := \nu_\delta^* \Lambda_{\delta-1} \subset \mathbb{P}(H^0(Y, \mathcal{O}(2))^*)$. For every $j = 1, \dots, \delta - 1$ the divisor $q_1^j + q_2^j$ belongs to Λ by construction. To prove that also $q_1^\delta + q_2^\delta$ belongs to Λ we repeat the same construction with respect to a different ordering of the nodes of X, for example by switching n_δ with n_1. As Λ is uniquely determined by H_X, and as $\delta \geq 3$, we are done.

(iv)⇒(i). Set $M = \mathcal{O}_Y(q_1^j + q_2^j)$ (for all j). If $Y \neq \mathbb{P}^1$ we have $h^0(Y, M) = 2$ and $h^0(Y, M - q_1^j - q_2^j) = 1$, so the proof is a straightforward iterated application of Lemma 5.1.3(2).

If $Y = \mathbb{P}^1$ we have $h^0(Y, M) = 3$ and M has no base point. Let $\nu_1 : Y \to X_1$ be the map that glues only one pair of branches, say q_1^δ, q_2^δ, so that $p_a(X_1) = 1$. Then for any $M_1 \in \operatorname{Pic} X_1$ such that $\nu_1^* M_1 = M$ we have $h^0(X_1, M_1) = 2$. Pick $M_1 = \mathcal{O}_{X_1}(q_1^1 + q_2^1)$ (abusing notation); we claim that for every $j = 2, \dots, \delta - 1$ we have $\mathcal{O}_{X_1}(q_1^j + q_2^j) \cong M_1$. This follows from the fact that, on Y, the divisors $q_1^j + q_2^j$ all belong to the same g_2^1, Λ. Indeed, recall that a line bundle on X_1 is uniquely determined by its pull-back to Y, M, and by the constant $c \in K^*$ gluing the two fibers $M_{q_1^\delta} \xrightarrow{\cdot c} M_{q_2^\delta}$ via the multiplication by c. Furthermore, if $s \in H^0(Y, M)$ does not vanish at q_1^δ and q_2^δ, set $c(s) = s(q_2^\delta)/s(q_1^\delta)$; then $c(s)$ determines a unique line bundle L_s which pulls back to M and such that the section s descends to a section $\overline{s} \in H^0(X, L_s)$. Now, for every $j = 1, \dots, \delta$, let $s_j \in H^0(Y, M)$ be such that $\operatorname{div}(s_j) = q_1^j + q_2^j$. Then M_1 is uniquely determined by $c(s_1)$. By hypothesis, the δ sections s_j span a two-dimensional subspace of $H^0(Y, M)$ and $s_\delta(q_1^\delta) = s_\delta(q_2^\delta) = 0$; therefore we have $c(s_j) = c(s_1)$ for every $j \le \delta - 1$, proving that $\mathcal{O}_{X_1}(q_1^j + q_2^j) \cong M_1$ if $j \le \delta - 1$ (indeed, $\operatorname{div}(\overline{s_j}) = q_1^j + q_2^j$).

The claim enables us to complete the argument, again by Lemma 5.1.3(2).

(ii)⇒iii). If $X \in \overline{H_g}$ there exists a family of hyperelliptic curves specializing to X. Up to a finite base change, we get a family $f : \mathcal{X} \to B$ where B is some smooth curve with a marked point $b_0 \in B$ such that the fiber X_b, $b \neq b_0$, is smooth and hyperelliptic, and the fiber over b_0 is X. Moreover, we get a line bundle $\mathcal{H}$ on $\mathcal{X} \smallsetminus X$ whose restriction to X_b is the hyperelliptic line bundle on X_b. The data $(\mathcal{X} \to B, \mathcal{H})$ induce a canonical map μ from $B \smallsetminus b_0$ to Pic^2_f such that $\mu(b) \in \operatorname{Pic}^2 X_b$ is the hyperelliptic class of X_b for all $b \in B \smallsetminus b_0$. As B is a smooth curve, μ extends to a regular map $\mu : B \to \overline{P^2_f}$ (see 5.1.2).

We claim that $\mu(b_0) \in \operatorname{Pic}^2 X \subset \overline{P^2_X} \subset \overline{P^2_f}$. By contradiction, suppose $\mu(b_0)$ is a boundary point of $\overline{P^2_X}$. Then $\mu(b_0) = [M, S]$ where $S \subset X_{\text{sing}}$ with $\delta_S = \#S \ge 1$ and $M \in \operatorname{Pic}^{2-\delta_S} Y_S$. Since $\deg M \le 1$ we have $h^0(Y_S, M) \le 1$. By Lemma 4.2.5 we get $h^0(\widehat{X}_S, \widehat{M}) \le 1$ for any representative $\widehat{M}$ for $[M, S]$. But $\widehat{M}$ is the specialization of line bundles having $h^0 \ge 2$, so this is impossible. The claim is thus proved, and so is the implication (ii)⇒(iii).

Finally, we prove that (iv)⇒(ii). Let us denote by $G \subset \overline{M}_g$ the locus of curves satisfying (iv). We claim that G is irreducible of dimension $2g - \delta - 1$. Assume first $\delta \le g - 1$; then G is the locus of irreducible curves X with δ nodes such that on the normalization Y we have $h^0(Y, q_1^j + q_2^j) = 2$ and if $\delta = g - 1$ we need to impose also $q_1^j + q_2^j \sim q_1^{j'} + q_2^{j'}$. Thus a curve in G is determined by its normalization Y and by the choice of δ linearly equivalent divisors of degree 2 on Y. As $\dim H_{g-\delta} = 2(g-\delta) - 1$ we get $\dim G = \dim H_{g-\delta} + \delta = 2g - \delta - 1$. Moreover, G is irreducible because so is $H_{g-\delta}$. If $\delta = g$, i.e. $Y = \mathbb{P}^1$, an element in G is determined by a g_2^1 on $\mathbb{P}^1$ and by δ divisors in it, everything up to automorphisms. This yields $\dim G = 2 + \delta - 3 = \delta - 1$.

Now denote by Δ_0^δ the closure in $\overline{M}_g$ of the locus of irreducible curves with δ nodes. It is well known that $\operatorname{codim}_{\overline{M}_g} \Delta_0^\delta = \delta$. Therefore $\dim(\overline{H_g} \cap \Delta_0^\delta) \ge 2g - 1 - \delta$ (as

$\dim \overline{H_g} = 2g - 1$). Note that (ii)$\Rightarrow$(iv) (we proved (ii)$\Rightarrow$(iii)$\Rightarrow$(i)$\Rightarrow$(iv)), hence $\overline{H_g} \cap \Delta_0^\delta \subseteq G$. As $\dim \overline{H_g} \cap \Delta_0^\delta \geq \dim G$, this inclusion is an equality and we are done. □

The next lemma is easy to prove for smooth curves (cf. [ACGH, p. 13]); our proof of the generalization is elementary and maybe known, we include it for completeness.

Lemma 5.2.3. *Let X be a hyperelliptic irreducible curve of genus $g \geq 3$; let d and r be such that $2 \leq d \leq g$ and $0 < 2r \leq d$. Then $\dim W_d^r(X) = d - 2r$.*

Proof. By definition, X is the specialization of some family of smooth hyperelliptic curves. The variety $W_d^r(C)$ of a smooth hyperelliptic curve C is irreducible of dimension $d - 2r$. Therefore, by the construction of 5.1.4, $W_d^r(X)$ has dimension at least $d - 2r$. So, it suffices to prove that every component of $W_d^r(X)$ has dimension at most $d - 2r$ and that there exists one component for which equality holds. Furthermore, using the "residuation" isomorphism

$$W_d^r(X) \xrightarrow{\cong} W_{2g-2-d}^{g-d+r-1}(X), \qquad L \mapsto K_X \otimes L^{-1}, \tag{70}$$

we can reduce ourselves to proving the result for $d \leq g - 1$.

Consider the partial normalization $\nu_n : Y_n \to X$ of one node n of X and let $\rho_r : W_d^r(X) \to W_d^r(Y_n)$ be the pull-back map. By Proposition 5.2.1 we have $\nu^* H_X = H_{Y_n} = \mathcal{O}_{Y_n}(q_1 + q_2)$, where $\nu_n^{-1}(n) = \{q_1, q_2\}$.

We use induction on δ. Suppose $\delta = 1$. We omit the subscript n (i.e. $Y = Y_n$); now $g_Y = g - 1$ and Y is a smooth hyperelliptic curve. $W_d^r(Y)$ is irreducible of dimension $d - 2r$. Let $U \subset W_d^r(Y)$ be the open dense subset $U = W_d^r(Y) \smallsetminus W_d^{r+1}(Y)$. Pick $M \in U$. Then ([ACGH, p. 13]) $M = H_Y^{\otimes r}(\sum_{i=1}^{d-2r} p_i)$ with $h^0(Y, p_i + p_j) = 1$ for all $i \neq j$. By Lemma 5.1.3, $W_M^r(X)$ is either empty or a single point; more precisely, $W_M^r(X)$ is not empty exactly when neither q_1 nor q_2 appear among the p_i (as $h^0(M - q_1 - q_2) = h^0(M \otimes H_Y^{-1}) = h^0(M) - 1$). In this case every $\nu(p_i)$ is a smooth point of X, which we call again p_i; observe that $h^0(X, H_X^{\otimes r}(\sum_{i=1}^{d-2r} p_i)) = r + 1$, therefore we necessarily have $W_M^r(X) = \{H_X^{\otimes r}(\sum_{i=1}^{d-2r} p_i)\}$. We conclude that ρ_r dominates U; more precisely, $W_d^r(X)$ has a unique irreducible component of dimension equal to $d - 2r$ dominating U. We have also found that $\rho_r^{-1}(U)$ consists of line bundles of the form $H_X^r(\sum_{i=1}^{d-2r} p_i)$ with $h^0(X, p_i + p_j) = 1$ for all $i \neq j$.

The complement $W_d^{r+1}(Y)$ of U has dimension $d - 2r - 2$ and the generic fiber of ρ_r over it is a k^*. Hence $\dim \rho_r^{-1}(W_d^{r+1}(Y)) = d - 2r - 1$, so we are done.

Now assume $\delta \geq 2$. By the induction hypothesis, $W_d^r(Y_n)$ is irreducible of dimension $d - 2r$ and $W_d^{r+1}(Y_n)$ is either empty or irreducible of dimension $d - 2r - 2$. We proceed as for $\delta = 1$; set $U = W_d^r(Y_n) \smallsetminus W_d^{r+1}(Y_n)$ so that U is irreducible of dimension $d - 2r$. By what we proved before, U contains a non-empty open subset U_n consisting of line bundles M of the form $M = H_{Y_n}^{\otimes r}(\sum_{i=1}^{d-2r} p_i)$ with $h^0(Y_n, p_i + p_j) = 1$ for all $i \neq j$. By a trivial dimension count we can disregard $U \smallsetminus U_n$ and concentrate on U_n.

Let $U_n' \subset U_n$ be the open subset of M having neither q_1 nor q_2 as base points; by Lemma 5.1.3, $W_M^r(X)$ is a single point for every $M \in U_n'$, and $W_M^r(X) = \emptyset$ if $M \notin U_n'$.

Therefore $W^r_M(X)$ has a unique irreducible component of dimension $d - 2r$ dominating U_n. The rest of the proof is the same as for $\delta - 1$. □

The next result is well known if X is non-singular.

Theorem 5.2.4. *Let X be irreducible of genus $g \geq 3$. Then*

$$\dim W^1_{g-1}(X) = \begin{cases} g-3 & \text{if } X \text{ is hyperelliptic,} \\ g-4 & \text{otherwise.} \end{cases}$$

Proof. If X is hyperelliptic this is a special case of Lemma 5.2.3, so we will assume X is not hyperelliptic. Now, for every smooth curve C of genus $g \geq 3$, every irreducible component of $W^1_{g-1}(C)$ has dimension at least $g-4$ (and equality holds if and only if C is not hyperelliptic). Therefore by 5.1.4, $\dim W^1_{g-1}(X) \geq g-4$, hence it suffices to prove that

$$\dim W^1_{g-1}(X) \leq g-4 \tag{71}$$

(i.e. every irreducible component has dimension at most $g-4$).

If $g = 3$ then $W^1_2(X)$ is empty; this follows immediately from Proposition 5.2.1 (namely, from the fact that if $W^1_2(X) \neq \emptyset$ then X is hyperelliptic). So we shall assume $g \geq 4$ from now on. Since X is not hyperelliptic, by Proposition 5.2.1 there exists a node n of X such that, denoting by $\nu : Y \to X$ the normalization of X at only n and $\{q_1, q_2\} = \nu^{-1}(n)$, we have

$$h^0(Y, q_1 + q_2) = 1. \tag{72}$$

Let us fix such a normalization, denote by $g_Y = g - 1$ the genus of Y and consider the pull-back map

$$\rho_1 : W^1_{g-1}(X) \to W^1_{g-1}(Y) = W^1_{g_Y}(Y)$$

defined by $\rho_1(L) = \nu^* L$. Recall that $W^1_{g_Y}(Y) \cong W^0_{g_Y-2}(Y)$ (by (70)), hence

$$\dim W^1_{g_Y}(Y) = \dim W^0_{g_Y-2}(Y) = g_Y - 2 = g - 3. \tag{73}$$

The fibers of ρ_1 have obviously dimension at most 1. Set $\operatorname{Im} \rho_1 = I_0 \dot\cup I_1$ where

$$I_j = \{M \in \operatorname{Im} \rho_1 : \dim \rho_1^{-1}(M) = j\}, \qquad j = 0, 1.$$

We shall prove (71) by showing that

$$\dim I_0 \leq g-4, \tag{74}$$

$$\dim I_1 \leq g-5. \tag{75}$$

To prove (74) we begin by observing that (72) is equivalent to

$$h^0(Y, \omega_Y(-q_1 - q_2)) = g_Y - 2. \tag{76}$$

Now it is easy to check that there exists a dense open subset $U \subset W^0_{g_Y-2}(Y)$ such that $h^0(Y, \omega_Y(-q_1 - q_2) \otimes N^{-1}) = 0$ for all $N \in U$ (using 2.2.5). Equivalently

$$h^0(Y, N(q_1 + q_2)) = 1, \qquad \forall N \in U. \tag{77}$$

This implies that the map

$$u : W^0_{g_Y-2}(Y) \to \operatorname{Pic}^{g_Y} Y, \qquad N \mapsto N(q_1+q_2), \tag{78}$$

satisfies

$$\dim\big(u(W^0_{g_Y-2}(Y)) \cap W^1_{g_Y}(Y)\big) < \dim W^0_{g_Y-2}(Y) = g_Y - 2. \tag{79}$$

Now by Lemma 5.1.3 we have

$$I_0 = \{M \in W^1_{g_Y}(Y) : h^0(M-q_1-q_2) = h^0(M-q_h) = 1,\ h = 1,2\}. \tag{80}$$

Therefore $I_0 \subset u(W^0_{g_Y-2}(Y)) \cap W^1_{g_Y}(Y)$; by (79) we obtain $\dim I_0 \leq g_Y - 3 = g - 4$, proving (74). To prove (75) we apply Lemma 5.1.3 to get

$$I_1 = \{M \in W^1_{g_Y}(Y) : h^0(M-q_1) = h^0(M-q_2) = h^0(M)\} \cup W^2_{g_Y}(Y); \tag{81}$$

so we set $I_1 = J_a \cup J_b$ where $J_a := \{M : h^0(M-q_h) = h^0(M) \geq 2,\ h = 1,2\}$ and $J_b := W^2_{g_Y}(Y)$.

The residuation isomorphism (70) gives

$$W^1_{g_Y-2}(Y) \cong W^2_{g_Y}(Y) = J_b. \tag{82}$$

Assume Y is hyperelliptic. Then by Lemma 5.2.3 we get $\dim J_b = g_Y - 4 = g - 5$ as wanted. Furthermore, we have an injective map

$$J_a \hookrightarrow W^1_{g_Y-2}(Y), \qquad M \mapsto M(-q_1-q_2), \tag{83}$$

hence again by Lemma 5.2.3 we derive $\dim J_a \leq \dim W^1_{g_Y-2}(Y) = g_Y - 4 = g - 5$, finishing the proof when Y is hyperelliptic. To conclude, observe that if (75) holds in the special case of Y hyperelliptic, it necessarily holds in the generic case when Y is not hyperelliptic, so we are done. ◻

Example 5.2.5. The irreducibility hypothesis on X cannot be removed from Theorem 5.2.4. To see that, let $X = C_1 \cup C_2$ be the union of two smooth curves meeting in one node n of X; let $q_i \in C_i$ be the point corresponding to that node. Recall that X is hyperelliptic if and only if $h^0(C_i, 2q_i) = 2$ for $i = 1, 2$ (cf. [CH]).

For any such X, a description of $\overline{P^{g-1}_X}$ and of its theta divisor has been given in Example 4.2.7. We identify $\overline{P^{g-1}_X} = \operatorname{Pic}^{(g_1-1,g_2-1)} C_1 \dot\cup C_2 = \operatorname{Pic}^{g_1-1} C_1 \times \operatorname{Pic}^{g_2-1} C_2$ and $\Theta(X) = (W_{g_1-1}(C_1) \times \operatorname{Pic}^{g_2-1} C_2) \cup (\operatorname{Pic}^{g_1-1} C_1 \times W_{g_2-1}(C_2))$. Thus we naturally define

$$W^1_{(g-1)}(X) = W^1_{(g_1-1,g_2-1)}(C_1 \dot\cup C_2) \subset \Theta(X).$$

Let us pick C_1 hyperelliptic of genus $g_1 \geq 3$ and C_2 non-hyperelliptic of genus $g_2 \geq 3$. Hence X is not hyperelliptic. Now we claim $W^1_{g-1}(X)$ has a component of dimension $g-3$. Indeed, consider $W^1_{g_1-1}(C_1) \times \operatorname{Pic}^{g_2-1} C_2$. Since C_1 is hyperelliptic, $\dim W^1_{g_1-1}(C_1) = g_1 - 3$, hence

$$\dim(W^1_{g_1-1}(C_1) \times \operatorname{Pic}^{g_2-1} C_2) = g_1 - 3 + g_2 = g - 3.$$

On the other hand, it is clear that $W^1_{g_1-1}(C_1) \times \mathrm{Pic}^{g_2-1} C_2 \subset W^1_{(g-1)}(X)$ (indeed, for every $M \in W^1_{g_1-1}(C_1) \times \mathrm{Pic}^{g_2-1} C_2$ we have $h^0(C_1 \dot{\cup} C_2, M) \geq 2$).

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