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The $K(\pi, 1)$ problem for the affine Artin group of type $\widetilde{B}_n$ and its cohomology

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Abstract. We prove that the complement to the affine complex arrangement of type $\widetilde{B}_n$ is a $K(\pi, 1)$ space. We also compute the cohomology of the affine Artin group $G_{\widetilde{B}_n}$ (of type $\widetilde{B}_n$) with coefficients in interesting local systems. In particular, we consider the module $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$, where the first n standard generators of $G_{\widetilde{B}_n}$ act by $(-q)$-multiplication while the last generator acts by $(-t)$-multiplication. Such a representation generalizes the analogous 1-parameter representation related to the bundle structure over the complement to the discriminant hypersurface, endowed with the monodromy action of the associated Milnor fibre. The cohomology of $G_{\widetilde{B}_n}$ with trivial coefficients is derived from the previous one.

Keywords. Affine Artin groups, twisted cohomology, group representations

1. Introduction

Let (W, S) be a Coxeter system, so a presentation for W is

$$\langle s \in S \mid (ss')^{m(s,s')} = 1 \rangle$$

where $m(s, s') \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $s \neq s'$ and $m(s, s) = 1$ (see [Bou68], [Hum90]).

The Artin group G_W associated to (W, S) is the extension of W given by the presentation (see [BS72])

$$\langle g_s,\ s \in S \mid g_s g_{s'} g_s \cdots = g_{s'} g_s g_{s'} \ldots\ (s \neq s',\ m(s, s') \text{ factors}) \rangle.$$

One says that an Artin group G_W is of *finite type* when W is finite. We are interested in *finitely generated* Artin groups, that is, when S is finite. In this case, W can be geometrically represented as a linear reflection group in $\mathbb{R}^n$ (for example, by using the *Tits representation* of W, see [Bou68]). Let $\mathcal{A}^{\mathbb{R}}$ be the arrangement of hyperplanes given by

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the mirrors of the reflections in W and let its complement be $\mathbf{Y}(\mathcal{A}^{\mathbb{R}}) := \mathbb{R}^n \setminus \bigcup_{\mathbf{H}^{\mathbb{R}} \in \mathcal{A}^{\mathbb{R}}} \mathbf{H}^{\mathbb{R}}$. The connected components of the complement $\mathbf{Y}(\mathcal{A}^{\mathbb{R}})$ are called the *chambers* of $\mathcal{A}^{\mathbb{R}}$.

Consider (for finite type) the arrangement $\mathcal{A}$ in $\mathbb{C}^n$ obtained by complexifying the hyperplanes of $\mathcal{A}^{\mathbb{R}}$ and let $\mathbf{Y}(\mathcal{A})$ be its complement. We have an induced action of W on $\mathbf{Y}(\mathcal{A})$ and it turns out that the *orbit space* $\mathbf{Y}(\mathcal{A})/W$ has the Artin group G_W as fundamental group (see [Bri73]). Moreover, it follows from a theorem by Deligne ([Del72]) that $\mathbf{Y}(\mathcal{A})/W$ is a $K(\pi, 1)$ space. Indeed, the theorem concerns a more general situation. Recall that a real arrangement $\mathcal{A}^{\mathbb{R}}$ is said to be *simplicial* if all its chambers consist of simplicial cones; reflection arrangements are known to be simplicial [Bou68].

Theorem 1.1 ([Del72]). *Let $\mathcal{A}^{\mathbb{R}}$ be a finite central arrangement, and $\mathbf{Y}(\mathcal{A})$ the complement of its complexification. If $\mathcal{A}^{\mathbb{R}}$ is simplicial, then $\mathbf{Y}(\mathcal{A})$ is a $K(\pi, 1)$ space.* □

Infinite type Coxeter groups are represented (by the Tits representation; see also [Vin71] for more general constructions) as groups of linear, not necessarily orthogonal, reflections with respect to the walls of a polyhedral cone C of maximal dimension in $\mathbf{V} = \mathbb{R}^n$. It can be shown that the union $U = \bigcup_{w \in W} wC$ of W-translates of C is a convex cone and that W acts properly on the interior U^0 of U. We may now rephrase the construction used in the finite case as follows. Let $\mathcal{A}$ be the complexified arrangement of the mirrors of the reflections in W and consider $I := \{v \in \mathbf{V} \otimes \mathbb{C} \mid \Re(v) \in U^0\}$. Then W acts freely on $\mathbf{Y} = I \setminus \bigcup_{\mathbf{H} \in \mathcal{A}} \mathbf{H}$ and we can form the orbit space $\mathbf{X} := \mathbf{Y}/W$. It is known ([vdL83]; see also [Sal94]) that G_W is indeed the fundamental group of $\mathbf{X}$, but in general it is only conjectured that $\mathbf{X}$ is a $K(\pi, 1)$. This conjecture is known to be true for: 1) Artin groups of large type ([Hen85]); 2) Artin groups satisfying the FC condition and "two-dimensional" Artin groups ([CD95]); 3) affine Artin groups of type $\widetilde{A}_n$, $\widetilde{C}_n$ ([Oko79]). In this note, we extend this result to the affine Artin group of type $\widetilde{B}_n$, showing:

Theorem 1.2. *$\mathbf{Y}(\widetilde{B}_n)$, and hence $\mathbf{X}(\widetilde{B}_n)$, are $K(\pi, 1)$ spaces.*

The idea of proof can be described in few words: up to a $\mathbb{C}^*$ factor, the orbit space is presented (through the exponential map) as a covering of the complement to a finite simplicial arrangement, so we apply Theorem 1.1.

We just digress a bit on the peculiarity of affine Artin groups. In this case the associated Coxeter group is an affine Weyl group W_a and, as such, it can be geometrically represented as a group generated by affine (orthogonal) reflections in a real vector space. This geometric representation and that given by the Tits cone are linked in a precise manner; indeed, it turns out that U_0 for an affine Weyl group is an open half-space in $\mathbf{V}$ and that W_a acts as a group of affine orthogonal reflections on a hyperplane section E of U_0. The representation on E coincides with the geometric representation and $\mathbf{Y}(W_a)$ is homotopic to the complement of the complexified affine reflection arrangement.

Our second main result is the computation of the cohomology of the group $G_{\widetilde{B}_n}$ (so, by Theorem 1.2, of $\mathbf{X}(\widetilde{B}_n)$). We consider cohomologies with interesting local coefficients, deriving from these the cohomology with trivial rational coefficients (Theorem 4.6). We take the 2-parameter representations of $G_{\widetilde{B}_n}$ over the ring $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ and over the module $\mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]]$ defined by sending the standard generator corresponding to the last node of the Dynkin diagram to $(-t)$-multiplication and the other standard generators to $(-q)$-multiplication (the minus sign is only for technical reasons). Such representations are quite natural, as we briefly explain here.

First, take any finite irreducible Coxeter group W and the corresponding arrangement $\mathcal{A} = \{\mathbf{H} \subset \mathbb{C}^n\}$ of complexified reflection hyperplanes. Consider the polynomial map defining the arrangement

$$\delta : \mathbb{C}^n \to \mathbb{C}$$

given by the product $\prod_{\mathbf{H}\in\mathcal{A}} l_{\mathbf{H}}^2$, where $l_{\mathbf{H}}$ is a linear functional defining $\mathbf{H}$. The map δ is invariant with respect to the action of the group W so it induces a map Δ on the *orbit space* $\mathbb{C}^n/W$ (an affine space in this case) such that the *discriminant* $\Sigma := \Delta^{-1}(0)$ is the image of the arrangement $\mathcal{A}$ under the projection $\pi : \mathbb{C}^n \to \mathbb{C}^n/W$. The map Δ induces a fibering over $\mathbb{C}^*$ with total space the complement $\mathbf{X}(W)$ of the discriminant, and *Milnor fibre* $\mathbf{F}_1 := \Delta^{-1}(1)$. It follows from the associated homotopy sequence that $\mathbf{F}_1$ is a $K(\pi, 1)$ space (when $\mathbf{X}(W)$ is a $K(\pi, 1)$); also, $\pi_1(\mathbf{F}_1)$ is the commutator subgroup of the Artin group G_W when the rank of the abelianization of G_W is one (cases A_n, D_n, E_i, H_i, $I_2(2p+1)$). It turns out that $\mathbf{F}_1$ is homotopy equivalent to an infinite cyclic covering of $\mathbf{X}(W)$. Let $\mathbb{Q}[q^{\pm 1}]$ be the G_W-module where standard generators act by $(-q)$-multiplication. From standard results in group cohomology it follows that the cohomology of $\mathbf{X}(W)$ with coefficients in the module $\mathbb{Q}[q^{\pm 1}]$ equals the cohomology of $\mathbf{F}_1$ with rational coefficients, where the q-action here corresponds to the natural action of the monodromy over the cohomology (for several computations in these cases see for example [Fre88], [CS98], [DPSS99], [DPS01], [Cal05]).

One can generalize this construction to 2-parameter representations when the roots have two different lengths (even in the affine case). In general, one obtains a fibration only up to homotopy: the cohomology of the orbit space $\mathbf{X}(W)$ with coefficients in $\mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]]$ equals the cohomology with rational coefficients both of the homotopy Milnor fibre $\mathbf{F}$ and of the corresponding abelian covering of $\mathbf{X}(W)$. When $\mathbf{X}(W)$ is a $K(\pi, 1)$ space, such cohomology equals also that of the fundamental group of $\mathbf{F}$: in our case, this is the commutator subgroup of G_W.

The main tool to perform computations is an algebraic complex which was discovered in [Sal94], [DS96] by using topological methods (and independently, by algebraic methods in [Squ94]). The cohomology factorizes into two parts (see also [DPSS99]): the *invariant* part reduces to that of the Artin group of finite type B_n, whose 2-parameter cohomology was computed in [CMS06]; for the *anti-invariant* part we use suitable filtrations and the associated spectral sequences.

Let φ_d be the d-th cyclotomic polynomial in the variable q. We define the quotient rings

$$\begin{aligned}
\{1\}_i &= \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]/(1 + tq^i),\\
\{d\}_i &= \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]/(\varphi_d, 1 + tq^i),\\
\{\{d\}\}_j &= \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]/\Big(\varphi_d, \prod_{i=0}^{d-1} 1 + tq^i\Big)^j.
\end{aligned}$$

The final result is the following:

Theorem 1.3. *The cohomology* $H^{n-s}(G_{\widetilde{B}_n}, \mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]])$ *is given by*

$$\mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]] \quad \textit{for } s = 0,$$

$$\bigoplus_{h>0} \{\{2h\}\}_{f(n,h)} \quad \textit{for } s = 1,$$

$$\bigoplus_{\substack{h>2 \\ i\in I(n,h)}} \{2h\}_i^{c(n,h,s)} \oplus \bigoplus_{\substack{d\mid n \\ 0\le i\le d-2}} \{d\}_i \oplus \{1\}_{n-1} \quad \textit{for } s = 2,$$

$$\bigoplus_{\substack{h>2 \\ i\in I(n,h)}} \{2h\}_i^{c(n,h,s)} \oplus \bigoplus_{\substack{d\mid n \\ 0\le i\le d-2 \\ d\le n/(j+1)}} \{d\}_i \quad \textit{for } s = 2+2j,$$

$$\bigoplus_{\substack{h>2 \\ i\in I(n,h)}} \{2h\}_i^{c(n,h,s)} \oplus \bigoplus_{\substack{d\nmid n \\ d\le n/(j+1)}} \{d\}_{n-1} \quad \textit{for } s = 3+2j,$$

where $c(n, h, s) = \max(0, \lfloor n/2h \rfloor - s)$*,* $f(n, h) = \lfloor (n + h - 1)/2h \rfloor$ *and* $I(n, h) = \{n, \ldots, n+h-2\}$ *if* $n \equiv 0, 1, \ldots, h \bmod (2h)$ *and* $I(n, h) = \{n+h-1, \ldots, n+2h-1\}$ *if* $n \equiv h+1, h+2, \ldots, 2h-1 \bmod (2h)$.

The paper is organized as follows. In Section 2 we recall some results and notations about Coxeter and Artin groups, including a 2-parameter Poincaré series which we need in the boundary operators of the above mentioned algebraic complex. In Section 3 we prove Theorem 1.2. In Section 4 we use a suitable filtration of the algebraic complex, reducing computation of the cohomology mainly to:

- calculation of generators of certain subcomplexes for the Artin group of type D_n (whose cohomology was known from [DPSS99], but we need explicit suitable generators);
- analysis of the associated spectral sequence to deduce the cohomology of $\widetilde{B}_n$ with local coefficients;
- use of some exact sequences for the cohomology with costant coefficients.

In this paper we prove that the complement to the affine complex arrangement of type $\widetilde{B}_n$ is a $K(\pi, 1)$ space. We also compute the cohomology of the affine Artin group $G_{\widetilde{B}_n}$ (of type $\widetilde{B}_n$) with coefficients in several interesting local systems. In particular, we consider the module $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$, where the first n standard generators of $G_{\widetilde{B}_n}$ act by $(-q)$-multiplication while the last generator acts by $(-t)$-multiplication. Such a representation generalizes the analogous 1-parameter representation related to the bundle structure over the complement to the discriminant hypersurface, endowed with the monodromy action of the associated Milnor fibre. The cohomology of $G_{\widetilde{B}_n}$ with trivial coefficients is derived from the previous one.

2. Preliminary results

In this section we fix the notation and recall some preliminary results. We will use classical facts ([Bou68], [Hum90]) without further reference.

2.1. *Coxeter groups and Artin braid groups*

A *Coxeter graph* is a finite undirected graph, whose edges are labelled with integers ≥ 3 or with the symbol ∞.

Let S, E be respectively the vertex and edge set of a Coxeter graph. For every edge $\{s, t\} \in E$ let $m_{s,t}$ be its label. If $s, t \in S$ $(s \neq t)$ are not joined by an edge, set by convention $m_{s,t} = 2$. Let also $m_{s,s} = 1$.

Two groups are associated to a Coxeter graph (as in the Introduction): the *Coxeter group* W defined by

$$W = \langle s \in S \mid (st)^{m_{s,t}} = 1 \ \forall s, t \in S \text{ such that } m_{s,t} \neq \infty \rangle$$

and the *Artin braid group* G_W defined by (see [BS72], [Bri73], [Del72]):

$$G = \langle s \in S \mid \underbrace{stst\ldots}_{m_{s,t} \text{ terms}} = \underbrace{tsts\ldots}_{m_{s,t} \text{ terms}} \ \forall s, t \in S \text{ such that } m_{s,t} \neq \infty \rangle.$$

There is a natural epimorphism $\pi : G_W \to W$ and, by Matsumoto's Lemma [Mat64], π admits a canonical set-theoretic section $\psi : W \to G_W$.

2.2. *Some reflection arrangements*

In this paper, we are primarily interested in Artin braid groups associated to Coxeter graphs of type B_n, $\widetilde{B}_n$ and D_n (see Table 1).

The associated Coxeter groups can be described as reflection groups with respect to an arrangement of hyperplanes (or mirrors). Let $x_1, \ldots, x_n$ be the standard coordinates in $\mathbb{R}^n$. Consider the linear hyperplanes:

$$\mathbf{H}_k = \{x_k = 0\}, \quad \mathbf{L}_{ij}^{\pm} = \{x_i = \pm x_j\},$$

and, for an integer $a \in \mathbb{Z}$, their affine translates:

$$\mathbf{H}_k(a) = \{x_k = a\}, \quad \mathbf{L}_{ij}^{\pm}(a) = \{x_i = \pm x_j + a\}.$$

The Coxeter group B_n is identified with the group of reflections with respect to the mirrors in the arrangement

$$\mathcal{A}(B_n) := \{\mathbf{H}_k \mid 1 \leq k \leq n\} \cup \{\mathbf{L}_{ij}^{\pm} \mid 1 \leq i < j \leq n\}.$$

As such, it is the group of signed permutations of the coordinates in $\mathbb{R}^n$. Notice that B_n is generated by n basic reflections $s_1, \ldots, s_n$ having respectively as mirrors the $n - 1$

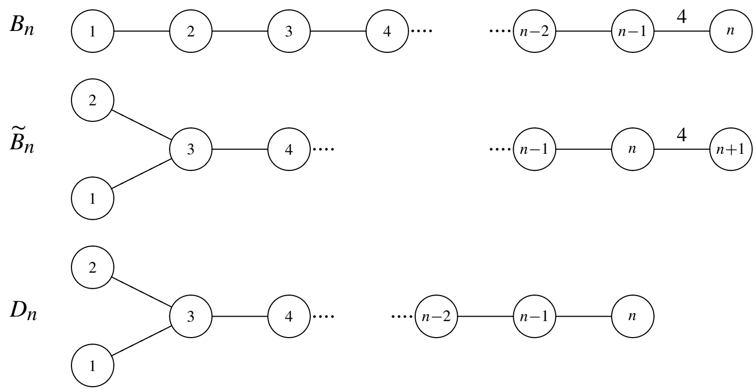

Table 1. Coxeter graphs of type B_n, $\widetilde{B}_n$, D_n.

hyperplanes $\mathbf{L}^+_{i,i+1}$ $(1 \le i \le n-1)$ and the hyperplane $\mathbf{H}_n$. This numbering of the reflections is consistent with the numbering of the vertices of the Coxeter graph for B_n shown in Table 1.

The affine Coxeter group $\widetilde{B}_n$ is the semidirect product of the Coxeter group B_n and the coroot lattice, consisting of integer vectors whose coordinates add up to an even number. The arrangement of mirrors is then the affine hyperplane arrangement:

$$\mathcal{A}(\widetilde{B}_n) := \{\mathbf{H}_k(a) \mid 1 \le k \le n,\ a \in \mathbb{Z}\} \cup \{\mathbf{L}^{\pm}_{ij}(a) \mid 1 \le i < j \le n,\ a \in \mathbb{Z}\}. \qquad (1)$$

It is generated by the basic reflections for B_n plus an extra affine reflection $\widetilde{s}$ having $\mathbf{L}^-_{12}(1)$ as mirror. The latter commutes with all the basic reflections of B_n but s_2, for which $(\widetilde{s}s_2)^3 = 1$. This accounts for the Coxeter graph of type $\widetilde{B}_n$ in the table, where, however, we chose for our convenience a somewhat unusual vertex numbering.

Finally, the group D_n has reflection arrangement

$$\mathcal{A}(D_n) := \{\mathbf{L}^{\pm}_{ij} \mid 1 \le i < j \le n\}$$

and it can be regarded as the group of signed permutations of the coordinates which involve an even number of sign changes. In particular, D_n is a subgroup of index 2 in B_n. The group is generated by n basic reflections with respect to the hyperplanes $\mathbf{L}^-_{12}$ and $\mathbf{L}^+_{i,i+1}$ $(1 \le i \le n-1)$.

2.3. Generalized Poincaré series

For future use in cohomology computations, we will need some analog of ordinary Poincaré series for Coxeter groups. Consider a domain R and let R^* be the group of units of R. Given an abelian representation

$$\eta : G_W \to R^*$$

of the Artin group G_W and a finite subset $U \subset W$, we may consider the η-Poincaré series

$$U(\eta) = \sum_{w \in U} (-1)^{\ell(w)} \eta(\psi w) \in R$$

where ℓ is the length in the Coxeter group and $\psi : W \to G_W$ is the canonical section. In particular, when W is finite, we say that $W(\eta)$ is the η-Poincaré series of the group. Notice that for $R = \mathbb{Q}[q^{\pm 1}]$ we may consider the representation η_q that sends the standard generators of G_W into $(-q)$-multiplication; in this situation we recover the ordinary Poincaré series:

$$W(\eta_q) = W(q).$$

Further, for the Artin group of type $W = B_n,\ \widetilde{B}_n$ we are interested in the representation

$$\eta_{q,t} : G_W \to \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$$

defined by sending the last standard generator (the one lying in the tree leaf labelled 4) to $(-t)$-multiplication and the remaining ones to $(-q)$-multiplication. The associated Poincaré series $B_n(q,t) := B_n(\eta_{q,t})$ will be called the (q,t)-*weighted Poincaré series* for B_n.

In order to recall closed formulas for Poincaré series, we first fix some notations that will be adopted throughout the paper. We define the q-analog of a positive integer m to be the polynomial

$$[m]_q := 1 + q + \cdots + q^{m-1} = \frac{q^m - 1}{q - 1}.$$

It is easy to see that $[m]_q = \prod_{i|m} \varphi_m(q)$. Moreover, we define the q-factorial and double factorial inductively as:

$$[m]_q! := [m]_q \cdot [m-1]_q!, \qquad [m]_q!! := [m]_q \cdot [m-2]_q!!,$$

where it is understood that $[1]! = [1]!! = [1]$ and $[2]!! = [2]$. A q-analog of the binomial $\binom{m}{i}$ is given by the polynomial

$$\begin{bmatrix} m \\ i \end{bmatrix}_q := \frac{[m]_q!}{[i]_q![m-i]_q!}.$$

We can also define the (q,t)-analog of an even number

$$[2m]_{q,t} := [m]_q(1 + tq^{m-1})$$

and of the double factorial

$$[2m]_{q,t}!! := \prod_{i=1}^{m} [2i]_{q,t} \ = \ [m]_q! \prod_{i=0}^{m-1} (1 + tq^i).$$

Notice that specializing t to q, we recover the q-analogue of an even number and of its double factorial. Finally, we define the polynomial

$$\begin{bmatrix} m \\ i \end{bmatrix}'_{q,t} := \frac{[2m]_{q,t}!!}{[2i]_{q,t}!![m-i]_q!} = \begin{bmatrix} m \\ i \end{bmatrix}_q \prod_{j=i}^{m-1}(1+tq^j). \tag{2}$$

With this notation the ordinary Poincaré series for D_n and B_n may be written as

$$D_n(q) := \sum_{w \in D_n} q^{\ell(w)} = [2(n-1)]_q!! \cdot [n]_q, \tag{3}$$

$$B_n(q) := \sum_{w \in B_n} q^{\ell(w)} = [2n]_q!!, \tag{4}$$

while the (q,t)-*weighted Poincaré series* for B_n is given by (see e.g. [Rei93])

$$B_n(q,t) = [2n]_{q,t}!!. \tag{5}$$

3. The $K(\pi, 1)$ problem for the affine Artin group of type $\widetilde{B}_n$

Using the explicit description of the reflection mirrors in (1), the complement of the complexified affine reflection arrangement of type $\widetilde{B}_n$ is given by

$$\mathbf{Y} := \mathbf{Y}(\widetilde{B}_n) = \{x \in \mathbb{C}^n \mid x_i \pm x_j \notin \mathbb{Z} \text{ for all } i \neq j,\ x_k \notin \mathbb{Z} \text{ for all } k\}.$$

On $\mathbf{Y}$ we have, by standard facts, a free action by translations of the coweight lattice Λ, identified with the standard lattice $\mathbb{Z}^n \subset \mathbb{C}^n$.

Proof of Theorem 1.2. We first explicitly describe the covering $\mathbf{Y} \to \mathbf{Y}/\Lambda$ applying the exponential map $y = \exp(2\pi \mathrm{i} x)$ componentwise to $\mathbf{Y}$:

$$\begin{aligned} \mathbf{Y} \xrightarrow{\pi} \mathbf{Y}/\Lambda \simeq \{y \in \mathbb{C}^n \mid y_i \neq y_j^{\pm 1},\ y_k \neq 0, 1\}, \\ (x_1, \ldots, x_n) \mapsto (\exp(2\pi \mathrm{i} x_1), \ldots, \exp(2\pi \mathrm{i} x_n)). \end{aligned}$$

Notice now that the function

$$\mathbb{C} \setminus \{0,1\} \ni y \mapsto g(y) = \frac{1+y}{1-y} \in \mathbb{C} \setminus \{\pm 1\}$$

satisfies $g(y^{-1}) = -g(y)$. Further, g is invertible, its inverse being given by $z \mapsto \frac{z-1}{z+1}$. Therefore applying g componentwise to $\mathbf{Y}/\Lambda$, we have

$$\mathbf{Y}/\Lambda \simeq \{z \in \mathbb{C}^n \mid z_i \neq \pm z_j,\ z_k \neq \pm 1\}$$

Consider now the arrangement $\mathcal{A}$ in $\mathbb{R}^{n+1}$ consisting of the hyperplanes $\mathbf{L}_{ij}^{\pm}$ for $1 \leq i < j \leq n+1$ and $\mathbf{H}_1$, and let $\mathbf{Y}(\mathcal{A})$ be the complement of its complexification. We have a homeomorphism

$$\eta : \mathbb{C}^* \times \mathbf{Y}/\Lambda \to \mathbf{Y}(\mathcal{A})$$

defined by

$$\eta\big(\lambda, (z_1, \dots, z_n)\big) = (\lambda, \lambda z_1, \dots, \lambda z_n).$$

To show that $\mathbf{Y}/\Lambda$ is a $K(\pi, 1)$, it is then sufficient to show that $\mathbf{Y}(\mathcal{A})$ is a $K(\pi, 1)$. We will show in Lemma 3.1 below that $\mathcal{A}$ is simplicial, and therefore the result follows from Deligne's Theorem 1.1. □

Remark. By the same exponential argument one may recover the results of [Oko79] for the affine Artin group of type $\widetilde{A}_n$, $\widetilde{C}_n$ (for further applications we refer to [All02]).

Lemma 3.1. *Let $\mathcal{A}$ be the real arrangement in $\mathbb{R}^{n+1}$ consisting of the hyperplanes $\mathbf{L}^{\pm}_{ij}$ for $1 \le i < j \le n+1$ and $\mathbf{H}_1$. Then $\mathcal{A}$ is simplicial.*

Proof. Notice that $\mathcal{A}$ is the union of the reflection arrangement $\mathcal{A}(D_{n+1})$ of type D_{n+1} and the hyperplane $\mathbf{H}_1 = \{x_1 = 0\}$. Hence we study how the chambers of $\mathcal{A}(D_{n+1})$ are cut by the hyperplane $\mathbf{H}_1$. Since the Coxeter group D_{n+1} acts transitively on the collection of chambers, it is enough to consider how the fundamental chamber $\mathbf{C}_0$ of $\mathcal{A}(D_{n+1})$ is cut by the D_{n+1}-translates of the hyperplane $\mathbf{H}_1$, i.e. by the coordinate hyperplanes $\mathbf{H}_k$ for $k = 1, \dots, n+1$.

We may choose

$$\mathbf{C}_0 = \{-x_2 < x_1 < x_2 < \cdots < x_n < x_{n+1}\}$$

as fundamental chamber. Of course, this is a simplicial cone. Notice that the coordinates of a point in $\mathbf{C}_0$ are all positive except (possibly) the first. Thus it is clear that for $k \ge 2$ the hyperplanes $\mathbf{H}_k$ do not cut $\mathbf{C}_0$.

A quick check shows instead that $\mathbf{H}_1$ cuts $\mathbf{C}_0$ into two simplicial cones $\mathbf{C}_1$, $\mathbf{C}_2$ given precisely by

$$\mathbf{C}_1 = \{0 < x_1 < x_2 < \cdots < x_n < x_{n+1}\},$$
$$\mathbf{C}_2 = \{0 < -x_1 < x_2 < \cdots < x_n < x_{n+1}\}.$$

□

4. Cohomology

In this section we will compute the cohomology groups

$$H^*(G_{\widetilde{B}_n}, \mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]]_{q,t})$$

where $\mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]]_{q,t}$ is the local system over the module of Laurent series $\mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]]$ and the action is $(-q)$-multiplication for the standard generators associated to the first n nodes of the Dynkin diagram, while it is $(-t)$-multiplication for the generator associated to the last node.

4.1. Algebraic complexes for Artin groups

As a main tool for cohomological computations we use the algebraic complex described in [Sal94] (see the Introduction); the algebraic generalization of this complex by De Concini–Salvetti [DS96] provides an effective way to determine the cohomology of the orbit space $X(W)$ with values in an arbitrary G_W-module. When $X(W)$ is a $K(\pi, 1)$ space, of course, we get the cohomology of the group G_W.

For simplicity, we restrict ourselves to the abelian representations considered in Section 2.3. Let (W, S) be a Coxeter system. Given a representation $\eta : G_W \to R^*$, let M_η be the induced structure of G_W-module on an arbitrary (even non-free) R-module M. We may describe a cochain complex $C^*(W)$ for the cohomology $H^*(X(W); M_\eta)$ as follows. The cochains in dimension k form the free R-module indexed by the finite parabolic subgroups of W:

$$C^k(W) := \bigoplus_{\substack{\Gamma : |\Gamma|=k \\ |W_\Gamma|<\infty}} M.e_\Gamma \tag{6}$$

and the coboundary maps are completely described by the formula

$$\mathrm{d}(e_\Gamma) = \sum_{\substack{\Gamma'\supset\Gamma \\ |\Gamma'|=|\Gamma|+1 \\ |W_{\Gamma'}|<\infty}} (-1)^{\alpha(\Gamma,\Gamma')} \frac{W_{\Gamma'}(\eta)}{W_\Gamma(\eta)} e_{\Gamma'} \tag{7}$$

where $W_\Gamma(\eta)$ is the η-Poincaré series of the parabolic subgroup W_Γ and $\alpha(\Gamma, \Gamma')$ is an incidence index depending on a fixed linear order of S. For $\Gamma' \setminus \Gamma = \{s'\}$ it is defined as

$$\alpha(\Gamma, \Gamma') := |\{s \in \Gamma \mid s < s'\}|.$$

We identify (consistently with Table 1) the generating reflections set S for $\widetilde{B}_n$ with the set $\{1, \ldots, n+1\}$. It is useful to represent a subset $\Gamma \subset S$ by its characteristic function. For example the subset $\{1, 3, 5, 6\}$ for $\widetilde{B}_6$ may be represented as the binary string

$$\begin{smallmatrix}0\\1\end{smallmatrix}10110$$

To determine the cohomology of $G_{\widetilde{B}_n}$, it will be necessary to give a close look at the cohomology of G_{D_n}. It is convenient to number the vertices of D_n as in Table 1 and to regard parabolic subgroups as binary strings as before.

4.2. Change of coefficients

Let R be the ring of Laurent polynomials $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ and M be the R-module of Laurent series $\mathbb{Q}[[q^{\pm 1}, t^{\pm 1}]]$ and let $R_{q,t}$, $M_{q,t}$ be the corresponding local systems, with action $\eta_{q,t}$. Our main interest is to compute the cohomology with trivial rational coefficients of the group

$$Z_{\widetilde{B}_n} = \ker(G_{\widetilde{B}_n} \to \mathbb{Z}^2)$$

that is the commutator subgroup of $G_{\widetilde{B}_n}$ (see the Introduction for some motivations). By the Shapiro Lemma (see [Bro82]) we have the following isomorphism:

$$H^*(Z_{\widetilde{B}_n}, \mathbb{Q}) \simeq H^*(G_{\widetilde{B}_n}, M_{q,t})$$

and the second term of the equality is computed by the Salvetti complex $C^*(\widetilde{B}_n)$ over the module $M_{q,t}$. Notice that the finite parabolic subgroups of $W_{\widetilde{B}_n}$ are in 1-1 correspondence with the proper subsets of the set S of simple roots.

We can define an *augmented* Salvetti complex $\widehat{C}^*(\widetilde{B}_n)$ as follows:

$$\widehat{C}^*(\widetilde{B}_n) = C^*(\widetilde{B}_n) \oplus (M_{q,t}).e_S.$$

We need to define the boundary map for the n-dimensional generators. Let us first define a quasi-Poincaré polynomial for $G_{\widetilde{B}_n}$. We set

$$\widehat{W}_S(q,t) = \widehat{W}_{\widetilde{B}_n}(q,t) = [2(n-1)]!!\,[n] \prod_{i=0}^{n-1}(1+tq^i).$$

It is easy to verify that $\widehat{W}_{\widetilde{B}_n}(q,t)$ is the least common multiple of all $W_\Gamma(q,t)$ for $\Gamma \subset S$ with $|\Gamma| = n$. This allows us to define the boundary map for the generators e_Γ with $|\Gamma| = n$:

$$d(e_\Gamma) = (-1)^{\alpha(\Gamma,S)} \frac{\widehat{W}_{\widetilde{B}_n}(q,t)}{W_\Gamma(q,t)} e_S,$$

and it is straightforward to verify that $\widehat{C}^*(\widetilde{B}_n)$ is still a chain complex. Moreover, we have the following relations between the cohomologies of $C^*(\widetilde{B}_n)$ and $\widehat{C}^*(\widetilde{B}_n)$:

$$H^i(C^*(\widetilde{B}_n)) = H^i(\widehat{C}^*(\widetilde{B}_n))$$

for $i \neq n, n+1$ and we have the short exact sequence

$$0 \to H^n(\widehat{C}^*(\widetilde{B}_n), M_{q,t}) \to H^n(C^*(\widetilde{B}_n), M_{q,t}) \to M_{q,t} \to 0.$$

Finally, one can prove that the complex $\widehat{C}^*(\widetilde{B}_n)$ with coefficients in the local system $R_{q,t}$ is *well filtered* (as defined in [Cal05]) with respect to the variable t and so it gives the same cohomology, modulo an index shifting, of the complex with coefficients in the module $\mathbb{Q}[t^{\pm 1}][[q^{\pm 1}]]$. Another index shifting can be proved with a slight improvement of the results in [Cal05], allowing one to pass to the module M. Hence we have the following

Proposition 4.1.

$$H^i(Z_{\widetilde{B}_n}, \mathbb{Q}) \simeq H^i(\widehat{C}^*(\widetilde{B}_n), M_{q,t}) \simeq H^{i+2}(\widehat{C}^*(\widetilde{B}_n), R_{q,t}) \simeq H^{i+2}(G_{\widetilde{B}_n}, R_{q,t})$$

for $i \neq n, n+1$ *and*

$$H^n(Z_{\widetilde{B}_n}, \mathbb{Q}) \simeq H^n(G_{\widetilde{B}_n}, M_{q,t}) \simeq M, \qquad H^{n+1}(Z_{\widetilde{B}_n}, \mathbb{Q}) \simeq H^{n+1}(G_{\widetilde{B}_n}, M_{q,t}) \simeq 0. \quad \square$$

From now on we deal only with the complex $\widehat{C}^*(\widetilde{B}_n)$ with coefficients in the local system $R_{q,t}$.

4.3. Splitting of the complex

For Coxeter groups of type $W = D_n, \widetilde{B}_n$ the Salvetti complex C^*W exhibits an involution σ defined by

$$\begin{aligned} {}^0_0A &\mapsto {}^0_0A, & \qquad {}^1_1A &\mapsto -{}^1_1A, \\ {}^0_1A &\mapsto {}^1_0A, & \qquad {}^1_0A &\mapsto {}^0_1A. \end{aligned}$$

Let I^*W be the module of σ-invariants and K^*W the module of σ-anti-invariants. We may then split the complex into

$$C^*W = I^*W \oplus K^*W.$$

In particular, the computation of the cohomology of C^*W may be performed by analyzing separately the two subcomplexes.

4.4. Cohomology of K^*D_n

The cohomology of the anti-invariant subcomplex for D_n was completely determined in [DPSS99]. However, we will need generators for the cohomology groups, which are not easily deduced from the argument in the original paper. So we briefly recall this result.

Let G_n^1 be the subcomplex of $C(D_n)$ generated by the strings of type ${}^0_1A$ and ${}^1_1A$. It is easy to see that G_n^1 is isomorphic (as a complex) to $K(D_n)$.

Define the set

$$S_n = \{h \in \mathbb{N} \mid 2h \mid n, \text{ or } h \mid n-1 \text{ and } 2h \nmid (n-1)\}.$$

Note that h appears in S_n if and only if $n = 2\lambda h$ (i.e. n is an even multiple of h) or $n = (2\lambda + 1)h + 1$ (n is an odd multiple of h incremented by 1).

We introduce the following notation, similar to the one given in the first section: we write

$$\{h\}$$

without a subscript for the module

$$\mathbb{Q}[q^{\pm 1}]/(\varphi_h).$$

Proposition 4.2 ([DPSS99]). *The top cohomology of G_n^1 is*

$$H^n G_n^1 = \bigoplus_{h \in S_n} \{2h\},$$

whereas for $s > 0$ one has

$$H^{n-2s} G_n^1 = \bigoplus_{\substack{h \in S_n \\ 1 < h < n/2s}} \{2h\}, \qquad H^{n-2s+1} G_n^1 = \bigoplus_{\substack{h \in S_n \\ 1 < h \leq n/2s}} \{2h\}. \qquad \square$$

We need a description of the generators for these modules. First we define the following basic binary strings:

$$o_\mu[h] = \begin{cases} \begin{smallmatrix}0\\1\end{smallmatrix} 1^{h-1} & \text{for } \mu = 0, \\ \begin{smallmatrix}1\\1\end{smallmatrix} 1^{2\mu h-2}01^h & \text{for } \mu \geq 1, \end{cases}$$

$$e_\mu[h] = \begin{smallmatrix}1\\1\end{smallmatrix} 1^{(2\mu-1)h-1}01^{h-2} \quad \text{for } \mu \geq 1,$$

$$s_h = 01^{h-2}, \qquad l_h = 01^h.$$

A set of candidate cohomology generators is given by the following cocycles:

$$o_{\mu,2i}[h] = \frac{1}{\varphi_{2h}} d(o_\mu[h](s_h l_h)^i), \qquad e_{\mu,2i}[h] = \frac{1}{\varphi_{2h}} d(e_\mu[h](l_h s_h)^i),$$

$$o_{\mu,2i+1}[h] = \frac{1}{\varphi_{2h}} d(o_\mu[h](s_h l_h)^i s_h), \qquad e_{\mu,2i+1}[h] = \frac{1}{\varphi_{2h}} d(e_\mu[h](l_h s_h)^i l_h).$$

Indeed, these cocycles account for all the generators:

Proposition 4.3. (1) *Let $n = 2\lambda h$. Then for $0 \leq s < \lambda$ the summand of $H^{n-2s}(G_n^1)$ isomorphic to $\{2h\}$ is generated by $e_{\lambda-s,2s}[h]$. Similarly for $0 \leq s < \lambda$ the summand of $H^{n-2s-1}(G_n^1)$ is generated by $o_{\lambda-s-1,2s+1}[h]$.*

(2) *Let $n = (2\lambda + 1)h + 1$. Then for $0 \leq s \leq \lambda$ the summand of $H^{n-2s}(G_n^1)$ isomorphic to $\{2h\}$ is generated by $o_{\lambda-s,2s}[h]$. For $0 \leq s < \lambda$ the summand of $H^{n-2s-1}(G_n^1)$ is generated by $e_{\lambda-s,2s+1}[h]$.*

Proposition 4.3 is best proven by induction on n, recovering in particular the quoted result from [DPSS99].

Proof. We filter the complex G_n^1 from the right and use the associated spectral sequence. Let

$$F_k G_n^1 = \langle A1^k \rangle$$

be the subcomplex generated by binary strings ending with at least k ones. We have a filtration

$$G_n^1 = F_0 G_n^1 \supset F_1 G_n^1 \supset \cdots \supset F_{n-2} G_n^1 \supset F_{n-1} G_{n-1}^1 \supset 0$$

in which the subsequent quotients for $k = 1, \ldots, n-3$,

$$\frac{F_k G_n^1}{F_{k+1} G_n^1} = \langle A01^k \rangle \simeq G_{n-k-1}^1[k],$$

are isomorphic to the complex for G_{n-k-1}^1 shifted in degree by k, while

$$\frac{F_{n-2} G_n^1}{F_{n-1} G_n^1} = \left\langle \begin{smallmatrix}0\\1\end{smallmatrix} 1^{n-2} \right\rangle \simeq R[n-1], \qquad F_{n-1} G_n^1 = \left\langle \begin{smallmatrix}1\\1\end{smallmatrix} 1^{n-2} \right\rangle \simeq R[n].$$

Therefore the columns of the E_1 term of the spectral sequence are either the module R or are given by the cohomology of $G^1_{n'}$ with $n' < n$. Reasoning by induction, we may thus suppose that their cohomology has the generators prescribed by the proposition. Since there can be no non-zero maps between the modules $\{2h\}$, $\{2h'\}$ for $h \neq h'$, we may separately detect the φ_{2h}-torsion in the cohomology.

Fix an integer $h > 1$. Then the relevant modules for the φ_{2h}-torsion in the E_1 term are suggested in Table 2. We will call a column *even* if it is relative to $G^1_{2\mu h}$, and *odd* if it is relative to $G^1_{(2\mu+1)h+1}$ for some μ. The differential d_1 is zero everywhere but

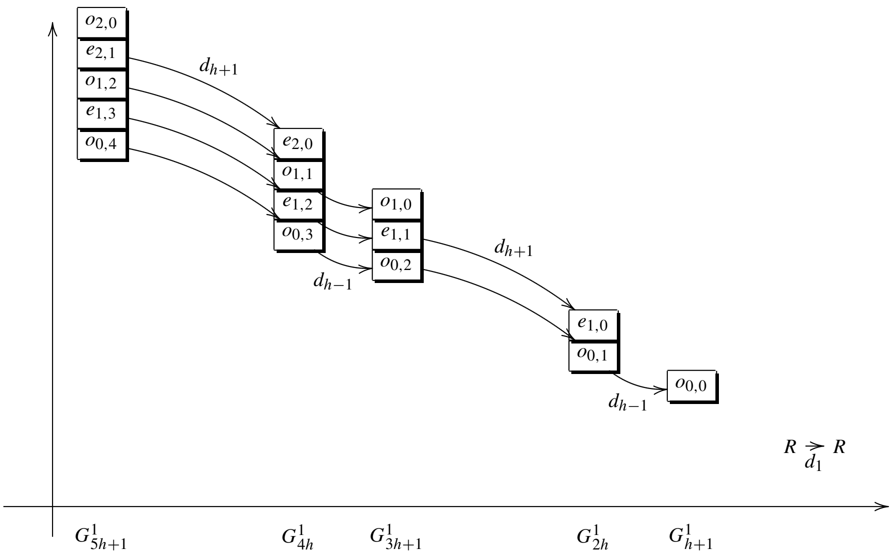

Table 2. Spectral sequence for G^1_n.

$d_1 : E_1^{(n-2,1)} \to E_1^{(n-1,1)}$ where it is given by multiplication by $[2(n-1)]!!/[n-1]!$. Thus the E_2 term differs from the E_1 only in positions $(n-2, 1)$ and $(n-1, 1)$, where

$$E_2^{(n-2,1)} = 0, \qquad E_2^{(n-1,1)} = \frac{R}{[2(n-1)]!!/[n-1]!}.$$

Then all other differentials are zero up to d_{h-2}.

It is now useful to distinguish among four cases according to the remainder of n mod$(2h)$:

(a) $n = 2\lambda h + c$ for $1 \leq c \leq h$,
(b) $n = (2\lambda + 1)h + 1$,
(c) $n = (2\lambda + 1)h + 1 + c$ for $1 \leq c \leq h - 2$,
(d) $n = 2\lambda h$.

In case (a), note the first column relevant for φ_{2h}-torsion is even (see also Table 3).

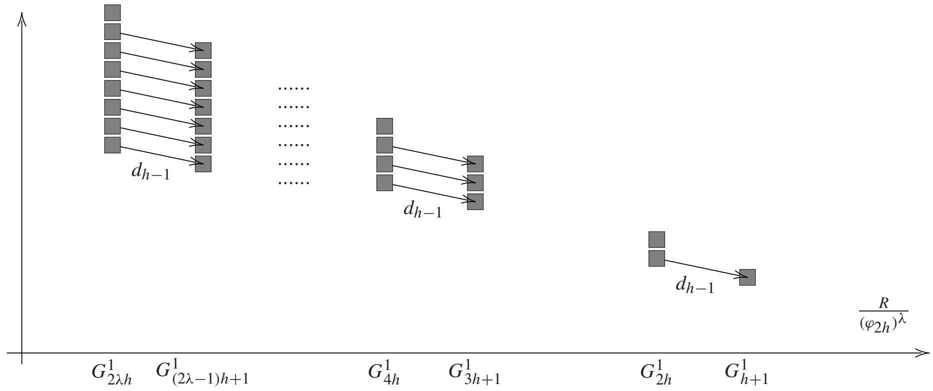

Table 3. E_{h-1} term of the spectral sequence for G^1_n in case (a).

The differential d_{h-1} maps the modules of positive codimension of an even column $G^1_{2\mu h}$ $(1 \leq \mu \leq \lambda)$ to those in the odd column $G^1_{(2\mu-1)h+1}$. Using the suitable generators of type $e_{\cdot,\cdot}[h]$, $o_{\cdot,\cdot}[h]$, the map d_{h-1} may be identified with multiplication by

$$\begin{bmatrix} n-(2\mu-1)h-1 \\ h-1 \end{bmatrix} = \begin{bmatrix} 2(\lambda-\mu)+c+h-1 \\ h-1 \end{bmatrix}. \tag{8}$$

Since this polynomial is not divisible by φ_{2h}, the restriction of d_{h-1} to positive codimension elements in even columns is injective. It follows that in the E_h term the only survivors are in positions $(c+2(\lambda-\mu)h-1, 2\mu h)$, generated by $e_{\mu,0}[h]$ and

$$E_h^{(n-1,1)} \simeq E_2^{(n-1,1)} = \frac{R}{[2(n-1)]!!/[n-1]!}.$$

Note that in $E_h^{(n-1,1)}$ the only torsion of type φ^l_{2h} is given by the summand

$$\frac{R}{(\varphi_{2h})^{\lambda}}$$

The setup is summarized in Table 4. In the table the survivors are in dark grey boxes while annihilated terms are in light grey.

Further, using the generators and up to an invertible, we may identify the differential $d_{2\mu h} : E_{2\mu h}^{(c+2(\lambda-\mu)h-1,2\mu h)} \to E_{2\mu h}^{n-1,1}$ with multiplication by $\varphi_{2h}^{\lambda-\mu}$ $(1 \leq \mu \leq \lambda)$. Thus, for example, in the E_{2h+1} term the module in position $(c+2(\lambda-1)h-1, 2h)$ vanishes and the φ_{2h}-torsion in $E_{2h+1}^{(n-1,1)}$ is reduced to $R/(\varphi_{2h})^{\lambda-1}$. Continuing in this way, all φ_{2h}-torsion vanishes. In summary there is no φ_{2h}-torsion in the cohomology of G^1_n; this ends case (a).

For case (b), the first column in the spectral sequence relevant for φ_{2h} is still even. The differential d_{h-1} may be identified again as multiplication as in formula (8), but now it vanishes, since the polynomial is divisible by φ_{2h}.

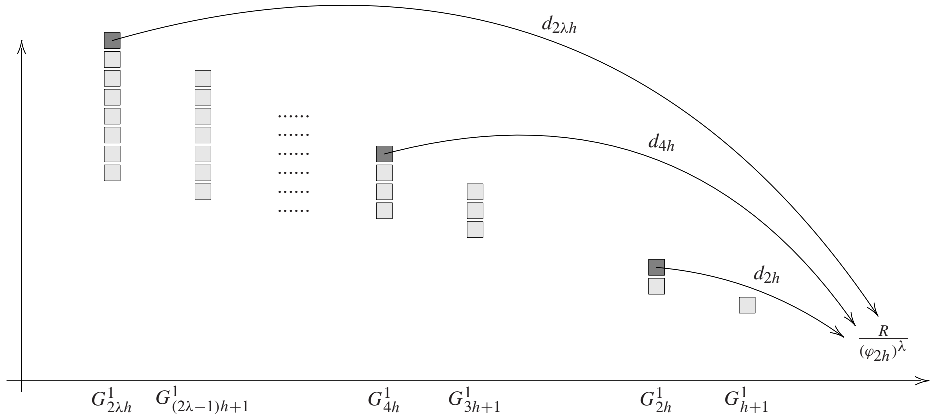

Table 4. Setup for the higher degree terms in the spectral sequence for G^1_n in case (a).

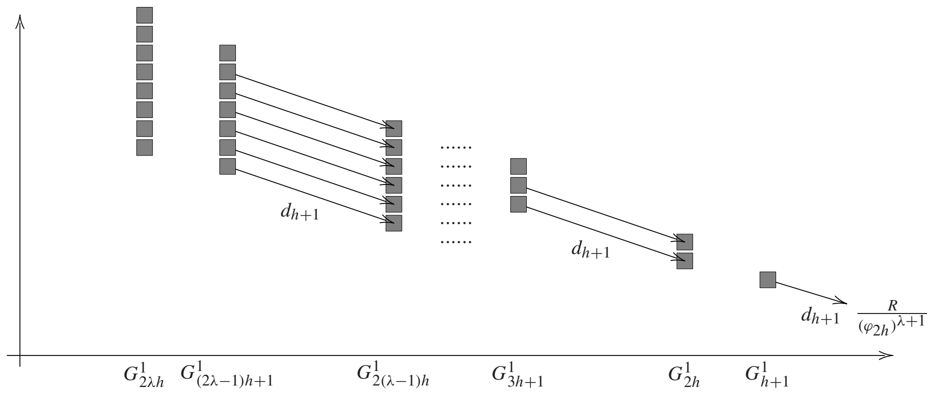

Table 5. E_{h-1} term of the spectral sequence for G^1_n in case (b).

The next non-vanishing differential is d_{h+1}. See Table 5. It takes the module in positive codimension in an odd column $G^1_{(2\mu+1)h+1}$ to the elements in the even column $G^1_{2\mu h}$ (for $1 \leq \mu \leq \lambda - 1$). Via generators, it may be identified with multiplication by

$$\begin{bmatrix} n - 2\mu h \\ h+1 \end{bmatrix} = \begin{bmatrix} 2(\lambda - \mu)h + h + 1 \\ h+1 \end{bmatrix} \tag{9}$$

and it is therefore injective when restricted to modules in positive codimension in odd columns. Further, d_{h+1} is also non-zero as a map $E_{h+1}^{(2\lambda h-1,h+1)} \to E_{h+1}^{(n-1,1)}$. Actually, the term

$$E_{h+1}^{(n-1,1)} \simeq E_2^{(n-1,1)} \simeq \frac{R}{[2(n-1)]!!/[n-1]!}$$

has $R/(\varphi_{2h})^{\lambda+1}$ as the only summand with torsion of type φ_{2h}^l. It is easy to check that the relative map can be identified with multiplication by φ_{2h}^{λ}.

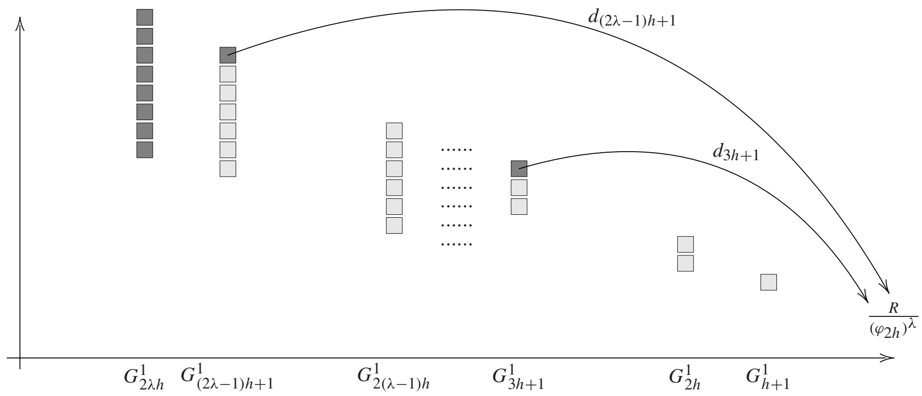

Table 6. Setup for the higher degree terms in the spectral sequence for G^1_n in case (b).

Thus, the only survivors in the E_{2h} term are the first even column, the top modules in the odd columns, generated in positions $(2(\lambda - \mu)h - 1, (2\mu + 1)h + 1)$ by $o_{\mu,0}$ for $1 \le \mu \le \lambda - 1$, as well as $E_{2h}^{(n-1,1)}$ which has $R/(\varphi_{2h})^\lambda$ as summand.

Note that the higher differentials vanish when restricted to the first even column. Actually we may lift the generators of type $e_{\lambda-s,2s}[h]$ to global generators $e_{\lambda-s,2s+1}[h]$ for $0 \le s < \lambda$. Similarly for $0 \le s < \lambda$ we may lift $o_{\lambda-s-1,2s+1}[h]$ to the global generator $o_{\lambda-s-1,2s+2}[h]$. Finally, as in case (a), the modules in positions $(2(\lambda - \mu)h - 1, (2\mu + 1)h + 1)$ for $1 \le \mu \le \lambda - 1$ vanish in the higher terms of the spectral sequence while the module in position $(n - 1, 1)$ has eventually R/φ_{2h} as summand. Clearly the coboundary $o_{\lambda,0}[h]$ projects onto a generator of the latter.

Cases (c) and (d) present no new complications and are omitted. □

4.5. *Spectral sequence for* $G_{\widetilde{B}_n}$

We can now compute the cohomology $H^*(G_{\widetilde{B}_n}, R_{q,t})$. We will do this by means of the Salvetti complex $\widehat{C}^*\widetilde{B}_n$.

As in Section 4.3, let $\widehat{I\widetilde{B}_n}$ be the module of σ-invariant elements and $\widehat{K}\widetilde{B}_n$ the module of σ-anti-invariant elements. We can split our module $\widehat{C^*}\widetilde{B}_n$ into the direct sum

$$\widehat{C^*}\widetilde{B}_n = \widehat{I}\widetilde{B}_n \oplus \widehat{K}\widetilde{B}_n.$$

Using the map $\beta : C^*B_n \to \widehat{C^*}\widetilde{B}_n$ defined by

$$0A \mapsto {}^0_0A, \quad 1A \mapsto {}^1_0A + {}^0_1A,$$

one can see that the submodule $\widehat{I\widetilde{B}_n}$ is isomorphic (as a differential complex) to C^*B_n. Its cohomology has been computed in [CMS06]. We recall the result:

Theorem 4.4 ([CMS06]).

$$H^i(G_{B_n}, R_{q,t}) = \begin{cases} \bigoplus_{d|n,\, 0\le i\le d-2}\{d\}_i \oplus \{1\}_{n-1} & \textit{if } i = n, \\ \bigoplus_{d|n,\, 0\le i\le d-2,\, d\le n/(j+1)}\{d\}_i & \textit{if } i = n-2j, \\ \bigoplus_{d\nmid n,\, d\le n/(j+1)}\{d\}_{n-1} & \textit{if } i = n-2j-1. \end{cases} \qquad \square$$

Hence we only need to compute the cohomology of $\widehat{K}\widetilde{B}_n$. In order to do this we make use of the results of Section 4.4. First consider the subcomplex of $\widehat{C^*}\widetilde{B}_n$ defined as

$$L_n^1 = \left\langle {0 \atop 1} A,\ {1 \atop 1} A \right\rangle.$$

We define the map $\kappa : L_n^1 \to \widehat{K}\widetilde{B}_n$ by

$${0 \atop 1} A \mapsto {0 \atop 1} A - {1 \atop 0} A, \qquad {1 \atop 1} A \mapsto 2 {1 \atop 1} A.$$

It is easy to check that κ gives an isomorphism of differential complexes. Now we define a filtration $\mathcal{F}$ on the complex L_n^1:

$$\mathcal{F}_i L_n^1 = \left\langle {0 \atop 1} A1^i,\ {1 \atop 1} A1^i \right\rangle.$$

The quotient $\mathcal{F}_i L_n^1/\mathcal{F}_{i+1} L_n^1$ is isomorphic to the complex $(G_{n-i}^1[t^{\pm 1}])[i]$ (see Proposition 4.2) with trivial action on the variable t. Hence we use the spectral sequence defined by the filtration $\mathcal{F}$ to compute the cohomology of the complex L_n^1.

The E_0 term of the spectral sequence is given by

$$E_0^{i,j} = \frac{(\mathcal{F}_i L_n^1)^{(i+j)}}{(\mathcal{F}_{i+1} L_n^1)^{(i+j)}} = ((G_{n-i}^1)^{(i+j)}[t^{\pm 1}])[i] = (G_{n-i}^1)^j[t^{\pm 1}]$$

for $0 \le i \le n-2$. Finally,

$$E_0^{n-1,1} = R, \qquad E_0^{n,1} = R,$$

and all the other terms are zero. The differential $d_0 : E_0^{i,j} \to E_0^{i,j+1}$ corresponds to the differential on the complex G_{n-i}^1. It follows that the E^1 term is given by the cohomology of the complexes G_{n-i}^1:

$$E_{i,j}^1 = H^j(G_{n-i}^1)[t^{\pm 1}] \quad \text{for } 0 \le i \le n-2, \qquad E_1^{n-1,1} = R, \qquad E_1^{n,1} = R.$$

As in Section 4.4, we can separately consider, in the spectral sequence E_*, the modules with torsion of type φ_{2h}^l for an integer $h \ge 1$.

For a fixed integer $h > 0$, let $c \in \{0, \dots, 2h-1\}$ be the congruence class of $n \bmod (2h)$ and let λ be an integer such that $n = c + 2\lambda h$. We consider the two cases:

(a) $0 \le c \le h$;
(b) $h + 1 \le c \le 2h - 1$.

In case (a) the modules of φ_{2h}-torsion are:

$$E_1^{c+2\mu h, 2(\lambda-\mu)h-2i} \simeq \{2h\}[t^{\pm 1}] \quad \text{with } 0 \leq \mu \leq \lambda - 1,\ 0 \leq i \leq \lambda - \mu - 1,$$

generated by $e_{\lambda-\mu-i,2i}[h]01^{c+2\mu h}$;

$$E_1^{c+2\mu h, 2(\lambda-\mu)h-2i-1} \simeq \{2h\}[t^{\pm 1}] \quad \text{with } 0 \leq \mu \leq \lambda - 1,\ 0 \leq i \leq \lambda - \mu - 1,$$

generated by $o_{\lambda-\mu-i-1,2i+1}[h]01^{c+2\mu h}$;

$$E_1^{c+2\mu h+h-1, 2(\lambda-\mu)h-h+1-2i} \simeq \{2h\}[t^{\pm 1}] \quad \text{with } 0 \leq \mu \leq \lambda - 1,\ 0 \leq i \leq \lambda - \mu - 1,$$

generated by $o_{\lambda-\mu-i-1,2i}[h]01^{c+2\mu h+h-1}$;

$$E_1^{c+2\mu h+h-1, 2(\lambda-\mu)h-h+1-2i-1} \simeq \{2h\}[t^{\pm 1}] \quad \text{with } 0 \leq \mu \leq \lambda - 2,\ 0 \leq i \leq \lambda - \mu - 2,$$

generated by $e_{\lambda-\mu-i-1,2i+1}[h]01^{c+2\mu h+h-1}$.

In case (b) the modules of φ_{2h}-torsion are:

$$E_1^{c+2\mu h, 2(\lambda-\mu)h-2i} \simeq \{2h\}[t^{\pm 1}] \quad \text{with } 0 \leq \mu \leq \lambda - 1,\ 0 \leq i \leq \lambda - \mu - 1,$$

generated by $e_{\lambda-\mu-i,2i}[h]01^{c+2\mu h}$;

$$E_1^{c+2\mu h, 2(\lambda-\mu)h-2i-1} \simeq \{2h\}[t^{\pm 1}] \quad \text{with } 0 \leq \mu \leq \lambda - 1,\ 0 \leq i \leq \lambda - \mu - 1,$$

generated by $o_{\lambda-\mu-i-1,2i+1}[h]01^{c+2\mu h}$;

$$E_1^{c+2\mu h-h-1, 2(\lambda-\mu)h+h+1-2i} \simeq \{2h\}[t^{\pm 1}] \quad \text{with } 0 \leq \mu \leq \lambda - 1,\ 0 \leq i \leq \lambda - \mu - 1,$$

generated by $o_{\lambda-\mu-i,2i}[h]01^{c+2\mu h-h-1}$;

$$E_1^{c+2\mu h-h-1, 2(\lambda-\mu)h+h+1-2i-1} \simeq \{2h\}[t^{\pm 1}] \quad \text{with } 0 \leq \mu \leq \lambda - 1,\ 0 \leq i \leq \lambda - \mu - 1,$$

generated by $e_{\lambda-\mu-i,2i+1}[h]01^{c+2\mu h-h-1}$.

In the E_1 term of the spectral sequence, the only non-trivial map is $d_1 : E_1^{n-1,1} \to E_1^{n,1}$, corresponding to multiplication by the polynomial

$$\frac{\widehat{W}_{\widetilde{B}_n}[q,t]}{W_{B_n}[q,t]} = \prod_{i=1}^{n-1}(1+q^i) = \prod_{h \leq n} \varphi_{2h}^{\lfloor (n-1)/h \rfloor - \lfloor (n-1)/2h \rfloor}.$$

Then in E_2 we have

$$E_2^{n-1,1} = 0 \quad \text{and} \quad E_2^{n,1} = \bigoplus R/(\varphi_{2h}^{\lfloor (n-1)/h \rfloor - \lfloor (n-1)/2h \rfloor}).$$

Notice that the integer $f(n,h) = \lfloor (n-1)/h \rfloor - \lfloor (n-1)/2h \rfloor$ corresponds to λ in case (a) and to $\lambda + 1$ in case (b).

Now we consider the higher differentials in the spectral sequence. The first possibly non-trivial maps are d_{h-1} and d_{h+1}. In case (a) the map d_{h-1} is given by multiplication by

$$\prod_{i=n}^{n+h-2}(1+tq^i)$$

and d_{h+1} is the null map. The maps

$$d_{2(\lambda-\mu)h} : \{2h\}[t^{\pm 1}] = E_{2(\lambda-\mu)h}^{c+2\mu h, 2(\lambda-\mu)h} \to E_{2(\lambda-\mu)h}^{n,1},$$

where μ goes from $\lambda-1$ to 0, correspond, up to invertibles, modulo φ_{2h}, to multiplication by

$$\varphi_{2h}^{\mu}\Big(\prod_{i=0}^{2h-1}(1+tq^i)\Big)^{\lambda-\mu}.$$

Moreover, they are all injective and the term $E_{2(\lambda)h+1}^{n,1}$ is given by the quotient

$$R/\Big(\varphi_{2h}^{\lambda}, \varphi_{2h}^{\lambda-1}\prod_{i=0}^{2h-1}(1+tq^i), \ldots, \Big(\prod_{i=0}^{2h-1}(1+tq^i)\Big)^{\lambda}\Big) = R/\Big(\varphi_{2h}, \prod_{i=0}^{2h-1}(1+tq^i)\Big)^{\lambda}.$$

In case (b) the map d_{h-1} is null and d_{h+1} is multiplication by the polynomial

$$\prod_{i=n+h-1}^{n+2h-1}(1+tq^i).$$

The maps

$$d_{2(\lambda-\mu)h+h+1} : \{2h\}[t^{\pm 1}] = E_{2(\lambda-\mu)h+h+1}^{c+2\mu h+h-1, 2(\lambda-\mu)h-h} \to E_{2(\lambda-\mu)h+h+1}^{1,n},$$

where μ goes from λ to 0, correspond, up to invertibles, modulo φ_{2h}, to multiplication by

$$\varphi_{2h}^{\mu}\Big(\prod_{i=0}^{2h-1}(1+tq^i)\Big)^{\lambda-\mu+1}.$$

Hence they are all injective and the term $E_{2(\lambda)h+h+2}^{n,1}$ is given by the quotient

$$R/\Big(\varphi_{2h}, \prod_{i=0}^{2h-1}(1+tq^i)\Big)^{\lambda+1}.$$

Since all the generators lift to global cocycles, it turns out that all the other differentials are null. Hence we have proved the following:

Theorem 4.5.

$$H^{n+1}(\widehat{K}\widetilde{B}_n) \simeq \bigoplus_{h>0} \{\{2h\}\}_{f(n,h)},$$

and for $s \geq 0$*,*

$$H^{n-s}(\widehat{K}\widetilde{B}_n) \simeq \bigoplus_{\substack{h>2 \\ i\in I(n,h)}} \{2h\}_i^{\oplus \max(0, \lfloor n/2h \rfloor - s)}$$

with $I(n,h) = \{n, \ldots, n+h-2\}$ *if* $n \simeq 0, 1, \ldots, h \bmod(2h)$*,* $f(n,h) = \lfloor (n+h-1)/2h \rfloor$ *and* $I(n,h) = \{n+h-1, \ldots, n+2h-1\}$ *if* $n \simeq h+1, h+2, \ldots, 2h-1 \bmod(2h)$*.* □

Putting together the results of Theorems 4.4 and 4.5, we get Theorem 1.3.

As a corollary, we use the long exact sequences associated to

$$0 \to \mathbb{Q}[[t^{\pm 1}]] \xrightarrow{m(q)} M \xrightarrow{1+q} M \to 0$$

and

$$0 \to \mathbb{Q} \xrightarrow{m(t)} \mathbb{Q}[[t^{\pm 1}]] \xrightarrow{1+t} \mathbb{Q}[[t^{\pm 1}]] \to 0$$

to get the constant coefficients cohomology for $G_{\widetilde{B}_n}$. Here $m(x)$ is multiplication by the series

$$\sum_{i\in\mathbb{Z}} (-x)^i.$$

We give only the result, omitting the details which come from non-difficult analysis of the above mentioned sequences and recalling that the Euler characteristic of the complex is 1 for n even, and -1 for n odd.

Theorem 4.6.

$$H^i(G_{\widetilde{B}_n}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0, \\ \mathbb{Q}^2 & \text{if } 1 \leq i \leq n-2, \\ \mathbb{Q}^{2+\lfloor n/2 \rfloor} & \text{if } i = n-1, n, \end{cases}$$

where the t *and* q *actions correspond to multiplication by* -1*.* □

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