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Formal power series rings over a π-domain

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Abstract. Let R be an integral domain, $\mathcal{X}$ be a set of indeterminates over R, and $R[[\mathcal{X}]]_3$ be the full ring of formal power series in $\mathcal{X}$ over R. We show that the Picard group of $R[[\mathcal{X}]]_3$ is isomorphic to the Picard group of R. An integral domain is called a π-domain if every principal ideal is a product of prime ideals. An integral domain is a π-domain if and only if it is a Krull domain that is locally a unique factorization domain. We show that $R[[\mathcal{X}]]_3$ is a π-domain if $R[[X_1, \ldots, X_n]]$ is a π-domain for every $n \geq 1$. In particular, $R[[\mathcal{X}]]_3$ is a π-domain if R is a Noetherian regular domain. We extend these results to rings with zero-divisors. A commutative ring R with identity is called a π-ring if every principal ideal is a product of prime ideals. We show that $R[[\mathcal{X}]]_3$ is a π-ring if R is a Noetherian regular ring.

Keywords. Krull domain, π-domain, unique factorization domain, formal power series ring, invertible ideal, class group, Picard group

1. Introduction

The question whether the power series ring over a unique factorization domain (UFD) is a UFD had remained open for a long time until Samuel constructed a Noetherian counterexample in [17]. However, there do exist UFDs R such that $R[[X_1, \ldots, X_n]]$ is a UFD for every n, e.g., regular UFDs, and more specifically principal ideal domains. All of these examples are Noetherian. Non-Noetherian examples were constructed by Deckard [7], Cashwell and Everette [5] as well as Nishimura [16] (see also Deckard and Durst [8]) exploring power series rings in an infinite number of variables. Recall that in the case of an infinite number of variables, there are several types of power series rings. Let $\mathcal{X} = \{X_\lambda\}_{\lambda\in\Lambda}$ be a set of indeterminates over R, and S be the weak direct sum of the additive abelian semigroup $\mathbb{N}$ with itself $|\Lambda|$ times, where $\mathbb{N}$ is the set of nonnegative integers. Following the notation and definition in [10, 11], the *full ring of formal power series* in $\mathcal{X}$ over R is defined to be the set of all functions $f : S \to R$, where $(f+g)(s) = f(s)+g(s)$ and $(fg)(s) = \sum_{t+u=s} f(t)g(u)$ for any $s \in S$, the notation $\sum_{t+u=s}$ indicating that the sum is taken over all ordered pairs (t, u) of elements of S with sum s. The full ring of formal power series in $\mathcal{X}$ over R is denoted by $R[[\mathcal{X}]]_3$ while $R[[\mathcal{X}]]_2$ denotes the $(\mathcal{X})$-adic completion of the polynomial ring $R[\mathcal{X}]$. The aforementioned authors showed that

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for an infinite set $\mathcal{X}$ of indeterminates over a domain R, $R[[\mathcal{X}]]_3$ is a UFD as long as $R[[X_1, \ldots, X_n]]$ is a UFD for every n. Thus $R[[\mathcal{X}]]_3$ is a non-Noetherian UFD such that its arbitrary power series extensions are UFDs provided that $R[[X_1, \ldots, X_n]]$ is a UFD for every n. As for the divisor class groups, Samuel's example [17] shows that $\mathrm{Cl}(R[[X]]) \ncong \mathrm{Cl}(R)$ in general. Indeed, $\mathrm{Cl}(R[[X]]) \cong \mathrm{Cl}(R) \oplus \mathrm{Cl}(R[[X]]_{R^*})$ for a Noetherian Krull domain R. However, for the Picard group of an integral domain R, $\mathrm{Pic}(R[[X]]) \cong \mathrm{Pic}(R)$. From this fact, it follows that for a UFD R, $R[[X]]$ is a UFD if and only if $R[[X]]_{R^*}$ is a UFD if and only if $R[[X]]$ is a π-domain.

These results motivate us to investigate power series extensions of a π-domain. An integral domain is called a *π-domain* if every principal ideal is a product of prime ideals. Recall that an integral domain is a π-domain if and only if it is a Krull domain that is locally a UFD. Thus π-domains lie between Krull domains and UFDs. Gilmer [10] showed that for a Krull domain R, $R[[\mathcal{X}]]_3$ is a Krull domain. So it is natural to ask if $R[[\mathcal{X}]]_3$ is a π-domain when R is a π-domain such that $R[[X_1, \ldots, X_n]]$ is a π-domain for each $n \geq 1$. The main purpose of this paper is to answer this question. To help the readers better understand π-domains, we quote a couple of their characterizations. An integral domain is a π-domain if and only if every t-ideal is an invertible ideal if and only if every principal ideal is a product of prime ideals if and only if every nonzero prime ideal contains an invertible prime ideal [1, 2, 12]. The most well-known examples of π-domains are Noetherian regular domains. A regular local ring R is a UFD (the Auslander–Buchsbaum theorem [3]), each $R[[X_1, \ldots, X_n]]$ is a regular local ring, and in this case we have already mentioned that its arbitrary power series extension $R[[\mathcal{X}]]_3$ is also a UFD. However, nothing is known about the global case, i.e., about $R[[\mathcal{X}]]_3$ for a Noetherian regular domain R. One of our main results is that $R[[\mathcal{X}]]_3$ is a π-domain for a Noetherian regular domain R. This provides examples of non-Noetherian π-domains whose full rings of formal power series extensions are also π-domains. In fact, for an integral domain R, we prove the stronger result that $R[[\mathcal{X}]]_3$ is a π-domain if and only if $R[[X_1, \ldots, X_n]]$ is a π-domain for every $n \geq 1$. In the process, we prove that $\mathrm{Pic}(R[[\mathcal{X}]]_3) \cong \mathrm{Pic}(R)$ for an integral domain R. We extend the results on integral domains to rings with zero-divisors. A commutative ring R with identity is called a *π-ring* if every principal ideal is a product of prime ideals. We show that $R[[\mathcal{X}]]_3$ is a π-ring if R is a Noetherian regular ring.

2. Preliminaries

In this section we introduce concepts and basic properties of power series rings which are needed in what follows. Our general reference for power series rings is Brewer's [4]. Throughout this paper (except in Section 5), R will be an integral domain and $\mathcal{X}$ will stand for a set of indeterminates. Note that an element f of $R[[\mathcal{X}]]_3$ can be written as the formal sum $f = \sum c(a_{i_1}, \ldots, a_{i_n}) X_{i_1}^{a_{i_1}} \cdots X_{i_n}^{a_{i_n}}$, where $\{X_{i_1}, \ldots, X_{i_n}\}$ ranges over the finite subsets of $\mathcal{X}$, a_{i_j}'s are nonnegative integers, and $c(a_{i_1}, \ldots, a_{i_n})$ belongs to R. For a finite subset $\mathcal{F}$ of $\mathcal{X}$, define a map $\pi_{\mathcal{F}}$ from $R[[\mathcal{X}]]_3$ onto $R[[\mathcal{F}]]$ by evaluating at $X_i = 0$ for all $X_i \notin \mathcal{F}$. Clearly the map is a ring epimorphism, called the *canonical projection* with respect to $\mathcal{F}$. For each $\alpha \in R[[\mathcal{X}]]_3$, define the *$\mathcal{F}$-projection* of α to be the element

$\mathcal{F}(\alpha)$, which is the image of α under the canonical projection with respect to $\mathcal{F}$. For a subset I of $R[[\mathcal{X}]]_3$, we denote by $\mathcal{F}(I)$ the set of all $\mathcal{F}$-projections of elements of I. In particular, if $\mathcal{F} = \emptyset$, then the $\emptyset$-projections of α and I will be denoted by $\alpha(0)$ and $I_0 := \emptyset(I)$, respectively.

Given a finite subset $\mathcal{F}$ of $\mathcal{X}$, we define a *sequence over* $\mathcal{F}$ to be a function which assigns to every finite subset $\mathcal{K}$ of $\mathcal{X}$ containing $\mathcal{F}$ a definite element $\alpha_{\mathcal{K}}$ of $R[[\mathcal{K}]]$. A sequence over $\mathcal{F}$ is called *projective* if $\mathcal{K}(\alpha_{\mathcal{L}}) = \alpha_{\mathcal{K}}$ whenever $\mathcal{F} \subseteq \mathcal{K} \subseteq \mathcal{L}$. A sequence over $\mathcal{F}$ is called *associative* if $\mathcal{K}(\alpha_{\mathcal{L}}) \sim \alpha_{\mathcal{K}}$ in $R[[\mathcal{K}]]$ whenever $\mathcal{F} \subseteq \mathcal{K} \subseteq \mathcal{L}$. Here $\beta \sim \gamma$ means $\beta = \gamma\nu$ for some unit ν of the indicated domain.

We quote useful results from [5], adding a couple of minor new results.

Theorem 2.1. *Let $\mathcal{X}$ be a set of indeterminates, and $\mathcal{F}$ be a finite subset of $\mathcal{X}$. Then, for $\alpha, \beta \in R[[\mathcal{X}]]_3$,*

(1) $\mathcal{F}(\alpha\beta) = \mathcal{F}(\alpha)\mathcal{F}(\beta)$ *and* $\mathcal{F}(\alpha + \beta) = \mathcal{F}(\alpha) + \mathcal{F}(\beta)$,
(2) $\mathcal{F}(\mathcal{K}(\alpha)) = \mathcal{F}(\alpha)$ *if $\mathcal{K}$ is a finite subset of $\mathcal{X}$ containing $\mathcal{F}$,*
(3) *if $\mathcal{F}(\alpha)$ is a unit of $R[[\mathcal{F}]]$, then α is a unit of $R[[\mathcal{X}]]_3$,*
(4) *if $\mathcal{K}(\alpha) = \mathcal{K}(\beta)$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$, then $\alpha = \beta$,*
(5) *every projective sequence $\{\alpha_{\mathcal{K}}\}$ over $\mathcal{F}$ is of the form $\alpha_{\mathcal{K}} = \mathcal{K}(\alpha)$ for some $\alpha \in R[[\mathcal{X}]]_3$,*
(6) *for every associative sequence $\{\alpha_{\mathcal{K}}\}$ over $\mathcal{F}$, there exists an element $\alpha \in R[[\mathcal{X}]]_3$ such that $\mathcal{K}(\alpha) \sim \alpha_{\mathcal{K}}$ in $R[[\mathcal{K}]]$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$,*
(7) *if I and J are ideals of $R[[\mathcal{X}]]_3$, then $\mathcal{F}(I)$ and $\mathcal{F}(J)$ are ideals of $R[[\mathcal{F}]]$; moreover, $\mathcal{F}(I + J) = \mathcal{F}(I) + \mathcal{F}(J)$ and $\mathcal{F}(IJ) = \mathcal{F}(I)\mathcal{F}(J)$,*
(8) *for an ideal I of $R[[\mathcal{X}]]_3$, we have $\mathcal{F}(\mathcal{K}(I)) = \mathcal{F}(I)$ if $\mathcal{K}$ is a finite subset of $\mathcal{X}$ containing $\mathcal{F}$.*

Proof. (1)–(6) are proved in [5]. From (1), (2), and the fact that the projection map is a ring epimorphism, (7) and (8) follow. □

We briefly review the definition of a divisorial ideal and relations between a Krull domain and a π-domains.

Let R be an integral domain with quotient field K. By a *fractional ideal* F we mean an R-submodule of K. By F^{-1} (the inverse of F) we mean the set of all x in K with $xF \subseteq R$. Note that F^{-1} is also a fractional ideal. We say that F is *invertible* if $FF^{-1} = R$. For a fractional ideal F, we denote by F_v the fractional ideal $(F^{-1})^{-1}$. The fractional ideal $\sum\{I_v \mid I$ is a finitely generated fractional ideal contained in $F\}$ is denoted by F_t. If $F = F_v$, then F is called a *divisorial ideal*. If $F = F_t$, then F is called a *t-ideal*. It is well known that every divisorial ideal of a Krull domain R is a v-product of minimal prime ideals of R—namely, for a divisorial ideal I, $I = (P_1 \dots P_k)_v$, where P_i's are (not necessarily distinct) minimal prime ideals of R. We refer the readers to [9, 11, 13] for more information on divisorial ideals.

An integral domain in which every principal ideal is a product of finitely many prime ideals is called a *π-domain*, which is a generalization of the unique factorization domain (UFD). The following is a relation between Krull domains and π-domains. For a proof, the readers are referred to [1, 2, 9, 11, 12, 14].

Theorem 2.2. *The following are equivalent for a domain R:*

(1) *R is a π-domain.*
(2) *R_M is a UFD for each maximal ideal M of R, and every minimal prime ideal of R is finitely generated.*
(3) *R is a Krull domain, and every minimal prime ideal of R is invertible.*
(4) *R is a Krull domain, and the product of any finite number of divisorial ideals is a divisorial ideal.*
(5) *Every divisorial ideal is a finite product of prime ideals.*
(6) *Every t-ideal is a finite product of prime ideals.*
(7) *Every t-ideal is invertible.*

If R is a π-domain, then it follows from Theorem 2.2 that every divisorial ideal of R is a product of a finite number of minimal prime ideals of R—namely, for a divisorial ideal I of R,

$$I = P_1^{n_1} \cdots P_k^{n_k},$$

where P_i's are distinct minimal prime ideals of R.

3. Invertible ideals of a power series ring

In this section, we investigate a relation between invertible ideals of an integral domain and those of its power series ring. Recall that the $\emptyset$-projection of an ideal I of a power series ring is denoted by $I_0 := \emptyset(I)$, which is the same as the set of all the constant terms of elements in I.

Lemma 3.1. *Let $\Phi : D \to R$ be a ring epimorphism between integral domains. If I is an invertible ideal of D and $\Phi(I) \neq (0)$, then $\Phi(I)$ is an invertible ideal of R.*

Proof. Choose $0 \neq a \in I$ such that $\Phi(a) \neq 0$. Then $aD = I(aI^{-1})$ and $aI^{-1} \subseteq D$. We have

$$\Phi(a)R = \Phi(aD) = \Phi(I(aI^{-1})) = \Phi(I)\Phi(aI^{-1}).$$

So $\Phi(I)$ is an invertible ideal of R. □

The next result is well known. For easy reference, we include its proof.

Theorem 3.2. *Let R be an integral domain with quotient field K, and I be an invertible ideal of $R[[X]]$, where X is an indeterminate over R. Then*

(i) *if $I_0 \neq (0)$, then I_0 is also an invertible ideal of R,*
(ii) *there exists an invertible ideal I' of R such that $I = fI'[[X]]$, where $f \in K[[X]]$; moreover, if $I_0 \neq (0)$, then we can take I_0 for I'.*

Proof. Choose an integer $n \geq 0$ such that $I \subseteq (X^n)$ and $I \nsubseteq (X^{n+1})$. Put $J = X^{-n}I$. Then J is an invertible ideal of $R[[X]]$ such that $J_0 \neq (0)$. Therefore it suffices to show that if I is an invertible ideal of $R[[X]]$ such that $I_0 \neq (0)$, then I_0 is invertible and $I = fI_0[[X]]$ for some $f \in K[[X]]$.

By Lemma 3.1, I_0 is an invertible ideal of R. Put $J = I(I_0^{-1}[[X]])a$, where $0 \neq a \in I_0$. Since $a \in I_0$, J is an ideal of $R[[X]]$. Applying the $\emptyset$-projection of J, we have

$$J_0 = \emptyset(J) = \emptyset(I(I_0^{-1}[[X]])a) = \emptyset(I)\emptyset(I_0^{-1}[[X]])\emptyset(a) = I_0I_0^{-1}a = aR.$$

Since $a \in J_0$, we can choose an element $j \in J$ such that $j(0) = a$. Thus $J_0 = j(0)R$. Since for each $g \in J$, $g = g(0) + Xg_1$, where $g_1 \in R[[X]]$, we have $g(0) \in J_0$. Hence $J \subseteq J_0 + XR[[X]] = j(0)R + XR[[X]] \subseteq jR[[X]] + XR[[X]]$. If $g \in J$, then $g = jh + Xg_2$, where $h, g_2 \in R[[X]]$. Since $g, j \in J$, we have $g_2 \in (J : X)$. Therefore, $J = jR[[X]] + X(J : X)$.

We claim that $J = (J : X)$. If $g \in (J : X)$, then $Xg \in J = I(I_0^{-1}[[X]])a$. Thus $XgI_0[[X]] \subseteq aI$. Since aI is an invertible ideal of $R[[X]]$, we have $XgI_0[[X]] = (aI)I'$ for some ideal I' of $R[[X]]$. Note that $aI \nsubseteq (X)$ because $I_0 = \emptyset(I) \neq (0)$. Since (X) is an invertible prime ideal, we have $I' \subseteq (X)$, i.e., $I' = XI''$ for some ideal I'' of $R[[X]]$. Hence $XgI_0[[X]] = (aI)I' = aXII''$. By cancellation, we have $gI_0[[X]] = (aI)I''$. Therefore, $g \in (aI)I_0^{-1}[[X]]I'' = JI'' \subseteq J$. Thus the claim is proved.

Since $J = (J : X)$, we have $J = jR[[X]] + XJ$, where $j \in J$. Note that since J is invertible, J is a finitely generated $R[[X]]$-module. Since X is in the Jacobson radical of $R[[X]]$, by the Nakayama lemma, we have $J = jR[[X]] = I(I_0^{-1}[[X]])a$. Therefore, $I = a^{-1}jI_0[[X]] = fI_0[[X]]$, where $f = a^{-1}j \in K[[X]]$. □

We generalize Theorem 3.2 to a finite number of indeterminates.

Corollary 3.3. *Let R be an integral domain with quotient field K, and I be an invertible ideal of $R[[\mathcal{F}]]$, where $\mathcal{F}$ is a finite set of indeterminates over R. Then there exists an invertible ideal I' of R such that*

$$I = fI'[[\mathcal{F}]] \quad \textit{for some } f \in K[[\mathcal{F}]].$$

Moreover, if $\emptyset(I) = I_0 \neq (0)$, then $I = fI_0[[\mathcal{F}]]$ for some $f \in K[[\mathcal{F}]]$.

In a π-domain, every principal ideal is a product of prime ideals instead of principal prime ideals. So we have to consider a projective sequence of ideals instead of elements.

Definition 3.4. *Let R be an integral domain and $\mathcal{X}$ be a set of indeterminates over R. Let $\mathcal{F}$ be a finite subset of $\mathcal{X}$ and $I_{\mathcal{K}}$ be an ideal of $R[[\mathcal{K}]]$ for each finite subset $\mathcal{K}$ of $\mathcal{X}$ containing $\mathcal{F}$. We call a sequence $\{I_{\mathcal{K}}\}_{\mathcal{K}\supseteq\mathcal{F}}$ of ideals a* projective sequence *over $\mathcal{F}$ if $\mathcal{K}(I_{\mathcal{L}}) = I_{\mathcal{K}}$ whenever $\mathcal{F} \subseteq \mathcal{K} \subseteq \mathcal{L}$.*

Theorem 3.5. *Let R be an integral domain and $J \subseteq I$ be ideals of $R[[\mathcal{X}]]_3$, where I is an invertible ideal of $R[[\mathcal{X}]]_3$. If $\mathcal{K}(I) = \mathcal{K}(J) \neq (0)$ for some finite subset $\mathcal{K}$ of $\mathcal{X}$, then $\mathcal{K}(I)$ is invertible, and $I = J$.*

Proof. Suppose that I is an invertible ideal of $R[[\mathcal{X}]]_3$ containing an ideal J such that $\mathcal{K}(I) = \mathcal{K}(J) \neq (0)$ for some finite subset $\mathcal{K}$ of $\mathcal{X}$. Then $J = IJ'$, where $J' := I^{-1}J$ is an ideal of $R[[\mathcal{X}]]_3$. Since $\mathcal{K}(I) = \mathcal{K}(J)$, it follows from the $\mathcal{K}$-projection of J that

$$\mathcal{K}(I) = \mathcal{K}(J) = \mathcal{K}(I)\mathcal{K}(J').$$

Since $\mathcal{K}(I)$ is an invertible ideal of $R[[\mathcal{K}]]$ by Lemma 3.1, we cancel $\mathcal{K}(I)$ and obtain the equality $\mathcal{K}(J') = R[[\mathcal{K}]]$. Therefore we can choose $f \in J'$ such that $\mathcal{K}(f) = 1$. By Theorem 2.1(3), f is a unit in $R[[\mathcal{X}]]_3$. Thus $J' = R[[\mathcal{X}]]_3$. Hence $J = IJ' = I$. □

Theorem 3.6. *Let I and J be ideals of $R[[\mathcal{X}]]_3$, where R is an integral domain, such that $\mathcal{F}(I) = \mathcal{F}(J)$ for every finite subset $\mathcal{F}$ of $\mathcal{X}$. If I is an invertible ideal, then $I = J$.*

Proof. Choose $0 \neq a \in I$. Put $H := (aI^{-1})J$, which is an ideal of $R[[\mathcal{X}]]_3$. Note that since $0 \neq a \in I$, there exists a finite subset $\mathcal{F}$ of $\mathcal{X}$ such that $\mathcal{K}(a) \neq 0$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. Thus $\mathcal{K}(I) \neq 0$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. We have $\mathcal{K}(H) = \mathcal{K}(aI^{-1})\mathcal{K}(J)$. We show $\mathcal{K}(aI^{-1}) = \mathcal{K}(a)\mathcal{K}(I)^{-1}$. By taking the $\mathcal{K}$-projection of the equation $(a) = (aI^{-1})I$, we get $(\mathcal{K}(a)) = \mathcal{K}(aI^{-1})\mathcal{K}(I)$. By Lemma 3.1, $\mathcal{K}(I)$ is an invertible ideal of $R[[\mathcal{K}]]$, so $\mathcal{K}(aI^{-1}) = \mathcal{K}(a)\mathcal{K}(I)^{-1}$. Now

$$\mathcal{K}(H) = \mathcal{K}(a)\mathcal{K}(I)^{-1}\mathcal{K}(J) = \mathcal{K}(a)\mathcal{K}(I)^{-1}\mathcal{K}(I) = (\mathcal{K}(a)).$$

Thus $\mathcal{K}(H) = (\mathcal{K}(a))$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. We show $H \subseteq (a)$. Let $h \in H$. Choose $\gamma_{\mathcal{K}} \in R[[\mathcal{K}]]$ such that $\mathcal{K}(h) = \gamma_{\mathcal{K}}\mathcal{K}(a)$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. It is easy to see that $\{\gamma_{\mathcal{K}}\}_{\mathcal{K} \supseteq \mathcal{F}}$ is a projective sequence. By Theorem 2.1(5), there exists $\gamma \in R[[\mathcal{X}]]_3$ such that $\mathcal{K}(\gamma) = \gamma_{\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. Now $\mathcal{K}(h) = \mathcal{K}(\gamma)\mathcal{K}(a) = \mathcal{K}(\gamma a)$. By Theorem 2.1(4), $h = \gamma a$ and hence $H \subseteq (a)$. Applying Theorem 3.5, we deduce $H = (a)$, whence $aI^{-1}J = (a)$. Cancelling (a) from this equation, we get the equality $I^{-1}J = R[[\mathcal{X}]]_3$, and hence $I = J$. □

We present an example which justifies the condition that "I is an invertible ideal" in Theorems 3.5 and 3.6.

Example 3.7. Let R be an integral domain and $\mathcal{X}$ be an infinite set of indeterminates. Put

$$\begin{cases} (\mathcal{X})_2 := \{f \mid f \in R[[\mathcal{X}]]_2,\ f(0) = 0\}, \\ (\mathcal{X})_3 := \{f \mid f \in R[[\mathcal{X}]]_3,\ f(0) = 0\}. \end{cases}$$

Put $J := (\mathcal{X})_2 R[[\mathcal{X}]]_3$ and $I := (\mathcal{X})_3$. It is clear that $J \subseteq I$. Let $f \in I$. Then for every finite subset $\mathcal{K}$ of $\mathcal{X}$, we have $\mathcal{K}(f) \in (\mathcal{X})_2$ and $\mathcal{K}(f) = \mathcal{K}(\mathcal{K}(f))$, hence $\mathcal{K}(f) \in \mathcal{K}(J)$. So $\mathcal{K}(I) \subseteq \mathcal{K}(J)$. Thus $\mathcal{K}(I) = \mathcal{K}(J)$ for every finite subset $\mathcal{K}$ of $\mathcal{X}$. We claim that $I \neq J$. Put $f := X_1 + X_2 + \cdots = \sum_{i=1}^{\infty} X_i$. Thus $f \in I$. We claim $f \notin J$. Suppose that $f \in J = (\mathcal{X})_2 R[[\mathcal{X}]]_3$. Then

$$f = X_1 + X_2 + \cdots = f_1 g_1 + \cdots + f_n g_n,$$

where $f_i \in (\mathcal{X})_2$ and $g_i \in R[[\mathcal{X}]]_3$ for $i = 1, \ldots, n$. Since $f_i(0) = 0$ for each i, comparing the first degree terms, we have

$$f = X_1 + X_2 + \cdots = f_{11}g_{10} + \cdots + f_{n1}g_{n0},$$

where f_{i1} is the first degree form of f_i and $g_{i0} = g_i(0)$ for each i. But since $f_i \in (\mathcal{X})_2$, f_{i0} has only finitely many monomials of degree 1. Thus $f = X_1 + X_2 + \cdots$ has only finitely many monomials of degree 1, which is a contradiction.

Next we study when the condition "$\mathcal{F}(J) \subseteq \mathcal{F}(I)$ for all $\mathcal{F}$" forces $J \subseteq I$.

Theorem 3.8. *Let I and J be ideals of $R[[\mathcal{X}]]_3$, where R is an integral domain, such that $\mathcal{F}(J) \subseteq \mathcal{F}(I)$ for every finite subset $\mathcal{F}$ of $\mathcal{X}$. If I is an invertible ideal, then $J \subseteq I$.*

Proof. Let $0 \neq a \in J$. Then there exists a finite subset $\mathcal{F}$ such that $\mathcal{K}(a) \neq 0$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. Since $0 \neq \mathcal{K}(a) \in \mathcal{K}(J) \subseteq \mathcal{K}(I)$, by Lemma 3.1, $\mathcal{K}(I)$ is an invertible ideal of $R[[K]]$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. Choose $0 \neq b \in I$. Put $H := abI^{-1}$, which is an ideal of $R[[\mathcal{X}]]_3$. By taking the $\mathcal{K}$-projection of the equation $(b) = (bI^{-1})I$, we get $(\mathcal{K}(b)) = \mathcal{K}(bI^{-1})\mathcal{K}(I)$. Since $\mathcal{K}(I)$ is an invertible ideal of $R[[\mathcal{K}]]$, we have $\mathcal{K}(bI^{-1}) = \mathcal{K}(b)\mathcal{K}(I)^{-1}$. By applying the $\mathcal{K}$-projection of H, since $\mathcal{K}(a) \in \mathcal{K}(I)$, we have

$$\mathcal{K}(H) = \mathcal{K}(a)\mathcal{K}(bI^{-1}) = \mathcal{K}(a)\mathcal{K}(b)\mathcal{K}(I)^{-1} \subseteq (\mathcal{K}(b)).$$

Thus $\mathcal{K}(H) \subseteq (\mathcal{K}(b))$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. We show $H \subseteq (b)$. Let $h \in H$. Choose $\gamma_{\mathcal{K}} \in R[[\mathcal{K}]]$ such that $\mathcal{K}(h) = \gamma_{\mathcal{K}}\mathcal{K}(b)$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. It is easy to see that $\{\gamma_{\mathcal{K}}\}_{\mathcal{K} \supseteq \mathcal{F}}$ is a projective sequence. By Theorem 2.1(5), there exists $\gamma \in R[[\mathcal{X}]]_3$ such that $\mathcal{K}(\gamma) = \gamma_{\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. Now $\mathcal{K}(h) = \mathcal{K}(\gamma)\mathcal{K}(b) = \mathcal{K}(\gamma b)$. By Theorem 2.1(4), $h = \gamma b$ and hence $H = abI^{-1} \subseteq (b)$. Thus $a \in I$. Therefore $J \subseteq I$. $\square$

Cashwell and Everett [5] showed that a projective sequence of principal ideals has a primitive ideal in the sense that the projective sequence is the projection of an ideal in $R[[\mathcal{X}]]_3$. We show that a projective sequence of invertible ideals also has a primitive ideal in $R[[\mathcal{X}]]_3$.

Theorem 3.9. *Let R be an integral domain with quotient field K, and $\mathcal{X}$ be a set of indeterminates over R. If $\{I_{\mathcal{F}}\}$ is a projective sequence of invertible ideals of $R[[\mathcal{F}]]$ over $\emptyset$, then there exists a unique invertible ideal I of the form $fI_0[[\mathcal{X}]]_3$ of $R[[\mathcal{X}]]_3$, where $f \in K[[\mathcal{X}]]_3$ and $I_0 := I_{\emptyset}$ is an invertible ideal of R, such that $\mathcal{F}(I) = I_{\mathcal{F}}$ in $R[[\mathcal{F}]]$ for every finite subset $\mathcal{F}$ of $\mathcal{X}$.*

Proof. Let $\{I_{\mathcal{F}}\}_{\mathcal{F} \supseteq \emptyset}$ be a projective sequence of invertible ideals of $R[[\mathcal{F}]]$ over $\emptyset$. Put $I_0 := I_{\emptyset}$. Since $I_{\mathcal{F}}$ is an invertible ideal of $R[[\mathcal{F}]]$ for each finite subset $\mathcal{F}$ of $\mathcal{X}$, it follows from Corollary 3.3 that

$$I_{\mathcal{F}} = f_{\mathcal{F}} I_0[[\mathcal{F}]], \quad \text{where} \quad f_{\mathcal{F}} \in K[[\mathcal{F}]].$$

Since $I_0 \neq (0)$, choose $0 \neq b \in I_0$ so that $bf_{\mathcal{F}} \in R[[\mathcal{F}]]$ for every $\mathcal{F}$.

We claim that $\{bf_{\mathcal{F}}\}_{\mathcal{F}\supseteq\emptyset}$ is an associative sequence. For any pair of finite subsets $\mathcal{F}$ and $\mathcal{K}$ of $\mathcal{X}$ such that $\mathcal{F} \subseteq \mathcal{K}$, we have $bI_{\mathcal{K}} = bf_{\mathcal{K}}I_0[[\mathcal{K}]]$ and $bI_{\mathcal{F}} = bf_{\mathcal{F}}I_0[[\mathcal{F}]]$. Since $b \in I_0$ and $\{I_{\mathcal{F}}\}_{\mathcal{F}\supseteq\emptyset}$ is projective, it follows from the $\mathcal{F}$-projection of $bI_{\mathcal{K}}$ that

$$bf_{\mathcal{F}}I_0[[\mathcal{F}]] = bI_{\mathcal{F}} = \mathcal{F}(bI_{\mathcal{K}}) = \mathcal{F}(bf_{\mathcal{K}}I_0[[\mathcal{K}]]) = \mathcal{F}(bf_{\mathcal{K}})I_0[[\mathcal{F}]].$$

Since I_0 is an invertible ideal of R, $I_0[[\mathcal{F}]]$ is also an invertible ideal of $R[[\mathcal{F}]]$. We cancel $I_0[[\mathcal{F}]]$ and obtain the equality $bf_{\mathcal{F}}R[[\mathcal{F}]] = \mathcal{F}(bf_{\mathcal{K}})R[[\mathcal{F}]]$. Therefore, $bf_{\mathcal{F}} \sim \mathcal{F}(bf_{\mathcal{K}})$ for any pair of finite subsets $\mathcal{F}$ and $\mathcal{K}$ of $\mathcal{X}$ such that $\mathcal{F} \subseteq \mathcal{K}$. Thus the claim is proved.

Since $\{bf_{\mathcal{F}}\}_{\mathcal{F}\supseteq\emptyset}$ is an associative sequence, by Theorem 2.1(6) there exists an element $f \in R[[\mathcal{X}]]_3$ such that $\mathcal{F}(f) \sim bf_{\mathcal{F}}$ for every finite subset $\mathcal{F}$ of $\mathcal{X}$. Thus $\mathcal{F}(f)R[[\mathcal{F}]] = bf_{\mathcal{F}}R[[\mathcal{F}]]$. Put $I = b^{-1}fI_0[[\mathcal{X}]]_3$. Since $\mathcal{F}(f) \sim bf_{\mathcal{F}}$ for every finite subset $\mathcal{F}$ of $\mathcal{X}$, we have $\mathcal{F}(I) = b^{-1}\mathcal{F}(f)I_0[[\mathcal{F}]] = f_{\mathcal{F}}I_0[[\mathcal{F}]] = I_{\mathcal{F}}$. Thus $\mathcal{F}(I) = I_{\mathcal{F}}$ for every finite subset $\mathcal{F}$ of $\mathcal{X}$. So $I \subseteq R[[\mathcal{X}]]_3$. It is clear that $I = b^{-1}fI_0[[\mathcal{X}]]_3$ is an invertible ideal of $R[[\mathcal{X}]]_3$. The uniqueness of I follows from Theorem 3.6. □

Corollary 3.10. *Let R be an integral domain with quotient field K, and $\mathcal{X}$ be a set of indeterminates over R. If a sequence $\{I_{\mathcal{K}}\}_{\mathcal{K}\supseteq\mathcal{F}}$ of invertible ideals of $R[[\mathcal{K}]]$ over $\mathcal{F}$ is projective, then there exists a unique invertible ideal I of the form $fI'[[\mathcal{X}]]_3$ in $R[[\mathcal{X}]]_3$, where $f \in K[[\mathcal{X}]]_3$ and I' is an invertible ideal of R, such that $\mathcal{K}(I) = I_{\mathcal{K}}$ in $R[[\mathcal{F}]]$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$.*

Proof. Put $D := R[[\mathcal{F}]]$. Note that $R[[\mathcal{X}]]_3 = D[[\mathcal{X}']]_3$, where $\mathcal{X}' = \mathcal{X} \setminus \mathcal{F}$. Then $\{I_{\mathcal{K}}\}_{\mathcal{K}\supseteq\mathcal{F}} = \{I_{\mathcal{K}'}\}_{\mathcal{K}'\supseteq\emptyset}$ is a projective sequence of invertible ideals of $D[[\mathcal{K}']]$. By Theorem 3.9, there exists an invertible ideal $I := gI_{\mathcal{F}}[[\mathcal{X}']]_3$ of $D[[\mathcal{X}']]_3 = R[[\mathcal{X}]]_3$ such that $\mathcal{K}(I) = I_{\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. Since $I_{\mathcal{F}}$ is an invertible ideal of $D = R[[\mathcal{F}]]$, by applying Corollary 3.3 to $I_{\mathcal{F}}$, we have

$$I = g(hI'[[\mathcal{F}]])[[\mathcal{X} \setminus \mathcal{F}]]_3 = fI'[[\mathcal{X}]]_3,$$

where $f = gh$ and I' is an invertible ideal of R. Note that $f \in K[[\mathcal{X}]]_3$ since $af \in I \subseteq R[[\mathcal{X}]]_3$, where $0 \neq a \in I'$. □

We establish a relation between invertible ideals of $R[[\mathcal{X}]]_3$ and their projections.

Theorem 3.11. *Let R be an integral domain with quotient field K. If I is an invertible ideal of $R[[\mathcal{X}]]_3$, then $I = fI'[[\mathcal{X}]]_3$, where I' is an invertible ideal of R and $f \in K[[\mathcal{X}]]_3$. Moreover, if $I_0 = \emptyset(I) \neq (0)$, then $I = fI_0[[\mathcal{X}]]_3$, where $f \in K[[\mathcal{X}]]_3$.*

Proof. Let I be an invertible ideal of $R[[\mathcal{X}]]_3$. Put $I_0 = \emptyset(I)$. We will divide the proof into two cases; either $I_0 \neq (0)$ or $I_0 = (0)$.

Case I : $I_0 \neq (0)$ (note that I_0 is an invertible ideal of R by Lemma 3.1).

By Lemma 3.1, $\mathcal{K}(I)$ is an invertible ideal of $R[[\mathcal{K}]]$ for every finite subset $\mathcal{K}$ of $\mathcal{X}$. Since $\{\mathcal{K}(I)\}_{\mathcal{K}\supseteq\emptyset}$ is a projective sequence of invertible ideals, by Theorem 3.9 there exists an invertible ideal $I' = fI_0[[\mathcal{X}]]_3$ of $R[[\mathcal{X}]]_3$, where $f \in K[[\mathcal{X}]]_3$ such that $\mathcal{K}(I') = \mathcal{K}(f)I_0[[\mathcal{K}]] = \mathcal{K}(I)$ for every finite subset $\mathcal{K}$ of $\mathcal{X}$. We claim $I \subseteq I' = fI_0[[\mathcal{X}]]_3$. Let $i \in I$. Since $\mathcal{K}(I) = \mathcal{K}(f)I_0[[\mathcal{K}]]$, we have $\mathcal{K}(i) = \mathcal{K}(f)a_{\mathcal{K}}$, where $a_{\mathcal{K}} \in I_0[[\mathcal{K}]]$. Note that $\{a_{\mathcal{K}}\}_{\mathcal{K}\supseteq\emptyset}$ is a projective sequence. By Theorem 2.1(5), there exists a power series $a \in R[[\mathcal{X}]]_3$ such that $\mathcal{K}(a) = a_{\mathcal{K}}$ for every finite subset $\mathcal{K}$ of $\mathcal{X}$. Clearly $a \in I_0[[\mathcal{X}]]_3$ since $\mathcal{K}(a) = a_{\mathcal{K}} \in I_0[[\mathcal{K}]]$ for every finite subset $\mathcal{K}$ of $\mathcal{X}$. Therefore, for every finite subset $\mathcal{K}$ of $\mathcal{X}$, we have

$$\mathcal{K}(i) = \mathcal{K}(f)a_{\mathcal{K}} = \mathcal{K}(f)\mathcal{K}(a) = \mathcal{K}(fa).$$

By Theorem 2.1(4), $i = fa \in fI_0[[\mathcal{X}]]_3$. Thus the claim is proved. Since $I' = fI_0[[\mathcal{X}]]_3$ is an invertible ideal of $R[[\mathcal{X}]]_3$ containing I, and $\mathcal{K}(I) = \mathcal{K}(I')$, we can apply Theorem 3.5 and obtain the equality $I = I' = fI_0[[\mathcal{X}]]_3$.

Case II : $\mathcal{K}(I) \neq (0)$ for some finite subset $\mathcal{K}$ of $\mathcal{X}$ (note that $\mathcal{K}(I)$ is an invertible ideal of $R[[\mathcal{K}]]$ by Lemma 3.1).

Let $D = R[[\mathcal{K}]]$. Then $R[[\mathcal{X}]]_3 = D[[\mathcal{X}']]_3$, where $\mathcal{X}' = \mathcal{X} \setminus \mathcal{K}$. By Case I, $I = gJ[[\mathcal{X}']]_3$, where $J = \mathcal{K}(I)$ is an invertible ideal of $R[[\mathcal{K}]]$. By Corollary 3.3, $J = hI'[[\mathcal{K}]]$, where I' is an invertible ideal of R. Therefore, $I = ghI'[[\mathcal{K} \cup \mathcal{X}']]_3 = fI'[[\mathcal{X}]]_3$, where $f = gh$. Note that $f \in K[[\mathcal{X}]]_3$ since $af \in I \subseteq R[[\mathcal{X}]]_3$, where $0 \neq a \in I'$. □

Let R be an integral domain. The set $\mathcal{I}(R)$ of invertible fractional ideals of R is a group under ideal multiplication. The set $\mathcal{P}(R)$ of principal fractional ideals of R forms a subgroup of $\mathcal{I}(R)$. The factor group $\mathcal{I}(R)/\mathcal{P}(R)$ is called the *Picard group* of R. In the case when R is a Krull domain, let $\mathcal{D}(R)$ be the group of divisorial ideals of R. Then $\mathcal{D}(R)/\mathcal{P}(R)$ is called the *divisor class group* of R. We denote the divisor class group and Picard group of R by $\mathrm{Cl}(R)$ and $\mathrm{Pic}(R)$, respectively.

It is well known that $\mathrm{Pic}(R) \cong \mathrm{Pic}(R[[X]])$ in the single variable case. This result can be generalized to an arbitrary number of variables using Theorem 3.11.

Corollary 3.12. *Let R be an integral domain. The natural mapping $J \mapsto J[[\mathcal{X}]]_3$, where J is an ideal of R, induces an isomorphism* $\mathrm{Pic}(R) \cong \mathrm{Pic}(R[[\mathcal{X}]]_3)$.

4. $R[[\mathcal{X}]]_3$ is a π-domain if R is a formally stable π-domain

Let R be an integral domain and $\mathcal{X}$ be a set of indeterminates over R. If $R[[X_1, \dots, X_n]]$ is a π-domain for each finite set $\{X_1, \dots, X_n\}$ of indeterminates over R, then we say that R is a *formally stable π-domain*. Note that if R is a formally stable π-domain, then R is also a π-domain. In this section, we will prove that if R is a formally stable π-domain, then $R[[\mathcal{X}]]_3$ is also a π-domain.

If R is a π-domain, then every principal ideal (a) of R is a product of a finite number of prime ideals of R. Since the prime ideals in this factorization are invertible, the factorization is unique. We denote by $\|a\|$ and $N(a)$ the number of prime ideals and distinct prime ideals in the ideal factorization of (a), respectively.

Lemma 4.1. *Let δ be a nonunit in $R[[\mathcal{X}]]_3$, where R is a formally stable π-domain. Then*

(i) $\|\mathcal{K}(\delta)\| \leq \|\mathcal{F}(\delta)\|$ *whenever $\mathcal{F} \subseteq \mathcal{K}$, where $\mathcal{F}$ and $\mathcal{K}$ are finite subsets of $\mathcal{X}$,*

(ii) *there exists a finite subset $\mathcal{F}_\delta$, depending on δ, of $\mathcal{X}$ such that for every finite subset $\mathcal{K} \supseteq \mathcal{F}_\delta$ of $\mathcal{X}$, $\|\mathcal{K}(\delta)\| = \|\mathcal{F}_\delta(\delta)\|$; furthermore, if $(\mathcal{F}_\delta(\delta)) = P_{1,\mathcal{F}_\delta} \cdots P_{n,\mathcal{F}_\delta}$ and $(\mathcal{K}(\delta)) = P_{1,\mathcal{K}} \cdots P_{n,\mathcal{K}}$, where $P_{i,\mathcal{F}}$'s and $P_{i,\mathcal{K}}$'s are prime ideals of $R[[\mathcal{F}]]$ and $R[[\mathcal{K}]]$, respectively, then $\mathcal{F}_\delta(P_{i,\mathcal{K}}) = P_{i,\mathcal{F}_\delta}$ for $i = 1, \ldots, n$ after rearranging the indices.*

Proof. (i) Since δ is a nonunit, it follows from Theorem 2.1 that $\mathcal{F}(\delta)$ is a nonunit in $R[[\mathcal{F}]]$ for any finite subset $\mathcal{F}$ of $\mathcal{X}$. Thus $\|\mathcal{F}(\delta)\| \geq 0$. Let $\mathcal{F} \subseteq \mathcal{K}$ be any pair of finite subsets of $\mathcal{X}$ such that $\|\mathcal{F}(\delta)\| = n$, and $\|\mathcal{K}(\delta)\| = m$. Suppose that $n < m$. Since R is a formally stable π-domain, $(\mathcal{F}(\delta)) = P_{1,\mathcal{F}} \cdots P_{n,\mathcal{F}}$ and $(\mathcal{K}(\delta)) = P_{1,\mathcal{K}} \cdots P_{m,\mathcal{K}}$, where $P_{i,\mathcal{F}}$'s and $P_{j,\mathcal{K}}$'s are (not necessarily distinct) prime ideals of $R[[\mathcal{F}]]$ and $R[[\mathcal{K}]]$, respectively. From Theorem 2.1, we obtain

$$\mathcal{F}(\mathcal{K}(\delta)) = \mathcal{F}(P_{1,\mathcal{K}} \cdots P_{m,\mathcal{K}}) = \mathcal{F}(P_{1,\mathcal{K}}) \cdots \mathcal{F}(P_{m,\mathcal{K}}) = P_{1,\mathcal{F}} \cdots P_{n,\mathcal{F}} = (\mathcal{F}(\delta)).$$

Since $P_{1,\mathcal{F}}$ is a prime ideal of $R[[\mathcal{F}]]$, we may assume that $\mathcal{F}(P_{1,\mathcal{K}}) \subseteq P_{1,\mathcal{F}}$. Moreover, it follows from the invertibility of $P_{1,\mathcal{F}}$ that $\mathcal{F}(P_{1,\mathcal{K}}) = P_{1,\mathcal{F}} I_{1,\mathcal{F}}$ for some ideal $I_{1,\mathcal{F}}$ of $R[[\mathcal{F}]]$. Note that $I_{1,\mathcal{F}}$ need not be a proper ideal. By cancellation, we get

$$I_{1,\mathcal{F}} \mathcal{F}(P_{2,\mathcal{K}}) \cdots \mathcal{F}(P_{m,\mathcal{K}}) = P_{2,\mathcal{F}} P_{3,\mathcal{F}} \cdots P_{n,\mathcal{F}}. \tag{1}$$

Since $n < m$, if we continue this process on $P_{i,\mathcal{F}}$'s for $i = 2, \ldots, n$, then there exists at least one $P_{j,\mathcal{K}}$ such that $R[[\mathcal{F}]] = \mathcal{F}(P_{j,\mathcal{K}}) I$, where I is an ideal of $R[[\mathcal{F}]]$. Since $P_{j,\mathcal{K}}$ is a proper ideal of $R[[\mathcal{K}]]$, $\mathcal{F}(P_{j,\mathcal{K}})$ is a proper ideal of $R[[\mathcal{F}]]$. This is a contradiction. Therefore, $\|\mathcal{K}(\delta)\| \leq \|\mathcal{F}(\delta)\|$.

(ii) Suppose that there are no finite subsets of $\mathcal{X}$ satisfying the condition above. Then there exists an infinite chain $\{\mathcal{F}_i\}$ of finite subsets of $\mathcal{X}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ and $\|\mathcal{F}_1(\delta)\| > \|\mathcal{F}_2(\delta)\| > \cdots$, which contradicts the finiteness of $\|\mathcal{F}_1(\delta)\|$. Hence there exists a finite subset $\mathcal{F}_\delta$, depending on δ, of $\mathcal{X}$ such that for every finite subset $\mathcal{K} \supseteq \mathcal{F}_\delta$ of $\mathcal{X}$, $\|\mathcal{K}(\delta)\| = \|\mathcal{F}_\delta(\delta)\|$.

Let $(\mathcal{F}_\delta(\delta)) = P_{1,\mathcal{F}_\delta} \cdots P_{n,\mathcal{F}_\delta}$ and $(\mathcal{K}(\delta)) = P_{1,\mathcal{K}} \cdots P_{n,\mathcal{K}}$. Since $\|\mathcal{K}(\delta)\| = \|\mathcal{F}_\delta(\delta)\|$, by the same argument as in the proof of (i), we may assume that $\mathcal{F}_\delta(P_{i,\mathcal{K}}) \subseteq P_{i,\mathcal{F}_\delta}$ and $\mathcal{F}_\delta(P_{i,\mathcal{K}}) = P_{i,\mathcal{F}_\delta} I_{i,\mathcal{F}_\delta}$ for $i = 1, \ldots, n$, after rearranging the indices. By cancellation, we can get

$$I_{1,\mathcal{F}_\delta} \cdots I_{n,\mathcal{F}_\delta} = R[[\mathcal{F}_\delta]]. \tag{2}$$

Since $I_{i,\mathcal{F}_\delta}$ is an ideal of $R[[\mathcal{F}_\delta]]$, $I_{i,\mathcal{F}_\delta} = R[[\mathcal{F}_\delta]]$ and $\mathcal{F}_\delta(P_{i,\mathcal{K}}) = P_{i,\mathcal{F}_\delta}$ for $i = 1, \ldots, n$. □

Remark. (1) For any finite subset $\mathcal{K}$ of $\mathcal{X}$ containing $\mathcal{F}_\delta$, if $P_\mathcal{K}$ is a prime ideal of $R[[\mathcal{K}]]$ appearing in the ideal factorization of $\mathcal{K}(\delta)$, then $\mathcal{F}_\delta(P_\mathcal{K})$ is a prime ideal of $R[[\mathcal{F}_\delta]]$ appearing in the ideal factorization of $\mathcal{F}_\delta(\delta)$.

(2) Although $P_\mathcal{K}$ and $P'_\mathcal{K}$ are prime ideals of $R[[\mathcal{K}]]$ appearing in the ideal factorization of $\mathcal{K}(\delta)$ such that $\mathcal{F}_\delta(P_\mathcal{K}) = \mathcal{F}_\delta(P'_\mathcal{K})$, $P_\mathcal{K}$ and $P'_\mathcal{K}$ need not be the same. Thus it may happen that $N(\mathcal{K}(\delta)) \neq N(\mathcal{F}_\delta(\delta))$ even if $\|\mathcal{K}(\delta)\| = \|\mathcal{F}_\delta(\delta)\|$.

Lemma 4.2. *Let δ be a nonunit in $R[[\mathcal{X}]]_3$, where R is a formally stable π-domain. Then*

(i) *$N(\mathcal{F}_\delta(\delta)) \leq N(\mathcal{K}(\delta))$ for any finite subset $\mathcal{K}$ of $\mathcal{X}$ containing $\mathcal{F}_\delta$,*

(ii) *there exists a finite subset $\mathcal{F}_0$ of $\mathcal{X}$ such that for every finite subset $\mathcal{K}$ of $\mathcal{X}$ containing $\mathcal{F}_0$, $N(\mathcal{F}_0(\delta)) = N(\mathcal{K}(\delta))$ and $\|\mathcal{F}_0(\delta)\| = \|\mathcal{K}(\delta)\|$.*

Proof. (i) Let $P_{\mathcal{K}}$ be a prime ideal of $R[[\mathcal{K}]]$ in the ideal factorization of $\mathcal{K}(\delta)$. Since $\mathcal{F}_\delta(P_{\mathcal{K}})$ is a prime ideal of $R[[\mathcal{F}]]$ containing $\mathcal{F}_\delta(\delta)$, it is exactly one of prime ideals of $R[[\mathcal{F}]]$ in the ideal factorization of $\mathcal{F}_\delta(\delta)$. Therefore, $N(\mathcal{F}_\delta(\delta)) \leq N(\mathcal{K}(\delta))$.

(ii) For any pair of finite subsets $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{X}$ containing $\mathcal{F}_\delta$ such that $\mathcal{K} \subseteq \mathcal{L}$, we have $N(\mathcal{F}_\delta(\delta)) \leq N(\mathcal{K}(\delta)) \leq N(\mathcal{L}(\delta))$ and $\|\mathcal{F}_\delta(\delta)\| = \|\mathcal{K}(\delta)\| = \|\mathcal{L}(\delta)\|$. Since $N(\mathcal{K}(\delta)) \leq \|\mathcal{K}(\delta)\| = \|\mathcal{F}_\delta(\delta)\|$, and $\|\mathcal{F}_\delta(\delta)\|$ is finite, we can choose a finite subset $\mathcal{F}_0$ of $\mathcal{X}$ containing $\mathcal{F}_\delta$ such that for any finite subset $\mathcal{K} \supseteq \mathcal{F}_0$ of $\mathcal{X}$, $N(\mathcal{F}_0(\delta)) = N(\mathcal{K}(\delta))$ and $\|\mathcal{F}_0(\delta)\| = \|\mathcal{K}(\delta)\|$. □

Suppose that $(\mathcal{F}_0(\delta)) = P_{1,\mathcal{F}_0}^{m_1} \cdots P_{n,\mathcal{F}_0}^{m_n}$, where $P_{i,\mathcal{F}_0}$'s are distinct prime ideals of $R[[\mathcal{F}_0]]$. Since $N(\mathcal{F}_0(\delta)) = N(\mathcal{K}(\delta))$ and $\|\mathcal{F}_0(\delta)\| = \|\mathcal{K}(\delta)\|$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}_0$ of $\mathcal{X}$, we may assume that $(\mathcal{K}(\delta)) = P_{1,\mathcal{K}}^{m_1} \cdots P_{n,\mathcal{K}}^{m_n}$ and $\mathcal{F}_0(P_{i,\mathcal{K}}) = P_{i,\mathcal{F}_0}$ for each i, by rearranging the indices.

The following is an immediate consequence of the observation above.

Proposition 4.3. *Let δ be a nonunit in $R[[\mathcal{X}]]_3$, where R is a formally stable π-domain. Suppose that $(\mathcal{F}_0(\delta)) = P_{1,\mathcal{F}_0}^{m_1} \cdots P_{n,\mathcal{F}_0}^{m_n}$ is the factorization of $\mathcal{F}_0(\delta)$, where $\mathcal{F}_0$ is a finite subset of $\mathcal{X}$ defined in Lemma 4.2. Then, for every finite subset $\mathcal{K}$ of $\mathcal{X}$ containing $\mathcal{F}_0$, $(\mathcal{K}(\delta)) = P_{1,\mathcal{K}}^{m_1} \cdots P_{n,\mathcal{K}}^{m_n}$, and $\{P_{i,\mathcal{K}}\}_{\mathcal{K}\supseteq\mathcal{F}}$ is a projective sequence for $i = 1, \ldots, n$.*

Remark. In Proposition 4.3, $\{P_{i,\mathcal{K}}\}_{\mathcal{K}\supseteq\mathcal{F}_0}$ is a projective sequence of invertible prime ideals of $R[[\mathcal{K}]]$. According to Corollary 3.10, there exists an invertible ideal Q_i of the form $fP_i'[[\mathcal{X}]]_3$ in $R[[\mathcal{X}]]_3$, where $f \in K[[\mathcal{X}]]_3$ and P_i' is an invertible ideal of R, such that $\mathcal{K}(Q_i) = P_{i,\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}_0$ of $\mathcal{X}$.

From a projective sequence of prime ideals, we will produce a prime ideal.

Lemma 4.4. *Let R be an integral domain and $\mathcal{F}$ be a finite subset of $\mathcal{X}$. Suppose that $\{P_{\mathcal{K}}\}_{\mathcal{K}\supseteq\mathcal{F}}$ is a projective sequence of prime ideals, where $\mathcal{K}$ is an arbitrary finite subset of $\mathcal{X}$ containing $\mathcal{F}$. Then the set $\{\alpha \in R[[\mathcal{X}]]_3 \mid \mathcal{K}(\alpha) \in P_{\mathcal{K}}$ for every finite $\mathcal{K} \supseteq \mathcal{F}\}$ is a prime ideal of $R[[\mathcal{X}]]_3$.*

Proof. Put $P_\infty = \{\alpha \in R[[\mathcal{X}]]_3 \mid \mathcal{K}(\alpha) \in P_{\mathcal{K}}$ for every finite $\mathcal{K} \supseteq \mathcal{F}\}$. Since $0 \in P_\infty$, P_∞ is a nonempty set. Since the map $\alpha \mapsto \mathcal{K}(\alpha)$ is a ring homomorphism, P_∞ is an ideal of $R[[\mathcal{X}]]_3$. Suppose that $\alpha \notin P_\infty$ and $\beta \notin P_\infty$. Since $\{P_{\mathcal{K}}\}_{\mathcal{K}\supseteq\mathcal{F}}$ is a projective sequence, we can choose finite subsets $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ of $\mathcal{X}$ containing $\mathcal{F}$ such that $\mathcal{F}_\alpha(\alpha) \notin P_{\mathcal{F}_\alpha}$ and $\mathcal{F}_\beta(\beta) \notin P_{\mathcal{F}_\beta}$. Note that $\mathcal{K}(\alpha) \notin P_{\mathcal{K}}$ and $\mathcal{L}(\beta) \notin P_{\mathcal{L}}$ for any finite subsets $\mathcal{K} \supseteq \mathcal{F}_\alpha$ and $\mathcal{L} \supseteq \mathcal{F}_\beta$. Put $\mathcal{F}' = \mathcal{F}_\alpha \cup \mathcal{F}_\beta$. Note that $\mathcal{F} \subseteq \mathcal{F}'$, $\mathcal{K}(\alpha) \notin P_{\mathcal{K}}$, and $\mathcal{K}(\beta) \notin P_{\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}'$ of $\mathcal{X}$. Since $P_{\mathcal{K}}$ is a prime ideal, $\mathcal{K}(\alpha\beta) = \mathcal{K}(\alpha)\mathcal{K}(\beta) \notin P_{\mathcal{K}}$. Thus $\alpha\beta \notin P_\infty$. Therefore P_∞ is a prime ideal of $R[[\mathcal{X}]]_3$. □

Now we are ready to show that $R[[\mathcal{X}]]_3$ is a π-domain over a formally stable π-domain R.

Theorem 4.5. *Let R be an integral domain and $\mathcal{X}$ be a set of indeterminates. If $R[[X_1, \ldots, X_n]]$ is a π-domain for every integer $n \geq 1$, then $R[[\mathcal{X}]]_3$ is a π-domain.*

Proof. Let δ be a nonunit element of $R[[\mathcal{X}]]_3$. By Proposition 4.3, there exists a finite subset $\mathcal{F}$, depending on δ, of $\mathcal{X}$ such that for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$,

$$(\mathcal{F}(\delta)) = P_{1,\mathcal{F}}^{m_1} \cdots P_{n,\mathcal{F}}^{m_n} \quad \text{and} \quad (\mathcal{K}(\delta)) = P_{1,\mathcal{K}}^{m_1} \cdots P_{n,\mathcal{K}}^{m_n}, \tag{3}$$

where $P_{i,\mathcal{F}}$'s and $P_{i,\mathcal{K}}$'s are distinct prime ideals of $R[[\mathcal{F}]]$ and $R[[\mathcal{K}]]$, respectively. Moreover, $\{P_{i,\mathcal{K}}\}_{\mathcal{K}\supseteq\mathcal{F}}$ is a projective sequence of invertible prime ideals for each i. Put $P_{i,\infty} = \{\alpha \in R[[\mathcal{X}]]_3 \mid \mathcal{K}(\alpha) \in P_{i,\mathcal{K}}$ for every finite $\mathcal{K} \supseteq \mathcal{F}\}$. Note that $\mathcal{K}(P_{i,\infty}) \subseteq P_{i,\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. Since $\delta \in P_{i,\infty}$, it follows from Lemma 4.4 that $P_{i,\infty}$ is a prime ideal of $R[[\mathcal{X}]]_3$ containing δ.

We claim that $(\delta) = P_{1,\infty}^{m_1} \cdots P_{n,\infty}^{m_n}$.

Let $x \in P_{1,\infty}^{m_1} \cdots P_{n,\infty}^{m_n}$. Since $\mathcal{K}(P_{i,\infty}) \subseteq P_{i,\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$ and each i, it follows from Theorem 2.1(7) that

$$\mathcal{K}(x) \in \mathcal{K}(P_{1,\infty}^{m_1} \cdots P_{n,\infty}^{m_n}) \subseteq P_{1,\mathcal{K}}^{m_1} \cdots P_{n,\mathcal{K}}^{m_n} = (\mathcal{K}(\delta)). \tag{4}$$

Thus $\mathcal{K}(x) = \mathcal{K}(\delta)\gamma_{\mathcal{K}}$ for some $\gamma_{\mathcal{K}} \in R[[\mathcal{K}]]$. For any pair of finite subsets $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{X}$ such that $\mathcal{F} \subseteq \mathcal{K} \subseteq \mathcal{L}$, we have $\mathcal{L}(x) = \mathcal{L}(\delta)\gamma_{\mathcal{L}}$ and $\mathcal{K}(x) = \mathcal{K}(\delta)\gamma_{\mathcal{K}}$. Since $\mathcal{K}(\mathcal{L}(x)) = \mathcal{K}(x)$ and $\mathcal{K}(\mathcal{L}(\delta)) = \mathcal{K}(\delta)$, we have $\mathcal{K}(\gamma_{\mathcal{L}}) = \gamma_{\mathcal{K}}$. Therefore, $\{\gamma_{\mathcal{K}}\}_{\mathcal{K}\supseteq\mathcal{F}}$ is a projective sequence. In view of Theorem 2.1(5), there exists $\gamma \in R[[\mathcal{X}]]_3$ such that $\mathcal{K}(\gamma) = \gamma_{\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. Then $\mathcal{K}(x) = \mathcal{K}(\delta)\gamma_{\mathcal{K}} = \mathcal{K}(\delta)\mathcal{K}(\gamma) = \mathcal{K}(\delta\gamma)$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$. It follows from Theorem 2.1(4) that $x = \delta\gamma$. Therefore $P_{1,\infty}^{m_1} \cdots P_{n,\infty}^{m_n} \subseteq (\delta)$.

It follows from the remark just after Proposition 4.3 that there exist invertible ideals Q_i in $R[[\mathcal{X}]]_3$ such that $\mathcal{K}(Q_i) = P_{i,\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$ and each i. Note that $Q_i \subseteq P_{i,\infty}$ for each i. Since $\mathcal{K}(Q_i) = P_{i,\mathcal{K}}$ for every finite subset $\mathcal{K} \supseteq \mathcal{F}$ of $\mathcal{X}$, we have

$$\begin{cases} Q_1^{m_1} \cdots Q_n^{m_n} \subseteq P_{1,\infty}^{m_1} \cdots P_{n,\infty}^{m_n} \subseteq (\delta), \\ \mathcal{K}(Q_1^{m_1} \cdots Q_n^{m_n}) = \mathcal{K}(Q_1)^{m_1} \cdots \mathcal{K}(Q_n)^{m_n} = P_{1,\mathcal{K}}^{m_1} \cdots P_{n,\mathcal{K}}^{m_n} = (\mathcal{K}(\delta)). \end{cases}$$

Thus $Q_1^{m_1} \cdots Q_n^{m_n} \subseteq (\delta)$ and their $\mathcal{K}$-projections are the same. Therefore, by Theorem 3.5, $Q_1^{m_1} \cdots Q_n^{m_n} = (\delta)$ and hence $(\delta) = P_{1,\infty}^{m_1} \cdots P_{n,\infty}^{m_n}$. □

Remark. Since $P_{i,\infty}$ is an invertible ideal of $R[[\mathcal{X}]]_3$ containing Q_i and $\mathcal{K}(P_{i,\infty}) = \mathcal{K}(Q_i)$ for a finite subset $\mathcal{K}$, it follows from Theorem 3.5 that $P_{i,\infty} = Q_i = fP_i'[[\mathcal{X}]]_3$ in $R[[\mathcal{X}]]_3$, where $f \in K[[\mathcal{X}]]_3$ and P_i' is an invertible ideal of R.

If R is a Noetherian regular domain, then $R[[X_1, \ldots, X_n]]$ is also a Noetherian regular domain. Since a Noetherian regular domain is a π-domain, R is a formally stable π-domain. In particular, if R is a Dedekind domain, then R is a formally stable π-domain. The following is an immediate application of Theorem 4.5.

Corollary 4.6. *Let R be an integral domain and $\mathcal{X}$ be a set of indeterminates over R.*

(i) *If R is a Noetherian regular domain, then $R[[\mathcal{X}]]_3$ is a π-domain.*
(ii) *If R is a Dedekind domain, then $R[[\mathcal{X}]]_3$ is a π-domain.*

In [6], Claborn showed that if R is a Noetherian regular domain, then $\mathrm{Cl}(R) \cong \mathrm{Cl}(R[[X_1, \dots, X_n]])$ canonically and $\mathrm{Cl}(R[[X_1, \dots, X_n]]_{R^*}) = 0$, where R^* is the set of nonzero elements of R. Since a regular domain is a π-domain, our next result is a generalization of Claborn's results to a formally stable π-domain as well as to an arbitrary set of indeterminates.

Theorem 4.7. *If R is a formally stable π-domain and $\mathcal{X}$ is a set of indeterminates over R, then*

(i) $\mathrm{Cl}(R) \cong \mathrm{Cl}(R[[\mathcal{X}]]_3)$ *canonically,*
(ii) $\mathrm{Cl}((R[[\mathcal{X}]]_3)_{R^*}) = 0$*; thus* $(R[[\mathcal{X}]]_3)_{R^*}$ *is a UFD.*

Proof. (i) By Theorem 4.5, $R[[\mathcal{X}]]_3$ is a π-domain. By Theorem 2.2(6), every divisorial ideal of $R[[\mathcal{X}]]_3$ is invertible. Since $\mathrm{Cl}(R[[\mathcal{X}]]_3) = \mathrm{Pic}(R[[\mathcal{X}]]_3)$, it follows from Corollary 3.12 that $\mathrm{Cl}(R) \cong \mathrm{Cl}(R[[\mathcal{X}]]_3)$.

(ii) Let J be a divisorial ideal of $(R[[\mathcal{X}]]_3)_{R^*}$. It is easy to see that $J = I_{R^*}$, where I is a divisorial ideal of $R[[\mathcal{X}]]_3$ [12]. By Theorem 3.11, $I = fI'[[\mathcal{X}]]_3$, where I' is an invertible ideal of R. Therefore, $I_{R^*} = (fI'[[\mathcal{X}]]_3)_{R^*} = f(R[[\mathcal{X}]]_3)_{R^*}$. Thus $J = I_{R^*}$ is a principal ideal. □

Corollary 4.8. *If R is a Noetherian regular domain and $\mathcal{X}$ is a set of indeterminates over R, then*

(i) $\mathrm{Cl}(R) \cong \mathrm{Cl}(R[[\mathcal{X}]]_3)$ *canonically,*
(ii) $\mathrm{Cl}((R[[\mathcal{X}]]_3)_{R^*}) = 0$*; thus* $(R[[\mathcal{X}]]_3)_{R^*}$ *is a UFD.*

5. Extension to rings with zero-divisors

Let R be a commutative ring with identity. We use the notation $\dim R$ for the Krull dimension of R. R is called a *π-ring* if every principal ideal is a product of prime ideals. If $R[[X_1, \dots, X_n]]$ is a π-ring for each finite set $\{X_1, \dots, X_n\}$ of indeterminates over R, then R is called a *formally stable π-ring*. Let $\mathcal{X}$ be a set of indeterminates over R. In this section, we will prove that if R is a formally stable π-ring, then $R[[\mathcal{X}]]_3$ is also a π-ring. As a corollary, it will follow that $R[[\mathcal{X}]]_3$ is a π-ring if R is a Noetherian regular ring. We abbreviate a Noetherian regular ring as a *regular ring*.

A *special primary ring* R is a quasi-local ring with maximal ideal M such that each proper ideal of R is a power of M. It is clear that if R is a special primary ring, then $\dim R = 0$.

Theorem 5.1 ([11]). *A quasi-local π-ring R with dimension greater than zero is an integral domain.*

Lemma 5.2 ([11]). *If $\{R_i\}_{i=1}^n$ is a finite set of ideals of R such that $R = R_1 \oplus \cdots \oplus R_n$, then R is a π-ring if and only if each R_i is a π-ring.*

Theorem 5.3 ([11, 15]). *R is a π-ring if and only if R is a finite direct sum of π-domains and special primary rings.*

Theorem 5.4. *If R is a formally stable π-ring, then $R[[\mathcal{X}]]_3$ is a π-ring.*

Proof. Replacing R by $R[[X]]$, we may assume that R is a π-ring. By Theorem 5.3, R can be written as

$$R = R_1 \oplus \cdots \oplus R_n,$$

where R_i is either a π-domain or a special primary ring. A special primary ring is clearly a π-ring. So each R_i is a π-ring. Since R is a formally stable π-ring, $R_1[[X]] \oplus \cdots \oplus R_n[[X]] = R[[X]]$ is a π-ring. By Lemma 5.2, each $R_i[[X]]$ is a π-ring. We will show that each R_i is in fact a π-domain. Assume that R_i is a special primary ring. Since $R_i[[X]]$ is a quasi-local π-ring and $\dim R_i[[X]] \geq 1$, $R_i[[X]]$ is an integral domain by Theorem 5.1 and hence R_i is an integral domain. So R_i is a π-domain.

Now we have $R = R_1 \oplus \cdots \oplus R_n$ and each R_i is a π-domain. Since R is a formally stable π-ring, $\bigoplus_{i=1}^n R_i[[X_1, \ldots, X_m]] = R[[X_1, \ldots, X_m]]$ is a π-ring for each $m \geq 1$. By Lemma 5.2, each $R_i[[X_1, \ldots, X_m]]$ is a π-domain. By Theorem 4.5, each $R_i[[\mathcal{X}]]_3$ is a π-domain. Therefore, by Theorem 5.3, $R[[\mathcal{X}]]_3 = R_1[[\mathcal{X}]]_3 \oplus \cdots \oplus R_n[[\mathcal{X}]]_3$ is a π-ring. □

It is well known that every regular ring is a finite direct sum of regular domains. Since a regular domain is a π-domain, a regular ring is a π-ring by Theorem 5.3. Since $R[[X]]$ is also a regular ring if R is a regular ring, a regular ring is a formally stable π-ring.

Corollary 5.5. *If R is a Noetherian regular ring and $\mathcal{X}$ is a set of indeterminates over R, then $R[[\mathcal{X}]]_3$ is a π-ring.*

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