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On the Lenstra constant associated to the Rosen continued fractions

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Abstract. The purpose of this paper is to describe the relation between the Legendre and the Lenstra constants. Indeed we show that they are equal whenever the Legendre constant exists; in particular, this holds for both Rosen continued fractions and α -continued fractions. We also give the explicit value of the entropy of the Rosen map with respect to the absolutely continuous invariant probability measure.

Keywords. Continued fractions, Hecke group, geodesic flows

1. Introduction

Let x be an irrational number in $[0, 1]$. We denote by p_n/q_n the n -th principal regular continued fraction convergent of x and recall the following.

Theorem (Legendre). *Suppose p and q (> 0) are relatively prime integers and $|x - p/q| < 1/2q^2$. Then p/q is a principal convergent to x . On the other hand, for any $c > 1/2$, there exist x and p/q , which is not a principal convergent, such that $|x - p/q| < c/q^2$.*

In this situation, we say that $1/2$ is the *Legendre constant of regular continued fractions*. Now we consider the error of the principal convergents. We put

$$\Theta_n = q_n^2 \left| x - \frac{p_n}{q_n} \right|.$$

The following fact was proved in [2].

Theorem (Bosma, Jager and Wiedijk, 1983). *For a.e. x , the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{\infty} \{1 \leq j \leq n : \Theta_j \leq t\}$$

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exists for any t , $0 \leq t \leq 1$, and is equal to

$$F(t) = \begin{cases} \frac{t}{\ln 2}, & 0 \leq t \leq 1/2, \\ \frac{1}{\ln 2}(1 - t + \ln 2t), & 1/2 \leq t \leq 1. \end{cases}$$

We note that $F(t)$ is linear in $0 \leq t \leq 1/2$ and not in $1/2 \leq t \leq 1$. We say that $1/2$ is the *Lenstra constant of regular continued fractions* because this fact was conjectured by Doeblin in 1940 [4] implicitly, and then proved by H. W. Lenstra in 1981; see [7, p. 252]. Legendre and Lenstra constants can be defined for other types of continued fraction expansions in a similar manner (e.g. for α -expansions, [16], [2]). In general, it is not hard to show that the Lenstra constant exists and is at least as large as the Legendre constant for each type of continued fraction expansion whenever the Legendre constant exists. However, a number of examples indicate that these constants may be equal; for example see [8], [9]. In this paper, we show that indeed the Lenstra constant is equal to the Legendre constant. Let us explain why their equality is important. The classical Hurwitz theorem states that for every irrational number x there are infinitely many rationals p/q for which

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}} \frac{1}{q^2}$$

and that the constant $1/\sqrt{5}$ is the maximum value for which this property holds. On the other hand, a result of Borel states that for every x and every $n \geq 1$ one has $\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < 1/\sqrt{5}$. Now Hurwitz' result is an immediate consequence of the result by Borel and Legendre stated above. Here we should note that $1/\sqrt{5} < 1/2$. In fact, Legendre's result implies that if one wants to approximate irrational numbers x by rationals p/q , one is—due to Legendre's result—'forced' to use continued fraction convergents. Furthermore, Borel's result (and also the Lenstra conjecture) shows that there are in fact many good rational approximations, which can be easily obtained via the Gauss map which induces the continued fraction expansions for irrational numbers $0 < x < 1$. Since the Lenstra constant can be calculated in a straightforward way, due to the result in this paper, one knows the Legendre constant quite easily.

Motivation of this paper is the metrical theory of the Rosen continued fractions associated to Hecke groups. It is possible to define the Legendre constants in this case even though the convergents of the Rosen continued fractions are not rational numbers. Indeed, in 1985 J. Lehner [13] claimed that the Legendre constant of Rosen continued fractions is greater than or equal to $1/2$ for Hecke groups of any indices. However, the proof was not correct (see [14], actually the correct value is less than $1/2$) and a lower estimate was given in [21] (where the constants depend on the indices of Hecke groups). On the other hand, the Lenstra constant was given in [3] (and [17] for the even indices case). Also [17] claimed (without proof) that the Lenstra constant and the Legendre constant are the same for each Hecke group of even index. Here we note that Corollary 4.1 of [3] did not say that the constant is the best possible one (which means it is the Lenstra constant), but it is not hard to see that this is indeed the case. We refer to [10] for this point.

In the following, we show that the Lenstra constant is equal to the Legendre constant for the Rosen continued fractions. In the next section, we introduce a generalized Diophantine approximation problem associated to a zonal Fuchsian group and give a law of large numbers for solutions of a Diophantine inequality. This ensures the existence of the Lenstra constant whenever the Legendre constant exists, and also implies that the Lenstra constant is at least as large as the Legendre constant. In §3, we prove the equality of these two constants, by showing that the Lenstra constant cannot be larger than the Legendre constant. As a corollary (of the proof), we get the explicit value of the entropy of the Rosen map. Finally, we note that the same result holds for α -continued fractions, $0 < \alpha \leq 1/2$. For $1/2 \leq \alpha \leq 1$, this result was shown by C. Kraaikamp [9] in a different way. Recently, R. Natsui [19] showed the existence of the Legendre constant for any $0 < \alpha \leq 1/2$ and thus we can apply the method of this paper to show the equality of these two constants. We also note that S. Ito [8] determined the Legendre constant of median convergents. We stress the difference between the concepts of the two constants. The Legendre constant is determined by a property which holds for all x , without exceptional point; on the other hand, the Lenstra constant comes from a metrical property which only holds for almost all points.

Finally, we note the relation between the Legendre constant and the Hurwitz constant. In [11], the authors derive a Borel type result for the Rosen continued fractions. Since the Legendre constant for the Rosen fractions is smaller than the Borel constant found by Kraaikamp et al., the result of these authors does *not* imply the Hurwitz-type result for the Rosen fractions, which was previously obtained by Haas and Series. In [10], the authors look at a different map associated with the Rosen fractions, and for this map the Legendre constant is larger than the Borel constant, and—as in the case of the regular continued fraction expansion—the result by Haas and Series follows.

2. Generalized Diophantine approximation

Let Γ be a finitely generated Fuchsian group acting on the complex upper half-plane \mathbb{H}^2 , L the set of limit points of Γ , and P the set of parabolic points. We assume that $\infty \in P$.

An element $g \in \Gamma$ can be viewed as a 2×2 real matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of determinant 1. We write $a = a(g)$, $b = b(g)$, $c = c(g)$ and $d = d(g)$.

J. Lehner [12] proved that there exists a positive number t depending on Γ such that

$$\#\{g(\infty) : |x - g(\infty)| < t/c^2(g), g \in \Gamma\} = \infty$$

for any $x \in L \setminus P$. He also proved that if Γ is of the first kind ($L = \mathbb{R}$), then for any sequence $\{\varepsilon_n\}$ of positive numbers and a.e. $x \in L \setminus P$, there exists a sequence $\{g_n\}$ in Γ such that

$$|x - g_n(\infty)| < \varepsilon_n/c^2(g_n).$$

Moreover, S. J. Patterson [20] proved a kind of Khinchin's theorem when Γ is of the first kind: for example, his result implies that

$$\#\{g(\infty) : |x - g(\infty)| < (\ln |c(g)|)/c^2(g), g \in \Gamma\} = \infty$$

for a.e. $x \in \mathbb{R} \setminus P$.

We shall estimate the asymptotic number of solutions $g(\infty)$ of

$$|x - g(\infty)| < t/c^2(g), \quad g \in \Gamma,$$

for some positive real number t and a.e. $x \in \mathbb{R} \setminus P$. To do this, we consider a relation among the Diophantine inequality, geodesics of \mathbb{H}^2 , and geodesics of \mathbb{H}^2/Γ . We show that the ergodicity of the geodesic flow on \mathbb{H}^2/Γ with the hyperbolic measure is closely related to the quantitative theory of Diophantine approximation on Γ (see [22] for the qualitative theory). The relation between the Diophantine inequality and geodesics of \mathbb{H}^2 has also been considered by A. Haas [5] and A. Haas and C. Series [6] to discuss the Lagrange spectrum of the approximation on Γ . The "height" of the Γ -congruent family of geodesics plays an important role in their discussion.

This idea is also applicable to the theory of Diophantine approximations for complex numbers, where we have to consider \mathbb{H}^3 (see [18]).

Since $\infty \in P$, there exists

$$U_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in \Gamma, \quad \lambda \in \mathbb{R}_+,$$

such that

$$\{U_\lambda^k : k \in \mathbb{Z}\} = \Gamma_\infty,$$

where Γ_∞ denotes the subgroup of Γ that fixes ∞ . We define the fundamental region \mathcal{F} of Γ by

$$\mathcal{F} = \{z = x + iy : -\lambda/2 < x < \lambda/2, y > 0\} \cap \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \{z : |c(g)z + d(g)| > 1\}.$$

Then \mathcal{F} is a hyperbolic polygon and each side of it is an arc of the isometric circle of an element $g \in \Gamma$. The image of this side under g is also a side of \mathcal{F} , which is an arc of the isometric circle of g^{-1} . We identify all such pairs and obtain a hyperbolic surface. It is well-known that the hyperbolic metric $ds = \sqrt{dx^2 + dy^2}/y$ and the hyperbolic measure $d\mu = dx dy/y^2$ on \mathbb{H}^2 are invariant under the Γ -action over \mathbb{H}^2 .

Theorem 1. *Let*

$$t_0 = \frac{1}{2} \min_{g \in \Gamma \setminus \Gamma_\infty} |c(g)|.$$

Then

$$\lim_{N \rightarrow \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| \leq N, g \in \Gamma\}}{\ln N} = \frac{4\lambda t}{\pi \mu(\mathcal{F})}$$

for any $t, 0 < t < t_0$ (a.e. $x \in \mathbb{R} \setminus P$).

The proof of this theorem is basically the same as that of the main result in [18] for the imaginary quadratic field case with the hyperbolic upper half-space. So we only give a sketch. We start with some lemmas.

We denote by $\gamma(x, y)$ the geodesic curve with initial point x and terminal point y for $(x, \beta) \in (\mathbb{R} \cup \{\infty\})^2 \setminus \{\text{diagonal}\}$. We also denote by

$$F_t(g(\infty)), \quad t > 0,$$

the circle which is tangent to the real line at $a(g)/c(g)$ with radius $t/c^2(g)$ for $g \notin \Gamma_\infty$ and $\{x + iy : y = 1/2t\}$ for $g \in \Gamma_\infty$.

We show the following:

Lemma 1. *If $t > 0$, then*

$$h(F_t(g(\infty))) = F_t(hg(\infty)) \quad \text{for any } h \text{ and } g \in \Gamma.$$

Proof. This follows from the fact that $F_t(g(\infty))$ is an image of $\{x + iy : y = 1/2t\}$, which forms an invariant family of circles under the Γ -action. \square

Lemma 2. *For any $k > 0$,*

$$|x - g(\infty)| < t/c^2(g)$$

if and only if $\gamma(\infty, x)$ and $\bigcap F_t(g(\infty))$ do not cross each other.

If $t < t_0$, then we see that $\{F_t(g(\infty))\}$ is a disjoint family of circles, that is,

$$F_t(g(\infty)) \cap F_t(h(\infty)) = \emptyset \quad \text{if } g(\infty) \neq h(\infty).$$

Thus we have the following:

Lemma 3. *If $0 < t < t_0$, then every point of $F_t(g(\infty)) \setminus (\mathbb{R} \cup \{\infty\})$ is congruent to some point of $F_t(\infty) \cap (\mathcal{F} \setminus \{\infty\})$. In particular, if $p \in F_t(g(\infty)) \setminus (\mathbb{R} \cup \{\infty\})$, then there exists $h \in \Gamma$ such that $h(p) = x + iy$, $-\lambda/2 < x < \lambda/2$ and $y = 1/2t$.*

Proof of Theorem 1. Let $\mathbf{T}(\mathbb{H}^2)$ and $\mathbf{T}(\mathcal{F})$ be the unit tangent bundles of \mathbb{H}^2 and \mathcal{F} , respectively. We consider the geodesic flows f_s and \hat{f}_s on $\mathbf{T}(\mathbb{H}^2)$ and $\mathbf{T}(\mathcal{F})$, respectively. For $\omega^* \in \mathbf{T}(\mathbb{H}^2)$, there is a unique geodesic (x, β) passing tangentially through ω^* . If $x \neq \infty$ and $\beta \neq \infty$, then we denote by s the (signed) hyperbolic length from the top of the geodesic arc (x, β) to ω , which is the base point of ω^* . If $x = \infty$ (or $\beta = \infty$), then we denote by s the hyperbolic length from the point $\beta + i$ or $x + i$ to ω , respectively. Thus we can parameterize $\omega^* \in \mathbf{T}(\mathbb{H}^2)$ by $(x, \beta, s) \in ((\mathbb{R} \cup \{\infty\})^2 \setminus \{\text{diagonal}\}) \times \mathbb{R}$. So if $0 < t < t_0$, we see from Lemmas 2 and 3 that

$$\left| \#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| \leq N, g \in \Gamma\} \right. \\ \left. - \#\left\{s : f_s(\infty, x, -\ln(4t_0 + 1)) \text{ crosses a circle } F_t(g(\infty)) \text{ from} \right. \right. \\ \left. \left. \text{outside at time } s, 0 < s \leq \ln(4t_0 + 1) - \ln t + 2 \ln N \right\} \right| \leq 1$$

and

$$\begin{aligned} & \# \left\{ s : \begin{array}{l} f_s(\infty, x, -\ln(4t_0 + 1)) \text{ crosses a circle } F_t(g(\infty)) \text{ from} \\ \text{outside at time } s, 0 < s \leq \ln(4t_0 + 1) - \ln t + 2 \ln N \end{array} \right\} \\ &= \# \left\{ s : \begin{array}{l} \hat{f}_s(\omega^*) \text{ crosses } F_t(\infty) \text{ from below at time } s, \\ 0 < s \leq \ln(4t_0 + 1) - \ln t + 2 \ln N \end{array} \right\}, \end{aligned}$$

where $\omega^* \in \mathbf{T}(\mathcal{F})$ is the point corresponding to $(\infty, x, -\ln(4t_0 + 1)) \in \mathbf{T}(\mathbb{H}^2)$.

Now we apply the individual ergodic theorem for $(\mathbf{T}(\mathcal{F}), \hat{f}, \hat{\mu})$ to our problem. Here, the hyperbolic measure $\hat{\mu}$ on $\mathbf{T}(\mathcal{F})$ induced from μ is given by

$$\hat{\mu} = \frac{dx d\beta ds}{(x - \beta)^2}$$

if we parameterize a point in $\mathbf{T}(\mathcal{F})$ by (x, β, s) .

Proposition 1. *If we fix t , $0 < t < t_0$, then*

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{\# \{s : f_s(\omega^*) \text{ crosses } F_t(\infty) \text{ from below, } 0 < s < u\}}{u} \\ = \frac{\mu\{x + iy \in \mathcal{F} : y > 1/2t\}}{\pi \mu(\mathcal{F})} \end{aligned}$$

for a.e. $\omega^* \in \mathbf{T}(\mathcal{F})$.

Moreover, by using an approximation method on t , we have

Proposition 2. *For a.e. $\omega^* \in \mathbf{T}(\mathcal{F})$,*

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{\# \{s : f_s(\omega^*) \text{ crosses } F_t(\infty) \text{ from below, } 0 < s < u\}}{u} \\ = \frac{\mu\{x + iy \in \mathcal{F} : y > 1/2t\}}{\pi \mu(\mathcal{F})} \end{aligned}$$

for any t , $0 < t < t_0$.

Furthermore, it is possible to show that if $\omega^* = (x, \beta, s) \in \mathbf{T}(\mathcal{F})$ has the above property, then for any $x' \in \mathbb{R} \cup \{\infty\}$ and $s' \in \mathbb{R}$, $\omega^{**} = (x', \beta, s')$ also has the same property. Since the hyperbolic length between $x + (4t_0 + 1)i$ and $x + (1/N)i$ is equal to $\ln N + \ln(4t_0 + 1)$, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\# \{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| \leq N, g \in \Gamma\}}{\ln N} \\ = 2 \frac{\mu\{x + iy \in \mathcal{F} : y > 1/2t\}}{\pi \mu(\mathcal{F})} = \frac{4\lambda t}{\pi \mu(\mathcal{F})} \end{aligned}$$

for any t , $0 < t < t_0$ and a.e. $x \in \mathbb{R} \setminus P$. □

We apply the theorem to the Hecke group of index k , G_k , $3 \leq k \leq \infty$, and its congruent subgroups $G_k(m)$. Here G_k is the group generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \lambda_k \\ 0 & 1 \end{pmatrix}$$

where $\lambda_k = 2 \cos(\pi/k)$ for $k \geq 3$ (and $= 2$ when $k = \infty$), and $G_k(m)$ the subgroup of G_k defined by

$$G_k(m) = \left\{ g \in G_k : g \equiv \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \pmod{m \cdot \lambda_k} \right\}$$

where $(m \cdot \lambda_k)$ denotes the ideal generated by $m \cdot \lambda_k$ with a positive integer m .

A fundamental region \mathcal{F}_k of G_k is given by

$$\mathcal{F}_k = \{x + iy : -\cos \pi n < x \leq \cos \pi k, x^2 + y^2 > 1, y > 0\}.$$

Thus we see that G_k is of the first kind and

$$P = P_k = G_k(\infty) = \{g(\infty) : g \in G_k\}$$

if $k \neq \infty$. In this case, we have

$$\lim_{N \rightarrow \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| \leq N, g \in G_k\}}{\ln N} = \frac{4k\lambda_k t}{(k-2)\pi^2}$$

for any t , $0 < t < 1/2$, and a.e. $x \in \mathbb{R} \setminus P_k$. We can also apply Theorem 1 to $G_k(m)$. Then, in this case, the set of parabolic points of G_k is divided into a finite number of disjoint sets. We put

$$t_m = \frac{1}{2} \min_{g \in G_k(m) \setminus G_k(m)_\infty} |c(g)|.$$

There exists a constant $C > 0$ such that

$$\lim_{N \rightarrow \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| \leq N, g \in G_k(m)\}}{\ln N} = Ct$$

for any t , $0 < t < t_m$, and a.e. $x \in \mathbb{R}$. Since $G_k(m)$ is a subgroup of G_k , for each cusp of the fundamental region of $G_k(m)$ there exists $g_\eta \in G_k$ such that $g_\eta(\eta) = \infty$. It is obvious that $g_\eta \mathcal{F}$ is a fundamental region of $g_\eta G_k(m) g_\eta^{-1}$. Since $G_k(m)$ is normal, $g_\eta \mathcal{F}$ is a fundamental region of $G_k(m)$. This means the ‘‘width’’ of the cusp η is the same as that of ∞ . Thus we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| \leq N, g \in G_k, \exists \hat{g} \in G_k(m), g(\infty) = \hat{g}(\eta)\}}{\ln N} \\ = Ct \end{aligned}$$

for any t , $0 < t < t_m$, and a.e. $x \in \mathbb{R}$. Moreover, it turns out that $t_m \rightarrow \infty$ as $m \rightarrow \infty$. This shows the following:

Corollary. For a.e. $x \in \mathbb{R}$,

$$\lim_{N \rightarrow \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| \leq N, g \in G_k\}}{\ln N} = \frac{4k\lambda_k t}{(k-2)\pi^2}$$

for any $t > 0$.

Remark. The above proof (of this corollary) shows the equidistribution property (a.e.) of solutions associated to cusps. We refer to R. Moeckel [15] for the original idea of this method.

3. Rosen continued fractions

Given any element of G_k of the form

$$\begin{pmatrix} p & \cdot \\ q & \cdot \end{pmatrix},$$

we have $g(\infty) = p/q$; moreover, for any $\tilde{g} \in G_k$ with $\tilde{g}(\infty) = p/q$ we have

$$\tilde{g} = \begin{pmatrix} p & \cdot \\ q & \cdot \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -p & \cdot \\ -q & \cdot \end{pmatrix}.$$

So, for any parabolic point of G_k , p and $q > 0$ are uniquely determined.

We define the λ_k -nearest continued fraction transformation of $[-\lambda_k/2, \lambda_k/2)$ onto itself by

$$S(x) = \begin{cases} \left\lfloor \frac{1}{|x|} \right\rfloor - \left[\left\lfloor \frac{1}{|x|} \right\rfloor \right]_k & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

where $[w]_k = b\lambda_k$, $b \in \mathbb{Z}$, when $w \in [b - \lambda_k/2, b + \lambda_k/2)$. We put

$$\varepsilon_n = \varepsilon_n(x) = \text{sgn}(S^{n-1}(x)) \quad \text{and} \quad a_n = a_n(x) = \left[\left| \frac{1}{S^{n-1}(x)} \right| \right]_k$$

for any $n \geq 1$ and have a continued fraction expansion

$$x = \frac{\varepsilon_1}{|a_1|} + \frac{\varepsilon_2}{|a_2|} + \dots$$

We call this expansion the *Rosen continued fraction expansion* of x . In the case of $k = 3$, it is the classical nearest integer continued fraction expansion. In general, if a continued fraction, either finite or infinite, is given by some x as its Rosen continued fraction expansion, we call it a *Rosen continued fraction*. We define the principal convergent p_n/q_n , $n \geq 0$, by

$$\begin{pmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_2 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{pmatrix} \quad \text{for } n \geq 1.$$

It is easy to see that $q_n > 0$ for any $n \geq 0$.

Lemma 4. *We have*

$$\frac{1}{q_n(q_{n+1} + q_n)} \leq \left| x - \frac{p_n}{q_n} \right| \leq \begin{cases} \frac{1}{q_n^2(1 - \lambda_k/2)} & \text{if } k \text{ is even,} \\ \frac{1}{q_n^2(1/R_k - \lambda_k/2)} & \text{if } k \text{ is odd,} \end{cases}$$

where R_k is the positive root of $X^2 + (2 - \lambda)X - 1 = 0$.

Proof. We have

$$T^n(x) = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}^{-1}(x),$$

and therefore (see also [7]),

$$x = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} (T^n(x)) = \frac{p_{n-1}T^n x + p_n}{q_{n-1}T^n x + q_n}.$$

Then we have (see e.g. [7, p. 27]),

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{q_n^2} \frac{1}{q_{n+1}/q_n + T^{n+1}x}.$$

Since $q_{n+1}/q_n > 1$ if k is even (and $q_{n+1}/q_n > 1/R_k$ if k is odd), the result follows. \square

Lemma 5. *For a.e. $x \in \mathbb{I}$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n$$

exists and is equal to half the entropy h_k of the Rosen map with respect to the absolutely continuous invariant probability measure.

Proof. Let μ be the absolutely continuous invariant probability measure for T . Since (T, μ) is ergodic (see [3]), the Shannon–McMillan–Breiman theorem implies that the entropy h_k of the Rosen map is given by

$$h_k = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mu(\Delta[\varepsilon_1 : r_1, \dots, \varepsilon_n : r_n]) \quad \text{a.e.}$$

We can replace μ by the normalized Lebesgue measure m because μ has a positive density function bounded away from 0 and bounded from above (see [3]), that is,

$$h_k = \lim_{n \rightarrow \infty} \frac{1}{n} \ln m(\Delta[\varepsilon_1 : r_1, \dots, \varepsilon_n : r_n]) \quad \text{a.e.}$$

From Lemma 4, it turns out that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln m(\Delta[\varepsilon_1 : r_1, \dots, \varepsilon_n : r_n]) = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n$$

if the limit on the left hand side exists. Thus we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n = \frac{h_k}{2}$$

for a.e. $x \in \mathbb{I}$. □

Now we denote by $\mathcal{L}g_k$ the *Legendre constant of Rosen continued fractions* of index k , that is, the following hold:

- (1) for $c \leq \mathcal{L}g_k$ and $x \in [-\lambda_k/2, \lambda_k/2)$, and $(\frac{p}{q} \cdot) \in G_k$, $q \neq 0$, if $|x - p/q| < c/q^2$, then $p/q = p_n/q_n$ for some $n \geq 0$,
- (2) for $c > \mathcal{L}g_k$, there exist $x \in [-\lambda_k/2, \lambda_k/2)$ and $(\frac{p}{q} \cdot) \in G_k$, $q \neq 0$, such that $|x - p/q| < c/q^2$ and $p/q \neq p_n/q_n$ for any $n \geq 0$.

As mentioned in the introduction, the existence of $\mathcal{L}g_k$ was shown in [21].

On the other hand, we denote by $\mathcal{L}e_k$ the *Lenstra constant of Rosen continued fractions* of index k . This means that for almost every $x \in [-\lambda_k/2, \lambda_k/2)$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n : 1 \leq n \leq N, \Theta_n(x) < t\} \begin{cases} = C_k t & \text{for any } 0 < t \leq \mathcal{L}e_k, \\ < C_k t & \text{for any } t > \mathcal{L}e_k, \end{cases}$$

where C_k is an absolute constant, which is given in [3], and

$$\Theta_n = q_n^2 \left| x - \frac{p_n}{q_n} \right|.$$

We will prove the following:

Theorem 2. *For any $k \geq 3$, we have $\mathcal{L}g_k = \mathcal{L}e_k$.*

Remark. As a direct consequence of Theorem 2 and Corollary 4.1 of [3], we obtain the Legendre constant of Rosen continued fractions as follows:

$$\mathcal{L}g_k = \mathcal{L}e_k = \begin{cases} \frac{\lambda_k}{\lambda_k + 2} & \text{if } k \geq 4 \text{ is an even integer,} \\ \frac{R_k}{R_k + 1} & \text{if } k \geq 3 \text{ is an odd integer.} \end{cases}$$

The proof of Theorem 2 is based upon the following two propositions.

Proposition 3. *For any $k \geq 3$, we have $\mathcal{L}g_k \leq \mathcal{L}e_k$.*

Proof. Suppose $0 \leq t \leq \mathcal{L}g_k$. From Corollary of §2, we have, for a.e. $x \in [-\lambda_k/2, \lambda_k/2)$,

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| = q_n, g \in G_k, \text{ for some } 0 \leq n \leq N\}}{\ln q_N} \\ &= \lim_{N \rightarrow \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| \leq q_N, g \in G_k\}}{\ln q_N} = \frac{4k\lambda_k t}{(k-2)\pi^2}. \end{aligned}$$

We note that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| = q_n, g \in G_k, \text{ for some } 0 \leq n \leq N\}}{\ln q_N} \\ &= \lim_{N \rightarrow \infty} \frac{\frac{1}{N} \#\{g(\infty) : |x - g(\infty)| < t/c^2(g), |c(g)| = q_n, g \in G_k, \text{ for some } 0 \leq n \leq N\}}{\frac{1}{N} \ln q_N}. \end{aligned}$$

From Lemma 5, the denominator of the right hand side converges to $h_k/2$ (a.e.), and we see that the numerator converges (a.e.) to

$$\frac{4k\lambda_k t h_k}{2(k-2)\pi^2}.$$

This means

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n : 1 \leq n \leq N, \Theta < t\} = \frac{4k\lambda_k t h_k}{2(k-2)\pi^2} \quad \text{a.e.} \quad (1)$$

for $0 \leq t \leq \mathcal{L}g_k$. \square

Proposition 4. For any $k \geq 3$, we have $\mathcal{L}g_k \geq \mathcal{L}e_k$.

Proof. Suppose that $t > \mathcal{L}g_k$. Then there exist $x \in [-\lambda_k/2, \lambda_k/2)$ and $(\frac{p}{q} \cdot) \in G_k$ such that

$$\left| x - \frac{p}{q} \right| < \frac{t}{q^2} \quad \text{and} \quad \frac{p}{q} \neq \frac{p_n}{q_n} \quad \text{for any } n \geq 0. \quad (2)$$

From this inequality, there exists $\varepsilon > 0$ such that

$$\left| x - \frac{p}{q} \right| < \frac{t}{q^2} - \varepsilon.$$

If $y \in [-\lambda_k/2, \lambda_k/2)$ is sufficiently close to x , i.e.

$$|x - y| < \frac{\varepsilon}{2}, \quad (3)$$

then

$$\left| y - \frac{p}{q} \right| < \frac{t}{q^2} - \frac{\varepsilon}{2}. \quad (4)$$

Moreover, there exists a positive integer M_0 such that the equality

$$(\varepsilon_i(x), a_i(x)) = (\varepsilon_i(y), a_i(y)) \quad \text{for } 1 \leq i \leq M_0$$

implies that (2) holds.

Now we look at the expansions of p/q and x . Since p/q is a parabolic point of G_k , it is easy to see that p/q has a finite Rosen expansion, say,

$$\frac{p}{q} = \frac{\hat{\varepsilon}_1}{|\hat{a}_1|} + \cdots + \frac{\hat{\varepsilon}_n}{|\hat{a}_n|}.$$

This means there exists m , $1 \leq m \leq n$, such that

$$(\varepsilon_i, a_i) = (\hat{\varepsilon}_1, \hat{a}_1) \quad \text{for } 1 \leq i \leq m-1 \quad \text{and} \quad (\varepsilon_m, a_m) \neq (\hat{\varepsilon}_m, \hat{a}_m)$$

with

$$x = \frac{\varepsilon_1}{|a_1|} + \cdots + \frac{\varepsilon_m}{|a_m|} + \frac{\varepsilon_{m+1}}{|a_{m+1}|} + \cdots + \frac{\varepsilon_n}{|a_n|} + \cdots.$$

Here we may assume that $M_0 > n$. We choose a positive integer M_1 sufficiently large and define

$$\hat{a}_0 = M_1 \lambda_k.$$

We see that

$$\frac{\hat{\varepsilon}_0}{|\hat{a}_0|} + \frac{\hat{\varepsilon}_1}{|\hat{a}_1|} + \cdots + \frac{\hat{\varepsilon}_n}{|\hat{a}_n|}$$

is a Rosen continued fraction, where $\hat{\varepsilon}_0 = +1$. We fix any finite Rosen continued fraction

$$\frac{\varepsilon'_1}{|b_1|} + \cdots + \frac{\varepsilon'_l}{|b_l|}$$

so that

$$\frac{\varepsilon'_1}{|b_1|} + \cdots + \frac{\varepsilon'_l}{|b_l|} + \frac{\hat{\varepsilon}_0}{|\hat{a}_0|} + \frac{\hat{\varepsilon}_1}{|\hat{a}_1|} + \cdots + \frac{\hat{\varepsilon}_n}{|\hat{a}_n|}$$

and

$$\frac{\varepsilon'_1}{|b_1|} + \cdots + \frac{\varepsilon'_l}{|b_l|} + \frac{\hat{\varepsilon}_0}{|\hat{a}_0|} + \frac{\varepsilon_1}{|a_1|} + \cdots + \frac{\varepsilon_m}{|a_m|} + \frac{\varepsilon_{m+1}}{|a_{m+1}|} + \cdots + \frac{\varepsilon_n}{|a_n|} + \cdots + \frac{\varepsilon_{M_0}}{|a_{M_0}|} + \cdots$$

are also Rosen continued fractions. Suppose that $y_0 \in [-\lambda_k/2, \lambda_k/2)$ with the Rosen expansion of the form

$$\frac{\varepsilon'_1}{|b_1|} + \cdots + \frac{\varepsilon'_l}{|b_l|} + \frac{\hat{\varepsilon}_0}{|\hat{a}_0|} + \frac{\varepsilon_1}{|a_1|} + \cdots + \frac{\varepsilon_m}{|a_m|} + \frac{\varepsilon_{m+1}}{|a_{m+1}|} + \cdots + \frac{\varepsilon_n}{|a_n|} + \cdots + \frac{\varepsilon_{M_0}}{|a_{M_0}|} + (\text{free})$$

and

$$\frac{P}{Q} = \frac{\varepsilon'_1}{|b_1|} + \cdots + \frac{\varepsilon'_l}{|b_l|} + \frac{\hat{\varepsilon}_0}{|\hat{a}_0|} + \frac{\hat{\varepsilon}_1}{|\hat{a}_1|} + \cdots + \frac{\hat{\varepsilon}_{n-1}}{|\hat{a}_{n-1}|} + \frac{\hat{\varepsilon}_n}{|\hat{a}_n|}.$$

We note that P and Q are uniquely determined by $(\frac{P}{Q} \cdot) \in G_k$ and $Q > 0$. We also note that $y := S^{l+1}(y_0)$ has Rosen continued fraction

$$\frac{\varepsilon_1}{|a_1|} + \cdots + \frac{\varepsilon_m}{|a_m|} + \frac{\varepsilon_{m+1}}{|a_{m+1}|} + \cdots + \frac{\varepsilon_n}{|a_n|} + \cdots + \frac{\varepsilon_{M_0}}{|a_{M_0}|} + (\text{free})$$

which satisfies (2). We put

$$\begin{pmatrix} P_{n+l} & P_{n+l+1} \\ Q_{n+l} & Q_{n+l+1} \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon'_1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon'_l \\ 1 & b_l \end{pmatrix} \begin{pmatrix} 0 & \hat{\varepsilon}_0 \\ 1 & \hat{a}_0 \end{pmatrix} \begin{pmatrix} 0 & \hat{\varepsilon}_1 \\ 1 & \hat{a}_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & \hat{\varepsilon}_n \\ 1 & \hat{a}_n \end{pmatrix},$$

which implies $(P, Q) = (P_{n+l+1}, Q_{n+l+1})$, and we will estimate

$$\left| y_0 - \frac{P_{n+l+1}}{Q_{n+l+1}} \right|.$$

We also define

$$\begin{aligned} \begin{pmatrix} P_{l-1} & P_l \\ Q_{l-1} & Q_l \end{pmatrix} &= \begin{pmatrix} 0 & \varepsilon'_1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon'_l \\ 1 & b_l \end{pmatrix}, \\ \begin{pmatrix} P_l & P_{l+1} \\ Q_l & Q_{l+1} \end{pmatrix} &= \begin{pmatrix} 0 & \varepsilon'_1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon'_l \\ 1 & b_l \end{pmatrix} \begin{pmatrix} 0 & \hat{\varepsilon}_0 \\ 1 & \hat{a}_0 \end{pmatrix}. \end{aligned}$$

We denote by U the linear fractional transformation defined by $\begin{pmatrix} P_l & P_{l+1} \\ Q_l & Q_{l+1} \end{pmatrix}$. Then it is easy to see that

$$U\left(\frac{p}{q}\right) = \frac{P}{Q} \quad \text{and} \quad U(y) = y_0.$$

Thus

$$\left| y_0 - \frac{P_{n+l+1}}{Q_{n+l+1}} \right| = \left| U\left(\frac{p}{q}\right) - U(y) \right|$$

and the following holds:

$$\begin{aligned} \left| U\left(\frac{p}{q}\right) - U(y) \right| &= \left| \frac{P_l y + P_{l+1}}{Q_l y + Q_{l+1}} - \frac{P_l \frac{p}{q} + P_{l+1}}{Q_l \frac{p}{q} + Q_{l+1}} \right| \\ &= \left| \left(\frac{P_l y + P_{l+1}}{Q_l y + Q_{l+1}} - \frac{P_{l+1}}{Q_{l+1}} \right) + \left(\frac{P_{l+1}}{Q_{l+1}} - \frac{P_l \frac{p}{q} + P_{l+1}}{Q_l \frac{p}{q} + Q_{l+1}} \right) \right| \\ &= \left| \frac{y}{Q_{l+1}(Q_l y + Q_{l+1})} + \frac{\frac{p}{q}}{Q_{l+1}(Q_l \frac{p}{q} + Q_{l+1})} \right| \\ &\leq \frac{|y - \frac{p}{q}|}{|Q_{l+1}(Q_l y + Q_{l+1})|} \\ &\quad + \left| \frac{\frac{p}{q}}{Q_{l+1}(Q_l \frac{p}{q} + Q_{l+1})} - \frac{\frac{p}{q}}{Q_{l+1}(Q_l y + Q_{l+1})} \right| \\ &\leq \frac{1}{|Q_{l+1}(Q_l y + Q_{l+1})|} \left(\frac{t}{q^2} - \frac{\varepsilon}{2} \right) \\ &\quad + \left| \frac{p}{q} \right| \left| \frac{1}{Q_{l+1}(Q_l \frac{p}{q} + Q_{l+1})} - \frac{1}{Q_{l+1}(Q_l y + Q_{l+1})} \right|. \end{aligned}$$

From the definition of Q_{n+l+1} , we see

$$q_{l+n+1} = pQ_l + qQ_{l+1}.$$

Since $|y| < 1$, $|p/q|$ cannot be large, $Q_{l+1} = \hat{a}_0 Q_l + \hat{\varepsilon}_0 Q_{l-1}$, and Q_{l-1}/Q_l is bounded (see [3]), we see that

$$\left| \frac{Q_l y}{Q_{l+1}} \right| \quad \text{and} \quad \left| \frac{Q_l \frac{p}{q}}{Q_{l+1}} \right|$$

can be arbitrarily small and

$$\frac{qQ_{l+1}}{Q_{l+n+1}}$$

can be sufficiently close to 1 when we choose M_1 sufficiently large (note that $\hat{a}_0 = M_1 \lambda_k$).

In the above discussion, the choice of M_1 can be independent of $(\varepsilon'_1, b_1), \dots, (\varepsilon'_l, b_l)$. Thus we get

$$\left| y_0 - \frac{P}{Q} \right| < \frac{t}{Q^2}.$$

It is obvious from the construction that

$$\frac{P}{Q} \neq \frac{p_u}{q_u}$$

for any $u \geq 0$. Now we pick up a “generic point” $w \in [-\lambda_k/2, \lambda_k/2)$. Then the ergodicity of S with respect to μ_k yields

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq u \leq N : (\varepsilon_u(w), a_u(w)) = (\hat{\varepsilon}_0, \hat{a}_0), (\varepsilon_{u+1}(w), a_{u+1}(w)) = (\varepsilon_1, a_1), \dots, \\ & \quad (\varepsilon_{u+n}(w), a_{u+n}(w)) = (\varepsilon_n, a_n), \dots, (\varepsilon_{u+M_0}(w), a_{u+M_0}(w)) = (\varepsilon_{M_0}, a_{M_0})\} \\ &= \mu_k(\{w : (\varepsilon_1(w), a_1(w)) = (\hat{\varepsilon}_0, \hat{a}_0), (\varepsilon_2(w), a_2(w)) = (\varepsilon_1, a_1), \dots, \\ & \quad (\varepsilon_{M_0+1}(w), a_{M_0+1}(w)) = (\varepsilon_{M_0}, a_{M_0})\}) \\ &> 0. \end{aligned}$$

Finally, we look at

$$\begin{aligned} & \frac{1}{\ln Q} \#\left\{1 \leq q \leq Q : \left| w - \frac{p}{q} \right| < \frac{t}{q^2} \right\} \\ &= \frac{1}{\ln Q} \#\left\{1 \leq q \leq Q : \left| w - \frac{p}{q} \right| < \frac{t}{q^2}, \frac{p}{q} = \frac{p_n}{q_n} \text{ for some } n \geq 1 \right\} \\ & \quad + \frac{1}{\ln Q} \#\left\{1 \leq q \leq Q : \left| w - \frac{p}{q} \right| < \frac{t}{q^2}, \frac{p}{q} \neq \frac{p_n}{q_n} \text{ for some } n \geq 1 \right\}. \end{aligned}$$

From the above discussion, the second term has a positive “liminf” and then the first term cannot converge to $4k\lambda_k t / (k-2)\pi^2$. Since this calculation holds for a.e. w , t is larger than $\mathcal{L}e_k$. \square

Consequently, we have shown the assertion of Theorem 2. The method of the proof in the above yields the following generalization:

Claim. *Suppose that T is a map of an interval onto itself that induces continued fraction expansions for real numbers in the domain interval. Moreover, assume*

- (i) *T has an absolutely invariant probability measure.*
- (ii) *There exists a real number $M > 0$ such that for any possible coefficient value c larger than M (or $|c| > M$), one can concatenate any admissible sequence after c as an admissible sequence of continued fractions arising from T .*
- (iii) *The Legendre constant of T exists.*
- (iv) *t_0 in Theorem 1 is larger than the Legendre constant.*

Then the Lenstra constant exists and the Legendre and the Lenstra constants are equal.

From (1) in the proof of Proposition 3 together with Corollary 4.1 of [3], we have the explicit value of the entropy of the Rosen map.

Corollary. *The entropy of the Rosen map with respect to the absolutely continuous invariant probability measure is equal to*

$$\frac{C(k-2)\pi^2}{2k}$$

with

$$C = \begin{cases} \frac{1}{\ln\{(1 + \cos \frac{\pi}{k}) / \sin \frac{\pi}{k}\}} & \text{if } k \text{ is even,} \\ \frac{1}{\ln(1 + R_k)} & \text{if } k \text{ is odd.} \end{cases}$$

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