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Convergence and sharp thresholds for propagation in nonlinear diffusion problems

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Abstract. We study the Cauchy problem

$$u_t = u_{xx} + f(u) \quad (t > 0, x \in \mathbb{R}^1), \quad u(0, x) = u_0(x) \quad (x \in \mathbb{R}^1),$$

where $f(u)$ is a locally Lipschitz continuous function satisfying $f(0) = 0$. We show that any non-negative bounded solution with compactly supported initial data converges to a stationary solution as $t \rightarrow \infty$. Moreover, the limit is either a constant or a symmetrically decreasing stationary solution. We also consider the special case where f is a bistable nonlinearity and the case where f is a combustion type nonlinearity. Examining the behavior of a parameter-dependent solution u_λ , we show the existence of a sharp threshold between extinction (i.e., convergence to 0) and propagation (i.e., convergence to 1). The result holds even if f has a jumping discontinuity at $u = 1$.

Keywords. Nonlinear diffusion equation, asymptotic behavior, omega limit set, Cauchy problem, Allen–Cahn, combustion, sharp threshold

1. Introduction

We consider the Cauchy problem

$$\begin{cases} u_t = u_{xx} + f(u), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^1, \\ u(0, x) = u_0(x) \geq 0, & x \in \mathbb{R}^1, \end{cases} \quad (1.1)$$

where $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}^1 = (-\infty, \infty)$, and $f : [0, \infty) \rightarrow \mathbb{R}^1$ is a locally Lipschitz continuous function satisfying

$$f(0) = 0. \quad (1.2)$$

Later we will also consider the case where f has a discontinuity at $u = 1$.

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The initial data u_0 is assumed to be in $L^\infty(\mathbb{R}^1)$, but we do not necessarily assume its continuity. Thus the initial condition in (1.1) is understood in the following sense:

$$\lim_{t \rightarrow 0} u(t, x) = u_0(x) \quad \text{for a.e. } x \in \mathbb{R}^1, \quad \limsup_{t \rightarrow 0} \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^1)} < \infty.$$

It is well-known that, for each bounded nonnegative initial data u_0 , (1.1) possesses a unique nonnegative local-in-time solution that has the above property. Moreover, the solution is smooth for $t > 0$, and is defined as long as it stays finite.

The stationary problem for (1.1) is written as

$$v'' + f(v) = 0, \quad x \in \mathbb{R}^1. \quad (1.3)$$

The solutions of (1.3) can be classified into the following categories:

- (1) constant solutions; in other words, zeros of f ;
- (2) periodic solutions;
- (3) *symmetrically decreasing* solutions, that is, for some $a \in \mathbb{R}^1$,

$$v(x) \equiv v(2a - x) \quad \text{and} \quad v'(x) < 0 \quad \text{for } x \in (a, \infty);$$

- (4) *symmetrically increasing* solutions, that is, for some $a \in \mathbb{R}^1$,

$$v(x) \equiv v(2a - x) \quad \text{and} \quad v'(x) > 0 \quad \text{for } x \in (a, \infty);$$

- (5) strictly increasing or decreasing solutions.

Let us introduce some more notations. For each $\phi \in L^\infty(\mathbb{R}^1)$, we define its *support* $\text{spt}(\phi)$ as the smallest closed set $A \subset \mathbb{R}^1$ such that $\phi = 0$ a.e. in $\mathbb{R}^1 \setminus A$. We use the notation $[\text{spt}(\phi)]$ to denote the convex hull of $\text{spt}(\phi)$. This is the smallest closed interval containing $\text{spt}(\phi)$.

Our first main result is the following:

Theorem 1.1 (Convergence theorem). *Let f satisfy (1.2) and let $u_0 \in L^\infty(\mathbb{R}^1)$ be nonnegative and compactly supported. Suppose that the solution $u(t, x)$ of (1.1) is globally defined for $t \geq 0$ and remains bounded as $t \rightarrow \infty$. Then u converges to a stationary solution as $t \rightarrow \infty$ locally uniformly in \mathbb{R}^1 . Moreover, the limit is either a constant solution or a symmetrically decreasing solution of (1.3).*

Remark 1.2. From the above theorem we see that u never converges to a periodic stationary solution. We also remark that certain types of constant solutions of (1.3) can never be the limit of u ; see Lemma 3.5 for details.

Note that we assume no sign condition nor differentiability of f . The condition (1.2) is all we need for the theorem to hold. A convergence theorem for such a general nonlinearity f has been known for problems on a bounded interval or a circle (see, for example, [Ze, M, C1, CM]), but it is new for an unbounded spatial domain such as \mathbb{R}^1 .

Our second main result is concerned with the existence of a “sharp threshold” between extinction and propagation. We consider two types of nonlinearities:

(f_B) the bistable nonlinearity, (f_C) the combustion nonlinearity.

In the bistable case (f_B), we assume that f is locally Lipschitz and satisfies

$$f(0) = f(\theta_0) = f(1) = 0, \quad f(s) \begin{cases} < 0 & \text{in } (0, \theta_0) \cup (1, \infty), \\ > 0 & \text{in } (\theta_0, 1), \end{cases} \quad (1.4)$$

for some $\theta_0 \in (0, 1)$ and

$$\int_0^1 f(s) ds > 0. \quad (1.5)$$

Such a nonlinearity appears in various applications including mathematical ecology, population genetics and physics. See, for example, [F, NAY, AC, AW1, AW2, FM] and the references therein.

In the combustion case (f_C), we assume that f is locally Lipschitz and satisfies

$$f(s) = 0 \quad \text{in } [0, \theta_0], \quad f(s) > 0 \quad \text{in } (\theta_0, 1), \quad f(s) \leq 0 \quad \text{in } [1, \infty) \quad (1.6)$$

for some $\theta_0 \in (0, 1)$, and that there exists $\delta_0 > 0$ such that

$$f(s) \text{ is nondecreasing in } (\theta_0, \theta_0 + \delta_0). \quad (1.7)$$

(The case $\theta_0 = 0$ will be discussed in Remark 4.12.) Such a nonlinearity appears, typically, as a model for combustion; see [ZFK, K, BN, Z] and the references therein. The value θ_0 is called the “ignition temperature”. (The case $\theta_0 = 0$ was first investigated by Fisher [F] and Kolmogorov–Petrovskii–Piskunov [KPP], and is known as Fisher’s equation or the KPP equation, which was used to describe the propagation of advantageous genes in a population.)

Both in case (f_B) and in case (f_C), the following typical behaviors are known to occur:

$$\text{Extinction: } \lim_{t \rightarrow \infty} u(t, x) = 0; \quad \text{Propagation: } \lim_{t \rightarrow \infty} u(t, x) = 1.$$

Our goal in the second half of the paper is to show that there is a sharp transition between extinction and propagation when we vary the initial data.

To put the sharp transition phenomenon into perspective, let us look at the set of bounded nonnegative solutions of (1.3) with f of type (f_B) or (f_C). First, in the case of the bistable nonlinearity (f_B), it is easy to see that the bounded nonnegative solutions of (1.3) consist of the following:

- (1 $_B$) constant solutions: $0, \theta_0, 1$;
- (2 $_B$) a family of periodic solutions satisfying $0 < \min v < \theta_0 < \max v < \theta^*$;
- (3 $_B$) symmetrically decreasing solutions $U(\cdot - a)$, $a \in \mathbb{R}^1$, where

$$U'' + f(U) = 0 \quad \text{in } \mathbb{R}^1, \quad U(0) = \theta^*, \quad U'(0) = 0, \quad (1.8)$$

which necessarily satisfies $\lim_{|x| \rightarrow \infty} U(x) = 0$.

Here $\theta^* \in (\theta_0, 1)$ is the constant determined by the condition

$$\int_0^{\theta^*} f(s) ds = 0.$$

In order to describe the threshold in the sharp transition, we introduce a one-parameter family of initial data $u_0 = \phi_\lambda$ ($\lambda > 0$) satisfying the following conditions, where $X^+ := \{\phi \in L^\infty(\mathbb{R}^1) : \phi \geq 0, \text{ spt}(\phi) \text{ is compact}\}$:

- (Φ_1) $\phi_\lambda \in X^+$ for every $\lambda > 0$, and the map $\lambda \mapsto \phi_\lambda$ is continuous from \mathbb{R}_+ to $L^1(\mathbb{R}^1)$;
- (Φ_2) if $0 < \lambda_1 < \lambda_2$, then $\phi_{\lambda_1} \leq \phi_{\lambda_2}$ and $\phi_{\lambda_1} \not\equiv \phi_{\lambda_2}$ in the a.e. sense;
- (Φ_3) $\lim_{\lambda \rightarrow 0} \phi_\lambda(x) = 0$ a.e. in \mathbb{R}^1 .

Theorem 1.3 (Threshold for the bistable case). *Let f be a locally Lipschitz continuous function satisfying (1.4), (1.5) and let $\{\phi_\lambda\}_{\lambda>0}$ satisfy (Φ_1) through (Φ_3). Denote by $u_\lambda(t, x)$ the solution of (1.1) with initial data $u_0 = \phi_\lambda$. Then one of the following holds:*

- (a) $\lim_{t \rightarrow \infty} u_\lambda(t, x) = 0$ uniformly in \mathbb{R}^1 for every $\lambda > 0$;
- (b) there exist $\lambda^* > 0$ and $x_0 \in \mathbb{R}^1$ such that

$$\lim_{t \rightarrow \infty} u_\lambda(t, x) = \begin{cases} 0 & \text{uniformly in } \mathbb{R}^1 & (0 < \lambda < \lambda^*), \\ U(x - x_0) & \text{uniformly in } \mathbb{R}^1 & (\lambda = \lambda^*), \\ 1 & \text{locally uniformly in } \mathbb{R}^1 & (\lambda > \lambda^*). \end{cases}$$

Next we consider the case of combustion nonlinearity (f_C). In this case, bounded non-negative solutions of (1.3) are the following constants:

$$v(x) \equiv \theta \in [0, \theta_0] \cup \{1\} \cup S_1,$$

where S_1 denotes the set of all zeros of f that are larger than 1 (possibly empty). The corresponding threshold theorem is stated as follows:

Theorem 1.4 (Threshold for the combustion case). *Let f be a locally Lipschitz continuous function satisfying (1.6), (1.7) and let $\{\phi_\lambda\}_{\lambda>0}$ satisfy (Φ_1) through (Φ_3). Denote by $u_\lambda(t, x)$ the solution of (1.1) with initial data $u_0 = \phi_\lambda$. Then one of the following holds:*

- (a) $\lim_{t \rightarrow \infty} u_\lambda(t, x) = 0$ uniformly in \mathbb{R}^1 for every $\lambda > 0$;
- (b) there exists $\lambda^* > 0$ such that

$$\lim_{t \rightarrow \infty} u_\lambda(t, x) = \begin{cases} 0 & \text{uniformly in } \mathbb{R}^1 & (0 < \lambda < \lambda^*), \\ \theta_0 & \text{locally uniformly in } \mathbb{R}^1 & (\lambda = \lambda^*), \\ 1 & \text{locally uniformly in } \mathbb{R}^1 & (\lambda > \lambda^*). \end{cases}$$

Remark 1.5. The conclusions of Theorems 1.3 and 1.4 remain valid even if $f(u)$ has a discontinuous jump at $u = 1$. This answers a question raised by Brezis; see Section 6 for details.

Let us give two typical examples of ϕ_λ satisfying the conditions (Φ_1) through (Φ_3):

Example 1.6. If $\phi \in L^\infty(\mathbb{R}^1)$ is an arbitrary nonnegative function with nonempty compact support, then

$$\phi_\lambda(x) = \lambda\phi(x) \tag{1.9}$$

clearly satisfies (Φ_1) to (Φ_3) . Furthermore, the alternative (b) of Theorems 1.3 and 1.4 always holds if f is globally Lipschitz. In fact, under this assumption on f , we can show

$$u_\lambda(t, x) \geq \lambda e^{-Mt} w(t, x),$$

where w is the solution of $w_t = w_{xx}$ with initial data ϕ and M is the Lipschitz coefficient of f . To use the above inequality, we simply fix any $t > 0$, choose λ sufficiently large and then apply Lemmas 4.2 and 4.8.

Example 1.7. Given arbitrary points $x_1 < \dots < x_k$ and constants $\sigma_i > 0$ ($i = 1, \dots, k$), we set

$$\phi_\lambda(x) = \sum_{i=1}^k \sigma_i \chi_{[x_i - \lambda, x_i + \lambda]}(x), \tag{1.10}$$

where χ_A denotes the characteristic function of a set $A \subset \mathbb{R}^1$. Then (Φ_1) to (Φ_3) are clearly satisfied. Moreover, by Lemmas 4.2 and 4.8, the alternative (b) occurs if $\sum_{i=1}^k \sigma_i > \theta_0$. On the other hand, if $\sum_{i=1}^k \sigma_i < \theta_0$, one can easily check that alternative (a) occurs.

The present work was motivated by a recent paper of Zlatoš [Z], where the conclusions of Theorems 1.3 and 1.4 were shown to hold for the particular family of initial data $u_0(x) = \chi_{[-\lambda, \lambda]}(x)$, $\lambda \in (0, \infty)$. The results of [Z] gave a positive answer to a question left open in the work of Kanel [K] and Aronson–Weinberger [AW1]. In [K], it was shown that in the combustion case with $u_0(x) = \chi_{[-\lambda, \lambda]}(x)$, there exists $0 < \lambda_0 \leq \lambda_1$ such that extinction occurs for $\lambda < \lambda_0$, and propagation occurs for $\lambda > \lambda_1$. The same was shown to hold for the bistable case in [AW1]. However, the question of whether $\lambda_0 = \lambda_1$ or not remained open until [Z]. While the methods used in [Z] can be extended to treat more general initial data u_0 than the characteristic function of an interval, the arguments there depend crucially on the symmetry and monotonicity properties of $u_0(x)$. Our theorems, on the other hand, are applicable to a much wider class of initial data.

The proof of Theorem 1.1 is based on a zero number argument, the reflection method and estimates near $|x| = \infty$. In the special case of bistable nonlinearity (f_B) , one can also use a Lyapunov functional (the energy) to partly simplify the argument (see Remark 4.6). However, unlike the problems in a bounded domain, this approach does not work under the general assumption (1.2), since the energy may tend to $-\infty$ as $t \rightarrow \infty$ for most solutions. For this reason we have avoided the use of the energy and based our argument on the zero number properties.

To be more precise, the proof of Theorem 1.1 consists of two steps. First we show that any element of $\omega(u)$ is a solution of (1.3). Here $\omega(u)$ denotes the ω -limit set of the solution u . Secondly, we show that $\omega(u)$ contains only one element. A key observation for establishing the first claim is that, for any $w \in \omega(u)$ and for any solution v of (1.3) which is positive in \mathbb{R}^1 or has compact positive support, either $w(x) - v(x)$ has only

simple zeros, or $w(x) \equiv v(x)$. This can be shown by studying the number of zeros of $u(t, x) - v(x)$ as t approaches ∞ . The second claim is established partly by showing that there exists $x_0 \in \mathbb{R}^1$ such that

$$w'(x_0) = 0 \quad \text{for any } w \in \omega(u).$$

This is done by using the reflection argument found in [C1, CM].

The key point in Theorems 1.3 and 1.4 is the assertion that the situation $u_\lambda \not\rightarrow 0, 1$ ($t \rightarrow \infty$) occurs only for a single value of $\lambda = \lambda^*$. The rest follows easily from Theorem 1.1 and the comparison principle. The proof of this sharp threshold result differs between the case of bistable nonlinearity (f_B) and the case of combustion nonlinearity (f_C), but in either case a key step is established by making use of the following proposition. Since this proposition holds in any space dimension, we state it in a rather general framework:

Proposition 1.8. *Suppose that $g : [0, \infty) \rightarrow \mathbb{R}^1$ is locally Lipschitz continuous, $g(0) = 0$, and that $u_0 \in L^\infty(\mathbb{R}^N)$ is nonnegative and has nonempty compact support $\text{spt}(u_0)$. Then there exist $\delta, R > 0$ (depending on g and u_0) such that the solution of the Cauchy problem*

$$\begin{cases} u_t = \Delta u + g(u), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.11}$$

satisfies $u_t(t, x) > 0$ for all $t \in (0, \delta]$ and $|x| \geq R$.

The above proposition follows easily from the following identity, which we will prove in Section 5:

$$\lim_{t \downarrow 0, |x| \rightarrow \infty} \left(\frac{t}{|x|} \right)^2 \frac{u_t(t, x)}{u(t, x)} = \frac{1}{4}. \tag{1.12}$$

We now give a brief overview of related research. The front propagation in (1.1) for the cases (f_B) and (f_C) has been studied extensively in the past few decades, and various sufficient conditions on the initial data u_0 for ‘extinction’ ($\lim_{t \rightarrow \infty} u(t, x) = 0$) and for ‘propagation’ ($\lim_{t \rightarrow \infty} u(t, x) = 1$) have been found. For example, it is easy to show that if $0 \leq u_0(x) < \theta_0$ ($x \in \mathbb{R}^N$), then $u(t, x) \rightarrow 0$ as $t \rightarrow \infty$. In [AW1], it is shown that there exists a continuous function ϕ_0 with compact support such that if $u_0 \geq \phi_0$, then $u(t, x) \rightarrow 1$ as $t \rightarrow \infty$.

If f is C^1 and $f'(0) < 0, f'(1) < 0$, then much more is known. For example, [FM] shows that, for each $\theta > \theta_0$, there exists $L = L(\theta) > 0$ such that if $u_0(x) \geq \theta$ on some interval I with length $|I| \geq L$, then $u(t, x) \rightarrow 1$ as $t \rightarrow \infty$. (See Lemma 4.2 below for a slight extension of this result.) Moreover, the existence, uniqueness and stability of traveling fronts connecting 0 and 1 are proved in [FM]. Under the same assumption on f , [J] has studied radially symmetric solutions of a higher dimensional problem

$$u_t = \Delta u + f(u), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N, \tag{1.13}$$

and obtained sharp estimates on the speed of propagation.

More recently, under the assumption that $f \in C^1$, $f(0) = 0$, $f'(0) < 0$, [BJP] has proved the convergence of bounded solutions of (1.13) under the condition that $u(t, x)$ decays to 0 as $|x| \rightarrow \infty$ uniformly in $t > 0$. This last condition is automatically fulfilled if f is a bistable nonlinearity (f_B) (plus $f \in C^1$ and $f'(0) < 0$), if $u_0(x)$ is nonnegative and decays to 0 exponentially fast as $|x| \rightarrow \infty$, and if $u \rightarrow 1$ as $t \rightarrow \infty$. Their approach makes substantial use of the condition $f'(0) < 0$ and relies on a dynamical system method based on the properties of invariant manifolds in a neighborhood of equilibria, which requires $f \in C^1$.

Our approach is very different and need not assume such conditions as

$$f(u) \leq -\sigma u \quad \text{for some } \sigma > 0 \text{ and all small } u > 0, \tag{1.14}$$

which is commonly used in the literature including [FM, J, BJP].

In [FP], the convergence problem for the periodic counterpart of (1.1) was studied, where $f = f(t, u)$ is C^1 and τ -periodic in t for some $\tau > 0$, and satisfies

$$f(t, 0) = 0, \quad f_u(t, 0) < 0. \tag{1.15}$$

However, they only consider solutions satisfying the following conditions:

- (i) $u(t, \cdot) \in C_0(\mathbb{R}^1) := \{\phi \in C(\mathbb{R}^1) : \lim_{|x| \rightarrow \infty} \phi(x) = 0\}$ for all $t > 0$,
- (ii) $\{u(t, \cdot) : t \in \mathbb{R}_+\}$ is relatively compact in $C_0(\mathbb{R}^1)$, namely, for any $t_n > 0$, $\{u(t_n, \cdot)\}$ has a subsequence that converges to some $u_\infty \in C_0(\mathbb{R}^1)$ uniformly in \mathbb{R}^1 .

These restrictions (especially (ii)) force the ω -limit set of a given solution to be an element of a set consisting of (up to translation in x) at most three elements $0, u_+$ and u_- , where u_+ (when it exists) is the unique positive periodic solution with property (i), and u_- (when it exists) is the unique negative periodic solution with property (i). The approach in [FP] used zero number arguments as well as dynamical system arguments based on properties of invariant manifolds in a neighborhood of equilibria, which requires $f \in C^1$. In contrast, though our convergence theorem (Theorem 1.1) deals with the autonomous problem (1.1), it does not have such implicit restrictions on the ω -limit set, and it applies to more general f .

The paper is organized as follows. In Sections 2 and 3, we prove Theorem 1.1. In Section 4, we consider parameterized initial data and prove Theorems 1.3 and 1.4, assuming the validity of Proposition 1.8, whose proof is given in Section 5. Finally, in Section 6, we extend Theorems 1.1, 1.3 and 1.4 to cover certain important cases with a discontinuous nonlinearity.

2. Preliminaries

In this section, we collect some basic facts which will be needed in our proof of Theorem 1.1 in the next section.

2.1. Behavior near $|x| = \infty$

Lemma 2.1. *Let $u(t, x)$ be a solution of (1.1) satisfying the assumptions of Theorem 1.1 and put $[\text{spt}(u_0)] = [L_1, L_2]$. Then*

$$u_x > 0 \text{ for } x < L_1, t > 0; \quad u_x < 0 \text{ for } x > L_2, t > 0. \tag{2.1}$$

Consequently,

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^1)} = \|u(t, \cdot)\|_{L^\infty((L_1, L_2))}. \tag{2.2}$$

Proof. Fix any $x_0 \in (-\infty, L_1)$ and define $w(t, x) = u(t, x) - u(t, 2x_0 - x)$. Then, since u is bounded and f is locally Lipschitz continuous, we can write

$$f(u(t, x)) - f(u(t, 2x_0 - x)) = c(t, x)w(t, x)$$

with some bounded function $c(t, x)$. Thus w satisfies

$$\begin{cases} w_t = w_{xx} + c(t, x)w & (t > 0, -\infty < x < x_0), \\ w(t, x_0) = 0 & (t > 0), \\ w(0, x) \leq 0 & (-\infty < x < x_0) \end{cases}$$

and $w(0, x) \not\equiv 0$. Therefore the strong maximum principle yields $w(t, x) < 0$ for $t > 0$ and $x < x_0$. It then follows from the Hopf boundary lemma that $w_x(t, x_0) > 0$, that is, $2u_x(t, x_0) > 0$ for all $t > 0$. The case $x_0 > L_2$ can be argued analogously. The lemma is proved. \square

Lemma 2.2. *Any solution of (1.1) as given in Theorem 1.1 satisfies*

$$\lim_{|x| \rightarrow \infty} u(t, x) = 0 \text{ uniformly in } t \in [0, T] \text{ for each } T > 0. \tag{2.3}$$

Proof. Since f is locally Lipschitz continuous and $f(0) = 0$, and $u(t, x)$ is bounded, there exists a constant $M \geq 0$ such that $f(u(t, x)) \leq Mu(t, x)$; hence by the comparison principle, we have

$$u(t, x) \leq e^{Mt} \bar{u}(t, x), \tag{2.4}$$

where \bar{u} is the solution of $\bar{u}_t = \bar{u}_{xx}$ with initial data $\bar{u}(0, x) = u_0(x)$. Since $\bar{u}(t, x) \rightarrow 0$ as $|x| \rightarrow \infty$, we obtain (2.3). \square

2.2. Zero number properties

In what follows, $\mathcal{Z}(w)$ will denote the number of sign changes of a continuous function $w(x)$ defined on \mathbb{R}^1 . Note that, if the zeros of w are all simple, then $\mathcal{Z}(w)$ coincides with the number of zeros of w . We also use the notation $\mathcal{Z}_I(w)$ to denote the number of sign changes of w on a given interval I .

The following well-known lemma is a refined version of the Sturmian theorem for one-dimensional parabolic equations.

Lemma 2.3. *Let $w(t, x) \not\equiv 0$ be a solution of the equation*

$$w_t = w_{xx} + c(t, x)w \quad (t \in (t_1, t_2), x \in \mathbb{R}^1), \tag{2.5}$$

where the coefficient function c is bounded. Then, for each $t \in (t_1, t_2)$, the zeros of the function $x \mapsto w(t, x)$ do not accumulate in \mathbb{R}^1 . Furthermore,

- (i) $\mathcal{Z}(w(t, \cdot))$ is nonincreasing in t ;
- (ii) if $w(t^*, x^*) = w_x(t^*, x^*) = 0$ for some $t^* \in (t_1, t_2)$, $x^* \in \mathbb{R}^1$, then

$$\mathcal{Z}(w(t, \cdot)) > \mathcal{Z}(w(s, \cdot)) \quad \text{for all } t \in (t_1, t^*), s \in (t^*, t_2)$$

whenever $\mathcal{Z}(w(s, \cdot)) < \infty$.

The same assertions hold for $\mathcal{Z}_I(w(t, \cdot))$ for any interval $I \subset \mathbb{R}^1$, provided that either w never vanishes on the boundary of I or $w \equiv 0$ on the boundary of I .

Most of the conclusions in Lemma 2.3 are contained in or are easy consequences of [An], but statement (i) does not follow directly from [An] when I is an infinite interval. However, it can be easily shown by combining the argument in [M] and the Phragmén–Lindelöf type maximum principle.

Corollary 2.4. *Let w be a solution of (2.5). If $w(t, x)$ has a constant sign near $x = -\infty$ and near $x = \infty$, then*

$$\mathcal{Z}(w(t, \cdot)) < \infty;$$

if $\mathcal{Z}(w(t_0, \cdot)) < \infty$ then

$$\mathcal{Z}(w(t, \cdot)) < \infty \quad \text{for all } t > t_0.$$

Corollary 2.5. *Let w be a solution of (2.5) and suppose that there exists $x_0 \in \mathbb{R}^1$ such that $w(t, x_0) = w_x(t, x_0) = 0$ for every t in some interval $t_1 < t < t_2$. Then $w \equiv 0$.*

Let us recall other basic properties of \mathcal{Z} . Since \mathcal{Z} denotes the number of sign changes, the pointwise convergence $w_n(x) \rightarrow w(x)$ implies

$$w \equiv 0 \quad \text{or} \quad \mathcal{Z}(w) \leq \liminf_{n \rightarrow \infty} \mathcal{Z}(w_n). \tag{2.6}$$

Moreover, if I is a compact interval, if $w_n \rightarrow w$ in $C^1(I)$, if every zero of w in I is simple, and finally if $w \neq 0$ on the boundary of I , then

$$\mathcal{Z}_I(w) = \lim_{n \rightarrow \infty} \mathcal{Z}_I(w_n). \tag{2.7}$$

Lemma 2.6. *Let $w_n(t, x)$ be a sequence of functions converging to $w(t, x)$ in $C^1((t_1, t_2) \times I)$, where I is an open interval in \mathbb{R}^1 . Assume that, for every $t \in (t_1, t_2)$ and $n \in \mathbb{N}$, the function $x \mapsto w_n(t, x)$ has only simple zeros in I , and that $w(t, x)$ satisfies an equation of the form (2.5) on $(t_1, t_2) \times I$. Then for every $t \in (t_1, t_2)$, either $w(t, x) \equiv 0$ on I , or $w(t, x)$ has only simple zeros in I .*

Here and in what follows, we use the usual convention that u_n converges to u in $C^m(\Omega)$ if and only if $u_n \rightarrow u$ in $C^m(K)$ for any compact subset K of Ω .

Proof. Suppose that there exists $t^* \in (t_1, t_2)$ such that $w(t^*, x)$ has a degenerate zero, say x^* , in I and that $w(t^*, x) \not\equiv 0$. By Lemma 2.3, the zeros of $w(t^*, x)$ are isolated, therefore we can choose $\tau, \sigma > 0$ sufficiently small so that x^* is the only zero of $w(t^*, x)$ in the interval $J := \{|x - x^*| \leq \sigma\} \subset I$ and that

$$w(t, x^* \pm \sigma) \neq 0 \quad \text{for } t \in [t^* - \tau, t^* + \tau].$$

Since x^* is a degenerate zero, by Lemma 2.3 we have

$$\mathcal{Z}_J(w(t^* - \tau, \cdot)) > \mathcal{Z}_J(w(t^* + \tau, \cdot)).$$

Here we can assume without loss of generality that the zeros of $w(t^* \pm \tau, x)$ in the interval J are all simple, since degenerate zeros appear at discrete time moments by Lemma 2.3(ii). In view of this and (2.7), we see that

$$\begin{aligned} \mathcal{Z}_J(w_n(t^* - \tau, \cdot)) &> \mathcal{Z}_J(w_n(t^* + \tau, \cdot)), \\ w_n(t, x^* \pm \sigma) &\neq 0 \quad \text{for } t \in [t^* - \tau, t^* + \tau] \end{aligned}$$

for all large n , which implies that $w_n(t, x)$ has a degenerate zero in the region $(t, x) \in (t^* - \tau, t^* + \tau) \times J$, contradicting the assumption of the lemma. This contradiction proves the lemma. \square

2.3. Reflection of the solution

Lemma 2.7. *Let u be as in Lemma 2.1 and, for each $a \in \mathbb{R}^1$, define*

$$w_a(t, x) := u(t, x) - u(t, 2a - x).$$

Then for every $a \neq (L_1 + L_2)/2$ and $t > 0$, the function $x \mapsto w_a(t, x)$ changes sign at most finitely many times.

Proof. Let us write $w(t, x) = w_a(t, x)$ for simplicity. Since both $u(t, x)$ and $u(t, 2a - x)$ satisfy the same equation (1.1), w satisfies a linear parabolic equation of the form

$$w_t = w_{xx} + c(t, x)w,$$

where c is bounded because of the local Lipschitz continuity of f .

Denote by $\bar{u}(t, x)$ the solution of the following Cauchy problem:

$$\bar{u}_t = \bar{u}_{xx} \quad (t > 0, x \in \mathbb{R}^1), \quad \bar{u}(0, x) = u_0(x) \quad (x \in \mathbb{R}^1).$$

Since u is bounded and f is locally Lipschitz, $-Mu \leq f(u) \leq Mu$ for some constant $M \geq 0$ and all $t \geq 0, x \in \mathbb{R}^1$. Therefore a simple comparison argument shows that

$$e^{-Mt}\bar{u}(t, x) \leq u(t, x) \leq e^{Mt}\bar{u}(t, x) \quad \text{for all } (t, x) \in [0, \infty) \times \mathbb{R}^1.$$

Thus we have

$$\begin{cases} w(t, x) \geq e^{-Mt} \bar{u}(t, x) - e^{Mt} \bar{u}(t, 2a - x), \\ w(t, x) \leq e^{Mt} \bar{u}(t, x) - e^{-Mt} \bar{u}(t, 2a - x). \end{cases} \quad (2.8)$$

Note also that

$$\begin{aligned} \bar{u}(t, x) &= (4\pi t)^{-1/2} \int_{L_1}^{L_2} \exp\left(-\frac{(x-y)^2}{4t}\right) u_0(y) dy, \\ \bar{u}(t, 2a-x) &= (4\pi t)^{-1/2} \int_{2a-L_2}^{2a-L_1} \exp\left(-\frac{(x-y)^2}{4t}\right) u_0(2a-y) dy. \end{aligned} \quad (2.9)$$

In what follows we consider only the case $a < (L_1 + L_2)/2$, since the other case can be treated the same way. In this case, we claim that

$$\lim_{x \rightarrow -\infty} \frac{\bar{u}(t, 2a-x)}{\bar{u}(t, x)} = \infty, \quad \lim_{x \rightarrow \infty} \frac{\bar{u}(t, 2a-x)}{\bar{u}(t, x)} = 0. \quad (2.10)$$

To see this, we use (2.9). Choose L_* with $2a - L_2 < L_* < L_1$. Then for some constants $C_1, C_2 > 0$ we have

$$\begin{aligned} \bar{u}(t, x) &\leq C_1 \exp\left(-\frac{(x-L_1)^2}{4t}\right) \quad \text{for } x \leq L_1, \\ \bar{u}(t, 2a-x) &\geq (4\pi t)^{-1/2} \int_{2a-L_2}^{L_*} \exp\left(-\frac{(x-y)^2}{4t}\right) u_0(2a-y) dy \\ &\geq C_2 \exp\left(-\frac{(x-L_*)^2}{4t}\right) \quad \text{for } x \leq 2a - L_2, \end{aligned}$$

from which the first assertion of (2.10) follows. The second assertion of (2.10) can be shown similarly. Consequently,

$$\begin{cases} e^{Mt} \bar{u}(t, x) - e^{-Mt} \bar{u}(t, 2a-x) < 0 & \text{for } x \ll -1, \\ e^{-Mt} \bar{u}(t, x) - e^{Mt} \bar{u}(t, 2a-x) > 0 & \text{for } x \gg 1. \end{cases}$$

This and (2.8) imply that, for each $t_0 > 0$, there exists $\ell > 0$ such that

$$w(t_0, x) < 0 \quad \text{for } x \leq -\ell, \quad w(t_0, x) > 0 \quad \text{for } x \geq \ell.$$

Thus by Corollary 2.4,

$$\mathcal{Z}(w(t, \cdot)) < \infty \quad \text{for all } t > t_0.$$

Since $t_0 > 0$ is arbitrary, the conclusion of the lemma follows. □

2.4. Position of peaks

By Lemma 2.1, the positions of the local maxima and minima of the function $x \mapsto u(t, x)$ are confined to the interval $[L_1, L_2]$ for every $t > 0$. The following lemma implies, roughly speaking, that the positions of the local maxima and minima do not oscillate indefinitely.

Lemma 2.8. *For every $x \neq (L_1 + L_2)/2$, the following limit exists:*

$$\lim_{t \rightarrow \infty} \operatorname{sgn}(u_x(t, x)) \neq 0,$$

where $\operatorname{sgn}(w) := 1, -1, 0$ depending on whether $w > 0$, $w < 0$ or $w = 0$.

Proof. We use an argument introduced in [C1, CM]. Fix $a \neq (L_1 + L_2)/2$ arbitrarily and put $w(t, x) := u(t, x) - u(t, 2a - x)$. Then by Lemmas 2.7 and 2.3, $\mathcal{Z}(w(t, \cdot))$ is finite for every $t > 0$ and it is nonincreasing in t . Therefore $\mathcal{Z}(w(t, \cdot))$ is constant for all large t . This implies, by Lemma 2.3(ii), that the function $x \mapsto w(t, x)$ has only simple zeros in \mathbb{R}^1 for all large t . In particular, since $w(t, a) = 0$ for all $t > 0$, we have

$$w_x(t, a) \neq 0 \quad \text{for all large } t > 0.$$

In view of this and the fact that $w_x(t, a)$ is continuous in t , we see that $w_x(t, a)$ does not change sign for all large $t > 0$. The conclusion of the lemma now follows since $w_x(t, a) = 2u_x(t, a)$. \square

Now we define x_0 as follows. (Roughly speaking, x_0 denotes the limit position of the leftmost local maximum of the function $x \mapsto u(t, x)$.)

$$x_0 := \inf\{x \in \mathbb{R}^1 : \lim_{t \rightarrow \infty} \operatorname{sgn}(u_x(t, x)) = -1\}. \quad (2.11)$$

It is clear from (2.1) that

$$L_1 \leq x_0 \leq L_2. \quad (2.12)$$

2.5. Intersection with a stationary solution

If we solve the ordinary differential equation (1.3) under the initial condition

$$v(x_0) = \alpha, \quad v'(x_0) = 0 \quad (2.13)$$

for some $\alpha > 0$ and $x_0 \in \mathbb{R}^1$, then $v(x)$ is symmetric about $x = x_0$ and either $v(x) > 0$ for all $x \in \mathbb{R}^1$ or there exists $R > 0$ such that

$$v(x) > 0 \quad \text{for } |x - x_0| < R, \quad v(x_0 \pm R) = 0.$$

In the latter case we call v a solution of (1.3) with *compact positive support*, and define $\operatorname{spt}^+(v) := [x_0 - R, x_0 + R]$. In the former case, we define $\operatorname{spt}^+(v) := \mathbb{R}^1$.

Lemma 2.9. *Let u be as in Lemma 2.1. Suppose that v is either a positive solution of (1.3) in \mathbb{R}^1 or a solution of (1.3) with compact positive support. Then*

$$\mathcal{Z}_I(u(t, \cdot) - v) < \infty \quad \text{for } t > 0, \tag{2.14}$$

and it is nonincreasing in t , where $I = \text{spt}^+(v)$.

Proof. First we assume that $v > 0$ on \mathbb{R}^1 . Let $L > 0$ be such that $\text{spt}(u_0) \subset (-L, L)$. Then we can choose $\delta > 0$ sufficiently small so that

$$0 < u(t, \pm L) < v(\pm L) \quad \text{for } 0 \leq t \leq \delta.$$

Since $u(0, x) < v(x)$ for $|x| \geq L$, the comparison principle implies

$$u(t, x) - v(x) < 0 \quad \text{for } 0 \leq t \leq \delta, |x| \geq L.$$

Thus, by Lemma 2.3, we have

$$\mathcal{Z}(u(t, \cdot) - v) < \infty \quad \text{for } 0 < t \leq \delta.$$

Since $\mathcal{Z}(u(t, \cdot) - v)$ is nonincreasing in t , (2.14) holds for all $t > 0$.

Next we consider the case where v is a compactly supported nonnegative solution. Put $[L^-, L^+] := \text{spt}^+(v)$. Then we have

$$u(t, L^\pm) > v(L^\pm) = 0 \quad \text{for } t > 0.$$

This and Lemma 2.3 yield (2.14). The lemma is proved. □

2.6. Property of symmetrically decreasing solutions

Lemma 2.10. *Let $v(x)$ be a positive solution of (1.3) such that*

$$v'(x_0) = 0, \quad v'(x) < 0 \quad \text{for } x > x_0.$$

Then there exists a sequence $\{v_n\}_{n=1}^\infty$ of positive periodic solutions such that

$$v'_n(x_0) = 0 \quad (n = 1, 2, \dots), \quad v_n(x_0) \nearrow v(x_0) \text{ as } n \rightarrow \infty.$$

Proof. Clearly the solution $v(x)$ is symmetric with respect to $x = x_0$, and $f(v(x_0)) > 0$. Put $a := \lim_{|x| \rightarrow \infty} v(x)$ and $b = v(x_0)$. Then multiplying (1.3) by v' and integrating it over $x_0 < x < \infty$, we obtain

$$\int_a^b f(s) ds = 0, \quad \int_c^b f(s) ds > 0 \quad \text{for } c \in (a, b).$$

Choose $\varepsilon_0 > 0$ sufficiently small so that $f(s) > 0$ for $s \in [b - \varepsilon_0, b]$. Then for any $\varepsilon \in (0, \varepsilon_0]$, we can find $a_\varepsilon \in (a, b - \varepsilon)$ such that

$$\int_{a_\varepsilon}^{b-\varepsilon} f(s) ds = 0, \quad \int_c^{b-\varepsilon} f(s) ds > 0 \quad \text{for any } c \in (a_\varepsilon, b - \varepsilon).$$

As is easily seen, the solution v_ε of (1.3) satisfying $v_\varepsilon(x_0) = b - \varepsilon$, $v'_\varepsilon(x_0) = 0$ is either a symmetrically decreasing solution converging to a_ε as $|x| \rightarrow \infty$, or a periodic solution oscillating between $b - \varepsilon$ and a_ε , depending on whether $f(a_\varepsilon) = 0$ or $f(a_\varepsilon) \neq 0$. Note that a_ε can also be characterized as

$$F(a_\varepsilon) = m_\varepsilon, \quad \text{where} \quad F(u) := \int_u^b f(s) ds, \quad m_\varepsilon := F(b - \varepsilon).$$

By Sard's theorem, $F'(a_\varepsilon) (= -f(a_\varepsilon)) \neq 0$ for almost every value of m_ε . It follows that this holds for a.e. $\varepsilon \in (0, \varepsilon_0]$, since $m_\varepsilon = F(b - \varepsilon)$ and $\frac{d}{d\varepsilon} F(b - \varepsilon) = -f(b - \varepsilon) \neq 0$. This means that v_ε is a periodic solution for almost every $\varepsilon \in (0, \varepsilon_0]$. The lemma is proved. \square

3. Proof of the convergence theorem

3.1. Basic properties of the ω -limit set

Let $u(t, x)$ be as given in Theorem 1.1. Denote by $\omega(u)$ the ω -limit set of $u(t, \cdot)$ in the topology of $L^\infty_{\text{loc}}(\mathbb{R}^1)$. Thus a function $w(x)$ belongs to $\omega(u)$ if and only if there exists a sequence $0 < t_n \nearrow \infty$ such that

$$\lim_{n \rightarrow \infty} u(t_n, x) = w(x) \quad \text{locally uniformly in } \mathbb{R}^1. \tag{3.1}$$

By local parabolic estimates, we see that the convergence (3.1) implies convergence in the $C^2(\mathbb{R}^1)$ topology. Thus the definition of $\omega(u)$ remains unchanged if the topology of $L^\infty_{\text{loc}}(\mathbb{R}^1)$ is replaced by that of $C^2(\mathbb{R}^1)$.

It is well-known that $\omega(u)$ is compact and connected, and it is an invariant set. This means that for any $w \in \omega(u)$ there exists an entire orbit (namely a solution of (1.1) defined for all $t \in \mathbb{R}^1$) passing through w . Choosing a suitable sequence $0 < t_n \nearrow \infty$, we can find such an entire solution $W(t, x)$ with $W(0, x) = w(x)$ as follows:

$$u(t + t_n, x) \rightarrow W(t, x) \quad \text{as } n \rightarrow \infty. \tag{3.2}$$

Here the convergence is understood in the L^∞_{loc} sense in $(t, x) \in \mathbb{R}^1 \times \mathbb{R}^1$, but, by parabolic regularity, it takes place in the $C^{1,2}(\mathbb{R}^1 \times \mathbb{R}^1)$ sense.

The following lemma is an immediate consequence of Lemma 2.1.

Lemma 3.1. *Let $w = w(x)$ be any element of $\omega(u)$. Then*

$$w'(x) \geq 0 \quad \text{for } x < L_1, \quad w'(x) \leq 0 \quad \text{for } x > L_2.$$

Consequently, $\omega(u)$ contains no periodic solution of (1.3) and no symmetrically increasing solution.

The next lemma follows easily from Lemma 2.8.

Lemma 3.2. *Let x_0 be the point defined in (2.11). Then any element w of $\omega(u)$ satisfies $w'(x_0) = 0$.*

Proof. By the definition of x_0 and Lemma 2.8, we have $\lim_{t \rightarrow \infty} \operatorname{sgn}(u_x(t, x)) = 1$ for all $x < x_0$ except possibly $x = (L_1 + L_2)/2$, and there exists a sequence of points $x_n \searrow x_0$ such that $\lim_{t \rightarrow \infty} \operatorname{sgn}(u_x(t, x_n)) = -1$. Consequently,

$$w'(x) \geq 0 \quad \text{for } x < x_0, \quad w'(x_n) \leq 0 \quad \text{for } n = 1, 2, \dots$$

This implies $w'(x_0) = 0$ and the lemma is proved. □

Lemma 3.3. *Let $v(x)$ be a bounded positive solution of (1.3), and let w be any element of $\omega(u)$. Then*

$$\text{either } w \equiv v \quad \text{or} \quad \mathcal{Z}(w - v) < \infty \tag{3.3}$$

Proof. The lemma follows immediately from Lemma 2.9 and (2.6). □

3.2. Further properties of the ω -limit set

Lemma 3.4. *Let u be as in Theorem 1.1. Then $\omega(u)$ consists only of stationary solutions.*

Proof. Let w be any element of $\omega(u)$ and $W(t, x)$ be as given in (3.2). Since W is a nonnegative solution of the equation in (1.1) and since $f(0) = 0$, by the strong maximum principle we have either $W(t, x) > 0$ for all $t \in \mathbb{R}^1, x \in \mathbb{R}^1$, or $W \equiv 0$. In the latter case we have $w \equiv 0$, so the conclusion of the lemma holds. In what follows we assume the former; thus $w > 0$.

Let x_0 be as in Lemma 3.2 and let $v(x)$ be the solution of the following initial value problem:

$$v'' + f(v) = 0, \quad v(x_0) = w(x_0), \quad v'(x_0) = 0.$$

Then v is symmetric about $x = x_0$. Since $w(x_0) > 0$, v is either a positive solution of (1.3) in \mathbb{R}^1 , or a solution of (1.3) with compact positive support. Then by Lemma 2.9, $\mathcal{Z}_I(u(t, \cdot) - v) < \infty$ for $t > 0$, where $I := \operatorname{spt}^+(v)$. Since this quantity is nonincreasing in t , it is constant for all large t . Consequently, by Lemma 2.3, the function $u(t, x) - v(x)$ has only simple zeros on I for all large t . In view of this and the fact that

$$\lim_{n \rightarrow \infty} u(t + t_n, x) - v(x) = W(t, x) - v(x) \quad \text{in } C^1(\mathbb{R}^1 \times I),$$

and applying Lemma 2.6, we see that for each $t \in \mathbb{R}^1$, either $W(t, x) - v(x) \equiv 0$ on I , or $W(t, x) - v(x)$ has only simple zeros on I . However, the latter is impossible for $t = 0$, since $W(0, x) - v(x) = w(x) - v(x)$ has a degenerate zero at $x = x_0$. Consequently, $W(0, x) \equiv v(x)$, that is, w is a stationary solution. The lemma is proved. □

3.3. Completion of the proof of Theorem 1.1

It remains to show that $\omega(u)$ consists of only one element. Suppose that it contains more than one element. Then, since $\omega(u)$ is connected and compact in the topology of $L^\infty_{loc}(\mathbb{R}^1)$, and since every element w of $\omega(u)$ satisfies $w'(x_0) = 0$ by Lemma 3.2, there exist $0 \leq \gamma^- < \gamma^+$ such that $\omega(u)$ consists of solutions v_α ($\alpha \in [\gamma^-, \gamma^+]$) of (1.3) defined by

$$v_\alpha(x_0) = \alpha, \quad v'_\alpha(x_0) = 0.$$

By Lemma 3.1, for each $\alpha \in [\gamma^-, \gamma^+]$, v_α is neither a periodic solution nor a symmetrically increasing solution. In view of this and Lemma 2.10, we see that, for each $\alpha \in (\gamma^-, \gamma^+]$, v_α is not a symmetrically decreasing solution. Consequently, v_α is a constant solution for every $\alpha \in (\gamma^-, \gamma^+]$, that is, $v_\alpha \equiv \alpha$. By continuity, the same holds for $\alpha = \gamma^-$. Fix α with $\gamma^- < \alpha < \gamma^+$. Then, since γ^- belongs to $\omega(u)$, there is a sequence $t_n \nearrow \infty$ such that

$$u(t_n, x) \rightarrow \gamma^- \quad \text{as } n \rightarrow \infty \text{ locally uniformly in } \mathbb{R}^1.$$

In view of this and (2.1), we see that

$$u(t_n, x) \leq \alpha \quad \text{for } x \in \mathbb{R}^1 \text{ for sufficiently large } n.$$

Then by the comparison principle, we have $u(t, x) \leq \alpha$ for all large $t > 0$, which contradicts the assumption that γ^+ belongs to $\omega(u)$. This contradiction proves that $\omega(u)$ is a singleton.

Finally, as we have already noted, the element of $\omega(u)$ is either a constant or a symmetrically decreasing solution by virtue of Lemma 3.1. The proof of Theorem 1.1 is complete.

3.4. A further remark

We remark that certain types of constant solutions of (1.3) can never belong to the ω -limit set of u .

Lemma 3.5. *Let u, f be as in Theorem 1.1 and β be a positive zero of f . Then $\beta \notin \omega(u)$ if one of the following conditions holds for some $\varepsilon > 0$:*

- (a) $f(s) \leq 0$ for $s \in [\beta - \varepsilon, \beta + \varepsilon]$, (b) $(\beta - s)f(s) < 0$ for $s \in [\beta - \varepsilon, \beta) \cup (\beta, \beta + \varepsilon]$.

Proof. Let us first assume (a). Suppose that there exists a sequence $t_n \nearrow \infty$ such that $u(t_n, x)$ converges to β as $n \rightarrow \infty$ locally uniformly in \mathbb{R}^1 . Then by (2.2), there exists $t_0 \geq 0$ such that

$$u(t_0, x) \leq \beta + \varepsilon \quad \text{for } x \in \mathbb{R}^1.$$

Let $v(t, x)$ be a solution of the following problem:

$$v_t = v_{xx} \quad (t \geq t_0, x \in \mathbb{R}^1), \quad v(t_0, x) = \max\{u(t_0, x), \beta - \varepsilon\} \quad (x \in \mathbb{R}^1).$$

Then $\beta - \varepsilon \leq v(t, x) \leq \beta + \varepsilon$ for all $t \geq t_0$, $x \in \mathbb{R}^1$, and hence, by condition (a), v is a supersolution of (1.1). Consequently $v \geq u$ for all $t \geq t_0$. On the other hand, by (2.3), $v(t_0, x)$ equals $\beta - \varepsilon$ for all large $|x|$. It follows that $v(t, x) \rightarrow \beta - \varepsilon$ as $t \rightarrow \infty$, and

$$\beta = \lim_{t \rightarrow \infty} u(t, x) \leq \lim_{t \rightarrow \infty} v(t, x) = \beta - \varepsilon < \beta.$$

This contradiction shows that $\lim_{t \rightarrow \infty} u \neq \beta$.

Next we assume (b). Then we easily see that there exists a positive periodic solution $v(x)$ of (1.3) satisfying $\min v < \beta < \max v$, which implies that $\mathcal{Z}(\beta - v) = \infty$. Hence we can apply Lemma 3.3 to conclude that $\beta \notin \omega(u)$. The lemma is proved. \square

4. Sharp threshold results

In this section we consider two specific classes of nonlinearities—those of the bistable type and those of the combustion type—and prove Theorems 1.3 and 1.4.

4.1. Bistable nonlinearity

Throughout this subsection we assume that

$$f \text{ is locally Lipschitz and satisfies (1.4) and (1.5).} \tag{f_B}$$

Let u be the solution of (1.1) with u_0 as in Theorem 1.1. Since $f(s) < 0$ for $s > 1$, a simple comparison argument shows that u is bounded; hence, by Theorem 1.1, $u(t, x)$ converges to a solution of the stationary problem (1.3) as $t \rightarrow \infty$. Recall that solutions of (1.3) which are constants or symmetrically decreasing can only be:

$$0, \quad \theta_0, \quad 1, \quad U(x - a) \quad (a \in \mathbb{R}^1),$$

where U is determined by (1.8). Since θ_0 intersects each periodic solution of (1.3) infinitely many times, we see from Lemma 3.3 that $\theta_0 \notin \omega(u)$. Therefore

$$\omega(u) = \{0\}, \quad \{1\} \quad \text{or} \quad \{U(\cdot - a)\} \text{ for some } a \in \mathbb{R}^1, \tag{4.1}$$

where the ω -limit set is defined with respect to the topology of $L_{loc}^\infty(\mathbb{R}^1)$.

Lemma 4.1. *Suppose that (Φ_1) through (Φ_3) hold. Then the set*

$$\Sigma_0 := \{\lambda > 0 : u_\lambda(t, x) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ locally uniformly in } x \in \mathbb{R}^1\}$$

is an open interval $(0, \lambda^0)$ for some $\lambda^0 \in (0, \infty]$.

Proof. We first observe that

$$\lambda \in \Sigma_0 \Leftrightarrow \max_{L_1^\lambda \leq x \leq L_2^\lambda} u_\lambda(t_0, x) < \theta_0 \text{ for some } t_0 > 0, \tag{4.2}$$

where $[L_1^\lambda, L_2^\lambda] = [\text{spt}(\phi_\lambda)]$. Indeed, the assertion ‘ \Rightarrow ’ is obvious. To prove ‘ \Leftarrow ’, suppose that the right-hand side of (4.2) holds. Then, by (2.2), we have

$$0 \leq u_\lambda(t_0, x) \leq \theta_0 - \varepsilon \quad (x \in \mathbb{R}^1) \quad \text{for some } \varepsilon > 0.$$

Since $f(s) < 0$ for $s \in (0, \theta_0)$, we see from a comparison argument that $u_\lambda(t, x) \rightarrow 0$ as $t \rightarrow \infty$ uniformly in $x \in \mathbb{R}^1$, which proves (4.2).

We next show that $\lambda \in \Sigma_0$ if $\lambda > 0$ is sufficiently small. Fix $t_0 > 0$ arbitrarily. By (Φ_1) and (Φ_2) , ϕ_λ remains uniformly bounded on \mathbb{R}^1 as λ varies in a finite interval. This and (Φ_3) imply that $\phi_\lambda \rightarrow 0$ in $L^p(\mathbb{R}^1)$ for any $p > 1$ as $\lambda \rightarrow 0$. By standard interior parabolic estimates, $u_\lambda(t_0, x) \rightarrow 0$ as $\lambda \rightarrow 0$ uniformly in $x \in [L_1^\delta, L_2^\delta]$ for each $\delta > 0$; hence, by (4.2), we have $\lambda \in \Sigma_0$ for sufficiently small $\lambda > 0$. In particular, $\Sigma_0 \neq \emptyset$.

Put $\lambda^0 := \sup \Sigma_0$. Then $0 < \lambda^0 \leq \infty$, and the comparison theorem and (Φ_2) yield

$$(0, \lambda^0) \subset \Sigma_0.$$

It remains to show that $\lambda^0 \notin \Sigma_0$ if $\lambda^0 < \infty$. Suppose the contrary. Then the right-hand side of (4.2) holds for $\lambda = \lambda^0$ and some $t_0 > 0$. Then, since the mapping $\lambda \mapsto u_\lambda(t_0, \cdot)$ is continuous from $[0, \lambda^0 + \delta]$ to $C([L_1^{\lambda^0 + \delta}, L_2^{\lambda^0 + \delta}])$ ($\delta > 0$) by virtue of (Φ_3) and parabolic estimates, we see that the right-hand side of (4.2) holds for $\lambda = \lambda^0 + \delta$ if $\delta > 0$ is sufficiently small. This contradicts the definition of λ^0 , and the lemma is proved. \square

The following lemma gives a sufficient condition for $\omega(u) = \{1\}$.

Lemma 4.2. *For any given constant $\alpha \in (\theta_0, 1)$, there exists $L_\alpha > 0$ depending on α, f such that the solution $u(t, x)$ of (1.1) with initial data $\alpha \chi_{[-L_\alpha, L_\alpha]}$ converges to 1 as $t \rightarrow \infty$ locally uniformly in $x \in \mathbb{R}^1$.*

Proof. If f is C^1 and $f'(0) < 0, f'(1) < 0$ then the conclusion follows from Theorem 3.2 of [FM]. Clearly we can choose a function $g \in C^1([0, \infty))$ such that

$$\begin{aligned} f(s) &\geq g(s) \quad \text{for } s \geq 0, & g(0) &= g(\xi_0) = g(\xi_1) = 0, \\ g(s) &< 0 \quad \text{for } s \in (0, \xi_0) \cup (\xi_1, \infty), & g(s) &> 0 \quad \text{for } s \in (\xi_0, \xi_1), \\ g'(0) &< 0, \quad g'(\xi_1) < 0, & \int_0^{\xi_1} g(s) ds &> 0 \end{aligned}$$

for some $\xi_0 \in (0, \theta_0), \xi_1 \in (\theta_0, 1)$. Let v be the solution of the problem

$$v_t = v_{xx} + g(v) \quad (t > 0, x \in \mathbb{R}^1), \quad v(0, x) = \alpha \chi_{[-L_0, L_0]}(x) \quad (x \in \mathbb{R}^1).$$

Applying [FM, Theorem 3.2] to the above problem, we get $\lim_{t \rightarrow \infty} v(t, x) = \xi_1$ if L_0 is chosen sufficiently large. Since $f \geq g$, the comparison principle yields

$$\lim_{t \rightarrow \infty} u(t, x) \geq \lim_{t \rightarrow \infty} v(t, x) = \xi_1.$$

This and (4.1) imply $\omega(u) = \{1\}$. The lemma is proved. \square

Lemma 4.3. *Suppose that (Φ_1) through (Φ_3) hold. Then*

$$\Sigma_1 := \{\lambda > 0 : u_\lambda(t, x) \rightarrow 1 \text{ as } t \rightarrow \infty \text{ locally uniformly in } x \in \mathbb{R}^1\}$$

is either empty or an open interval (λ^1, ∞) for some $\lambda^1 > 0$.

Proof. Fix constants $\alpha \in (\theta_0, 1)$ and $L_\alpha > 0$ as in Lemma 4.2. Then

$$\lambda \in \Sigma_1 \Leftrightarrow \min_{-L_\alpha \leq x \leq L_\alpha} u_\lambda(t_0, x) > \alpha \text{ for some } t_0 > 0. \tag{4.3}$$

Indeed, ‘ \Rightarrow ’ is obvious, and ‘ \Leftarrow ’ follows easily from Lemma 4.2, the comparison principle and (4.1). Now assume $\Sigma_1 \neq \emptyset$, and put $\lambda^1 := \inf \Sigma_1$. Then the comparison theorem and (Φ_2) imply

$$(\lambda^1, \infty) \subset \Sigma_1.$$

It remains to show that $\lambda^1 \notin \Sigma_1$; but this follows easily from (4.3), by an argument analogous to that in the last part of the proof of Lemma 4.1. The lemma is proved. \square

The following lemma shows the sharpness of transition between extinction and propagation.

Lemma 4.4. *If $\lim_{t \rightarrow \infty} u_{\lambda^*} = U(x - a_0)$ for some $\lambda^* > 0$ and $a_0 \in \mathbb{R}^1$, then $\lim_{t \rightarrow \infty} u_\lambda = 1$ for every $\lambda > \lambda^*$.*

We will prove the above assertion using the following lemma.

Lemma 4.5. *Let ϕ_1, ϕ_2 be bounded nonnegative functions on \mathbb{R}^1 satisfying*

$$\phi_1 \leq \phi_2, \quad \phi_1 \not\equiv \phi_2, \quad [\text{spt}(\phi_2)] \subset [-L, L]$$

for some $L > 0$, and let u_i be the solution of (1.1) with $u_0 = \phi_i$, $i = 1, 2$. Then there exist positive constants t_0, δ and ε such that

$$u_2(t + \delta, x) > u_1(t, x + a) \quad \text{for } t \geq t_0, \ x \in \mathbb{R}^1 \text{ and } |a| \leq \varepsilon.$$

Proof. By Proposition 1.8, there exists $t_0 > 0$ such that $u_i(t, x)$ is increasing in t for $t \in (0, 2t_0]$ and $|x| \geq 1/(2t_0)$. We may assume $L_0 := 1/(2t_0) > L$.

By the strong maximum principle, we have $u_2(t, x) > u_1(t, x)$ for all $t > 0$ and $x \in \mathbb{R}^1$. Thus, by uniform continuity, we can find constants $\delta \in (0, t_0]$ and $\varepsilon > 0$ such that

$$u_2(t_0 + \delta, x) > u_1(t_0, x + a) \quad \text{for } x \in [-L_0, L_0], \ |a| \leq \varepsilon. \tag{4.4}$$

Now the inequality $u_2(t, x) > u_1(t, x)$ and the monotonicity of $u_2(t, \pm L_0)$ in t imply

$$u_2(t + \delta, x) > u_1(t, x) \quad \text{for } t \in [0, t_0], \ x = \pm L_0. \tag{4.5}$$

Since $L_0 > L$, $u_1(t, x)$ is continuous up to $t = 0$ in a neighborhood of $x = \pm L_0$. Therefore, if we choose a sufficiently small $\varepsilon > 0$, then from (4.5) we see that

$$u_2(t + \delta, x) > u_1(t, x + a) \quad \text{for } t \in [0, t_0], \ x = \pm L_0, \ |a| \leq \varepsilon. \tag{4.6}$$

Also, since $u_1(0, x) = 0$ for $|x| > L$, we have

$$u_2(\delta, x) > u_1(0, x + a) \quad \text{for } |x| \geq L_0, |a| \leq \varepsilon, \tag{4.7}$$

provided that ε is chosen small enough so that $\varepsilon < L_0 - L$. Combining this and (4.6), and applying the comparison principle, we obtain

$$u_2(t_0 + \delta, x) > u_1(t_0, x + a) \quad \text{for } |x| \geq L_0.$$

This and (4.4) yield

$$u_2(t_0 + \delta, x) > u_1(t_0, x + a) \quad \text{for } x \in \mathbb{R}^1, |a| \leq \varepsilon.$$

The conclusion of the lemma now follows from this and the comparison principle. \square

Proof of Lemma 4.4. Let $\lambda^* < \lambda$ be as in the lemma. Then, by Lemma 4.5, we have

$$u_\lambda(t + \delta, x) > u_{\lambda^*}(t, x + a) \quad \text{for } t \geq t_0, x \in \mathbb{R}^1 \text{ and } |a| \leq \varepsilon.$$

Put $v(x) := \lim_{t \rightarrow \infty} u_\lambda(t, x)$. Then v satisfies

$$v(x) \geq U(x - a_0 + a) \quad \text{for } x \in \mathbb{R}^1, |a| \leq \varepsilon.$$

The only solution of (1.3) satisfying this is $v = 1$. The lemma is proved. \square

Now we are ready to prove the main theorem of this subsection.

Proof of Theorem 1.3. By Lemma 4.1, we have $\Sigma_0 = (0, \lambda^0)$ for some $\lambda^0 \in (0, \infty]$. If $\lambda^0 = \infty$, then the alternative (a) of the theorem holds. On the other hand, if $\lambda^0 < \infty$, then by Lemma 4.3, $\lambda^0 \notin \Sigma_0 \cup \Sigma_1$, hence

$$u_{\lambda^0}(t, x) \rightarrow U(x - a_0) \quad \text{as } t \rightarrow \infty \quad \text{for some } a_0 \in [L_1, L_2].$$

By Lemma 4.4, we have $\lambda \in \Sigma_1$ for all $\lambda > \lambda^0$. The proof of the theorem is complete. \square

Remark 4.6. In the bistable case (f_B), the convergence results of Theorem 1.1 can be proved more easily by using the energy functional

$$J(u(t, \cdot)) := \int_{\mathbb{R}^1} \left[\frac{1}{2} u_x^2(t, x) - \int_0^{u(t,x)} f(s) ds \right] dx.$$

Indeed, if $\lim_{t \rightarrow \infty} u(t, x) \neq 1$, then by Lemmas 2.1 and 4.2 there exists $L > 0$ with $\text{spt}(u_0) \subset [-L, L]$ such that $u(t, x) \leq \theta^*$ for all $t \geq 0, |x| \geq L$. This and the comparison principle imply

$$u(t, x) \leq U(x + L) \quad \text{for } x \leq -L, \quad u(t, x) \leq U(x - L) \quad \text{for } x \geq L.$$

From the above estimates one sees that $J(u(t, \cdot))$ is bounded from below as $t \rightarrow \infty$; hence $\omega(u)$ is contained in the set of nonnegative solutions of (1.3) decaying to 0 as $|x| \rightarrow \infty$. One can then use Lemma 2.8 to conclude that either $\omega(u) = \{0\}$ or $\omega(u) = \{U(\cdot - a)\}$ for some $a \in \mathbb{R}^1$.

4.2. Combustion nonlinearity

In this subsection we assume that

$$f \text{ is locally Lipschitz and satisfies (1.6) and (1.7).} \tag{f_C}$$

Let u be as in Theorem 1.4. Since $f(s) \leq 0$ for $s \in [1, \infty)$, we see by a comparison argument that u remains bounded as $t \rightarrow \infty$. Thus, by Theorem 1.1, u converges to a stationary solution as $t \rightarrow \infty$. As we have mentioned in the introduction, any nonnegative solution of (1.3) is constant. This and Lemma 3.5 imply

$$\omega(u) = \{0\}, \quad \{\theta_0\} \quad \text{or} \quad \{1\}. \tag{4.8}$$

Our proof of Theorem 1.4 is for the most part analogous to that of Theorem 1.3 in the previous subsection, though differences arise in some details, particularly in the proof of Lemma 4.10 (sharpness of the transition).

Lemma 4.7. *Suppose that (Φ_1) through (Φ_3) hold. Then the set*

$$\Sigma_0 := \{\lambda > 0 : u_\lambda(t, x) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ locally uniformly in } x \in \mathbb{R}^1\}$$

is an open interval $(0, \lambda^0)$ for some $\lambda^0 \in (0, \infty]$.

Proof. We first prove (4.2). The assertion ‘ \Rightarrow ’ is obvious. To prove ‘ \Leftarrow ’, suppose that the right-hand side of (4.2) holds. Then, by (2.2), we have

$$0 \leq u_\lambda(t_0, x) \leq \theta_0 - \varepsilon \quad (x \in \mathbb{R}^1) \quad \text{for some } \varepsilon > 0.$$

Since $\theta_0 - \varepsilon$ is a stationary solution, the comparison principle yields

$$\lim_{t \rightarrow \infty} u(t, x) \leq \theta_0 - \varepsilon.$$

This and (4.8) imply $u(t, x) \rightarrow 0$ as $t \rightarrow \infty$, establishing (4.2). The rest of the proof is identical to that of Lemma 4.1. □

Lemma 4.8. *For any given constant $\alpha \in (\theta_0, 1)$, there exists $L_\alpha > 0$ depending on α , f such that the solution $u(t, x)$ of (1.1) with initial data $\alpha \chi_{[-L_\alpha, L_\alpha]}$ converges to 1 as $t \rightarrow \infty$ locally uniformly in $x \in \mathbb{R}^1$.*

Proof. By modifying the value of f in the range $(0, \theta_0) \cup (1, \infty)$, we can construct a function \tilde{f} satisfying (f_B) and

$$\tilde{f}(s) \leq f(s) \quad \text{for } 0 \leq s \leq 1.$$

Let $\tilde{u}(t, x)$ be the solution of the problem

$$\tilde{u}_t = \tilde{u}_{xx} + \tilde{f}(\tilde{u}) \quad (t > 0, x \in \mathbb{R}^1), \quad \tilde{u}(0, x) = \alpha \chi_{[-L_\alpha, L_\alpha]}(x) \quad (x \in \mathbb{R}^1).$$

By Lemma 4.2, $\tilde{u}(t, x) \rightarrow 1$ as $t \rightarrow \infty$, if L_α is chosen large enough. Since $0 \leq \tilde{u} \leq 1$ and $\tilde{f}(s) \leq f(s)$ for $s \in [0, 1]$, \tilde{u} is a subsolution for (1.1). Therefore, $\lim_{t \rightarrow \infty} u(t, x) \geq \lim_{t \rightarrow \infty} \tilde{u}(t, x) = 1$. By (4.8), $\lim_{t \rightarrow \infty} u(t, x) = 1$. The lemma is proved. □

Lemma 4.9. *Suppose that (Φ_1) through (Φ_3) hold. Then*

$$\Sigma_1 := \{\lambda > 0 : u_\lambda(t, x) \rightarrow 1 \text{ as } t \rightarrow \infty \text{ locally uniformly in } x \in \mathbb{R}^1\}$$

is either empty or an open interval (λ^1, ∞) for some $\lambda^1 > 0$.

Proof. Fix constants $\alpha \in (\theta_0, 1)$ and $L_\alpha > 0$ as in Lemma 4.8. Then (4.3) holds. Indeed, ‘ \Rightarrow ’ is obvious, and ‘ \Leftarrow ’ is also clear from Lemma 4.8, the comparison principle and (4.8). Once we have (4.3), the rest of the proof is identical to that of Lemma 4.3. \square

The following lemma shows the sharpness of transition between extinction and propagation.

Lemma 4.10. *If $\lim_{t \rightarrow \infty} u_{\lambda^*} = \theta_0$ for some $\lambda^* > 0$, then $\lim_{t \rightarrow \infty} u_\lambda = 1$ for every $\lambda > \lambda^*$.*

The above lemma will follow easily from the next lemma.

Lemma 4.11. *Let ϕ_1, ϕ_2 be bounded nonnegative functions on \mathbb{R}^1 satisfying*

$$\phi_1 \leq \phi_2, \quad \phi_1 \not\equiv \phi_2, \quad [\text{spt}(\phi_2)] \subset [-L, L]$$

for some $L > 0$, and let u_i be the solution of (1.1) with $u_0 = \phi_i$, $i = 1, 2$. Assume that

$$\|u_1(t, \cdot)\|_{L^\infty} \leq \theta_0 + \delta_0/2 \quad \text{for all large } t \geq 0,$$

where θ_0 is the constant in (1.7). Then there exist constants $T > 0$, $\delta > 0$ and $\sigma \in (0, 1)$ such that

$$u_2(t + \delta, x) > \sigma^{-1} u_1(\sigma t, \sqrt{\sigma} x) \quad \text{for } x \in \mathbb{R}^1, t \geq T.$$

Proof. Choose $T \geq 1$ such that

$$\|u_1(t, \cdot)\|_{L^\infty} \leq \theta_0 + \delta_0/2 \quad \text{for } t \geq T - 1. \tag{4.9}$$

Arguing as in Lemma 4.5, we find that there exists $\delta > 0$ such that

$$u_2(T + \delta, x) > u_1(T, x) \quad \text{for } x \in \mathbb{R}^1. \tag{4.10}$$

Now, given a constant $\sigma \in (0, 1)$, we define

$$v^\sigma(t, x) := \sigma^{-1} u_1(\sigma t, \sqrt{\sigma} x).$$

Then, by (4.9), we can choose $\sigma_0 \in (0, 1)$ close enough to 1 so that

$$\|v^\sigma(t, \cdot)\|_{L^\infty} \leq \theta_0 + \delta_0 \quad \text{for every } t \geq T, \sigma \in [\sigma_0, 1]. \tag{4.11}$$

Observe that v^σ satisfies the equation

$$v_t^\sigma = v_{xx}^\sigma + f(\sigma v^\sigma).$$

By (4.11), (1.6) and (1.7), we have $f(\sigma v^\sigma) \leq f(v^\sigma)$, therefore v^σ is a subsolution of (1.1) for $t \geq T$.

To complete the proof of the lemma, in view of the comparison principle, we only have to show that

$$u_2(T + \delta, x) \geq v^\sigma(T, x) \quad \text{for } x \in \mathbb{R}^1. \tag{4.12}$$

To do so, we first derive the estimate

$$\lim_{|x| \rightarrow \infty} \frac{u_2(T + \delta, x)}{v^\sigma(T, x)} = \infty \quad \text{uniformly in } \sigma \in [\sigma_0, 1]. \tag{4.13}$$

Since f is locally Lipschitz continuous with $f(0) = 0$ and since u_1, u_2 are bounded, there exists a constant $M \geq 0$ such that $|f(u_i)| \leq Mu_i$ ($i = 1, 2$). Therefore

$$e^{-Mt} \bar{u}_i(t, x) \leq u_i(t, x) \leq e^{Mt} \bar{u}_i(t, x) \quad (i = 1, 2),$$

where \bar{u}_i is the solution of $u_t = u_{xx}$ with initial data ϕ_i , $i = 1, 2$. Consequently,

$$\frac{u_2(T + \delta, x)}{v^\sigma(T, x)} = \frac{u_2(T + \delta, x)}{\sigma^{-1}u_1(\sigma T, \sqrt{\sigma}x)} \geq C_\sigma \frac{\bar{u}_2(T + \delta, x)}{\bar{u}_1(\sigma T, \sqrt{\sigma}x)} \geq C_\sigma \frac{\bar{u}_2(T + \delta, x)}{\bar{u}_2(\sigma T, \sqrt{\sigma}x)},$$

where $C_\sigma := \sigma e^{-M(\sigma T + T + \delta)}$. The last term is calculated as

$$\tilde{C}_\sigma \int_{-L}^L \exp\left(-\frac{(x-y)^2}{4(T+\delta)}\right) \phi_2(y) dy \bigg/ \int_{-L}^L \exp\left(-\frac{(x-\sqrt{\sigma^{-1}}y)^2}{4T}\right) \phi_2(y) dy,$$

with $\tilde{C}_\sigma = C_\sigma \sqrt{\sigma T / (T + \delta)}$. Since $\delta > 0$, the above quantity tends to ∞ as $|x| \rightarrow \infty$, thus proving (4.13). Consequently, there exists $R > 0$ such that

$$u_2(T + \delta, x) \geq v^\sigma(T, x) \quad \text{for } |x| \geq R, \sigma \in [\sigma_0, 1].$$

Choosing σ sufficiently close to 1, we also have, by virtue of (4.10),

$$u_2(T + \delta, x) \geq v^\sigma(T, x) \quad \text{for } |x| \leq R.$$

This establishes (4.12), completing the proof of the lemma. □

Proof of Lemma 4.10. Let $\lambda^* < \lambda$ be as in the lemma. The convergence $u_{\lambda^*}(t, x) \rightarrow \theta_0$ as $t \rightarrow \infty$ and (2.2) imply that there exists $T > 1$ such that

$$\|u_{\lambda^*}(t, \cdot)\|_{L^\infty} \leq \theta_0 + \delta_0/2 \quad \text{for } t \geq T - 1.$$

Therefore, by Lemma 4.11, we have, for some $\sigma \in (0, 1)$,

$$u_\lambda(t + \delta, x) > \sigma^{-1}u_{\lambda^*}(\sigma t, \sqrt{\sigma}x) \quad \text{for } x \in \mathbb{R}^1, t \geq T.$$

Letting $t \rightarrow \infty$, we obtain

$$\lim_{t \rightarrow \infty} u_\lambda(t, x) \geq \sigma^{-1}\theta_0.$$

In view of (4.8) and $\sigma^{-1}\theta_0 > \theta_0$, we see that $\lim_{t \rightarrow \infty} u_\lambda = 1$. The lemma is proved. □

Proof of Theorem 1.4. The proof is identical to that of Theorem 1.3, except that Lemmas 4.1, 4.3 and 4.4 are now replaced by Lemmas 4.7, 4.9 and 4.10. □

Remark 4.12. If $\theta_0 = 0$, the situation is slightly different. Lemma 4.9 and its proof remain valid, hence the alternatives (a), (b) of Theorem 1.4 still hold, except that we may have $\lambda^* = 0$ in the alternative (b):

$$\lim_{t \rightarrow \infty} u_\lambda(t, x) = \begin{cases} 0 & \text{uniformly in } \mathbb{R}^1 & (0 \leq \lambda \leq \lambda^*), \\ 1 & \text{locally uniformly in } \mathbb{R}^1 & (\lambda > \lambda^*), \end{cases}$$

for some $\lambda^* \geq 0$. Whether $\lambda^* = 0$ (known as the *hair-trigger effect*) or $\lambda^* > 0$ depends on the behavior of $f(s)$ near $s = 0$. More precisely, $\lambda^* = 0$ if $f(s) \geq Cs^p$ for some $p < 3$ and all small $s > 0$, while $\lambda^* > 0$ if $f(s) \leq Cs^p$ for some $p > 3$ and all small $s > 0$; see Section 3 of [AW2] and the Remark after Theorem 1 of [Z] for further details. Thus we may call $p = 3$ a threshold value. For problems in \mathbb{R}^N , the threshold value of p is $1 + 2/N$ (see [AW2]).

5. Proof of Proposition 1.8

Since g is locally Lipschitz continuous and $u_0 \in L^\infty(\mathbb{R}^N)$, by the standard parabolic theory, $u(t, x)$ is defined for $x \in \mathbb{R}^N$ on some maximal time interval of existence $(0, T(u_0))$, with $0 < T(u_0) \leq \infty$. Moreover, for each $T \in (0, T(u_0))$ there exists $C_T > 0$ such that $0 < u(t, x) \leq C_T$ for all $t \in (0, T]$ and $x \in \mathbb{R}^N$.

Fix $T \in (0, T(u_0))$. Then there exists $M > 0$ such that

$$|g(s_1) - g(s_2)| \leq M|s_1 - s_2| \quad \text{for all } s_1, s_2 \in [0, C_T].$$

Therefore, due to $g(0) = 0$,

$$-Ms \leq g(s) \leq Ms \quad \text{for all } s \in [0, C_T].$$

Let $\bar{u}(t, x)$ be the solution of the following problem:

$$\bar{u}_t = \Delta \bar{u} \quad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N, \quad \bar{u}(0, x) = u_0(x) \quad \text{for } x \in \mathbb{R}^N. \quad (5.1)$$

Then

$$\bar{u}(t, x) = \int_{\Omega} (4\pi t)^{-N/2} \exp\left(-\frac{|x-y|^2}{4t}\right) u_0(y) dy, \quad (5.2)$$

where $\Omega = \text{spt}(u_0)$. One easily checks that $e^{Mt}u$ and $e^{-Mt}u$ are, respectively, super- and subsolutions of (5.1) for $(t, x) \in (0, T] \times \mathbb{R}^N$; hence

$$e^{-Mt}\bar{u}(t, x) \leq u(t, x) \leq e^{Mt}\bar{u}(t, x) \quad \text{for all } (t, x) \in (0, T] \times \mathbb{R}^N. \quad (5.3)$$

Now we choose an arbitrary sequence $(t_n, x_n) \in (0, \infty) \times \mathbb{R}^N$ satisfying

$$t_n \rightarrow 0, \quad |x_n| \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad (5.4)$$

and define

$$w_n(s, z) = \frac{u(t_n + \varepsilon_n^2 s, x_n + \varepsilon_n z)}{\bar{u}(t_n, x_n)}, \quad \bar{w}_n(s, z) = \frac{\bar{u}(t_n + \varepsilon_n^2 s, x_n + \varepsilon_n z)}{\bar{u}(t_n, x_n)},$$

where $\varepsilon_n = t_n/|x_n|$. These functions are well-defined if $0 < t_n + \varepsilon_n^2 s < T$, or, equivalently, if $-|x_n|^2/t_n < s < (T - t_n)|x_n|^2/t_n^2$. By (5.3), we have

$$e^{-M(t_n + \varepsilon_n^2 s)} \bar{w}_n(s, z) \leq w_n(s, z) \leq e^{M(t_n + \varepsilon_n^2 s)} \bar{w}_n(s, z) \tag{5.5}$$

as long as $0 < t_n + \varepsilon_n^2 s \leq T$. Using the expression (5.2), we obtain

$$\bar{w}_n(s, z) = \left(1 + \frac{\varepsilon_n}{|x_n|} s\right)^{-N/2} \frac{\int_{\Omega} \exp\left(-\frac{|x_n + \varepsilon_n z - y|^2}{4(t_n + \varepsilon_n^2 s)}\right) u_0(y) dy}{\int_{\Omega} \exp\left(-\frac{|x_n - y|^2}{4t_n}\right) u_0(y) dy}.$$

Considering that Ω is bounded, a careful calculation shows that

$$\frac{|x_n - y|^2}{4t_n} - \frac{|x_n + \varepsilon_n z - y|^2}{4(t_n + \varepsilon_n^2 s)} = \frac{s}{4} - \frac{1}{2} e_n \cdot z + O(|x_n|^{-1}),$$

as long as s, z, y vary in a bounded region, where $e_n = x_n/|x_n|$; we thus obtain

$$\bar{w}_n(s, z) = \exp\left(\frac{s}{4} - \frac{1}{2} e_n \cdot z\right) + O(|x_n|^{-1}).$$

By passing to a subsequence, we may assume that $e_n \rightarrow \theta \in S^{N-1}$. This and (5.5) yield

$$w_n(s, z) = \exp\left(\frac{s}{4} - \frac{1}{2} \theta \cdot z\right) + O(|x_n|^{-1}) + O(t_n), \tag{5.6}$$

uniformly as $n \rightarrow \infty$ and as (s, z) varies in a bounded subset of $\mathbb{R}^1 \times \mathbb{R}^N$.

Observe that w_n satisfies the equation

$$(w_n)_s = \Delta_z(w_n) + \frac{\varepsilon_n^2}{\alpha_n} g(\alpha_n w_n) \quad (\text{where } \alpha_n := \bar{u}(t_n, x_n))$$

in the region where $0 < t_n + \varepsilon_n^2 s \leq T$, $z \in \mathbb{R}^N$ and that the functions

$$w \mapsto \frac{\varepsilon_n^2}{\alpha_n} g(\alpha_n w) \quad (n = 1, 2, \dots)$$

are Lipschitz continuous, uniformly in n , as long as $0 \leq w \leq C_T/\alpha_n$. They also converge to 0 uniformly for w in any bounded interval $[0, C]$. Consequently, by interior parabolic estimates (see [LSU] or [L]), we see that $\{w_n\}$ is bounded in $C^{1+\mu/2, 2+\mu}(\bar{\Omega})$ for any bounded domain $\Omega \subset \mathbb{R}^1 \times \mathbb{R}^N$. Therefore we can use a standard diagonal process to extract a further subsequence, still denoted by $\{w_n\}$, such that w_n converges to some w_0 in the sense of $C^{1,2}(\mathbb{R}^1 \times \mathbb{R}^N)$. By (5.6), necessarily

$$w_0(s, z) = \exp\left(\frac{s}{4} - \frac{1}{2} \theta \cdot z\right).$$

Hence

$$\lim_{n \rightarrow \infty} \frac{\partial}{\partial s} w_n(s, z) = \frac{\partial}{\partial s} w_0(s, z) = \frac{1}{4} w_0(s, z)$$

locally uniformly in (s, z) . It follows that

$$\lim_{n \rightarrow \infty} \frac{t_n^2}{|x_n|^2} \frac{u_t(t_n, x_n)}{\bar{u}(t_n, x_n)} = \lim_{n \rightarrow \infty} \frac{\partial}{\partial s} w_n(0, 0) = \frac{1}{4}.$$

The assertion (1.12) now follows from this and (5.3), since (t_n, x_n) is an arbitrary sequence in $(0, \infty) \times \mathbb{R}^N$ satisfying (5.4). This completes the proof of Proposition 1.8.

6. Nonlinearity with a jumping discontinuity

In this section we consider the combustion and the bistable type nonlinearities with a jumping discontinuity at $u = 1$ and show that all the previous main results continue to hold.

6.1. The combustion case

We begin with the case where f satisfies (f_C) except that it has the following jumping discontinuity at $u = 1$:

$$f(1 - 0) > 0 \geq f(1 + 0). \tag{6.1}$$

Such a case often occurs in applications (see [BKS]). We will show that for such f , the conclusions of Theorems 1.1 and 1.4 remain valid. The same is true for the following type of discontinuity:

$$f(1 - 0) = 0 > f(1 + 0).$$

Since this latter case can be treated precisely the same way (and in some cases more easily), we will focus on (6.1). Thus we assume that

$$(\hat{f}_C) \begin{cases} f \text{ is Lipschitz in } [0, 1) \text{ and locally Lipschitz in } [1, \infty); \\ f \text{ satisfies (1.6), (1.7) and (6.1).} \end{cases}$$

Note that the jumping discontinuity in f can create a situation that is markedly different from the regular case. For example, some solutions can take the value 1 on an open subset of $\mathbb{R}_+ \times \mathbb{R}^1$ without being identically equal to 1. This means that the zeros of $u - 1$ are not necessarily isolated, and moreover degenerate zeros of $u - 1$ do not necessarily disappear instantly. Therefore part of our previous argument that relied on Lemma 2.3 has to be modified.

Let us first look at what essential properties of f are retained. Since the discontinuity is away from $u = 0$, and $f(u)$ jumps down when u increases across $u = 1$, it is easy to see that for any given $C > 0$ there exists $M = M_C > 0$ such that, for $0 \leq u \leq v \leq C$,

$$-Mu \leq f(u) \leq Mu, \quad f(v) - f(u) \leq M(v - u). \tag{6.2}$$

As we will see below, thanks to (6.2), most of our previous arguments for locally Lipschitz continuous f carry over to the current discontinuous case.

Lemma 6.1. *Let the above assumptions on f be satisfied and $u_0 \in L^\infty(\mathbb{R}^1)$ be non-negative. Then (1.1) has a unique (weak) solution u , which is defined for all $t > 0$. Furthermore, if u_0 has compact support, then*

$$\begin{cases} \limsup_{t \rightarrow \infty} u(t, x) \leq 1 & \text{for fixed } x \in \mathbb{R}^1, \\ \lim_{|x| \rightarrow \infty} u(t, x) = 0 & \text{uniformly in } t \in [0, T], \end{cases} \quad (6.3)$$

where $T > 0$ is an arbitrary positive constant.

Proof. For $\varepsilon > 0$ small, we define a Lipschitz approximation of $f(u)$ as follows:

$$f_\varepsilon(u) = \begin{cases} f(u), & u \in [0, 1), \\ f(1 - 0) + \varepsilon^{-1}[f(1 + \varepsilon) - f(1 - 0)](u - 1), & u \in [1, 1 + \varepsilon], \\ f(u), & u > 1 + \varepsilon. \end{cases}$$

Clearly $f_\varepsilon(u) \geq f(u)$ for all $u \geq 0$ and all sufficiently small $\varepsilon > 0$. Furthermore, $f_\varepsilon \rightarrow f$ in $L^p_{loc}(\mathbb{R}_+)$ for any $p > 1$ as $\varepsilon \rightarrow 0$.

Let $u_\varepsilon(t, x)$ denote the solution of the equation

$$u_t = u_{xx} + f_\varepsilon(u) \quad (6.4)$$

on $\mathbb{R}_+ \times \mathbb{R}^1$ with initial data $u_0(x)$. Then $u_\varepsilon(t, x)$ is defined for all $t > 0$. Since f_ε is nondecreasing in ε , the comparison argument shows that u_ε is nondecreasing in ε . Therefore the pointwise limit

$$u(t, x) := \lim_{\varepsilon \rightarrow 0} u_\varepsilon(t, x)$$

exists for every $(t, x) \in [0, \infty) \times \mathbb{R}^1$. By standard parabolic estimates, this convergence takes place in $C^{(1+\mu)/2, 1+\mu}_{loc}(\mathbb{R}_+ \times \mathbb{R}^1)$ for any $\mu \in (0, 1)$. Hence u is a weak solution of (1.1).

Next we prove the uniqueness assertion. Suppose that u and v are both weak solutions of (1.1). Then, since $f(0) = 0$ and $f(s) \leq 0$ for $s > 1$, we see by the maximum principle that both u and v are bounded. We claim that $u \leq v$ for all $t > 0$ and $x \in \mathbb{R}^1$. Suppose the contrary. Then $\Omega := \{(t, x) \in (0, \infty) \times \mathbb{R}^1 : u(t, x) > v(t, x)\}$ is a nonempty open set. By (6.2), there exists a constant $M_0 > 0$ such that

$$f(u(t, x)) - f(v(t, x)) \leq M_0(u(t, x) - v(t, x))$$

whenever $u(t, x) > v(t, x)$. This and the fact that $u = v$ at $t = 0$ imply that $w := u - v$ satisfies (in the weak sense)

$$\begin{cases} w_t - w_{xx} \leq M_0 w, & (t, x) \in \Omega, \\ w = 0, & (t, x) \in \partial\Omega. \end{cases}$$

From the maximum principle (of the Phragmén–Lindelöf type) we deduce $w \leq 0$ in Ω , a contradiction to the definition of Ω . Hence $u \leq v$. Similarly $v \leq u$. This implies $u \equiv v$, and the uniqueness assertion is proved.

Finally, suppose u_0 has compact support. Then by Theorems 1.1, 1.4 and the definition of f_ε , we have $\lim_{t \rightarrow \infty} u_\varepsilon(t, x) \leq 1_\varepsilon$, where $1_\varepsilon \in (1, 1 + \varepsilon]$ is the minimal zero of f_ε in $[1, 1 + \varepsilon]$. This and the inequality $0 \leq u \leq u_\varepsilon$ imply

$$\limsup_{t \rightarrow \infty} u(t, x) \leq \lim_{t \rightarrow \infty} u_\varepsilon(t, x) \leq 1_\varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we obtain the first assertion in (6.3).

By (6.2), $-Ms \leq f(s) \leq Ms$ for all $s \in [0, \sup u_\varepsilon]$ and some $M > 0$. Hence we can deduce (2.3) as before. This proves the second assertion in (6.3). The proof of the lemma is now complete. \square

As we have mentioned above, the unique weak solution of (1.1) belongs to the space $C_{\text{loc}}^{(1+\mu)/2, 1+\mu}(\mathbb{R}_+ \times \mathbb{R}^1)$ for any $0 < \mu < 1$.

We have the following weaker version of Lemma 2.1.

Lemma 6.2. *Let $u(t, x)$ be the weak solution of (1.1) as in Lemma 6.1 and put $[\text{spt}(u_0)] = [L_1, L_2]$. Then*

$$u_x \geq 0 \quad \text{for } x < L_1, t > 0; \quad u_x \leq 0 \quad \text{for } x > L_2, t > 0.$$

Proof. By Lemma 6.1 and its proof, we know that

$$u(t, x) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(t, x) \quad \text{and} \quad u_x(t, x) = \lim_{\varepsilon \rightarrow 0} (u_\varepsilon)_x(t, x)$$

locally uniformly in $(0, \infty) \times \mathbb{R}^1$. Since Lemma 2.1 applies to u_ε , we have

$$(u_\varepsilon)_x > 0 \quad \text{for } x < L_1, t > 0; \quad (u_\varepsilon)_x < 0 \quad \text{for } x > L_2, t > 0.$$

Letting $\varepsilon \rightarrow 0$, we obtain the required inequalities. \square

Lemma 6.3. *Let u and u_0 be as in Lemma 6.1, and suppose that u_0 is not identically 0. Then (1.12) holds.*

Proof. This follows by a careful examination of the proof of Proposition 1.8. Letting $g = f$, we find that there exists $M > 0$ such that

$$-Ms \leq g(s) \leq Ms \quad \text{for all } s \in [0, \sup u].$$

Therefore we still have (5.3) and (5.6). Since $\alpha_n \rightarrow 0$ and $g(u)$ is continuous in $[0, 1)$, for all large n , the functions

$$w \mapsto \frac{\varepsilon_n^2}{\alpha_n} g(\alpha_n w)$$

have all the properties stated in the proof of Proposition 1.8. Hence (1.12) can be proved by the same argument. \square

Lemma 6.4. *Let J be an interval in \mathbb{R}^1 (finite or infinite) and let ∂J denote its boundary. Let u and \tilde{u} satisfy the equation (1.1) in the weak sense on the domain $(t_1, t_2) \times J$ and suppose that $u \neq \tilde{u}$ on $(t_1, t_2) \times \partial J$. Then $\mathcal{Z}_J(u(t, \cdot) - \tilde{u}(t, \cdot))$ is nonincreasing in $t \in (t_1, t_2)$.*

Proof. It suffices to show that, for any $t_0 \in (t_1, t_2)$, the following holds:

$$\mathcal{Z}_J(u(t_0, \cdot) - \tilde{u}(t_0, \cdot)) \geq \mathcal{Z}_J(u(t, \cdot) - \tilde{u}(t, \cdot)) \quad \text{for } t \in [t_0, t_2].$$

Let f_ε be the Lipschitz approximation of f given in the proof of Lemma 6.1, and let $u_\varepsilon, \tilde{u}_\varepsilon$ be the solution of (6.4) on the domain $[t_0, t_2] \times J$ with the following initial and the boundary conditions:

$$\begin{aligned} u_\varepsilon(t_0, x) &= u(t_0, x) \quad (x \in \bar{J}), & u_\varepsilon &= u \quad \text{on } (t_0, t_2) \times \partial J, \\ \tilde{u}_\varepsilon(t_0, x) &= \tilde{u}(t_0, x) \quad (x \in \bar{J}), & \tilde{u}_\varepsilon &= \tilde{u} \quad \text{on } (t_0, t_2) \times \partial J. \end{aligned}$$

(Here, the boundary condition applies only if $\partial J \neq \emptyset$.)

Then by Lemma 2.3, we have

$$\mathcal{Z}_J(u(t_0, \cdot) - \tilde{u}(t_0, \cdot)) = \mathcal{Z}_J(u_\varepsilon(t_0, \cdot) - \tilde{u}_\varepsilon(t_0, \cdot)) \geq \mathcal{Z}_J(u_\varepsilon(t, \cdot) - \tilde{u}_\varepsilon(t, \cdot))$$

for $t \in [t_0, t_2]$. Arguing as in the proof of Lemma 6.1, we see that u_ε and \tilde{u}_ε converge to u and \tilde{u} , respectively. Letting $\varepsilon \rightarrow 0$ in the above inequality and using (2.6) (which is easily seen to hold if \mathcal{Z} is replaced by \mathcal{Z}_J), we obtain the desired inequality. \square

Lemma 6.5. *Let u be the solution of (1.1) given in Lemma 6.1, and v a solution of (1.3). Suppose that $J = (a, b)$ is any finite open interval in \mathbb{R}^1 such that $0 < v(x) \neq 1$ in J , $u(t, x) - v(x) \neq 0$ for $x \in \{a, b\}$ and for t in some finite or infinite interval $(t_1, t_2) \subset [0, \infty)$. Then*

$$z(t) := \mathcal{Z}_J(u(t, \cdot) - v)$$

is finite and nonincreasing in $t \in (t_1, t_2)$. Furthermore, if $u(t_0, x) - v(x)$ has a degenerate zero in J for some $t_0 \in (t_1, t_2)$, then $z(t_0 - 0) > z(t_0 + 0)$.

Proof. Fix $t_0 \in (t_1, t_2)$. Suppose $u(t_0, x) - v(x) = 0$ for some $x \in J$ and let x_0 be any such point. Then, since $u \neq v$ for $x = a, b$ and since both u and v are continuous and $v \neq 1$, we can find a small constant $\varepsilon > 0$ and an interval J_{x_0} with $x_0 \in J_{x_0} \subset J$ such that

$$\begin{aligned} (u(t, x) - 1)(v(x) - 1) &> 0 \quad \text{for } (t, x) \in [t_0 - \varepsilon, t_0 + \varepsilon] \times J_{x_0}, \\ u(x, t) &\neq v(x) \quad \text{for } (t, x) \in [t_0 - \varepsilon, t_0 + \varepsilon] \times \partial J_{x_0}. \end{aligned}$$

In this region $u(t, x)$ is away from the discontinuity point of f , and $u - v$ satisfies an equation of the form (2.5). Thus we can apply Lemma 2.3 to conclude that x_0 is an isolated zero of $u(t_0, x) - v(x)$. The fact that any zero of $u(t_0, x) - v(x)$ in J is isolated and that the end points of J are not zeros imply that there are only finitely many zeros of $u(t_0, x) - v(x)$ in J .

That $z(t)$ is nonincreasing follows from Lemma 6.4. Furthermore, if $u(t_0, x) - v(x)$ has a degenerate zero in J , say at $x = x_0$, then by Lemma 2.3,

$$\mathcal{Z}_{J_{x_0}}(u(t_0 - 0, \cdot) - v) > \mathcal{Z}_{J_{x_0}}(u(t_0 + 0, \cdot) - v),$$

where J_{x_0} is as above. On the other hand, by Lemma 6.4,

$$\mathcal{Z}_{J \setminus J_{x_0}}(u(t_0 - 0, \cdot) - v) \geq \mathcal{Z}_{J \setminus J_{x_0}}(u(t_0 + 0, \cdot) - v).$$

Hence $z(t_0 - 0) > z(t_0 + 0)$. The lemma is proved. \square

We now examine the stationary solutions of (1.1) with f satisfying (\hat{f}_C) . We say that v is a *stationary solution* of (1.1) if the (unique) weak solution u of (1.1) with initial data $u_0 = v$ satisfies $u \equiv v$. Let us first observe that any stationary solution v satisfies

$$v'' + f(v - 0) \geq 0, \quad v'' + f(v + 0) \leq 0 \tag{6.5}$$

in the weak sense. To see this, let $f_\varepsilon (\geq f)$ denote the Lipschitz approximation of f defined in the proof of Lemma 6.1. Then v is a lower solution of the approximate problem, that is, $v'' + f_\varepsilon(v) \geq 0$ in the weak sense. Letting $\varepsilon \rightarrow 0$, we obtain the first inequality of (6.5). The second inequality can be shown similarly by using a Lipschitz approximation of f from below.

From (6.5) we see that $v \equiv 1$ is a stationary solution even though it does not satisfy (1.3) when $f(1 - 0) > 0 > f(1 + 0)$. It is also clear that any stationary solution v satisfies (1.3) at every $x \in \mathbb{R}^1$ where $v(x) \neq 1$. In view of this, one can show that nonnegative bounded stationary solutions of (1.1) consist of constant functions $v \equiv c$ with $c \in [0, \theta_0] \cup \{1\}$ as in the case (f_C) .

Next we consider solutions of (1.3) satisfying

$$v(x_0) = \alpha \in (0, 1), \quad v'(x_0) = 0 \tag{6.6}$$

for some $x_0 \in \mathbb{R}^1$. Since f satisfies (\hat{f}_C) , it is easily seen that v is either a constant solution (when $\alpha \in (0, \theta_0]$), or a solution with compact positive support (when $\alpha \in (\theta_0, 1)$). Thus in all the possible cases, we have

$$v(x) < 1 \quad \text{whenever it is defined.}$$

In what follows, the symbol $\text{spt}^+(v)$ will denote the positive support of a function v as defined in Subsection 2.5.

Lemma 6.6. *Let u be the solution of (1.1) given in Lemma 6.1, and v a solution of (1.3) satisfying (6.6). Then*

$$\mathcal{Z}_I(u(t, \cdot) - v) < \infty \quad \text{for } t > 0,$$

and it is nonincreasing in t , where $I = \text{spt}^+(v)$.

Proof. The proof is the same as that of Lemma 2.9. Here we use Lemma 6.5 instead of Lemma 2.3. □

We are now ready to prove the convergence theorem.

Theorem 6.7 (Convergence for the discontinuous combustion case). *Assume (\hat{f}_C) and let u be the weak solution of (1.1). Then $\lim_{t \rightarrow \infty} u(t, x)$ exists in the topology of $L^\infty_{\text{loc}}(\mathbb{R}^1)$, and the limit is a constant $c \in \{0, \theta_0, 1\}$.*

Proof. We first show that $\omega(u)$ consists of stationary solutions. This is done by slightly modifying the proof of Lemma 3.4. Let w be any element of $\omega(u)$. By Lemma 6.1, we have $0 \leq w \leq 1$. If $w \equiv 1$, then the proof is over, so we suppose that $w \not\equiv 1$. Then there exists a sequence $x_n \in \mathbb{R}^1$ ($n = 1, 2, \dots$) such that

$$w(x_n) \rightarrow \beta := \inf_{x \in \mathbb{R}^1} w(x), \quad w'(x_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Denote by $v_n(x)$ ($n = 1, 2, \dots$) a sequence of solutions of (1.3) satisfying

$$v_n(x_n) = w(x_n), \quad v'_n(x_n) = w'(x_n).$$

Here the domain of definition of v_n is taken to be the maximal interval that contains x_n and on which $0 \leq v_n < 1$. Now define

$$H(v, p) := \frac{1}{2}p^2 + F(v), \quad F(v) := \int_0^v f(s) ds.$$

Then $H(v_n(x), v'_n(x))$ is independent of x and $H(v_n(x_n), v'_n(x_n)) \rightarrow H(\beta, 0) = F(\beta)$ as $n \rightarrow \infty$. Since $F(v) = 0$ for $0 \leq v \leq \theta_0$ and $F(v)$ is strictly increasing in $\theta_0 < v < 1$, we have $F(\beta) < F(\gamma) \leq H(\gamma, p)$, where γ is any point satisfying $\max\{\beta, \theta_0\} < \gamma < 1$. Fix such a γ . Then, for sufficiently large n ,

$$H(v_n, v'_n) < H(\gamma, p) \quad (\forall p \in \mathbb{R}),$$

which implies that $v_n(x)$ cannot attain the value γ ; therefore $\max v_n < \gamma < 1$.

Hence v_n is either a constant solution of (1.3) or a solution with compact positive support and $\max v_n < 1$. Thus Lemma 6.6 applies for such $v = v_n$. Arguing as in the proof of Lemma 3.4 (while using Lemma 6.6 instead of Lemma 2.9, and Lemma 6.5 instead of Lemma 2.3), we find that w coincides with such v_n . Therefore w is either a constant or a solution of (1.3) with compact positive support. But the latter is impossible since w is smooth and nonnegative. Hence w is a constant function with value in the interval $[0, \theta_0]$, provided that $w \not\equiv 1$.

It remains to show that $\omega(u)$ is a singleton, and that its element is either 0, θ_0 or 1. Since the proof of Lemma 3.5 is valid once we replace Lemma 2.1 by Lemma 6.2, we have $\omega(u) \subset \{0, \theta_0, 1\}$. The connectedness of $\omega(u)$ then implies that $\omega(u)$ is a singleton. This completes the proof of the theorem. \square

Theorem 6.8 (Threshold for the discontinuous combustion case). *The conclusions in Theorem 1.4 remain valid under the assumption (\hat{f}_C) .*

Proof. We first note that the usual comparison principle continues to hold. Indeed, this follows from the same argument as in the proof of Lemma 6.1 for uniqueness. The theorem can then be shown by repeating the proof of Theorem 1.4 in Subsection 4.2 with only minor modifications. First, the proofs of Lemmas 4.7, 4.8 and 4.9 go without change. The proof of Lemma 4.11 carries over when Lemma 6.3 is used instead of Proposition 1.8 (which was used to derive (4.10)), and when we notice that only the inequality $|f(u_i)| \leq Mu_i$ was needed rather than the Lipschitz continuity of f . The conclusion of Lemma 4.10 is then proved in the same way except that Theorem 6.7 is used instead of Theorem 1.1. \square

6.2. The bistable case

In this subsection we consider the case where f satisfies (f_B) with a jumping discontinuity at $u = 1$. As we have mentioned in the previous subsection, discontinuity of the type

$f(1 - 0) = 0 > f(1 + 0)$ can be treated precisely the same way as the type $f(1 - 0) > 0 \geq f(1 + 0)$, so we focus on the latter. Thus we assume:

$$(\hat{f}_B) \begin{cases} f \text{ is Lipschitz in } [0, 1) \text{ and locally Lipschitz in } [1, \infty); \\ f \text{ satisfies (1.4), (1.5) and (6.1).} \end{cases}$$

Our main results in this subsection are the following:

Theorem 6.9 (Convergence for the discontinuous bistable case). *Assume (\hat{f}_B) and let u be the weak solution of (1.1). Then $\lim_{t \rightarrow \infty} u(t, x)$ exists in the topology of $L_{\text{loc}}^\infty(\mathbb{R}^1)$, and the limit is either 0, 1, or a symmetrically decreasing stationary solution $U(x - a)$ for some $a \in \mathbb{R}$, where U is as in (1.8).*

Theorem 6.10 (Threshold for the discontinuous bistable case). *The conclusions in Theorem 1.3 remain valid under the assumption (\hat{f}_B) .*

Proof of Theorem 6.9. We first show that $\omega(u)$ consists only of stationary solutions. The proof is exactly the same as the corresponding part of the proof of Theorem 6.7, except that the function v_n is now either the constant solution θ_0 , a periodic solution satisfying $0 < v_n < 1$, a symmetrically decreasing solution with $\max v_n < 1$, or a solution with compact positive support and $\max v_n < 1$. In all these cases, Lemma 6.6 applies, so one can just repeat the proof of Theorem 6.7 to show that w coincides with v_n and that v_n is either 0 or a positive stationary solution.

It remains to show that $\omega(u)$ is a singleton. We first observe that Lemma 3.1 still holds because of Lemma 6.2. Therefore any element of $\omega(u)$ is either a symmetrically decreasing solution of the form $U(x - a)$, or one of the constant solutions 0, θ_0 , 1. But the constant θ_0 cannot belong to $\omega(u)$ by virtue of Lemma 3.3, since θ_0 intersects periodic solutions infinitely many times. (Note that Lemma 3.3 remains valid by virtue of Lemma 6.6.) Thus

$$\omega(u) \subset \{0, 1\} \cup \{U(x - a) : a \in \mathbb{R}\}.$$

If $\omega(u)$ contains 1, then by the connectedness of $\omega(u)$ we have $\omega(u) = \{1\}$, and the proof is over. Next suppose $\omega(u) \subset \{0\} \cup \{U(x - a) : a \in \mathbb{R}\}$. Then by Lemma 6.2, we have $\max_x u(t, x) < 1$ for all sufficiently large t ; hence u is a classical solution of (1.1) for large t . Consequently, we can apply Lemma 2.8 to derive Lemma 3.2. This implies that $\omega(u)$ is a singleton, and the proof of the theorem is complete. \square

Proof of Theorem 6.10. The proof is identical to that of Theorem 1.3 except that Lemma 4.5 is derived from Lemma 6.3 rather than from Proposition 1.8. The details are omitted. \square

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