



Bernhard Ruf · P. N. Srikanth

Singularly perturbed elliptic equations with solutions concentrating on a 1-dimensional orbit

Received June 7, 2007 and in revised form April 2, 2008

Abstract. We consider a singularly perturbed elliptic equation with superlinear nonlinearity on an annulus in \mathbb{R}^4 , and look for solutions which are invariant under a fixed point free 1-parameter group action. We show that this problem can be reduced to a nonhomogeneous equation on a related annulus in dimension 3. The ground state solutions of this equation are single peak solutions which concentrate near the inner boundary. Transforming back, these solutions produce a family of solutions which concentrate along the orbit of the group action near the inner boundary of the domain.

1. Introduction

We consider the following superlinear elliptic boundary value problem on the annulus $A = \{x \in \mathbb{R}^4 \mid 0 < a < |x| < b\}$:

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } A, \\ u(x) > 0, & \text{in } A, \\ u(x) = 0, & \text{on } \partial A. \end{cases} \quad (1)$$

Here $p > 1$, and ε^2 is a singular perturbation parameter.

In the pioneering papers [10–13] qualitative properties of the *least energy solution* for this singularly perturbed equation (with varying boundary conditions) have been studied. In particular, W.-M. Ni and J. Wei showed in [13] that the least energy solutions of equations of form (1) concentrate, for $\varepsilon \rightarrow 0$, to single peak solutions, whose maximum points x_ε converge to a point x_0 with maximal distance from the boundary $\partial\Omega$. Furthermore, Ni and Wei gave precise decay estimates for these solutions.

Another type of concentrating solutions was studied by A. Ambrosetti, A. Malchiodi and W.-M. Ni in [1] (see also [5]); they consider solutions which concentrate on *spheres*, i.e. on $N - 1$ -dimensional manifolds. Such solutions are of particular interest for applica-

B. Ruf: Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini, 50, 20123 Milano, Italy; e-mail: ruf@mat.unimi.it

P. N. Srikanth: TIFR Centre for Applicable Mathematics, Bangalore 560 065, India; e-mail: srikanth@math.tifrbng.res.in

tions of the equation to models of activator-inhibitor systems in biology (see the survey [14]). In their paper, Ambrosetti, Malchiodi and Ni consider equation (1) in the presence of a potential $V(r)$, i.e.

$$\begin{cases} -\varepsilon^2 \Delta u + V(|x|)u = u^p & \text{in } A \subset \mathbb{R}^N, \\ u(x) = 0 & \text{on } \partial A. \end{cases} \quad (2)$$

They showed that if the function $M(r) = r^{n-1}V^\theta(r)$ with $\theta = (p+1)/(p-1) - 1/2$ satisfies $M'(b) < 0$ (resp. $M'(a) > 0$), then there exists a family of radial solutions which concentrate on $|x| = r_\varepsilon$ with $r_\varepsilon \rightarrow b$ (resp. $r_\varepsilon \rightarrow a$) as $\varepsilon \rightarrow 0$.

It has been conjectured that for $N \geq 3$ there could also exist solutions concentrating on some manifolds of dimension k with $1 \leq k \leq N-2$.

In this paper we will prove

Theorem 1. *Let $1 < p < 5$. Then for problem (1) there exists a family of positive solutions which concentrate on a 1-dimensional orbit $T_\tau x_\varepsilon$, $\tau \in [0, 2\pi)$, where x_ε satisfies $|x_\varepsilon| \rightarrow a$, and T_τ denotes a continuous and fixed point free group action on A .*

These solutions will be obtained by introducing a suitable group action T_τ on A ; this symmetry is the natural one induced by the Hopf fibration of S^3 . Then, looking for T_τ -invariant solutions, one can reduce the problem to an equivalent nonhomogeneous equation in an annulus $B \subset \mathbb{R}^3$. To this equation the results of Ni and Wei [13] can be applied, producing single peak solutions. Adapting the methods of del Pino–Felmer [3] we show that the peaks converge to the inner boundary of B . Transforming back to the original problem then yields the result.

Our restriction to \mathbb{R}^4 is due to the fact that we can define an explicit fixed point free group action (see Remark B below). Since every smooth action on a $2m$ -sphere has fixed points, extensions of our result to odd dimensions seem impossible. On the other hand, extensions to higher even dimensions should be possible; however, since the explicit form of the reduced *nonhomogeneous* equation is required in order to study the behavior of the concentrating solutions, we need an explicit $2m$ -dimensional analogue of the coordinate system (4) (see below) together with a fixed point free action, as well as an explicit reduction procedure to an equation in \mathbb{R}^{2m-1} ; and this seems not easy to achieve.

Remark A. 1) Note that the natural limitation for p due to the Sobolev embedding theorem in \mathbb{R}^4 is $1 < p < (N+2)/(N-2) = 3$; however, since by the above mentioned group invariance the problem will be reduced to a problem in three dimensions, we can allow $1 < p < 5$.

2) It is known that the single peak solutions for equation (1) concentrate at a point P_0 with $|P_0| = \max_{P \in A} d(P, \partial A) = (a+b)/2$ (see [13]). On the other hand, the solutions concentrating on spheres found by Ambrosetti–Malchiodi–Ni [1] concentrate for $V(|x|) \equiv 1$ on the inner boundary of A (since $M(r) = r^{n-1}$). Our result yields concentration orbits which also converge to the inner boundary of A .

2. The group action

Consider the equation

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } A, \\ u = 0 & \text{on } \partial A, \\ u > 0 & \text{in } A, \end{cases} \quad (3)$$

where $A = \{x \in \mathbb{R}^4 \mid 0 < a < |x| < b\}$ is an annular domain in \mathbb{R}^4 , and $1 < p < 5$. Let $H_{0,\text{rad}}^1 \subset H_0^1(\Omega)$ denote the subspace of $H_0^1(\Omega)$ consisting of radial functions.

We consider the following coordinate system in \mathbb{R}^4 :

$$\begin{aligned} x_1 &= r \sin \theta_1 \sin \theta_3, & a < r < b, \\ x_2 &= r \cos \theta_1 \sin \theta_3, & 0 \leq \theta_i < 2\pi \ (i = 1, 2), \\ x_3 &= r \sin \theta_2 \cos \theta_3, & 0 \leq \theta_3 < \pi/2, \\ x_4 &= r \cos \theta_2 \cos \theta_3, \end{aligned} \quad (4)$$

where $0 \leq \theta_i \leq 2\pi$ ($i = 1, 2$) denote the angles between (x_1, x_2) in the x_1x_2 -plane and between (x_3, x_4) in the x_3x_4 -plane, and $0 \leq \theta_3 < \pi/2$ denotes the angle between the planes x_1x_2 and x_3x_4 . A direct calculation gives the volume element in the $(r, \theta_1, \theta_2, \theta_3)$ -coordinates:

$$r^3 \sin \theta_3 \cos \theta_3 dr d\theta_1 d\theta_2 d\theta_3. \quad (5)$$

A simple but tedious computation shows that the Laplacian Δ takes in the coordinate system (4) the form

$$\begin{aligned} \Delta u &= u_{rr} + \frac{3}{r}u_r + u_{\theta_1\theta_1} \frac{1}{r^2 \sin^2 \theta_3} + u_{\theta_2\theta_2} \frac{1}{r^2 \cos^2 \theta_3} \\ &+ \frac{1}{r^2} u_{\theta_3} \left(-\frac{\sin \theta_3}{\cos \theta_3} + \frac{\cos \theta_3}{\sin \theta_3} \right) + \frac{1}{r^2} u_{\theta_3\theta_3}. \end{aligned} \quad (6)$$

Consider the following group action T_τ on A : for

$$z = (r, \theta_1, \theta_2, \theta_3), \quad a < r < b, \quad 0 \leq \theta_i < 2\pi \ (i = 1, 2), \quad 0 \leq \theta_3 < \pi/2,$$

let

$$T_\tau z = (r, \theta_1 + \tau, \theta_2 + \tau, \theta_3), \quad \tau \in [0, 2\pi),$$

and define the subspace $H_{0,\#}^1(A) \subset H_0^1(A)$ of functions which are invariant under this action, i.e.

$$u \in H_{0,\#}^1 \quad \text{if} \quad u(T_\tau x) = u(x), \quad \forall \tau \in [0, 2\pi).$$

Remark B. 1) Note that the action defined above is *fixed point free*. This is important, since otherwise the solutions might concentrate on fixed points and thus would not yield a concentrating orbit.

2) The usual coordinate system in \mathbb{R}^4 is

$$\begin{aligned} x_1 &= r \sin \theta_3 \sin \theta_2 \sin \theta_1, & 0 \leq \theta_1 < 2\pi, \\ x_2 &= r \sin \theta_3 \sin \theta_2 \cos \theta_1, & 0 \leq \theta_2 < \pi, \\ x_3 &= r \sin \theta_3 \cos \theta_2, & 0 \leq \theta_3 \leq \pi, \\ x_4 &= r \cos \theta_3, & a < r < b. \end{aligned} \quad (7)$$

Since only the variable θ_1 varies in $[0, 2\pi)$, the only obvious way to define a group action is $T_\tau u(r, \theta_1, \theta_2, \theta_3) = u(r, \theta_1 + \tau, \theta_2, \theta_3)$. However, this action has the fixed points $x_r = (0, 0, 0, r) \in \mathbb{R}^4$, $a < r < b$.

The following properties connected with the group action T_τ are easily verified:

Lemma 2. 1) $H_{0,\#}^1(A)$ is a closed subspace of $H_0^1(A)$.

2) $H_{0,\#}^1(A)$ is invariant under the Laplacian, i.e.

$$u \in H_{0,\#}^1(A) \Rightarrow \Delta u \in H_{0,\#}^1(A).$$

3) $H_{0,\text{rad}}^1(A) \subset H_{0,\#}^1(A)$. □

For the moment, let us assume that $1 < p < 3$, so that the following functional $J(u)$ is well defined in $H_{0,\#}^1$:

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_\Omega u^2 - \frac{1}{p+1} \int_\Omega (u^+)^{p+1}.$$

Using the Mountain-Pass Lemma [2], one finds a critical level for $J_\varepsilon(u)$, and one knows that at this level there exists a solution which has Morse index less than or equal to 1 (see [4, 7, 8]).

We now show that for $\varepsilon > 0$ sufficiently small, this solution cannot be independent of the variable θ_3 .

Lemma 3. For ε^2 sufficiently small, the mountain pass solution u_ε with Morse index 1 is not independent of the variable θ_3 .

Proof. Suppose to the contrary that u_ε is independent of θ_3 . From

$$-\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon - u_\varepsilon^p = 0 \quad \text{in } A \quad (8)$$

we get

$$-\varepsilon^2 \Delta u_\varepsilon - (p u_\varepsilon^{p-1} - 1) u_\varepsilon = -(p-1) u_\varepsilon^p.$$

On the left of the equation, we have the linearization of the operator in (8) in direction u_ε , and hence we deduce from

$$(-\varepsilon^2 \Delta u_\varepsilon - (p u_\varepsilon^{p-1} - 1) u_\varepsilon, u_\varepsilon) = -(p-1) (u_\varepsilon^p, u_\varepsilon) < 0$$

that u_ε contributes 1 to the Morse index.

Next, consider the function $u_\varepsilon \cos(2\theta_3)$. Note that $u_\varepsilon \cos(2\theta_3)$ is orthogonal to u_ε on A , i.e.

$$\int_A u_\varepsilon \cos(2\theta_3) u_\varepsilon r^3 \sin \theta_3 \cos \theta_3 dr d\theta_1 d\theta_2 d\theta_3 = 0,$$

since by our assumption u_ε is independent of θ_3 and since $\int_0^{\pi/2} \cos(2\theta_3) \sin \theta_3 \cos \theta_3 d\theta_3 = 0$.

We now show that $u_\varepsilon \cos(2\theta_3)$ also contributes 1 to the Morse index. We calculate $-\varepsilon^2 \Delta[u_\varepsilon \cos(2\theta_3)]$, using (6):

$$\begin{aligned} -\varepsilon^2 \Delta[u_\varepsilon \cos(2\theta_3)] &= (u_\varepsilon^p - u_\varepsilon) \cos(2\theta_3) \\ &\quad - \frac{\varepsilon^2}{r^2} u_\varepsilon \left(-\frac{\sin \theta_3}{\cos \theta_3} + \frac{\cos \theta_3}{\sin \theta_3} \right) [\cos(2\theta_3)]_{\theta_3} - \frac{\varepsilon^2}{r^2} u_\varepsilon [\cos(2\theta_3)]_{\theta_3 \theta_3} \\ &= (u_\varepsilon^p - u_\varepsilon) \cos(2\theta_3) + \frac{4\varepsilon^2}{r^2} u_\varepsilon \cos(2\theta_3) + \frac{4\varepsilon^2}{r^2} u_\varepsilon \cos(2\theta_3) \\ &= (pu_\varepsilon^{p-1} - 1)u_\varepsilon \cos(2\theta_3) - (p-1)u_\varepsilon^p \cos(2\theta_3) + \frac{8\varepsilon^2}{r^2} u_\varepsilon \cos(2\theta_3). \end{aligned}$$

Multiplying by $u_\varepsilon \cos(2\theta_3)$ and integrating over A , we see that the last two terms give

$$\begin{aligned} &\int_A \left(-(p-1)u_\varepsilon^p \cos(2\theta_3) + \frac{8\varepsilon^2}{r^2} u_\varepsilon \cos(2\theta_3) \right) u_\varepsilon \cos(2\theta_3) dx \\ &= \int_0^{\pi/2} \frac{1}{2} \sin(2\theta_3) \cos^2(2\theta_3) \int_a^b \int_0^{2\pi} \int_0^{2\pi} \left(-(p-1)u_\varepsilon^{p+1} + \frac{8\varepsilon^2}{r^2} u_\varepsilon^2 \right) d\theta_1 d\theta_2 r^3 dr, \end{aligned}$$

which is negative for ε^2 small, since $\int_A u_\varepsilon^{p+1} = \int_A (\varepsilon^2 |\nabla u_\varepsilon|^2 + u_\varepsilon^2) > \int_A u_\varepsilon^2$.

Thus, we have shown that any solution which is independent of θ_3 has Morse index ≥ 2 ; hence the mountain-pass solution, which has Morse index 1, cannot be independent of θ_3 . \square

We now use invariance under the group T_τ to reduce problem (3) to an equation in three dimensions.

3. Reduction to a problem in three dimensions

In the variables $(r, \theta_1, \theta_2, \theta_3)$ the first term of the functional J_ε takes the form

$$\begin{aligned} &\frac{\varepsilon^2}{2} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} \int_a^b \left(|u_r|^2 + |u_{\theta_1}|^2 \frac{1}{r^2 \sin^2 \theta_3} \right. \\ &\quad \left. + |u_{\theta_2}|^2 \frac{1}{r^2 \cos^2 \theta_3} + |u_{\theta_3}|^2 \frac{1}{r^2} \right) r^3 \sin \theta_3 \cos \theta_3 dr d\theta_1 d\theta_2 d\theta_3. \quad (9) \end{aligned}$$

We now rewrite the functional $J_\varepsilon(u)$, taking into account the invariance along the orbit T_τ : for $u \in H_{0,\#}^1(A)$ we can consider

$$T_{-\theta_2}u(t, \theta_1, \theta_2, \theta_3) = u(r, \theta_1 - \theta_2, 0, \theta_3) =: v(r, \theta, \theta_3),$$

where we have introduced the new variable $\theta := \theta_1 - \theta_2$. Note that $v(r, \theta, \theta_3)$ is well defined: if $\theta = \theta'_1 - \theta'_2$, then $\theta'_1 = \theta_1 + \sigma$ for some $\sigma \in [0, 2\pi)$, and hence $\theta'_2 = \theta'_1 - \theta = \theta_1 + \sigma - \theta = (\theta_2 + \theta) + \sigma - \theta = \theta_2 + \sigma$.

Next, we calculate the derivative $v_\theta(r, \theta, \theta_3)$: we have

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{1}{h} (v(r, \theta + h, \theta_3) - v(r, \theta, \theta_3)) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(u \left(r, \theta_1 + \frac{h}{2}, \theta_2 - \frac{h}{2}, \theta_3 \right) - u(r, \theta_1, \theta_2, \theta_3) \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{1}{h} \left(u \left(r, \theta_1 + \frac{h}{2}, \theta_2 - \frac{h}{2}, \theta_3 \right) - u \left(r, \theta_1, \theta_2 - \frac{h}{2}, \theta_3 \right) \right) \right. \\ & \quad \left. + \frac{1}{h} \left(u \left(r, \theta_1, \theta_2 - \frac{h}{2}, \theta_3 \right) - u(r, \theta_1, \theta_2, \theta_3) \right) \right) \\ &= \frac{1}{2} u_{\theta_1}(r, \theta_1, \theta_2, \theta_3) - \frac{1}{2} u_{\theta_2}(r, \theta_1, \theta_2, \theta_3) = u_{\theta_1}(r, \theta_1, \theta_2, \theta_3) = -u_{\theta_2}(r, \theta_1, \theta_2, \theta_3); \end{aligned}$$

the last identities follow by considering the constant function

$$f(\tau) = u(r, \theta_1 + \tau, \theta_2 + \tau, \theta_3),$$

which yields

$$0 = f_\tau(0) = u_{\theta_1}(r, \theta_1, \theta_2, \theta_3) + u_{\theta_2}(r, \theta_1, \theta_2, \theta_3).$$

With this, we can now write the functional J_ε on the space $H_{0,\#}^1(A)$ in the form

$$\begin{aligned} J_\varepsilon(v) &= 2\pi \int_0^{\pi/2} \int_0^{2\pi} \int_a^b \left(\frac{\varepsilon^2}{2} |v_r|^2 + \frac{\varepsilon^2}{2} |v_\theta|^2 \frac{1}{r^2} \left(\frac{1}{\sin^2 \theta_3} + \frac{1}{\cos^2 \theta_3} \right) + \frac{\varepsilon^2}{2} |v_{\theta_3}|^2 \frac{1}{r^2} \right. \\ & \quad \left. + \frac{1}{2} |v|^2 - \frac{1}{p+1} |v|^{p+1} \right) r^3 \sin \theta_3 \cos \theta_3 \, dr \, d\theta \, d\theta_3. \quad (10) \end{aligned}$$

We make the change of variables $\varphi = 2\theta_3$, and use that $\sin \theta_3 \cos \theta_3 = \frac{1}{2} \sin \varphi$, to obtain

$$\begin{aligned} J_\varepsilon(v) &= 2\pi \int_0^\pi \int_0^{2\pi} \int_a^b \left(\frac{\varepsilon^2}{2} |v_r|^2 + \frac{\varepsilon^2}{2} |v_\theta|^2 \frac{4}{r^2 \sin^2 \varphi} + \frac{\varepsilon^2}{2} |v_\varphi|^2 \frac{4}{r^2} \right. \\ & \quad \left. + \frac{1}{2} |v|^2 - \frac{1}{p+1} |v|^{p+1} \right) \frac{r^3}{4} \sin \varphi \, dr \, d\theta \, d\varphi \\ &= 2\pi \int_0^\pi \int_0^{2\pi} \int_a^b \left(\frac{\varepsilon^2}{2} |v_r|^2 \frac{r^3}{4} \, dr + \frac{\varepsilon^2}{2} |v_\theta|^2 \frac{r \, dr}{\sin^2 \varphi} + \frac{\varepsilon^2}{2} |v_\varphi|^2 r \, dr \right. \\ & \quad \left. + \frac{1}{2} |v|^2 \frac{r^3}{4} \, dr - \frac{1}{p+1} |v|^{p+1} \frac{r^3}{4} \, dr \right) \sin \varphi \, d\theta \, d\varphi. \end{aligned}$$

Now, setting $s = \frac{1}{2}r^2$ we get

$$\begin{aligned} J_\varepsilon(v) &= 2\pi \int_0^\pi \int_0^{2\pi} \int_{a^2/2}^{b^2/2} \left(\frac{\varepsilon^2}{2} |v_s|^2 s^2 ds + \frac{\varepsilon^2}{2} |v_\theta|^2 \frac{ds}{\sin^2 \varphi} + \frac{\varepsilon^2}{2} |v_\varphi|^2 ds \right. \\ &\quad \left. + \frac{1}{2} |v|^2 \frac{s ds}{2} - \frac{1}{p+1} |v|^{p+1} \frac{s ds}{2} \right) \sin \varphi d\theta d\varphi \\ &= 2\pi \int_0^\pi \int_0^{2\pi} \int_{a^2/2}^{b^2/2} \left(\frac{\varepsilon^2}{2} |v_s|^2 + \frac{\varepsilon^2}{2} |v_\theta|^2 \frac{1}{s^2 \sin^2 \varphi} + \frac{\varepsilon^2}{2} |v_\varphi|^2 \frac{1}{s^2} \right. \\ &\quad \left. + \frac{1}{2} |v|^2 \frac{1}{2s} - \frac{1}{p+1} |v|^{p+1} \frac{1}{2s} \right) s^2 ds \sin \varphi d\theta d\varphi. \end{aligned}$$

We note that this functional is now defined in the usual polar coordinates (r, θ, φ) in \mathbb{R}^3 , and we may rewrite it after a standard change of variables in cartesian coordinates:

$$J_\varepsilon(v) = \int_B \left(\frac{\varepsilon^2}{2} |\nabla v|^2 + \frac{1}{2} |v|^2 \frac{1}{2|x|} - \frac{1}{p+1} |v|^{p+1} \frac{1}{2|x|} \right) dx \quad (11)$$

where $B = \{x \in \mathbb{R}^3 : a^2/2 < |x| < b^2/2\}$. Critical points of this functional now correspond to the following “reduced” equation in $B \subset \mathbb{R}^3$:

$$\begin{cases} -\varepsilon^2 \Delta v + \frac{1}{2|x|} v - \frac{1}{2|x|} v^p = 0 & \text{in } B, \\ v = 0 & \text{on } \partial B. \end{cases} \quad (12)$$

It is clear that if we have a solution of equation (12) in $B \subset \mathbb{R}^3$, then by reversing the above transformations, we will obtain a solution of equation (3) in $A \subset \mathbb{R}^4$. We remark in particular that if we find, for ε^2 small, solutions v_ε which converge to a single peak solution of equation (12), then the corresponding solutions u_ε will concentrate on a 1-dimensional curve given by the orbit under T_τ of v_ε .

4. Profile of the solution

W.-M. Ni and J. Wei studied in [13] the equation

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega \subset \mathbb{R}^n, \\ u > 0 & \text{in } \Omega, \quad u = 0 & \text{on } \partial\Omega, \end{cases} \quad (13)$$

and proved (Theorem 2.2 in [13]) that the least energy solutions of (13) have, for $\varepsilon > 0$ sufficiently small, at most one local maximum. This result is proved by a blow-up procedure which leads to the following limiting equation in \mathbb{R}^n :

$$\begin{cases} -\Delta z + z - z^p = 0 & \text{on } \mathbb{R}^n, \\ z(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases} \quad (14)$$

The following proposition is an adaptation of the above mentioned theorem by Ni and Wei to the nonhomogeneous equation (12).

Proposition 4. *Let u_ε be a least energy solution to (12). Then, for ε^2 sufficiently small:*

- (i) u_ε has at most one local maximum and it is achieved at exactly one point p_ε in B .
 Moreover, $u_\varepsilon(\cdot + p_\varepsilon) \rightarrow 0$ in $C^1_{\text{loc}}((B - p_\varepsilon) \setminus \{0\})$ where $B - p_\varepsilon = \{x - p_\varepsilon \mid x \in B\}$.
- (ii) $(1/\varepsilon)d(p_\varepsilon, \partial B) \rightarrow +\infty$ as $\varepsilon \rightarrow \infty$.

The blow-up procedure leads in this case to the limit equation

$$\begin{cases} -\Delta w + \frac{1}{2d}w - \frac{1}{2d}w^p = 0 & \text{in } \mathbb{R}^3, \\ w(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \tag{15}$$

with $d = |p_0|$.

Following the work of del Pino–Felmer [3], who simplified the proof of Ni–Wei (and at the same time removed a delicate “nondegeneracy condition”), we will prove

Theorem 5. *For $\varepsilon \rightarrow 0$ the maximum point p_ε converges to the inner boundary of B , i.e.*

$$|p_\varepsilon| \rightarrow a^2/2 \quad \text{as } \varepsilon \rightarrow 0.$$

This is in contrast to the homogeneous equation (13) for which Ni–Wei proved concentration at a point with maximal distance from the boundary.

5. Concentrating solutions

In this section we point out the necessary modifications in the proof of Theorem 2.2 by Ni and Wei when applied to the nonhomogeneous equation (12) to obtain concentrating solutions.

1) Proof of Proposition 4. This corresponds to the proof of Theorem 2.2(i) in Ni–Wei [13]. Lemmas 3.1–3.3 there require no change.

Step 1 (p. 737). We first prove (ii). Assume on the contrary that there exists $c > 0$ and a sequence $\varepsilon_k \rightarrow 0$ such that

$$d(p_\varepsilon, \partial B) \leq c\varepsilon \quad \text{for } \varepsilon = \varepsilon_k.$$

By passing to a subsequence, we may assume that $p_\varepsilon \rightarrow p_0 \in \partial B$, i.e. $|p_0| = a^2/2$ or $|p_0| = b^2/2$. Writing $p_\varepsilon = p_0 + \tilde{p}_\varepsilon$, we have $\tilde{p}_\varepsilon \rightarrow 0$. By “boundary straightening” around the point p_0 we obtain for $w_\varepsilon(z) := u_\varepsilon(\mathcal{G}(q_\varepsilon + \varepsilon z))$ the equation

$$\sum_{i,j} a_{i,j}^\varepsilon \frac{\partial^2 w_\varepsilon}{\partial z_i \partial z_j} + \varepsilon \sum_i b_j^\varepsilon \frac{\partial w_\varepsilon}{\partial z_j} - \frac{w_\varepsilon}{2|p_0 + \mathcal{G}(q_\varepsilon + \varepsilon z)|} + \frac{w_\varepsilon^p}{2|p_0 + \mathcal{G}(q_\varepsilon + \varepsilon z)|},$$

where $\mathcal{G} : \bar{B}_{\kappa/\varepsilon} \cap \{z_3 \geq -\alpha_\varepsilon\} \subset \mathbb{R}^3 \rightarrow B$ is the “straightening map” with $\mathcal{G}(q_\varepsilon) = p_\varepsilon$; $\kappa > 0$ is a small constant and $\alpha_\varepsilon > 0$ is bounded. In the limit $\varepsilon \rightarrow 0$ we find, since

$q_\varepsilon \rightarrow 0$, that $w_0 = \lim w_\varepsilon$ satisfies the equation (cf. (3.7) in [13])

$$\begin{cases} \Delta u - \frac{u}{2|p_0|} + \frac{u^p}{2|p_0|} = 0, & u > 0, \quad \text{on } \mathbb{R}_{\alpha,+}^3, \\ u = 0 & \text{on } \partial\mathbb{R}_{\alpha,+}^3, \end{cases}$$

where $\mathbb{R}_{\alpha,+}^3 = \{z \in \mathbb{R}^3 \mid z_3 \geq -\alpha\}$ with $\alpha = \lim \alpha_\varepsilon$.

By Theorem 1.1 of [6] one concludes that $w_0 \equiv 0$; but this contradicts that $w_0(0) = \lim w_\varepsilon(q_\varepsilon) = \lim u_\varepsilon(p_\varepsilon) \geq \bar{u}$.

Steps 2 and 3 are the same, provided Proposition 3.4 works (see below). This gives the conclusion of Theorem 2.2(i) of [13].

2) Profile of u_ε . The statement and proof of Proposition 3.4 in [13] need some modifications. For convenience, we state it here:

Proposition 3.4'. *Let $\tilde{v}_\varepsilon(y) = u_\varepsilon(p_\varepsilon + \varepsilon y)$.*

(i) *For given $\eta > 0$ there exist $\varepsilon_0 > 0$ and $k_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,*

$$B_{2k_0\varepsilon}(p_\varepsilon) \subset B \quad \text{and} \quad \|\tilde{v}_\varepsilon - w_d\|_{C^2(B_{k_0}(0))} < \eta,$$

where w_d is the unique solution of equation (15), with

$$w_d(x) := w\left(\frac{x}{\sqrt{2d}}\right) \tag{16}$$

and w the unique solution of (14) satisfying $w(0) = \max_{x \in \mathbb{R}^3} w(x)$ and $w(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

(ii) *For any $0 < \delta < 1$ there exists a constant C such that*

$$\tilde{v}_\varepsilon(y) \leq C e^{-\frac{\sqrt{1-\delta}}{b}|y|} \quad \text{for } y \in \tilde{B}_\varepsilon := \{y \in \mathbb{R}^3 \mid p_\varepsilon + \varepsilon y \in B\},$$

where b is the outer radius of the annulus A .

Proof. (i): As in Ni–Wei [13].

(ii): Comparison with the known solution $w_d(r)$ of equation (15): By Theorem 2 in Gidas–Ni–Nirenberg [9] one knows that for the solution $w(r)$ of (14) one has $w(r) \leq C_0 e^{-r}$, and hence by (16),

$$w_d(r) \leq C_0 e^{-r/\sqrt{2d}} \quad \text{for } r \geq 0.$$

For given $\eta > 0$ choose $R > 0$ such that $\eta = C_0 e^{-R/\sqrt{2d}}$. By (i) there exists $\varepsilon_0 > 0$ such that $\|\tilde{v}_\varepsilon - w_d\|_{C^2(\tilde{B}_{2R}(0))} \leq \eta$ for $0 < \varepsilon < \varepsilon_0$. Thus

$$\tilde{v}_\varepsilon(y) \leq w(y) + \eta \leq C_0 e^{-R/\sqrt{2d}} + \eta = 2\eta \quad \text{for } |y| = R.$$

Hence (see Ni–Wei [13])

$$v_\varepsilon \leq 2\eta \quad \text{in } B_\varepsilon^{(o)} := B \setminus B_{R\varepsilon}(p_\varepsilon).$$

Note that \tilde{v}_ε satisfies

$$\begin{cases} \Delta \tilde{v}_\varepsilon - \frac{1}{2|p_\varepsilon + \varepsilon y|} (1 - \tilde{v}_\varepsilon^{p-1}) \tilde{v}_\varepsilon = 0 & \text{in } \tilde{B}_\varepsilon^{(o)} := \tilde{B}_\varepsilon \setminus B_R(0), \\ \tilde{v}_\varepsilon|_{\partial B_R(0)} \leq 2\eta, \quad \tilde{v}_\varepsilon = 0 & \text{on } \partial \tilde{B}_\varepsilon, \end{cases}$$

and that $|p_\varepsilon + \varepsilon y| \leq b^2/2$. Now choose η such that $1 - s^{p-1} > 1 - \delta$ for $s < 2\eta$, and hence

$$\frac{1}{2|p_\varepsilon + \varepsilon y|} (1 - \tilde{v}_\varepsilon^{p-1}) > \frac{1}{b^2} (1 - \delta) \quad \text{in } \tilde{B}_\varepsilon^{(o)}.$$

Let $G_0(|y|)$ denote the Green’s function for $-\Delta + 1$ on \mathbb{R}^3 , and

$$\bar{v}(y) = \frac{2\eta G_0(|y|\sqrt{1-\delta}/b)}{G_0(R\sqrt{1-\delta}/b)}.$$

Then \bar{v} satisfies

$$\Delta \bar{v} - \frac{1-\delta}{b^2} \bar{v} = 0, \quad \bar{v} = 2\eta \quad \text{on } \partial B_R(0).$$

By the maximum principle on $\tilde{B}_\varepsilon^{(o)}$ we have

$$\tilde{v}_\varepsilon(y) \leq \bar{v}(y) \quad \text{on } \tilde{B}_\varepsilon^{(o)},$$

and hence

$$\tilde{v}_\varepsilon(y) \leq C e^{-\frac{\sqrt{1-\delta}}{b}|y|} \quad \text{for all } y \in \tilde{B}_\varepsilon^{(o)}.$$

6. Localizing the concentration points

In this section we show that the concentration points p_ε for the least energy solution of the nonhomogeneous equation (12) converge to the inner boundary of B , i.e. they satisfy

$$|p_\varepsilon| \rightarrow a^2/2 \quad \text{as } \varepsilon \rightarrow 0.$$

We follow the paper of del Pino–Felmer [3], obtaining precise upper and lower estimates for the least energy level c_ε .

6.1. Upper bound

Consider

$$I_d(w_d) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w_d|^2 dx + \frac{1}{2d} \int_{\mathbb{R}^3} \frac{1}{2} |w_d|^2 dx - \frac{1}{2d} \int_{\mathbb{R}^3} \frac{1}{p+1} |w_d|^{p+1} dx$$

where w_d is the solution of (15), i.e.

$$\begin{cases} -\Delta w + \frac{1}{2d}w - \frac{1}{2d}w^p = 0 & \text{in } \mathbb{R}^3, \\ w(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases} \quad (17)$$

We first note that

$$I_d(w_d) = \sqrt{2d} I(z) \quad (18)$$

where z is the solution of the equation

$$-\Delta z + z - z^p = 0 \quad \text{on } \mathbb{R}^3,$$

and $I(z)$ the corresponding functional. Note that the unique radial solution of (17) is given by

$$w(r) = z\left(\frac{r}{\sqrt{2d}}\right). \quad (19)$$

Then we calculate, denoting by ω_2 the surface volume of $S^2 \subset \mathbb{R}^3$:

$$\begin{aligned} I_d(w_d) &= \frac{1}{2} \int |\nabla w_d|^2 - \frac{1}{2d} \int \frac{1}{2} |w_d|^2 - \frac{1}{2d} \int \frac{1}{p+1} |w_d|^{p+1} \\ &= \frac{\omega_2}{2} \int_0^\infty \left| \nabla_r z\left(\frac{r}{\sqrt{2d}}\right) \right|^2 r^2 dr - \frac{\omega_2}{2} \frac{1}{2d} \int_0^\infty \left| z\left(\frac{r}{\sqrt{2d}}\right) \right|^2 r^2 dr \\ &\quad - \frac{\omega_2}{p+1} \frac{1}{2d} \int_0^\infty \left| z\left(\frac{r}{\sqrt{2d}}\right) \right|^{p+1} r^2 dr \\ &= \frac{\omega_2}{2} \int_0^\infty \left| \nabla_s z(s) \frac{ds}{dr} \right|^2 2d\sqrt{2d} s^2 ds - \frac{\omega_2}{2} \frac{1}{\sqrt{2d}} \int_0^\infty |z(s)|^2 2d\sqrt{2d} s^2 ds \\ &\quad - \frac{\omega_2}{p+1} \frac{1}{2d} \int_0^\infty |z(s)|^{p+1} 2d\sqrt{2d} s^2 ds \\ &= \sqrt{2d} I(z). \end{aligned}$$

Next, we derive an upper bound on the least energy value c_ε .

Lemma 6. Denote by u_ε the minimal energy solution of equation (12), with the corresponding functional

$$I_{\varepsilon,|x|}(u) = \frac{\varepsilon^2}{2} \int_B |\nabla u|^2 + \frac{1}{2} \int_B \frac{1}{2|x|} |u|^2 - \frac{1}{p+1} \int_B \frac{1}{2|x|} |u|^{p+1}$$

and set

$$c_\varepsilon = I_{\varepsilon,|x|}(u_\varepsilon).$$

Then, for given $\delta > 0$,

$$c_\varepsilon \leq \varepsilon^3 (\sqrt{a^2 + \delta} I(z) + \varepsilon c) \quad \text{for } \varepsilon > 0 \text{ sufficiently small.}$$

Proof. Consider a point x_ε which lies close to the inner boundary, i.e. with $\text{dist}(x_\varepsilon, a^2/2) = \delta/2$, and the ball $B_{\delta/2}(x_\varepsilon)$. Since the least energy values for $I_{\varepsilon,|x|}$ in B and $B_{\delta/2}(x_\varepsilon) \subset B$ are ordered with respect to the domain we get

$$c_\varepsilon \leq I_{\varepsilon,|x|}(u_{\delta/2}) =: c_{\varepsilon,\delta/2},$$

where $u_{\delta/2}$ is the least energy solution on $B_{\delta/2}(x_\varepsilon)$. Recall that

$$I_{\varepsilon,|x|}(u_{\delta/2}) = \inf_{u \in S_{\varepsilon,|x|}} I_{\varepsilon,|x|}(u), \tag{20}$$

where

$$S_{\varepsilon,|x|} = \left\{ u \in H_0^1(B_{\delta/2}(x_\varepsilon)) \mid \int_{B_{\delta/2}(x_\varepsilon)} \left(\varepsilon^2 |\nabla u|^2 + \frac{|u|^2}{2|x|} \right) = \int_{B_{\delta/2}(x_\varepsilon)} \frac{|u|^{p+1}}{2|x|} \right\}.$$

Next, let $\tilde{u}_{\delta/2}$ denote the least energy solution of

$$-\varepsilon^2 \Delta u + \frac{1}{2d_\varepsilon} u = \frac{1}{2d_\varepsilon} u^p \quad \text{on } B_{\delta/2}(x_\varepsilon),$$

where $d_\varepsilon = |x_\varepsilon|$. Then $\tilde{u}_{\delta/2}$ is radially symmetric, and by (20),

$$I_{\varepsilon,|x|}(u_{\delta/2}) \leq I_{\varepsilon,|x|}(\tilde{t}\tilde{u}_{\delta/2}),$$

where \tilde{t} is such that $\tilde{t}\tilde{u}_{\delta/2} \in S_{\varepsilon,|x|}$. We estimate

$$\begin{aligned} & I_{\varepsilon,|x|}(\tilde{t}\tilde{u}_{\delta/2}) \\ &= \frac{\varepsilon^2}{2} \int_{B_{\delta/2}(x_\varepsilon)} |\nabla \tilde{t}\tilde{u}_{\delta/2}|^2 + \frac{1}{2} \int_{B_{\delta/2}(x_\varepsilon)} \frac{|\tilde{t}\tilde{u}_{\delta/2}|^2}{2|x_\varepsilon + (x - x_\varepsilon)|} \\ &\quad - \frac{1}{p+1} \int_{B_{\delta/2}(x_\varepsilon)} \frac{|\tilde{t}\tilde{u}_{\delta/2}|^{p+1}}{2(|x_\varepsilon + (x - x_\varepsilon)|)} \\ &\leq \frac{\varepsilon^2}{2} \int_{B_{\delta/2}(x_\varepsilon)} |\nabla \tilde{t}\tilde{u}_{\delta/2}|^2 + \frac{1}{2} \int_{B_{\delta/2}(x_\varepsilon)} \frac{|\tilde{t}\tilde{u}_{\delta/2}|^2}{2|x_\varepsilon| - 2|x - x_\varepsilon|} \\ &\quad - \frac{1}{p+1} \int_{B_{\delta/2}(x_\varepsilon)} \frac{|\tilde{t}\tilde{u}_{\delta/2}|^{p+1}}{2|x_\varepsilon| + 2|x - x_\varepsilon|} \\ &= \frac{\varepsilon^2}{2} \int_{B_{\delta/2}(x_\varepsilon)} |\nabla \tilde{t}\tilde{u}_{\delta/2}|^2 + \frac{1}{2} \int_{B_{\delta/2}(x_\varepsilon)} \frac{|\tilde{t}\tilde{u}_{\delta/2}|^2}{2|x_\varepsilon|} - \frac{1}{p+1} \int_{B_{\delta/2}(x_\varepsilon)} \frac{|\tilde{t}\tilde{u}_{\delta/2}|^{p+1}}{2|x_\varepsilon|} \\ &\quad + \frac{1}{2} \int_{B_{\delta/2}(x_\varepsilon)} \frac{|\tilde{t}\tilde{u}_{\delta/2}|^2|x - x_\varepsilon|}{2|x_\varepsilon||x - x_\varepsilon|} + \frac{1}{p+1} \int_{B_{\delta/2}(x_\varepsilon)} \frac{|\tilde{t}\tilde{u}_{\delta/2}|^{p+1}|x - x_\varepsilon|}{2|x_\varepsilon||x - x_\varepsilon|}. \tag{21} \end{aligned}$$

We now consider the problem

$$-\Delta u + \frac{1}{2d_\varepsilon} u = \frac{1}{2d_\varepsilon} u^p \quad \text{in } B_\rho, \quad u > 0 \quad \text{in } B_\rho, \quad u = 0 \quad \text{on } \partial B_\rho, \tag{22}$$

with the associated functional $J_{\rho,d_\varepsilon} : H_0^1(B_\rho) \rightarrow \mathbb{R}$, where $d_\varepsilon = |x_\varepsilon|$.

With the change of variable $x = x_\varepsilon + \varepsilon y$, setting $\rho = \delta/2\varepsilon$ and $v_{\delta/2}(y) = \tilde{t}\tilde{u}_{\delta/2}(x_\varepsilon + \varepsilon y)$, we obtain from (21)

$$\begin{aligned} I_{\varepsilon,|x|}(\tilde{t}\tilde{u}_{\delta/2}) &\leq \varepsilon^3 \left(J_{\rho,d_\varepsilon}(v_{\delta/2}) + \varepsilon c \int_{B_\rho(x_\varepsilon)} |v_{\delta/2}|^2 |y| dy + \varepsilon c \int_{B_\rho(x_\varepsilon)} |v_{\delta/2}|^{p+1} |y| dy \right) \\ &\leq \varepsilon^3 (J_{\rho,d_\varepsilon}(v_{\delta/2}) + \varepsilon c) \leq \varepsilon^3 (J_{\rho,d_\varepsilon}(\bar{t}v_{\delta/2}) + \varepsilon c), \end{aligned}$$

where \bar{t} is such that $\bar{t}v_{\delta/2} \in S_{\rho,|x_\varepsilon|}$, i.e. $\bar{t}v_{\delta/2}$ solves (22). Next, we use Lemma 2.1 in [3] to get

$$J_{\rho,d_\varepsilon}(\bar{t}v_{\delta/2}) \leq I_{d_\varepsilon}(w) + e^{-2\rho(1+o(1))} = \sqrt{2d_\varepsilon} I(z) + e^{-\frac{\delta}{\varepsilon}(1+o(1))},$$

and hence finally

$$c_\varepsilon \leq \varepsilon^3 (\sqrt{2d_\varepsilon} I(z) + \varepsilon c) = \varepsilon^3 (\sqrt{a^2 + \delta} I(z) + \varepsilon c). \quad \square$$

6.2. Lower bound

In this section we show

Lemma 7. *Let p_ε denote the maximum point of the least energy solution u_ε of equation (12). If $|p_\varepsilon| \geq a^2/2 + \delta$ for all $\varepsilon > 0$, then there exists a positive constant (independent of ε) such that*

$$c_\varepsilon \geq \varepsilon^3 (\sqrt{a^2 + 2\delta} I(z) - \varepsilon c).$$

Proof. First note that

$$c_\varepsilon = \max_{t>0} I_{\varepsilon,|x|}(tu_\varepsilon) \geq I_{\varepsilon,|x|}(tu_\varepsilon) \quad \text{for (say) } t \in [0, 2].$$

Consider $B_\delta(p_\varepsilon) \subset B$, and fix $0 < \delta' < \delta$. Setting $x = p_\varepsilon + \varepsilon y$, $\rho = \delta/\varepsilon$, $v_\varepsilon(y) = u_\varepsilon(p_\varepsilon + \varepsilon y)$ and J_{ρ,d_ε} as in (22), we obtain, by a similar estimate to (21),

$$I_{\varepsilon,|x|}(tu_\varepsilon) \geq \varepsilon^3 (J_{\rho,d_\varepsilon}(tv_\varepsilon) - \varepsilon c).$$

Next, choose $\eta \in C^1([0, \delta], \mathbb{R}^+)$ with $\eta(s) = 1$ for $0 \leq s \leq \delta'$ and $\eta(s) = 0$ for $s \geq \delta$, and such that $|\eta'(s)| \leq c$. Set $\tilde{v}_\varepsilon(y) = v_\varepsilon(y)\eta(|y|)$. Then, as for (2.6) in [3], we get

$$\varepsilon^3 (J_{\rho,d_\varepsilon}(tv_\varepsilon) - \varepsilon c) \geq \varepsilon^3 (J_{\rho,d_\varepsilon}(t\tilde{v}_\varepsilon) - e^{-2\delta'/\varepsilon} - \varepsilon c).$$

At this point we choose $t = t^*$ such that $J_{\rho,d_\varepsilon}(t^*v_\varepsilon) = \max_{t \geq 0} J_{\rho,d_\varepsilon}(tv_\varepsilon)$, and conclude that

$$J_{\rho,d_\varepsilon}(t^*v_\varepsilon) \geq J_{\rho,d_\varepsilon}(w_\varepsilon), \quad \text{where } w_\varepsilon \text{ solves (22).}$$

Finally, using the estimate of Lemma 2.1 in [3], we get

$$J_{\rho,d_\varepsilon}(w_\varepsilon) \geq I_{d_\varepsilon}(w) - e^{-2\delta'/\varepsilon}.$$

Joining the above estimates we obtain

$$c_\varepsilon \geq \varepsilon^3 (I_{d_\varepsilon}(w) - c\varepsilon) = \varepsilon^3 (\sqrt{a^2 + 2\delta} I(z) - c\varepsilon). \quad \square$$

7. Proof of Theorem 1

From Lemmas 6 and 7 it follows that, for $\varepsilon > 0$ sufficiently small, one has

$$|p_\varepsilon| \leq a^2/2 + \delta.$$

Since $\delta > 0$ is arbitrary, we conclude that $|p_\varepsilon|$ converges to the inner boundary of B . Note that Theorem 4 says that this convergence cannot be too fast, since

$$\frac{1}{\varepsilon} d(p_\varepsilon, \partial B) \rightarrow +\infty.$$

Finally, reversing the steps in Section 3 we see that the solutions u_ε concentrating on the single peak p_ε , which we obtained in the previous sections, become solutions which concentrate on the orbits $T_\tau p_\varepsilon$ of the group action T_τ which converge to the inner boundary a of A for $\varepsilon \rightarrow 0$. \square

Acknowledgments. The authors wish to thank the referee for his very helpful comments.

References

- [1] Ambrosetti, A., Malchiodi, A., Ni, W.-N.: Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, part II. *Indiana Univ. Math. J.* **53**, 297–329 (2004) Zbl 1081.35008 MR 2056434
- [2] Ambrosetti, A., Rabinowitz, P. H.: Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14**, 349–381 (1973) Zbl 0273.49063 MR 0370183
- [3] del Pino, M., Felmer, P.: Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting. *Indiana Univ. Math. J.* (2004) **48**, 883–898 (1999) Zbl 0932.35080 MR 1736974
- [4] Ekeland, I., Ghoussoub, N.: Selected new aspects of the calculus of variations in the large. *Bull. Amer. Math. Soc. (N.S.)* **39**, 207–265 (2002) Zbl 1064.35054 MR 1886088
- [5] Esposito, P., Mancini, G., Santra, S., Srikanth, P. N.: Asymptotic behavior of radial solutions for a semilinear elliptic problem on an annulus through Morse index. *J. Differential Equations* **239**, 1–15 (2007) Zbl 1143.35039 MR 2341546
- [6] Esteban, M. J., Lions, P.-L.: Existence and nonexistence results for semilinear elliptic problems in unbounded domains. *Proc. Roy Soc. Edinburgh Sect. A* **93**, 1–14 (1982) Zbl 0506.35035 MR 0688279
- [7] Fang, G., Ghoussoub, N.: Morse-type information on Palais–Smale sequences obtained by Min–Max principles. *Comm. Pure Appl. Math.* **47**, 1595–1653 (1994) Zbl 0829.58008 MR 1303222
- [8] Ghoussoub, N.: *Duality and Perturbation Methods in Critical Point Theory*. Cambridge Tracts in Math. 107, Cambridge Univ. Press (1993) Zbl 0790.58002 MR 1251958
- [9] Gidas, B., Ni, W.-M., Nirenberg, L.: Symmetry of positive solutions of nonlinear elliptic equations in \mathbb{R}^n . In: *Mathematical Analysis and Applications*, Adv. Math. Suppl. Stud. 7A, Academic Press, 369–402 (1981) Zbl 0469.35052 MR 0634248
- [10] Lin, C. S., Ni, W.-M., Takagi, I.: Large amplitude stationary solutions to a chemotaxis system. *J. Differential Equations* **72**, 1–27 (1988) Zbl 0676.35030 MR 0929196

-
- [11] Ni, W.-M., Takagi, I.: On the shape of least energy solutions to a semilinear Neumann problem. *Comm. Pure Appl. Math.* **44**, 819–851 (1991) Zbl 0754.35042 MR 1115095
 - [12] Ni, W.-M., Takagi, I.: Locating the peaks of least energy solutions to a semilinear Neumann problem. *Duke Math. J.* **70**, 247–281 (1993) Zbl 0796.35056 MR 1219814
 - [13] Ni, W.-M., Wei, J.: On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems. *Comm. Pure Appl. Math.* **68**, 731–768 (1995) Zbl 0838.35009 MR 1342381
 - [14] Ni, W.-M., Wei, J.: Diffusion, cross-diffusion, and their spike-layer steady states. *Notices Amer. Math. Soc.* **45**, 9–18 (1998) Zbl 0917.35047 MR 1490535