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Symplectic critical surfaces in Kähler surfaces

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Abstract. Let M be a Kähler surface and Σ be a closed symplectic surface which is smoothly immersed in M. Let α be the Kähler angle of Σ in M. We first deduce the Euler–Lagrange equation of the functional $L = \int_\Sigma \frac{1}{\cos\alpha}\, d\mu$ in the class of symplectic surfaces. It is $\cos^3\alpha\, H = (J(J\nabla\cos\alpha)^\top)^\perp$, where H is the mean curvature vector of Σ in M, and J is the complex structure compatible with the Kähler form ω in M; it is an elliptic equation. We call a surface satisfying this equation a symplectic critical surface. We show that, if M is a Kähler–Einstein surface with nonnegative scalar curvature, each symplectic critical surface is holomorphic. We also study the topological properties of symplectic critical surfaces. By our formula and Webster's formula, we deduce that the Kähler angle of a compact symplectic critical surface is constant, which is not true for noncompact symplectic critical surfaces.

Keywords. Symplectic surface, holomorphic curve, Kähler surface

1. Introduction

Suppose that M is a Kähler surface. Let ω be the Kähler form on M and let J be a complex structure compatible with ω. The Riemannian metric $\langle\, ,\rangle$ on M is defined by

$$\langle U, V\rangle = \omega(U, JV).$$

For a compact oriented real surface Σ which is smoothly immersed in M, one defines, following [4], the *Kähler angle* α of Σ in M by

$$\omega|_\Sigma = \cos\alpha\, d\mu_\Sigma \tag{1.1}$$

where $d\mu_\Sigma$ is the area element of Σ of the metric induced from $\langle\, ,\rangle$. We say that Σ is a *holomorphic curve* if $\cos\alpha \equiv 1$; Σ is a *Lagrangian surface* if $\cos\alpha \equiv 0$, and a *symplectic surface* if $\cos\alpha > 0$.

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It was conjectured by Tian [8] that every embedded orientable closed symplectic surface in a compact Kähler–Einstein surface is isotopic to a symplectic minimal surface in a suitable sense.

If the Kähler–Einstein surface is of nonnegative scalar curvature, a symplectic minimal surface is holomorphic. However, if the scalar curvature is negative, there are symplectic minimal surfaces which are not holomorphic ([1]). A symplectic minimal surface is a critical point of the area of surfaces, which is symplectic. It may be more natural to consider directly the critical point of the functional

$$L = \int_\Sigma \frac{1}{\cos\alpha}\, d\mu_\Sigma,$$

in the class of symplectic surfaces in a Kähler surface. It is clear that holomorphic curves minimize the functional. A critical point of this functional is called a *symplectic critical surface*.

In the paper, we first calculate the Euler–Lagrange equation of the functional L.

Theorem 1.1. *Let M be a Kähler surface. The Euler–Lagrange equation of the functional L is*

$$\cos^3\alpha\, H - (J(J\nabla\cos\alpha)^\top)^\perp = 0,$$

where H is the mean curvature vector of Σ in M, $()^\top$ is the tangential component of $()$, and $()^\perp$ is the normal components of $()$.

We will check that the above equation is elliptic. We also derive an equation for the Kähler angle of a symplectic critical surface in a Kähler–Einstein surface.

Theorem 1.2. *If M is a Kähler–Einstein surface and Σ is a symplectic critical surface, then*

$$\Delta\cos\alpha = \frac{3\sin^2\alpha - 2}{\cos\alpha}|\nabla\alpha|^2 - K_0\cos^3\alpha\sin^2\alpha,$$

where K_0 is the scalar curvature of M.

As a corollary, we see that, if the scalar curvature K_0 of the Kähler–Einstein surface M is nonnegative, then a symplectic critical surface in M is holomorphic.

It is not difficult to see that a nonholomorphic symplectic critical surface in a Kähler surface has at most finitely many complex points. Moreover, we can show:

Theorem 1.3. *Suppose that Σ is a nonholomorphic symplectic critical surface in a Kähler surface M. Then*

$$\chi(\Sigma) + \chi(\nu) = -P - \frac{1}{2\pi}\int_\Sigma \frac{|\nabla\alpha|^2}{\cos^2\alpha}\, d\mu,$$

and

$$c_1(M)([\Sigma]) = -P - \frac{1}{2\pi}\int_\Sigma \frac{|\nabla\alpha|^2}{\cos^3\alpha}\, d\mu,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, $\chi(\nu)$ is the Euler characteristic of the normal bundle of Σ in M, $c_1(M)$ is the first Chern class of M, $[\Sigma] \in H_2(M, \mathbb{Z})$ is the homology class of Σ in M, and P is the number of complex tangent points.

From Theorem 1.3, and Webster's formula ([9, Proposition 1]), we have the following theorem.

Theorem 1.4. *Suppose that Σ is a symplectic critical surface in a Kähler surface M. Then*

$$\chi(\Sigma) + \chi(\nu) = c_1(M)([\Sigma]),$$

and Σ is a minimal surface with constant Kähler angle.

If Σ is not compact, the theorem is not true. The rotational symmetric surface $z = -\frac{1}{2}\log(x^2 + y^2)$ in $\mathbb{C}^2$ is a symplectic critical surface which is not minimal.

Recall that a minimal surface with constant Kähler angle is an infinitesimally holomorphic immersion introduced by Micallef–Wolfson [6]. The theorem shows that we may use the variation of the functional L to find infinitesimally holomorphic immersions which are holomorphic in many cases.

Let g be the genus of Σ, I_Σ the self-intersection number of Σ, and D_Σ the number of double points of Σ. Then

$$\chi(\Sigma) = 2 - 2g, \qquad \chi(\nu) = I_\Sigma - 2D_\Sigma.$$

Setting

$$c_1(\Sigma) = c_1(M)([\Sigma]),$$

we have

Theorem 1.5. *Suppose that Σ is a symplectic critical surface in a Kähler surface M. Then*

$$2 - 2g - c_1(\Sigma) + I_\Sigma - 2D_\Sigma = 0.$$

In forthcoming papers, we will use a variational approach and the flow method to study the existence of symplectic critical surfaces in a Kähler surface.

It is natural to conjecture that, *in each homotopy class of symplectic surfaces in a Kähler surface, there is a symplectic critical surface.*

As a starting point for the study of the gradient flow of the function L, we derive the evolution equation of $\cos\alpha$ along the flow, which implies that the symplectic property is preserved.

2. The Euler–Lagrange equation

Let $\{\phi_t\}_{0\le t\le 1}$ be a one-parameter family of immersions $\Sigma \to M$ such that $\phi_0 = F$ and $\Sigma_t = \phi_t(\Sigma)$ are symplectic. Also, let X denote the initial velocity vector for ϕ_t, i.e, $X = \frac{\partial\phi_t}{\partial t}\Big|_{t=0}$. We denote by $\bar\nabla$ the covariant derivative and by K the Riemannian curvature tensor on M. Furthermore, ∇, R denote the covariant derivative and the Riemannian curvature tensor of the induced metric g on the surface Σ_0.

We start by computing the first variation of the area for this one-parameter family of surfaces, which is in fact well-known.

Proposition 2.1. *The variation of the area of Σ_t is*

$$\frac{\partial}{\partial t}\,d\mu_t\bigg|_{t=0} = (\operatorname{div} X^\top - X\cdot H)\,d\mu_t.$$

Proof. The proof is routine (cf. [7]). Fix $p\in\Sigma$. Let $\{x^i\}$ be a normal coordinate system for Σ at p. Around p we choose a local orthonormal frame $\{\tilde e_1,\tilde e_2,\tilde v_3,\tilde v_4\}$ on M along Σ_t such that $\{\tilde e_1=\partial\phi_t/\partial x^i,\tilde e_2=\partial\phi_t/\partial x^2\}$ and $\{\tilde v_3,\tilde v_4\}$ are in the tangent bundle and in the normal bundle of Σ_t respectively. For simplicity, we denote $\tilde e_i(0)=\partial F/\partial x^i$ by e_i and identify it with $\partial_i=\partial/\partial x^i$, $i=1,2$. We also denote $\tilde v_\alpha(0)$ by v_α, $\alpha=3,4$. Furthermore, we assume that $\nabla_{e_i}e_j=0$ at p. Suppose that in this frame X takes the form $X=X^ie_i+X^\alpha v_\alpha$ and $(g_t)_{ij}=\langle\partial\phi_t/\partial x^i,\partial\phi_t/\partial x^j\rangle$. Then

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t}\bigg|_{t=0} &= \frac{\partial}{\partial t}\bigg|_{t=0}\langle\partial\phi_t/\partial x^i,\partial\phi_t/\partial x^j\rangle=\langle\bar\nabla_{e_i}X,e_j\rangle+\langle e_i,\bar\nabla_{e_j}X\rangle\\ &=\langle\bar\nabla_{e_i}(X^ke_k+X^\alpha v_\alpha),e_j\rangle+\langle e_i,\bar\nabla_{e_j}(X^ke_k+X^\alpha v_\alpha)\rangle\\ &=X_{j,i}-X_\alpha h^\alpha_{ij}+X_{i,j}-X_\alpha h^\alpha_{ij}.\end{aligned}$$

It is easy to see that

$$\frac{\partial}{\partial t}\,d\mu_t\bigg|_{t=0}=\frac12 g^{ij}(X_{j,i}-X_\alpha h^\alpha_{ij}+X_{i,j}-X_\alpha h^\alpha_{ij})\,d\mu_t=(\operatorname{div}X^\top-X\cdot H)\,d\mu.\quad\square$$

Theorem 2.2. *Let M be a Kähler surface. The first variational formula of the functional L is, for any smooth vector field X on Σ,*

$$\delta_X L=-2\int_\Sigma\frac{X\cdot H}{\cos\alpha}\,d\mu+2\int_\Sigma\frac{X\cdot(J(J\nabla\cos\alpha)^\top))^\perp}{\cos^4\alpha}\,d\mu,\tag{2.1}$$

where H is the mean curvature vector of Σ in M, $()^\top$ is the tangential component of $()$, and $()^\perp$ is the normal component of $()$. The Euler–Lagrange equation of the functional L is

$$\cos^3\alpha\,H-(J(J\nabla\cos\alpha)^\top)^\perp=0.\tag{2.2}$$

Proof. We use the frame that we have chosen in Proposition 2.1. From the definition of Kähler angle (1.1) we have

$$\cos\alpha_t=\frac{\omega(\partial\phi_t/\partial x^1,\partial\phi_t/\partial x^2)}{\sqrt{\det(g_t)}},$$

where $\det(g_t)$ is the determinant of the metric (g_t). So, the functional can be written as

$$L_t=L(\phi_t)=\int_\Sigma\frac{\det(g_t)}{\omega(\partial\phi_t/\partial x^1,\partial\phi_t/\partial x^2)}\,dx^1\wedge dx^2.$$

Thus using Proposition 2.1 we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} L_t &= \int_\Sigma \left(\frac{\partial_t g_{ij}|_{t=0} g^{ij}}{\omega(e_1, e_2)} - \frac{\partial_t \omega(\partial\phi_t/\partial x^1, \partial\phi_t/\partial x^2)|_{t=0}}{\omega^2(e_1, e_2)} \right) \det(g)\, dx^1 \wedge dx^2 \\ &= \int_\Sigma \frac{2 \operatorname{div} X^\top - 2X \cdot H}{\cos\alpha} \sqrt{\det(g)}\, dx^1 \wedge dx^2 \\ &\quad - \int_\Sigma \frac{\omega(\bar\nabla_{e_1} X, e_2) + \omega(e_1, \bar\nabla_{e_2} X)}{\cos^2\alpha}\, dx^1 \wedge dx^2 \\ &= I + II.\end{aligned}$$

Since Σ is closed, applying the Stokes formula, we obtain

$$I = \int_\Sigma \left(\frac{2\langle X, e_1\rangle \nabla_{e_1} \cos\alpha + 2\langle X, e_2\rangle \nabla_{e_2} \cos\alpha}{\cos^2\alpha} - \frac{2X \cdot H}{\cos\alpha} \right) \sqrt{\det(g)}\, dx^1 \wedge dx^2.$$

The second term is

$$\begin{aligned} II &= -\int_\Sigma \frac{\nabla e_1(\omega(X, e_2)) - \omega(X, \bar\nabla_{e_1} \bar\nabla_{e_2} F)}{\cos^2\alpha}\, dx^1 \wedge dx^2 \\ &\quad - \int_\Sigma \frac{\nabla_{e_2}(\omega(e_1, X)) - \omega(\bar\nabla_{e_2} \bar\nabla_{e_1} F, X)}{\cos^2\alpha}\, dx^1 \wedge dx^2 \\ &= -\int_\Sigma \frac{\nabla_{e_1}(\omega(X, e_2)) + \nabla_{e_2}(\omega(e_1, X))}{\cos^2\alpha}\, dx^1 \wedge dx^2 \\ &= -2 \int_\Sigma \frac{\omega(X, e_2) \nabla_{e_1} \cos\alpha + \omega(e_1, X) \nabla_{e_2} \cos\alpha}{\cos^3\alpha}\, dx^1 \wedge dx^2, \end{aligned}$$

where we have used the fact that ω is parallel. In the following, we compute pointwise so we assume the frame is orthonormal. Note that

$$\begin{aligned} \omega(X^\top, e_2) &= -\langle X^\top, Je_2\rangle = -\langle X, e_1\rangle\langle e_1, Je_2\rangle = \langle X, e_1\rangle \cos\alpha, \\ \omega(e_1, X^\top) &= \langle X^\top, Je_1\rangle = \langle X, e_2\rangle\langle e_2, Je_1\rangle = \langle X, e_2\rangle \cos\alpha. \end{aligned}$$

We separate the second term into two parts,

$$\begin{aligned} II = &-2 \int_\Sigma \left(\frac{\langle X, e_1\rangle \nabla_{e_1} \cos\alpha}{\cos^2\alpha} + \frac{\langle X, e_2\rangle \nabla_{e_2} \cos\alpha}{\cos^2\alpha} \right) d\mu \\ &- 2 \int_\Sigma \frac{\omega(X^\perp, e_2) \nabla_{e_1} \cos\alpha + \omega(e_1, X^\perp) \nabla_{e_2} \cos\alpha}{\cos^3\alpha}\, d\mu. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} L_t = &-2 \int_\Sigma \frac{X \cdot H}{\cos\alpha}\, d\mu - 2 \int_\Sigma \frac{\omega(X^\perp, e_2) \nabla_{e_1} \cos\alpha}{\cos^3\alpha}\, d\mu \\ &- 2 \int_\Sigma \frac{\omega(e_1, X^\perp) \nabla_{e_2} \cos\alpha}{\cos^3\alpha}\, d\mu. \end{aligned}$$

Because

$$\begin{aligned}(J\nabla\cos\alpha)^\top &= (Je_1\nabla_{e_1}\cos\alpha + Je_2\nabla_{e_2}\cos\alpha)^\top\\ &= \langle Je_1, e_2\rangle e_2\nabla_{e_1}\cos\alpha + \langle Je_2, e_1\rangle e_1\nabla_{e_2}\cos\alpha\\ &= (e_2\nabla_{e_1}\cos\alpha - e_1\nabla_{e_2}\cos\alpha)\cos\alpha,\end{aligned}$$

we have

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} L_t &= -2\int_\Sigma \frac{X\cdot H}{\cos\alpha}\,d\mu - 2\int_\Sigma \frac{\omega(X^\perp, (J\nabla\cos\alpha)^\top)}{\cos^4\alpha}\,d\mu\\ &= -2\int_\Sigma \frac{X\cdot H}{\cos\alpha}\,d\mu + 2\int_\Sigma \frac{X\cdot(J(J\nabla\cos\alpha)^\top))^\perp}{\cos^4\alpha}\,d\mu.\end{aligned}$$

This completes the proof of the theorem. □

For a later purpose, and to understand the equation, we express $(J(J\nabla\cos\alpha)^\top))^\perp$ at a fixed point p in a local frame. Let $\{e_1, e_2, v_3, v_4\}$ be an orthonormal frame around $p\in\Sigma$ that is normal at p and such that ω, J take the forms (cf. [3], [4])

$$\omega = \cos\alpha u_1\wedge u_2 + \cos\alpha\, u_3\wedge u_4 + \sin\alpha\, u_1\wedge u_3 - \sin\alpha\, u_2\wedge u_4 \tag{2.3}$$

where $\{u_1, u_2, u_3, u_4\}$ is the dual frame of $\{e_1, e_2, v_3, v_4\}$, and

$$J = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0\\ -\cos\alpha & 0 & 0 & -\sin\alpha\\ -\sin\alpha & 0 & 0 & \cos\alpha\\ 0 & \sin\alpha & -\cos\alpha & 0\end{pmatrix}. \tag{2.4}$$

Then

$$\begin{aligned}(J(J\nabla\cos\alpha)^\top))^\perp &= (J(\cos\alpha\,\partial_1\cos\alpha\, e_2 - \cos\alpha\,\partial_2\cos\alpha\, e_1))^\perp\\ &= -\cos\alpha\sin\alpha\,\partial_1\cos\alpha\, v_4 - \cos\alpha\sin\alpha\,\partial_2\cos\alpha\, v_3\\ &= \cos\alpha\sin^2\alpha\,\partial_1\alpha\, v_4 + \cos\alpha\sin^2\alpha\,\partial_2\alpha\, v_3.\end{aligned}$$

Furthermore,

$$\begin{aligned}\partial_1\cos\alpha &= \omega(\bar\nabla_{e_1}e_1, e_2) + \omega(e_1, \bar\nabla_{e_1}e_2) = h^\alpha_{11}\langle Jv_\alpha, e_2\rangle + h^\alpha_{12}\langle Je_1, v_\alpha\rangle\\ &= (h^4_{11} + h^3_{12})\sin\alpha.\end{aligned}$$

Similarly, we can get

$$\partial_2\cos\alpha = (h^3_{22} + h^4_{12})\sin\alpha.$$

Set $V = \partial_2\alpha\, v_3 + \partial_1\alpha\, v_4$. Then

$$V = -(h^3_{22} + h^4_{12})v_3 - (h^4_{11} + h^3_{12})v_4. \tag{2.5}$$

Consequently, the Euler–Lagrange equation of the functional L is

$$\cos^2\alpha\, H - \sin^2\alpha\, V = 0, \tag{2.6}$$

from which we see that $H = \sin\alpha \cdot g$ and g is a smooth vector field on $N\Sigma$, or equivalently,

$$H - \sin^2\alpha\,(H + V) = 0.$$

It is not difficult to see that, roughly, the symbol of the equation is

$$\sigma := \begin{pmatrix} (1-\sin^2\alpha)\xi^2 + \eta^2 & (\sin^2\alpha)\xi\eta \\ (\sin^2\alpha)\xi\eta & \xi^2 + (1-\sin^2\alpha)\eta^2 \end{pmatrix},$$

which makes one believe that the equation (2.2) is elliptic.

In the following, we give a detailed proof.

Theorem 2.3. *The equation* (2.2) *is elliptic.*

Proof. Assume that Σ is immersed in M by F. Let $\{x, y\}$ be a coordinate system around $p \in \Sigma$. Since Σ is smooth, by the implicit function theorem we can write Σ as the graph of two functions f, g in a small neighborhood U of p, i.e., $F = (x, y, f(x, y), g(x, y))$ in U. Suppose that the complex structure in the neighborhood of $F(p)$ is standard, i.e.,

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We choose $e_1 = \partial F/\partial x = (1, 0, f_x, g_x)$, $e_2 = \partial F/\partial y = (0, 1, f_y, g_y)$ and $v_3 = (-f_x, -f_y, 1, 0)$, $v_4 = (-g_x, -g_y, 0, 1)$. Then $\{e_1, e_2, v_3, v_4\}$ is a basis of M. The metric of Σ in this basis is

$$(g_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 1 + f_x^2 + g_x^2 & f_x f_y + g_x g_y \\ f_x f_y + g_x g_y & 1 + f_y^2 + g_y^2 \end{pmatrix}$$

and the inverse matrix is

$$(g^{ij})_{1\le i,j\le 2} = \frac{1}{\det(g_{ij})}\begin{pmatrix} 1 + f_y^2 + g_y^2 & -f_x f_y - g_x g_y \\ -f_x f_y - g_x g_y & 1 + f_x^2 + g_x^2 \end{pmatrix}.$$

Moreover, the metric on the normal bundle $T^\perp\Sigma$ is

$$(g_{\alpha\beta})_{3\le\alpha,\beta\le 4} = \begin{pmatrix} 1 + f_x^2 + f_y^2 & f_x g_x + f_y g_y \\ f_x g_x + f_y g_y & 1 + g_x^2 + g_y^2 \end{pmatrix}$$

and the inverse matrix is

$$(g^{\alpha\beta})_{3\le\alpha,\beta\le 4} = \frac{1}{\det(g_{\alpha\beta})}\begin{pmatrix} 1 + g_x^2 + g_y^2 & -f_x g_x - f_y g_y \\ -f_x g_x - f_y g_y & 1 + f_x^2 + f_y^2 \end{pmatrix}.$$

By a direct computation, we have

$$\det(g_{ij}) = 1 + f_x^2 + f_y^2 + g_x^2 + g_y^2 + (f_x g_y - f_y g_x)^2$$

and

$$\det(g_{\alpha\beta}) = 1 + f_x^2 + f_y^2 + g_x^2 + g_y^2 + (f_x g_y - f_y g_x)^2 = \det(g_{ij}).$$

From now on, we denote

$$\det(g) = \det(g_{ij}) = \det(g_{\alpha\beta}).$$

Using our choice of J and the basis, we have

$$\begin{aligned} Je_1 &= (0, 1, -g_x, f_x), \quad & Je_2 &= (-1, 0, -g_y, f_y), \\ Jv_3 &= (f_y, -f_x, 0, 1), \quad & Jv_4 &= (g_y, -g_x, -1, 0). \end{aligned}$$

It is easy to see that

$$\cos\alpha = \frac{w(e_1, e_2)}{\sqrt{\det(g)}} = \frac{\langle Je_1, e_2\rangle}{\sqrt{\det(g)}} = \frac{1 + f_x g_y - f_y g_x}{\sqrt{\det(g)}}$$

and

$$\sin^2\alpha = \frac{\det(g) - \langle Je_1, e_2\rangle^2}{\det(g)} = \frac{(f_x - g_y)^2 + (f_y + g_x)^2}{\det(g)}.$$

We now express the Euler–Lagrangian equation (2.2) of L explicitly. One checks that

$$\begin{aligned} \nabla\cos\alpha &= g^{ij}\langle\nabla\cos\alpha, e_i\rangle e_j \\ &= \left(g^{11}\frac{\partial\cos\alpha}{\partial x} + g^{12}\frac{\partial\cos\alpha}{\partial y}\right)e_1 + \left(g^{12}\frac{\partial\cos\alpha}{\partial x} + g^{22}\frac{\partial\cos\alpha}{\partial y}\right)e_2 \\ &=: Ae_1 + Be_2. \end{aligned}$$

Note that

$$\begin{aligned} (Je_1)^\top &= g^{ij}\langle Je_1, e_i\rangle e_j = g^{2j}\langle Je_1, e_2\rangle e_j = \langle Je_1, e_2\rangle(g^{21}e_1 + g^{22}e_2), \\ (Je_2)^\top &= g^{ij}\langle Je_2, e_i\rangle e_j = g^{1j}\langle Je_2, e_1\rangle e_j = -\langle Je_1, e_2\rangle(g^{11}e_1 + g^{12}e_2). \end{aligned}$$

We obtain

$$\begin{aligned} (J\nabla\cos\alpha)^\top &= A(Je_1)^\top + B(Je_2)^\top \\ &= \langle Je_1, e_2\rangle[A(g^{21}e_1 + g^{22}e_2) - B(g^{11}e_1 + g^{12}e_2)] \\ &= \langle Je_1, e_2\rangle[(Ag^{21} - Bg^{11})e_1 + (Ag^{22} - Bg^{12})e_2)]. \end{aligned}$$

Moreover,

$$\begin{aligned} (Je_1)^\perp &= g^{\alpha\beta}\langle Je_1, v_\alpha\rangle v_\beta \\ &= (g^{33}\langle Je_1, v_3\rangle + g^{43}\langle Je_1, v_4\rangle)v_3 + (g^{34}\langle Je_1, v_3\rangle + g^{44}\langle Je_1, v_4\rangle)v_4, \\ (Je_2)^\perp &= g^{\alpha\beta}\langle Je_2, v_\alpha\rangle v_\beta \\ &= (g^{33}\langle Je_2, v_3\rangle + g^{43}\langle Je_2, v_4\rangle)v_3 + (g^{34}\langle Je_2, v_3\rangle + g^{44}\langle Je_2, v_4\rangle)v_4. \end{aligned}$$

It follows that

$$\begin{aligned}
(J(J\nabla\cos\alpha)^{\top})^{\perp} &= \langle Je_1, e_2\rangle[(Ag^{21}-Bg^{11})(Je_1)^{\perp}+(Ag^{22}-Bg^{12})(Je_2)^{\perp}] \\
&= \langle Je_1, e_2\rangle\{(Ag^{21}-Bg^{11})[(g^{33}\langle Je_1, v_3\rangle+g^{43}\langle Je_1, v_4\rangle)v_3 \\
&\quad +(g^{34}\langle Je_1, v_3\rangle+g^{44}\langle Je_1, v_4\rangle)v_4] \\
&\quad +(Ag^{22}-Bg^{12})[(g^{33}\langle Je_2, v_3\rangle+g^{43}\langle Je_2, v_4\rangle)v_3 \\
&\quad +(g^{34}\langle Je_2, v_3\rangle+g^{44}\langle Je_2, v_4\rangle)v_4]\} \\
&= \langle Je_1, e_2\rangle\{[(Ag^{21}-Bg^{11})(g^{33}\langle Je_1, v_3\rangle+g^{43}\langle Je_1, v_4\rangle) \\
&\quad +(Ag^{22}-Bg^{12})(g^{33}\langle Je_2, v_3\rangle+g^{43}\langle Je_2, v_4\rangle)]v_3 \\
&\quad +[(Ag^{21}-Bg^{11})(g^{34}\langle Je_1, v_3\rangle+g^{44}\langle Je_1, v_4\rangle) \\
&\quad +(Ag^{22}-Bg^{12})(g^{34}\langle Je_2, v_3\rangle+g^{44}\langle Je_2, v_4\rangle)]v_4\}, \qquad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
H &= H^{\alpha}v_{\alpha} = g^{ij}h^{\alpha}_{ij}v_{\alpha} = g^{ij}\left(\frac{\partial^2 F}{\partial x_i\partial x_j}\right)^{\perp} \\
&= g^{ij}g^{\alpha\beta}\left\langle\frac{\partial^2 F}{\partial x_i\partial x_j}, v_{\alpha}\right\rangle v_{\beta} \\
&= \left(g^{33}g^{ij}\left\langle\frac{\partial^2 F}{\partial x_i\partial x_j}, v_3\right\rangle+g^{43}g^{ij}\left\langle\frac{\partial^2 F}{\partial x_i\partial x_j}, v_4\right\rangle\right)v_3 \\
&\quad +\left(g^{34}g^{ij}\left\langle\frac{\partial^2 F}{\partial x_i\partial x_j}, v_3\right\rangle+g^{44}g^{ij}\left\langle\frac{\partial^2 F}{\partial x_i\partial x_j}, v_4\right\rangle\right)v_4 \\
&= \left[g^{33}\left(g^{11}\left\langle\frac{\partial^2 F}{\partial x^2}, v_3\right\rangle+2g^{12}\left\langle\frac{\partial^2 F}{\partial x\partial y}, v_3\right\rangle+g^{22}\left\langle\frac{\partial^2 F}{\partial y^2}, v_3\right\rangle\right)\right. \\
&\quad \left.+g^{43}\left(g^{11}\left\langle\frac{\partial^2 F}{\partial x^2}, v_4\right\rangle+2g^{12}\left\langle\frac{\partial^2 F}{\partial x\partial y}, v_4\right\rangle+g^{22}\left\langle\frac{\partial^2 F}{\partial y^2}, v_4\right\rangle\right)\right]v_3 \\
&\quad +\left[g^{34}\left(g^{11}\left\langle\frac{\partial^2 F}{\partial x^2}, v_3\right\rangle+2g^{12}\left\langle\frac{\partial^2 F}{\partial x\partial y}, v_3\right\rangle+g^{22}\left\langle\frac{\partial^2 F}{\partial y^2}, v_3\right\rangle\right)\right. \\
&\quad \left.+g^{44}\left(g^{11}\left\langle\frac{\partial^2 F}{\partial x^2}, v_4\right\rangle+2g^{12}\left\langle\frac{\partial^2 F}{\partial x\partial y}, v_4\right\rangle+g^{22}\left\langle\frac{\partial^2 F}{\partial y^2}, v_4\right\rangle\right)\right]v_4. \qquad (2.8)
\end{aligned}$$

By (2.7) and (2.8) we obtain

$$\begin{aligned}
&\cos^3\alpha\left[g^{33}\left(g^{11}\left\langle\frac{\partial^2 F}{\partial x^2}, v_3\right\rangle+2g^{12}\left\langle\frac{\partial^2 F}{\partial x\partial y}, v_3\right\rangle+g^{22}\left\langle\frac{\partial^2 F}{\partial y^2}, v_3\right\rangle\right)\right. \\
&\qquad\qquad \left.+g^{43}\left(g^{11}\left\langle\frac{\partial^2 F}{\partial x^2}, v_4\right\rangle+2g^{12}\left\langle\frac{\partial^2 F}{\partial x\partial y}, v_4\right\rangle+g^{22}\left\langle\frac{\partial^2 F}{\partial y^2}, v_4\right\rangle\right)\right] \\
&-\langle Je_1, e_2\rangle[(Ag^{12}-Bg^{11})(g^{33}\langle Je_1, v_3\rangle+g^{43}\langle Je_1, v_4\rangle) \\
&\qquad\qquad +(Ag^{22}-Bg^{12})(g^{33}\langle Je_2, v_3\rangle+g^{43}\langle Je_2, v_4\rangle)] = 0, \qquad (2.9)
\end{aligned}$$

and

$$\cos^3\alpha\left[g^{34}\left(g^{11}\left\langle\frac{\partial^2 F}{\partial x^2}, v_3\right\rangle + 2g^{12}\left\langle\frac{\partial^2 F}{\partial x\partial y}, v_3\right\rangle + g^{22}\left\langle\frac{\partial^2 F}{\partial y^2}, v_3\right\rangle\right)\right.$$
$$\left. + g^{44}\left(g^{11}\left\langle\frac{\partial^2 F}{\partial x^2}, v_4\right\rangle + 2g^{12}\left\langle\frac{\partial^2 F}{\partial x\partial y}, v_4\right\rangle + g^{22}\left\langle\frac{\partial^2 F}{\partial y^2}, v_4\right\rangle\right)\right]$$
$$- \langle Je_1, e_2\rangle[(Ag^{12} - Bg^{11})(g^{34}\langle Je_1, v_3\rangle + g^{44}\langle Je_1, v_4\rangle)$$
$$+ (Ag^{22} - Bg^{12})(g^{34}\langle Je_2, v_3\rangle + g^{44}\langle Je_2, v_4\rangle)] = 0. \quad (2.10)$$

Set $c = 1 + f_x g_y - f_y g_x = \sqrt{\det(g)}\cos\alpha$, $a = f_y + g_x$, $b = f_x - g_y$. Recalling that

$$F = (x, y, f(x, y), g(x, y)),$$

we have

$$\left\langle\frac{\partial^2 F}{\partial x^2}, v_3\right\rangle = f_{xx}, \quad \left\langle\frac{\partial^2 F}{\partial x\partial y}, v_3\right\rangle = f_{xy}, \quad \left\langle\frac{\partial^2 F}{\partial y^2}, v_3\right\rangle = f_{yy},$$
$$\left\langle\frac{\partial^2 F}{\partial x^2}, v_4\right\rangle = g_{xx}, \quad \left\langle\frac{\partial^2 F}{\partial x\partial y}, v_4\right\rangle = g_{xy}, \quad \left\langle\frac{\partial^2 F}{\partial y^2}, v_4\right\rangle = g_{yy}.$$

Moreover,

$$\langle Je_1, v_3\rangle = -f_y - g_x = -a, \quad \langle Je_1, v_4\rangle = f_x - g_y = b,$$
$$\langle Je_2, v_3\rangle = f_x - g_y = b, \quad \langle Je_2, v_4\rangle = f_y + g_x = a,$$
$$\langle Je_1, e_2\rangle = \sqrt{\det(g)}\cos\alpha = c.$$

Then equations (2.9) and (2.10) can be written as

$$\frac{c^2}{(\sqrt{\det(g)})^3}[g^{33}(g^{11}f_{xx} + 2g^{12}f_{xy} + g^{22}f_{yy}) + g^{43}(g^{11}g_{xx} + 2g^{12}g_{xy} + g^{22}g_{yy})]$$
$$-\{[g^{12}(g^{34}b - g^{33}a) + g^{22}(g^{33}b + g^{34}a)]A - [g^{11}(g^{34}b - g^{33}a) + g^{12}(g^{33}b + g^{34}a)]B\} = 0$$

and

$$\frac{c^2}{(\sqrt{\det(g)})^3}[g^{34}(g^{11}f_{xx} + 2g^{12}f_{xy} + g^{22}f_{yy}) + g^{44}(g^{11}g_{xx} + 2g^{12}g_{xy} + g^{22}g_{yy})]$$
$$-\{[g^{12}(g^{44}b - g^{34}a) + g^{22}(g^{34}b + g^{44}a)]A - [g^{11}(g^{44}b - g^{34}a) + g^{12}(g^{34}b + g^{44}a)]B\} = 0.$$

Recalling that

$$A = g^{11}\frac{\partial\cos\alpha}{\partial x} + g^{12}\frac{\partial\cos\alpha}{\partial y}, \quad B = g^{12}\frac{\partial\cos\alpha}{\partial x} + g^{22}\frac{\partial\cos\alpha}{\partial y},$$

we have

$$[g^{12}(g^{34}b - g^{33}a) + g^{22}(g^{33}b + g^{34}a)]A - [g^{11}(g^{34}b - g^{33}a) + g^{12}(g^{33}b + g^{34}a)]B$$
$$= (g^{11}g^{22} - g^{12}g^{12})(g^{33}b + g^{34}a)\frac{\partial \cos\alpha}{\partial x} + (g^{11}g^{22} - g^{12}g^{12})(g^{33}a - g^{34}b)\frac{\partial \cos\alpha}{\partial y},$$

and

$$[g^{12}(g^{44}b - g^{34}a) + g^{22}(g^{34}b + g^{44}a)]A - [g^{11}(g^{44}b - g^{34}a) + g^{12}(g^{34}b + g^{44}a)]B$$
$$= (g^{11}g^{22} - g^{12}g^{12})(g^{34}b + g^{44}a)\frac{\partial \cos\alpha}{\partial x} + (g^{11}g^{22} - g^{12}g^{12})(g^{34}a - g^{44}b)\frac{\partial \cos\alpha}{\partial y}.$$

Therefore, equations (2.9) and (2.10) can be rewritten as

$$\frac{c^2}{(\sqrt{\det(g)})^3}[g^{33}(g^{11}f_{xx} + 2g^{12}f_{xy} + g^{22}f_{yy}) + g^{43}(g^{11}g_{xx} + 2g^{12}g_{xy} + g^{22}g_{yy})]$$
$$-\left[(g^{11}g^{22}-g^{12}g^{12})(g^{33}b+g^{34}a)\frac{\partial \cos\alpha}{\partial x}+(g^{11}g^{22}-g^{12}g^{12})(g^{33}a-g^{34}b)\frac{\partial \cos\alpha}{\partial y}\right] = 0 \tag{2.11}$$

and

$$\frac{c^2}{(\sqrt{\det(g)})^3}[g^{34}(g^{11}f_{xx} + 2g^{12}f_{xy} + g^{22}f_{yy}) + g^{44}(g^{11}g_{xx} + 2g^{12}g_{xy} + g^{22}g_{yy})]$$
$$-\left[(g^{11}g^{22}-g^{12}g^{12})(g^{34}b+g^{44}a)\frac{\partial \cos\alpha}{\partial x}+(g^{11}g^{22}-g^{12}g^{12})(g^{34}a-g^{44}b)\frac{\partial \cos\alpha}{\partial y}\right] = 0. \tag{2.12}$$

On the other hand, we have

$$\begin{aligned}
\frac{\partial \cos\alpha}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{1 + f_x g_y - f_y g_x}{\sqrt{\det(g)}}\right] \\
&= \frac{1}{(\sqrt{\det(g)})^3}[f_{xx}(g_y - f_x + f_y g_x g_y + f_x f_y g_y + g_y f_y^2 + g_y g_x^2 + g_y^3 - f_x g_y^2) \\
&\quad + f_{xy}(-g_x - f_y + f_y g_x g_y - f_x f_y g_y - g_x f_x^2 - g_x g_y^2 - g_x^3 - f_y g_x^2) \\
&\quad + g_{xx}(-g_x - f_y + f_x f_y g_y - f_x g_x g_y - f_y f_x^2 - f_y g_y^2 - f_y^3 - g_x f_y^2) \\
&\quad + g_{xy}(f_x - g_y + f_x f_y g_x + f_y g_x g_y + f_x g_x^2 + f_x f_y^2 + f_x^3 - g_y f_x^2)] \\
&= \frac{1}{(\sqrt{\det(g)})^3}[f_{xx}(g_{12}a - g_{22}b) + f_{xy}(g_{12}b - g_{11}a) \\
&\quad + g_{xx}(-g_{22}a - g_{12}b) + g_{xy}(g_{11}b + g_{12}a)].
\end{aligned} \tag{2.13}$$

Similarly, one checks that

$$\begin{aligned}\frac{\partial\cos\alpha}{\partial y}=\frac{1}{(\sqrt{\det(g)})^3}&[f_{yy}(-g_{11}a+g_{12}b)+f_{xy}(g_{12}a-g_{22}b)\\&+g_{yy}(g_{11}b+g_{12}a)+g_{xy}(-g_{22}a-g_{12}b)].\end{aligned}\tag{2.14}$$

By (2.11)–(2.14), we see that the system (2.2) is

$$\begin{aligned}&f_{xx}[g^{11}g^{33}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{34}a+g^{33}b)(g_{12}a-g_{22}b)]\\&+f_{xy}[2g^{33}g^{12}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{34}a+g^{33}b)(g_{12}b-g_{11}a)\\&\qquad-(g^{11}g^{22}-g^{12}g^{12})(g^{33}a-g^{34}b)(g_{12}a-g_{22}b)]\\&+f_{yy}[g^{22}g^{33}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{33}a-g^{34}b)(-g_{11}a+g_{12}b)]\\&+g_{xx}[g^{11}g^{34}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{34}a+g^{33}b)(-g_{22}a-g_{12}b)]\\&+g_{xy}[2g^{34}g^{12}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{34}a+g^{33}b)(g_{11}b+g_{12}a)\\&\qquad-(g^{11}g^{22}-g^{12}g^{12})(g^{33}a-g^{34}b)(-g_{22}a-g_{12}b)]\\&+g_{yy}[g^{22}g^{34}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{33}a-g^{34}b)(g_{11}b+g_{12}a)]=0,\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}&f_{xx}[g^{11}g^{34}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{44}a+g^{34}b)(g_{12}a-g_{22}b)]\\&+f_{xy}[2g^{34}g^{12}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{44}a+g^{34}b)(g_{12}b-g_{11}a)\\&\qquad-(g^{11}g^{22}-g^{12}g^{12})(g^{34}a-g^{44}b)(g_{12}a-g_{22}b)]\\&+f_{yy}[g^{22}g^{34}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{34}a-g^{44}b)(-g_{11}a+g_{12}b)]\\&+g_{xx}[g^{11}g^{44}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{44}a+g^{34}b)(-g_{22}a-g_{12}b)]\\&+g_{xy}[2g^{44}g^{12}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{44}a+g^{34}b)(g_{11}b+g_{12}a)\\&\qquad-(g^{11}g^{22}-g^{12}g^{12})(g^{34}a-g^{44}b)(-g_{22}a-g_{12}b)]\\&+g_{yy}[g^{22}g^{44}c^2-(g^{11}g^{22}-g^{12}g^{12})(g^{34}a-g^{44}b)(g_{11}b+g_{12}a)]=0.\end{aligned}\tag{2.16}$$

For simplicity, we write the system as

$$\begin{aligned}A_{11}f_{xx}+A_{12}f_{xy}+A_{22}f_{yy}+B_{11}g_{xx}+B_{12}g_{xy}+B_{22}g_{yy}=0,\\C_{11}f_{xx}+C_{12}f_{xy}+C_{22}f_{yy}+D_{11}g_{xx}+D_{12}g_{xy}+D_{22}g_{yy}=0,\end{aligned}$$

where A_{ij}, B_{ij}, C_{ij} and D_{ij} are defined clearly ($i, j = 1, 2$). So the symbol of the system is

$$\sigma:=\begin{pmatrix}A_{11}\xi^2+A_{12}\xi\eta+A_{22}\eta^2 & B_{11}\xi^2+B_{12}\xi\eta+B_{22}\eta^2\\C_{11}\xi^2+C_{12}\xi\eta+C_{22}\eta^2 & D_{11}\xi^2+D_{12}\xi\eta+D_{22}\eta^2.\end{pmatrix}$$

A direct computation yields

$$\begin{aligned}
\det(\sigma) &= c^4(g^{33}g^{44}-g^{34}g^{34})[(g^{11})^2\xi^4+(g^{22})^2\eta^4+4g^{11}g^{12}\xi^3\eta+4g^{12}g^{22}\xi\eta^3\\
&\quad +2g^{11}g^{22}\xi^2\eta^2+4(g^{12})^2\xi^2\eta^2]\\
&\quad +c^2(a^2+b^2)(g^{11}g^{22}-g^{12}g^{12})(g^{33}g^{44}-g^{34}g^{34})(g^{11}g_{22}\xi^4+g^{22}g_{11}\eta^4\\
&\quad +2g^{12}g_{22}\xi^3\eta-2g^{11}g_{12}\xi^3\eta+2g^{12}g_{11}\xi\eta^3-2g^{22}g_{12}\xi\eta^3+g^{11}g_{11}\xi^2\eta^2\\
&\quad +g^{22}g_{22}\xi^2\eta^2-4g^{12}g_{12}\xi^2\eta^2)\\
&= c^4(g^{33}g^{44}-g^{34}g^{34})(g^{11}\xi^2+2g^{12}\xi\eta+g^{22}\eta^2)^2\\
&\quad +\frac{c^2(a^2+b^2)}{\det(g)}(g^{11}g^{22}-g^{12}g^{12})(g^{33}g^{44}-g^{34}g^{34})(g_{22}^2\xi^4+g_{11}^2\eta^4\\
&\quad -4g_{12}g_{22}\xi^3\eta-4g_{11}g_{12}\xi\eta^3+2g_{11}g_{22}\xi^2\eta^2+4g_{12}^2\xi^2\eta^2)\\
&= \frac{c^4}{\det(g)}(g^{11}\xi^2+2g^{12}\xi\eta+g^{22}\eta^2)^2+\frac{c^2(a^2+b^2)}{(\det(g))^3}(g_{22}\xi^2-2g_{12}\xi\eta+g_{11}\eta^2)^2\\
&= \frac{c^2(a^2+b^2+c^2)}{(\det(g))^3}(g_{22}\xi^2-2g_{12}\xi\eta+g_{11}\eta^2)^2\\
&= \frac{c^2}{(\det g)^2}(g_{22}\xi^2-2g_{12}\xi\eta+g_{11}\eta^2)^2=\frac{\cos^2\alpha}{\det g}(g_{22}\xi^2-2g_{12}\xi\eta+g_{11}\eta^2)^2.
\end{aligned}$$

Because (g_{ij}) is positive definite and $\cos\alpha > 0$, we see that

$$\det(\sigma) > 0 \quad \text{if } (\xi,\eta) \neq (0,0),$$

which implies that equation (2.2) is elliptic. This proves the theorem. □

3. Equations of the Kähler angle of a symplectic critical surface

In the following, we always choose the orthonormal basis $\{e_1, e_2, v_3, v_4\}$ on M along Σ such that $\{e_1, e_2\}$ is a basis of Σ and ω takes the form (2.3), and the complex structure J on M takes the form (2.4).

Let $T\Sigma$, $N\Sigma$ be the tangent and normal bundles of Σ in M respectively. The second fundamental form $A : T\Sigma \times T\Sigma \to N\Sigma$ is defined by $A(X,Y) = (\bar\nabla_X Y)^\perp$ for any tangent vector fields X, Y. The operator $B : T\Sigma \times N\Sigma \to T\Sigma$ is defined by $B(X,N) = (\bar\nabla_X N)^\top$, $N \in N\Sigma$. Here $()^\top$ denotes the projection from TM onto $T\Sigma$ and $()^\perp$ denotes the projection onto $N\Sigma$. Evidently,

$$\langle A(X,Y), N\rangle = -\langle Y, B(X,N)\rangle.$$

Proposition 3.1. *Let M be a Kähler surface with Kähler form ω and J be the complex structure compatible with ω on M. If Σ is a symplectic surface which is smoothly immersed in M with Kähler angle α, then*

$$\begin{aligned}
\Delta\cos\alpha &= \cos\alpha\,(-|h^3_{1k}-h^4_{2k}|^2-|h^4_{1k}+h^3_{2k}|^2)\\
&\quad +\sin\alpha\,(H^4_{,1}+H^3_{,2})-\frac{\sin^2\alpha}{\cos\alpha}(K_{1212}+K_{1234}).
\end{aligned}\tag{3.1}$$

where K is the curvature operator of M and $H^\alpha_{,i} = \langle\bar\nabla^N_{e_i}H, v_\alpha\rangle$.

Proof. It is evident that

$$\Delta \cos\alpha = \Delta \frac{\omega(e_1, e_2)}{\sqrt{\det(g)}} = \Delta\omega(e_1, e_2) - \frac{1}{2}\cos\alpha\, \Delta g_{ij} g^{ij}. \tag{3.2}$$

Using the property that $\bar{\nabla}\omega = 0$, we obtain

$$\begin{aligned}
\Delta\omega(e_1, e_2) &= \bar{\nabla}_{e_k}\bar{\nabla}_{e_k}\omega(e_1, e_2) \\
&= \omega(\bar{\nabla}_{e_k}\bar{\nabla}_{e_k}e_1, e_2) - \omega(\bar{\nabla}_{e_k}\bar{\nabla}_{e_k}e_2, e_1) + 2\omega(\bar{\nabla}_{e_k}e_1, \bar{\nabla}_{e_k}e_2) \\
&= \omega(\bar{\nabla}_{e_k}(\nabla_{e_k}e_1 + A(e_k, e_1)), e_2) - \omega(\bar{\nabla}_{e_k}(\nabla_{e_k}e_2 + A(e_k, e_2)), e_1) \\
&\quad + 2\omega(A(e_k, e_1), A(e_k, e_2)) \\
&= \omega(\nabla_{e_k}\nabla_{e_k}e_1, e_2) + \omega(A(e_k, \nabla_{e_k}e_1), e_2) + \omega(\bar{\nabla}_{e_k}A(e_k, e_1), e_2) \\
&\quad - \omega(\nabla_{e_k}\nabla_{e_k}e_2, e_1) + \omega(A(e_k, \nabla_{e_k}e_2), e_1) - \omega(\bar{\nabla}_{e_k}A(e_k, e_2), e_1) \\
&\quad + 2\omega(A(e_k, e_1), A(e_k, e_2)) \\
&= \cos\alpha\,\langle\nabla_{e_k}\nabla_{e_k}e_1, e_1\rangle + \cos\alpha\,\langle\nabla_{e_k}\nabla_{e_k}e_2, e_2\rangle \\
&\quad + \omega(\bar{\nabla}_{e_k}A(e_k, e_1), e_2) - \omega(\bar{\nabla}_{e_k}A(e_k, e_2), e_1) \\
&\quad + 2\omega(A(e_k, e_1), A(e_k, e_2)).
\end{aligned}$$

It is not hard to check that

$$\begin{aligned}
\frac{1}{2}\cos\alpha\,\Delta g_{ij} g^{ij} &= \frac{1}{2}\cos\alpha\,\Delta\langle e_i, e_j\rangle g^{ij} = \cos\alpha\,\langle\nabla_{e_k}\nabla_{e_k}e_i, e_j\rangle g^{ij} \\
&= \cos\alpha\,\langle\nabla_{e_k}\nabla_{e_k}e_1, e_1\rangle + \cos\alpha\,\langle\nabla_{e_k}\nabla_{e_k}e_2, e_2\rangle.
\end{aligned}$$

Putting the last two identities into (3.2) and using (2.3), we obtain

$$\begin{aligned}
\Delta\cos\alpha &= \omega(\bar{\nabla}_{e_k}A(e_k, e_1), e_2) - \omega(\bar{\nabla}_{e_k}A(e_k, e_2), e_1) \\
&\quad + 2\omega(A(e_k, e_1), A(e_k, e_2)) \\
&= \omega(\bar{\nabla}_{e_k}(h^\alpha_{1k}v_\alpha), e_2) - \omega(\bar{\nabla}_{e_k}(h^\alpha_{2k}v_\alpha), e_1) + 2\omega(h^\alpha_{1k}v_\alpha, h^\beta_{2k}v_\beta) \\
&= \omega(h^\alpha_{k1,k}v_\alpha - h^\alpha_{k1}h^\alpha_{kl}e_l, e_2) - \omega(h^\alpha_{k2,k}v_\alpha - h^\alpha_{k2}h^\alpha_{kl}e_l, e_1) \\
&\quad + 2\omega(h^\alpha_{1k}v_\alpha, h^\beta_{2k}v_\beta) \\
&= \cos\alpha\,(-(h^\alpha_{1k})^2 - (h^\alpha_{2k})^2 + 2h^3_{1k}h^4_{2k} - 2h^4_{1k}h^3_{2k}) \\
&\quad + \omega(h^\alpha_{kk,1}v_\alpha - K_{\alpha k1k}v_\alpha, e_2) - \omega(h^\alpha_{kk,2}v_\alpha - K_{\alpha k2k}v_\alpha, e_1) \\
&= \cos\alpha\,(-(h^\alpha_{1k})^2 - (h^\alpha_{2k})^2 + 2h^3_{1k}h^4_{2k} - 2h^4_{1k}h^3_{2k}) \\
&\quad + \sin\alpha\,(H^4_{,1} + H^3_{,2}) - \sin\alpha\,(K_{4k1k} + K_{3k2k}) \\
&= \cos\alpha\,(-(h^\alpha_{1k})^2 - (h^\alpha_{2k})^2 + 2h^3_{1k}h^4_{2k} - 2h^4_{1k}h^3_{2k}) \\
&\quad + \sin\alpha\,(H^4_{,1} + H^3_{,2}) - \sin\alpha\,(K_{1213} - K_{1224}).
\end{aligned} \tag{3.3}$$

Since J is integrable, we have

$$\begin{aligned}
K_{1212} &= K(e_1, e_2, Je_1, Je_2) = K(e_1, e_2, \cos\alpha\, e_2 + \sin\alpha\, v_3, -\cos\alpha\, e_1 - \sin\alpha\, v_4) \\
&= \cos^2\alpha\, K_{1212} - \sin^2\alpha K_{1234} + \sin\alpha\cos\alpha\,(K_{1213} - K_{1224}).
\end{aligned}$$

We therefore obtain

$$\sin\alpha\,(K_{1212}+K_{1234})=\cos\alpha\,(K_{1213}-K_{1224}). \tag{3.4}$$

Then adding (3.4) into (3.3), we get

$$\begin{aligned}\Delta\cos\alpha &= \cos\alpha\,(-(h^{\alpha}_{1k})^2-(h^{\alpha}_{2k})^2+2h^3_{1k}h^4_{2k}-2h^4_{1k}h^3_{2k})\\ &\quad+\sin\alpha\,(H^4_{,1}+H^3_{,2})-\frac{\sin^2\alpha}{\cos\alpha}(K_{1212}+K_{1234}).\end{aligned}$$

This proves the proposition. □

Lemma 3.2. *Suppose that M is a Kähler surface. Then*

$$\mathrm{Ric}(Je_1,e_2)=\frac{1}{\cos\alpha}(K_{1212}+K_{1234}). \tag{3.5}$$

Proof. From the Bianchi identity we see that

$$\begin{aligned}\mathrm{Ric}(Je_1,e_2) &= \sum_{A=1}^{4}K(Je_1,e_A,e_2,e_A)=\sum_{A=1}^{4}K(Je_1,e_A,Je_2,Je_A)\\ &= -\sum_{A=1}^{4}K(Je_1,Je_2,Je_A,e_A)-\sum_{A=1}^{4}K(Je_1,Je_A,e_A,Je_2)\\ &= \sum_{A=1}^{4}K(e_1,e_2,e_A,Je_A)-\sum_{A=1}^{4}K(Je_1,Je_A,e_2,Je_A)\\ &= \sum_{A=1}^{4}K(e_1,e_2,e_A,Je_A)-\mathrm{Ric}(Je_1,e_2),\end{aligned}$$

where we have used the fact that $\{Je_A\}$ is also an orthonormal basis of M. Using (3.4) we get

$$\begin{aligned}\mathrm{Ric}(Je_1,e_2) &= \frac{1}{2}K(e_1,e_2,e_A,Je_A)=\sin\alpha\,(K_{1213}-K_{1224})+\cos\alpha\,(K_{1212}+K_{1234})\\ &= \frac{1}{\cos\alpha}(K_{1212}+K_{1234}).\end{aligned}$$

□

Theorem 3.3. *Suppose that M is a Kähler surface and Σ is a symplectic critical surface in M with Kähler angle α. Then $\cos\alpha$ satisfies*

$$\begin{aligned}\Delta\cos\alpha &= \frac{3\sin^2\alpha-2}{\cos\alpha}|\nabla\alpha|^2-\cos\alpha\sin^2\alpha\,(K_{1212}+K_{1234})\\ &= \frac{3\sin^2\alpha-2}{\cos\alpha}|\nabla\alpha|^2-\cos^2\alpha\sin^2\alpha\ \mathrm{Ric}(Je_1,e_2).\end{aligned} \tag{3.6}$$

Proof. If Σ is a symplectic critical surface, then $H = \frac{\sin^2\alpha}{\cos^2\alpha} V$. It is easy to check that

$$(h^3_{1k} - h^4_{2k})^2 + (h^4_{1k} + h^3_{2k})^2 = |H|^2 + 2|V|^2 + 2H \cdot V = \left(\frac{\sin^4\alpha}{\cos^4\alpha} + 2 + 2\frac{\sin^2\alpha}{\cos^2\alpha}\right)|V|^2$$
$$= \frac{1 + \cos^4\alpha}{\cos^4\alpha}|\nabla\alpha|^2,$$

and

$$H^4_{,1} + H^3_{,2} = \partial_1\left(\frac{\sin^2\alpha}{\cos^2\alpha}\partial_1\alpha\right) + \partial_2\left(\frac{\sin^2\alpha}{\cos^2\alpha}\partial_2\alpha\right) = \frac{\sin^2\alpha}{\cos^2\alpha}\Delta\alpha + 2\frac{\sin\alpha}{\cos^3\alpha}|\nabla\alpha|^2$$
$$= \frac{\sin\alpha}{\cos^2\alpha}(-\Delta\cos\alpha - \cos\alpha|\nabla\alpha|^2) + 2\frac{\sin\alpha}{\cos^3\alpha}|\nabla\alpha|^2.$$

Putting these two equations into (3.1), we obtain

$$\Delta\cos\alpha = \frac{-1 - \cos^4\alpha + 2\sin^2\alpha - \sin^2\alpha\cos^2\alpha}{\cos^3\alpha}|\nabla\alpha|^2 - \frac{\sin^2\alpha}{\cos^2\alpha}\Delta\cos\alpha$$
$$- \frac{\sin^2\alpha}{\cos\alpha}(K_{1212} + K_{1234}).$$

Lemma 3.2 now yields (3.6). □

Corollary 3.4. *If M is a Kähler–Einstein surface with scalar curvature K_0, and Σ is a symplectic critical surface in M with Kähler angle α, we have*

$$\Delta\cos\alpha = \frac{3\sin^2\alpha - 2}{\cos\alpha}|\nabla\alpha|^2 - K_0\cos^3\alpha\sin^2\alpha.$$

If, in addition, we assume that $K_0 \geq 0$, then the symplectic critical surface Σ is a holomorphic curve.

Proof. Suppose that M is a Kähler–Einstein surface with scalar curvature K_0. Then we have

$$\mathrm{Ric}(Je_1, e_2) = \cos\alpha\, K_0,$$

and the identity in the corollary follows. The second statement of the corollary follows from the maximum principle. □

4. Topology of the symplectic critical surfaces

In this section we will analyze the topological properties of compact symplectic critical surfaces. At a point $p \in \Sigma$ with $\alpha(p) = 0$ the tangent plane $T_p\Sigma$ of M at p is a complex line in $T_{F(p)}M$. So such a point is called a *complex tangent point*. We recall some equations obtained by Wolfson in [11] (see also [10]). We write the metric of M as

$$ds^2 = \sum_{\beta=-1,1} \omega_\beta\bar{\omega}_\beta.$$

The induced metric on Σ can be written as

$$ds_\Sigma^2 = \phi \circ \bar{\phi},$$

where ϕ is a complex valued 1-form defined up to a complex factor of norm one; furthermore, one can assume that

$$\omega_1 = \cos(\alpha/2)\phi, \quad \omega_{-1} = \sin(\alpha/2)\bar{\phi},$$

where α is the Kähler angle.

Suppose that the complex second fundamental form of Σ in M is

$$II^C = a\phi^2 + 2b\phi\bar{\phi} + c\bar{\phi}^2.$$

Relative to the coframe field ω_{-1}, ω_1, there is a unitary connection $\omega_{\beta\gamma}$ which satisfies

$$d\omega_\beta = \omega_{\beta\gamma} \wedge \omega_\gamma, \quad \omega_{\beta\gamma} + \omega_{\gamma\beta} = 0.$$

We set

$$\cos(\alpha/2)\omega_1 + \sin(\alpha/2)\bar{\omega}_{-1} = \theta_1 + \sqrt{-1}\,\theta_2,$$
$$\sin(\alpha/2)\omega_1 - \cos(\alpha/2)\bar{\omega}_{-1} = \theta_3 + \sqrt{-1}\,\theta_4,$$

where $\theta_k, k = 1, \ldots, 4$, is an orthonormal coframe of the Riemannian structure of M. So, along Σ we have

$$\sin(\alpha/2)\omega_1 - \cos(\alpha/2)\bar{\omega}_{-1} = 0.$$

It follows that ([10, (1.6)], [11, (2.18)])

$$\frac{1}{2}(d\alpha + \sin\alpha(\omega_{-1\overline{-1}} + \omega_{1\bar{1}})) = a\phi + b\bar{\phi}. \tag{4.1}$$

The relation between the real second fundamental form and the complex second fundamental form is given in [11, Section 2],

$$\begin{pmatrix} 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix} = \begin{pmatrix} h_{11}^3 & h_{12}^3 \\ h_{12}^3 & h_{22}^3 \end{pmatrix} + \sqrt{-1} \begin{pmatrix} h_{11}^4 & h_{12}^4 \\ h_{12}^4 & h_{22}^4 \end{pmatrix}.$$

We therefore have

$$b = \frac{1}{4}(H^3 + \sqrt{-1}\,H^4). \tag{4.2}$$

Using (4.1), (4.2) and (2.6), we see that on the symplectic critical surface we have

$$\frac{\partial \sin\alpha}{\partial \bar{\zeta}} = (\sin\alpha)h,$$

where h is a smooth complex function, and ζ is a local complex coordinate on Σ. By Bers' result [2], we have

Proposition 4.1. *A nonholomorphic symplectic critical surface in a Kähler surface has at most finitely many complex points.*

Set

$$g(\alpha) = \ln(\sin^2 \alpha).$$

Then using (3.6), we obtain

$$\begin{aligned}\Delta g(\alpha) &= -2|\nabla\alpha|^2 - 2\frac{\cos\alpha}{\sin^2\alpha}\Delta\cos\alpha - 4\frac{\cos^2\alpha}{\sin^2\alpha}|\nabla\alpha|^2 \\ &= -4|\nabla\alpha|^2 + 2\cos^2\alpha\,(K_{1212} + K_{1234}). \end{aligned} \tag{4.3}$$

This equation is valid away from the complex tangent points of M. By the Gauss equation and Ricci equation, we have

$$R_{1212} = K_{1212} + h^{\alpha}_{11}h^{\alpha}_{22} - (h^{\alpha}_{12})^2, \qquad R_{1234} = K_{1234} + h^3_{1k}h^4_{2k} - h^4_{1k}h^3_{2k},$$

where R_{1212} is the curvature of $T\Sigma$ and R_{1234} is the curvature of $N\Sigma$. Adding these two equations, we get

$$\begin{aligned} K_{1212} + K_{1234} &= R_{1212} + R_{1234} - \frac{1}{2}|H|^2 + \frac{1}{2}((h^3_{1k} - h^4_{2k})^2 + (h^4_{1k} + h^3_{2k})^2) \\ &= R_{1212} + R_{1234} + |V|^2 + H \cdot V = R_{1212} + R_{1234} + \frac{1}{\cos^2\alpha}|\nabla\alpha|^2. \end{aligned}$$

Thus,

$$R_{1212} + R_{1234} = \frac{1}{2}\frac{\Delta g(\alpha)}{\cos^2\alpha} + \frac{|\nabla\alpha|^2}{\cos^2\alpha}.$$

Integrating the above equality over Σ we have

$$2\pi(\chi(T\Sigma) + \chi(N\Sigma)) = -2\pi P - \int_{\Sigma}\frac{|\nabla\alpha|^2}{\cos^2\alpha}\,d\mu_{\Sigma},$$

where $\chi(T\Sigma)$ is the Euler characteristic of Σ, $\chi(N\Sigma)$ is the Euler characteristic of the normal bundle of Σ in M, and P is the sum of the orders of the complex tangent points. We have thus proved the following theorem.

Theorem 4.2. *Let Σ be a nonholomorphic symplectic critical surface in a Kähler surface M. Let P denote the sum of the orders of the complex tangent points. Then*

$$\chi(T\Sigma) + \chi(N\Sigma) = -P - \frac{1}{2\pi}\int_{\Sigma}\frac{|\nabla\alpha|^2}{\cos^2\alpha}\,d\mu_{\Sigma}.$$

Similarly, we can show

Theorem 4.3. *Let Σ be a nonholomorphic symplectic critical surface in a Kähler surface M. Then*

$$F^*c_1(M)[\Sigma] = -P - \frac{1}{2\pi}\int_\Sigma \frac{|\nabla\alpha|^2}{\cos^3\alpha}\,d\mu_\Sigma.$$

where $c_1(M)$ is the first Chern class of M and $[\Sigma]$ is the fundamental homology class of Σ.

Proof. By (4.3) we also get

$$\Delta g(\alpha) = -4|\nabla\alpha|^2 + 2\cos^3\alpha\ \mathrm{Ric}(Je_1, e_2).$$

Note that $\mathrm{Ric}(Je_1, e_2)\,d\mu_\Sigma$ is the pull back to Σ, by the immersion F, of the Ricci 2-form of M, i.e.,

$$F^*(\mathrm{Ric}^M) = \mathrm{Ric}(Je_1, e_2)\,d\mu_\Sigma.$$

Thus,

$$F^*(\mathrm{Ric}^M) = \left(\frac{1}{2}\frac{\Delta g(\alpha)}{\cos^3\alpha} + 2\frac{|\nabla\alpha|^2}{\cos^3\alpha}\right)d\mu_\Sigma.$$

Integrating it over Σ, we obtain

$$2\pi F^*c_1(M)[\Sigma] = -2\pi P - \int_\Sigma \frac{|\nabla\alpha|^2}{\cos^3\alpha}\,d\mu_\Sigma.$$

This proves the theorem. □

Corollary 4.4. *Suppose that Σ is a symplectic critical surface in a Kähler surface M. Then*

$$\chi(T\Sigma) + \chi(N\Sigma) = F^*c_1(M)[N],$$

and Σ is a minimal surface with constant Kähler angle.

Proof. By Proposition 4.1, Webster's formula ([9, Proposition 1]), and Theorem 4.2, we have

$$\int_\Sigma \frac{|\nabla\alpha|^2}{\cos^2\alpha}\,d\mu_\Sigma = 0.$$

That means $|\nabla\alpha| = 0$, which implies that $\alpha \equiv$ constant. By (2.2), we see that $H = 0$. □

Remark 4.5. If Σ is not compact, the corollary above is not true. In fact, the rotational symmetric surface $z = -\frac{1}{2}\log(x^2 + y^2)$ in $\mathbb{R}^3$ considered as a surface in $\mathbb{C}^2$ is a symplectic critical surface which is not minimal.

Remark 4.6. By Lemma 4.1 and Remark 4.1 in [5], one checks that a minimal surface with constant Kähler angle is an infinitesimally holomorphic immersion ([6, p. 253, Definition]). So we have a variational approach for infinitesimally holomorphic immersions which are, in many cases, holomorphic.

Let g be the genus of Σ, I_Σ the self-intersection number of Σ, and D_Σ the number of double points of Σ. Then

$$\chi(\Sigma) = 2 - 2g, \qquad \chi(\nu) = I_\Sigma - 2D_\Sigma.$$

Setting $c_1(\Sigma) = c_1(M)([\Sigma])$, we have

Corollary 4.7. *Suppose that Σ is a symplectic critical surface in a Kähler surface M. Then*

$$2 - 2g - c_1(\Sigma) + I_\Sigma - 2D_\Sigma = 0.$$

Proof. This follows easily from Theorems 4.2 and 4.3. We omit the details. □

5. The gradient flow

In this section we consider the gradient flow of the function L, i. e,

$$\frac{dF}{dt} = \cos^2\alpha\, H - \frac{1}{\cos\alpha}(J(J\nabla\cos\alpha)^\top)^\perp. \tag{5.1}$$

We set

$$f = \cos^2\alpha\, H - \frac{1}{\cos\alpha}(J(J\nabla\cos\alpha)^\top)^\perp.$$

It is clear that if $f = 0$, then Σ is a symplectic critical surface.

By the first variational formula of the functional L (Theorem 2.2), we see that, along the flow,

$$\begin{aligned}\frac{dL}{dt} &= -2\int_\Sigma \frac{1}{\cos^3\alpha}\left|\cos^2\alpha H - \frac{1}{\cos\alpha}(J(J\nabla\cos\alpha)^\top)^\perp\right|^2 d\mu \\ &= -2\int_\Sigma \frac{1}{\cos^3\alpha}|f|^2\, d\mu.\end{aligned} \tag{5.2}$$

By Theorem 2.3, we know that (5.1) is a parabolic equation, and the short time existence can be shown by a standard argument. We set $\Sigma_t = F(\Sigma, t)$ with $\Sigma_0 = \Sigma$.

Using the same local frame as in Section 2, we can write (5.1) as

$$\frac{dF}{dt} = \cos^2\alpha\, H - \sin^2\alpha\, V \equiv f. \tag{5.3}$$

We compute the evolution of the area element of Σ_t along the flow.

Lemma 5.1.

$$\frac{d}{dt}\, d\mu_t = (-\cos^2\alpha\, |H|^2 + \sin^2\alpha\, V\cdot H)\, d\mu_t. \tag{5.4}$$

Proof. It is easy to check that

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right\rangle = 2\left\langle \frac{\partial f}{\partial x^i}, \frac{\partial F}{\partial x^j}\right\rangle = -2\left\langle f, \frac{\partial^2 F}{\partial x^i \partial x^j}\right\rangle \\ &= 2(-\cos^2\alpha\, H^\alpha + \sin^2\alpha\, V^\alpha) h^\alpha_{ij}.\end{aligned}$$

Here we have used the fact that $\nabla_{e_i} e_j = 0$ at the fixed point. Therefore,

$$\frac{d}{dt} d\mu_t = -f \cdot H \, d\mu_t = (-\cos^2\alpha \, |H|^2 + \sin^2\alpha \, V \cdot H) \, d\mu_t. \qquad \square$$

Now we derive the evolution equation of $\cos\alpha$ along the flow (5.3), which can be seen as a starting point for the study of the flow.

Theorem 5.2. *Let M be a Kähler surface. Assume that α is the Kähler angle of Σ_t which evolves by the flow* (5.3)*. Then $\cos\alpha$ satisfies the equation*

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)\cos\alpha &= \cos^3\alpha \, (|h^3_{1k} - h^4_{2k}|^2 + |h^3_{2k} + h^4_{1k}|^2) + \cos^2\alpha \sin^2\alpha \ \mathrm{Ric}(Je_1, e_2) \\ &\quad + \cos\alpha \sin^2\alpha \, |H|^2 - \cos\alpha \sin^2\alpha \, |V + H|^2, \qquad (5.5)\end{aligned}$$

where $\{e_1, e_2, v_3, v_4\}$ is an orthonormal basis of T_pM such that ω, J take the form (2.3), (2.4).

Proof. Using the fact that $\bar\nabla\omega = 0$ and Lemma 5.1, we have

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha &= \frac{\partial}{\partial t}\frac{\omega(e_1, e_2)}{\sqrt{\det(g_t)}} = \omega(\bar\nabla_{e_1} f, e_2) - \omega(\bar\nabla_{e_2} f, e_1) - \frac{1}{2}\cos\alpha \frac{\partial}{\partial t} g_{ij} g^{ij} \\ &= \omega(\bar\nabla_{e_1} f, e_2) - \omega(\bar\nabla_{e_2} f, e_1) + \cos\alpha \ f \cdot H.\end{aligned}$$

By breaking $\bar\nabla_{e_1} f$ and $\bar\nabla_{e_2} f$ into the normal and tangent parts, we get

$$\begin{aligned}&\omega(\bar\nabla_{e_1} f, e_2) - \omega(\bar\nabla_{e_2} f, e_1) \\ &\quad = \omega(\bar\nabla^N_{e_1} f, e_2) - \omega(\bar\nabla^N_{e_2} f, e_1) + \omega(\bar\nabla^T_{e_1} f, e_2) - \omega(\bar\nabla^N_{e_2} f, e_1) \\ &\quad = \omega(\bar\nabla^N_{e_1} f, e_2) - \omega(\bar\nabla^N_{e_2} f, e_1) + \omega(B(e_1, f), e_2) - \omega(B(e_2, f), e_1) \\ &\quad = \omega(\bar\nabla^N_{e_1} f, e_2) - \omega(\bar\nabla^N_{e_2} f, e_1) + \cos\alpha \, (\langle B(e_1, f), e_1\rangle + \langle B(e_2, f), e_2\rangle) \\ &\quad = \omega(\bar\nabla^N_{e_1} f, e_2) - \omega(\bar\nabla^N_{e_2} f, e_1) - \cos\alpha \ f \cdot H.\end{aligned}$$

Combining these two identities we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\cos\alpha &= \omega(\bar\nabla^N_{e_1} f, e_2) - \omega(\bar\nabla^N_{e_2} f, e_1) \\ &= \omega(\bar\nabla^N_{e_1}(\cos^2\alpha \, H - \sin^2\alpha \, V), e_2) - \omega(\bar\nabla^N_{e_2}(\cos^2\alpha \, H - \sin^2\alpha \, V), e_1) \\ &= \cos^2\alpha \, (\omega(\bar\nabla^N_{e_1} H, e_2) - \omega(\bar\nabla^N_{e_2} H, e_1)) \\ &\quad + \omega(\partial_1 \cos^2\alpha \, H, e_2) - \omega(\partial_2 \cos^2\alpha \, H, e_1) \\ &\quad - \omega(\partial_1 \sin^2\alpha \, V, e_2) + \omega(\partial_2 \sin^2\alpha \, V, e_1) \\ &\quad - \sin^2\alpha \, (\omega(\bar\nabla^N_{e_1} V, e_2) - \omega(\bar\nabla^N_{e_2} V, e_1)) \\ &=: I + II + III + IV. \qquad (5.6)\end{aligned}$$

By Theorem 3.1 we have

$$
\begin{aligned}
I &= \cos^2\alpha \sin\alpha\,(H^4_{,1} - H^3_{,2}) \\
&= \cos^2\alpha\,(\Delta\cos\alpha + \cos\alpha\,|h^3_{1k} - h^4_{2k}|^2 + \cos\alpha\,|h^3_{2k} + h^4_{1k}|^2 + \sin^2\alpha\,\mathrm{Ric}(Je_1, e_2)).
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
II &= H^4 \sin\alpha\,\partial_1 \cos^2\alpha + H^3 \sin\alpha\,\partial_2 \cos^2\alpha = -2\sin^2\alpha\cos\alpha\,(H^4\partial_1\alpha + H^3\partial_2\alpha) \\
&= -2\cos\alpha\sin^2\alpha\, H\cdot V.
\end{aligned}
$$

From the definition of V, we can see that

$$
\begin{aligned}
III &= -\sin\alpha\,\partial_1\sin^2\alpha\,\partial_1\alpha - \sin\alpha\,\partial_2\sin^2\alpha\,\partial_2\alpha = -2\sin^2\alpha\cos\alpha\,|\nabla\alpha|^2 \\
&= -2\sin^2\alpha\cos\alpha|V|^2.
\end{aligned}
$$

Similarly, one obtains

$$
\begin{aligned}
IV &= -\sin^2\alpha\,\omega(\bar\nabla^N_{e_1}(\partial_2\alpha\, v_3 + \partial_1\alpha\, v_4), e_2) + \sin^2\alpha\,\omega(\bar\nabla^N_{e_2}(\partial_2\alpha\, v_3 + \partial_1\alpha\, v_4), e_1) \\
&= -\sin^2\alpha\,(\omega(\partial_1\partial_1\alpha\, v_4, e_2) - \omega(\partial_2\partial_2\alpha\, v_3, e_1)) = -\sin^3\alpha\,\Delta\alpha \\
&= \sin^2\alpha\,\Delta\cos\alpha + \sin^2\alpha\cos\alpha\,|\nabla\alpha|^2 = \sin^2\alpha\,\Delta\cos\alpha + \sin^2\alpha\cos\alpha\,|V|^2.
\end{aligned}
$$

Putting these equations into (5.6), we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\cos\alpha &= \Delta\cos\alpha + \cos^3\alpha\,|h^3_{1k} - h^4_{2k}|^2 + \cos^3\alpha\,|h^3_{2k} + h^4_{1k}|^2 \\
&\quad + \cos^2\alpha\sin^2\alpha\ \mathrm{Ric}(Je_1, e_2) - 2\cos\alpha\sin^2\alpha\, H\cdot V - \cos\alpha\sin^2\alpha\,|V|^2.
\end{aligned}
$$

This proves the theorem. ◻

Theorem 5.3. *Let M be a Kähler–Einstein surface with scalar curvature K_0. Assume that α is the Kähler angle of Σ_t which evolves by the flow* (5.3). *Then* $\cos\alpha$ *satisfies the equation*

$$
\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)\cos\alpha &= \cos^3\alpha\,(|h^3_{1k} - h^4_{2k}|^2 + |h^3_{2k} + h^4_{1k}|^2) + K_0\cos^3\alpha\sin^2\alpha \\
&\quad + \cos\alpha\sin^2\alpha\,|H|^2 - \cos\alpha\sin^2\alpha\,|V + H|^2,
\end{aligned} \tag{5.7}
$$

where $\{e_1, e_2, v_3, v_4\}$ is an orthonormal basis of T_pM such that ω, J take the form (2.3), (2.4). *Consequently, if Σ is symplectic, then along the flow* (5.3), *at each time t, Σ_t is symplectic.*

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