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Sagbi bases of Cox–Nagata rings

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Abstract. We degenerate Cox–Nagata rings to toric algebras by means of sagbi bases induced by configurations over the rational function field. For del Pezzo surfaces, this degeneration implies the Batyrev–Popov conjecture that these rings are presented by ideals of quadrics. For the blow-up of projective n-space at $n + 3$ points, sagbi bases of Cox–Nagata rings establish a link between the Verlinde formula and phylogenetic algebraic geometry, and we use this to answer questions due to D'Cruz–Iarrobino and Buczyńska–Wiśniewski. Inspired by the zonotopal algebras of Holtz and Ron, our study emphasizes explicit computations, and offers a new approach to Hilbert functions of fat points.

1. Powers of linear forms

We fix n vector fields on a d-dimensional space with coordinates $(z_1, \dots, z_d)$:

$$\ell_j = \sum_{i=1}^{d} a_{ij} \frac{\partial}{\partial z_i} \quad \text{for } j = 1, \dots, n.$$

Here the coefficients a_{ij} are scalars in a field K, which we assume to have characteristic zero. We regard $\ell_1, \dots, \ell_n$ as linear forms in the polynomial ring $S = K[\partial_1, \dots, \partial_d]$, where $\partial_i = \partial/\partial z_i$. We further assume that $\{\ell_1, \dots, \ell_n\}$ spans the space S_1 of all linear forms in S. This implies $d \le n$. We are interested in the family of zero-dimensional polynomial ideals

$$I_u := \langle \ell_1^{u_1+1}, \dots, \ell_n^{u_n+1} \rangle \subset S,$$

where $u = (u_1, \dots, u_n)$ runs over the set $\mathbb{N}^n$ of non-negative integer vectors. The ideal I_u represents a system of linear partial differential equations with constant coefficients, and we consider its solution space

$$(I_u)^\perp = \bigoplus_{r=0}^{\infty} I_{(r,u)}^\perp \subset K[z_1, \dots, z_d] =: K[z].$$

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Here $I^{\perp}_{(r,u)}$ is the space of all polynomials that are homogeneous of degree r and are annihilated by $u_i + 1$ repeated applications of the vector field ℓ_i. Leibniz' rule for taking derivatives of products implies

$$I_u^{\perp} \cdot I_v^{\perp} = I_{u+v}^{\perp}. \tag{1}$$

This identity is responsible for the algebra structure which will emerge from Proposition 2.1 below. Our principal object of interest is the *counting function*

$$\psi : \mathbb{N}^{n+1} \to \mathbb{N}, \qquad (r, u) \mapsto \dim_K I^{\perp}_{(r,u)}. \tag{2}$$

Note that $\psi(r, u)$ is the value of the Hilbert function of S/I_u at the integer r. Here are some examples.

Example 1.1 ($n = d$). If the given linear forms are linearly independent then we can assume $\ell_1 = z_1, \ldots, \ell_d = z_d$, so that $I^{\perp}_{(r,u)}$ is spanned by all monomials $z^v = z_1^{v_1} \cdots z_d^{v_d}$ whose degree $v_1 + \cdots + v_d$ equals r and $v_i \leq u_i$ for all i. The number $\psi(r, u)$ of these monomials equals the coefficient of q^r in the generating function

$$\frac{(1 - q^{u_1+1}) \cdots (1 - q^{u_d+1})}{(1 - q)^d}.$$

Note that $\psi : \mathbb{N}^{d+1} \to \mathbb{N}$ is a piecewise polynomial function of degree $d - 1$.

Example 1.2 ($d = 2$). For pairwise linearly independent binary linear forms,

$$\psi(r, u) = \Big(r + 1 - \sum_{i=1}^{n} (r - u_i)_+\Big)_+, \tag{3}$$

where $(x)_+ = \max\{x, 0\}$, so $\psi : \mathbb{N}^{n+1} \to \mathbb{N}$ is a piecewise linear function.

The formulas in Examples 1.1 and 1.2 are well-known. Formula (3) can be found, for instance, in the article by Kuttler and Wallach [20, p. 361]. The next example shows the new type of formulas to be derived in the present paper.

Example 1.3 ($d = 3, n = 5$). For five general linear forms in $K[\partial_1, \partial_2, \partial_3]$, the piecewise quadratic function $\psi : \mathbb{N}^6 \to \mathbb{N}$ is given by counting lattice points in the following convex polygon. The value $\psi(r, u_1, \ldots, u_5)$ equals

$$\begin{aligned} \#\{(x, y) \in \mathbb{Z}^2 : &\max(r - u_1, r - u_2) \leq x \leq \min(r, u_3 + u_4 + u_5 - r) \text{ and} \\ &\max(0, 2r - u_1 - u_2 - u_3) \leq y \leq \min(u_4, u_5) \text{ and} \\ &2r - u_1 - u_2 \leq x + y \leq u_4 + u_5 \text{ and } 0 \leq x - y \leq u_3\}. \end{aligned}$$

The derivation of this Ehrhart-type formula is presented in Section 4.

We now discuss the organization of this paper and we summarize its main contributions. In Section 2 we introduce the Cox–Nagata ring, and we express ψ as the multigraded Hilbert function of that ring. For $d \geq 3$, this is the Cox ring of the variety gotten from $\mathbb{P}^{d-1}$ by blowing up the n points dual to the linear forms ℓ_i, and we review some relevant material on fat points with a fixed support, and on Weyl groups generated by Cremona transformations.

In Section 3 we present our technique, which is to examine the flat family induced by taking linear forms defined over $K = \mathbb{Q}(t)$. Under favorable circumstances, this degeneration has desirable combinatorial properties, and the minimal generators of the Cox–Nagata ring form a sagbi basis. In Sections 4, 5 and 6 we demonstrate that this happens for del Pezzo surfaces, which arise by blowing up $n \leq 8$ general points in $\mathbb{P}^2$. We present a proof, found independently from that given in [33], of the Batyrev–Popov conjecture [3, 21, 29] which states that the presentation ideals of these Cox–Nagata rings are quadratically generated. Explicit formulas like that in Example 1.2 are obtained for the number of sections of line bundles on all del Pezzo surfaces.

In Section 7 we treat the case of $n = d + 2$ points in $\mathbb{P}^{d-1}$. We resolve a problem left open by Buczyńska and Wiśniewski in [6], by showing that the phylogenetic varieties in [6, 32] arise as flat limits of the Cox–Nagata rings constructed by Castravet and Tevelev in [8]. We also prove a conjecture of D'Cruz and Iarrobino [10] on Hilbert functions of fat points. A key tool is an interpretation of the Verlinde formula [23] suggested to us by Jenia Tevelev.

In Section 8 we turn to the original motivation which started this project, namely, the work on zonotopal algebra by Holtz and Ron [18]. In their setting, the linear forms ℓ_i represent all the hyperplanes that are spanned by a configuration of vectors, and when ψ is restricted to a certain linear subspace, matroid theory yields a particularly beautiful formula for its values.

2. Cox–Nagata rings

Let G be the space of linear relations on the linear forms ℓ_i. Thus G is the subspace of K^n which consists of all vectors $\lambda = (\lambda_1, \dots, \lambda_n)$ that satisfy

$$\lambda_1 \ell_1(z) + \cdots + \lambda_n \ell_n(z) = 0.$$

The K-vector space G has dimension $n - d$. We regard it is an additive group. This group acts on the polynomial ring in $2n$ variables, $R = K[x_1, \dots, x_n, y_1, \dots, y_n]$, by the following *Nagata action* (cf. [8, 24, 25]):

$$x_i \mapsto x_i \quad \text{and} \quad y_i \mapsto y_i + \lambda_i x_i \quad \text{for all } \lambda \in G.$$

Let R^G denote the subring of R consisting of all polynomials that are fixed by this action. The *Cox–Nagata ring* R^G is a multigraded ring with respect to the grading which is induced by the following $\mathbb{Z}^{n+1}$-grading on R:

$$\deg(x_i) = e_i \quad \text{and} \quad \deg(y_i) = e_0 + e_i \quad \text{for } i = 1, \dots, n.$$

Here $e_0, e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{Z}^{n+1}$, and $R^G_{(r,u)}$ is the finite-dimensional space of invariant polynomials of multidegree $re_0 + \sum_{i=1}^n u_i e_i$. We start out by presenting an elementary proof of the following result.

Proposition 2.1. *There exists a natural isomorphism of K-vector spaces*

$$I^{\perp}_{(r,u)} \simeq R^G_{(r,u)}. \tag{4}$$

Hence ψ is the $\mathbb{Z}^{n+1}$-graded Hilbert function of the Cox–Nagata ring R^G.

Proof. We introduce the polynomial ring $K[Y] = K[Y_1, \ldots, Y_n]$ and we let L_G denote the ideal generated by all linear forms $\lambda_1 Y_1 + \cdots + \lambda_n Y_n$, where λ runs over G. Then $K[Y]/L_G$ is isomorphic to $S = K[z]$ under the K-algebra homomorphism which sends Y_i to $\ell_i(z)$, and this induces an isomorphism

$$S/I_u \simeq K[Y]/(L_G + \langle Y_1^{u_1+1}, \ldots, Y_n^{u_n+1} \rangle).$$

Therefore the solution space $I^{\perp}_{(r,u)}$ is isomorphic to the space of homogeneous polynomials f of degree r in $K[Y]$ that are invariant under the G-action $Y_i \mapsto Y_i + \lambda_i$ and that satisfy $\deg_{Y_i}(f) \leq u_i$ for $i = 1, \ldots, n$. The map

$$f \mapsto f\left(\frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n}\right) \cdot x_1^{u_1} \cdots x_n^{u_n}$$

defines an isomorphism to the component $R^G_{(r,u)}$ of the Cox–Nagata ring. □

The isomorphism (4) has the following explicit description. Let (a_{ij}) be the $d \times n$-matrix over K such that $\ell_j = \sum_{i=1}^d a_{ij} z_i$ for $j = 1, \ldots, n$. The map

$$g(z_1, \ldots, z_d) \mapsto g\left(\sum_{j=1}^n a_{1j} \frac{y_j}{x_j}, \ldots, \sum_{j=1}^n a_{dj} \frac{y_j}{x_j}\right) x_1^{u_1} \cdots x_n^{u_n} \tag{5}$$

takes the solution space $I^{\perp}_{(r,u)}$ bijectively onto the graded component $R^G_{(r,u)}$. This map can be inverted precisely for those elements of R that lie in the invariant ring R^G. In this manner, every choice of a homogeneous K-basis of the $\mathbb{N}^{n+1}$-graded K-algebra R^G determines a basis of $I^{\perp}_{(r,u)}$ for all r, u.

We now explain why we chose the name *Cox–Nagata ring* for R^G. In 1959 Nagata resolved Hilbert's 14th problem whether the ring of polynomial invariants of any matrix group is finitely generated [25]. He showed that R^G is not finitely generated when G is a generic subspace of K^d and $d = 3, n = 16$. The final and precise statement along these lines is due to Mukai [24, p. 1].

Theorem 2.2 ([8, 24]). *Suppose that G is a generic subspace of codimension d in K^n. Then the Cox–Nagata ring R^G is finitely generated if and only if*

$$\frac{1}{2} + \frac{1}{d} + \frac{1}{n-d} > 1. \tag{6}$$

The name of David Cox is attached to the ring R^G because of the following geometric interpretation. Let $\mathbb{P}^{d-1}$ denote the projective space whose points are equivalence classes of linear forms in $S = K[\partial_1, \ldots, \partial_d]$ modulo scaling. Our linear form $\ell_j = \sum_{i=1}^n a_{ij}\partial_i$ is represented by the point $(a_{1j} : \cdots : a_{dj})$ in $\mathbb{P}^{d-1}$. Let P_j denote the homogeneous prime ideal in S of that point, that is, P_j is the ideal generated by the 2×2-minors of the $2 \times n$-matrix

$$\begin{pmatrix} \partial_1 & \cdots & \partial_d \\ a_{1j} & \cdots & a_{dj} \end{pmatrix}.$$

The following classical fact relates our problem to the study of ideals of fat points [10]. We learned Lemma 2.3 from Emsalem and Iarrobino [14, Theorem 1].

Lemma 2.3. *The polynomial solutions of degree r to the linear partial differential equations with constant coefficients expressed by I_u are precisely those polynomials that vanish of order $\geq r - u_j$ at the point ℓ_j for all j. In symbols,*

$$I^{\perp}_{(r,u)} = \Big(\bigcap_{j\,:\,r-u_j>0} P_j^{r-u_j} \Big)_r. \tag{7}$$

Suppose $d \geq 3$ and let X_G denote the rational variety gotten from $\mathbb{P}^{d-1}$ by blowing up the points $\ell_1, \ldots, \ell_n$. We write L for the pullback of the hyperplane class from $\mathbb{P}^{d-1}$ to X_G and $E_1, \ldots, E_n$ for the exceptional divisors of the blow-up. The vector space (7) can be rewritten as a space of sections:

$$I^{\perp}_{(r,u)} = H^0(X_G, rL + (u_1 - r)E_1 + \cdots + (u_n - r)E_n). \tag{8}$$

Taking the direct sum of these spaces for all (r, u) in $\mathbb{Z}^{n+1}$, which we identify with the Picard group of X_G, we obtain the *Cox ring* of the blow-up:

$$\mathrm{Cox}(X_G) = \bigoplus_{(r,u)\in\mathbb{Z}^{n+1}} H^0(X_G, rL + (u_1 - r)E_1 + \cdots + (u_n - r)E_n).$$

We summarize our discussion as follows:

Corollary 2.4. *If $d \geq 3$ then the Cox–Nagata ring R^G equals the Cox ring of the variety X_G which is gotten from $\mathbb{P}^{d-1}$ by blowing up the points $\ell_1, \ldots, \ell_n$.*

The Cox–Nagata ring R^G has received a considerable amount of attention in the recent literature in algebraic geometry. Some relevant references are [3, 8, 12, 21, 24, 29, 31, 33]. These papers are primarily concerned with the case when $\ell_1, \ldots, \ell_n$ are generic. In this case, the function ψ is invariant under the action of the Weyl group of the Dynkin diagram $T_{2,d,n-d}$ with three legs of length 2, d and $n - d$. The action of this Weyl group on $\mathbb{Z}^{n+1}$ is generated by permutations of $(u_1, \ldots, u_n)$ and the transformation $(r, u) \mapsto (r', u')$ where

$$r' = \sum_{i=1}^{d} u_i - r, \quad u'_j = u_j \quad \text{for } j = 1, \ldots, d,$$

$$u'_j = \sum_{i=1}^{d} u_i - 2r + u_j \quad \text{for } j = d+1, \ldots, n.$$

While the invariance $\psi(r, u_1, \ldots, u_n) = \psi(r, u_{\pi(1)}, \ldots, u_{\pi(n)})$ for all permutations π is entirely obvious from the definition of ψ, the second invariance

$$\psi(r, u_1, \ldots, u_n) = \psi(r', u'_1, \ldots, u'_n) \tag{9}$$

is less obvious and due to Nagata [25]. He proved (9) by applying a Cremona transformation to the points $\ell_1, \ldots, \ell_d$ in $\mathbb{P}^{d-1}$. This induces an isomorphism of the blow-up X_G which replaces the divisor class $r \cdot L + \sum_{j=1}^{n}(r - u_j)E_j$ by the divisor class $r' \cdot L + \sum_{j=1}^{n}(r' - u'_j)E_j$. An alternative proof using representation theory of $SL_2(K)$ was given by Kuttler and Wallach [20]. The reader may find it instructive to verify (9) in Examples 1.1–1.3.

Our primary objective is to describe nice bases for R^G which express ψ as the number of lattice points in a $(d-1)$-dimensional polytope parametrized by (r, u). The volume of this polytope is the leading form of ψ. Algebraic geometers know this as the *volume of the line bundle* $r \cdot L + \sum_{i=1}^{n}(u_i - r)E_i$ on X_G. For instance, the area of the polygon in Example 1.3 would be an extension of the formula in [5, Example 3.5] to blowing up $n = 5$ points.

The *support semigroup* $\mathcal{S}^G$ of the Cox–Nagata ring R^G is the subsemigroup of $\mathbb{N}^{n+1}$ which consists of all multidegrees (r, u) for which $\psi(r, u) > 0$. The *support cone* $\mathcal{C}^G$ is the cone in $\mathbb{R}^{n+1}$ generated by the support semigroup.

Remark 2.5. The support cone $\mathcal{C}^G$ is a full-dimensional cone in $\mathbb{R}^{n+1}$.

If $\ell_1, \ldots, \ell_n$ are generic then the Weyl group $T_{2,n-d,d}$ acts on the semigroup $\mathcal{S}^G$ and its cone $\mathcal{C}^G$. It is known that $T_{2,n-d,d}$ is finite if and only if inequality (6) holds if and only if R^G is finitely generated if and only if its semigroup $\mathcal{S}^G$ is finitely generated [8, 24]. In this finite situation, the cone $\mathcal{C}^G$ is the cone over an n-dimensional polytope $\mathcal{P}^G$, which we call the *support polytope*. For $d = 3$ and $n = 6, 7, 8$ our Weyl group is E_6, E_7, E_8 respectively, and the support polytopes are the *Gosset polytopes* [16] associated with these exceptional groups. However, in our view, the support polytope and all the other concepts introduced in this section are combinatorially interesting even if the ℓ_i are not generic. Here is an example to illustrate this perspective.

Example 2.6. We consider the six special points in the plane $\mathbb{P}^2$ given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}.$$

These are the intersection points of four general lines in $\mathbb{P}^2$. Here $d = 3$, $n = 6$, $G = \text{kernel}(A)$, and $(\ell_1, \ldots, \ell_6) = (z_1, z_2, z_3) \cdot A$. The Cox–Nagata ring equals

$$R^G = K[x_1, x_2, x_3, x_4, x_5, x_6, L_{124}, L_{135}, L_{236}, L_{456}, M_{16}, M_{25}, M_{34}],$$

where the L_{ijk} and M_{ij} represent lines spanned by triples and pairs of points:

$$\begin{aligned}
L_{124} &= \underline{y_3x_5x_6} + x_3y_5x_6 - x_3x_5y_6 && \text{in degree (1001011)},\\
L_{135} &= \underline{y_2x_4x_6} - x_2y_4x_6 + x_2x_4y_6 && \text{in degree (1010101)},\\
L_{236} &= \underline{y_1x_4x_5} + x_1y_4x_5 - x_1x_4y_5 && \text{in degree (1100110)},\\
L_{456} &= \underline{y_1x_2x_3} + x_1y_2x_3 + x_1x_2y_3 && \text{in degree (1111000)},\\
M_{16} &= \underline{y_2x_3x_4x_5} + x_2y_3x_4x_5 - x_2x_3y_4x_5 + x_2x_3x_4y_5 && \text{in degree (1011110)},\\
M_{25} &= \underline{y_1x_3x_4x_6} + x_1y_3x_4x_6 + x_1x_3y_4x_6 - x_1x_3x_4y_6 && \text{in degree (1101101)},\\
M_{34} &= \underline{y_1x_2x_5x_6} + x_1y_2x_5x_6 - x_1x_2y_5x_6 + x_1x_2x_5y_6 && \text{in degree (1110011)}.
\end{aligned}$$

These generators are derived from [7, Lemma 7.3]. The 7-dimensional support semigroup $\mathcal{S}^G$ is spanned by the seven listed vectors together with the unit vectors $e_i = \deg(x_i)$ for $i = 1, \dots, 6$. The 6-dimensional support polytope $\mathcal{P}^G$ has 13 vertices. Its f-vector equals (13, 69, 186, 260, 168, 38). The seven underlined monomials together with $x_1, \dots, x_6$ are a sagbi basis of R^G, as defined in the next section. The algorithm explained in Section 4 now computes an Ehrhart-type formula for the Hilbert function ψ of R^G. It outputs that $\psi(r, u)$ is the number of integer vectors $(x, y) \in \mathbb{Z}^2$ satisfying

$$\begin{gathered}
x \le \min(u_3, u_5, u_6), \qquad y \le \min(u_2, u_4), \qquad 2x + y \le \min(u_3 + u_6, u_5 + u_6),\\
3x + y \le u_3 + u_5 + 2u_6 - r, \qquad r - u_2 - u_4 \le x - y \le u_3 + u_5 - r,\\
r - u_1 \le x + y \le \min(r, u_2 + u_5 + u_6 - r, u_3 + u_4 + u_6 - r).
\end{gathered}$$

This piecewise quadratic function counts the sections of line bundles on a singular surface X_G which is known classically as *Cayley's cubic surface*.

3. Sagbi bases

In this section we introduce sagbi bases into the study of Cox–Nagata rings, and we give a complete classification of such bases when $n = d + 1$. The basic idea is as follows. Let K be the field $\mathbb{Q}(t)$ of rational functions in one variable t. As always in tropical geometry [30], any field with a non-trivial non-archimedean valuation would work as well, but to keep things as simple and Gröbner-friendly as possible, we take $K = \mathbb{Q}(t)$ to be our field of definition.

The *order* of a scalar $c(t) \not\equiv 0$ in K is the unique integer ω such that $t^{-\omega} \cdot c(t)$ has neither a pole nor a zero at $t = 0$. Likewise, any non-zero element in a polynomial ring $K[x_1, \dots, x_m]$ over the field K has a lowest order ω in the unknown t. We define the *initial form* $\text{in}(f)$ of the polynomial f to be the coefficient of that lowest power t^ω in f. In symbols,

$$\text{in}(f) := (t^{-\omega} \cdot f)|_{t=0} \in \mathbb{Q}[x_1, \dots, x_m].$$

A subset $\mathcal{F}$ of $K[x_1, \dots, x_m]$ is called *moneric* if $\text{in}(f)$ is a monomial for all $f \in \mathcal{F}$. For any K-subalgebra U of the polynomial ring $K[x_1, \dots, x_m]$, the *initial algebra* $\text{in}(U)$

is, by definition, the $\mathbb{Q}$-subalgebra of $\mathbb{Q}[x_1, \ldots, x_m]$ generated by the initial forms $\mathrm{in}(f)$ where f runs over all polynomials f in U. Even if U is a finitely generated K-algebra, it will often happen that the $\mathbb{Q}$-algebra $\mathrm{in}(U)$ is not finitely generated. A subset $\mathcal{F}$ of U is called a *sagbi basis* if $\mathcal{F}$ is moneric and the initial algebra $\mathrm{in}(U)$ is generated as a $\mathbb{Q}$-algebra by the monomials $\mathrm{in}(f)$ for $f \in \mathcal{F}$. The sagbi basis $\mathcal{F}$ can be infinite even if U is finitely generated. The acronym *sagbi* was coined by Robbiano and Sweedler [28]. It stands for "subalgebra analogue to Gröbner bases for ideals". Our definition of sagbi bases is more general than the definition usually given in the computer algebra literature [15, 19, 22]. There one uses a monomial order to define initial monomials and the initial algebra, but this situation can be modeled by introducing an extra variable t as in [13, §15.8] to get to the situation described above. See also [9, §1].

Remark 3.1. Throughout this paper we make frequent use of the basics concerning the construction and properties of sagbi bases which remain valid in our setting. Such basics are: (a) sagbi bases induce binomial initial ideals for partial term orders (as in [22, Theorem 14.16]), (b) the lifting of all binomial syzygies implies the sagbi property (as in [9, Proposition 1.3]), (c) a multigraded algebra R and its initial algebra $\mathrm{in}(R)$ share the Hilbert function.

Example 3.2. The example of a sagbi basis best known among computer algebraists is the set of elementary symmetric polynomials. Let $n = 3$ and $\mathcal{F} = \{x_1 + tx_2 + t^2x_3, x_1x_2 + tx_1x_3 + t^2x_2x_3, x_1x_2x_3\}$. The initial algebra of $U = K[\mathcal{F}]$ is $\mathrm{in}(U) = \mathbb{Q}[x_1, x_1x_2, x_1x_2x_3]$. If we wish to extend to the invariants of the alternating group, then U' is the K-algebra generated by $\mathcal{F}$ and the discriminant $(x_1 - tx_2)(x_1 - t^2x_3)(x_2 - tx_3)$. Now, the initial algebra $\mathrm{in}(U')$ is not finitely generated. See the work of Göbel [15] for details.

We now fix the ambient polynomial ring $R = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Our aim is to construct explicit sagbi bases of Cox–Nagata subrings R^G of R. A first example was seen in Example 2.6 where the 13 generators of R^G form a sagbi basis with the underlined initial monomials. The following example underlines that our sagbi bases do not generally come from term orders.

Example 3.3. Let $d = 2, n = 4$ and suppose that G is a generic two-dimensional linear subspace of K^4. Then $R^G = K[x_1, x_2, x_3, x_4, E_1, E_2, E_3, E_4]$, where

$$\begin{aligned}
E_1 &= p_{23}x_2x_3y_4 - p_{24}x_2y_3x_4 + p_{34}y_2x_3x_4,\\
E_2 &= p_{13}x_1x_3y_4 - p_{14}x_1y_3x_4 + p_{34}y_1x_3x_4,\\
E_3 &= p_{12}x_1x_2y_4 - p_{14}x_1y_2x_4 + p_{24}y_1x_2x_4,\\
E_4 &= p_{12}x_1x_2y_3 - p_{13}x_1y_2x_3 + p_{23}y_1x_2x_3.
\end{aligned}$$

The coefficients p_{ij} are non-zero scalars in K that satisfy $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. They represent the Plücker coordinates of the subspace G as in (12) (in Section 4). The ring R^G has Krull dimension 6 and is presented by the ideal

$$I^G = \langle p_{14}x_1E_1 - p_{24}x_2E_2 + p_{34}x_3E_3,\ p_{13}x_1E_1 - p_{23}x_2E_2 + p_{34}x_4E_4 \rangle.$$

Among the $81 = 3^4$ choices of one term from each E_i, only 12 are induced by a term order on R, and these are all equivalent under permuting $\{1, 2, 3, 4\}$. A representative is given by taking the first term in each E_i as listed above. These leading terms specify a sagbi basis both in the classical sense and in our sense. However, there is another symmetry class (also having 12 elements) which specifies a sagbi basis in our sense but **not** via a term order in (x, y). A representative is given by taking the second term in each E_i. The corresponding initial toric algebras of the Cox–Nagata ring R^G are

$$\begin{aligned}\mathrm{in}(R^G) &= \mathbb{Q}[x_1, x_2, x_3, x_4, x_2x_3y_4, x_1x_3y_4, x_1x_2y_4, x_1x_2y_3]\\ &= \mathbb{Q}[x_1, \dots, x_4, E_{11}, \dots, E_{41}]/\langle x_1E_{11} - x_2E_{21}, x_2E_{21} - x_3E_{31}\rangle\\ &\qquad\text{e.g. for } (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) = (1, 2, t, 1, t, t),\\ \mathrm{in}(R^G) &= \mathbb{Q}[x_1, x_2, x_3, x_4, x_2y_3x_4, x_1y_3x_4, x_1y_2x_4, x_1y_2x_3]\\ &= \mathbb{Q}[x_1, \dots, x_4, E_{12}, \dots, E_{42}]/\langle x_1E_{12} - x_2E_{22}, x_3E_{32} - x_4E_{42}\rangle\\ &\qquad\text{e.g. for } (p_{12}, \dots, p_{34}) = (t^2 - t^4, t - t^5, 1 - t^6, t^2 - t^4, t - t^5, t^2 - t^4).\end{aligned}$$

For both types we verify the sagbi property via [9, Proposition 1.3], by observing that all the binomial relations lift to polynomials in I^G.

Remark 3.4. For $d \geq 3$, the Cox–Nagata ring R^G serves as the affine coordinate ring of the *universal torsor* [12] over the variety X_G. The geometric interpretation of sagbi bases is that the universal torsor degenerates to a toric variety. By restricting to the graded components of R^G, this induces a simultaneous degeneration of each projective embedding of X_G to a projective toric variety. For instance, each lattice polygon in Example 1.3 represents a projective toric surface along with a deformation to a del Pezzo surface.

We now present a classification of sagbi bases for the Cox–Nagata ring when $n = d + 1$ and G is generic. We fix non-zero scalars $\alpha_1, \dots, \alpha_n$ in K and the one-dimensional linear subspace $G = \mathrm{span}_K\{(\alpha_1, \dots, \alpha_n)\}$ of K^n.

Theorem 3.5. *Let $\mathcal{F}$ be the set consisting of the 2×2-minors of the matrix*

$$\mathcal{M} := \begin{bmatrix} \alpha_1x_1 & \alpha_2x_2 & \cdots & \alpha_nx_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

and the variables $x_1, \dots, x_n$. Then $\mathcal{F}$ is moneric if and only if the orders of the α_i are distinct. In this case $\mathcal{F}$ is a sagbi basis of the Cox–Nagata ring R^G.

Proof. After relabeling we may assume $\mathrm{ord}(\alpha_1) > \cdots > \mathrm{ord}(\alpha_n)$. The 2×2-minors have the initial monomials x_iy_j for $1 \leq i < j \leq n$, and

$$\mathrm{in}(\mathcal{F}) = \{x_1, \dots, x_n, y_1x_2, y_1x_3, \dots, y_{n-1}x_n\}. \tag{10}$$

It was shown in [22, §14.3] that these monomials generate the initial algebra with respect to a term order. Since the monomials (10) lie in the initial algebra $\mathrm{in}(K[\mathcal{F}])$, they generate a $\mathbb{Q}$-subalgebra which has the same $\mathbb{Z}^{n+1}$-graded Hilbert function as $K[\mathcal{F}]$. From this it follows that they generate $\mathrm{in}(K[\mathcal{F}])$. This establishes the sagbi basis property in the sense defined above. Clearly, the elements of $\mathcal{F}$ are invariant under the action of G on R. The result that R^G is generated by $\mathcal{F}$ is classical [8, Remark 3.9]. □

The Cox–Nagata ring R^G is isomorphic to the algebra generated by the 2×2-minors of a general $2 \times (n+1)$-matrix, that is, the coordinate ring of the Grassmannian $\mathrm{Gr}(2, n+1)$ of lines in $\mathbb{P}^n$. Theorem 3.5 represents the familiar sagbi degeneration of the Grassmannian $\mathrm{Gr}(2, n+1)$ to a toric variety.

We seek good formulas for evaluating the Hilbert function ψ of the Cox–Nagata ring R^G. For the case $n = d+1$ discussed here, we can utilize standard tools from algebraic combinatorics. Recall (e.g. from [22, §14.4]) that a *two-row Gelfand–Tsetlin pattern* is a non-negative integer $2 \times n$-matrix (λ_{ij}) that satisfies $\lambda_{2n} = 0$, $\lambda_{1,j+1} \geq \lambda_{2,j}$ and $\lambda_{i,j} \geq \lambda_{i,j+1}$ for $i = 1, 2$ and $j = 1, \ldots, n-1$. Two-row Gelfand–Tsetlin patterns are identified with monomials in our initial algebra $\mathrm{in}(R^G)$ via the isomorphism of [22, Theorem 14.23]. Here the monomials in (10) correspond to two-row partitions as in [22, Example 14.21]. Using this isomorphism, our sagbi basis implies the following formula:

Proposition 3.6. *For $n = d+1$, $\psi(r, u)$ is the number of two-rowed Gelfand–Tsetlin patterns with $\lambda_{21} = r$ and $\lambda_{1j} + \lambda_{2j} = u_j + \cdots + u_n$ for $j = 1, \ldots, n$.*

This formula shows that ψ is a piecewise polynomial function of degree $n-2$ on the $(n+1)$-dimensional support cone $\mathcal{C}^G$. The underlying n-dimensional support polytope $\mathcal{P}^G$ is affinely isomorphic to the second hypersimplex $\Delta(n+1, 2) = \mathrm{conv}\{e_i + e_j : 0 \leq i < j \leq n\}$. The $\mathbb{Z}^{n+1}$-grading specifies a linear map from a Gelfand–Tsetlin polytope onto the second hypersimplex. Its fibers are $(n-2)$-polytopes. They represent toric degenerations of all projective embeddings of the blow-up of $\mathbb{P}^{n-2}$ at n points.

Example 3.7. Suppose that $n = d+1$. In [10, end of §4.1] the authors asked the question whether

$$\varphi(j, \ldots, j) := \sum_{r=0}^{\infty} \psi(r, j, \ldots, j) = \dim_K(S/I_{(j,\ldots,j)})$$

is a polynomial function in the one variable j. We find a negative answer to this question by examining the case $d = 2$ and $n = 3$. By Proposition 3.6, $\varphi(j, j, j)$ equals the Ehrhart quasi-polynomial of the rational quadrangle

$$\{(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) \in \mathbb{R}^4_{\geq 0} : \lambda_{11} + \lambda_{21} = 3,\ \lambda_{12} + \lambda_{22} = 2,$$
$$\lambda_{11} \geq \lambda_{12} \geq 1,\ \lambda_{21} \geq \lambda_{22},\ \lambda_{12} \geq \lambda_{21},\ 1 \geq \lambda_{22}\}.$$

The number of lattice points in j times this quadrangle equals

$$\varphi(j, j, j) = \sum_{r \geq 0} (r + 1 - 3(r-j)_+)_+ = \begin{cases} (3j^2 + 6j + 3)/4 & \text{if } j \text{ is odd,} \\ (3j^2 + 6j + 4)/4 & \text{if } j \text{ is even.} \end{cases}$$

This is not a polynomial in j but it is a quasi-polynomial with period two.

4. Five points in the plane

Let G be a linear subspace of codimension d in K^n. We call the subspace G *moneric* if the Cox–Nagata ring R^G has a minimal generating set $\mathcal{F}$ which is moneric, and we say that G is *sagbi* if R^G has a minimal generating set $\mathcal{F}$ which is also a sagbi basis. These definitions make sense for any d, n and G, even if R^G is not finitely generated. In general, it is possible that R^G has two distinct minimal generating sets $\mathcal{F}$ and $\mathcal{F}'$ where $\mathcal{F}$ is moneric but $\mathcal{F}'$ is not. However, this phenomenon does not happen in the cases studied in Sections 4–6 of this paper because here the minimal generators of R^G are uniquely characterized (up to a scalar multiple) by their $\mathbb{Z}^{n+1}$-degrees.

Two moneric subspaces G and G' are called *equivalent* if the initial subalgebras $\mathrm{in}(R^G)$ and $\mathrm{in}(R^{G'})$ are identical, and we call their common equivalence class a *moneric class*. A moneric class can either be sagbi or not sagbi, depending on whether the subspaces in that class are sagbi or not. In this language, Theorem 3.5 says that for $n = d+1$, there is precisely one moneric class up to permutations and this class is sagbi.

We fix $n = d + 2$ and we consider a generic linear subspace

$$G = \mathrm{rowspan}\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \end{bmatrix} \subset K^n. \tag{11}$$

Genericity means in particular that the $\binom{n}{2}$ Plücker coordinates

$$p_{ij} := b_{1i}b_{2j} - b_{1j}b_{2i} \quad (1 \le i < j \le n) \tag{12}$$

are all non-zero. These Plücker coordinates already appeared in Example 3.3, where the following result was established: *There are precisely* 24 *moneric classes of* 2*-dimensional linear subspaces of* K^4*. Modulo permutations of* $\{1, 2, 3, 4\}$ *the number of classes is two, and both of them are sagbi classes.* Our main result in this section is the analogous classification in the next case.

Theorem 4.1. *There are precisely* 600 *moneric classes of* 3*-dimensional generic linear subspaces of* K^5*. All but* 60 *of these classes are sagbi. Modulo permutations of the indices* $\{1, \dots, 5\}$*, the number of moneric classes is seven and the number of sagbi classes is six.*

Proof. The Cox–Nagata ring R^G is minimally generated by a distinguished set $\mathcal{F}$ of 16 polynomials [3, 8, 31]. First, the five variables x_1, x_2, x_3, x_4, x_5 represent the exceptional divisors of the blow-up. Second, there are ten generators corresponding to the lines passing through pairs of points:

$$L_{ijk} := p_{ij} \cdot x_i x_j y_k - p_{ik} \cdot x_i y_j x_k + p_{jk} \cdot y_i x_j x_k \quad \text{for } 1 \le i < j < k \le 5.$$

And, finally, there is one generator for the quadric through the five points:

$$Q_{12345} := p_{12}p_{13}p_{23}p_{45} \cdot x_1x_2x_3y_4y_5 - p_{12}p_{14}p_{24}p_{35} \cdot x_1x_2y_3x_4y_5 \pm \cdots .$$

The ten monomials $x_i x_j x_k y_l y_m$ in this expression correspond to the ten splits $\{i, j, k\} \cup \{l, m\}$ of the index set $\{1, 2, 3, 4, 5\}$. There are $3^{10} \times 10$ ways to choose one monomial from each generator. To identify those choices that arise from moneric subspaces G, we examine the *tropical Plücker coordinates*

$$d_{ij} := -\mathrm{ord}(p_{ij}) \quad \text{for } 1 \le i < j \le 5.$$

After adding a positive constant, these quantities are non-negative, and they represent the distances in a finite metric space. We order them as follows:

$$d = (d_{12}, d_{13}, d_{14}, d_{15}, d_{23}, d_{24}, d_{25}, d_{34}, d_{35}, d_{45}).$$

The metrics d which arise from 2-dimensional subspaces G of K^5 are the lattice points in the *tropical Grassmannian* Trop(Gr(2, 5)). This is a 7-dimensional fan in $\mathbb{R}^{10}$ whose combinatorial structure is the Petersen graph [30]. Each of its 15 maximal cones is described as follows up to relabeling:

$$d_{ij} + d_{kl} = d_{ik} + d_{jl} \ge d_{il} + d_{jk} \quad \text{for } 1 \le i < j < k < l \le 5. \tag{13}$$

A computation reveals that the choice of leading terms for the generating set $\mathcal{F}$ of R^G divides the cone (13) into 160 smaller convex cones, and thus we obtain a finer fan structure with 2400 maximal cones on Trop(Gr(2, 5)). These 2400 cones determine 600 distinct monomial sets in($\mathcal{F}$). Modulo permuting the indices {1, 2, 3, 4, 5}, the 600 moneric classes come in seven types:

Type 1: $d = (1, 2, 3, 4, 3, 4, 5, 1, 2, 3)$, $\mathrm{in}(\mathcal{F}) = \{x_1, x_2, x_3, x_4, x_5, y_1x_2x_3, y_1x_2x_4, y_1x_2x_5, x_1y_3x_4, x_1y_3x_5, x_1y_4x_5, x_2y_3x_4, x_2y_3x_5, x_2y_4x_5, y_3x_4x_5, y_1x_2y_3x_4x_5\}$, **120**.
Type 2: $d = (5, 3, 5, 6, 4, 6, 7, 2, 3, 5)$, $\mathrm{in}(\mathcal{F}) = \{x_1, x_2, x_3, x_4, x_5, x_1x_2y_3, y_1x_2x_4, y_1x_2x_5, x_1y_3x_4, x_1y_3x_5, x_1y_4x_5, x_2y_3x_4, x_2y_3x_5, x_2y_4x_5, y_3x_4x_5, y_1x_2y_3x_4x_5\}$, **120**.
Type 3: $d = (2, 1, 3, 4, 3, 5, 6, 2, 3, 5)$, $\mathrm{in}(\mathcal{F}) = \{x_1, x_2, x_3, x_4, x_5, y_1x_2x_3, y_1x_2x_4, y_1x_2x_5, x_1y_3x_4, x_1y_3x_5, y_1x_4x_5, x_2y_3x_4, x_2y_3x_5, x_2y_4x_5, y_3x_4x_5, y_1x_2y_3x_4x_5\}$, **120**.
Type 4: $d = (1, 1, 4, 4, 2, 5, 5, 3, 3, 6)$, $\mathrm{in}(\mathcal{F}) = \{x_1, x_2, x_3, x_4, x_5, y_1x_2x_3, y_1x_2x_4, y_1x_2x_5, x_1y_3x_4, x_1y_3x_5, y_1x_4x_5, x_2y_3x_4, x_2y_3x_5, y_2x_4x_5, y_3x_4x_5, y_1x_2y_3x_4x_5\}$, **60**.
Type 5: $d = (1, 4, 5, 5, 5, 6, 6, 7, 7, 8)$, $\mathrm{in}(\mathcal{F}) = \{x_1, x_2, x_3, x_4, x_5, y_1x_2x_3, y_1x_2x_4, y_1x_2x_5, y_1x_3x_4, y_1x_3x_5, y_1x_4x_5, y_2x_3x_4, y_2x_3x_5, y_2x_4x_5, y_3x_4x_5, y_1y_3x_2x_4x_5\}$, **60**.
Type 6: $d = (1, 1, 2, 3, 2, 3, 4, 1, 2, 1)$, $\mathrm{in}(\mathcal{F}) = \{x_1, x_2, x_3, x_4, x_5, y_1x_2x_3, y_1x_2x_4, y_1x_2x_5, x_1y_3x_4, x_1y_3x_5, x_1y_4x_5, x_2y_3x_4, x_2y_3x_5, x_2y_4x_5, x_3y_4x_5, y_1x_2x_3y_4x_5\}$, **60**.
Type 7: $d = (1, 2, 3, 3, 3, 4, 4, 5, 5, 6)$, $\mathrm{in}(\mathcal{F}) = \{x_1, x_2, x_3, x_4, x_5, y_1x_2x_3, y_1x_2x_4, y_1x_2x_5, y_1x_3x_4, y_1x_3x_5, y_1x_4x_5, y_2x_3x_4, y_2x_3x_5, y_2x_4x_5, y_3x_4x_5, y_1y_2x_3x_4x_5\}$, **60**.

We now check for each of the seven moneric types whether it is sagbi. This is done as follows. We first compute the *toric ideal* of algebraic relations among the 16 monomial generators. For Types 1–6, this toric ideal is minimally generated by 20 quadratic binomials, while Type 7 requires 21 binomial generators. This immediately implies that Type 7 is not sagbi, because one the 21 binomials does not lift to a relation on R^G, thus violating the sagbi criterion in [9, Proposition 1.3]. For Types 1–6 the 20 binomials are distributed over ten different multidegrees. For instance, for Type 6 the minimal generators form a Gröbner basis (with leading terms listed first):

Degree	Two generators in that degree	
(2, 1, 1, 1, 1, 2)	$L_{125}L_{345} - x_5Q_{12345}$,	$L_{145}L_{235} - L_{135}L_{245}$
(2, 1, 1, 1, 2, 1)	$L_{124}L_{345} - x_4Q_{12345}$,	$L_{145}L_{234} - L_{134}L_{245}$
(2, 1, 1, 2, 1, 1)	$L_{123}L_{345} - x_3Q_{12345}$,	$L_{135}L_{234} - L_{134}L_{235}$
(2, 1, 2, 1, 1, 1)	$L_{123}L_{245} - x_2Q_{12345}$,	$L_{125}L_{234} - L_{124}L_{235}$
(2, 2, 1, 1, 1, 1)	$L_{123}L_{145} - x_1Q_{12345}$,	$L_{125}L_{134} - L_{124}L_{135}$
(2, 0, 1, 1, 1, 1)	$x_3L_{245} - x_2L_{345}$,	$x_5L_{234} - x_4L_{235}$
(2, 1, 0, 1, 1, 1)	$x_3L_{145} - x_1L_{345}$,	$x_5L_{134} - x_4L_{135}$
(2, 1, 1, 0, 1, 1)	$x_2L_{145} - x_1L_{245}$,	$x_5L_{124} - x_4L_{125}$
(2, 1, 1, 1, 0, 1)	$x_2L_{135} - x_1L_{235}$,	$x_5L_{123} - x_3L_{125}$
(2, 1, 1, 1, 1, 0)	$x_2L_{134} - x_1L_{234}$,	$x_4L_{123} - x_3L_{124}$

Each of these binomials lifts to a relation in the Cox–Nagata ring. For instance, the relation $x_2L_{134} - x_1L_{234}$ among the underlined leading monomials of $L_{134} = p_{14} \cdot \underline{x_1y_3x_4} + \cdots$ and $L_{234} = p_{24} \cdot \underline{x_2y_3x_4} + \cdots$ lifts to the relation $p_{24}x_2L_{134} - p_{14}x_1L_{234} - p_{34}x_3L_{124}$ in the prime ideal I^G of relations on R^G. Indeed, the twenty generators of I^G and their degrees are well-known, and the existence of these quadrics verifies the sagbi property for Types 1–6. □

Remark 4.2. The existence of a nice sagbi basis implies desirable commutative algebra properties. The reason is that the algebra under consideration is a flat deformation of its initial toric algebra [9]. For instance, the Cox–Nagata ring R^G is normal, Gorenstein and Koszul whenever the toric algebra $\mathrm{in}(R^G)$ has these properties. These properties for $d = 3$, $n = 5$ and G generic are known by work of Popov [27], but we can now easily derive combinatorial certificates from our sagbi bases in Theorem 4.1. The Gröbner basis for the toric ideal of Type 6 is squarefree and quadratic. We conclude that the toric algebra $\mathrm{in}(R^G)$ is normal and Koszul, and hence so is its flat deformation R^G. The Gorenstein property of R^G holds because the toric algebras $\mathrm{in}(R^G)$ of Types 1–5 are Gorenstein with the same $\mathbb{Z}^6$-graded Betti numbers as R^G.

The Cox–Nagata ring R^G and its initial algebras $\mathrm{in}(R^G)$ share the same Hilbert function ψ. Each of the six sagbi types specifies a formula for the piecewise quadratic function ψ. For instance, the formula in Example 1.3 comes from Type 6. We explain how this works. The variables y_2 and y_5 are absent from $\mathrm{in}(\mathcal{F})$. Hence every monomial in the subalgebra $\mathrm{in}(R^G)$ of $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ has the form $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}y_1^{b_1}y_3^{b_3}y_4^{b_4}$ and corresponds to a lattice point $(a, b) = (a_1, a_2, a_3, a_4, a_5, b_1, b_3, b_4)$ in $\mathbb{Z}^8$. We now perform a change of variables $(a, b) \mapsto (r, u, x, y)$ in the lattice $\mathbb{Z}^8$ as follows:

$$r = b_1 + b_3 + b_4, \quad x = b_3 + b_4, \quad y = b_4, \quad u_1 = a_1 + b_1, \quad u_2 = a_2,$$
$$u_3 = a_3 + b_3, \quad u_4 = a_4 + b_4, \quad u_5 = a_5.$$

Let Γ denote the convex cone in $\mathbb{R}^8$ generated by the 16 vectors (r, u, x, y) corresponding to the 16 monomials in $\mathrm{in}(\mathcal{F})$. Then $\Gamma \cap \mathbb{Z}^8$ is the normal semigroup whose semigroup algebra equals $\text{in}(R^G)$. Our degree function is now the projection $(r, u, x, y) \mapsto (r, u)$ onto the first six coordinates, and the Hilbert function of our Type 6 semigroup $\Gamma \cap \mathbb{Z}^8$ for this grading equals

$$\psi(r, u) = \#\{(x, y) \in \mathbb{Z}^2 : (r, u, x, y) \in \Gamma\}. \tag{14}$$

What is left to do is to compute the cone Γ. The software `polymake` reveals that the f-vector of Γ equals $(16, 80, 180, 216, 148, 58, 12)$, and it generates the twelve facet-defining inequalities of Γ. These are precisely the twelve inequalities, such as $y - x \leq u_4 + u_5 - r$, which are listed in Example 1.3.

We had chosen Type 6 for the formula in Example 1.3 because it gives the fewest linear inequalities (only 12). For the other types, the 8-dimensional cone Γ has ≥ 14 facets. More precisely, $f(\Gamma) = (16, 84, 200, 253, 180, 71, 14)$ for Type 1, and $f(\Gamma) = (16, 87, 221, 301, 229, 94, 18)$ for Types 2, 3, 4 and 5.

The 540 cones Γ all share the same image under the linear map

$$\text{degree} : \mathbb{R}^8 \to \mathbb{R}^6, \qquad (r, u, x, y) \mapsto (r, u).$$

That image is the 6-dimensional support cone $\mathcal{C}^G = \text{degree}(\Gamma)$, with f-vector $(16, 80, 160, 120, 26)$. The underlying support polytope $\mathcal{P}^G$ is the *demicube*

$$\mathcal{P}^G = \text{conv}\{(10000), \ldots, (00001), (11100), (11010), \ldots, (00111), (11111)\}.$$

Each of our 540 sagbi bases specifies a 7-dimensional polytope Π with a distinguished projection $\Pi \to \mathcal{P}^G$ onto the 5-dimensional demicube. Its fibers over the interior of $\mathcal{P}^G$ are 2-dimensional polygons, one for each ample line bundle on the del Pezzo surface X_G. The corresponding projective embedding of X_G degenerates to the toric surface associated with that polygon.

5. The cubic surface

Let G be a generic linear subspace of dimension 3 in K^6 and let p_{ijk} denote its dual Plücker coordinates, i.e., p_{ijk} is the 3×3-minor with column indices $(i.j, k)$ of a 3×6-matrix A whose kernel equals G. The Cox–Nagata ring R^G has 27 minimal generators. They correspond to the 27 lines in the cubic surface gotten from $\mathbb{P}^2$ by blowing up the points in the columns of A. There are six generators $E_i = x_i$ representing the exceptional divisors, 15 generators F_{ij} representing the lines through pairs of points, and six generators G_i representing the quadrics through any five of the points. We can express these as polynomials in x and y whose coefficients are expressions in the p_{ijk}. For instance, the generator for the line through points 1 and 2 equals

$$F_{12} = p_{123} \cdot y_3x_4x_5x_6 + p_{124} \cdot x_3y_4x_5x_6 + p_{125} \cdot x_3x_4y_5x_6 + p_{126} \cdot x_3x_4x_5y_6,$$

and the generator for the quadric through 1, 2, 3, 4, 5 equals

$$
\begin{aligned}
G_6 = {} & p_{123}p_{124}p_{125}p_{345}\cdot y_1y_2x_3x_4x_5x_6^2 - p_{132}p_{135}p_{134}p_{245}\cdot y_1y_3x_2x_4x_5x_6^2\\
& + p_{142}p_{143}p_{145}p_{235}\cdot y_1y_4x_2x_3x_5x_6^2 - p_{152}p_{153}p_{154}p_{234}\cdot y_1y_5x_2x_3x_4x_6^2\\
& + p_{231}p_{234}p_{235}p_{145}\cdot y_2y_3x_1x_4x_5x_6^2 - p_{241}p_{243}p_{245}p_{135}\cdot y_2y_4x_1x_3x_5x_6^2\\
& + p_{251}p_{253}p_{254}p_{134}\cdot y_2y_5x_1x_3x_4x_6^2 + p_{341}p_{342}p_{345}p_{125}\cdot y_3y_4x_1x_2x_5x_6^2\\
& - p_{351}p_{352}p_{354}p_{124}\cdot y_3y_5x_1x_2x_4x_6^2 + p_{451}p_{452}p_{453}p_{123}\cdot y_4y_5x_1x_2x_3x_6^2\\
& + (p_{124}p_{235}p_{136}p_{145} - p_{123}p_{245}p_{146}p_{135})\cdot y_1y_6x_2x_3x_4x_5x_6\\
& + (p_{124}p_{135}p_{236}p_{245} - p_{123}p_{145}p_{246}p_{235})\cdot y_2y_6x_1x_3x_4x_5x_6\\
& + (p_{134}p_{125}p_{236}p_{345} + p_{123}p_{145}p_{346}p_{235})\cdot y_3y_6x_1x_2x_4x_5x_6\\
& + (p_{124}p_{135}p_{346}p_{245} - p_{134}p_{125}p_{246}p_{345})\cdot y_4y_6x_1x_2x_3x_5x_6\\
& + (p_{125}p_{134}p_{356}p_{245} - p_{135}p_{124}p_{256}p_{345})\cdot y_5y_6x_1x_2x_3x_4x_6\\
& + (p_{124}p_{135}p_{236}p_{456} - p_{123}p_{145}p_{246}p_{356})\cdot y_6^2x_1x_2x_3x_4x_5.
\end{aligned}
$$

Note that $\deg(F_{12}) = (1,0,0,1,1,1,1)$ and $\deg(G_6) = (2,1,1,1,1,1,2)$.

We shall now construct a toric model for the cubic surface X_G as follows.

Theorem 5.1. *There exists a* 3*-dimensional sagbi subspace G of K^6 whose toric ideal* $\mathrm{in}(I^G)$ *is generated by quadrics and has a squarefree Gröbner basis.*

Proof. Let G denote the kernel of the matrix

$$
A = \begin{bmatrix} t^1 & t^{11} & t^{11} & t^{13} & t^7 & t^7\\ t^6 & 1 & t^{13} & t^{10} & t^{15} & t^{15}\\ t^6 & t^5 & 1 & t^{15} & t^5 & t \end{bmatrix}.
$$

This subspace induces the following initial monomials for the 27 generators:

E_1	E_2	E_3	E_4	E_5	E_6	F_{12}	F_{13}	F_{14}	F_{15}
x_1	x_2	x_3	x_4	x_5	x_6	$y_3x_4x_5x_6$	$y_2x_4x_5x_6$	$x_2y_3x_5x_6$	$y_2x_3x_4x_6$

F_{16}	F_{23}	F_{24}	F_{25}	F_{26}	F_{34}	F_{35}
$y_2x_3x_4x_5$	$y_1x_4x_5x_6$	$x_1y_3x_5x_6$	$y_1x_3x_4x_6$	$y_1x_3x_4x_5$	$y_1x_2x_5x_6$	$x_1y_2x_4x_6$

F_{36}	F_{45}	F_{46}	F_{56}	G_6	G_5
$x_1y_2x_4x_5$	$y_1x_2x_3x_6$	$y_1x_2x_3x_5$	$x_1y_2x_3x_4$	$x_1y_2y_3x_4x_5x_6^2$	$x_1y_2y_3x_4x_5^2x_6$

G_4	G_3	G_2	G_1
$x_1y_2y_3x_4^2x_5x_6$	$x_1y_2y_3x_3x_4x_5x_6$	$x_1x_2y_2y_3x_4x_5x_6$	$x_1^2y_2y_3x_4x_5x_6$.

The toric ideal of relations among these 27 monomials is minimally generated by 81 quadrics. These generators occur as triples in 27 distinct degrees. For example, in degree $(1,1,1,1,0,1,1)$ we find the three generators $E_3F_{34} - E_6F_{46}$, $E_1F_{14} - E_2F_{24}$, $E_3F_{34} - E_5F_{45}$, in degree $(3,2,1,2,2,2,2)$ we find the three minimal generators $F_{12}G_1 - F_{24}G_4$, $F_{23}G_3 - F_{25}G_5$, $F_{25}G_5 - F_{26}G_6$ etc.

Each of these 81 binomial relations among our 27 monomials lifts to a quadratic relation in the presentation ideal I^G of the Cox–Nagata ring R^G. The lifting property can be checked either directly, by writing down a quadratic polynomial in I^G whose initial form is the given binomial, or indirectly, by computing the values of ψ on each of the 27 observed degrees:

$$\psi(1,1,1,1,0,1,1) = 2, \quad \psi(3,2,1,2,2,2,2) = 2, \quad \ldots.$$

These values are obtained easily in `Macaulay 2`, namely, by computing the $\mathbb{Z}$-graded Hilbert function of the artinian algebra $K[\partial_1, \ldots, \partial_d]/I_u$ in the Introduction. Of course, only one such computation suffices if we use the fact that all 27 degrees are in a single orbit under the Cremona action (9) of the Weyl group E_6. Using [9, Proposition 1.3] we conclude that $\mathrm{in}(R^G)$ is generated by the 81 monomials listed above, and that the toric ideal $\mathrm{in}(I^G)$ is generated by the 81 quadratic binomials $E_3F_{34} - E_6F_{46}, \ldots, F_{25}G_5 - F_{26}G_6, \ldots$.

A computation shows that the reduced Gröbner basis of the toric ideal $\mathrm{in}(I^G)$ with respect to the reverse lexicographic order has squarefree initial monomials. This property ensures that the toric algebra $\mathrm{in}(R^G)$ is normal, and we can expect a nice polyhedral formula for its Hilbert function ψ. □

Just as in Section 4, our sagbi basis automatically implies a polyhedral formula for ψ, namely, we need to compute the facet inequalities of the cone Γ underlying the normal toric algebra $\mathrm{in}(R^G)$. Using `polymake`, we find

$$f(\Gamma) = (27, 216, 747, 1287, 1191, 603, 162, 21).$$

The 21 facet inequalities translate into a formula for ψ if we perform the following change of variables $(a, b) \mapsto (r, u, x, y)$ in the lattice $\mathbb{Z}^9$:

$$r = b_1 + b_2 + b_3, \quad x = b_2, \quad y = b_3, \quad u_1 = a_1 + b_1, \quad u_2 = a_2 + b_2,$$
$$u_3 = a_3 + b_3, \quad u_4 = a_4, \quad u_5 = a_5, \quad u_6 = a_6.$$

After this change of coordinates, Γ is the convex cone in $\mathbb{R}^9$ generated by the 27 vectors (r, u, x, y) corresponding to the 27 monomial generators

$$x^a y^b = x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} x_5^{a_5} x_6^{a_6} y_1^{b_1} y_2^{b_2} y_3^{b_3}$$

of $\mathrm{in}(R^G)$. Then formula (14) holds, and we find:

Corollary 5.2. *If $d = 3$, $n = 6$ and the linear forms $\ell_1, \ldots, \ell_6$ are generic then $\psi(r, u_1, \ldots, u_6)$ equals the number of lattice points (x, y) satisfying*

$$\max(0, 2r-u_2-u_3-u_4, 2r-u_2-u_3-u_6) \le x \le \min(u_1, u_5, u_1+u_4+u_5+u_6-2r),$$
$$\max(0, 2r-u_1-u_4-u_6, 2r-u_4-u_5-u_6) \le y \le \min(u_2, u_3, u_1+u_2+u_3+u_5-2r),$$
$$\max(r-u_4, r-u_6, 3r-u_2-u_3-u_4-u_6) \le x+y$$
$$\le \min(r, u_1+u_3+u_5-r, u_1+u_2+u_5-r),$$
$$r-u_2-u_3 \le x-y, \quad 2x+y \le u_1+u_5, \quad 2r-u_4-u_6 \le x+2y.$$

Proof. Theorem 5.1 establishes this formula for the subspace G which is defined by the specific 3×6-matrix A chosen above. However, we can multiply each entry of A by a generic complex number, and the initial monomials of the 27 generators will not change. Hence G lies in the open dense subset of the Grassmannian where the $\mathbb{Z}^7$-graded Hilbert function of R^G is constant, and our formula gives that Hilbert function. □

Remark 5.3. Using the formula above, we can rapidly compute the dimension of the space of sections (7) of any line bundle on the del Pezzo surface X_G of degree three. In particular, we can check that $\psi(3, 2, 2, 2, 2, 2, 2) = 4$, which corresponds to the familiar embedding of X_G as a cubic surface in $\mathbb{P}^3$.

We conjecture that the formula of Corollary 5.2 is optimal, in the sense that the number 21 of linear inequalities is minimal. Equivalently, the number of facets of the 9-dimensional cone Γ arising from any three-dimensional sagbi subspace G of K^6 is at least 21. A proof of this conjecture might require the complete classification of the equivalence classes of subspaces, as was done for $n = 5$ in Section 4. We suggest this as a research problem:

Problem 5.4. Determine all equivalence classes of 3-dimensional sagbi subspaces of K^6, i.e. extend the classification of Theorem 4.1 from K^5 to K^6.

In such a classification, the role of the tropical Grassmannian Trop(Gr(2, 5)) would be played by a suitable tropical model of the moduli space of cubic surfaces. We expect this model to be a variant of the fan constructed by Hacking, Keel and Tevelev [17]. The Naruki coordinates on that moduli space correspond to the coefficients of the generators of R^G. An example of such a coordinate is the coefficient $p_{124}p_{135}p_{236}p_{456} - p_{123}p_{145}p_{246}p_{356}$ appearing in the generator G_6 listed prior to Theorem 5.1. The task of Problem 5.4 is to divide that variant of the Hacking–Keel–Tevelev fan into many small cones, one for each system of choices of leading terms for the 27 generators of R^G.

6. Del Pezzo surfaces of degree one and two

The computational result of the previous section extends to $n = 7$ and $n = 8$:

Theorem 6.1. *Let $4 \leq n \leq 8$. Then there exists a generic 3-dimensional sagbi subspace G of K^n whose toric ideal* $\mathrm{in}(I^G)$ *is generated by quadrics.*

Since $\mathrm{in}(I^G)$ is a flat degeneration of I^G, Theorem 6.1 implies in particular that the presentation ideal I^G of the Cox–Nagata ring R^G is generated by quadrics. This furnishes a computational proof of the following result which was obtained independently also by Testa, Várilly-Alvarado and Velasco [33].

Corollary 6.2. *The presentation ideal of the Cox ring of a del Pezzo surface gotten by blowing up at most 8 general points in $\mathbb{P}^2$ is generated by quadrics.*

Proof. The proof of Theorem 6.1 below establishes the claimed quadratic presentation for the subspace G which is defined by the specific 3×7-matrix A in (15). However, we can multiply each entry of A by a generic complex number, and the initial monomials of the 56 generators will not change. Hence G lies in the open dense subset of the Grassmannian where the Betti numbers in the minimal free resolution of R^G as an R-module are constant. □

This quadratic generation result had been conjectured by Batyrev and Popov in [3], and it was proved for $n \leq 7$ in the subsequent papers [11, 21, 29].

Proof of Theorem 6.1. The case $n = 4$ is covered by Theorem 3.5 and the cases $n = 5$ and $n = 6$ are dealt with in Theorems 4.1 and 5.1 respectively. In what follows we present choices of sagbi subspaces G for $n = 7$ and $n = 8$.

First consider the case $n = 7$ where X_G is the del Pezzo surfaces of degree two. The Cox–Nagata ring R^G has 56 minimal generators. There are seven generators $E_i = x_i$ for the exceptional divisors, 21 generators F_{ij} representing the lines through pairs $\{\ell_i, \ell_j\}$ of points, 21 generators G_{ij} representing the quadrics through any five of the points, missing $\{\ell_i, \ell_j\}$, and seven generators C_i representing the cubics through all seven points, where ℓ_i is a double point.

Let G be the 4-dimensional subspace of K^7 which is the kernel of

$$A = \begin{bmatrix} t^3 & t^{10} & 1 & t^6 & t^{17} & t^{12} & t^{11} \\ t^{18} & t^{15} & t^8 & t^4 & t^6 & t^7 & t \\ t^{10} & t^{16} & t^2 & 1 & t^6 & t^4 & t^9 \end{bmatrix}. \tag{15}$$

This subspace induces the following initial monomials for the 56 generators:

E_1	E_2	E_3	E_4	E_5	E_6	E_7	F_{12}	F_{13}	F_{14}
x_1	x_2	x_3	x_4	x_5	x_6	x_7	$x_3x_5x_6x_7y_4$	$x_2x_4x_5x_6y_7$	$x_2x_3x_5x_6y_7$

F_{15}	F_{16}	F_{17}	F_{23}	F_{24}	F_{25}
$x_2x_3x_6x_7y_4$	$x_2x_3x_4x_5y_7$	$x_2x_3x_5x_6y_4$	$x_1x_4x_5x_6y_7$	$x_1x_3x_5x_6y_7$	$x_1x_3x_6x_7y_4$
F_{26}	F_{27}	F_{34}	F_{35}	F_{36}	F_{37}
$x_1x_3x_4x_5y_7$	$x_1x_3x_5x_6y_4$	$x_1x_2x_5x_6y_7$	$x_1x_2x_6x_7y_4$	$x_1x_2x_4x_5y_7$	$x_1x_2x_5x_6y_4$
F_{45}	F_{46}	F_{47}	F_{56}	F_{57}	F_{67}
$x_1x_2x_6x_7y_3$	$x_1x_2x_5x_7y_3$	$x_1x_2x_5x_6y_3$	$x_1x_2x_4x_7y_3$	$x_1x_2x_4x_6y_3$	$x_1x_2x_4x_5y_3$

G_{67}	G_{57}	G_{56}	G_{47}
$x_1x_2x_3x_5x_6^2x_7y_4y_7$	$x_1x_2x_3x_4x_5^2x_6y_7^2$	$x_1x_2x_3x_5^2x_6^2y_4y_7$	$x_1x_2x_3x_4x_5x_6x_7y_4y_7$
G_{46}	G_{45}	G_{37}	G_{36}
$x_1x_2x_3x_5x_6^2x_7y_4^2$	$x_1x_2x_3x_4x_5^2x_6y_4y_7$	$x_1x_2x_3x_5x_6x_7^2y_3y_4$	$x_1x_2x_3x_5x_6^2x_7y_3y_4$
G_{35}	G_{34}	G_{27}	G_{26}
$x_1x_2x_3x_5^2x_6x_7y_3y_4$	$x_1x_2x_3x_4x_5x_6x_7y_3y_4$	$x_1x_2^2x_4x_5x_6x_7y_3y_7$	$x_1x_2^2x_4x_5x_6^2y_3y_7$

$$
\begin{array}{cccc}
G_{25} & G_{24} & G_{23} & G_{17} \\
x_1x_2^2x_4x_5^2x_6y_3y_7 & x_1x_2^2x_4^2x_5x_6y_3y_7 & x_1x_2^2x_3x_4x_5x_6y_3y_7 & x_1^2x_2x_4x_5x_6x_7y_3y_7 \\[1ex]
G_{16} & G_{15} & G_{14} & G_{13} \\
x_1^2x_2x_4x_5x_6^2y_3y_7 & x_1^2x_2x_4x_5^2x_6y_3y_7 & x_1^2x_2x_4^2x_5x_6y_3y_7 & x_1^2x_2x_3x_4x_5x_6y_3y_7 \\[1ex]
G_{12} & C_1 & C_2 & C_3 \\
x_1^2x_2^2x_4x_5x_6y_3y_7 & x_1x_2^2x_3x_4x_5^2x_6^2x_7y_3y_4y_7 & x_1^2x_2x_3x_4x_5^2x_6^2x_7y_3y_4y_7 & x_1^2x_2^2x_4^2x_5^2x_6^2y_3y_7^2 \\[1ex]
C_4 & C_5 & C_6 & C_7 \\
x_1^2x_2^2x_3x_4x_5^2x_6^2y_3y_7^2 & x_1^2x_2^2x_3x_4x_5x_6^2x_7y_3y_4y_7 & x_1^2x_2^2x_3x_4^2x_5^2x_6y_3y_7^2 & x_1^2x_2^2x_3x_4x_5^2x_6^2y_3y_4y_7.
\end{array}
$$

The toric ideal of relations among these 56 monomials has 529 minimal generators. All generators are quadratic binomials. This computation is performed most efficiently using the software `4ti2` due to Hemmecke et al. [1]. The 56 monomials involve 10 variables. We input these data as a 10×56-matrix into `4ti2` and apply the command `markov`. The resulting *Markov basis* consists of 529 vectors, each representing a quadratic binomial in 56 unknowns.

To check that the subspace G is sagbi, it remains to verify that each of the 529 quadratic binomials lifts to a relation in I^G. This is done in a self-contained manner by computing $\psi(r, u)$—using `Macaulay`—for each degree (r, u) that occurs among the 529 binomials in the output from `4ti2`. To verify the lifting condition of [9, Proposition 1.3], it suffices to check that $\psi(r, u)$ plus the number of binomials of degree (r, u) equals the number of all monomials of degree (r, u) in the quantities E_i, F_{ij}, G_{ij} and C_i.

Of course, this last computational step would be unnecessary if we allow ourselves to apply prior results from the literature. Indeed, it was already shown by Derenthal [11, §4] that I^G contains precisely 529 linearly independent quadrics, so we just need to compare the degrees output by `4ti2` with the degrees in Derenthal's generators, and the sagbi property follows. However, we wish to emphasize that our approach is independent of any prior work, as we can verify the sagbi property of G by computing a few values of ψ. We conclude that $\mathrm{in}(I^G)$ is generated by 529 quadrics and hence so is I^G.

We now come to the hardest case, $n = 8$, where X_G is the del Pezzo surface of degree one. Derenthal [11, Lemma 15] computed that the ideal I^G contains 17399 linearly independent quadrics, and he showed that these quadrics generate I^G up to radical. Testa, Várilly-Alvarado and Velasco [33] applied the methods of [21] to this situation, and they succeeded in proving that these 17399 quadrics do indeed generate the prime ideal I^G. Our sagbi approach gives an alternative proof which is computational and elementary. The only prior knowledge we are using is that R^G has 242 minimal generators.

Let G be the 5-dimensional subspace of K^8 which is the kernel of

$$
A = \begin{bmatrix} t^6 & t^{10} & t^3 & t^{10} & t^1 & t^4 & t^{10} & t^2 \\ t^{10} & t^3 & t^8 & t^6 & t^8 & t^1 & t^8 & t^8 \\ t^4 & t^8 & t^7 & t^7 & t^8 & t^5 & t & t^9 \end{bmatrix}. \tag{16}
$$

The initial form of each minimal generator of R^G is found to be a monomial, so the subspace G is moneric. Consider this list of 242 monomials in $x_1, \ldots, x_8, y_1, \ldots, y_8$.

Actually, of the eight y-variables only three appear among these monomials, so we are facing a list of 242 monomials in 11 variables, and our task is to compute the toric ideal of algebraic relations among these monomials. We write the list of monomials as a 11×242-matrix of non-negative integers, we input that matrix into the software `4ti2`, and we apply the command `markov` to compute minimal generators of the toric ideal.

After two days or so, the computation terminates. The output is an integer matrix with 17399 rows and 242 columns. Each row represents a binomial in our toric ideal, and we check that all 17399 binomials are quadrics. Using the same technique as in the proof for $n = 7$, namely computing a few values of ψ, we verify that all these 17399 quadratic binomials lift to polynomials in I^G. This proves that both $\mathrm{in}(I^G)$ and I^G are generated by quadrics. □

Remark 6.3. In the proofs of Theorems 5.1 and 6.1, the matrices A play an important role. We found these matrices by random search. We generated matrices A whose entries are random integer powers of t, and in each case we checked whether the subspace $G = \mathrm{kernel}(A)$ is sagbi. This process terminates surprisingly rapidly. For instance, even for $n = 8$ it takes only about five iterations on average to get a matrix A whose kernel G is sagbi.

Remark 6.4. All input and output files for the software `4ti2` used in this proof are posted at the website http://lsec.cc.ac.cn/~xuzq/cox.html.

Our proof of the Batyrev–Popov conjecture for del Pezzo surfaces of degree one is a fairly automatic process, albeit computationally intensive. Given standard computer algebra tools to carry out the verification of the sagbi property, the proof only amounts to exhibiting the matrices (15) and (16). Naturally, it would be extremely interesting to explore the moduli space of possibilities for such matrices, along the lines suggested in Problem 5.4.

Our motivation for this study was to find a formula for the counting function ψ. The sagbi matrices (15) and (16) specify explicit Ehrhart-type formulas for ψ, just as in Example 1.3 and Corollary 5.2. The shape of that formula appears in (14), and it is made completely explicit by listing the facets of the convex polyhedral cone Γ associated with the toric algebra $\mathrm{in}(R^G)$. For $n = 7$ the cone Γ is 10-dimensional and has 56 rays. For $n = 8$ the cone Γ is 11-dimensional and has 240 rays. An even higher-dimensional example is discussed in Example 8.4. See also Corollary 7.12 for $n = d + 2$.

When the Cox–Nagata ring R^G is not finitely generated, (e.g. when the inequality (6) does not hold), the convex cone Γ for $\mathrm{in}(R^G)$ still exists but it will no longer be polyhedral. The equation (14) remains valid, but the question how to make this formula explicit and useful requires further study.

7. Phylogenetic algebraic geometry

In this section we fix $n = d + 2$ and we assume that G is the row space of a $2 \times n$-matrix (b_{kl}) as in (11) whose Plücker coordinates $p_{ij} = b_{1i}b_{2j} - b_{1j}b_{2i}$ are all non-zero. For any subset $\{i_0 < i_1 < \cdots < i_{2k}\}$ of $\{1, \ldots, n\}$ having odd cardinality we define $Q_{i_0 i_1 \cdots i_{2k}}$ to be the determinant of the matrix

$$\begin{bmatrix} b_{1i_0}^k x_{i_0} & b_{1i_1}^k x_{i_1} & b_{1i_2}^k x_{i_2} & b_{1i_3}^k x_{i_3} & \cdots & b_{1i_{2k}}^k x_{i_{2k}} \\ b_{1i_0}^{k-1} y_{i_0} & b_{1i_1}^{k-1} y_{i_1} & b_{1i_2}^{k-1} y_{i_2} & b_{1i_3}^{k-1} y_{i_3} & \cdots & b_{1i_{2k}}^{k-1} y_{i_{2k}} \\ b_{1i_0}^{k-1} b_{2i_0} x_{i_0} & b_{1i_1}^{k-1} b_{2i_1} x_{i_1} & b_{1i_2}^{k-1} b_{2i_2} x_{i_2} & b_{1i_3}^{k-1} b_{2i_3} x_{i_3} & \cdots & b_{1i_{2k}}^{k-1} b_{2i_{2k}} x_{i_{2k}} \\ b_{1i_0}^{k-2} b_{2i_0} y_{i_0} & b_{1i_1}^{k-2} b_{2i_1} y_{i_1} & b_{1i_2}^{k-2} b_{2i_2} y_{i_2} & b_{1i_3}^{k-2} b_{2i_3} y_{i_3} & \cdots & b_{1i_{2k}}^{k-2} b_{2i_{2k}} y_{i_{2k}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2i_0}^k x_{i_0} & b_{2i_1}^k x_{i_1} & b_{2i_2}^k x_{i_2} & b_{2i_3}^k x_{i_3} & \cdots & b_{2i_{2k}}^k x_{i_{2k}} \end{bmatrix}. \tag{17}$$

This is a homogeneous polynomial with

$$\deg(Q_{i_0 i_1 \cdots i_{2k}}) = ke_0 + e_{i_0} + e_{i_1} + \cdots + e_{i_{2k}}. \tag{18}$$

Moreover, $Q_{i_0 i_1 \cdots i_{2k}}$ is invariant under the Nagata action. Indeed, the action by any vector in G adds to each y-row in the matrix (17) a linear combination of the two adjacent x-rows. This leaves the determinant unchanged. We can write the coefficients of this polynomial as products of Plücker coordinates:

$$Q_{i_0 i_1 \cdots i_{2k}} = \sum \pm \Big(\prod_{i,j \in \{0,\ldots,k\}} p_{a_i a_j} \Big) \cdot \Big(\prod_{r,s \in \{1,\ldots,k\}} p_{b_r, b_s} \Big) \cdot x_{a_0} x_{a_1} \cdots x_{a_k} y_{b_1} \cdots y_{b_k} \tag{19}$$

where the sum is over all partitions

$$\{i_0, i_1, \ldots, i_{2k}\} = \{a_0, a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_k\}.$$

Theorem 7.1 (Castravet–Tevelev [8]). *The Cox–Nagata ring R^G is minimally generated by the 2^{n-1} invariants $Q_{i_0 i_1 \cdots i_{2k}}$ where $1 \le i_0 < \cdots < i_{2k} \le n$.*

This result is stated in [8, Theorem 1.1] for the case when one row of the matrix (b_{ij}) consists of ones. The general case easily follows because the del Pezzo variety X_G and its Cox ring remain unchanged if the blown-up points in $\mathbb{P}^{n-3}$ undergo a projective transformation. It is important to note, however, that our sagbi analysis will not work if the vector of ones lies in G.

Another key ingredient for this section is the celebrated *Verlinde formula* which lies at the interface of algebraic geometry and mathematical physics. We refer to equation (12.6) of Mukai's book [23, §12]. The following interpretation of that formula and its proof were suggested to us by Jenia Tevelev.

Theorem 7.2 (Verlinde formula). *For $n = d + 2$ and G generic we have*

$$\psi(dl, 2l, \ldots, 2l) = \frac{1}{2l+1} \sum_{j=0}^{2l} (-1)^{dj} \Big(\sin \frac{2j+1}{4l+2} \pi \Big)^{-d}.$$

Here l can be a half-integer if d is even but must be an integer if d is odd.

Proof. The right hand side is the number of sections of multiples of the canonical line bundle on the moduli space $\mathcal{N}_{0,n}$ of rank two stable quasiparabolic vector bundles on $\mathbb{P}^1$ with $n = d + 2$ marked points. See [23, §12.5]. A result due to Stefan Bauer [4] states that $\mathcal{N}_{0,n}$ and the blow-up X_G of $\mathbb{P}^{n-3}$ at n points are related by a sequence of flops. See [24, §2] and after [23, Theorem 12.56]. This implies that $\mathcal{N}_{0,n}$ and the blow-up X_G have the same Picard group, their Cox rings are isomorphic, and their canonical classes agree.

The anticanonical class on X_G equals $-K = dH - (d-2)(E_1 + \cdots + E_{d+2})$. This is a primitive element in the Picard group when d is odd but divisible by 2 when d is even [23, Remark 12.54]. The right hand sum above equals

$$\dim_K H^0(\mathcal{N}_{0,d+2}, \mathcal{O}(-lK)) = \dim_K H^0(X_G, \mathcal{O}(-lK)).$$

Our equation (7) implies that this dimension equals $\psi(r, u_1, \ldots, u_n)$ where $r = dl$, $u_i - r = l(2-d)$ and l is allowed to be a half-integer if d is even. □

Hilbert functions of ideals generated by powers of n general linear forms in $n - 2$ unknowns were studied by D'Cruz and Iarrobino in [10]. The previous two theorems establish both parts of their main conjecture on page 77 of [10].

Corollary 7.3. *If $n = 2k + 1$ and $d = 2k - 1$ then $\psi(k, 1, \ldots, 1) = 1$.*

Proof. From Theorem 7.1 we know that the generators of R^G are $Q_{i_0 \cdots i_{2k}}$ where $1 \leq i_0 < \cdots < i_{2k} \leq n$. Let $I_1, \ldots, I_m$ be subsets of $\{1, \ldots, n\}$ having odd cardinality and let $j_1, \ldots, j_m$ be positive integers such that $\deg(Q_{I_1}^{j_1} \cdots Q_{I_m}^{j_m}) = (k, 1, \ldots, 1)$. From (18), we find that $j_1 = \cdots = j_m = 1$. The equation $\deg(Q_{I_1} \cdots Q_{I_m}) = (k, 1, \ldots, 1)$ implies

$$k = (\#I_1 - 1)/2 + \cdots + (\#I_m - 1)/2 = (2k + 1)/2 - m/2.$$

We conclude that $m = 1$ and $I_1 = \{1, \ldots, n\}$. So, if $n = 2k+1$ then $Q_{12\cdots n}$ is the unique element (up to scaling) of the Cox–Nagata ring R^G in degree $(k, 1, \ldots, 1)$. Hence the vector space $R^G_{(k,1,\ldots,1)}$ is one-dimensional. □

Corollary 7.4. *If $n = 2k + 2$ and $d = 2k$ then $\psi(k, 1, \ldots, 1) = 2^k$.*

Proof. For $d = n - 2 = 2k$ and $l = 1/2$ the trigonometric sum in the Verlinde formula (Theorem 7.2) simplifies to 2^k. See also [23, page 483, line 4]. We are grateful to Jenia Tevelev for suggesting this derivation to us. □

We are now prepared to move towards the punchline of this section: *Sagbi bases connect the Verlinde formula to phylogenetic algebraic geometry*. Let $K[q]$ and $\mathbb{Q}[q]$ denote the polynomial rings over K and $\mathbb{Q}$ in 2^{n-1} variables $q_{i_0 i_1 \cdots i_{2k}}$, one for each odd subset of $\{1, \ldots, n\}$. We seek to compute the presentation ideal $I^G \subset K[q]$ of the Cox–Nagata ring R^G. Our approach to this problem is to identify subspaces G that are sagbi. This allows us to study I^G by way of its toric initial ideal $\mathrm{in}(I^G)$ in $\mathbb{Q}[q]$. We begin by introducing a class of toric ideals in $\mathbb{Q}[q]$. These represent statistical models in [6, 32].

Let T be a trivalent phylogenetic tree with leaves labeled by $\{1, \ldots, n\}$. Each interior edge of T corresponds to a *split*, by which we mean an unordered pair $\{A, B\}$ such that $A \cup B = \{1, \ldots, n\}$, $A \cap B = \emptyset$, $|A| \geq 2$ and $|B| \geq 2$. The set of all $n - 3$ splits of

T is denoted splits(T). By [26, Theorem 2.35], the combinatorial type of the tree T is uniquely specified by the set splits(T).

For each $\{A, B\} \in$ splits(T) we introduce two matrices whose entries are variables in $\mathbb{Q}[q]$. They are denoted $\mathbf{M}_{A,B}$ and $\mathbf{M}_{B,A}$. Their formats are $2^{|A|-1} \times 2^{|B|-1}$ and $2^{|B|-1} \times 2^{|A|-1}$ respectively. For the matrix $\mathbf{M}_{A,B}$, the rows are indexed by even subsets σ of A, the columns are indexed by odd subsets τ of B, and the entry in row σ and column τ is the variable $q_{\sigma\cup\tau}$. Similarly, the entries of $M_{B,A}$ are the variables $q_{\sigma\cup\tau}$ for $\sigma \subseteq B$ even and $\tau \subseteq A$ odd. We write $\mathbf{I}_T$ for the ideal in $\mathbb{Q}[q]$ which is generated by the 2×2-minors of all $2n-6$ matrices $\mathbf{M}_{A,B}$ and $\mathbf{M}_{B,A}$ where $\{A, B\}$ runs over splits(T). Known results in phylogenetic algebraic geometry [6] imply that $\mathbf{I}_T$ is a prime ideal:

Theorem 7.5. *The ideals $\mathbf{I}_T$ are toric and they all have the same Hilbert function with respect to the $\mathbb{Z}^{n+1}$-grading* $\deg(q_{i_0 i_1 \cdots i_{2k}}) = ke_0 + e_{i_0} + \cdots + e_{i_{2k}}$.

Proof. Let $\mathbb{Q}[q']$ be the polynomial ring whose variables $q'_{j_1 \cdots j_{2k}}$ are indexed by the even subsets $\{j_1, \ldots, j_{2k}\}$ of $\{1, \ldots, n\}$. We declare the leaf n to be the root of the tree T and we identify $\mathbb{Q}[q]$ with $\mathbb{Q}[q']$ by mapping $q_\sigma \mapsto q'_{\sigma\backslash\{n\}}$ if $n \in \sigma$ and $q_\sigma \mapsto q'_{\sigma\cup\{n\}}$ if $n \notin \sigma$. The image of $\mathbf{I}_T$ in $\mathbb{Q}[q']$ under this identification coincides with the prime ideal of the binary symmetric model [6] (called *Jukes–Cantor model* in [26, 32]) on the phylogenetic tree T. Indeed, it was shown in [32, §6.2] that the 2×2-minors of the above matrices form a Gröbner basis for $\mathbf{I}_T$ with respect to a suitable term order. Buczyńska and Wiśniewski [6, Theorem 3.26] proved that all toric ideals $\mathbf{I}_T$ for the various T have the same Hilbert function in the standard $\mathbb{Z}$-grading. However, their proof works verbatim for our finer $\mathbb{Z}^{n+1}$-grading as well. □

The result by Buczyńska and Wiśniewski [6] that the Hilbert function of $\mathbf{I}_T$ is invariant under any choice of trivalent tree T is remarkable because there are as many as $(2n-5)!!$ distinct trees T. In [6] the question is left open whether the toric ideals $\mathbf{I}_T$ all lie on the same irreducible component of the corresponding multigraded Hilbert scheme, and, if yes, what is the general point on that component. We here answer this question, by constructing sagbi deformations of the toric varieties in [6] to the projective variety with coordinate ring R^G. Here it is essential that we use the definition of sagbi bases given in Section 3. Sagbi bases for term orders in $K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ will not work.

Let $\mathcal{F}$ be the set of the 2^{n-1} minimal generators $Q_{i_0 i_1 \cdots i_{2k}}$ in Theorem 7.1. Suppose that G is a moneric subspace of codimension 2 in K^n. This means that the 2^{n-1} elements $\mathrm{in}(Q_{i_0 i_1 \cdots i_{2k}})$ in the set $\mathrm{in}(\mathcal{F})$ are all monomials. Let $\mathbf{J}_G$ denote the kernel of the ring map $\pi : \mathbb{Q}[q] \to \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ which takes the variable $q_{i_0 i_1 \cdots i_{2k}}$ to the monomial $\mathrm{in}(Q_{i_0 i_1 \cdots i_{2k}})$. In other words, $\mathbf{J}_G \subset \mathbb{Q}[q]$ is the toric ideal of algebraic relations among the initial monomials of $\mathcal{F}$. Suppose that T is a trivalent tree with leaves $1, \ldots, n$ and $\mathbf{I}_T$ its toric ideal as above. We say that the moneric subspace G *realizes* the tree T if $\mathbf{I}_T = \mathbf{J}_G$.

Example 7.6. Let G be the row space of the $2 \times n$-matrix

$$[b_{ij}] = \begin{bmatrix} 1 & t & t^2 & \cdots & t^{n-3} & t^{n-2} & t^{n-1} \\ t^{n-1} & t^{n-2} & t^{n-3} & \cdots & t^2 & t & 1 \end{bmatrix}.$$

Then G is moneric since, for each odd subset $\{i_0 < \cdots < i_{2k}\}$, we have

$$\mathrm{in}(Q_{i_0 i_1 \cdots i_{2k}}) = x_{i_0} y_{i_1} x_{i_2} y_{i_3} x_{i_4} \cdots y_{i_{2k-1}} x_{i_{2k}}. \tag{20}$$

This is seen either from the matrix (17), whose diagonal entries multiply to (20), or from the expansion (19). Let T be the *caterpillar tree* whose splits are

$$\mathrm{splits}(T) = \{\{A, B\} : \max(A) < \min(B)\}.$$

We claim that the subspace G realizes the tree T. For each split $\{A, B\}$ of T we consider the images of the matrices $\mathbf{M}_{A,B}$ and $\mathbf{M}_{B,A}$ under the map π. The matrix $\pi(\mathbf{M}_{A,B})$ equals the product of the column vector labeled by even sets $\sigma = \{\sigma_1 < \cdots < \sigma_{2i}\} \subseteq A$ and entries $x_{\sigma_1} y_{\sigma_2} x_{\sigma_3} \cdots y_{\sigma_{2i}}$, with the row vector labeled by odd sets $\tau = \{\tau_0 < \cdots < \tau_{2i}\} \subseteq A$ and entries $x_{\tau_0} y_{\tau_1} x_{\tau_2} \cdots y_{\tau_{2i}}$. This uses the assumption $\sigma_{2i} < \tau_0$. The matrix $\pi(\mathbf{M}_{B,A})$ is a similar product with the roles of B and A reversed. Hence the matrices $\mathbf{M}_{A,B}$ and $\mathbf{M}_{B,A}$ have rank one modulo $\mathbf{J}_G$, and this implies $\mathbf{J}_G \subseteq \mathbf{I}_T$. Since both ideals are prime of the same Krull dimension, namely $2n - 2$, it follows that $\mathbf{J}_G = \mathbf{I}_T$.

Our next lemma says that the caterpillar tree is not alone:

Lemma 7.7. *For any trivalent tree T there is a subspace G which realizes T.*

Proof. We proceed by induction on n. For $n \leq 5$ every trivalent tree is a caterpillar tree, so we are done by Example 7.6. Let $n \geq 6$ and fix any split of T. We cut the tree along the split into two smaller trees T' and T'', each having a leaf in the place of that split. By induction, the trees T' and T'' can be realized by subspaces G' and G''. Let d' and d'' be the corresponding metric spaces, given by negated orders of the Plücker coordinates of G' and G''. We build a new tree metric by joining the tree metrics d' and d'' along the split, but in such a way that the length of the split edge is **much larger** than any of the previous edge lengths. The effect of this choice is that any initial monomial $\mathrm{in}(Q_{i_0 i_1 \cdots i_{2k}})$ is the product of previous initial monomials coming from G' and G'' with a variable x_{split} removed where needed. Given the metric d we determine the initial monomials from the expansion (19). Our choice ensures that each matrix $\pi(M_{A,B})$ or $\pi(M_{B,A})$ is a product of a column vector times a row vector. As in Example 7.6 we conclude $\mathbf{J}_G \subseteq \mathbf{I}_T$ and, as both ideals are prime of Krull dimension $2n-2$, it follows that $\mathbf{J}_G = \mathbf{I}_T$. □

Our next two lemmas establish the relationship to the Verlinde formula.

Lemma 7.8. *Let u be a vector in $\{0, 1, 2\}^n$ which has i entries 0, j entries 2, and $2k + 2$ entries 1. Then $\psi(j + k, u) = 2^k$.*

Proof. If $n = 2k + 2$ then this is the content of Corollary 7.4. Next consider the case when all entries in u are 0 or 1. After relabeling we may assume $u = (1, \ldots, 1, 0, \ldots, 0)$, and we can take the linear forms corresponding to the 0's to be variables, say, $\ell_{2k+3} = z_{n-2k+1}, \ldots, \ell_n = z_{n-2}$. Then we have

$$I_u = \langle \ell_1^2, \ldots, \ell_{2k+2}^2, z_{2k+1}, \ldots, z_{n-2} \rangle,$$

and $\psi(k, 1, \ldots, 1, 0, \ldots, 0) = \psi(k, 1, \ldots, 1) = 2^k$ is clear from the definition of ψ. For the general case we use induction on j, and we apply the fact that ψ is invariant under the action of the Weyl group D_n by Cremona transformation. Using this action on our degree $(j + k, u)$, we can replace each entry 0 in u by an entry 2 while incrementing the first coordinate $j + k$ by one. □

Lemma 7.9. *Let u be a vector in $\{0, 1, 2\}^n$ with i entries 0, j entries 2, and $2k + 2$ entries 1. Then $\dim(\mathbb{Q}[q]_{(j+k,u)}) = 2^{2k}$ and $\dim((\mathbb{Q}[q]/\mathbf{I}_T)_{(j+k,u)}) = 2^k$.*

Proof. The monomials of degree $(j + k, u)$ in $\mathbb{Q}[q]$ are products $q_\sigma q_\tau$ where $|\sigma \cup \tau| = j+k$, $|\sigma \cap \tau| = j$, $u_s = 2$ for $s \in \sigma \cap \tau$, and $u_t = 0$ for $t \notin \sigma \cup \tau$. The set $(\sigma \backslash \tau) \cup (\tau \backslash \sigma)$ has $2k + 2$ elements, and the above products $q_\sigma q_\tau$ are in bijection with partitions of this set into two odd subsets. There are 2^{2k} such partitions, and this implies the first assertion $\dim(\mathbb{Q}[q]_{(j+k.u)}) = 2^{2k}$.

For the second assertion we may assume that T is the caterpillar tree, in light of Theorem 7.5, so we have $\mathbf{I}_T = \mathbf{J}_G$ as in Example 7.6. Let us first consider the case $i = j = 0$. A monomial in the algebra generated by (20) has degree $(k, u) = (k, 1, \ldots, 1)$ if and only if it can be factored in the form

$$x_1 \cdot (x_2 y_3 \text{ or } y_2 x_3) \cdot (x_4 y_5 \text{ or } y_4 x_5) \cdots (x_{2k} y_{2k+1} \text{ or } y_{2k} x_{2k+1}) \cdot x_{2k+2}. \tag{21}$$

The number of distinct such products equals 2^k as required. The general case is obtained by removing both variables x_s and y_s whenever $u_s = 0$, and by including both variables x_t and y_t in the above product when $u_t = 0$. □

We are now prepared to state and prove our main result in this section.

Theorem 7.10. *Every trivalent phylogenetic tree T is realized by a subspace G which is sagbi, and the counting function ψ equals the common $\mathbb{Z}^{d+3}$-graded Hilbert function of the toric algebras $\mathbb{Q}[q]/\mathbf{I}_T$ associated with the trees T.*

Proof. Using Lemma 7.7 we find a subspace G which realizes the given tree T. The ideal of algebraic relations among the initial monomials $\text{in}(\mathcal{F})$ equals the toric ideal $\mathbf{I}_T$. The generators of $\mathbf{I}_T$ are the 2×2-minors of the matrices $\mathbf{M}_{A,B}$ and $\mathbf{M}_{B,A}$ for $\{A, B\} \in \text{splits}(T)$. Lemma 7.9 describes the number of linearly independent generators of $\mathbf{I}_T$ in each multidegree (k, u). Lemmas 7.8 and 7.9 tell us that the Cox–Nagata ring $R^G = K[\mathcal{F}]$ has the same number of relations in each such degree (k, u). This means that the inclusion $\text{in}(K[\mathcal{F}]_{(k.u)}) \subseteq \mathbb{Q}[\text{in}(\mathcal{F})]_{(k,u)}$ is an equality for each syzygy degree (k, u). This means that each binomial relation on $\text{in}(\mathcal{F})$ lifts to a relation on $\mathcal{F}$. Using [9, Proposition 1.3], we conclude that $\mathcal{F}$ is a sagbi basis of R^G, which means that G is sagbi. The statement about the Hilbert function follows. □

Corollary 7.11. *The ideal I^G of the Cox–Nagata ring is generated by quadrics.*

Proof. The initial toric ideal $\text{in}(I^G) = \mathbf{J}_G = \mathbf{I}_T$ is generated by quadrics. □

We next derive an explicit piecewise polynomial formula for the function $\psi : \mathbb{N}^{n+1} \to \mathbb{N}$. Consider any trivalent tree T with n leaves and let ρ be a positive integer. We define a

T-decoration of order ρ to be an assignment of non-negative integer weights to the edges of T such that the half-weight of every interior node is an integer bounded above by ρ and bounded below by each adjacent edge weight. Here the *half-weight* of an interior node of T is defined as half the sum of the weights of the three adjacent edges.

Corollary 7.12. *Fix any trivalent phylogenetic tree T with n leaves. Then the value $\psi(r, u_1, \ldots, u_n)$ equals the number of T-decorations of order $\rho = u_1 + \cdots + u_n - 2r$ whose pendant edges have weights $u_1, \ldots, u_{n-1}, \rho - u_n$.*

Sketch of proof. This uses the realization of the Jukes–Cantor ideal $\mathbf{I}_T$ in the polynomial $\mathbb{Q}[q']$ whose variables $q'_{j_1 \cdots j_{2k}}$ are indexed by the even subsets $\{j_1, \ldots, j_{2k}\}$. The parametrization discussed in [6, 32] maps the variable $q'_{j_1 \cdots j_{2k}}$ to the unique collection of edges in T which connect the leaves $j_1, \ldots, j_{2k}$ pairwise by k edge-disjoint paths on T. The subsemigroup of $\mathbb{N}^{\mathrm{edges}(T)}$ generated by these sets of edges is saturated, and the linear inequalities describing the corresponding cone are precisely the inequalities in terms of half-weights of the interior nodes in our definition of T-decorations. $\square$

Example 7.13. Our count of tree decorations offers a piecewise polynomial version of the Verlinde formula: if T is any trivalent tree on $n = 2k + 2$ leaves then the number of T-decorations of order $4l$ whose pendant edges have weight $2l$ is the sum on the right hand side of Theorem 7.2. In particular, if $l = 1/2$ then the number of T-decorations equals $2^k = \psi(k, 1, \ldots, 1)$.

8. The zonotopal Cox ring

In this section we explain the genesis of the present project. Our point of departure was the work on *zonotopal algebra* by Holtz and Ron [18] which derives combinatorial formulas for ψ when the linear forms ℓ_j and the exponents u_j have the following special form. Fix a $d \times m$-matrix $C = (c_{ik})$ of rank d over K. Let $H_1, \ldots, H_n$ denote the hyperplanes in K^d which are spanned by subsets of the columns of C. We have $m \leq n \leq \binom{m}{d-1}$ and the upper bound is attained if the matrix C is generic. For each hyperplane H_j we fix a non-zero linear form $\ell_j \in K[z]$ that vanishes on H_j. Zonotopal algebra is concerned with these linear forms $\ell_1, \ldots, \ell_n$ and the ideals I_u generated by certain specific powers of the ℓ_i. As before, we write $A = (a_{ij})$ for the $d \times n$-matrix of coefficients of $\ell_1, \ldots, \ell_n$, and $G \subset K^n$ is the kernel of A.

The Cox–Nagata ring R^G has a special structure which depends on the given matrix C, and if that matrix is generic then R^G will depend only on the parameters d and m. Also in this special situation, the ring R^G may fail to be Noetherian. For instance, this happens when $d = 3$ and $m = 9$ because the resulting $n = 36$ linear forms will contain $m = 9$ general linear forms.

However, the situation becomes finite and very nice if we restrict the choice of u to translates of a certain m-dimensional sublattice of $\mathbb{Z}^n$. This sublattice is the image of the following $m \times n$-matrix $\mathbf{C}$ with entries in $\{0, 1\}$. The entry of $\mathbf{C}$ in row k and column j is zero if the kth column of C lies on the hyperplane H_j. Let $\mathbf{e}$ denote the vector $(1, \ldots, 1)$ in $\mathbb{Z}^n$. Holtz and Ron [18] establish formulas for $\psi(r, \mathbf{C}v)$, $\psi(r, \mathbf{C}v - \mathbf{e})$ and $\psi(r, \mathbf{C}v - 2\mathbf{e})$. If C is a unimodular matrix then their formulas are expressed via volumes and lattice points in zonotopes (whence the term *zonotopal algebra*), while in the general (non-unimodular) case they involve matroid theory. For instance, $\sum_r \psi(r, \mathbf{C}v)$ is the number of independent sets in the rank d matroid on $v_1 + \cdots + v_m$ elements obtained by duplicating the kth column of C exactly v_k times. Likewise, the quantity $\sum_r \psi(r, \mathbf{C}v - \mathbf{e})$ is the number of bases of that matroid. We shall propose an algebraic explanation of that result.

We define the *zonotopal Cox ring* of the matrix C to be the subalgebra Z^G of R^G which is the direct sum of all graded components $R_{r,u}$ where $r \in \mathbb{N}$ and u runs over the lattice points in the image of $\mathbf{C}$. We use the notation

$$Z^G \quad = \bigoplus_{(r,v) \in \mathbb{Z}^{m+1}} R^G_{(r, \mathbf{C}v)}.$$

Clearly, the Cox–Nagata ring R^G is a module over the zonotopal Cox ring Z^G, and for any fixed vector $\omega \in \mathbb{Z}^n$ we can also consider the submodule

$$Z^{G,w} \quad = \bigoplus_{(r,v) \in \mathbb{Z}^{m+1}} R^G_{(r, \mathbf{C}v)+w}.$$

We call $Z^{G,w}$ the *zonotopal Cox module* of *shift* w. Thus the results of Holtz and Ron give formulas for the $\mathbb{Z}^{m+1}$-graded Hilbert series of the zonotopal Cox ring and the zonotopal Cox modules with shifts $w = -\mathbf{e}$ and $w = -2\mathbf{e}$. Their results have been extended in recent work of Ardila and Postnikov [2].

The generators of the zonotopal Cox ring have the following description. For each $k \in \{1, \ldots, m\}$ let $\mathbf{c}_k$ denote the kth column of the matrix $\mathbf{C}$, and let $E_k = x^{\mathbf{c}_k}$ be the corresponding squarefree monomial. We denote by $f_k(z) = \sum_{i=1}^d c_{ik} z_i$ the linear form corresponding to the kth column of C, and we define $F_k(x, y)$ to be the image of $f_k(z)$ in the Cox–Nagata ring R^G. In other words, F_k is the numerator of the Laurent polynomial $\sum_{j=1}^n f_k(a_j) \cdot (y_j / x_j)$ where a_j is the jth column of the matrix a_j. We have

$$\deg(E_k) = (0, \mathbf{c}_k) \quad \text{and} \quad \deg(F_k) = (1, \mathbf{c}_k),$$

where $\mathbf{c}_k$ is the kth column of the matrix $\mathbf{C}$. In particular both E_k and F_k lie in the zonotopal Cox ring Z^G. These elements suffice to generate:

Theorem 8.1. *The zonotopal Cox ring equals* $Z^G = K[E_1, \ldots, E_m, F_1, \ldots, F_m]$.

Proof. This lemma is a reinterpretation of the result on exterior zonotopal algebra in [18], which implies that the K-vector space $(I^{\perp}_{\mathbf{C}v})_r$ is spanned by the products $f_1^{h_1} \cdots f_m^{h_m}$ where $h_1 + \cdots + h_m = r$ and $h_k \leq v_k$ for all k. The image of that product in Z^G equals $E_1^{v_1 - h_1} \cdots E_k^{v_k - h_k} F_1^{h_1} \cdots F_k^{h_k}$. □

Using constructions from matroid theory, one can select a subset of the products $f_1^{h_1} \cdots f_m^{h_m}$ which forms a basis of the space $(I^{\perp}_{\mathbf{C}v})_r \simeq Z^G_{(r, \mathbf{Cv})}$. The cardinality of that basis is computed as the value of a *multivariate Tutte polynomial*, as explained in [2]. In particular, we conclude that the Hilbert function $(r, v) \mapsto \psi(r, \mathbf{C}v)$ of the zonotopal Cox ring is a piecewise polynomial function. We shall now present an explicit formula for that function.

If μ is any multiset of positive integers and $s \in \mathbb{N}$ then we write $\Phi(\mu, s)$ for the coefficient of q^s in the expansion of $\prod_{\ell \in \mu}(\sum_{i=0}^{\ell-1} q^i)$. Thus Φ is a piecewise polynomial function of degree $|\mu| - 1$ in its $|\mu| + 1$ arguments. Let $M(C)$ be the rank d matroid on $\{1, \ldots, m\}$ defined by the matrix C. For any $J \subseteq \{1, \ldots, m\}$ we write $\mathrm{span}(J)$ for the flat of $M(C)$ spanned by J.

Corollary 8.2. *The Hilbert function of the zonotopal Cox ring Z^G equals*

$$\psi(r, \mathbf{C}v) = \sum_I \Phi\Big(\{v_i\}_{i \in I}, r - \sum_{j \notin \mathrm{span}(I \cap \{1,\ldots,m\})} v_j\Big), \tag{22}$$

where the sum is over all independent sets I of the matroid $M(C)$.

Proof. This formula is derived by applying [18, Theorem 4.2] to the rank d matroid on $v_1+\cdots+v_m$ elements which is obtained from $M(C)$ by duplicating the elements $1, \ldots, m$ respectively $v_1, \ldots, v_m$ times. □

The right hand side of (22) is obviously a piecewise polynomial function in (r, v) of degree $d - 1$. If we sum this expression over r from 0 to $\sum_{i=1}^m v_i$ then the piecewise nature disappears and we get a polynomial of degree d:

$$\sum_{r \geq 0} \psi(r, \mathbf{C}v) = \sum_I \prod_{i \in I} v_i. \tag{23}$$

We now illustrate formula (22) by relating it to earlier examples.

Example 8.3. Let $d = 3, m = 4$ and $C = \begin{bmatrix} 0&0&1&1 \\ 0&1&0&1 \\ 1&0&0&1 \end{bmatrix}$. Then $n = 6$ and A is the matrix in Example 2.6. The zonotopal Cox ring Z^G is the subalgebra of R^G generated by L_{124}, L_{135}, L_{236}, L_{456}, $x_1x_2x_3$, $x_1x_4x_5$, $x_2x_4x_6$ and $x_3x_5x_6$. The Hilbert function of Z^G is the following specialization of $\psi : \mathbb{N}^7 \to \mathbb{N}$:

$$\begin{aligned} &\psi(r, v_1+v_2, v_1+v_3, v_2+v_3, v_1+v_4, v_2+v_4, v_3+v_4) \\ &\quad = \Phi(\{v_1, v_2, v_3\}, r) + \Phi(\{v_1, v_2, v_4\}, r-v_3) + \Phi(\{v_1, v_3, v_4\}, r-v_2) \\ &\qquad + \Phi(\{v_2, v_3, v_4\}, r - v_1) + \sum_{|I| \leq 2} \Phi\Big(\{v_i\}_{i \in I}, r - \sum_{j \notin I} v_j\Big). \end{aligned}$$

If we sum this piecewise quadratic over all r then we get the cubic polynomial

$$(23) = (v_1+1)(v_2+1)(v_3+1)(v_4+1) - v_1v_2v_3v_4.$$

A problem that remains open and of interest is to give a description of the Cox–Nagata ring R^G for the configurations considered here. In algebraic geometry terms, we are concerned with blowing up the intersection points of a hyperplane arrangement. For instance, we may ask when R^G is finitely generated, and what its structure is as a module over the nice subring Z^G.

First results in this direction were obtained by Ana-Maria Castravet. She proved, for instance, that R^G is finitely generated when $d = 4$ and $m = 5$, that is, for the blow-up of $\mathbb{P}^3$ at the $n = 10$ intersection points determined by five general planes. Her proof is based on the methods developed in [7]. We close by presenting that example from our initial perspective in Section 1.

Example 8.4. Consider the system of linear partial differential equations

$$\begin{aligned} I_u = \langle &\partial_1^{u_1+1}, \partial_2^{u_2+1}, \partial_3^{u_3+1}, \partial_4^{u_4+1}, (\partial_1 - \partial_2)^{u_5+1}, (\partial_1 - \partial_3)^{u_6+1}, \\ &(\partial_1 - \partial_4)^{u_7+1}, (\partial_2 - \partial_3)^{u_8+1}, (\partial_2 - \partial_4)^{u_9+1}, (\partial_3 - \partial_4)^{u_{10}+1}\rangle. \end{aligned}$$

We shall compute the number $\psi(r, u)$ of linearly independent polynomial solutions of degree r. To this end, we fix the following matrix over $K = \mathbb{Q}(t)$:

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & t \\ 0 & 0 & 1 & 0 & t^2 \\ 0 & 1 & 0 & 0 & t^3 \\ 1 & 0 & 0 & 0 & t^4 \end{bmatrix},$$

and we let G be the corresponding linear subspace of codimension 4 in K^{10}. Note that the ten linear forms in I_u arise from the matrix C if we set $t = 1$. A computation with Castravet's generators reveals that the subspace G is sagbi. We find that the toric algebra $\mathrm{in}(R^G)$ is generated by the ten variables

$$\begin{matrix} 123 & 124 & 134 & 234 & 125 & 135 & 235 & 145 & 245 & 345 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{matrix} \tag{24}$$

which represent the intersection points $H_j \cap H_k \cap H_l$, and the 15 monomials

$$\begin{matrix} y_1x_2x_3x_4 & y_1x_5x_6x_7 & y_2x_5x_8x_9 & y_3x_6x_8x_{10} & y_4x_7x_9x_{10} \\ y_3x_4x_6x_7x_8x_9 & y_2x_4x_5x_7x_8x_{10} & y_1x_4x_5x_6x_9x_{10} & y_1x_2x_3x_7x_9x_{10} & y_2x_3x_5x_6x_9x_{10} \\ y_1x_3x_5x_7x_8x_{10} & y_1x_2x_4x_6x_8x_{10} & y_1x_2x_6x_7x_8x_9 & y_1x_3x_4x_5x_8x_9 & y_2x_3x_4x_5x_6x_7 \end{matrix}$$

which represent the planes spanned by the ten intersection points in $\mathbb{P}^3$. The toric ideal of relations among these 25 monomials has 55 minimal generators, and each of these lifts to a relation in I^G. We compute the $\mathbb{Z}^{11}$-graded Hilbert function of R^G using the technique explained in Sections 4 and 5, namely by listing the facets of the cone Γ spanned by the 25 monomials. This cone has

$$f(\Gamma) = (25, 261, 1536, 5790, 14935, 27309, 35985, 34247, 23276, 10989, 3419, 634, 56)$$

and `polymake` supplies an explicit 14×56 matrix M such that $\psi(r, u)$ is the number of integer solutions (x, y, z) to $(r, u_1, u_2, \ldots, u_{10}, x, y, z) \cdot M \geq 0$.

Remark 8.5. The 14×56 matrix M above and other supplementary materials for this paper are posted at http://lsec.cc.ac.cn/∼xuzq/cox.html.

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References

[1] 4ti2 team: 4ti2—A software package for algebraic, geometric and combinatorial problems on linear spaces. Available at www.4ti2.de

[2] Ardila, F., Postnikov, A.: Combinatorics and geometry of power ideals. Trans. Amer. Math. Soc., to appear

[3] Batyrev, V., Popov, O.: The Cox ring of a del Pezzo surface. In: Higher-Dimensional Algebraic Varieties (Palo Alto, CA, 2002), Progr. Math. 226, Birkhäuser, Boston, 85–103 (2004) Zbl 1075.14035 MR 2029863

[4] Bauer, S.: Parabolic bundles, elliptic surfaces and $SU(2)$-representation spaces of genus zero Fuchsian groups. Math. Ann. **290**, 509–526 (1991) Zbl 0752.14035 MR 1116235

[5] Bauer, T., Küronya, A., Szemberg, T.: Zariski chambers, volumes, and stable base loci. J. Reine Angew. Math. **576**, 209–233 (2004) Zbl 1055.14007 MR 2099205

[6] Buczyńska, W., Wiśniewski, J.: On the geometry of binary symmetric models of phylogenetic trees. J. Eur. Math. Soc. **9**, 609–635 (2007) Zbl 1147.14027 MR 2314109

[7] Castravet, A.-M.: The Cox ring of $\bar{M}_{0,6}$. Trans. Amer. Math. Soc. **361**, 3851–3878 (2009) Zbl pre05567279 MR 2491903

[8] Castravet, A.-M., Tevelev, J.: Hilbert's 14th problem and Cox rings. Compositio Math. **142**, 1479–1498 (2006) Zbl 1117.14048 MR 2278756

[9] Conca, A., Herzog, J., Valla, G.: Sagbi bases with applications to blow-up algebras. J. Reine Angew. Math. **474**, 113–138 (1996) Zbl 0866.13010 MR 1390693

[10] D'Cruz, C., Iarrobino, A.: High-order vanishing ideals at $n+3$ points of $\mathbb{P}^n$. J. Pure Appl. Algebra **152**, 75–82 (2000) Zbl 0971.14038 MR 1783987

[11] Derenthal, U.: On the Cox ring of Del Pezzo surfaces. arXiv:math/0603111

[12] Derenthal, U.: Universal torsors of Del Pezzo surfaces and homogeneous spaces. Adv. Math. **213**, 849–864 (2007) Zbl 1125.14020 MR 2332612

[13] Eisenbud, D.: Commutative Algebra with a View toward Algebraic Geometry. Grad. Texts in Math. 150, Springer, New York (1995) Zbl 0819.13001 MR 1322960

[14] Emsalem, J., Iarrobino, A.: Inverse system of a symbolic power I. J. Algebra **174**, 1080–1090 (1995) Zbl 0842.14002 MR 1337186

[15] Göbel, M.: Rings of polynomial invariants of the alternating group have no finite sagbi bases with respect to any admissible order. Information Process. Lett. **74**, 15–18 (2000) Zbl 1014.68228 MR 1761193

[16] Gosset, T.: On the regular and semi-regular figures in space of n dimensions. Messenger Math. **29**, 43–48 (1900) JFM 30.0494.02

[17] Hacking, P., Keel, S., Tevelev, J.: Stable pair, tropical, and log canonical compactifications of moduli spaces of del Pezzo surfaces. Invent. Math. **178**, 173–227 (2009) Zbl pre05602518 MR 2237265

[18] Holtz, O., Ron, A.: Zonotopal algebra. arXiv:0708.2632

[19] Kapur, D., Madlener, K.: A completion procedure for computing a canonical basis for a k-subalgebra. In: Computer and Mathematics (Cambridge, MA, 1989), Springer, 1–11 (1989) Zbl 0692.13001 MR 1005954

[20] Kuttler, J., Wallach, N.: Representations of SL_2 and the distribution of points in $\mathbb{P}^n$. In: Noncommutative Harmonic Analysis, Progr. Math. 220, Birkhäuser, Boston, 355–373 (2004) Zbl 1129.13004 MR 2036576

[21] Laface, A., Velasco, M.: Picard-graded Betti numbers and the defining ideals of Cox rings. J. Algebra **322**, 353–372 (2009) Zbl pre05610482 MR 2529093

[22] Miller, E., Sturmfels, B.: Combinatorial Commutative Algebra. Grad. Texts in Math. 227, Springer, New York (2004) Zbl 1066.13001 MR 2110098

[23] Mukai, S.: An Introduction to Invariants and Moduli. Cambridge Univ. Press (2003) Zbl 1033.14008 MR 2004218

[24] Mukai, S.: Finite generation of the Nagata invariant rings in A-D-E cases. RIMS Preprint # 1502 (2005)

[25] Nagata, M.: On the fourteenth problem of Hilbert. In: Proc. ICM Edinburgh 1958, Cambridge Univ. Press, 459–462 (1960) Zbl 0127.26302 MR 0116056

[26] Pachter, L., Sturmfels, B.: Algebraic Statistics for Computational Biology. Cambridge Univ. Press (2005) Zbl 1108.62118 MR 2205869

[27] Popov, O.: The Cox ring of a del Pezzo surface has rational singularities. math.AG/0402154

[28] Robbiano, Sweedler, L., M.: Subalgebra bases. In: Commutative Algebra (Salvador, 1988), W. Bruns and A. Simis (eds.), Lecture Notes in Math. 1430, Springer, 61–87 (1990) Zbl 0725.13013 MR 1068324

[29] Serganova, V., Skorobogatov, A.: Del Pezzo surfaces and representation theory. Algebra Number Theory **1**, 393–420 (2007) Zbl pre05529313 MR 2368955

[30] Speyer, D., Sturmfels, B.: The tropical Grassmannian. Adv. Geom. **4**, 389–411 (2004) Zbl 1065.14071 MR 2071813

[31] Stillman, M., Testa, D., Velasco, M.: Gröbner bases, monomial group actions, and the Cox rings of Del Pezzo surfaces. J. Algebra **316**, 777–801 (2007). Zbl 1129.14047 MR 2358614

[32] Sturmfels, B., Sullivant, S.: Toric ideals of phylogenetic invariants, J. Comput. Biol. **12**, 204–228 (2005)

[33] Testa, D., Várilly-Alvarado, A., Velasco, M.: Cox rings of degree one del Pezzo surfaces. Algebra Number Theory **3**, 729–761 (2009) Zbl pre05652990