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Maximum principles and the method of moving planes for a class of degenerate elliptic linear operators

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Abstract. We deal with maximum principles for a class of linear, degenerate elliptic differential operators of the second order. In particular the Weak and Strong Maximum Principles are shown to hold for this class of operators in bounded domains, as well as a Hopf type lemma, under suitable hypothesis on the degeneracy set of the operator. We derive, as consequences of these principles, some generalized maximum principles and an a priori estimate on the solutions of the Dirichlet problem for the linear equation. A good example of such an operator is the Grushin operator on $\mathbb{R}^{d+k}$, to which we devote particular attention. As an application of these tools in the degenerate elliptic setting, we prove a partial symmetry result for classical solutions of semilinear problems on bounded, symmetric and suitably convex domains, which is a generalization of the result of Gidas–Ni–Nirenberg [12], [13], and a nonexistence result for classical solutions of semilinear equations with subcritical growth defined on the whole space, which is a generalization of the result of Gidas–Spruck [14] and Chen–Li [6]. We use the method of moving planes, implemented just in the directions parallel to the degeneracy set of the Grushin operator.

Keywords. Maximum principles, degenerate elliptic linear differential operators, Grushin operator, moving planes

1. Introduction

Maximum principles are a well known and useful tool in the study of partial differential equations, particularly for equations of elliptic type. In fact, the presence of suitable maximum principles plays a key role in, for example, proving uniqueness theorems and symmetry results for classical solutions of boundary value problems, in obtaining a priori estimates for solutions of differential inequalities and in obtaining nonexistence results for nontrivial, classical solutions of equations defined on the whole space.

The point of this paper is to extend some of these important results from the uniformly elliptic setting to a wider class of linear differential operators of the second order, defined on a bounded domain $\Omega \subset \mathbb{R}^N$, focusing our attention on classical solutions.

Indeed, we will treat operators, which we will call degenerate elliptic, that may not be elliptic on the whole of Ω but may degenerate on suitable subsets of the domain having no interior points.

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The idea we follow is to apply classical arguments and techniques, usually employed for uniformly elliptic operators as one can see in [16], to treat also the degenerate elliptic setting. The main problem, then, is how to deal with the degeneracy set of the differential operator. This is achieved following the idea of Agmon–Nirenberg–Protter [1], by requiring a uniform ellipticity condition for the operator L throughout its domain in just one fixed direction and assuming suitable hypothesis on the degeneracy set Σ of the operator, rather then imposing conditions on the regularity of its coefficients in order to regard it as a Hörmander operator, as done in Bony [5]. See conditions (E_ξ) and (Σ) in Section 2, and also Remark 2.4.

Section 2 of this paper is devoted to the extension of classical forms of the maximum principles, such as the Weak and the Strong Maximum Principles and a Hopf type lemma for noncharacteristic points of the boundary of the domain of the equation, from the uniformly elliptic to the degenerate elliptic setting.

In Section 3 we derive some a priori estimates on the solutions of Dirichlet problems for linear degenerate elliptic equations of the second order on bounded domains from the results of Section 2, which in turn yield a uniqueness result for such problems. We also study some generalized maximum principles, which are extensions of the analogous results for uniformly elliptic linear operators. The techniques used here are, once again, essentially the same as in the uniformly elliptic setting, possibly after the use of a suitable elliptic regularization of the operator (cf. Theorem 3.2). This classical idea has been used with much success for second order linear differential operators having nonnegative characteristic form, as can be seen in Oleĭnik–Radkevich [24].

Our interest in maximum principles for degenerate elliptic linear operators of the second order comes from our interest in the Grushin operator, defined on $\mathbb{R}^{d+k}$ by setting $G_\gamma u = |y|^{2\gamma}\Delta_x u + \Delta_y u$. Such an operator arises in problems of embedding manifolds with nonnegative curvature. In addition, when $d, k = 1$, it is also connected with transonic fluid flow in the nonhyperbolic part of the domain (see Frankl' [9]). Notice also that, when $\gamma \in \mathbb{N}$, the Grushin operator is also related to the sub-Laplacian on a group of Heisenberg type (see also [3] and references therein).

We treat the case of the Grushin operator in more detail in Section 4. There we show that any linear operator of the second order having G_γ as principal part satisfies both the Weak and the Strong Maximum Principle on Ω, if the 0^{th} order term is nonpositive on the domain. We also prove a refined Hopf's Lemma which, in the case of the Grushin operator, covers some of the cases when the boundary of the domain is characteristic.

It is known that, in general, boundary value problems involving linear second order differential operators having nonnegative characteristic form may not admit any classical solution (see for instance §6.6 of [15]), while often one can find a suitably defined weak solution, which belongs to an appropriate weighted Sobolev space.

We have addressed the problem of studying maximum principles for the class of operators considered here in a setting compatible with a suitable notion of weak solution in a work which will appear elsewhere (see [22]).

Several results are known about existence, uniqueness and regularity of suitably defined weak solutions of Dirichlet and Neumann problems involving a second order linear differential operator L, having nonnegative characteristic form on a bounded domain

$\Omega \subset \mathbb{R}^N$. See for instance the work of Oleĭnik–Radkevich [24] and references therein for a comprehensive overview of such results until 1973.

We notice however that, by standard elliptic regularity theory, a (suitably defined) weak solution of a degenerate elliptic linear differential equation is regular in each open connected component of $\Omega \setminus \Sigma$, provided that the coefficients of the equation are regular enough. Thus, if the degeneracy set of the operator L lies on the boundary of the domain, any weak solution of the equation will be a classical one in the interior.

Finally, in Section 5 we use the results obtained in the previous sections to study two semilinear problems for the Grushin operator. In both cases, we rely heavily on the technique of moving planes, with which one can exploit maximum principles and symmetries and invariances of the operator to get symmetries for the solutions of semilinear problems related to the operator itself.

The first of the two applications is concerned with the problem

$$\begin{cases} G_\gamma u + f(u) = 0 & \text{in } \Omega \\ u > 0 \text{ in } \Omega, \quad u \equiv 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

It is known from the pioneering works of Gidas–Ni–Nirenberg [12] and [13] that the analogous problem for the Laplace operator

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } B_R(0), \\ u > 0 \text{ in } B_R(0), \quad u \equiv 0 \text{ on } \partial B_R(0), \end{cases}$$

admits only radially symmetric solutions when f is a sufficiently regular function. The proof of this fact is based upon the technique of moving planes, which was introduced by Aleksandrov [2] and Serrin [27] and later perfected by these authors, and relies upon some generalized maximum principles and upon the invariance of the Laplace operator with respect to translations and reflections about hyperplanes in $\mathbb{R}^N$.

Later Berestycki–Nirenberg [4] presented a much simplified approach, yielding improved results. Their approach relied on improved forms of the maximum principles for uniformly elliptic operators in "narrow domains", again in order to exploit the technique of moving planes.

The Grushin operator G_γ is not invariant with respect to translations and reflections about hyperplanes *in all the directions* of $\mathbb{R}^{d+k}$, and hence, as remarked by Monti–Morbidelli [20], it is not possible to apply the technique of moving planes to this operator.

It must be noticed, however, that G_γ is actually invariant with respect to translations and reflections about hyperplanes *at least in some directions* of $\mathbb{R}^{d+k}$, namely in $\mathbb{R}^d \times \{0\}$, i.e. in those directions which are parallel to the degeneracy set Σ of the operator.

Then, one can prove a similar symmetry result for the Grushin operator, exploiting its invariances and suitable maximum principles and Hopf's Lemma specially tailored for this class of operators. Thus, following the "narrow domains" idea of Berestycki–Nirenberg [4], one can show that every classical solution of problem (1) is radially symmetric in the $x \in \mathbb{R}^d$ variables about some point if the bounded domain Ω is strictly convex and symmetric in the directions of $\mathbb{R}^d \times \{0\}$ (see Theorem 5.1).

The second application concerns the problem

$$\begin{cases} G_\gamma u + u^p = 0 & \text{in } \mathbb{R}^{d+k}, \\ u \geq 0 \quad \text{in } \mathbb{R}^{d+k}, & u \in C^2(\mathbb{R}^{d+k}). \end{cases} \tag{2}$$

The analogous problem for the Laplace operator

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \mathbb{R}^N, \\ u \geq 0 \quad \text{in } \mathbb{R}^N, & u \in C^2(\mathbb{R}^N), \end{cases}$$

was first studied by Gidas–Spruck [14], who proved that any solution of this problem vanishes identically if $N \geq 3$ and $1 < p < (N+2)/(N-2)$. In the same paper they also showed that for $p = (N+2)/(N-2)$ any solution must be radially symmetric about some point in $\mathbb{R}^N$, and hence takes the form

$$u(x) = [N(N-2)\lambda^2]^{(N-2)/4}/(\lambda^2 + |x - x_0|^2)^{(N-2)/2},$$

with $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$.

Note that $(N+2)/(N-2) = 2^* - 1$, where 2^* is the critical exponent in the Sobolev embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, which is compact for $1 \leq p < 2^*$, but it is only continuous when $p = 2^*$.

The result of [14] is a consequence of nonlinear energy estimates, which are obtained by applying the Divergence Theorem to a suitable vector field in $\mathbb{R}^N$, depending both on the solution u and on a cutoff function.

Later the same result was proved also in the work of Chen–Li [6], exploiting the invariance of the Laplace operator with respect to the Kelvin transform and then again applying the technique of moving planes "from infinity".

When $N \geq 3$ and $1 < p \leq N/(N-2)$, a stronger result is also known from the work of Gidas [11], who showed that in those cases the problem

$$\begin{cases} \Delta u + u^p \leq 0 & \text{in } \mathbb{R}^N, \\ u \geq 0 \quad \text{in } \mathbb{R}^N, & u \in C^2(\mathbb{R}^N), \end{cases}$$

has no nontrivial solution. We recall that $p = N/(N-2)$ is also known as the Serrin critical exponent for the Laplace operator.

An analogous result for the Grushin operator has been proven by D'Ambrosio–Lucente [7], who exploited a nonlinear capacity argument to show that any nonnegative solution of

$$G_\gamma u + u^p \leq 0 \quad \text{in } \mathbb{R}^{d+k}$$

vanishes identically if $d, k \in \mathbb{N}$ and $1 < p \leq Q/(Q-2)$, where $Q = (1+\gamma)d + k$ plays for the Grushin operator the same role as the Euclidean dimension N of the ambient space does for the Laplace operator.

On the other hand, explicit solutions of problem (2) are known for some values of $\gamma > 0$ when $p = (Q+2)/(Q-2)$, which for any $d, k \in \mathbb{N}$ and $\gamma > 0$ is a critical

Sobolev exponent in the embedding of a suitably weighted version of $H_0^1(\Omega)$ into $L^p(\Omega)$ (see also Franchi–Lanconelli [8]).

In the paper of Monti–Morbidelli [20] it is also shown that any solution of problem (2) with $p = (Q+2)/(Q-2)$ must exhibit a kind of "spherical symmetry", by exploiting the invariance of the equation with respect to a suitable conformal inversion, i.e. the Kelvin transform for the Grushin operator (see also Lupo–Payne [18] for further details).

The question then arises if there are nontrivial solutions to problem (2) for $p \in (Q/(Q-2), (Q+2)/(Q-2))$, i.e. for p between the Serrin critical exponent and the Sobolev critical exponent for the Grushin operator, minus 1.

We address this problem in Theorem 5.2, following the ideas of Chen–Li [6], and thus exploiting the invariance of the equation with respect to the Kelvin transform and then implementing the technique of moving planes, with respect to directions which are parallel to the degeneracy set of the operator.

This technique makes use of maximum principles and Hopf's Lemma for the Grushin operator and once again of the invariance of the operator with respect to translations and reflections in suitable directions of $\mathbb{R}^{d+k}$. In this way we can "move the hyperplanes from infinity", and thus prove the symmetry of the solutions only in directions of $\mathbb{R}^d \times \{0\}$, as noted before.

For this approach to work, however, in the course of the proof we need to rely on an auxiliary function g, satisfying suitable conditions (see (24) for further details). We are able to produce such a function only if $d, k \in \mathbb{N}$ and $0 < \gamma < 1$ or if $d \in \mathbb{N}, k \in \mathbb{N}\setminus\{1, 2\}$ and $\gamma > 0$.

In this way, it is possible to show that any solution u of problem (2) for $p \in (Q/(Q-2), (Q+2)/(Q-2))$ must be radially symmetric in the $x \in \mathbb{R}^d$ variables about every point of $\mathbb{R}^d$. Thus the solution u is actually independent of the $x \in \mathbb{R}^d$ variables. Then we can reduce problem (2) to the classical one for the Laplace operator in $\mathbb{R}^k$, and thus we conclude that any solution must vanish identically on the whole space.

We also note that if $p = (Q+2)/(Q-2)$, our result states that any solution of problem (2) must be radially symmetric in the $x \in \mathbb{R}^d$ variables about some point.

Finally notice that the analogue of problem (2) when $p = (Q+2)/(Q-2)$ for the sub-Laplacian on groups of Heisenberg type has been studied by Garofalo–Vassilev (see [10] and references therein), following the ideas of Chen–Li [6] and exploiting the technique of moving planes.

2. Weak and strong maximum principles

Let $\Omega \subset \mathbb{R}^N$ be a bounded and connected domain, let $n(x)$ be the outward normal unit vector at each sufficiently regular boundary point $x \in \partial\Omega$ and consider the linear differential operator L in Ω

$$Lu \equiv a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u$$

for $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$. We assume $b_i, c \in L^\infty(\Omega)$, $a_{ij} \in \mathcal{C}(\overline{\Omega})$ and $a_{ij} = a_{ji}$ for every $i, j = 1, \dots, N$ and that L has nonnegative characteristic form in Ω, i.e.

$$a_{ij}(x)\xi_i\xi_j \geq 0 \quad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^N. \tag{3}$$

Here and throughout it is assumed that expressions with repeated indices are summed from 1 to N.

Our principal interest will be in cases where L is degenerate elliptic, and condition (3) will be suitably strengthened.

Lemma 2.1. *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be such that $Lu > 0$ in Ω and let $c \leq 0$ in Ω. Then if u has a nonnegative maximum in $\overline{\Omega}$, it cannot attain this maximum in Ω.*

Proof. Let $x_0 \in \Omega$ be such that $u(x_0) = \max_{x \in \overline{\Omega}} u(x) \geq 0$. Then $\nabla u(x_0) = 0$ and $B := [D_{ij}u(x_0)]$ is a nonpositive definite matrix. By condition (3) the matrix $A := [a_{ij}(x_0)]$ is nonnegative definite, hence AB has nonpositive trace. Then

$$\begin{aligned} Lu(x_0) &= a_{ij}(x_0)D_{ij}u(x_0) + b_i(x_0)D_iu(x_0) + c(x_0)u(x_0) \\ &\leq a_{ij}(x_0)D_{ij}u(x_0) = \operatorname{trace}(AB) \leq 0, \end{aligned}$$

which contradicts the hypothesis. □

Remark 2.1. If $c(x) \equiv 0$ in Ω then u cannot have local maxima in Ω, i.e. we can remove the nonnegativity hypothesis on $\max_{x \in \overline{\Omega}} u(x)$ in Lemma 2.1.

Remark 2.2. Lemma 2.1 implies that $\sup_\Omega u \leq \sup_{\partial\Omega} u^+$.

Before proceeding, we need to make a further assumption on the operator L,

$$\begin{gathered} \exists \beta > 0,\ \xi \in \mathbb{R}^N \text{ with } |\xi| = 1 \text{ such that} \\ \langle \xi, A(x)\xi \rangle_{\mathbb{R}^N} := a_{ij}(x)\xi_i\xi_j \geq \beta > 0 \quad \forall x \in \Omega, \end{gathered} \tag{E_ξ}$$

which states that, even if L is not uniformly elliptic on its domain, it is in fact uniformly elliptic at least in a given direction ξ on the whole of Ω. In this case we will call ξ a *noncharacteristic direction* for the operator L in Ω.

Theorem 2.1 (Weak Maximum Principle). *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be such that $Lu \geq 0$ with $c(x) \leq 0$ in Ω and let condition (E_ξ) hold. Then u attains on $\partial\Omega$ its nonnegative maximum, i.e.* $\sup_\Omega u \leq \sup_{\partial\Omega} u^+$.

Proof. Let $h(x) := e^{\alpha(\sum_{k=1}^N \xi_k x_k)}$ with $\alpha > 0$ to be chosen later. Then h is strictly positive and bounded on $\overline{\Omega}$, and we have

$$Lh(x) = h(x)[\alpha^2 a_{ij}(x)\xi_i\xi_j + \alpha b_i(x)\xi_i + c(x)] \geq h(x)[\beta\alpha^2 - M_1\alpha - M_2] > 0$$

if we choose $\alpha > 0$ large enough, where M_1, M_2 are suitable positive constants which bound the L^∞-norm of b_i and c in Ω respectively. Now let $w(x) := u(x) + \varepsilon h(x)$ with $\varepsilon > 0$. Then we have $Lw = Lu + \varepsilon Lh > 0$ in Ω. By Lemma 2.1, w attains its nonnegative maximum only on $\partial\Omega$ and hence

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\Omega} w(x) \leq \sup_{x\in\partial\Omega} w^+(x) \leq \sup_{x\in\partial\Omega} u^+(x) + \varepsilon \sup_{x\in\partial\Omega} h(x)$$

for every $\varepsilon > 0$. The conclusion follows by letting $\varepsilon \to 0$. □

Remark 2.3. If $c(x) \equiv 0$ in Ω then we find $\sup_\Omega u \leq \sup_{\partial\Omega} u$.

Remark 2.4. Theorem 2.1 also holds if instead of condition (E_ξ) we assume that there exists $\beta > 0$ and a conservative vector field $\xi \in \mathcal{C}_b^1(\Omega)$ with potential $U \in \mathcal{C}_b^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $a_{ij}(x)\xi_i(x)\xi_j(x) \geq \beta > 0$ for all $x \in \Omega$, where for any $k \in \mathbb{N}$, $\mathcal{C}_b^k(\Omega)$ is the space of functions with continuous and bounded derivatives up to order k in the domain Ω.

In this case, in fact, we can repeat the above argument exploiting the auxiliary function $h(x) = e^{\alpha U(x)}$ to get the weak maximum principle on Ω.

Notice that this possible generalization of condition (E_ξ) is invariant under the action of diffeomorphisms of class $C^2(\overline{\Omega})$ having the property that they transform conservative vector fields into conservative vector fields.

Remark 2.5. Notice that if we suppose that $A(x) = [a_{ij}(x)] \geq 0$ is a nonnegative definite matrix which is not 0 for every $x \in \overline{\Omega}$, then condition (E_ξ) is automatically satisfied by L in Ω for a suitable direction $\xi \in \mathbb{R}^N$, if $d := \operatorname{diam}(\Omega)$ is small enough.

In fact, for every $x \in \overline{\Omega}$ the matrix $A(x)$ is real, symmetric and nonnegative definite, hence it is diagonalizable with nonnegative eigenvalues. Let $\lambda_N(x) := \sup_{|\xi|=1} a_{ij}(x)\xi_i\xi_j \geq 0$ be the largest one. Since $A(x)$ is non-null, $\lambda_N(x) > 0$ for every $x \in \overline{\Omega}$. By the regularity assumptions on $[a_{ij}(x)]$, the function $\lambda_N(x)$ is continuous on $\overline{\Omega}$ and thus there exists a maximum point $x_0 \in \overline{\Omega}$. Next we can find $\xi_0 \in \mathbb{R}^N$ with $|\xi_0| = 1$ and $A(x_0)\xi_0 = \lambda_N(x_0)\xi_0$. Then $\langle \xi_0, A(x_0)\xi_0\rangle_{\mathbb{R}^N} = \lambda_N(x_0) > 0$.

Exploiting again the continuity of $[a_{ij}(x)]$ on $\overline{\Omega}$, we can find $\delta > 0$ such that for all $x \in \overline{\Omega}$ satisfying $|x - x_0| < \delta$ we have $\|A(x) - A(x_0)\| < \frac{1}{2}\lambda_N(x_0)$. It is easy to see that for those x,

$$|\langle \xi_0, A(x)\xi_0\rangle_{\mathbb{R}^N} - \langle \xi_0, A(x_0)\xi_0\rangle_{\mathbb{R}^N}| \leq \|A(x) - A(x_0)\| < \frac{1}{2}\lambda_N(x_0)$$

and hence

$$a_{ij}(x)\xi_{0,i}\xi_{0,j} = \langle \xi_0, A(x)\xi_0\rangle_{\mathbb{R}^N} > \frac{1}{2}\lambda_N(x_0) > 0.$$

Condition (E_ξ) is now satisfied by L in Ω with $\beta = \frac{1}{2}\lambda_N(x_0)$ in the direction ξ_0, if $d < \delta$.

Lemma 2.2 (Hopf's Lemma). *Let $B := B_R(P) \subset \mathbb{R}^N$ be the open ball centered at P and with radius $R > 0$ and let $x_0 \in \partial B$. Let L be a second order linear operator satisfying the nonnegative characteristic form assumption* (3) *and such that $c \leq 0$ in B. Let $u \in \mathcal{C}^2(B) \cap \mathcal{C}(\overline{B})$ be such that $Lu \geq 0$ in B. Finally suppose that $u(x) < u(x_0)$ for all $x \in B$, $u(x_0) \geq 0$ and*

$$\langle (x_0 - P), A(x_0)(x_0 - P)\rangle_{\mathbb{R}^N} > 0.$$

Then for every outward direction ν at x_0, i.e. such that $\langle \nu, n(x_0)\rangle_{\mathbb{R}^N} > 0$, one has

$$\liminf_{t\to 0^+} \frac{u(x_0) - u(x_0 - t\nu)}{t} > 0.$$

Remark 2.6. The condition $\langle (x_0 - P), A(x_0)(x_0 - P)\rangle_{\mathbb{R}^N} > 0$ is the requirement that ∂B is noncharacteristic for L in x_0.

Remark 2.7. If $u \in \mathcal{C}^1(B \cup \{x_0\})$ then Lemma 2.2 implies that $D_\nu u(x_0) > 0$.

Proof of Lemma 2.2. We can assume that $u \in \mathcal{C}(\overline{B})$ and $u(x) < u(x_0)$ for every $x \in \overline{B} \setminus \{x_0\}$, otherwise we can pick a smaller ball contained in B and tangent to ∂B in x_0.

Since $\langle (x - P), A(x)(x - P)\rangle_{\mathbb{R}^N}$ is a nonnegative continuous function on $\overline{B}$ which is not zero at x_0, we find that for a suitable $\delta > 0$,

$$\langle (x - P), A(x)(x - P)\rangle_{\mathbb{R}^N} \geq \delta > 0 \tag{4}$$

if $x \in \overline{B}$ and $|x - x_0|$ is small enough. Then we can also assume that (4) holds on $\overline{B}$, otherwise we can construct a smaller ball contained in B and tangent to ∂B at x_0 with radius small enough so that our assumptions are satisfied.

Now let $h(x) := e^{-\alpha|x-P|^2} - e^{-\alpha R^2}$ and $\Omega := B_R(P) \cap B_r(x_0) \subset B$, with $\alpha > 0$ to be chosen later and $0 < r < R/2$. Then $h \geq 0$ in $\overline{\Omega}$ and $h \equiv 0$ on $\partial B_R(P)$. Hence in Ω we have

$$\begin{aligned} Lh &= e^{-\alpha|x-P|^2}\Big[\,4\alpha^2 a_{ij}(x-P)_i(x-P)_j - 2\alpha\Big(\sum_{i=1}^N a_{ii} + b_i(x-P)_i\Big) + c\Big] - ce^{-\alpha R^2} \\ &\geq e^{-\alpha|x-P|^2}[\,4\delta\alpha^2 - 2M_1\alpha - M_2] > 0 \end{aligned}$$

if $\alpha > 0$ is large enough, with M_1, M_2 suitable positive constants.

For any $\varepsilon > 0$, we can now apply Lemma 2.1 to the function $w := u + \varepsilon h$ in Ω, since we have $Lw = Lu + \varepsilon Lh > 0$.

On $\overline{\partial\Omega \cap B}$ one has $u(x) < u(x_0)$ by hypothesis, hence by compactness one has $u(x) < u(x_0) - \eta$ for a suitable $\eta > 0$. Since h is bounded on $\overline{\Omega}$ by continuity, we can choose $\varepsilon > 0$ small enough such that $\varepsilon h \leq \eta$ in $\overline{\Omega}$. Then

$$w(x) = u(x) + \varepsilon h(x) \leq u(x) + \eta < u(x_0) = w(x_0)$$

for every $x \in \overline{\partial\Omega \cap B}$.

On $\partial\Omega \cap \partial B$ one has $h(x) \equiv 0$ and so $w(x) = u(x) < u(x_0) = w(x_0)$, with $w(x_0) = u(x_0) \geq 0$ by hypothesis. Hence it follows from Lemma 2.1 that $w(x) < w(x_0)$ for every $x \in \overline{\Omega} \setminus \{x_0\}$, and thus

$$\frac{w(x_0) - w(x_0 - t\nu)}{t} \geq 0 \quad \forall t > 0 \text{ small enough.}$$

Then $\liminf_{t\to 0^+} (w(x_0) - w(x_0 - t\nu))/t \geq 0$, and hence

$$\liminf_{t\to 0^+} \frac{u(x_0) - u(x_0 - t\nu)}{t} \geq -\varepsilon D_\nu h(x_0) = 2\varepsilon\alpha|x_0 - P|\, e^{-\alpha|x_0-P|^2}\langle \nu, n(x_0)\rangle_{\mathbb{R}^N} > 0.$$

Thus we get the conclusion. □

Corollary 2.1. *Let L be a second order degenerate elliptic linear operator in a bounded domain $\Omega \subset \mathbb{R}^N$ such that $c \leq 0$ in Ω. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ be such that $Lu \geq 0$ in Ω. Suppose further that there exists a point $x_0 \in \partial\Omega$ such that $u(x) < u(x_0)$ for all $x \in \Omega$ and that $u(x_0) \geq 0$. Finally assume that Ω satisfies at x_0 the interior ball condition and that the boundary of the interior ball is not characteristic for the operator L at x_0. Then $D_\nu u(x_0) > 0$.*

Proof. This result is an immediate consequence of the Hopf Lemma 2.2. □

Our next aim is to prove a strong maximum principle for second order linear degenerate elliptic operators. In order to be able to proceed, following the idea of [1], we need to assume another condition on the degeneracy set of the operator L in the domain Ω. See also comments in Remarks 2.9 and 2.12.

For every $x \in \overline{\Omega}$ let $\lambda(x) := \min_{1 \leq j \leq N} \lambda_j(x)$, where $\lambda_1(x), \ldots, \lambda_N(x) \in \mathbb{R}$ are the eigenvalues of the real symmetric matrix $A(x) = [a_{ij}(x)]$. Then $\lambda \in \mathcal{C}(\overline{\Omega})$, $\lambda(x) \geq 0$ in $\overline{\Omega}$ and $a_{ij}(x)v_i v_j \geq \lambda(x)|v|^2 \geq 0$ for every $v \in \mathbb{R}^N$, $x \in \overline{\Omega}$.

Let $\Sigma := \{x \in \overline{\Omega} : \lambda(x) = 0\} \subset \overline{\Omega}$ be the degeneracy set of the operator L. Then also $\Sigma = \{x \in \overline{\Omega} : \det A(x) = 0\}$ and Σ is closed and bounded. We will assume that

(Σ)
- Σ has no interior points. We let $\Omega_1, \Omega_2, \ldots$ denote the connected components of $\Omega \setminus \Sigma$, which are at most countably many.
- $\Sigma \cap \Omega = \Sigma_1 \cup \Sigma_2$, where for all $x_0 \in \Sigma_1$ and Ω_m such that $x_0 \in \partial\Omega_m$ there is $\overline{B_r(x_1)} \subset \overline{\Omega}_m$ such that $x_0 \in \partial B_r(x_1)$, $\langle (x_0 - x_1), A(x_0)(x_0 - x_1)\rangle_{\mathbb{R}^N} > 0$ and $\overline{B_r(x_1)} \cap \Sigma = \{x_0\}$, while for all $x_0 \in \Sigma_2$ there exists $\overline{B_r(x_1)} \subset \Omega$ such that $x_0 \in \partial B_r(x_1)$, $\langle (x_0 - x_1), A(x_0)(x_0 - x_1)\rangle_{\mathbb{R}^N} > 0$ and $\overline{B_r(x_1)} \cap \Sigma_2 = \{x_0\}$.
- For every $i \in \mathbb{N}$ there exists a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ with $\sigma(1) = i$ such that for every $h \in \mathbb{N}$, $h \geq 2$, there exists $l \in \mathbb{N}$ with $1 \leq l \leq h - 1$ and $\Sigma_1 \cap \partial\Omega_{\sigma(h)} \cap \partial\Omega_{\sigma(l)} \neq \emptyset$.

We explicitly remark that condition (Σ) prevents the operator L from degenerating on any open subset of the domain Ω.

Remark 2.8. If $\Omega \setminus \Sigma$ has a finite number $m \in \mathbb{N}$ of connected components, we require σ to be a permutation of the set $\{1, \ldots, m\}$. If $\Omega \setminus \Sigma$ has only one component, the third part of condition (Σ) is not necessary.

Remark 2.9. Notice that:

(i) The set Σ_1 is made up of those points $x_0 \in \Sigma \cap \Omega$ such that every connected component of $\Omega \setminus \Sigma$ having x_0 on its boundary satisfies the interior ball condition at that point. Moreover the boundary of the interior ball must not be characteristic for the operator L at x_0.

(ii) The condition on Σ_2 is satisfied if that set has dimension small enough, for instance if it is a $\mathcal{C}^{2,\alpha}$ manifold with $\alpha > 0$ and with dimension less than or equal to $N - 2$. The set Σ_2 represents a sort of singular set for the manifold Σ, and it contains all the points of Σ where there is no unique tangent hyperplane to the manifold.

(iii) The third part of condition (Σ) states that whichever connected component is chosen as the first, it is then possible to order all the remaining ones in such a way that for each $h \in \mathbb{N}$ it is possible to pass from the union of the first h connected components to the $(h+1)^{\text{th}}$ through a point of Σ_1.

Remark 2.10. Condition (Σ) is invariant under the action of diffeomorphisms of class $\mathcal{C}^{2,\alpha}(\overline{\Omega})$.

Theorem 2.2 (Strong Maximum Principle). *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be such that $Lu \geq 0$ with $c \leq 0$ in Ω and assume conditions (E_ξ) and (Σ) hold. Then the nonnegative maximum of u in $\overline{\Omega}$ can be attained only on $\partial\Omega$, unless u is constant.*

Proof. Let $x_0 \in \Omega$ be such that $u(x_0) = \max_{\overline{\Omega}} u =: M \geq 0$. Then $\nabla u(x_0) = 0$. We divide the proof into three steps.

Step 1. Suppose $x_0 \notin \Sigma$. Then there exists a connected component $\Omega_1 \subset \Omega \setminus \Sigma$ such that $x_0 \in \Omega_1$ and $\lambda(x_0) > 0$. Since $\lambda \in \mathcal{C}(\overline{\Omega})$, we can find an open ball $B_r(x_0)$ such that $\overline{B_r(x_0)} \subset \Omega_1$ where the operator L is uniformly elliptic. Since $Lu \geq 0$, $c \leq 0$ and u attains its maximum at an interior point of $B_r(x_0)$, the strong maximum principle for uniformly elliptic linear operators implies that $u(x) \equiv M$ on $B_r(x_0)$.

Then the set $\{x \in \Omega_1 : u(x) = M\}$ is open, closed and nonempty in Ω_1, which is connected. Hence $u(x) \equiv M$ in Ω_1, and by continuity u is constantly equal to M on $\overline{\Omega}_1$.

The proof ends here if $\Sigma \subset \partial\Omega$, since in this case $x_0 \notin \Sigma$ for every $x_0 \in \Omega$ and $\overline{\Omega}_1 = \overline{\Omega \setminus \Sigma} = \overline{\Omega}$.

Step 2. Suppose $\Omega \setminus \Sigma = \Omega_1$ has only one connected component. Then $\overline{\Omega}_1 = \overline{\Omega}$ since by condition (Σ) the degeneracy set of L in Ω has no interior points.

Now if there exists a point $x_0 \in \Omega_1$ such that $u(x_0) = M$, by Step 1 we get $u(x) \equiv M$ on $\overline{\Omega}_1 = \overline{\Omega}$.

If there is a point $x_0 \in \Sigma_1$ such that $u(x_0) = M$, then $x_0 \in \overline{\Omega}_1$ and by condition (Σ) we can find a ball $\overline{B_r(x_1)} \subset \overline{\Omega}_1$ such that $x_0 \in \partial B_r(x_1)$, $\overline{B_r(x_1)} \cap \Sigma = \{x_0\}$ and $\langle (x_0 - x_1), A(x_0)(x_0 - x_1) \rangle_{\mathbb{R}^N} > 0$. It follows that $B_r(x_1) \subset \Omega_1$.

Now there are two possibilities: either there exists $x_2 \in B_r(x_1)$ such that $u(x_2) = M$ or else $u(x) < M = u(x_0)$ for every $x \in B_r(x_1)$.

In the first case we have $x_2 \in \Omega_1$, and by the preceding argument we get $u(x) \equiv M$ on $\overline{\Omega}_1 = \overline{\Omega}$. In the latter case we can apply Hopf's Lemma 2.2 to conclude that there exists a direction ν such that $D_\nu u(x_0) > 0$, which is impossible because x_0 is an interior maximum point for the function u in $\overline{\Omega}$.

Finally if $x_0 \in \Sigma_2$ and $u(x_0) = M$, by condition (Σ) we can find a ball $\overline{B_r(x_1)} \subset \Omega$ such that $x_0 \in \partial B_r(x_1)$, $\overline{B_r(x_1)} \cap \Sigma_2 = \{x_0\}$ and $\langle (x_0 - x_1), A(x_0)(x_0 - x_1) \rangle_{\mathbb{R}^N} > 0$. Thus we have $B_r(x_1) \subset \Omega_1 \cup \Sigma_1$. Then by the Hopf Lemma 2.2 we can find $x_2 \in B_r(x_1)$ such that $u(x_2) = M$, and the result now follows again from the preceding argument, since we have either $x_2 \in \Omega_1$ or $x_2 \in \Sigma_1$.

Hence if $u(x_0) = M$ for a point $x_0 \in \Omega$, then the function u is constant.

The proof ends here if $\Omega \setminus \Sigma$ has only one connected component.

Step 3. Since $\Omega \setminus \Sigma$ has at most countably many connected components $\Omega_1, \Omega_2, \dots$, if $u(x_0) = M$ for a point $x_0 \in \Omega \setminus \Sigma$, then there exists $\sigma(1) \in \mathbb{N}$ such that $x_0 \in \Omega_{\sigma(1)}$, and from Step 1 it follows that $u(x) \equiv M$ on $\overline{\Omega}_{\sigma(1)}$. By condition (Σ) there is another component, which we call $\Omega_{\sigma(2)}$, such that $\Sigma_1 \cap \partial\Omega_{\sigma(1)} \cap \partial\Omega_{\sigma(2)} \neq \emptyset$.

Now let x_1 be a point in this set, so that $x_1 \in \overline{\Omega}_{\sigma(1)}$ and $u(x_1) = M$. Since $x_1 \in \Sigma_1 \cap \partial\Omega_{\sigma(2)}$, there exists a ball $\overline{B_r(x_2)} \subset \overline{\Omega}_{\sigma(2)}$ such that $x_1 \in \partial B_r(x_2)$, $\overline{B_r(x_2)} \cap \Sigma = \{x_1\}$ and $\langle (x_1 - x_2), A(x_1)(x_1 - x_2)\rangle_{\mathbb{R}^N} > 0$. If $u(x) < u(x_1) = M$ for every $x \in B_r(x_2)$, by the Hopf Lemma 2.2 there is a direction ν such that $D_\nu u(x_1) > 0$, which is not possible since x_1 is an interior maximum point for u in $\overline{\Omega}$. Thus we can find a point in $B_r(x_2) \subset \Omega_{\sigma(2)}$ where u attains its nonnegative maximum value M, and hence $u(x) \equiv M$ in $\overline{\Omega}_{\sigma(2)}$, by Step 1. Thus we have $u \equiv M$ in $\overline{\Omega}_{\sigma(1)} \cup \overline{\Omega}_{\sigma(2)}$.

Exploiting condition (Σ) and the Hopf Lemma 2.2, if $\Omega \setminus \Sigma$ has $m \in \mathbb{N}$ connected components, after m steps one finds that $u \equiv M$ in $\overline{\Omega}_{\sigma(1)} \cup \dots \cup \overline{\Omega}_{\sigma(m)}$, where σ is a permutation of $\{1, \dots, m\}$.

On the other hand, if $\Omega \setminus \Sigma$ has countably many connected components, following the preceding argument and exploiting condition (Σ), one can prove that $u \equiv M$ in $\overline{\Omega}_{\sigma(1)} \cup \dots \cup \overline{\Omega}_{\sigma(h)}$ for every $h \in \mathbb{N}$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is a suitable bijective map.

Thus in every case, by the continuity of u, we finally get $u(x) \equiv M$ in $\overline{\bigcup_h \Omega_{\sigma(h)}} = \overline{\Omega}$.

Now suppose $x_0 \in \Sigma_1$ is such that $u(x_0) = M$. Then there is $h \in \mathbb{N}$ such that $x_0 \in \partial\Omega_h$. Since $x_0 \in \Sigma_1$, by condition (Σ) we can find $\overline{B_r(x_1)} \subset \overline{\Omega}_h$ satisfying $x_0 \in \partial B_r(x_1)$, $\overline{B_r(x_1)} \cap \Sigma = \{x_0\}$ and $\langle (x_0 - x_1), A(x_0)(x_0 - x_1)\rangle_{\mathbb{R}^N} > 0$. By the Hopf Lemma 2.2, it cannot happen that $u(x) < u(x_0)$ for every $x \in B_r(x_1)$, since we must have $\nabla u(x_0) = 0$. Hence there exists a point in $B_r(x_1) \subset \Omega_h$ where u attains its nonnegative maximum value M, and by Step 1 we get $u(x) \equiv M$ on $\overline{\Omega}_h$. Then, by the preceding argument, we find again that $u(x) \equiv M$ on $\overline{\Omega}$.

Finally, if there is a point $x_0 \in \Sigma_2$ such that $u(x_0) = M$, by condition (Σ) we can find a ball $\overline{B_r(x_1)} \subset \Omega$ satisfying $x_0 \in \partial B_r(x_1)$, $\langle (x_0 - x_1), A(x_0)(x_0 - x_1)\rangle_{\mathbb{R}^N} > 0$ and $\overline{B_r(x_1)} \cap \Sigma_2 = \{x_0\}$. By the Hopf Lemma 2.2, there is a point $x_2 \in B_r(x_1) \subset \Omega \setminus \Sigma_2$ with $u(x_2) = M$. Then either $x_2 \in \Omega \setminus \Sigma$ or $x_2 \in \Sigma_1$, and in both cases, following the preceding arguments, we find that u is constant and equal to M on $\overline{\Omega}$.

Thus, if $u(x_0) = M$ for a point $x_0 \in \Omega$, we get $u(x) \equiv M$ on $\overline{\Omega}$. □

Remark 2.11. The hypothesis $\sup_{\overline{\Omega}} u = M \geq 0$ can be dropped if $c(x) \equiv 0$ on $\overline{\Omega}$.

Remark 2.12. If $\Omega \setminus \Sigma$ has countably many connected components $\{\Omega_j\}_{j \in \mathbb{N}}$, one can easily prove the following results:

- $\partial\Omega_j \cap \Omega \neq \emptyset$ for every $j \in \mathbb{N}$,
- $\partial\Omega_j \cap \Omega \subset \Sigma$ for every $j \in \mathbb{N}$,
- for every $j \in \mathbb{N}$ there exists a point $x \in \partial\Omega_j \cap \Omega$ and $k \in \mathbb{N} \setminus \{j\}$ such that $x \in \Sigma' := \partial\Omega_j \cap \partial\Omega_k \cap \Omega \neq \emptyset$.

Thus, as proved in Theorem 2.2, if there is $x_0 \in \Omega_j$ such that $u(x_0) = M$ we have $u(x) \equiv M$ on $\overline{\Omega}_j$. One has to show now that $\Sigma' \cap \Sigma_1 \neq \emptyset$ in order to proceed with the above argument and conclude that $u(x) \equiv M$ on $\overline{\Omega_j \cup \Omega_k}$. Granting this property is the aim of the third part of condition (Σ).

3. Generalized maximum principles

Throughout the first part of this section we will assume that $Lu \equiv a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u$ is a degenerate elliptic linear operator satisfying conditions (E_ξ) and (Σ) on a bounded domain $\Omega \subset \mathbb{R}^N$, unless otherwise stated.

We will also write $b(x) := (b_1(x), \dots, b_N(x))$, $A(x) = [a_{ij}(x)]$ and $D(x) = \det A(x)$ for any $x \in \Omega$, omitting the dependence of the functions on the points of the domain when there is no ambiguity.

Proposition 3.1 (Comparison Principle 1). *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be such that $Lu \geq 0$ with $c \leq 0$ in Ω. Then, if $u \leq 0$ on $\partial\Omega$, one has $u \leq 0$ on $\overline{\Omega}$. Moreover either $u < 0$ in Ω or $u \equiv 0$ in $\overline{\Omega}$.*

Proof. This proposition is an easy consequence of the weak maximum principle, Theorem 2.1, and of the strong maximum principle, Theorem 2.2. □

The next result is an extension of Serrin's maximum principle for uniformly elliptic linear operators.

Proposition 3.2 (Comparison Principle 2). *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be such that $Lu \geq 0$ and $u \leq 0$ in Ω. Then either $u < 0$ in Ω or $u \equiv 0$ in $\overline{\Omega}$.*

Proof. Suppose there is a point $x_0 \in \Omega$ such that $u(x_0) = 0$. Then we want to prove that $u \equiv 0$ in $\overline{\Omega}$. Write $c(x) = c^+(x) - c^-(x)$, with $c^\pm(x) \geq 0$ in Ω. Then

$$\widetilde{L}u := a_{ij}(x)D_{ij}u + b_i(x)D_iu - c^-(x)u \geq -c^+(x)u \geq 0 \quad \text{in } \Omega,$$

with $\widetilde{L}$ satisfying conditions (E_ξ) and (Σ). Hence, by the strong maximum principle, Theorem 2.2, we get $u \equiv 0$ on $\overline{\Omega}$. □

Theorem 3.1 (Generalized Maximum Principle). *Suppose there exists a function $w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $w > 0$ on $\overline{\Omega}$ and $Lw \leq 0$ on Ω. Then, if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfies $Lu \geq 0$ in Ω, the function u/w cannot attain in Ω its nonnegative maximum on $\overline{\Omega}$, unless it is constant.*

Proof. Let $v = u/w$ in Ω. Then the function v satisfies

$$\widetilde{L}v := a_{ij}(x)D_{ij}v + B_i(x)D_iv + \frac{Lw(x)}{w(x)}v \geq 0 \quad \text{in } \Omega,$$

with $B_i := b_i + \frac{2}{w}a_{ij}D_jw$ and $\frac{Lw}{w} \leq 0$ in Ω. Hence by the strong maximum principle, Theorem 2.2, v can attain its nonnegative maximum only on $\partial\Omega$, unless it is constant. □

The following maximum principle is a generalization of a result for uniformly elliptic linear operators due to Varadhan.

Theorem 3.2 (Maximum Principle for Narrow Domains). *Let $r > 0$ and $x_0 \in \mathbb{R}^N$ be such that $|\langle x - x_0, \xi\rangle_{\mathbb{R}^N}| < r$ for every $x \in \overline{\Omega}$, where ξ is a noncharacteristic direction for the operator L in Ω. Then there exists $r_0 > 0$ such that the assumptions of Theorem* 3.1 *are satisfied for $0 < r \leq r_0$.*

Proof. Let us consider the function $w(x) := e^{\alpha r} - e^{\alpha\langle x-x_0,\xi\rangle}$ in $\overline{\Omega}$, with $\alpha > 0$ to be chosen later. Write $c(x) = c^+(x) - c^-(x)$, with $c^\pm(x) \geq 0$ in Ω. Then $w > 0$ in $\overline{\Omega}$ and

$$\begin{aligned}Lw &= -(\alpha^2 a_{ij}(x)\xi_i\xi_j + \alpha b_i(x)\xi_i)e^{\alpha\langle x-x_0,\xi\rangle} + c(x)(e^{\alpha r} - e^{\alpha\langle x-x_0,\xi\rangle})\\ &\leq -(\beta\alpha^2 + \alpha b_i(x)\xi_i - c^-(x))e^{\alpha\langle x-x_0,\xi\rangle} + c^+(x)e^{\alpha r}.\end{aligned}$$

Since by hypothesis $b_i, c \in L^\infty(\Omega)$, we can find $M > 0$ such that $c^\pm, |b_i| \leq M$ in Ω and thus

$$\beta\alpha^2 + \alpha b_i\xi_i - c^- \geq \beta\alpha^2 - NM\alpha - M \geq 2M$$

if $\alpha > 0$ is large enough. Hence

$$Lw \leq -2Me^{\alpha\langle x-x_0,\xi\rangle} + Me^{\alpha r} \leq -2Me^{-\alpha r} + Me^{\alpha r} = Me^{-\alpha r}(e^{2\alpha r} - 2) \leq 0$$

in Ω, provided that $e^{2\alpha r} - 2 \leq 0$, i.e. $0 < r \leq r_0 := \frac{\ln 2}{2\alpha}$. □

Remark 3.1. No assumptions were made on the sign of the function c, the 0^{th} order term of the operator L, in Proposition 3.2 and in Theorems 3.1 and 3.2.

Remark 3.2. The hypotheses of Theorem 3.2 are satisfied if the bounded domain Ω is narrow enough in a direction $\xi \in \mathbb{R}^N$ such that $\langle\xi, A(x)\xi\rangle_{\mathbb{R}^N} \geq \beta > 0$ for every $x \in \overline{\Omega}$, i.e. if $\mathrm{l}_\xi(\Omega) := \sup_{x,\,y\in\Omega} |\langle x-y, \xi\rangle_{\mathbb{R}^N}| < r_0$, where $\mathrm{l}_\xi(\Omega)$ is the width of the domain in the given direction ξ which is noncharacteristic for the operator L.

Remark 3.3. If the bounded domain Ω is narrow in a given direction ζ_1, then it is narrow also in any direction not too far from ζ_1. In fact, if $\zeta_2 \in \mathbb{R}^N$ with $|\zeta_2| = 1$, we have

$$|\langle x-y, \zeta_2\rangle_{\mathbb{R}^N}| \leq \mathrm{l}_{\zeta_1}(\Omega) + \mathrm{diam}(\Omega)|\zeta_1 - \zeta_2| < (1+\varepsilon)\mathrm{l}_{\zeta_1}(\Omega)$$

for any direction $\zeta_2 \in \mathbb{R}^N$ such that $|\zeta_1 - \zeta_2| < \varepsilon \mathrm{l}_{\zeta_1}(\Omega)/\mathrm{diam}(\Omega)$. Hence for such directions we have $\mathrm{l}_{\zeta_2}(\Omega) < (1+\varepsilon)\mathrm{l}_{\zeta_1}(\Omega)$.

From now till the end of the section we will assume that the degenerate elliptic linear operator L satisfies just condition (E_ξ) on the bounded domain $\Omega \subset \mathbb{R}^N$, unless otherwise stated.

Theorem 3.3. *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, $\varphi \in L^\infty(\partial\Omega)$ and $f \in L^\infty(\Omega)$ be such that $Lu \geq f$ with $c \leq 0$ in Ω and $u \leq \varphi$ on $\partial\Omega$. Let $d > 0$ and $x_0 \in \Omega$ be such that $\overline{\Omega} \subset \{x \in \mathbb{R}^N : |\langle x-x_0, \xi\rangle_{\mathbb{R}^N}| \leq d\}$. Then*

$$\sup_\Omega u \leq \|\varphi^+\|_{L^\infty(\partial\Omega)} + \eta\|f^-\|_{L^\infty(\Omega)}$$

with η a positive constant depending only on d, $\|b_i\|_{L^\infty(\Omega)}$, ξ and $\|a_{ij}\|_{L^\infty(\Omega)}$.

Remark 3.4. The hypothesis $\overline{\Omega} \subset \{x \in \mathbb{R}^N : |\langle x-x_0, \xi\rangle_{\mathbb{R}^N}| \leq d\}$ for suitable $d > 0$ and $x_0 \in \Omega$ is always satisfied by a bounded domain $\Omega \subset \mathbb{R}^N$.

Proof of Theorem 3.3. Let $F := \|f^-\|_{L^\infty(\Omega)}$, $\phi := \|\varphi^+\|_{L^\infty(\partial\Omega)}$, pick $\alpha > 0$ to be specified later and define $w(x) := \phi + [e^{2\alpha d} - e^{\alpha(\langle \xi, x-x_0\rangle + d)}]F$ for every $x \in \overline{\Omega}$. Since $0 \le e^{2\alpha d} - e^{\alpha(\langle \xi, x-x_0\rangle + d)} \le e^{2\alpha d} - 1$ in $\overline{\Omega}$, we have

$$0 \le \phi \le w(x) \le \phi + (e^{2\alpha d} - 1)F \quad \text{ in } \overline{\Omega}.$$

Hence, if we assume $\|b_i\|_{L^\infty(\Omega)} \le M$, we get

$$\begin{aligned} -Lw &= Fe^{\alpha(\langle \xi, x-x_0\rangle + d)}(\alpha^2 a_{ij}(x)\xi_i\xi_j + \alpha\xi_i b_i(x)) - c(x)w \\ &\ge Fe^{\alpha(\langle \xi, x-x_0\rangle + d)}(\beta\alpha^2 - NM\alpha) \;\ge\; F(\beta\alpha^2 - NM\alpha) \;\ge\; F \end{aligned}$$

if $\alpha > 0$ is large enough. Then

$$\begin{cases} L(u-w) \ge f + F \ge 0 & \text{in } \Omega, \\ (u-w) \le \varphi - \phi \le 0 & \text{on } \partial\Omega, \end{cases}$$

and by the weak maximum principle, Theorem 2.1, we conclude that $\sup_\Omega (u-w) \le \sup_{\partial\Omega}(u-w)^+ = 0$, i.e. $u \le w$ in $\overline{\Omega}$. But then

$$\sup_\Omega u \le \sup_\Omega w \le \phi + (e^{2\alpha d} - 1)F = \|\varphi^+\|_{L^\infty(\partial\Omega)} + \eta\|f^-\|_{L^\infty(\Omega)}$$

with $\eta = \eta(d, \|b_i\|_{L^\infty(\Omega)}, \xi, \|a_{ij}\|_{L^\infty(\Omega)}) := e^{2\alpha d} - 1 > 0$. □

Proposition 3.3. *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, $\varphi \in L^\infty(\partial\Omega)$ and $f \in L^\infty(\Omega)$ be such that $Lu \le f$ with $c \le 0$ in Ω and $u \ge \varphi$ on $\partial\Omega$. Let $d > 0$ and $x_0 \in \Omega$ be such that $\overline{\Omega} \subset \{x \in \mathbb{R}^N : |\langle x - x_0, \xi\rangle_{\mathbb{R}^N}| \le d\}$. Then*

$$\inf_\Omega u \ge -\|\varphi^-\|_{L^\infty(\partial\Omega)} - \eta\|f^+\|_{L^\infty(\Omega)}$$

where η is the same positive constant depending only on d, $\|b_i\|_{L^\infty(\Omega)}$, ξ and $\|a_{ij}\|_{L^\infty(\Omega)}$ which appears in Theorem 3.3.

Proof. This result can be easily obtained by applying the preceding Theorem 3.3 to the function $v := -u$ in $\overline{\Omega}$. □

Proposition 3.4. *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, $\varphi \in L^\infty(\partial\Omega)$ and $f \in L^\infty(\Omega)$ be such that $Lu = f$ with $c \le 0$ in Ω and $u = \varphi$ on $\partial\Omega$. Let $d > 0$ and $x_0 \in \Omega$ be such that $\overline{\Omega} \subset \{x \in \mathbb{R}^N : |\langle x - x_0, \xi\rangle_{\mathbb{R}^N}| \le d\}$. Then*

$$\sup_\Omega |u| \le \|\varphi\|_{L^\infty(\partial\Omega)} + \eta\|f\|_{L^\infty(\Omega)} \tag{5}$$

where η is the same positive constant depending only on d, $\|b_i\|_{L^\infty(\Omega)}$, ξ and $\|a_{ij}\|_{L^\infty(\Omega)}$ which appears in Theorem 3.3.

Proof. Applying Theorem 3.3 and Proposition 3.3 to the functions u, φ and f yields inequality (5) immediately. □

Remark 3.5. The a priori estimate (5) given by Proposition 3.4 yields as a consequence a uniqueness result for $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ solutions of the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

for a second order linear degenerate elliptic operator L satisfying condition (E_ξ) and with 0^{th} order coefficient $c \leq 0$ on a bounded domain $\Omega \subset \mathbb{R}^N$, for functions $\varphi \in \mathcal{C}(\partial\Omega)$ and $f \in L^\infty(\Omega)$.

Proposition 3.5. *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be such that $Lu \geq 0$ in Ω. Let $d > 0$ and $x_0 \in \Omega$ be such that $\overline{\Omega} \subset \{x \in \mathbb{R}^N : |\langle x - x_0, \xi\rangle_{\mathbb{R}^N}| \leq d\}$. Then for every $\varepsilon \in (0, 1)$ there exists a $\delta > 0$ such that if $d \leq \delta$,*

$$(1 - \varepsilon) \sup_\Omega u \leq \max_{\partial\Omega} u^+.$$

Remark 3.6. This proposition is a kind of weak maximum principle for domains which are narrow in noncharacteristic directions for the operator, when no assumptions are made on the sign of c, the 0^{th} order coefficient of L in Ω.

Proof of Proposition 3.5. Write $c(x) = c^+(x) - c^-(x)$ with $c^\pm(x) \geq 0$ in Ω. Then

$$\widetilde{L}u := a_{ij}(x)D_{ij}u + b_i(x)D_iu - c^-(x)u \geq -c^+(x)u := f \quad \text{in } \Omega,$$

with $\widetilde{L}$ satisfying the assumptions of Theorem 3.3. Hence, if we assume $c^+(x) \leq M$ in Ω, from Theorem 3.3 we obtain

$$\begin{aligned} \sup_\Omega u &\leq \max_{\partial\Omega} u^+ + \eta \sup_\Omega [-c^+u]^- = \max_{\partial\Omega} u^+ + \eta \sup_\Omega [c^+u^+] \\ &\leq \max_{\partial\Omega} u^+ + M\eta \max_{\overline{\Omega}} u^+. \end{aligned} \tag{6}$$

Now if $u \leq 0$ in $\overline{\Omega}$, we obtain from (6) the trivial inequality $\sup_\Omega u \leq 0$, and the proof is complete. Otherwise we have $\max_{\overline{\Omega}} u^+ = \sup_\Omega u$. Recalling from Theorem 3.3 that we can choose $\eta = e^{2\alpha d} - 1$ for a suitable $\alpha > 0$, inequality (6) yields

$$\sup_\Omega u \leq \max_{\partial\Omega} u^+ + M(e^{2\alpha d} - 1) \sup_\Omega u.$$

Thus, given any $\varepsilon \in (0, 1)$, it is sufficient to choose $0 < d \leq \delta := \frac{\ln(1+\varepsilon/M)}{2\alpha}$ to conclude. □

Before stating the next theorem, we need to recall some classical results (see for instance [16]). For any function $u \in \mathcal{C}(\Omega)$ let

$$\Gamma^+ := \{x \in \Omega : u(z) \leq u(x) + \langle p, (z - x)\rangle_{\mathbb{R}^N} \ \forall z \in \Omega \text{ and for some } p = p(x) \in \mathbb{R}^N\}$$

be the *upper contact set* of u. Then u is concave if and only if $\Gamma^+ = \Omega$. If $u \in \mathcal{C}^1(\Omega)$ and $x \in \Gamma^+$, then $p(x) = \nabla u(x)$ and any support hyperplane must be tangent to the graph of the function u in $\mathbb{R}^{N+1}$. If $u \in \mathcal{C}^2(\Omega)$, then its Hessian matrix is nonpositive definite on Γ^+.

The following is a well known result due to Aleksandrov.

Theorem 3.4. *Let L be a second order linear elliptic operator with $c \leq 0$ on a bounded domain $\Omega \subset \mathbb{R}^N$ and let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be such that $Lu \geq f$ in Ω. Let* $\operatorname{diam}(\Omega) = d$, *$\omega_N$ be the volume of the unit ball in $\mathbb{R}^N$ and suppose that*

$$|b|/D^{1/N},\ f^-/D^{1/N} \in L^N(\Omega),$$

where $D(x) := \det A(x)$ is the determinant of the matrix of the coefficients of the second order derivatives in the operator L. Then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C\|f^-/D^{1/N}\|_{L^N(\Gamma^+)},$$

where Γ^+ is the upper contact set of the function u and where

$$C = C\left(N, d, \left\|\frac{b}{D^{1/N}}\right\|_{L^N(\Gamma^+)}\right) = d\left(\exp\left\{\frac{2^{N-2}}{\omega_N N^{N-1}}\left(\left\|\frac{b}{D^{1/N}}\right\|^N_{L^N(\Gamma^+)} + 1\right)\right\} - 1\right)^{1/N}.$$

Remark 3.7. The operator L in Theorem 3.4 is assumed to satisfy neither a uniform ellipticity condition nor condition (E_ξ) on Ω.

We are now ready to state and prove a maximum principle for domains with small volume, via Aleksandrov's maximum principle and via elliptic regularization of the degenerate elliptic linear operator L, which satisfies condition (E_ξ).

Theorem 3.5 (Maximum Principle for Domains with Small Volume). *Suppose that $u \in \mathcal{C}_b^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfies $Lu \geq 0$ in Ω, $u \leq 0$ on $\partial\Omega$ and let* $\operatorname{diam}(\Omega) \leq d$. *Then there exists $\delta > 0$, depending only on N, d, on the coefficients of the operator L and on the function u, such that if $|\Omega| < \delta$ then $u \leq 0$ in $\overline{\Omega}$.*

Remark 3.8. We remark that the positive constant δ of Theorem 3.5 depends also on the function u. Hence the requirement on the measure of the domain Ω is not uniform with respect to u, but it is sufficient to get the result only for any fixed function in $\mathcal{C}_b^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

Proof. If $c \leq 0$ in Ω, then the weak maximum principle, Theorem 2.1, holds on the domain. Hence $\sup_\Omega u \leq \sup_{\partial\Omega} u^+ = 0$ and the conclusion follows immediately, for any $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ and for any bounded domain Ω. If c is a generic bounded function in Ω, for any $\alpha > 0$ we have

$$(a_{ij}(x) + \delta_{ij})D_{ij}u + \left(b_i(x) + \alpha\frac{D_i u}{|\nabla u|}\right)D_i u + [c(x) + \alpha \operatorname{sign}(u)]u \geq \Delta u + \alpha(|\nabla u| + |u|)$$

where δ_{ij} is the Kronecker symbol. Hence

$$\begin{aligned}\widetilde{L}u &:= (a_{ij}(x) + \delta_{ij})D_{ij}u + \left(b_i(x) + \alpha\frac{D_i u}{|\nabla u|}\right)D_i u - [c(x) + \alpha \operatorname{sign}(u)]^- u \\ &\geq \Delta u + \alpha(|\nabla u| + |u|) - [c(x) + \alpha \operatorname{sign}(u)]^+ u \quad \text{ in } \Omega\end{aligned}$$

and $\widetilde{L}$ is a uniformly elliptic linear operator on Ω with nonpositive 0^{th} order coefficient. Notice that $\widetilde{L}u$ is well defined in Ω, also in the set where $\nabla u = 0$. With a slight abuse of notation, for every $i = 1, \dots, N$ we will denote by $D_i u/|\nabla u|$ the function

$$g_i(x) := \begin{cases} D_i u(x)/|\nabla u(x)| & \text{if } |\nabla u(x)| \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is bounded by 1 on the domain Ω.

Notice that on the upper contact set Γ^+ of the function u, the Hessian of u is nonpositive definite and hence on that set we have $\Delta u \leq 0$.

Notice also that if $\inf_{\Gamma^+}(|\nabla u| + |u|) = 0$, then one easily has $u \leq 0$ in Ω and hence the conclusion follows, no matter how large $|\Omega|$ is. In fact, in this case we could find a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \Gamma^+ \subset \Omega$ such that

$$\begin{gathered} u(x_n) \to 0, \qquad \nabla u(x_n) \to 0 \quad \text{as } n \to +\infty, \\ u(y) \leq u(x_n) + \langle \nabla u(x_n), (y - x_n)\rangle_{\mathbb{R}^N} \quad \forall y \in \Omega. \end{gathered}$$

Then passing to the limit as $n \to +\infty$ in the last inequality easily yields the conclusion $u \leq 0$ in $\overline{\Omega}$, given the boundedness of the domain Ω.

On the other hand, if $\inf_{\Gamma^+}(|\nabla u| + |u|) = \eta > 0$, we can choose $\alpha > 0$ large enough so that $\Delta u + \alpha(|\nabla u| + |u|) \geq 0$ in Γ^+, namely we need $\alpha \geq (1/\eta)\sup_{\Gamma^+}|\Delta u|$. Thus $\alpha = (1/\eta)\sup_\Omega |\Delta u|$ would do. Then it follows that

$$\begin{aligned} [\Delta u + \alpha(|\nabla u| + |u|) - (c(x) + \alpha \operatorname{sign}(u))^+ u]^- &\leq [-(c(x) + \alpha \operatorname{sign}(u))^+ u]^- \\ &= [c(x) + \alpha \operatorname{sign}(u)]^+ u^+ \quad \text{in } \Gamma^+. \end{aligned}$$

We also remark that, since the matrix $A(x) = [a_{ij}(x)]$ is nonnegative definite for every $x \in \overline{\Omega}$, each of the eigenvalues of $A(x) + I$ is greater than or equal to 1, thus $D(x) := \det[A(x) + I] \geq 1$ for every $x \in \overline{\Omega}$.

If we set $b := (b_1, \dots, b_N)$ and suppose that $|c|, |b| \leq M$ in Ω, then we have

$$\frac{1}{D}\left| b + \alpha \frac{\nabla u}{|\nabla u|}\right|^N \leq (M + \alpha)^N \quad \text{in } \Omega,$$

$$\begin{aligned} \frac{1}{D}|[\Delta u + \alpha(|\nabla u| + |u|) - (c(x) + \alpha \operatorname{sign}(u))^+ u]^-|^N &\leq ([c(x) + \alpha \operatorname{sign}(u)]^+ u^+)^N \\ &\leq (M + \alpha)^N \|u^+\|^N_{L^\infty(\Gamma^+)} \quad \text{in } \Gamma^+. \end{aligned}$$

Hence $D^{-1/N}\big|b + \alpha \nabla u/|\nabla u|\big|$, $D^{-1/N}[\Delta u + \alpha(|\nabla u| + |u|) - (c(x) + \alpha \operatorname{sign}(u))^+ u]^- \in L^N(\Gamma^+)$ and thus we can apply Aleksandrov's maximum principle 3.4 to the operator $\widetilde{L}$ to obtain

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C\left\| \frac{[\Delta u + \alpha(|\nabla u| + |u|) - (c(x) + \alpha \operatorname{sign}(u))^+ u]^-}{D^{1/N}}\right\|_{L^N(\Gamma^+)}$$

where

$$C := d\left(\exp\left\{\frac{2^{N-2}}{\omega_N N^{N-1}}\left(\left\|\frac{1}{D^{1/N}}\left(b+\alpha\frac{\nabla u}{|\nabla u|}\right)\right\|^N_{L^N(\Gamma^+)}+1\right)\right\}-1\right)^{1/N}.$$

But then, since $u \leq 0$ on $\partial\Omega$ by hypothesis, we get

$$\begin{aligned}\sup_\Omega u &\leq C\|[\Delta u+\alpha(|\nabla u|+|u|)-(c(x)+\alpha\,\mathrm{sign}(u))^+u]^-\|_{L^N(\Gamma^+)}\\ &\leq C\|[c(x)+\alpha\,\mathrm{sign}(u)]^+u^+\|_{L^N(\Gamma^+)} \leq C(M+\alpha)|\Omega|^{1/N}\sup_\Omega u^+,\end{aligned}$$

and this clearly implies $\sup_\Omega u \leq 0$ if $|\Omega|$ is small enough, i.e. if $|\Omega| < 1/(C^N(M+\alpha)^N)$. Hence $u \leq 0$ in $\overline{\Omega}$ if

$$|\Omega| < \delta := \begin{cases} +\infty & \text{if } \eta = \inf_{\Gamma^+}(|\nabla u|+|u|) = 0,\\ \dfrac{1}{C^N(M+\alpha)^N} & \text{if } \eta = \inf_{\Gamma^+}(|\nabla u|+|u|) > 0.\end{cases}$$ □

4. The case of the Grushin operator

The *Grushin operator* is the following linear partial differential operator:

$$G_\gamma u(z) = |y|^{2\gamma}\Delta_x u(z) + \Delta_y u(z) \tag{G_γ}$$

where $\gamma > 0$, $z := (x, y) \in \mathbb{R}^d \times \mathbb{R}^k$ with $d, k \in \mathbb{N}$ and $\Delta_x u(z) = \sum_{l=1}^d D_{x_l x_l} u(x, y)$, $\Delta_y u(z) = \sum_{m=1}^k D_{y_m y_m} u(x, y)$.

Notice that in this case $a_{ii}(z) = |y|^{2\gamma}$ if $i = 1, \ldots, d$, $a_{ii}(z) = 1$ if $i = d+1, \ldots, d+k$ and $a_{ij}(z) = 0$ if $i \neq j$. Then for every $z = (x, y) \in \mathbb{R}^{d+k}$ and $\xi \in \mathbb{R}^{d+k}$,

$$\langle \xi, A(z)\xi\rangle_{\mathbb{R}^{d+k}} = a_{ij}(z)\xi_i\xi_j \geq \min\{|y|^{2\gamma}, 1\}\,|\xi|^2 \geq 0.$$

According to the definition we gave in Section 2, the linear operator

$$Lu := G_\gamma u + b_l(z)D_{x_l}u + \widetilde{b}_m(z)D_{y_m}u + c(z)u \tag{7}$$

is degenerate elliptic on its domain $\Omega \subset \mathbb{R}^{d+k}$, with

$$\lambda(z) = \{\text{minimum eigenvalue of the real symmetric matrix } [a_{ij}(z)]\} = \min\{|y|^{2\gamma}, 1\}$$

and with degeneracy set $\Sigma = \lambda^{-1}\{0\} = \overline{\Omega} \cap (\mathbb{R}^d \times \{0\})$, which is closed and with no interior points.

Remark 4.1. The operator L defined above satisfies condition (E_ξ) with constant $\beta = 1$ in the direction $\xi = (0, \xi_y) \in \mathbb{R}^{d+k}$, for any $\xi_y \in \mathbb{R}^k$ with $|\xi_y| = 1$.

We are now interested in studying when a Hopf type lemma holds for the operator L defined in (7). Given a domain $\Omega \subset \mathbb{R}^{d+k}$ satisfying the interior ball condition, Lemma 2.2 holds at each noncharacteristic boundary point. The only case when a point $z_0 = (x_0, y_0)$ on its boundary can possibly be characteristic for the principal part of L is when z_0 belongs to the degeneracy set Σ, i.e. when $y_0 = 0$. Even in this case, $\partial\Omega$ is actually not characteristic for G_γ in z_0 if the normal versor to the boundary of the domain in z_0 is not parallel to Σ. Thus the only case when $\partial\Omega$ is characteristic for the principal part of L at a point z_0 is when $\partial\Omega$ and Σ have orthogonal intersection in z_0. Even in this case, anyway, if a suitable "convexity condition" on $\partial\Omega$ is satisfied, one can still recover a Hopf type lemma for the operator at z_0.

To make the statement more precise, we start by noticing that the Hopf Lemma 2.2 holds for the operator L on the ball $B_r(z_1) \subset \mathbb{R}^{d+k}$ with respect to the point $z_0 \in \partial B_r(z_1)$ if

$$|y_0|^{2\gamma}|x_0 - x_1|^2 + |y_0 - y_1|^2 = \langle (z_0 - z_1), A(z_0)(z_0 - z_1)\rangle_{\mathbb{R}^N} > 0, \tag{8}$$

where we set $z_0 = (x_0, y_0)$, $z_1 = (x_1, y_1)$. Condition (8) is clearly satisfied if

- $y_0 \neq 0$, or
- $y_0 = 0$ and $y_1 \neq 0$.

We cannot directly apply Lemma 2.2 only in the case when $z_0, z_1 \in \Sigma$.

Now define the following distance on $\mathbb{R}^{d+k}$:

$$d(z, z_1) := \left(|x - x_1|^2 + \frac{1}{(1+\gamma)^2}|y - y_1|^{2+2\gamma}\right)^{\frac{1}{2+2\gamma}} \tag{9}$$

for $z = (x, y)$, $z_1 = (x_1, y_1) \in \mathbb{R}^{d+k}$, and set

$$\widetilde{B}_r(z_1) := \{z = (x, y) \in \mathbb{R}^{d+k} : d(z, z_1) < r\}.$$

Using these balls, we can recover a Hopf lemma for the operator L defined in (7) also in some cases when $z_0, z_1 \in \Sigma$.

Lemma 4.1. *Let $B := \widetilde{B}_r(z_1)$ and let $u \in \mathcal{C}^2(B) \cap \mathcal{C}(B \cup \{z_0\})$, where $z_0, z_1 \in \mathbb{R}^d \times \{0\}$ and $z_0 \in \partial B$. Let also $u(z) < u(z_0)$ for every $z \in B$, $u(z_0) \geq 0$ and $Lu \geq 0$ in B, with $c \leq 0$ in B and with $\widetilde{b}_m, b_l/|y|^{2\gamma}, c/|y|^{2\gamma} \in L^\infty(B)$.*

Then for every outward direction ν at z_0, i.e. such that $\langle \nu, n(z_0)\rangle_{\mathbb{R}^{d+k}} > 0$, one has

$$\liminf_{t\to 0^+} \frac{u(z_0) - u(z_0 - t\nu)}{t} > 0.$$

Remark 4.2. If $u \in \mathcal{C}^1(B \cup \{z_0\})$ then we have $D_\nu u(z_0) > 0$.

Proof of Lemma 4.1. We may suppose $u \in \mathcal{C}^2(B) \cap \mathcal{C}(\overline{B})$ and that $u(z) < u(z_0)$ for every $z \in \overline{B} \setminus \{z_0\}$, as we did in the proof of the Hopf Lemma 2.2. Otherwise we can pick a smaller set $\widetilde{B}_{\check{r}}(\check{z}_1)$ contained in B, with $\check{z}_1 \in \mathbb{R}^d \times \{0\}$, and tangent to its boundary in z_0, where our assumptions are satisfied .

Now let $\alpha > 0$, $d(z) := d(z, z_1)$ and $h(z) := e^{-\alpha[d(z)]^{2+2\gamma}} - e^{-\alpha r^{2+2\gamma}}$. Notice that $h(z) \geq 0$ in $\overline{B}$.

To simplify the notation write $z_1 = (a, 0)$, $z_0 = (b, 0)$, $z = (x, y)$ with $a, b, x \in \mathbb{R}^d$ and with $y \in \mathbb{R}^k$. Then one has

$$\begin{aligned}Lh(z) = |y|^{2\gamma} e^{-\alpha[d(z)]^{2+2\gamma}} \Bigg[4\alpha^2 [d(z)]^{2+2\gamma} - 2\alpha \Bigg(\frac{Q + 2\gamma}{\gamma + 1} + (x_l - a_l) \frac{b_l}{|y|^{2\gamma}} \\ + y_m \frac{\widetilde{b}_m}{1+\gamma} \Bigg) + \frac{c}{|y|^{2\gamma}} \Bigg] - e^{-\alpha r^{2+2\gamma}} c \\ \geq |y|^{2\gamma} e^{-\alpha[d(z)]^{2+2\gamma}} \Bigg[4\alpha^2 [d(z)]^{2+2\gamma} - 2\alpha \Bigg(\frac{Q+2\gamma}{\gamma+1} + M_1 + M_2 \Bigg) - M_3 \Bigg],\end{aligned}$$

where we set $Q := (1 + \gamma)d + k$ and M_1, M_2, M_3 are suitable positive constants, since by our hypothesis $\widetilde{b}_m, b_l/|y|^{2\gamma}, c/|y|^{2\gamma} \in L^\infty(B)$.

Now let $\Omega = B \cap B_{r'}(z_0)$, with $r' = r^{\gamma+1}/4$. Then h is nonnegative and bounded in $\overline{\Omega}$, $h(z) \equiv 0$ on $\partial\Omega \cap \partial B$ and $d(z) \geq \delta > 0$ in $\overline{\Omega}$, since $z_1 \notin \overline{B}_{r'}(z_0)$. Thus we have

$$Lh(z) \geq |y|^{2\gamma} e^{-\alpha[d(z)]^{2+2\gamma}} \Bigg[4\alpha^2 \delta^{2+2\gamma} - 2\alpha \Bigg(\frac{Q+2\gamma}{\gamma+1} + M_1 + M_2 \Bigg) - M_3 \Bigg] \geq 0$$

in Ω, if we choose $\alpha > 0$ large enough.

Now, following the lines of the proof of the Hopf Lemma 2.2, one can use Theorem 2.1 and choose $\varepsilon > 0$ small enough so that

$$(u + \varepsilon h)(z_0) = \max_{\overline{\Omega}} (u + \varepsilon h).$$

Thus for any outward direction $\nu = (\nu_x, \nu_y) \in \mathbb{R}^{d+k}$ at z_0 with $\langle \nu, n(z_0) \rangle_{\mathbb{R}^{d+k}} > 0$, where we recall that $n(z_0)$ is the outward normal unit vector at the boundary point $z_0 \in \partial\Omega$, and for every $t > 0$ small enough we have

$$\frac{(u + \varepsilon h)(z_0) - (u + \varepsilon h)(z_0 - t\nu)}{t} \geq 0.$$

Hence passing to the lim inf as t tends to 0^+ and noting that, by our choice of the points z_0 and z_1, we have $n(z_0) = \frac{1}{|b-a|}((b-a), 0)$ we get

$$\begin{aligned}\liminf_{t \to 0^+} \frac{u(z_0) - u(z_0 - t\nu)}{t} &\geq -\varepsilon D_\nu h(z_0) \\ &= 2\alpha\varepsilon \Bigg[e^{-\alpha[d(z)]^{2+2\gamma}} \Bigg(\langle x - a, \nu_x \rangle_{\mathbb{R}^d} + \frac{|y|^{2\gamma}}{1+\gamma} \langle y, \nu_y \rangle_{\mathbb{R}^k} \Bigg) \Bigg]_{z=z_0} \\ &= 2\alpha\varepsilon e^{-\alpha r^{2+2\gamma}} \langle b - a, \nu_x \rangle_{\mathbb{R}^d} \\ &= 2\alpha\varepsilon |b-a| e^{-\alpha r^{2+2\gamma}} \langle n(z_0), \nu \rangle_{\mathbb{R}^{d+k}} > 0. \qquad \square\end{aligned}$$

Remark 4.3. The distance (9) on $\mathbb{R}^{d+k}$, which defines the ball $\widetilde{B}_r(z_1)$, is strictly related to the Grushin operator G_γ and satisfies the quasi homogeneity property

$$d((\lambda^{1+\gamma}x, \lambda y), 0) = \lambda d((x, y), 0) \quad \forall z = (x, y) \in \mathbb{R}^{d+k},\ \forall \lambda > 0.$$

As proven in [7], this distance is also related to the fundamental solution of G_γ,

$$\left(|x - \check{x}|^2 + \frac{1}{(\gamma+1)^2}|y|^{2\gamma+2}\right)^{-\frac{Q-2}{2\gamma+2}} \quad \text{with } Q = (\gamma+1)d + k, \tag{10}$$

at any point $\check{z} = (\check{x},\ 0)$ in $\Sigma = \mathbb{R}^d \times \{0\}$, where the operator degenerates, and also to the Kelvin transform for the Grushin operator (see [18] and also Section 5).

Notice also that the operator G_γ is the principal and nonsingular part of the Laplace–Beltrami operator on $\mathbb{R}^{d+k}$ endowed with the metric

$$\{g_{ij}(z)\} = \begin{pmatrix} |y|^{-2\gamma} & & & & & \\ & \ddots & & & 0 & \\ & & |y|^{-2\gamma} & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix} \tag{11}$$

which degenerates on Σ, as studied in [25].

Remark 4.4. The classical interior ball condition, which is usually required for the domain of a generic uniformly elliptic linear operator, can be substituted, in the case of a degenerate elliptic linear operator L having G_γ as principal part, satisfying the hypothesis of Lemma 4.1 and defined on $\Omega \subset \mathbb{R}^{d+k}$, with the following condition:

(B')
- for every $z \in \partial\Omega$, $z \notin \Sigma$ there exists a Euclidean ball $B_r(z_1) \subset \Omega$ such that $\partial B_r(z_1) \cap \partial\Omega = \{z\}$,
- for every $z \in \partial\Omega \cap \Sigma$ either
  - (i) there exists a Euclidean ball $B_r(z_1) \subset \Omega$ with $z_1 \notin \Sigma$ such that $\partial B_r(z_1) \cap \partial\Omega = \{z\}$, or
  - (ii) there exists a ball $\widetilde{B}_r(z_1) \subset \Omega$ in the topology defined by the distance (9) with $z_1 \in \Sigma$ such that $\partial\widetilde{B}_r(z_1) \cap \partial\Omega = \{z\}$.

Notice that (B') is more restrictive than the classical interior ball condition, since the sets $\widetilde{B}_r(z_1)$ with $z_1 \in \mathbb{R}^d \times \{0\}$ satisfy themselves the interior Euclidean ball condition at the points of $\partial\widetilde{B}_r(z_1) \cap (\mathbb{R}^d \times \{0\})$. On the other hand any Euclidean ball centered at a point in $\mathbb{R}^d \times \{0\}$ does not admit any interior ball of the topology defined by (9) tangent to its boundary at a point of $\partial B_r(z_1) \cap (\mathbb{R}^d \times \{0\})$.

Lemma 4.2. *The degenerate elliptic operator*

$$Lu := G_\gamma u + b_l(z)D_{x_l}u + \widetilde{b}_m(z)D_{y_m}u + c(z)u$$

defined on the bounded domain $\Omega \subset \mathbb{R}^{d+k}$ *satisfies condition* (Σ).

Proof. We begin by recalling that $\Sigma = \overline{\Omega} \cap (\mathbb{R}^d \times \{0\})$, and thus Σ is closed, with no interior points. We also note that $\Sigma_1 = \Sigma \cap \Omega$. In fact, if $z_0 = (x_0, 0) \in \Sigma \cap \Omega$, we see that $r := \text{dist}(z_0, \partial\Omega) > 0$. Hence, for every $y \in \mathbb{R}^k$ with $0 < |y| < r/3$, we have $\overline{B_{|y|}((x_0, y))} \subset \Omega$, $z_0 \in \partial B_{|y|}((x_0, y))$, $\Sigma \cap \overline{B_{|y|}((x_0, y))} = \{z_0\}$ and

$$\langle (x_0, 0) - (x_0, y), A((x_0, 0))((x_0, 0) - (x_0, y)) \rangle_{\mathbb{R}^{d+k}} = |y|^2 > 0.$$

Now suppose Ω_m is a connected component of $\Omega \setminus \Sigma$ such that $z_0 \in \partial\Omega_m$. Then we can choose $y \in \mathbb{R}^k$ with $|y|$ small enough so that $B_{|y|}((x_0, y)) \subset \Omega_m$, and thus $z_0 \in \Sigma_1$.

The third part of condition (Σ) is not required if $\Omega \setminus \Sigma$ has just one connected component, and thus in particular if $k \geq 2$.

So let $k = 1$ and let $\Omega \setminus \Sigma$ have at most countably many connected components $\Omega_1, \Omega_2, \ldots$. We conclude the proof of this lemma with the following *claim: L* satisfies the third part of condition (Σ) on Ω.

Let Ω_i be a connected component of $\Omega \setminus \Sigma$ and define $\widetilde{\sigma}(1) = i$. If there are countably many connected components, by induction it is now possible to construct a bijective map $\widetilde{\sigma} : \mathbb{N} \to \mathbb{N}$ such that for every $h \in \mathbb{N}$ with $h \geq 2$ there exists $l \in \{1, \ldots, h-1\}$ satisfying

$$\Sigma_1 \cap \partial\Omega_{\widetilde{\sigma}(h)} \cap \partial\Omega_{\widetilde{\sigma}(l)} \neq \emptyset. \tag{12}$$

Notice that if there are only $m \in \mathbb{N}$ connected components, our construction will yield a permutation $\widetilde{\sigma}$ of $\{1, \ldots, m\}$ with the required property.

First we want to prove that if $\Omega_{\widetilde{\sigma}(1)}, \ldots, \Omega_{\widetilde{\sigma}(j)}$ are distinct connected components satisfying (12) for every $1 \leq h \leq j$ and such that $\bigcup_{h=1}^{j} \Omega_{\widetilde{\sigma}(h)} \subsetneq \Omega \setminus \Sigma$, then we can find another connected component $\Omega_{\widetilde{\sigma}(j+1)}$ such that condition (12) is satisfied for every $1 \leq h \leq j+1$.

So define $\widetilde{\Omega} := \text{int}(\bigcup_{h=1}^{j} \overline{\Omega}_{\widetilde{\sigma}(h)}) \cap \Omega$. Then $\widetilde{\Omega}$ is a nonempty open set with $\widetilde{\Omega} \subset \Omega$. Moreover

(i) $\partial\widetilde{\Omega} \cap \Omega \neq \emptyset$, otherwise $\widetilde{\Omega}$ would be both open and closed in Ω, which is a connected set, and hence $\widetilde{\Omega} \equiv \Omega$, which is not possible;
(ii) $\partial\widetilde{\Omega} \cap \Omega \subset \Sigma_1$, since $\partial\Omega_l \cap \Omega \subset \Sigma \cap \Omega = \Sigma_1$ for each connected component Ω_l of $\Omega \setminus \Sigma$ and $\partial\widetilde{\Omega} \subset \bigcup_{h=1}^{j} \partial\Omega_{\widetilde{\sigma}(h)}$;
(iii) for each $z \in \Sigma \cap \Omega$ there exists a ball $B_r(z) \subset \Omega$ which intersects only a finite number of connected components of $\Omega \setminus \Sigma$ (just two due to the form of Σ, actually);
(iv) there exists a connected component, which we will call $\Omega_{\widetilde{\sigma}(j+1)}$, distinct from $\Omega_{\widetilde{\sigma}(1)}, \ldots, \Omega_{\widetilde{\sigma}(j)}$ and such that $\partial\Omega_{\widetilde{\sigma}(j+1)} \cap \partial\widetilde{\Omega} \cap \Omega \neq \emptyset$. In fact, otherwise one can easily prove that $(\widetilde{\Omega} \cup \partial\widetilde{\Omega}) \cap \Omega$ is both open and closed in Ω, which is not possible since Ω is connected and $\bigcup_{h=1}^{j} \Omega_{\widetilde{\sigma}(h)} \subsetneq \Omega$.

Thus there exists $h \in \{1, \ldots, j\}$ such that $\partial\Omega_{\widetilde{\sigma}(h)} \cap \partial\Omega_{\widetilde{\sigma}(j+1)} \cap \Omega \cap \Sigma_1 \neq \emptyset$.

Now if $\Omega \setminus \Sigma$ has a finite number m of connected components, we find the desired permutation of the set $\{1, \ldots, m\}$ after m steps. On the other hand, if $\Omega \setminus \Sigma$ has countably many connected components, the map $\widetilde{\sigma} : \mathbb{N} \to \mathbb{N}$ we constructed is injective but may not be surjective.

Now define

$$\Lambda := \left\{ A \subset \mathbb{N} : \begin{array}{l} \exists\, \sigma : \mathbb{N} \to \mathbb{N} \text{ an injective map satisfying } \sigma(1) = i, \\ \sigma(\mathbb{N}) = A \text{ and } \forall h \in \mathbb{N},\ h \geq 2\ \exists\, l \in \{1, \dots, h-1\} \\ \text{such that } \partial\Omega_{\sigma(h)} \cap \partial\Omega_{\sigma(l)} \cap \Sigma_1 \neq \emptyset \end{array} \right\}. \tag{13}$$

Then the set $\Lambda \subset \mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$, is partially ordered by inclusion and it is not empty, since it contains the set $\widetilde{\sigma}(\mathbb{N})$ obtained above. Now we prove that every totally ordered subset $\{A_r\}_{r\in I} \subset \Lambda$, where I is a set of indices, admits a maximal element in Λ. Indeed, let $\widehat{A} := \bigcup_{r\in I} A_r$. Clearly $A_r \subset \widehat{A} \subset \mathbb{N}$ for every $r \in I$ and we will construct an injective map $\widehat{\widehat{\sigma}} : \mathbb{N} \to \mathbb{N}$ with the properties described above to prove that $\widehat{A} \in \Lambda$.

Since $i \in A_r$ for every $r \in I$, we have $i \in \widehat{A}$ and we can define $\widehat{\sigma}(1) := i$. Now if $a_1 := \min\{\widehat{A} \setminus \{i\}\}$, we have $a_1 \in \widehat{A}$ and thus $a_1 \in A_{r_1}$ for a suitable $r_1 \in I$. Then we can find $\sigma_{r_1} : \mathbb{N} \to \mathbb{N}$ with the properties described in (13) such that $\sigma_{r_1}(\mathbb{N}) = A_{r_1}$, and $a_1 = \sigma_{r_1}(k_1)$ for a $k_1 \in \mathbb{N}$. Hence define $\widehat{\sigma}(j) = \sigma_{r_1}(j)$ for $2 \leq j \leq k_1$.

Now let $a_2 := \min\{\widehat{A} \setminus \{\widehat{\sigma}(1), \dots, \widehat{\sigma}(k_1)\}\}$. Then $a_2 \in \widehat{A}$ and there exist $r_2 \in I$ such that $a_2 \in A_{r_2}$ and a map $\sigma_{r_2} : \mathbb{N} \to \mathbb{N}$ satisfying (13) such that $\sigma_{r_2}(\mathbb{N}) = A_{r_2}$. Then $\sigma_{r_2}(k_2) = a_2$ for a $k_2 \in \mathbb{N}$. Hence define $\widehat{\sigma}(j + k_1) = \sigma_{r_2}(j+1)$ for $1 \leq j \leq k_2 - 1$.

Now by induction suppose we have defined $\widehat{\sigma}(j)$ for $1 \leq j \leq k_1 + \cdots + k_m - (m-1)$ and let $a_{m+1} := \min\{\widehat{A} \setminus \{\widehat{\sigma}(1), \dots, \widehat{\sigma}(k_1 + \cdots + k_m - m + 1)\}\}$. Since $a_{m+1} \in \widehat{A}$, we have $a_{m+1} \in A_{r_{m+1}}$ for a suitable $r_{m+1} \in I$ and we can find a map $\sigma_{r_{m+1}} : \mathbb{N} \to \mathbb{N}$ satisfying (13) such that $\sigma_{r_{m+1}}(k_{m+1}) = a_{m+1}$ for a $k_{m+1} \in \mathbb{N}$. Thus we can define

$$\widehat{\sigma}(j + k_1 + \cdots + k_m - m + 1) = \sigma_{r_{m+1}}(j+1) \quad \text{for } 1 \leq j \leq k_{m+1} - 1.$$

The map $\widehat{\sigma} : \mathbb{N} \to \mathbb{N}$ we obtain in this way may not be injective, thus we define $\widehat{\widehat{\sigma}} : \mathbb{N} \to \mathbb{N}$ by setting $\widehat{\widehat{\sigma}}(1) := i$ and by induction $\widehat{\widehat{\sigma}}(h+1) := \widehat{\sigma}(j_h)$ where

$$j_h := \min\{j \in \mathbb{N} : \widehat{\sigma}(j) \notin \{\widehat{\widehat{\sigma}}(1), \dots, \widehat{\widehat{\sigma}}(h)\}\}$$

for each $h \in \mathbb{N}$. Then by construction $\widehat{\widehat{\sigma}}$ is injective, $\widehat{\widehat{\sigma}}(1) = i$, $\widehat{\widehat{\sigma}}(\mathbb{N}) = \widehat{\sigma}(\mathbb{N}) = \widehat{A}$ and for each $h \in \mathbb{N}$ there exists $1 \leq l \leq h-1$ satisfying $\partial\Omega_{\widehat{\widehat{\sigma}}(h)} \bigcap \partial\Omega_{\widehat{\widehat{\sigma}}(l)} \cap \Omega \cap \Sigma_1 \neq \emptyset$. Hence $\widehat{A} \in \Lambda$ is a maximal element for $\{A_r\}_{r\in I} \subset \Lambda$.

By Zorn's lemma, the set Λ contains at least one maximal element, which we will call $\check{A}$. We *claim* that $\check{A} = \mathbb{N}$. Indeed, if $\check{A} \subsetneq \mathbb{N}$, we can construct a set $B \in \Lambda$ such that $\check{A} \subsetneq B$ in the following way.

Since $\check{A} \in \Lambda$ we can find an injective map $\check{\sigma} : \mathbb{N} \to \mathbb{N}$ satisfying conditions (13). Now let $\widetilde{\Omega} := \operatorname{int}(\overline{\bigcup_{m\in\check{A}} \Omega_m}) \cap \Omega$. Then $\widetilde{\Omega} \subsetneq \Omega$, since $\check{A} \subsetneq \mathbb{N}$, and hence $\partial\widetilde{\Omega} \cap \Omega \neq \emptyset$. Moreover $\partial\widetilde{\Omega} \cap \Omega \subset \Sigma \cap \Omega = \Sigma_1$, and one can easily also prove that there exists a connected component Ω_{j_0} of $\Omega \setminus \Sigma$ with $j_0 \in \mathbb{N} \setminus \check{A}$ such that $\partial\Omega_{j_0} \cap \partial\widetilde{\Omega} \cap \Omega \neq \emptyset$. Hence $\partial\Omega_{j_0} \cap \partial\Omega_{j_1} \cap \Sigma_1 \neq \emptyset$ for a suitable $j_1 \in \check{A}$.

Now let $B := \check{A} \cup \{j_0\}$ and define an injective map $\sigma_B : \mathbb{N} \to \mathbb{N}$ by setting

$$\sigma_B(j) = \begin{cases} \check{\sigma}(j) & \text{if } 1 \leq j \leq j_1, \\ j_0 & \text{if } j = j_1 + 1, \\ \check{\sigma}(j-1) & \text{if } j \geq j_1 + 2. \end{cases}$$

Then $\sigma_B(\mathbb{N}) = B$ and σ_B satisfies the conditions in (13). Hence $B \in \Lambda$ and $\check{A} \subsetneq B$, which contradicts the maximality of $\check{A} \in \Lambda$.

It follows that $\check{A} = \mathbb{N}$, thus $\mathbb{N} \in \Lambda$ and we can find a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(1) = i$ and for each $h \in \mathbb{N}$ with $h \geq 2$ there exists $1 \leq l \leq h-1$ satisfying $\partial\Omega_{\sigma(h)} \cap \partial\Omega_{\sigma(l)} \cap \Sigma_1 \neq \emptyset$.

Hence the domain $\Omega \subset \mathbb{R}^{d+k}$ satisfies condition (Σ). □

Remark 4.5. The strong maximum principle thus holds for a linear degenerate elliptic operator $Lu := G_\gamma u + b_l(z) D_{x_l} u + \widetilde{b}_m(z) D_{y_m} u + c(z)u$ on a bounded domain $\Omega \subset \mathbb{R}^{d+k}$.

5. Two applications of the moving planes to the Grushin operator

We recall the definition of the Grushin operator given in (G_γ) in Section 4:

$$G_\gamma u(z) = |y|^{2\gamma} \Delta_x u(z) + \Delta_y u(z), \qquad z \in \Omega,$$

where $\gamma > 0$, $z := (x, y) = (x_1, \ldots, x_d, y_1, \ldots, y_k) \in \Omega \subset \mathbb{R}^{d+k}$ with $d, k \in \mathbb{N}$ and $u : \Omega \subset \mathbb{R}^{d+k} \to \mathbb{R}$.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^{d+k}$ be a bounded domain and let G_γ be the Grushin operator. Suppose that if $z \in \partial\Omega$, then its symmetric point with respect to the hyperplane $T_0 := \{z = (x, y) \in \mathbb{R}^{d+k} : x_1 = 0\}$ also belongs to $\partial\Omega$ and that the segment having the two points as extremes lies in Ω. If $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a solution of*

$$\begin{cases} G_\gamma u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{14}$$

with f a locally Lipschitz function in $\mathbb{R}$, then u is symmetric with respect to direction x_1 and $D_{x_1} u(z) < 0$ for every $z \in \Omega$ with $x_1 > 0$ and with $y \neq 0$. If f is nondecreasing, then also $D_{x_1} u(z) < 0$ for every $z \in \Omega$ with $x_1 > 0$ and with $y = 0$.

Remark 5.1. With the hypothesis we made, Ω is convex in the direction x_1 and symmetric with respect to the hyperplane $T_0 := \{z = (x, y) \in \mathbb{R}^{d+k} : x_1 = 0\}$. Moreover $\partial\Omega$ does not have “flat portions” in the direction x_1.

Proof of Theorem 5.1. We will denote by $z = (x, y) = (x_1, \ldots, x_d, y_1, \ldots, y_k) = (x_1, \widetilde{x}, y)$ any point of $\mathbb{R}^{d+k}$. Let $a := \sup_{z \in \Omega} x_1$ and for $0 \leq \lambda < a$ define

$$\begin{aligned} \Omega_\lambda &= \{z \in \Omega : x_1 > \lambda\}, \qquad T_\lambda = \{z \in \mathbb{R}^{d+k} : x_1 = \lambda\}, \\ \Omega'_\lambda &= \{\text{the reflection of } \Omega_\lambda \text{ with respect to the hyperplane } T_\lambda\}, \\ z_\lambda &= (2\lambda - x_1, \widetilde{x}, y) \qquad \text{for any point } z = (x_1, \widetilde{x}, y) \in \Omega_\lambda. \end{aligned}$$

Hence z_λ is the symmetric point of $z \in \Omega_\lambda$ with respect to the hyperplane T_λ.

Now for any $\lambda \in (0, a)$ define in Ω_λ the function $w_\lambda(z) := u(z) - u(z_\lambda)$. Then $w_\lambda \in \mathcal{C}^2(\Omega_\lambda) \cap \mathcal{C}(\overline{\Omega}_\lambda)$ and one can easily prove that

$$\begin{aligned} &G_\gamma w_\lambda(z) + c(\lambda, z) w_\lambda(z) = 0 \quad && \text{in } \Omega_\lambda, \\ &w_\lambda \le 0 \quad \text{and} \quad w_\lambda \not\equiv 0 && \text{on } \partial\Omega_\lambda, \end{aligned} \tag{15}$$

by the mean value theorem, where $c(\lambda, z)$ is a bounded function of z on Ω_λ, since u is bounded on $\overline{\Omega}$ and f is locally Lipschitz.

Notice also that if $\lambda > 0$ we have $w_\lambda < 0$ on $\partial\Omega_\lambda \cap \partial\Omega$. Indeed, by hypothesis we have $u(z) = 0$ for every $z \in \partial\Omega$, and if $\lambda > 0$, by our assumptions on the domain Ω, $z_\lambda \in \Omega$ for every $z \in \partial\Omega$. Hence on $\partial\Omega_\lambda \cap \partial\Omega$ we have $w_\lambda(z) = u(z) - u(z_\lambda) = -u(z_\lambda) < 0$, by the positivity of u in Ω.

Now we want to prove that $w_\lambda < 0$ in Ω_λ for every $\lambda \in (0, a)$.

For any λ close enough to a, we have $w_\lambda < 0$ in Ω_λ by the maximum principle for narrow domains, Theorem 3.2, and the generalized maximum principle, Theorem 3.1. In fact, Ω_λ is narrow in direction x_1, provided that λ is sufficiently close to a. Since $\overline{\Omega}_\lambda \subset \overline{\Omega}$ is bounded, it is also narrow in any direction not too far from the direction x_1 (see Remarks 3.2 and 3.3). In particular we can choose a noncharacteristic direction ξ for the operator G_γ, i.e. a direction such that for every z in $\overline{\Omega}_\lambda$,

$$\langle \xi, A(z)\xi \rangle_{\mathbb{R}^{d+k}} = |y|^{2\gamma} |\xi_x|^2 + |\xi_y|^2 > 0,$$

and then apply the maximum principle for narrow domains, Theorem 3.2, and the generalized maximum principle, Theorem 3.1.

Let $\lambda_0 := \inf\{\mu \in (0, a) : w_\lambda < 0 \text{ in } \Omega_\lambda \ \forall \lambda \in (\mu, a)\}$, so that (λ_0, a) is the largest interval in $(0, a)$ satisfying $w_\lambda < 0$ in Ω_λ for every $\lambda \in (\lambda_0, a)$. We claim that $\lambda_0 = 0$.

If $\lambda_0 > 0$, by continuity we get $w_{\lambda_0} \le 0$ in Ω_{λ_0} and also $w_{\lambda_0} \not\equiv 0$ on $\partial\Omega_{\lambda_0}$. Then $w_{\lambda_0} < 0$ in Ω_{λ_0}, by the comparison principle, Proposition 3.2. Moreover recall that $w_{\lambda_0} < 0$ also on $\partial\Omega_{\lambda_0} \cap \partial\Omega$, since we assumed $\lambda_0 > 0$.

Now our aim is to prove that $w_{\lambda_0 - \varepsilon} < 0$ in $\Omega_{\lambda_0 - \varepsilon}$, if $\varepsilon > 0$ is sufficiently small.

Let $\delta > 0$ be a constant to be chosen later. Then since $w_{\lambda_0} < 0$ in $\overline{\Omega}_{\lambda_0} \cap \{x_1 > \lambda_0\}$ and since it is continuous, by compactness we have

$$w_{\lambda_0} \le -\eta < 0 \quad \text{in } K := \overline{\Omega}_{\lambda_0} \cap \{x_1 \ge \lambda_0 + \delta\},$$

for a suitable $\eta > 0$. Notice that $K = \overline{\Omega}_{\lambda_0 + \delta}$.

By continuity, we also have $w_{\lambda_0 - \varepsilon} \le -\eta/2 < 0$ in K for any $\varepsilon > 0$ small enough. Then we can choose $0 < \varepsilon < \delta$ so that $\Omega_{\lambda_0 - \varepsilon} \subset \Omega_{\lambda_0 - \delta}$, and then we fix δ small enough in such a way that we may apply the maximum principle for narrow domains, Theorem 3.2, and the generalized maximum principle, Theorem 3.1, in $\Omega_{\lambda_0 - \varepsilon} \setminus \Omega_{\lambda_0 + \delta}$, with respect to a suitable noncharacteristic direction for the operator G_γ. Notice that this domain is bounded and that its width in the x_1-direction is less than 2δ, hence it is also narrow in noncharacteristic directions not too far from direction x_1, provided that $\delta > 0$ is chosen small enough. Thus we obtain

$$w_{\lambda_0 - \varepsilon} \le 0 \quad \text{in } \Omega_{\lambda_0 - \varepsilon} \setminus K.$$

Now notice that $w_{\lambda_0-\varepsilon} \not\equiv 0$ on $\partial(\Omega_{\lambda_0-\varepsilon} \setminus K) \cap \partial\Omega$, since $w_{\lambda_0-\varepsilon} < 0$ on $\partial\Omega_{\lambda_0-\varepsilon} \cap \partial\Omega$ if $\varepsilon > 0$ is small enough. But then from the comparison principle, Proposition 3.2, it follows that $w_{\lambda_0-\varepsilon} < 0$ in $\Omega_{\lambda_0-\varepsilon} \setminus K$. Hence $w_{\lambda_0-\varepsilon} < 0$ in $\Omega_{\lambda_0-\varepsilon}$ if $\varepsilon > 0$ is small enough, and this contradicts the definition of λ_0.

Thus we have $\lambda_0 = 0$ and $w_\lambda < 0$ in Ω_λ for every $\lambda \in (0, a)$, i.e.

$$u(x_1, \widetilde{x}, y) = u(z) < u(z_\lambda) = u(2\lambda - x_1, \widetilde{x}, y) \quad \forall z = (x_1, \widetilde{x}, y) \in \Omega_\lambda.$$

By continuity, passing to the limit as λ tends to 0^+, we have $u(x_1, \widetilde{x}, y) \leq u(-x_1, \widetilde{x}, y)$ for every point $(x_1, \widetilde{x}, y) \in \overline{\Omega}_0$.

Repeating the same argument for the opposite direction $-x_1$, namely moving the planes T_λ from $-a$ toward the origin along the x_1-axis, we get the opposite inequality $u(-x_1, \widetilde{x}, y) \leq u(x_1, \widetilde{x}, y)$ for every point $z = (x_1, \widetilde{x}, y) \in \overline{\Omega}_0$, and hence we have the desired symmetry in the x_1-direction,

$$u(-x_1, \widetilde{x}, y) = u(x_1, \widetilde{x}, y) \quad \forall z = (x_1, \widetilde{x}, y) \in \Omega.$$

Now notice also that, since $w_\lambda < 0$ in Ω_λ for every $\lambda \in (0, a)$, the function w_λ attains its maximum value of 0 on $\overline{\Omega}_\lambda$ at each point of $\partial\Omega_\lambda \cap \Omega = T_\lambda \cap \Omega$. It is easy to see that the set Ω_λ also satisfies condition (B') of Remark 4.4 at every point of $\partial\Omega_\lambda \cap \Omega$. Hence, from (15) and from the Hopf Lemma 2.2, we get, for any such λ,

$$D_{x_1} u(\lambda, \widetilde{x}, y) = \frac{1}{2} D_{x_1} w_\lambda(\lambda, \widetilde{x}, y) < 0,$$

if $y \neq 0$, i.e. $D_{x_1} u(z) < 0$ for every $z \in \Omega$ with $x_1 > 0$ and $y \neq 0$.

If f is nondecreasing, then for any $\lambda \in (0, a)$ one also has $c(\lambda, z) \geq 0$ for every $z \in \Omega_\lambda$. Thus for any such λ we have $[c(\lambda, z)]^- \equiv 0$ on Ω_λ, and hence $[c(\lambda, z)]^-/|y|^{2\gamma} \in L^\infty(\Omega_\lambda)$. From (15) and from the Hopf Lemma 4.1, we get as before $D_{x_1} u(z) < 0$ also for every $z \in \Omega$ with $x_1 > 0$ and $y = 0$. The proof is now complete. □

Corollary 5.1. *If $u \in \mathcal{C}^2(B_1(0)) \cap \mathcal{C}(\overline{B_1(0)})$ is a solution of*

$$\begin{cases} G_\gamma u + f(u) = 0 & \text{in } B_1(0), \\ u > 0 & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0) \end{cases} \tag{16}$$

with $B_1(0) \subset \mathbb{R}^{d+k}$ and f is a locally Lipschitz, nondecreasing function in $\mathbb{R}$, then u is radially symmetric with respect to the $x \in \mathbb{R}^d$ variables about the origin. Moreover u is radially decreasing with respect to the $x \in \mathbb{R}^d$ variables.

Proof. This result is a straightforward application of Theorem 5.1. □

Our next aim is to prove a nonexistence result for the following problem on the whole space $\mathbb{R}^{d+k}$. Let the function u be a solution of

$$\begin{cases} G_\gamma u + u^p = 0 & \text{in } \mathbb{R}^{d+k}, \\ u \geq 0 & \text{in } \mathbb{R}^{d+k}, \quad u \in \mathcal{C}^2(\mathbb{R}^{d+k}), \end{cases} \tag{17}$$

with $Q := (\gamma + 1)d + k$ and $1 < p < (Q + 2)/(Q - 2)$.

Theorem 5.2. *If* $0 < \gamma < 1$ *and* $d, k \in \mathbb{N}$ *or if* $\gamma > 0$, $d \in \mathbb{N}$ *and* $k \in \mathbb{N} \setminus \{1, 2\}$, *then any solution* $u \in \mathcal{C}^2(\mathbb{R}^{d+k})$ *of problem* (17) *vanishes identically on* $\mathbb{R}^{d+k}$.

Before starting with the proof, we make a few remarks.

Remark 5.2. The corresponding problem for the Laplace operator

$$\begin{cases} \Delta u + u^p = 0 \quad \text{in } \mathbb{R}^N, \\ u \geq 0 \quad \text{in } \mathbb{R}^N, \quad u \in \mathcal{C}^2(\mathbb{R}^N), \end{cases} \tag{18}$$

for $N \geq 3$ admits no nontrivial solutions for $1 < p < (N+2)/(N-2) = 2^* - 1$, where 2^* is the Sobolev critical exponent. This has been proven first in [14], exploiting nonlinear energy estimates obtained by applying the Divergence Theorem to a suitable vector field, depending both on the solution u and on a cutoff function. This result has also been proven by the method of moving planes "from infinity" (see [6]), which exploits the invariances of the Laplace operator and maximum principles.

Remark 5.3. The number Q is the homogeneous dimension of the space $\mathbb{R}^{d+k}$ endowed with the distance (9), as it is related to the rate of growth of the Euclidean volume of the metric ball $\widetilde{B}_R$ with radius $R > 0$ as R tends to infinity. In fact

$$|\widetilde{B}_R| \sim cR^Q \quad \text{as } R \to +\infty$$

for a suitable constant $c = c(d, k, \gamma) > 0$.

We introduce the weighted gradient of a function $u \in \mathcal{C}^1(\mathbb{R}^{d+k})$ by setting

$$\begin{aligned} \widetilde{\nabla} u(z) &:= (|y|^\gamma \nabla_x u(z), \nabla_y u(z)) \\ &= (|y|^\gamma D_{x_1} u(z), \ldots, |y|^\gamma D_{x_d} u(z), D_{y_1} u(z), \ldots, D_{y_k} u(z)) \end{aligned} \tag{19}$$

for any point $z = (x, y) \in \mathbb{R}^{d+k}$.

We also introduce the weighted Sobolev spaces for any bounded domain $\Omega \subset \mathbb{R}^{d+k}$, by defining $\widetilde{W}^{1,p}(\Omega)$ as the completion of $\mathrm{Lip}(\overline{\Omega})$, the space of all Lipschitz-continuous functions on $\overline{\Omega}$, with respect to the norm

$$\|u\|_{\widetilde{W}^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \big\| |\widetilde{\nabla} u| \big\|_{L^p(\Omega)}.$$

Then $\widetilde{W}^{1,p}(\Omega)$ is a separable Banach space for any $1 \leq p < \infty$ and it is a separable Hilbert space for $p = 2$ with the scalar product and the equivalent norm

$$\langle u, v \rangle_{\widetilde{W}^{1,2}(\Omega)} := \langle u, v \rangle_{L^2(\Omega)} + \langle \widetilde{\nabla} u, \widetilde{\nabla} v \rangle_{L^2(\Omega)}, \qquad \|u\|^2_{\widetilde{W}^{1,2}(\Omega)} := \langle u, u \rangle.$$

Then for any $1 \leq p < \infty$ one can also prove that if $Q > p$ there are continuous embeddings

$$\widetilde{W}^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

for every $1 \leq q \leq pQ/(Q-p)$ and that those embeddings are compact if $1 \leq q < pQ/(Q-p)$ (see [8] and Proposition 2.4 of [17]). Hence the number $p^*(Q) := pQ/(Q-p)$ plays the role of the usual Sobolev critical exponent p^*, with the homogeneous dimension Q replacing the actual dimension of the Euclidean ambient space $\mathbb{R}^N$.

Remark 5.4. The problem

$$\begin{cases} G_\gamma u + u^p \le 0 \quad \text{in } \mathbb{R}^{d+k}, \\ u \ge 0 \quad \text{in } \mathbb{R}^{d+k}, \quad u \in \mathcal{C}^2(\mathbb{R}^{d+k}) \end{cases} \tag{20}$$

admits no solution if $1 < p \le Q/(Q-2)$ (see [7]). Hence $p_*(Q) := Q/(Q-2)$ plays for the Grushin operator the same role as the Serrin critical exponent for the Laplace operator, once again with the homogeneous dimension Q replacing the actual dimension of the Euclidean ambient space $\mathbb{R}^N$.

Remark 5.5. The homogeneous dimension Q appears also in the critical growth phenomenon exhibited by the Dirichlet problem

$$\begin{cases} G_\gamma u + u|u|^{p-1} = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{21}$$

where $\Omega \subset \mathbb{R}^{d+k}$ is a bounded, sufficiently regular domain. In fact, using standard arguments of calculus of variations, it is easy to see that this problem admits a suitably defined weak solution whenever $1 < p < (Q+2)/(Q-2)$. On the other hand, problem (21) does not admit nontrivial weak, and hence strong, solutions when $p > (Q+2)/(Q-2)$ and the domain Ω is starshaped with respect to the flow of a certain vector field, which is the infinitesimal generator of an anisotropic dilation with respect to which the Grushin operator G_γ is invariant. The key ingredient for this nonexistence result is a Pohožaev type identity for the operator G_γ in the domain Ω. For further details see for instance [21], [23] and [19].

Proof of Theorem 5.2. Suppose $u \in \mathcal{C}^2(\mathbb{R}^{d+k})$ is a solution of problem (17). We divide the proof into five steps.

Step 1 (*Reduction to the case of u strictly positive*). Notice that, by the strong maximum principle, Theorem 2.2, we have either $u \equiv 0$ or $u > 0$ in $\mathbb{R}^{d+k}$. In fact, if we can find a point $z_0 \in \mathbb{R}^{d+k}$ such that $u(z_0) = 0$, then we can apply the strong maximum principle to the function $-u$ in $B_R(z_0)$ for any $R > 0$. Thus we conclude that $u \equiv 0$ on that ball, since $-u \le 0$ and $G_\gamma(-u) = u^p \ge 0$ on $\mathbb{R}^{d+k}$. Since $R > 0$ is arbitrary, we get $u \equiv 0$ on the whole space.

From now on, we will suppose that u is a strictly positive solution of problem (17).

Step 2 (*Introduction of v, the Kelvin transform for G_γ of the function u*). For $z_1, z_2 \in \mathbb{R}^{d+k}$, let $d(z_1, z_2)$ denote the distance on $\mathbb{R}^{d+k}$ defined in (9) in Section 4. Then for any $z = (x, y) \in \mathbb{R}^{d+k}$ define

$$\rho(z) = \rho(x, y) := [d(z, 0)]^{2+2\gamma} = |x|^2 + \frac{1}{(1+\gamma)^2}|y|^{2\gamma+2}.$$

Now let

$$v(z) = v(x, y) := \frac{1}{[\rho(x, y)]^{\frac{Q-2}{2\gamma+2}}}\, u\left(\frac{x}{\rho(x, y)}, \frac{y}{[\rho(x, y)]^{\frac{1}{1+\gamma}}}\right) \tag{22}$$

be the Kelvin transform of the solution u with respect to the origin, which is defined for $z = (x, y) \in \mathbb{R}^{d+k} \setminus \{0\}$. We recall that the Kelvin transform (22) is an element of the symmetry group of the Grushin operator (see [18]). Then the function v satisfies

$$\begin{cases} G_\gamma v + \dfrac{1}{[\rho(z)]^{\frac{Q+2}{2\gamma+2} - p\frac{Q-2}{2\gamma+2}}} \, v^p = 0 \quad \text{in } \mathbb{R}^{d+k} \setminus \{0\}, \\ v > 0 \quad \text{in } \mathbb{R}^{d+k} \setminus \{0\}, \quad v \in \mathcal{C}^2(\mathbb{R}^{d+k} \setminus \{0\}). \end{cases} \tag{23}$$

Step 3 (*Symmetry of v in the x_1-direction: Introduction of the auxiliary function $\overline{w}_\lambda$*). Now denote by $z = (x, y) = (x_1, \ldots, x_d, y_1, \ldots, y_k)$ any point of $\mathbb{R}^{d+k}$ and define, for any $\lambda \le 0$,

$$\Omega_\lambda := \{z = (x, y) \in \mathbb{R}^{d+k} : x_1 < \lambda\}, \qquad T_\lambda := \partial\Omega_\lambda = \{z = (x, y) \in \mathbb{R}^{d+k} : x_1 = \lambda\}.$$

Let $z^\lambda := (2\lambda - x_1, \ldots, x_d, y_1, \ldots, y_k)$ be the reflection of any point $z = (x, y) \in \Omega_\lambda$ with respect to the hyperplane T_λ. Then define, in $\overline{\Omega}_\lambda$,

$$v_\lambda(z) := v(z^\lambda), \qquad w_\lambda(z) := v_\lambda(z) - v(z), \qquad \overline{w}_\lambda(z) := \frac{w_\lambda(z)}{g(z)}$$

where g is any function satisfying the following conditions:

- $g \in \mathcal{C}^2(\overline{\Omega}_0)$ and $g > 0$ on $\overline{\Omega}_0$.
- $D_{x_1} g \le 0$ in $\overline{\Omega}_0$, i.e. g is nonincreasing in the x_1-direction.
- For every $C > 0$ there exists $R > 0$, depending only on C and γ, such that $\frac{G_\gamma g(z)}{g(z)} + \frac{C}{[\rho(z)]^{4/(2+2\gamma)}} < 0$ for every $z \in \overline{\Omega}_0$ with $|z| > R$. (24)
- For every fixed $\lambda < 0$ we have $\overline{w}_\lambda(z) = w_\lambda(z)/g(z) \to 0$ in $\overline{\Omega}_\lambda$ as $|z| \to +\infty$.

We remark that, since v is singular in the origin, neither w_λ nor $\overline{w}_\lambda$ is well defined at the points $z = 0 \in \mathbb{R}^{d+k}$ and $z = z_\lambda := (2\lambda, \ldots, 0, 0, \ldots, 0)$. We note however that $z_\lambda \in \Omega_\lambda$, while $0 \notin \overline{\Omega}_\lambda$ for every $\lambda < 0$. Hence $\overline{w}_\lambda, w_\lambda \in \mathcal{C}^2(\Omega_\lambda \setminus \{z_\lambda\}) \cap \mathcal{C}^1(\overline{\Omega}_\lambda \setminus \{z_\lambda\})$ for any $\lambda < 0$.

Step 4 (*Symmetry of v in the x_1-direction*). This is the most difficult part of the proof of Theorem 5.2, and is based upon the technique of moving planes. Namely we want to prove that $\overline{w}_{\lambda_0} \equiv 0$ in $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$ for a suitable λ_0, by moving the hyperplane T_λ along the x_1-axis from ∞ towards the origin of $\mathbb{R}^{d+k}$. By the definition of $\overline{w}_{\lambda_0}$, this yields $v(z^{\lambda_0}) = v(z)$ for every $z \in \Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$, i.e. the function v is symmetric with respect to the hyperplane $T_{\lambda_0} = \{z \in \mathbb{R}^{d+k} : x_1 = \lambda_0\}$. We will go through the details of the proof of Step 4 later in this section.

Step 5 (*Reduction to the case of u independent of the $x \in \mathbb{R}^d$ variables, and conclusion*). By Step 4 of the proof, the function v defined in (22) is symmetric in the x_1-direction about a suitable hyperplane T_{λ_0} of $\mathbb{R}^{d+k}$. Since the direction x_1 can be chosen arbitrarily in $\mathbb{R}^d \times \{0\}$,[1] we conclude that v must be radially symmetric in the $x \in \mathbb{R}^d$ variables about some point.

[1] Or equivalently exploiting the invariance of problem (17) with respect to rotations in $\mathbb{R}^d \times \{0\}$.

Considering that $1 < p < (Q+2)/(Q-2)$, from (23) it follows that v can only be symmetric in the x variables with respect to the origin, if it is not identically zero. Hence, by its very definition, also the function u must be radially symmetric in the x variables about the origin.

Since the origin of the coordinate system can be chosen arbitrarily in $\mathbb{R}^d \times \{0\}$ when performing the Kelvin transform which defines the function v starting from u,[2] we find that the function u must be radially symmetric in the x variables with respect to any point of $\mathbb{R}^d$, and thus it is constant with respect to those variables.

Hence we have $u(x, y) = u(y)$ for every $z = (x, y) \in \mathbb{R}^{d+k}$. Problem (17) then becomes

$$\begin{cases} \Delta u + u^p = 0 \quad \text{in } \mathbb{R}^k, \\ u > 0 \quad \text{in } \mathbb{R}^k, \quad u \in \mathcal{C}^2(\mathbb{R}^k), \end{cases}$$

with $1 < p < (Q+2)/(Q-2)$.

Since $(Q+2)/(Q-2) < (k+2)/(k-2)$, the nonexistence result for this kind of problem, which has been proved in [14] and later via maximum principles in [6], shows that $u \equiv 0$ if $k \geq 3$.

If $k = 2$, then from (17) we have

$$\begin{cases} \Delta u = -u^p \leq 0 \quad \text{in } \mathbb{R}^2, \\ u > 0 \quad \text{in } \mathbb{R}^2, \quad u \in \mathcal{C}^2(\mathbb{R}^2), \end{cases}$$

and hence u is constant in $\mathbb{R}^2$ by the Liouville theorem. From the equation it then follows that $u \equiv 0$.

Finally, if $k = 1$, from (17) we get

$$\begin{cases} u'' = -u^p \leq 0 \quad \text{in } \mathbb{R}, \\ u > 0 \quad \text{in } \mathbb{R}, \quad u \in \mathcal{C}^2(\mathbb{R}), \end{cases}$$

and thus u is concave and bounded from below on $\mathbb{R}$. Hence it is constant, and once again from the equation it follows that $u \equiv 0$.

Thus, in every case, we find a contradiction with our assumptions on the function u, which we supposed in Step 1 of the proof to be strictly positive on the whole of $\mathbb{R}^{d+k}$.

Hence any solution u of problem (17) must vanish identically on $\mathbb{R}^{d+k}$, provided that Step 4 holds. □

Before proceeding with the proof of Step 4 of Theorem 5.2 we need to state and prove three lemmas. The first states that if $\overline{w}_\lambda$ is negative somewhere in its domain and if λ is negative enough, then the negative minimum of the function $\overline{w}_\lambda$ on $\Omega_\lambda \setminus \{z_\lambda\}$ is finite and is achieved. This lemma also states that, for any fixed $\lambda_0 < 0$, there is an a priori bound, which is uniform with respect to $\lambda \leq \lambda_0$, on the value of the $\mathbb{R}^{d+k}$-norm of the points of negative minimum of $\overline{w}_\lambda$.

[2] Or equivalently exploiting the invariance of problem (17) with respect to translations in the directions of $\mathbb{R}^d \times \{0\}$.

Lemma 5.1. (i) *If λ is negative enough and if* $\inf_{\Omega_\lambda \setminus \{z_\lambda\}} \overline{w}_\lambda < 0$*, then the infimum is achieved.*

(ii) *For every $\lambda_0 < 0$ there exists $R_0 > 0$, depending only on p, λ_0, γ and the function u, such that if z^0 is a minimum point for $\overline{w}_\lambda$ on $\Omega_\lambda \setminus \{z_\lambda\}$ with $\overline{w}_\lambda(z^0) < 0$ and $\lambda \le \lambda_0$, then $|z^0| \le R_0$.*

Proof. (i) The operator G_γ and suitable lower order perturbations satisfy the strong maximum principle, Theorem 2.2, on $\Lambda_m := B_1(0) \setminus \overline{B_{1/m}(0)}$, where we recall that $B_R(P)$ is the open Euclidean ball in $\mathbb{R}^{d+k}$ centered at P and with radius $R > 0$. We also note that for every $m \in \mathbb{N}$ one has

$$\partial \Lambda_m = \partial B_1(0) \cup \partial B_{1/m}(0), \qquad \Lambda_m \subset B_1(0) \setminus \{0\},$$

and that for every $z \in B_1(0) \setminus \{0\}$ there exists $N = N(z) \in \mathbb{N}$ such that $z \in \Lambda_m$ for every $m > N(z)$.

Now let $\varphi(z) := [\rho(z)]^{-(Q-2)/(2\gamma+2)}$. Then $\varphi \in \mathcal{C}^\infty(\overline{\Lambda}_m)$, φ is strictly positive in $\overline{\Lambda}_m$ and $G_\gamma \varphi \equiv 0$ in Λ_m for every $m \in \mathbb{N}$. We recall that, in fact, φ is the fundamental solution of G_γ at the origin (see formula (10)). Let $\varepsilon_0 := \inf_{\partial B_1(0)} v$. Then $\varepsilon_0 > 0$ since v is strictly positive on $\mathbb{R}^{d+k} \setminus \{0\}$. Now, recalling definition (19), for every $m \in \mathbb{N}$ one has in Λ_m

$$G_\gamma\left(\frac{\varepsilon_0 - v}{\varphi}\right) + \frac{2}{\varphi}\left\langle \widetilde{\nabla}\varphi, \widetilde{\nabla}\left(\frac{\varepsilon_0 - v}{\varphi}\right)\right\rangle_{\mathbb{R}^{d+k}} = -\frac{G_\gamma v}{\varphi} \ge 0. \tag{25}$$

By the strong maximum principle, Theorem 2.2, the function $(\varepsilon_0 - v)/\varphi$ cannot attain its nonnegative maximum in Λ_m unless it is constant, in particular

$$\sup_{\Lambda_m}\left(\frac{\varepsilon_0 - v}{\varphi}\right) \le \sup_{\partial\Lambda_m}\left(\frac{\varepsilon_0 - v}{\varphi}\right)^+ = \sup_{\partial B_{1/m}(0)}\left(\frac{\varepsilon_0 - v}{\varphi}\right)^+.$$

If $z \in \partial B_{1/m}(0)$ one has

$$\frac{\varepsilon_0 - v(z)}{\varphi(z)} \le \frac{\varepsilon_0}{\varphi(z)} = \varepsilon_0 [\rho(z)]^{\frac{Q-2}{2\gamma+2}} \le \varepsilon_0 \left(\frac{1}{m}\right)^{\frac{Q-2}{\gamma+1}}.$$

Then we get

$$\sup_{\partial B_{1/m}(0)}\left(\frac{\varepsilon_0 - v}{\varphi}\right)^+ \le \varepsilon_0 \left(\frac{1}{m}\right)^{\frac{Q-2}{\gamma+1}}$$

and for any $\delta > 0$ there exists $M_1 := (\delta/\varepsilon_0)^{-(\gamma+1)/(Q-2)}$ such that $\sup_{\partial B_{1/m}(0)} (\varepsilon_0 - v/\varphi)^+ < \delta$ for every $m > M_1$. Now let $z \in B_1(0) \setminus \{0\}$. Then there exists M_2 such that $z \in \Lambda_m$ for every $m > M_2$ and

$$\frac{\varepsilon_0 - v(z)}{\varphi(z)} \le \sup_{\Lambda_m}\left(\frac{\varepsilon_0 - v}{\varphi}\right) \le \sup_{\partial B_{1/m}(0)}\left(\frac{\varepsilon_0 - v}{\varphi}\right)^+ < \delta$$

for $m \in \mathbb{N}$ large enough, i.e. $m > \max\{M_1, M_2\}$. Hence $v(z) > \varepsilon_0 - \delta\varphi(z)$. Since $\delta > 0$ is arbitrary, we get $v(z) \geq \varepsilon_0$, and thus

$$\inf_{B_1(0)\setminus\{0\}} v \geq \varepsilon_0 > 0.$$

We remark that this kind of result is a maximum principle of Phragmén–Lindelöf type for the operator G_γ (see Theorem 19 of Section 9 in [26]).

Now observe that $v(z)$ tends to 0 as $|z|$ tends to $+\infty$. In fact, since

$$\rho(z) \geq \max\left\{|x|^2, \frac{1}{(\gamma+1)^2}|y|^{2\gamma+2}\right\} \to +\infty \quad \text{as } |z| \to +\infty,$$

one has

$$\left(\frac{x}{\rho(x,y)}, \frac{y}{[\rho(x,y)]^{\frac{1}{1+\gamma}}}\right) \to 0 \quad \text{as } |z| \to +\infty.$$

Hence, by the continuity of u, as $|z| \to +\infty$ we have

$$v(z) = \frac{1}{[\rho(z)]^{\frac{Q-2}{2\gamma+2}}}\, u\left(\frac{x}{\rho(z)}, \frac{y}{[\rho(z)]^{\frac{1}{1+\gamma}}}\right) \sim u(0)[\rho(z)]^{-\frac{Q-2}{2\gamma+2}} \to 0. \tag{26}$$

Then we can find $M > 0$ such that $0 < v(z) \leq \varepsilon_0$ for every z satisfying $|z| > M$.

If $-\lambda > (M+1)/2$ is fixed, we have $|z| > M$ for every $z \in B_1(z_\lambda)$, and hence for any $z \in B_1(z_\lambda) \setminus \{z_\lambda\}$ we get

$$\begin{aligned}\overline{w}_\lambda(z) = \frac{v_\lambda(z) - v(z)}{g(z)} &\geq \frac{\inf_{B_1(z_\lambda)\setminus\{z_\lambda\}} v_\lambda - \sup_{B_1(z_\lambda)\setminus\{z_\lambda\}} v}{g(z)} \\ &= \frac{\inf_{B_1(0)\setminus\{0\}} v - \sup_{B_1(z_\lambda)\setminus\{z_\lambda\}} v}{g(z)} \geq 0\end{aligned}$$

Moreover $\lim_{|z|\to+\infty} \overline{w}_\lambda(z) = 0$ by condition (24). Finally notice that, by definition, we have $\overline{w}_\lambda \equiv 0$ on $\partial\Omega_\lambda$ for every fixed $\lambda < 0$.

Thus, if $\inf_{\Omega_\lambda\setminus\{z_\lambda\}} \overline{w}_\lambda < 0$ and $\lambda < -(M+1)/2$, the infimum must be achieved at some point of $\Omega_\lambda \setminus B_1(z_\lambda)$ by the continuity of the function $\overline{w}_\lambda$.

(ii) Exploiting equation (23), it is easy to see that for any $\lambda < 0$ we have

$$G_\gamma v_\lambda(z) + \frac{[v_\lambda(z)]^p}{[\rho(z)]^{\frac{Q+2}{2\gamma+2} - p\frac{Q-2}{2\gamma+2}}} \leq 0 \quad \text{for every } z \in \Omega_\lambda \setminus \{z_\lambda\},$$

since $1 < p < (Q+2)/(Q-2)$ and $\rho(z^\lambda) = \rho(z) - 4\lambda x_1 + 4\lambda^2 \leq \rho(z)$ for any $z \in \overline{\Omega}_\lambda$. Then, by the mean value theorem, it follows that

$$G_\gamma w_\lambda(z) + c(z) w_\lambda(z) \leq 0 \quad \text{for every } z \in \Omega_\lambda \setminus \{z_\lambda\}, \tag{27}$$

where $c(z) = \frac{p}{[\rho(z)]^{(Q+2)/(2\gamma+2) - p(Q-2)/(2\gamma+2)}} \psi(z)^{p-1}$ and $\psi(z)$ is a real number between $v(z)$ and $v(z^\lambda)$. Thus $c(z)$ is positive for every $z \in \Omega_\lambda \setminus \{z_\lambda\}$.

By an easy calculation, one sees that

$$G_\gamma \overline{w}_\lambda + \frac{2}{g}\langle \widetilde{\nabla} g, \widetilde{\nabla} \overline{w}_\lambda \rangle_{\mathbb{R}^{d+k}} + \overline{w}_\lambda \frac{G_\gamma g}{g} - \frac{G_\gamma w_\lambda}{g} = 0 \quad \text{in } \Omega_\lambda \setminus \{z_\lambda\},$$

and hence, from (27), one has

$$G_\gamma \overline{w}_\lambda + \frac{2}{g}\langle \widetilde{\nabla} g, \widetilde{\nabla} \overline{w}_\lambda \rangle_{\mathbb{R}^{d+k}} + \overline{w}_\lambda \left(\frac{G_\gamma g}{g} + c \right) \leq 0 \quad \text{in } \Omega_\lambda \setminus \{z_\lambda\}. \tag{28}$$

Now notice that $u \in \mathcal{C}^2(\mathbb{R}^{d+k})$, u is positive and thus $u(x/\rho(z), y/[\rho(z)]^{1/(\gamma+1)}) \to u(0) > 0$ as $|z|$ tends to $+\infty$. Then for every $M > 0$ we can find positive constants $c_1 = c_1(M)$ and $c_2 = c_2(M)$ such that

$$0 < c_1 \leq u\left(\frac{x}{\rho(z)}, \frac{y}{[\rho(z)]^{\frac{1}{\gamma+1}}} \right) \leq c_2 \quad \text{for every } |z| \geq M. \tag{29}$$

Note that c_1 is nondecreasing and c_2 is nonincreasing in M, and it may happen that $c_1 \to 0$, $c_2 \to +\infty$ as M tends to 0^+, depending on the function u.

In particular, for every $M > 0$ there exist $c_1, c_2 > 0$ such that

$$0 < \frac{c_1}{[\rho(z)]^{\frac{Q-2}{2\gamma+2}}} \leq v(z) \leq \frac{c_2}{[\rho(z)]^{\frac{Q-2}{2\gamma+2}}} \quad \text{for every } |z| \geq M.$$

Now if $\lambda \leq \lambda_0 < 0$ and if z^0 is a negative minimum point for $\overline{w}_\lambda$ in $\Omega_\lambda \setminus \{z_\lambda\}$, that is, if $\overline{w}_\lambda(z^0) = \inf_{\Omega_\lambda \setminus \{z_\lambda\}} \overline{w}_\lambda < 0$, then it is easy to see that

- $|z^0| \geq -\lambda \geq -\lambda_0$, since $z^0 \in \Omega_\lambda \subset \Omega_{\lambda_0}$,
- $\widetilde{\nabla} \overline{w}_\lambda(z^0) = 0$, since $\overline{w}_\lambda(z^0) < 0$ and hence z^0 must be an interior point of $\Omega_\lambda \setminus \{z_\lambda\}$,
- $G_\gamma \overline{w}_\lambda(z_0) \geq 0$, since z^0 is a minimum point for $\overline{w}_\lambda$ lying in the interior of $\Omega_\lambda \setminus \{z_\lambda\}$,
- $0 < v((z^0)^\lambda) < v(z^0)$, since $\overline{w}_\lambda(z^0) < 0$.

Thus

$$0 < v((z^0)^\lambda) \leq \psi(z^0) \leq v(z^0) \leq \frac{\alpha}{[\rho(z^0)]^{\frac{Q-2}{2\gamma+2}}},$$

where $\alpha > 0$ is a suitable constant depending only on the function u and on λ_0, using the idea of (29). Hence

$$0 < c(z^0) = \frac{p}{[\rho(z^0)]^{\frac{Q+2}{2\gamma+2} - p\frac{Q-2}{2\gamma+2}}} (\psi(z^0))^{p-1} \leq \frac{p\alpha^{p-1}}{[\rho(z^0)]^{\frac{4}{2\gamma+2}}}. \tag{30}$$

From (28), it now follows that

$$\begin{aligned} 0 &\geq G_\gamma \overline{w}_\lambda(z^0) + \frac{2}{g(z^0)}\langle \widetilde{\nabla} g(z^0), \widetilde{\nabla} \overline{w}_\lambda(z^0) \rangle_{\mathbb{R}^{d+k}} + \overline{w}_\lambda(z^0) \left(\frac{G_\gamma g(z^0)}{g(z^0)} + c(z^0) \right) \\ &\geq \overline{w}_\lambda(z^0) \left(\frac{G_\gamma g(z^0)}{g(z^0)} + c(z^0) \right), \end{aligned}$$

and this last term is strictly positive if $|z^0|$ is large enough, by the hypotheses we made on the function g. In fact we have established that $\overline{w}_\lambda(z^0) < 0$, and by (30) and the assumptions on g it follows that

$$\frac{G_\gamma g(z^0)}{g(z^0)} + c(z^0) \le \frac{G_\gamma g(z^0)}{g(z^0)} + \frac{p\alpha^{p-1}}{[\rho(z^0)]^{\frac{4}{2\gamma+2}}} < 0$$

if $|z^0| > R_0$, where $R_0 > 0$ is a suitable constant depending only on α, p and γ, and hence depending only on the function u, on λ_0, on p and on γ, which is provided by condition (24).

Thus it follows that $|z^0| \le R_0$, with $R_0 = R_0(\lambda_0, p, u, \gamma) > 0$. □

The second lemma, which we are about to prove, states that for suitable values of the constants γ, d and k there exists a function g satisfying condition (24).

Lemma 5.2. (i) *If* $0 < \gamma < 1$ *and* $d, k \in \mathbb{N}$, *then there exists a function* g *which satisfies condition* (24).

(ii) *If* $\gamma > 0$, $d \in \mathbb{N}$ *and* $k \in \mathbb{N} \setminus \{1, 2\}$, *then there exists a function* g *satisfying condition* (24).

Proof. (i) For every $y \in \mathbb{R}^k$, $\widetilde{x} \in \mathbb{R}^{d-1}$ and $x_1 \le 0$, define $x = (x_1, \widetilde{x})$ and

$$g(z) = g(x, y) := \frac{1 - x_1}{\left((1 - x_1)^2 + |\widetilde{x}|^2 + \frac{1}{(\gamma+1)^2}(|y|^2 + 1)^{\gamma+1}\right)^\beta},$$

with $0 < \beta < \gamma/(2(\gamma + 1))$ to be chosen later. Then g is strictly positive on $\overline{\Omega}_0 = \overline{\mathbb{R}^-} \times \mathbb{R}^{d-1} \times \mathbb{R}^k$ and $g \in C^\infty(\overline{\Omega}_0)$. Moreover for every $z = (x_1, \widetilde{x}, y) \in \overline{\Omega}_0$ we have

$$D_{x_1} g(z) = \frac{(2\beta - 1)(1 - x_1)^2 - |\widetilde{x}|^2 - \frac{1}{(\gamma+1)^2}(|y|^2 + 1)^{\gamma+1}}{\left((1 - x_1)^2 + |\widetilde{x}|^2 + \frac{1}{(\gamma+1)^2}(|y|^2 + 1)^{\gamma+1}\right)^{\beta+1}} < 0,$$

since by our assumptions $\beta < 1/2$. Hence g is decreasing in the x_1-direction on $\overline{\Omega}_0$.

Claim 1. *For every* $\lambda < 0$ *we have*

$$\overline{w}_\lambda(z) = \frac{v(z^\lambda) - v(z)}{g(z)} \to 0 \quad \text{as } |z| \to +\infty \text{ in } \overline{\Omega}_\lambda.$$

We begin by noticing that, for $|z|$ large enough, we have

$$\begin{aligned}(1 - x_1)^2 + |\widetilde{x}|^2 &+ \frac{1}{(\gamma + 1)^2}(|y|^2 + 1)^{\gamma+1}\\ &\le 2^{\gamma+1}\left((1 - x_1)^2 + |\widetilde{x}|^2 + \frac{1}{(\gamma + 1)^2}|y|^{2\gamma+2}\right)\\ &\le 2^{\gamma+2}\left(|x|^2 + \frac{1}{(\gamma + 1)^2}|y|^{2\gamma+2}\right) = 2^{\gamma+2}\rho(z). \qquad (31)\end{aligned}$$

Now recall that by (26) we have

$$v(z) \sim u(0)[\rho(z)]^{-\frac{Q-2}{2\gamma+2}} \quad \text{as } |z| \to +\infty.$$

Hence, exploiting (31), we have, as $|z|$ tends to $+\infty$ in $\overline{\Omega}_0$,

$$\begin{aligned} 0 < \frac{v(z)}{g(z)} &\sim \frac{u(0)[\rho(z)]^{-\frac{Q-2}{2\gamma+2}}}{g(z)} \\ &= \frac{u(0)}{1-x_1}[\rho(z)]^{-\frac{Q-2}{2\gamma+2}}\left((1-x_1)^2 + |\widetilde{x}|^2 + \frac{1}{(\gamma+1)^2}(|y|^2+1)^{\gamma+1}\right)^{\beta} \\ &\le 2^{\beta(\gamma+2)}\frac{u(0)}{1-x_1}[\rho(z)]^{-\frac{Q-2}{2\gamma+2}+\beta} \\ &\le 2^{\beta(\gamma+2)}u(0)[\rho(z)]^{-\frac{Q-2}{2\gamma+2}+\beta} \to 0 \quad \text{as } |z| \to +\infty, \end{aligned}$$

since $x_1 \le 0$ in $\overline{\Omega}_0$, since $\rho(z)$ tends to $+\infty$ as $|z|$ tends to $+\infty$, and since, by our bounds on the choice of β, we have

$$-\frac{Q-2}{2\gamma+2} + \beta \le \frac{-\gamma}{2\gamma+2} + \beta < 0.$$

Now notice that, for any fixed $\lambda < 0$, we have $|z^\lambda|^2 = |z|^2 - 4\lambda x_1 + 4\lambda^2$ and that $\rho(z^\lambda) = \rho(z) - 4\lambda x_1 + 4\lambda^2$, hence $|z^\lambda| \sim |z|$ and $\rho(z^\lambda) \sim \rho(z)$ as $|z| \to +\infty$. Then

$$0 < \frac{v(z^\lambda)}{g(z)} \sim \frac{u(0)[\rho(z^\lambda)]^{-\frac{Q-2}{2\gamma+2}}}{g(z)} \sim \frac{u(0)[\rho(z)]^{-\frac{Q-2}{2\gamma+2}}}{g(z)} \to 0$$

as $|z|$ tends to $+\infty$ in $\overline{\Omega}_\lambda$. Hence

$$\overline{w}_\lambda(z) = \frac{v(z^\lambda)}{g(z)} - \frac{v(z)}{g(z)} \to 0 \quad \text{as } |z| \to +\infty \text{ in } \overline{\Omega}_\lambda,$$

as stated in Claim 1.

Claim 2. *For every $C > 0$ there exists $R > 0$, which depends only on γ and C, such that if $z \in \overline{\Omega}_0$ and $|z| > R$ then*

$$\frac{G_\gamma g(z)}{g(z)} + \frac{C}{[\rho(z)]^{\frac{4}{2\gamma+2}}} < 0.$$

To simplify the notation define $\widetilde{\rho}(z) := ((1-x_1)^2 + |\widetilde{x}|^2 + \frac{1}{(\gamma+1)^2}(|y|^2+1)^{\gamma+1})$ for every $z = (x_1, \widetilde{x}, y) \in \overline{\Omega}_0$. Then $\widetilde{\rho}(z) > 0$ and

$$\begin{aligned} \frac{G_\gamma g(z)}{g(z)} &= \frac{4\beta(\beta+1)}{(\gamma+1)^2}(|y|^2+1)^{2\gamma}|y|^{2\gamma}[\widetilde{\rho}(z)]^{-2}[(|y|^2)^{1-\gamma} - (|y|^2+1)^{1-\gamma}] \\ &- \frac{2\beta}{\gamma+1}(|y|^2+1)^{\gamma}[\widetilde{\rho}(z)]^{-1}\left[(d-2\beta)(\gamma+1)\left(\frac{|y|^2}{|y|^2+1}\right)^{\gamma} + k + 2\gamma\frac{|y|^2}{|y|^2+1}\right]. \end{aligned} \tag{32}$$

Now notice that for every $y \in \mathbb{R}^k$ we have

$$(|y|^2)^{1-\gamma} - (|y|^2+1)^{1-\gamma} < 0 \tag{33}$$

since $0 < \gamma < 1$. On the other hand

$$(d-2\beta)(\gamma+1)\left(\frac{|y|^2}{|y|^2+1}\right)^{\gamma} + k + 2\gamma\frac{|y|^2}{|y|^2+1} \geq 1 \tag{34}$$

since $d, k \geq 1$, $\beta < 1/2$ and $\gamma > 0$.

Exploiting inequalities (33) and (34), from (32) it follows that

$$\frac{G_\gamma g(z)}{g(z)} \leq -\frac{2\beta}{\gamma+1}[\widetilde{\rho}(z)]^{-1}. \tag{35}$$

Thus if $0 < \gamma < 1$ and $C > 0$, by (35), we have

$$\frac{G_\gamma g(z)}{g(z)} + \frac{C}{[\rho(z)]^{\frac{4}{2\gamma+2}}} \leq -\frac{2\beta}{1+\gamma}\,\frac{1}{[\widetilde{\rho}(z)]} + \frac{C}{[\rho(z)]^{\frac{4}{2\gamma+2}}},$$

and this last term is strictly negative if $|z|$ is large enough, i.e. if $|z| > R$ for a suitable $R > 0$ depending only on C and on γ. Indeed, $\widetilde{\rho}(z) \sim \rho(z) \to +\infty$ as $|z|$ tends to $+\infty$, and $4/(2\gamma+2) > 1$ by our assumptions on γ. Hence we get Claim 2.

(ii) In this case define

$$g(z) = g(x, y) := (|y|^2+1)^{-\alpha}$$

for any fixed α with $0 < \alpha < (k-2)/2$. Then g is strictly positive in $\mathbb{R}^{d+k}$, it is bounded and $g \in \mathcal{C}^\infty(\mathbb{R}^{d+k})$. Notice also that $D_{x_1} g \equiv 0$ in $\mathbb{R}^{d+k}$.

Claim 3. *For every $\lambda < 0$ we have*

$$\overline{w}_\lambda(z) = \frac{v(z^\lambda) - v(z)}{g(z)} \to 0 \quad \textit{as } |z| \to +\infty \textit{ in } \overline{\Omega}_\lambda.$$

Indeed, as $|z| \to +\infty$ we have $v(z) \sim u(0)[\rho(z)]^{-(Q-2)/(2\gamma+2)}$, as was shown in (26), and hence

$$\begin{aligned} 0 < \frac{v(z)}{g(z)} &\sim \frac{u(0)(|y|^2+1)^\alpha}{\left(|x|^2 + \frac{1}{(\gamma+1)^2}|y|^{2\gamma+2}\right)^{\frac{Q-2}{2\gamma+2}}} \\ &\leq C\min\left\{\frac{(|y|^2+1)^\alpha}{|y|^{Q-2}}, \frac{(|y|^2+1)^\alpha}{|x|^{\frac{Q-2}{\gamma+1}}}\right\} \to 0 \quad \text{as } |z| \to +\infty, \end{aligned}$$

where $C > 0$ is a suitable constant, since $2\alpha - Q + 2 < 2\alpha - k + 2 < 0$ by our choice of α.

Now recall that as $|z| \to +\infty$ one has $|z^\lambda| \sim |z|$ and also $\rho(z^\lambda) \sim \rho(z)$ for any fixed $\lambda < 0$. Hence $|z^\lambda| \to +\infty$ as $|z| \to +\infty$, and by (26) we get

$$0 < \frac{v(z^\lambda)}{g(z)} \sim \frac{u(0)[\rho(z^\lambda)]^{-\frac{Q-2}{2\gamma+2}}}{g(z)} \sim \frac{u(0)[\rho(z)]^{-\frac{Q-2}{2\gamma+2}}}{g(z)}$$
$$= \frac{u(0)(|y|^2+1)^\alpha}{\left(|x|^2 + \frac{1}{(\gamma+1)^2}|y|^{2\gamma+2}\right)^{\frac{Q-2}{2\gamma+2}}} \to 0 \quad \text{as } |z| \to +\infty \text{ in } \overline{\Omega}_\lambda,$$

as was shown before. But then

$$\overline{w}_\lambda(z) = \frac{v(z^\lambda)}{g(z)} - \frac{v(z)}{g(z)} \to 0 \quad \text{as } |z| \to +\infty \text{ in } \overline{\Omega}_\lambda,$$

and we get Claim 3.

Claim 4. *For every $C > 0$ there exists $R > 0$, depending only on γ, k and C, such that*

$$\frac{G_\gamma g(z)}{g(z)} + \frac{C}{[\rho(z)]^{\frac{4}{2\gamma+2}}} < 0 \tag{36}$$

whenever $z \in \overline{\Omega}_0$ and $|z| > R$.

Since $0 < \alpha < (k-2)/2$, we have

$$\frac{G_\gamma g(z)}{g(z)} = \frac{-2\alpha}{(|y|^2+1)^2}[k + (k-2\alpha-2)|y|^2] \le \frac{-2\alpha(k-2-2\alpha)}{|y|^2+1} < 0.$$

Then, recalling also that $k \ge 3$, for any $C > 0$ we have

$$\frac{G_\gamma g(z)}{g(z)} + \frac{C}{[\rho(z)]^{\frac{4}{2\gamma+2}}} \le \frac{-2\alpha(k-2-2\alpha)}{|y|^2+1} + \frac{C}{[\rho(z)]^{\frac{4}{2\gamma+2}}}$$
$$\le \frac{-2\alpha(k-2-2\alpha)}{|y|^2+1} + \frac{C(\gamma+1)^{\frac{4}{\gamma+1}}}{|y|^4} \tag{37}$$

and thus we can find $R_1 > 0$, depending only on γ, k and C, such that the right hand side of inequality (37) is strictly negative if $|y| > R_1$.

On the other hand, if $|y| \le R_1$, we have

$$\frac{G_\gamma g(z)}{g(z)} + \frac{C}{[\rho(z)]^{\frac{4}{2\gamma+2}}} \le \frac{-2\alpha(k-2-2\alpha)}{|y|^2+1} + \frac{C}{[\rho(z)]^{\frac{4}{2\gamma+2}}}$$
$$\le \frac{-2\alpha(k-2-2\alpha)}{R_1^2+1} + \frac{C}{|x|^{\frac{4}{\gamma+1}}} \tag{38}$$

and so we can find $R_2 > 0$, depending only on γ, k and C, such that the right hand side of (38) is strictly negative if $|x| > R_2$.

Hence property (36) holds if $|z| > R := \sqrt{R_1^2 + R_2^2}$, since in this case $|y| > R_1$ or $|y| \le R_1$ and $|x| > R_2$, and thus we get Claim 4. □

Lemma 5.3. *Let $\gamma > 0$ and $\lambda < 0$. If $w_\lambda > 0$ in $\Omega_\lambda \setminus \{z_\lambda\}$, then $D_{x_1} v(z) > 0$ for every $z \in T_\lambda$.*

Proof. Let $\lambda < 0$ and $\check{z} = (\check{x}, \check{y}) = (\lambda, \check{x}_2, \ldots, \check{x}_d, \check{y}_1, \ldots, \check{y}_k) \in T_\lambda = \partial\Omega_\lambda$. Then define $\hat{z} := (\hat{x}, \check{y}) = (\frac{9}{8}\lambda, \check{x}_2, \ldots, \check{x}_d, \check{y}_1, \ldots, \check{y}_k)$ and $r := |\lambda|/8$.

Now, if $\check{y} \neq 0$, we consider the Euclidean ball $B_r(\hat{z}) \subset \mathbb{R}^{d+k}$ centered at $\hat{z}$ and with radius r. Then

- $B_r(\hat{z}) \subset \Omega_\lambda \setminus \{z_\lambda\}$ and $\check{z} \in \partial B_r(\hat{z})$,
- by (27) we have $G_\gamma(-w_\lambda) \geq c(z) w_\lambda \geq 0$ in $B_r(\hat{z})$, since both $c(z)$ and w_λ are positive,
- $w_\lambda(z) > 0$ for every $z \in B_r(\hat{z})$ and $w_\lambda(\check{z}) = 0$, since $\check{z} \in T_\lambda$,
- $\langle(\check{z} - \hat{z}), A(\check{z})(\check{z} - \hat{z})\rangle_{\mathbb{R}^{d+k}} = \frac{\lambda^2}{64}|\check{y}|^{2\gamma} > 0$, where $A(z)$ is the matrix of the coefficients of the second order derivatives in the operator G_γ (see Section 4).

Hence we can apply the Hopf Lemma 2.2 to the function $-w_\lambda$ in $B_r(\hat{z})$ with respect to the point $\check{z}$, and we conclude that $-D_{x_1} w_\lambda(\check{z}) > 0$. Thus

$$D_{x_1} v(\check{z}) = -\frac{1}{2} D_{x_1} w_\lambda(\check{z}) > 0.$$

If $\check{y} = 0$, then $\check{z}, \hat{z} \in \mathbb{R}^d \times \{0\}$, which is the degeneracy set of the operator G_γ. Thus, instead of a Euclidean ball, we consider the set $\widetilde{B}_{\hat{r}}(\hat{z})$ defined in Section 4, with $\hat{r} = (|\lambda|/8)^{1/(1+\gamma)}$. Then again we find

- $\widetilde{B}_{\hat{r}}(\hat{z}) \subset \Omega_\lambda \setminus \{z_\lambda\}$ and $\check{z} \in \partial \widetilde{B}_{\hat{r}}(\hat{z})$,
- $G_\gamma(-w_\lambda) \geq c(z) w_\lambda \geq 0$ in $\widetilde{B}_{\hat{r}}(\hat{z})$ by (27),
- $w_\lambda(z) > 0$ for every $z \in \widetilde{B}_{\hat{r}}(\hat{z})$ and $w_\lambda(\check{z}) = 0$, since $\check{z} \in T_\lambda$.

Thus, by the Hopf Lemma for the Grushin operator, Lemma 4.1, also in this case we have

$$D_{x_1} v(\check{z}) = -\frac{1}{2} D_{x_1} w_\lambda(\check{z}) > 0. \qquad \square$$

We are now ready to prove Step 4 of the proof of Theorem 5.2.

Proof of Step 4. We want to prove that v is symmetric with respect to a suitable hyperplane T_{λ_0} in $\mathbb{R}^{d+k}$. This will be achieved by showing that $\overline{w}_{\lambda_0} \equiv 0$ on $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$ for a suitable λ_0.

We begin by noticing that Lemmas 5.1 and 5.2 together imply that for λ negative enough we have $\overline{w}_\lambda \geq 0$ in $\Omega_\lambda \setminus \{z_\lambda\}$.

Indeed, we first note that by Lemma 5.1(i) we can find $\lambda_1 < 0$ with the property that, if $\lambda < \lambda_1$ and $\inf_{\Omega_\lambda \setminus \{z_\lambda\}} \overline{w}_\lambda < 0$, then the infimum is achieved.

From Lemma 5.1(ii) it follows that there exists $R_1 > 0$, depending only on γ and λ_1, with the property that, if $\overline{w}_\lambda(\check{z}) = \inf_{\Omega_\lambda \setminus \{z_\lambda\}} \overline{w}_\lambda < 0$ and if $\lambda < \lambda_1$, then $|\check{z}| \leq R_1$.

Now if $\lambda \leq \min\{\lambda_1 - 1, -R_1 - 1\}$ and if $\inf_{\Omega_\lambda \setminus \{z_\lambda\}} \overline{w}_\lambda \geq 0$ does not hold, we get a contradiction. In fact, we could then find $\check{z} \in \Omega_\lambda \setminus \{z_\lambda\}$ such that $\overline{w}_\lambda(\check{z}) = \inf_{\Omega_\lambda \setminus \{z_\lambda\}} \overline{w}_\lambda < 0$. Since $\check{z} \in \Omega_\lambda \setminus \{z_\lambda\}$ we must have $|\check{z}| \geq |\lambda| > R_1$, and this contradicts the assumptions on R_1.

Now define

$$\lambda_0 := \sup\{\lambda \le 0 : \overline{w}_\mu \ge 0 \text{ in } \Omega_\mu \setminus \{z_\mu\} \text{ for every } \mu \le \lambda\}.$$

By the continuity of v we have $\overline{w}_{\lambda_0} \ge 0$ in $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$, and we remark that $\overline{w}_\lambda(z)$ and $w_\lambda(z)$ have the same sign for any $z \in \Omega_\lambda \setminus \{z_\lambda\}$ and for any $\lambda \le 0$.

Claim 1. *If* $\lambda_0 < 0$ *then* $\overline{w}_{\lambda_0} \equiv 0$ *in* $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$.

In fact suppose by contradiction that $\overline{w}_{\lambda_0} \not\equiv 0$ in $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$. Then $w_{\lambda_0} \ge 0$ and $w_{\lambda_0} \not\equiv 0$ in $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$, hence applying the strong maximum principle, Theorem 2.2, to (27) shows that both w_{λ_0} and $\overline{w}_{\lambda_0}$ are strictly positive in $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$. Now let $\{\lambda_h\}_{h\in\mathbb{N}}$ be a decreasing sequence with $0 > \lambda_h \searrow \lambda_0$ as h tends to ∞ and such that for every $h \in \mathbb{N}$ we have $\overline{w}_{\lambda_h}(z) < 0$ for some $z \in \Omega_{\lambda_h} \setminus \{z_{\lambda_h}\}$. Such a sequence exists by the very definition of λ_0 and since we assumed $\lambda_0 < 0$.

It follows that $\inf_{\Omega_{\lambda_h}\setminus\{z_{\lambda_h}\}} \overline{w}_{\lambda_h} < 0$ for every $h \in \mathbb{N}$.

We want to prove that for each $h \in \mathbb{N}$ large enough the infimum is achieved at some point $z_h \in \Omega_{\lambda_h} \setminus \{z_{\lambda_h}\}$, the sequence $\{z_h\}_{h\in\mathbb{N}}$ is bounded, and z_h stays uniformly away from z_{λ_h} for every $h \in \mathbb{N}$ large enough. To this end we will show that

$$\exists\, \varepsilon > 0,\ \delta > 0 \text{ such that } \overline{w}_{\lambda_0}(z) \ge \varepsilon \text{ for every } z \in B_\delta(z_{\lambda_0}) \setminus \{z_{\lambda_0}\}, \tag{39}$$

$$\inf_{B_\delta(z_\lambda)\setminus\{z_\lambda\}} \overline{w}_\lambda \ge \inf_{B_\delta(z_{\lambda_0})\setminus\{z_{\lambda_0}\}} \overline{w}_{\lambda_0} - \varepsilon/2 \ge \varepsilon/2 \quad \text{if } 0 > \lambda \ge \lambda_0 \text{ and } \lambda \text{ is close enough to } \lambda_0. \tag{40}$$

Proof of (39). Since $\lambda_0 < \lambda_h \le \lambda_1$ for every $h \in \mathbb{N}$, we have $B_{|\lambda_1|/2}(z_{\lambda_h}) \subset \Omega_{\lambda_h}$ for every $h \in \mathbb{N} \cup \{0\}$. Then we choose

$$\delta := \min\{1/2, |\lambda_1|/2\}, \qquad \varepsilon := \frac{\inf_{\partial B_\delta(z_{\lambda_0})} w_{\lambda_0}}{\sup_{\overline{B}_\delta(z_{\lambda_0})} g}. \tag{41}$$

Since by our assumptions $w_{\lambda_0} \ge 0$, by (27) we have $G_\gamma w_{\lambda_0} \le 0$ in $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$. Then, by a maximum principle of Phragmén–Lindelöf type as we showed in Lemma 5.1, we can prove that $w_{\lambda_0}(z) \ge \inf_{\partial B_\delta(z_{\lambda_0})} w_{\lambda_0} > 0$ for every $z \in B_\delta(z_{\lambda_0}) \setminus \{z_{\lambda_0}\}$. Hence for every $z \in B_\delta(z_{\lambda_0}) \setminus \{z_{\lambda_0}\}$ we will have

$$\overline{w}_{\lambda_0}(z) = \frac{w_{\lambda_0}(z)}{g(z)} \ge \frac{\inf_{\partial B_\delta(z_{\lambda_0})} w_{\lambda_0}}{\sup_{\overline{B}_\delta(z_{\lambda_0})} g} = \varepsilon > 0, \tag{42}$$

exploiting the continuity and positivity of g.

Indeed, let $\Lambda_m := B_\delta(z_{\lambda_0}) \setminus B_{\delta/m}(z_{\lambda_0})$ and let $\varphi(z) := [\rho(z^{\lambda_0})]^{-(Q-2)/(2+2\gamma)}$, which is the fundamental solution of G_γ centered at z_{λ_0}. Then $\varphi \in \mathcal{C}^\infty(\overline{\Lambda}_m)$, φ is strictly positive on $\overline{\Lambda}_m$ and $G_\gamma\varphi \equiv 0$ in Λ_m. Moreover $\Lambda_m \subset B_\delta(z_{\lambda_0}) \setminus \{z_{\lambda_0}\}$ for every $m \in \mathbb{N}$, and for each $z \in B_\delta(z_{\lambda_0}) \setminus \{z_{\lambda_0}\}$ there exists $N = N(z) \in \mathbb{N}$ such that $z \in \Lambda_m$ for every $m \ge N$.

If we set $\eta := \inf_{\partial B_\delta(z_{\lambda_0})} w_{\lambda_0}$, then $\eta > 0$ and for every $m \in \mathbb{N}$ we have, as in (25),

$$G_\gamma\left(\frac{\eta - w_{\lambda_0}}{\varphi}\right) + \frac{2}{\varphi}\left\langle \widetilde{\nabla}\varphi, \widetilde{\nabla}\left(\frac{\eta - w_{\lambda_0}}{\varphi}\right)\right\rangle_{\mathbb{R}^{d+k}} = -\frac{G_\gamma w_{\lambda_0}}{\varphi} \ge 0$$

in Λ_m. Thus by the strong maximum principle, Theorem 2.2, we have

$$\sup_{\Lambda_m} \frac{\eta - w_{\lambda_0}}{\varphi} \leq \sup_{\partial\Lambda_m} \left(\frac{\eta - w_{\lambda_0}}{\varphi} \right)^+ = \sup_{\partial B_{\delta/m}(z_{\lambda_0})} \left(\frac{\eta - w_{\lambda_0}}{\varphi} \right)^+ .$$

Since v is strictly positive and continuous in its domain and since $\overline{B_\delta(z_{\lambda_0})} \subset \mathbb{R}^{d+k} \setminus \{0\}$, for every $z \in \partial B_{\delta/m}(z_{\lambda_0})$ we have $z \in \overline{B_\delta(z_{\lambda_0})}$, and thus

$$\begin{aligned} \frac{\eta - w_{\lambda_0}(z)}{\varphi(z)} &= [\eta - (v(z^{\lambda_0}) - v(z))][\rho(z^{\lambda_0})]^{\frac{Q-2}{2\gamma+2}} \\ &\leq (\eta + v(z))[\rho(z^{\lambda_0})]^{\frac{Q-2}{2\gamma+2}} \leq (\eta + C)[\rho(z^{\lambda_0})]^{\frac{Q-2}{2\gamma+2}} \\ &\leq (\eta + C)|z - z_{\lambda_0}|^{\frac{Q-2}{\gamma+1}} \leq (\eta + C)\left(\frac{\delta}{m}\right)^{\frac{Q-2}{\gamma+1}}, \end{aligned}$$

where $C = \max_{\overline{B}_\delta(z_{\lambda_0})} v > 0$ and $\delta < 1$ is defined by (41). Then

$$\sup_{\partial B_{\delta/m}(z_{\lambda_0})} \left(\frac{\eta - w_{\lambda_0}}{\varphi} \right)^+ \leq (\eta + C)\left(\frac{\delta}{m}\right)^{\frac{Q-2}{\gamma+1}}$$

and for any $\mu > 0$ we can find

$$M_1 := \delta\left(\frac{\mu}{\eta + C}\right)^{-\frac{\gamma+1}{Q-2}} \quad \text{such that} \quad \sup_{\partial B_{\delta/m}(z_{\lambda_0})} \left(\frac{\eta - w_{\lambda_0}}{\varphi} \right)^+ < \mu \quad \text{for } m > M_1.$$

Now if $z \in B_\delta(z_{\lambda_0}) \setminus \{z_{\lambda_0}\}$ we can pick an $m > M_1$ large enough so that $z \in \Lambda_m$, and hence

$$\frac{\eta - w_{\lambda_0}(z)}{\varphi(z)} \leq \sup_{\Lambda_m} \frac{\eta - w_{\lambda_0}}{\varphi} \leq \sup_{\partial B_{\delta/m}(z_{\lambda_0})} \left(\frac{\eta - w_{\lambda_0}}{\varphi} \right)^+ < \mu.$$

It follows that $w_{\lambda_0}(z) > \eta - \mu\varphi(z)$, and since $\mu > 0$ is arbitrary we find finally $w_{\lambda_0}(z) \geq \eta = \inf_{\partial B_\delta(z_{\lambda_0})} w_{\lambda_0} > 0$ for every $z \in B_\delta(z_{\lambda_0}) \setminus \{z_{\lambda_0}\}$. Thus (39) holds.

Proof of (40). We exploit the uniform continuity of g and v on compact sets contained in their domains. Let $z \in B_\delta(z_\lambda) \setminus \{z_\lambda\}$ with $\lambda < 0$. Then we can write $z = z_\lambda + \hat{z}$ with $\hat{z} \in B_\delta(0) \setminus \{0\}$ and

$$\overline{w}_\lambda(z) = \overline{w}_\lambda(z_\lambda + \hat{z}) = \frac{v(\hat{z}^0) - v(z_\lambda + \hat{z})}{g(z_\lambda + \hat{z})}, \tag{43}$$

where the point $\hat{z}^0 = (-\hat{x}_1, \ldots, \hat{x}_d, \hat{y}_1, \ldots, \hat{y}_k)$ is symmetric to the point $\hat{z} = (\hat{x}_1, \ldots, \hat{x}_d, \hat{y}_1, \ldots, \hat{y}_k)$ with respect to the hyperplane T_0.

If $\lambda_0 \leq \lambda < \frac{4}{5}\lambda_0$, by our definition of δ we have, for every $\hat{z} \in B_\delta(0) \setminus \{0\}$,

$$2\lambda_0 - \delta \leq 2\lambda_0 + \hat{x}_1 \leq 2\lambda + \hat{x}_1 \leq 2\lambda + \delta \leq \frac{11}{10}\lambda_0 < \lambda_0 < 0 \tag{44}$$

and $-\delta \le \hat{x}_i \le \delta$, $-\delta \le \hat{y}_j \le \delta$ for every $i = 2, \dots, d$ and every $j = 1, \dots, k$. Hence for each $\hat{z} \in \overline{B_\delta(0)} \setminus \{0\}$ and $\lambda \in [\lambda_0, \frac{4}{5}\lambda_0)$ we have $z_\lambda + \hat{z}, z_{\lambda_0} + \hat{z} \in K$, with $K := [2\lambda_0 - \delta, \lambda_0] \times [-\delta, \delta]^{d-1} \times [-\delta, \delta]^k \subset \Omega_0$ compact.

Since v and g are continuous on K and since g is strictly positive, the function $-v/g$ is uniformly continuous on that set, thus we can find $\beta > 0$ such that if $a, b \in K$ and $|a - b| < \beta$ then

$$\left| \frac{-v(a)}{g(a)} - \frac{-v(b)}{g(b)} \right| < \frac{\varepsilon}{2}$$

with the same $\varepsilon > 0$ defined in (41). Hence if

$$\lambda_0 \le \lambda < \min\left\{\lambda_0 + \frac{\beta}{2}, \frac{4}{5}\lambda_0\right\} := \widetilde{\lambda}_0 \tag{45}$$

and if $\hat{z} \in B_\delta(0) \setminus \{0\}$, we have $z_\lambda + \hat{z}, z_{\lambda_0} + \hat{z} \in K$ and $|z_\lambda + \hat{z} - z_{\lambda_0} - \hat{z}| = 2|\lambda - \lambda_0| < \beta$. Thus

$$\frac{-v(z_\lambda + \hat{z})}{g(z_\lambda + \hat{z})} > -\frac{\varepsilon}{2} + \frac{-v(z_{\lambda_0} + \hat{z})}{g(z_{\lambda_0} + \hat{z})}\,. \tag{46}$$

Now notice that, by condition (24), for any fixed $\hat{z} \in B_\delta(0) \setminus \{0\}$ the function of λ

$$g(z_\lambda + \hat{z}) = g(2\lambda + \hat{x}_1, \hat{x}_2, \dots, \hat{x}_d, \hat{y}_1, \dots, \hat{y}_k)$$

is nonincreasing for $\lambda_0 \le \lambda < \widetilde{\lambda}_0$, since in this case $z_\lambda + \hat{z} \in \Omega_0$ by (44). Hence for $\lambda \in [\lambda_0, \widetilde{\lambda}_0)$ and $\hat{z} \in B_\delta(0) \setminus \{0\}$ we have

$$\frac{1}{g(z_\lambda + \hat{z})} \ge \frac{1}{g(z_{\lambda_0} + \hat{z})}\,. \tag{47}$$

Finally, recalling that v is positive, from (43) exploiting inequalities (46) and (47) we get, for every $\hat{z} \in B_\delta(0) \setminus \{0\}$,

$$\overline{w}_\lambda(z_\lambda + \hat{z}) = \frac{v(\hat{z}^0)}{g(z_\lambda + \hat{z})} - \frac{v(z_\lambda + \hat{z})}{g(z_\lambda + \hat{z})} \ge \frac{v(\hat{z}^0)}{g(z_{\lambda_0} + \hat{z})} - \frac{v(z_{\lambda_0} + \hat{z})}{g(z_{\lambda_0} + \hat{z})} - \frac{\varepsilon}{2} = \overline{w}_{\lambda_0}(z_{\lambda_0} + \hat{z}) - \frac{\varepsilon}{2}$$

and by (39) we have

$$\inf_{B_\delta(z_\lambda)\setminus\{z_\lambda\}} \overline{w}_\lambda \ge \inf_{B_\delta(z_{\lambda_0})\setminus\{z_{\lambda_0}\}} \overline{w}_{\lambda_0} - \frac{\varepsilon}{2} \ge \frac{\varepsilon}{2}$$

for any λ satisfying $\lambda_0 \le \lambda < \widetilde{\lambda}_0$, i.e. condition (40).

Now we want to prove that, if $h \in \mathbb{N}$ is large enough, we can find $z_h \in \Omega_{\lambda_h} \setminus \{z_{\lambda_h}\}$ such that $\overline{w}_{\lambda_h}(z_h) = \inf_{\Omega_{\lambda_h}\setminus\{z_{\lambda_h}\}} \overline{w}_{\lambda_h} < 0$.

For any fixed $h \in \mathbb{N}$, by condition (24) on g we know that $\overline{w}_{\lambda_h} \to 0$ as $|z|$ tends to $+\infty$ in $\overline{\Omega}_{\lambda_h}$. Hence for $|z|$ large enough one has $\overline{w}_{\lambda_h}(z) > \frac{1}{2} \inf_{\Omega_{\lambda_h}\setminus\{z_{\lambda_h}\}} \overline{w}_{\lambda_h}$. On the other hand, for $h \in \mathbb{N}$ large enough we have $\lambda_0 \le \lambda_h < \widetilde{\lambda}_0$, where $\widetilde{\lambda}_0$ is defined by (45). Thus, by (40), for such h one has $\overline{w}_{\lambda_h}(z) \ge \varepsilon/2 > 0$ for every $z \in B_\delta(z_{\lambda_h}) \setminus \{z_{\lambda_h}\}$. Finally notice that $\overline{w}_{\lambda_h} \equiv 0$ on T_{λ_h}.

Thus, since by our assumptions $\inf_{\Omega_{\lambda_h}\setminus\{z_{\lambda_h}\}} \overline{w}_{\lambda_h} < 0$, if $h \in \mathbb{N}$ is large enough the infimum must be achieved at some point z_h of $\Omega_{\lambda_h} \setminus B_\delta(z_{\lambda_h})$ by the continuity of the function $\overline{w}_{\lambda_h}$.

Now recall that $\{\lambda_h\}_{h\in\mathbb{N}}$ is decreasing, and hence $\lambda_h \leq \lambda_1 < 0$ for every $h \in \mathbb{N}$. Hence, by Lemma 5.1, we can find a suitable $R > 0$ independent of h such that $|z_h| < R$ for every $h \in \mathbb{N}$ large enough. Thus, up to a subsequence, the sequence z_h converges to a $z_0 \in \mathbb{R}^{d+k}$ as $h \to +\infty$.

Since $z_h \in \Omega_{\lambda_h}$ and since by construction $|z_h - z_{\lambda_h}| \geq \delta$ for every h, letting $h \to +\infty$ we get $z_0 \in \overline{\Omega}_{\lambda_0}$ and $|z_0 - z_{\lambda_0}| \geq \delta$. Hence $z_0 \in \overline{\Omega}_{\lambda_0} \setminus B_\delta(z_{\lambda_0})$.

By our assumptions $\overline{w}_{\lambda_h}(z_h) < 0$ for every h, so by the continuity of v and g, letting $h \to +\infty$ yields $\overline{w}_{\lambda_0}(z_0) = \lim_{h\to+\infty} \overline{w}_{\lambda_h}(z_h) \leq 0$.

Since $\overline{w}_{\lambda_0}$ is nonnegative in its domain, we have $\overline{w}_{\lambda_0}(z_0) = 0$. Thus $z_0 \in T_{\lambda_0}$, since by our assumptions $\overline{w}_{\lambda_0}$ is strictly positive in $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$. Then by Lemma 5.3, recalling that $\overline{w}_{\lambda_0}$ and w_{λ_0} have the same sign, we get

$$D_{x_1} v(z_0) > 0. \tag{48}$$

On the other hand, since for every h the function $\overline{w}_{\lambda_h}$ achieves its negative minimum $z_h \in \Omega_{\lambda_h} \setminus B_\delta(z_{\lambda_h})$, we have $\nabla\overline{w}_{\lambda_h}(z_h) = 0$. Taking the first entry of this gradient and letting h tend to $+\infty$, by the regularity of the functions considered we get

$$\begin{aligned}
0 = D_{x_1}\overline{w}_{\lambda_h}(z_h) &= \frac{-D_{x_1}v(z_h^{\lambda_h}) - D_{x_1}v(z_h)}{g(z_h)} - \frac{v(z_h^{\lambda_h}) - v(z_h)}{g^2(z_h)} D_{x_1} g(z_h) \\
&\to \frac{-D_{x_1}v(z_0^{\lambda_0}) - D_{x_1}v(z_0)}{g(z_0)} - \frac{v(z_0^{\lambda_0}) - v(z_0)}{g^2(z_0)} D_{x_1} g(z_0) \\
&= -2\frac{D_{x_1}v(z_0)}{g(z_0)},
\end{aligned}$$

since $z_0^{\lambda_0} = z_0$, and this clearly contradicts condition (48). Thus we finally get the claim, i.e. if $\lambda_0 < 0$ then $\overline{w}_{\lambda_0} \equiv 0$ in $\Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$.

This in turn implies that $v(z^{\lambda_0}) = v(z)$ for every $z \in \Omega_{\lambda_0} \setminus \{z_{\lambda_0}\}$, hence v is symmetric with respect to the hyperplane T_{λ_0} in $\mathbb{R}^{d+k}$.

On the other hand, if $\lambda_0 = 0$ we can repeat the preceding argument from the opposite direction, namely moving the hyperplane T_λ in direction x_1 from $+\infty$ toward the origin. If T_λ stops before reaching 0, we have once again the symmetry of v in the x_1-direction about some hyperplane.

If the hyperplane T_λ reaches 0 again, we can recover the symmetry result for v combining the two opposite inequalities on $\overline{w}_0$ obtained by moving the hyperplanes from the two opposite directions, and thus in this case v is symmetric with respect to the hyperplane T_0.

Hence v is symmetric in the x_1-direction about some hyperplane in $\mathbb{R}^{d+k}$.

The proof of Step 4, and hence of Theorem 5.2, is now complete. □

Remark 5.6. We remark that the gap in the set of $\gamma > 0$, $d, k \in \mathbb{N}$ left in the statement of Theorem 5.2 is due to the lack of a suitable function g satisfying condition (24) (see Lemma 5.2). If one could show the existence of g with those properties also when $\gamma \geq 1$, $d \in \mathbb{N}$ and $k \in \{1, 2\}$, then the nonexistence result for nontrivial solutions of problem (17) would also hold in those cases.

Remark 5.7. An explicit, nontrivial solution of problem (17) is known for some values of γ when $p = (Q + 2)/(Q - 2)$ (see for instance [7]). Our result in this case states that any solution of this problem must be radially symmetric in the x variables about some point in $\mathbb{R}^d$. A symmetry result for such solutions in the critical case with respect to a suitable inversion of $\mathbb{R}^{d+k}$ has been proven in [20].

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