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Link concordance, homology cobordism, and Hirzebruch-type defects from iterated p -covers

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Abstract. We obtain new invariants of topological link concordance and homology cobordism of 3-manifolds from Hirzebruch-type intersection form defects of towers of iterated p -covers. Our invariants can extract geometric information from an arbitrary depth of the derived series of the fundamental group, and can detect torsion which is invisible via signature invariants. Applications illustrating these features include the following: (1) There are infinitely many homology equivalent rational 3-spheres which are indistinguishable via multisignatures, η -invariants, and L^2 -signatures but have distinct homology cobordism types. (2) There is an infinite family of 2-torsion (amphichiral) knots with non-slice iterated Bing doubles; as a special case, we give the first proof of the conjecture that the Bing double of the figure eight knot is not slice. (3) There exist infinitely many torsion elements at any depth of the Cochran–Orr–Teichner filtration of link concordance.

Keywords. Link concordance, homology cobordism, iterated p -cover, intersection form defect, Bing double

1. Introduction and main results

In this paper we define invariants of 3-manifolds and links in S^3 from towers of iterated p -covers, where p is prime, and employ the invariants to study homology cobordism and link concordance. Essentially our invariants are defects of the (Witt classes of) twisted intersection forms of topological 4-manifolds bounded by the covers. The invariants have two remarkable merits: (i) they can extract geometric information from an arbitrary depth of the derived series of the fundamental group, and (ii) they can detect “torsion”, which is invisible via signature invariants. Also, in many interesting cases, the invariants can be computed by combinatorial algorithms as illustrated in our applications.

The above properties of our invariants may be discussed from the viewpoint of a geometric technique of producing new 3-manifolds and links, which is often referred to as “tying a knot”, “satellite construction”, or “infection”. Figure 1 illustrates infection on a link $L \subset S^3$ by the figure eight knot, which gives us the twice-iterated Bing double: given an unknotted circle $\alpha \subset S^3$ disjoint from L , by tying the figure eight knot along a 2-disk bounded by α , we obtain a new link. Or alternatively, the new link is obtained

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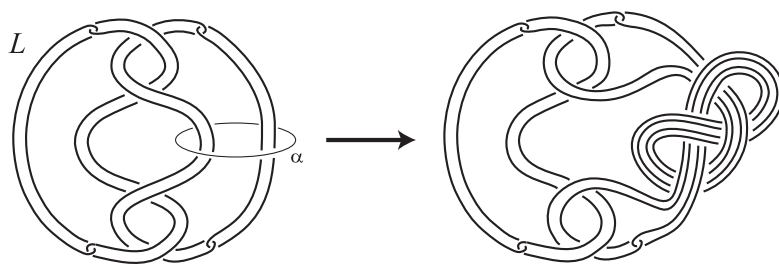


Fig. 1. Infection on a link L by the figure eight knot.

from L by filling in the exterior of α with the figure eight knot exterior. Infection on a 3-manifold is defined in a similar way. (For a precise definition, refer to Section 4.4.)

Roughly speaking, if α lies in the n th derived subgroup of the fundamental group, then by infection along α , further complication from the group of the infection knot appears in the n th derived subgroup. Based on this property, in many papers infection has been used as a primary source of examples revealing the structure of knot and link concordance. In particular, in their landmark paper [19] and subsequent works [20, 21], Cochran, Orr, and Teichner gave a new framework of a systematic study of concordance of knots and links in terms of a filtration related to the derived series, and used infection as a realization tool to illustrate the rich contents of their theory. (For definitions of concordance and the Cochran–Orr–Teichner filtration, see Sections 6 and 8.) There are several recent related results using infection, including works of Cochran, Taehee Kim, Harvey, Leidy, and the author [13, 18, 30, 31, 32, 26, 16, 17, 9]. Such reams of results lead us to study how to detect the effect of infection along a curve contained in a higher term of the derived series of the fundamental group, up to concordance and homology cobordism.

Our result provides a new method to detect the effect of infection. Our method is effective even when the infection knot K is *torsion*, that is, of finite order in the knot concordance group. So far, the von Neumann–Cheeger–Gromov L^2 -signature invariants have been used as the only available tool to detect infection when the infection curve α is in a higher term of the derived series. Roughly speaking, L^2 -signatures of the infected manifold or link associated to certain solvable coefficient systems reflect L^2 -signatures of the infection knot K associated to much simpler coefficient systems (e.g., abelian or metabelian ones). All recent works mentioned above [19, 20, 21, 13, 18, 30, 31, 32, 26, 16, 17, 9] depend on results of this type. When the infection knot K is torsion, however, those L^2 -signatures have failed to detect anything. Partly because of this limitation, many questions on torsion still remain unsolved. Our invariants can detect torsion in many interesting cases, as illustrated by the following applications:

- (1) There are homology equivalent rational homology 3-spheres which have vanishing previously known signature invariants but are not homology cobordant.
- (2) There are infinitely many 2-torsion (amphichiral) knots whose iterated Bing doubles are not slice; as a special case, we give the first proof of the conjecture that the Bing double of the figure eight knot is not slice.

- (3) There exist infinitely many 2-torsion elements in an arbitrarily deep level of the Cochran–Orr–Teichner filtration of link concordance.

Before discussing applications more extensively, we start with an overview of our invariants. In this paper all manifolds are oriented topological manifolds, and submanifolds are assumed to be locally flat.

1.1. Intersection form defects and homology cobordism

Our invariants are basically intersection form analogues of Hirzebruch-type signature defects of odd-dimensional closed manifolds. Let Γ be a group with $H_4(\Gamma) = 0$ and fix a homomorphism of the integral group ring $\mathbb{Z}\Gamma$ into a (skew-)field \mathcal{K} with involution. Suppose M is a closed 3-manifold endowed with a group homomorphism $\phi: \pi_1(M) \rightarrow \Gamma$ such that (M, ϕ) is null-bordant over Γ , i.e., $(M, \phi) = 0$ in the bordism group $\Omega_3^{\text{top}}(B\Gamma)$. Roughly speaking, we define an invariant $\lambda(M, \phi)$ to be the difference of the Witt class of “ \mathcal{K} -coefficient intersection form” of a topological 4-manifold bounded by M over ϕ and the Witt class of its ordinary intersection form. The value of $\lambda(M, \phi)$ lives in the Witt group $L^0(\mathcal{K})$ of nonsingular hermitian forms over \mathcal{K} . We prove that $\lambda(M, \phi)$ is well-defined along the lines of a standard bordism approach, appealing to an Atiyah-type result (Lemma 2.1) on the symmetric signatures of Mishchenko–Ranicki. We remark that $\lambda(M, \phi)$ can also be defined, as an element of $S^{-1}\mathbb{Z} \otimes_{\mathbb{Z}} L^0(\mathcal{K})$, under the weaker assumption that (M, ϕ) is S -torsion in $\Omega_4^{\text{top}}(B\Gamma)$ for a multiplicatively closed subset S of \mathbb{Z} . To simplify statements, in this section we will always consider the case that $S = \{1\}$. For the general case, see Section 2.

In order to extract homology cobordism invariants, we consider towers of abelian p -covers. Fix a prime p and let M_0 be M . Inductively choosing surjections $\phi_i: \pi_1(M_i) \rightarrow \Gamma_i$ with Γ_i an abelian p -group (that is, Γ_i is abelian and $|\Gamma_i| = p^a$), a tower

$$M_n \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_0 = M$$

of connected p -covers of M is constructed. We call it a p -tower determined by $\{\phi_i\}$. For $\phi_n: \pi_1(M_n) \rightarrow \Gamma_n$ with $\Gamma_n = \mathbb{Z}_d$ for some $d = p^a$, we consider the invariant $\lambda(M_n, \phi_n)$. Here, in order to define the invariant, \mathbb{Z}_d is endowed with the canonical map $\mathbb{Z}[\mathbb{Z}_d] \rightarrow \mathbb{Q}(\zeta_d)$ sending $1 \in \mathbb{Z}_d$ to $\zeta_d = \exp(2\pi\sqrt{-1}/d)$. The following result says that there is a bijection between p -towers of homology cobordant manifolds, and corresponding intersection form defect invariants have the same value:

Theorem 1.1. *Suppose that M and M' are homology cobordant, $\Gamma_0, \Gamma_1, \dots, \Gamma_{n-1}$ are abelian p -groups, and Γ_n is a cyclic p -group. Then for any p -tower*

$$M_n \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_0 = M$$

determined by $\{\phi_i: \pi_1(M_i) \rightarrow \Gamma_i\}$, there is a unique corresponding p -tower

$$M'_n \rightarrow M'_{n-1} \rightarrow \dots \rightarrow M'_0 = M'$$

which satisfies the following:

(1) For each $i = 0, 1, \dots, n$, there is a bijection

$$f_i : \text{Hom}(\pi_1(M_i), \Gamma_i) \xrightarrow{\cong} \text{Hom}(\pi_1(M'_i), \Gamma_i).$$

(2) M'_{i+1} is determined by $f_i(\phi_i)$ for $i = 0, 1, \dots, n - 1$.

(3) For any $\phi_n : \pi_1(M_n) \rightarrow \Gamma_n$, $\lambda(M_n, \phi_n)$ is well-defined if and only if so is $\lambda(M'_n, f_n(\phi_n))$, and in this case, $\lambda(M_n, \phi_n) = \lambda(M'_n, f_n(\phi_n))$ in $L^0(\mathbb{Q}(\zeta_d))$.

Indeed, Theorem 1.1 holds when M and M' are \mathbb{Z}_p -homology cobordant. See Section 3 for more details.

We remark that for any (possibly nonabelian) p -cover \tilde{M} of M , there is some p -tower which has \tilde{M} as its top cover M_n . Covers from which one can extract invariants using Theorem 1.1 are not limited to p -covers of M ; in general, for a p -tower considered in Theorem 1.1, M_n is not necessarily a regular cover of M . We also remark that in [10] metabelian towers were used to define signature invariants of links.

In many cases (e.g., for manifolds obtained by surgery along a link, see Corollary 6.5) there are infinitely many highly nontrivial p -towers so that Theorem 1.1 is not vacuous. To prove results of this type we reduce the problem to the case of another space that is easier to investigate, by appealing to the following proposition. For a group G , let \widehat{G} be the “algebraic closure of G with respect to $\mathbb{Z}_{(p)}$ -coefficients” in the sense of [8].

Proposition 1.2. *If X and Y are (of the homotopy type of) finite CW-complexes and $f : X \rightarrow Y$ is a map inducing an isomorphism $\pi_1(\widehat{X}) \rightarrow \pi_1(\widehat{Y})$, then f induces a one-to-one correspondence between p -towers of X and Y via pullback.*

For a more precise statement, the reader is referred to Definition 3.4 and Proposition 3.9. We note that a map which is 2-connected on the integral homology satisfies (the hypothesis of) Proposition 1.2, due to [8].

When we have a map of a simpler space X (e.g., a 1-complex) into a 3-manifold M satisfying the assumptions of Proposition 1.2, we can obtain p -towers of M from those of X which are easier to construct. Furthermore, for a 3-manifold obtained by infection, we can compute algorithmically the intersection form defect invariants in terms of some combinatorial data from the complex X and the infection knot. This algorithmic method applies to several interesting cases, including our applications that will be discussed below (e.g., see Sections 5 and 7).

To investigate the structure of the Witt group $L^0(\mathbb{Q}(\zeta_d))$ where our invariant lives, in this paper we use the signature and discriminant of a Witt class. In particular, the discriminant

$$\text{dis } \lambda \in \mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times / \{z \cdot \bar{z} \mid z \in \mathbb{Q}(\zeta_d)^\times\},$$

which is defined for $\lambda \in L^0(\mathbb{Q}(\zeta_d))$ as the determinant with a sign correction, plays an essential role in our results on torsion examples. We employ tools from algebraic number theory to detect nontrivial values of the discriminant; for $x \in \mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times$ and a prime (ideal) \mathfrak{p} of (the integer ring of) $\mathbb{Q}(\zeta_d + \zeta_d^{-1})$, the “norm residue symbol”

$$(x, D)_\mathfrak{p} \in \{1, -1\}, \quad \text{where } D = (\zeta_d + \zeta_d^{-1})^2 - 4,$$

is defined. For a rapid summary of algebraic properties on the norm residue symbol for nonexperts, see Section 4.5. (See also Chapter 3 of the author's monograph [7].) By the Hasse principle and local Artin reciprocity of algebraic number theory, if $x = z \cdot \bar{z}$ for some $z \in \mathbb{Q}(\zeta_d)$ then $(x, D)_p$ is trivial for all p (and the converse is true if we consider the archimedean valuations as well as nonarchimedean ones associated to primes). Since $(-, D)_p$ is multiplicative, it follows that $(-, D)_p$ induces a well-defined homomorphism

$$(-, D)_p: \mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times / \{z \cdot \bar{z} \mid z \in \mathbb{Q}(\zeta_d)^\times\} \rightarrow \{1, -1\}.$$

In the applications, we produce torsion examples realizing nontrivial values of the norm residue symbols.

1.2. Homology cobordism of rational homology 3-spheres

The work of Cappell and Shaneson [4] provides a surgery-theoretic strategy for the study of homology cobordism, which gives classification results in higher dimensions. For 3-manifolds, it is known that Cappell–Shaneson theory does not classify homology cobordism classes, however, it still gives a framework to understand some useful invariants, like homology surgery obstructions. For example, Wall's multisignature, or equivalently Atiyah–Singer's α -invariant or Casson–Gordon's invariant, is defined for a given M and a character $\chi: \pi_1(M) \rightarrow \mathbb{Z}_d$, as in [44]. Due to Gilmer and Livingston [24], multisignatures for prime power order characters are invariant under (topological) homology cobordism. Related works include the study of (topological and smooth) homology cobordism of (homology) lens spaces, due to Gilmer–Livingston [24], Ruberman [40, 41], Cappell–Ruberman [3], and Fintushel–Stern [22].

Levine studied the Atiyah–Patodi–Singer η -invariants of homology cobordant manifolds, focusing on applications to link concordance [36]. His result for the most general situation can be described in terms of the algebraic closure of groups; for the algebraic closure \widehat{G} of a group G with respect to \mathbb{Z} -coefficients in the sense of [8], there is a natural map $p_G: G \rightarrow \widehat{G}$. Denote by $R_k(G)$ the variety of k -dimensional unitary representations of G . Then, for M with $G = \pi_1(M)$ and for $\theta \in R_k(\widehat{G})$, there is defined the Atiyah–Patodi–Singer η -invariant $\tilde{\eta}(M, \theta \circ p_G) \in \mathbb{R}$. Roughly speaking, Levine showed the following: the function $\tilde{\eta}(M, - \circ p_G): R_k(\widehat{G}) \rightarrow \mathbb{R}$ depends only on the homology cobordism class of M , except for representations θ in some proper subvariety of $R_k(G)$ called “special” in [36]. We note that for homology cobordisms obtained from link concordances, it is more natural to consider the algebraic closure in the sense of Levine [35] instead of [8], as originally done in [36].

Recently, Harvey has proved the homology cobordism invariance of certain L^2 -signatures [26]. For a group G , let $G_H^{(n)}$ be the “torsion-free-derived-series” due to Harvey (for the definition, refer to [26, 15]). For M with $G = \pi_1(M)$, let $\rho_n(M)$ be the von Neumann L^2 -signature defect of a 4-manifold bounded by M over $G \rightarrow G/G_H^{(n)}$, or equivalently, the Cheeger–Gromov invariant of M associated to $G \rightarrow G/G_H^{(n)}$. In [26], it was shown that if M and M' are homology cobordant, then $\rho_n(M) = \rho_n(M')$.

As an application of our result, we show that our invariants distinguish “exotic homology cobordism types” of some rational homology spheres for which the known signature invariants always vanish.

Theorem 1.3. *There are infinitely many rational homology 3-spheres $\Sigma_0, \Sigma_1, \Sigma_2, \dots$ with the following properties:*

- (1) *There is a homology equivalence (i.e., a map inducing isomorphisms on homology groups) $\Sigma_i \rightarrow \Sigma_0$ for any i .*
- (2) *Each Σ_i has vanishing multisignatures.*
- (3) *For any finite-dimensional unitary representation θ of the algebraic closure of $\pi_1(\Sigma_i)$, $\tilde{\eta}(\Sigma_i, \theta \circ p_{\pi_1(\Sigma_i)})$ vanishes.*
- (4) *$\rho_n(\Sigma_i) = 0$ for any n .*
- (5) *Σ_i and Σ_j are not homology cobordant for $i \neq j$.*

In the proof of Theorem 1.3, to distinguish the homology cobordism classes of manifolds Σ_i we compute and compare our invariants defined from p -towers $M_n \rightarrow \dots \rightarrow M_0$ of the manifolds Σ_i with a fixed height n and with fixed deck transformation groups $\Gamma_0, \dots, \Gamma_{n-1}$. Roughly speaking, we construct primes \mathfrak{p}_i which are “algebraically dual” to the intersection form defect invariants of the Σ_i with respect to the norm residue symbols in the following sense:

- (1) If $i \neq j$, then for any $M_n \rightarrow \dots \rightarrow M_0 = \Sigma_j$ and any $\phi: \pi_1(M_n) \rightarrow \mathbb{Z}_d$, $(\text{dis } \lambda(M_n, \phi), D)_{\mathfrak{p}_i}$ is trivial.
- (2) For each i , there exist $M_n \rightarrow \dots \rightarrow M_0 = \Sigma_i$ and $\phi: \pi_1(M_n) \rightarrow \mathbb{Z}_d$ such that $(\text{dis } \lambda(M_n, \phi), D)_{\mathfrak{p}_i}$ is nontrivial.

From Theorem 1.1 and the existence of the algebraically dual primes \mathfrak{p}_i , it follows that Σ_i is not homology cobordant to Σ_j . See Section 5 for more details and further discussions.

1.3. Obstructions to being a slice link and applications to iterated Bing doubles

For a link L in S^3 , the *zero-surgery manifold* of L is obtained by performing surgery on S^3 along the zero-framing of each component of L . It is well-known that if two links in S^3 are concordant, then their zero-surgery manifolds are homology cobordant. Therefore the invariants introduced above give rise to link concordance invariants. In particular we show the following result as a consequence of Theorem 1.1:

Theorem 1.4. *If L is a slice link with zero-surgery manifold M , then for any p -tower $M_n \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_0 = M$ and for any $\phi: \pi_1(M_n) \rightarrow \mathbb{Z}_d$ with d a power of p , $\lambda(M_n, \phi) \in L^0(\mathbb{Q}(\zeta_d))$ is well-defined and vanishes.*

We apply Theorem 1.4 to iterated Bing doubles. Figure 2 illustrates the construction of the n th iterated Bing double $BD_n(K)$ of a knot K . In this paper Bing doubles are always untwisted, i.e., in Figure 2, the parallel strands passing through the box are twisted in such a way that their linking number is zero.

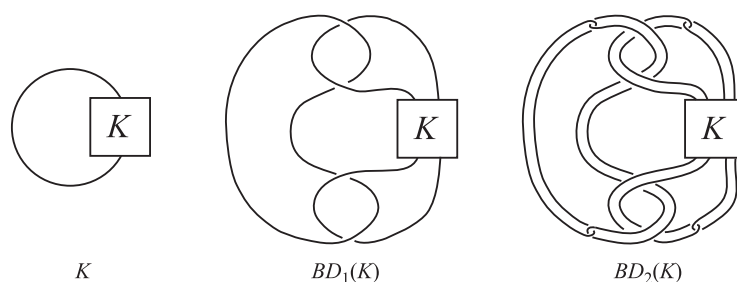


Fig. 2. Iterated Bing doubles of a knot K .

Recently the problem of detecting nonslice Bing doubles have been an interesting subject of research, partly motivated by the 4-dimensional topological surgery problem. It is known that many known link concordance invariants vanish for (iterated) Bing doubles. For an excellent discussion on this, the reader is referred to Cimasoni's paper [12]. Harvey [26] and Teichner (unpublished) proved, independently, that the integral of the Levine–Tristram signature function σ_K of K over the unit circle is zero if the Bing double $BD_1(K)$ is slice. Cimasoni proved a stronger result from a stronger assumption; namely, he proved that if $BD_1(K)$ is boundary slice, i.e., if there are disjoint 3-manifolds in the 4-ball bounded by the union of slice disks and Seifert surfaces of components of $BD_1(K)$, then K is algebraically slice [12]. Cochran, Harvey and Leidy announced a result that there exist algebraically slice knots K with nonslice $BD_n(K)$.

As illustrated in Figure 1 in case of the figure eight knot, $BD_n(K)$ is obtained from a trivial link by infection by K . It is folklore that the infection curve producing $BD_n(K)$ is in the n th derived subgroup (see Section 7 for more details). So for larger n , one needs to investigate geometric information from higher terms of the derived series. Using our invariants, we can detect the nonsliceness of $BD_n(K)$ in several interesting cases. First, we generalize the result of Harvey and Teichner:

Theorem 1.5. *For any n , the Levine–Tristram signature function σ_K is determined by the iterated Bing double $BD_n(K)$ of K . In particular, if σ_K is nontrivial, then for any n , $BD_n(K)$ is not slice.*

We remark that even for $n = 1$ Theorem 1.5 is stronger than the result of Harvey and Teichner since σ_K may be nontrivial even when the integral of σ_K is zero.

Because previously known obstructions are signatures, it has been unknown whether the Bing double of K can be nonslice for a torsion knot K (see also Remark 7.3). In particular, the following question on the second-simplest knot has been asked by several authors, including Schneiderman–Teichner [42], Cochran–Friedl–Teichner [14], and Cimasoni [12]: is the Bing double of the figure eight knot a slice link? We give a first answer to this question.

Theorem 1.6. *There are infinitely many amphichiral knots K , including the figure eight knot, such that $BD_n(K)$ is not slice for all n .*

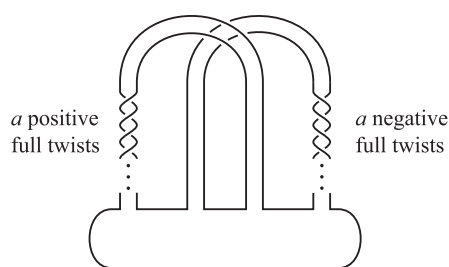


Fig. 3. Amphichiral knots K_a used for infection.

Our proof of Theorem 1.6 is constructive; we prove that there is a sequence $1 = a_1, a_2, \dots$ of integers such that $BD(K_{a_i})$ is not slice for any a_i and n , where K_a is the knot shown in Figure 3. The sequence $\{a_n\}$ is constructed by number-theoretic arguments based on an investigation of the norm residue symbols of discriminants.

Subsequent to this work, in [11] Livingston, Ruberman, and the author generalized the above results for $n = 1$ by proving that if $BD_1(K)$ is slice then K is algebraically slice, based on the ideas of rational knot concordance studied in a monograph of the author [7].

1.4. Torsion in the Cochran–Orr–Teichner filtration of link concordance

Recently, as part of the on-going effort to understand concordance, the structure of the solvable filtration has been studied actively. The notion of (n) -solvability of knots and links was first introduced by Cochran, Orr, and Teichner in [19]. Very roughly speaking, a knot or link is (n) -solvable if there is a topological 4-manifold bounded by the zero-surgery manifold such that the twisted intersection form of the cover associated to the n th derived subgroup of the fundamental group looks like that of a slice disk complement. They also defined $(n.5)$ -solvability as a refinement between (n) - and $(n + 1)$ -solvability. The sets $\mathcal{F}_{(n)}$ of (concordance classes of) (n) -solvable links form a filtration of the set of link concordance classes, which is often referred to as Cochran–Orr–Teichner’s *solvable filtration*.

It is known that there exist (infinitely many) nontrivial elements in any depth of the filtration, i.e., in $\mathcal{F}_{(n)}$ for any n . In case of knots, it was proved for $n = 2$ by Cochran–Orr–Teichner [19, 20], and for all higher n by Cochran–Teichner [21]. (For $n < 2$ the result is classical; e.g., see [34, 28].) Harvey first studied the case of links and proved a nontriviality result in the solvable filtration of links modulo “local knotting”, using her invariant ρ_n [26]. Note that (the sum of m copies of) the solvable filtration of knots injects into that of (m) -component links; Harvey’s result shows that the solvable filtration of links has its own peculiar complication even modulo the sophistication from knots. Taehee Kim considered a similar filtration for double-sliceness of knots, and proved an analogous nontriviality result for the filtration [32].

An open question is whether there is a nontrivial torsion element in $\mathcal{F}_{(n)}$ for higher n . Here, generalizing the notion of torsion knots, we say that a link L is *torsion* if for some

$r \neq 0$ a connected sum of r copies of L is slice for some choice of disk basings. (We remark that in order to construct a connect sum of links, one needs to choose a disk basing for each summand link in the sense of [25], or equivalently, choose a string link whose closure is the link. For knots a connected sum is well-defined independent of disk basings.) All previously known nontrivial elements in $\mathcal{F}_{(n)}$ for higher n are nontorsion, that is, of infinite order (even modulo $\mathcal{F}_{(n+h)}$ for some $h > 0$). L^2 -signatures have been used as the only computable obstruction to being (n) -solvable for higher n , but so far, they have failed to detect torsion. We remark that for lower n , classical invariants other than the L^2 -signatures were used to detect torsion; it is well known that there are torsion elements in $\mathcal{F}_{(0)}$ detected by the Alexander polynomial, and in [37] Livingston proved that there are torsion elements in $\mathcal{F}_{(1)}$ using the Casson–Gordon invariant.

The following refinement of Theorem 1.4 gives a new obstruction for a *link* to be (n) -solvable:

Theorem 1.7. *If L is an (n) -solvable link with zero-surgery manifold M , then for any p -tower $M_k \rightarrow M_{k-1} \rightarrow \cdots \rightarrow M_0 = M$ with height $k < n$ and for any $\phi: \pi_1(M_k) \rightarrow \mathbb{Z}_d$ with d a power of p , $\lambda(M_k, \phi) \in L^0(\mathbb{Q}(\zeta_d))$ is well-defined and vanishes.*

An analogue for 3-manifolds also holds (see Theorem 8.2.) As an application of Theorem 1.7, we prove a first result on the existence of “torsion” at an arbitrary depth of the solvable filtration.

Theorem 1.8. *For any n , there are infinitely many boundary links L_i in S^3 which are 2-torsion and (n) -solvable but not $(n+2)$ -solvable.*

The links L_i in Theorem 1.8 are constructed by taking the n th iterated Bing doubles of certain carefully chosen amphichiral knots.

We remark that it can be shown that the 2-torsion links L_i are “independent over \mathbb{Z}_2 ” modulo $\mathcal{F}_{(n+2)}$ for an *arbitrary* choice of disk basings, in the following sense: for any $a_i \in \{0, 1\}$, if a connected sum of the $a_i L_i$ is $(n+2)$ -solvable for some disk basings, then $a_i = 0$ for all i . A proof of this independence result will appear in a subsequent paper [6] because of additional technicalities required to treat the sophistication from disk basings.

We remark that there are some previously known results on the existence of independent infinite order elements in the Cochran–Orr–Teichner filtration. In the case of knots, for $n \leq 2$ there are infinitely many infinite order elements in $\mathcal{F}_{(n)}$ which are independent (over \mathbb{Z}) modulo $\mathcal{F}_{(n.5)}$, that is, $\mathcal{F}_{(n)}/\mathcal{F}_{(n.5)}$ is of infinite rank. (For $n \leq 1$, this is a classical result, and for $n = 2$, it was shown in [20].) In the case of links, Harvey considered the solvable filtration $\{\mathcal{BF}_{(n)}\}$ of the boundary string link concordance group instead of spherical links, so that the difficulty from disk basings is avoided, and proved that there are infinite order boundary string links which generate an infinite rank subgroup in (the abelianization of) $\mathcal{BF}_{(n)}/\mathcal{BF}_{(n+1)}$, using her invariant ρ_n [26]. It was generalized to $\mathcal{BF}_{(n)}/\mathcal{BF}_{(n.5)}$ in [16].

Theorem 1.7 also enables us to prove the following result on infinite order elements: there are infinitely many (n) -solvable boundary links L'_i which are not $(n+1)$ -solvable but have vanishing Harvey’s invariant ρ_n (see Corollary 8.6). It can also be shown that

these L'_i are “independent over \mathbb{Z} ” modulo $\mathcal{F}_{(n+1)}$ for an arbitrary choice of disk basings, in the sense that for any $a_i \in \mathbb{Z}$, if a connected sum of the $a_i L'_i$ is $(n+1)$ -solvable for some disk basings, then $a_i = 0$ for all i [6]. Considering the subgroup generated by string links whose closures are the L'_i , it can be shown that the kernel of ρ_n is large enough to contain a subgroup whose abelianization is isomorphic to \mathbb{Z}^∞ [6]. We also remark that previously known signature invariants of link concordance, including ρ_n , vanish on the 2-torsion links L_i given in Theorem 1.8. (See Remark 7.3.) So, the kernel of ρ_n also contains infinitely many independent 2-torsion links. We remark that Cochran–Harvey–Leidy have recently announced a proof that the kernel of ρ_n contains an infinite cyclic subgroup using a different technique.

In addition, our results also hold for the *grope filtration* of links which is defined and investigated in [19, 20, 21, 26]. In particular, for any $n \geq 2$ there are infinitely many 2-torsion elements in the n th term of the grope filtration of links (which are independent in an appropriate sense). For more details, see Section 8.2.

2. An Atiyah-type lemma and intersection form defects

In this section we define an invariant of 3-manifolds which is essentially a Witt class defect of intersection forms. We start with a discussion on an Atiyah-type result for Mishchenko–Ranicki’s symmetric signature of topological 4-manifolds, which will be used to show the well-definedness of the invariant.

2.1. An Atiyah-type lemma

We recall some terms of the L -theory which is used to state our Atiyah-type result in a general context. A standard reference for the terms is Ranicki’s book [39]. For a group π , we regard the integral group ring $\mathbb{Z}\pi$ as a ring with involution via $(\sum_{g \in \pi} n_g g)^- = \sum_{g \in \pi} n_g g^{-1}$. Let $L^n(\mathbb{Z}\pi)$ be the *symmetric L -group over $\mathbb{Z}\pi$* , that is, the abelian group of cobordism classes of symmetric Poincaré chain complexes of dimension n over $\mathbb{Z}\pi$. We say that X is a *geometric n -dimensional Poincaré space* if X is a finite CW-complex satisfying the Poincaré duality for any coefficients. The cellular chain complex $C_*(X; \mathbb{Z}\pi_1(X))$ endowed with the chain equivalence given by the cap product with the orientation class gives rise to an element $\sigma^*(X) \in L^n(\mathbb{Z}\pi_1(X))$ which is called the *Mishchenko–Ranicki symmetric signature*.

We say that a space X is *over a group Γ* if it is endowed with the homotopy class of a map $\phi: X \rightarrow B\Gamma$, where $B\Gamma$ denotes the classifying space for Γ , or equivalently, $K(\Gamma, 1)$. When the choice of ϕ is clearly understood from the context, we will not specify it explicitly. For a Poincaré space X of dimension n which is over Γ , we denote by $\sigma_\Gamma^*(X)$ the image of $\sigma^*(X)$ under the natural map

$$L^n(\mathbb{Z}\pi_1(X)) \rightarrow L^n(\mathbb{Z}\Gamma).$$

We will focus on the case of topological 4-manifolds. A closed topological 4-manifold W over Γ has a canonical homotopy type of a 4-dimensional Poincaré space, and therefore

$\sigma_\Gamma^*(W) \in L^4(\mathbb{Z}\Gamma)$ is defined. In particular, when Γ is a trivial group, we obtain an element in $L^4(\mathbb{Z}) \cong \mathbb{Z}$ which is equal to the ordinary signature $\sigma(W)$. Let

$$i_\Gamma^*: \mathbb{Z} = L^4(\mathbb{Z}) \rightarrow L^4(\mathbb{Z}\Gamma)$$

be the abelian group homomorphism induced by the inclusion of a trivial group into Γ .

Lemma 2.1. *Suppose $H_4(\Gamma) = 0$. Then for any closed topological 4-manifold W over Γ , $\sigma_\Gamma^*(W) = i_\Gamma^*\sigma(W)$ in $L^4(\mathbb{Z}\Gamma)$.*

Proof. Our proof follows the lines of the bordism-theoretic approach to Atiyah-type theorems, based on the fact that the association $W \rightarrow \sigma_\Gamma^*(W)$ gives rise to a homomorphism

$$\sigma_\Gamma^*: \Omega_4^{\text{top}}(B\Gamma) \rightarrow L^4(\mathbb{Z}\Gamma),$$

where $\Omega_4^{\text{top}}(B\Gamma)$ is the 4-dimensional topological bordism group over $B\Gamma$. First we claim that $\Omega_4^{\text{top}}(B\Gamma)$ is generated by $\mathbb{C}P^2$, which is automatically endowed with a constant map into $B\Gamma$, being a simply connected space. To show the claim, we consider the Atiyah–Hirzebruch spectral sequence for bordism groups:

$$E_{p,q}^2 = H_p(\Gamma; \Omega_q^{\text{top}}) \Rightarrow \Omega_n^{\text{top}}(B\Gamma)$$

where Ω_n^{top} denotes the topological cobordism group of n -manifolds. Note that $\Omega_0^{\text{top}} = \mathbb{Z}$, $\Omega_2^{\text{top}} = \Omega_3^{\text{top}} = 0$, and $\Omega_4^{\text{top}} = \mathbb{Z}$ generated by $\mathbb{C}P^2$. Also we have $E_{4,0}^2 = H_4(\Gamma; \mathbb{Z}) = 0$ by the hypothesis. So it follows that all the $E_{p,q}^2$ terms vanish for $p + q = 4$ but $E_{0,4}^2 = H_0(\Gamma; \Omega_4^{\text{top}}) \cong \Omega_4^{\text{top}}$. Since $d_{0,4}^r: E_{0,4}^r \rightarrow E_{-r,r+3}^r$ is obviously trivial, $E_{0,4}^r$ is a quotient of $E_{0,4}^2$ for all $r \geq 2$. (In fact, the only case that $d_{r,5-r}^r: E_{r,5-r}^r \rightarrow E_{0,4}^r$ is potentially nontrivial is when $r = 5$.) From this it follows that $E_{p,q}^\infty = 0$ for all $p + q = 4$ but $E_{0,4}^\infty$, which is a quotient of $\Omega_4^{\text{top}} \cong \mathbb{Z}$. This shows that $\Omega_4(B\Gamma)$ is generated by the bordism class of $\mathbb{C}P^2$.

By the claim, it suffices to consider the special case that W is simply connected. The symmetric signature $\sigma^*(W)$ of W is in $L^4(\mathbb{Z}\pi_1(W)) = L^4(\mathbb{Z}) = \mathbb{Z}$, and indeed equal to the ordinary signature $\sigma(W)$. Now by the definition, $\sigma_\Gamma^*(W) = i_\Gamma^*\sigma(W)$. □

2.2. Intersection form defects

We now define an invariant for 3-manifolds which lives in an L -group. For technical and computational convenience, we pass from the L -theory over $\mathbb{Z}\Gamma$ to that of a (skew-)field; let \mathcal{K} be a (skew-)field with involution, and fix a homomorphism $\mathbb{Z}\Gamma \rightarrow \mathcal{K}$ between rings with involutions, which induces a map

$$L^4(\mathbb{Z}\Gamma) \rightarrow L^4(\mathcal{K}).$$

Since \mathcal{K} is a skew-field, every \mathcal{K} -module is free. Also, $L^4(\mathcal{K})$ is canonically identified with $L^0(\mathcal{K})$, which is the Witt group of nonsingular hermitian forms on finitely generated (free) modules over \mathcal{K} . For a closed topological 4-manifold W over Γ , the image of

$\sigma_\Gamma^*(W)$ in $L^0(\mathcal{K})$ is represented by the intersection form of W with \mathcal{K} -coefficients. More precisely, on the \mathcal{K} -coefficient homology module

$$H_2(W; \mathcal{K}) = H_2(C_*(W; \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{K})$$

of W , the \mathcal{K} -coefficient intersection form

$$\lambda_{\mathcal{K}}(W): H_2(W; \mathcal{K}) \times H_2(W; \mathcal{K}) \rightarrow \mathcal{K}$$

is defined as a nonsingular hermitian form over \mathcal{K} . Its Witt class $[\lambda_{\mathcal{K}}(W)] \in L^0(\mathcal{K})$ is the image of $\sigma_\Gamma^*(W)$.

If W is not closed, then the above $\lambda_{\mathcal{K}}(W)$ is not necessarily nonsingular. However, it can be seen that, letting $\bar{H}_2(W; \mathcal{K})$ be the image of

$$H_2(W; \mathcal{K}) \rightarrow H_2(W, \partial W; \mathcal{K})$$

which is automatically finitely generated and \mathcal{K} -free, $\lambda_{\mathcal{K}}(W)$ gives rise to a nonsingular hermitian form

$$\bar{H}_2(W; \mathcal{K}) \times \bar{H}_2(W; \mathcal{K}) \rightarrow \mathcal{K}.$$

We will denote it by $\lambda_{\mathcal{K}}(W)$ as an abuse of notation. Indeed, it is well-defined and nonsingular since

$$H_2(W, \partial W; \mathcal{K}) \cong H^2(W; \mathcal{K}) \cong \text{Hom}(H_2(W; \mathcal{K}), \mathcal{K})$$

by the Poincaré duality and the universal coefficient theorem over the skew-field \mathcal{K} . Therefore, even when W is not closed, we can think of the Witt class $[\lambda_{\mathcal{K}}(W)] \in L^0(\mathcal{K})$.

Let S be a fixed nonempty multiplicatively closed subset of \mathbb{Z} .

Definition 2.2. Suppose M is a closed 3-manifold over Γ via $\phi: M \rightarrow B\Gamma$ such that the bordism class of (M, ϕ) is S -torsion in $\Omega^{\text{top}}(B\Gamma)$, i.e., $r(M, \phi) = 0$ in $\Omega^{\text{top}}(B\Gamma)$ for some $r \in S$. Then there is a compact topological 4-manifold W endowed with a map $W \rightarrow B\Gamma$ which is bounded by rM over Γ . Define

$$\lambda(M, \phi) = \frac{1}{r} \otimes ([\lambda_{\mathcal{K}}(W)] - i_{\mathcal{K}}^* \sigma(W)) \in S^{-1}\mathbb{Z} \otimes_{\mathbb{Z}} L^0(\mathcal{K})$$

where $i_{\mathcal{K}}^*: \mathbb{Z} = L^0(\mathbb{Z}) \rightarrow L^0(\mathcal{K})$ is the map induced by the canonical inclusion $\mathbb{Z} \rightarrow \mathcal{K}$.

Lemma 2.3. If $H_4(\Gamma) = 0$, then $\lambda(M, \phi)$ is well-defined for any (M, ϕ) which is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$.

Proof. Suppose W and W' are null-bordisms of rM and $r'M$ over Γ , respectively, where $r, r' \in S$. Let $V = r'W \cup_{rr'M} r(-W')$ be the manifold obtained by gluing $r'W$ and $r(-W')$ along $rr'M$. Then V is over Γ . The standard Novikov additivity argument shows

$$[\lambda_{\mathcal{K}}(V)] = r'[\lambda_{\mathcal{K}}(W)] - r[\lambda_{\mathcal{K}}(W')] \quad \text{and} \quad \sigma(V) = r'\sigma(W) - r\sigma(W').$$

By Lemma 2.1, $[\sigma_\Gamma^*(V)] = i_\Gamma^* \sigma(V)$, and therefore, $[\lambda_{\mathcal{K}}(V)] = i_{\mathcal{K}}^* \sigma(V)$. It follows that

$$r'([\lambda_{\mathcal{K}}(W)] - i_{\mathcal{K}}^* \sigma(W)) = r([\lambda_{\mathcal{K}}(W')] - i_{\mathcal{K}}^* \sigma(W')). \quad \square$$

Remark 2.4. (1) Our λ has naturality with respect to S in the following sense: Let \mathcal{M}_S be the collection of pairs (M, ϕ) which are S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$. Then $\lambda(-, -)$ can be viewed as a function on \mathcal{M}_S . For $S' \subset S$, we have $\mathcal{M}_{S'} \subset \mathcal{M}_S$ and there is a commutative diagram

$$\begin{array}{ccc}
 \mathcal{M}_{S'} & \hookrightarrow & \mathcal{M}_S \\
 \downarrow \lambda & & \downarrow \lambda \\
 S'^{-1}\mathbb{Z} \otimes L^0(\mathcal{K}) & \xrightarrow{f_{S',S}} & S^{-1}\mathbb{Z} \otimes L^0(\mathcal{K})
 \end{array}$$

where $f_{S',S}$ is induced by $S'^{-1}\mathbb{Z} \rightarrow S^{-1}\mathbb{Z}$. The kernel of $f_{S',S}$ is nontrivial in general. Indeed, it consists of the S -torsion part:

$$\text{Ker } f_{S',S} = \{x \in S'^{-1}\mathbb{Z} \otimes L^0(\mathcal{K}) \mid r \cdot x = 0 \text{ for some } r \in S\}.$$

Therefore, for a larger S , one has the advantage that $\lambda(M, \phi)$ is defined on a larger collection \mathcal{M}_S , but more (torsion) information is lost.

- (2) In particular when $S = \{1\}$ and (M, ϕ) is null-bordant, $\lambda(M, \phi)$ lives in $L^0(\mathcal{K})$, without losing any torsion information. This special case will be used in our applications discussed later.
- (3) If one wanted to extract information on the torsion-free part only (e.g., by considering signature-type invariants of $L^0(\mathcal{K})$), then $S = \mathbb{Z} - \{0\}$ could be used to define $\lambda(M, \phi) \in \mathbb{Q} \otimes L^0(\mathcal{K})$ for any (M, ϕ) which has finite order in $\Omega_3^{\text{top}}(B\Gamma)$.

3. p -towers and homology cobordism

Let R be an abelian group. We say that two closed 3-manifolds M and M' are R -homology cobordant if there is a compact 4-manifold W such that $\partial W = M \cup -M'$ and the inclusions of M and M' into W are R -homology equivalences, that is, the induced maps $H_i(M; R) \rightarrow H_i(W; R)$, $H_i(M'; R) \rightarrow H_i(W; R)$ are isomorphisms for all i . Such a manifold W is called an R -homology cobordism. When $R = \mathbb{Z}$, one usually says that M and M' are homology cobordant. We will often consider the case that $R = \mathbb{Z}_p$, the abelian group of residue classes modulo p , where p is prime.

The aim of this section is to show the homology cobordism invariance of the invariant $\lambda(M, \phi)$ which is defined in the previous section for $\phi: M \rightarrow B\Gamma$. As the first step, we investigate the case that Γ is a p -group and ϕ factors through W . As an abuse of notation, for a CW-complex X , we will regard (the homotopy class of) $\phi: X \rightarrow B\Gamma$ as a group homomorphism $\phi: \pi_1(X) \rightarrow \Gamma$. (For disconnected X , one may adopt the convention that $\pi_1(X)$ designates the free product of the fundamental groups of the components of X .)

As in the previous section, we assume $H_4(\Gamma) = 0$ and fix a multiplicatively closed subset S of \mathbb{Z} and a map $\mathbb{Z}\Gamma \rightarrow \mathcal{K}$. In addition, we assume that \mathcal{K} is of characteristic zero.

Proposition 3.1. *Suppose W is a \mathbb{Z}_p -homology cobordism between 3-manifolds M and M' , and $\psi: \pi_1(W) \rightarrow \Gamma$ is a group homomorphism into a p -group Γ .*

- (1) *Let $\phi: \pi_1(M) \rightarrow \Gamma$ and $\phi': \pi_1(M') \rightarrow \Gamma$ be the restrictions of ψ . Then (M, ϕ) is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$ if and only if so is (M', ϕ') . In that case, $\lambda(M, \phi) = \lambda(M', \phi')$ in $S^{-1}\mathbb{Z} \otimes L^0(\mathcal{K})$.*
- (2) *Let $W_\Gamma, M_\Gamma,$ and M'_Γ be the Γ -covers of $W, M,$ and M' determined by $\psi, \phi,$ and ϕ' , respectively. Then W_Γ is a \mathbb{Z}_p -homology cobordism between M_Γ and M'_Γ .*

In proving Proposition 3.1 and other results in this section, a crucial role is played by the following result which is essentially due to Levine (see the proofs of [36, p. 95, Proposition 3.2] and [36, p. 89, Lemma 4.3]; see also [15]).

Lemma 3.2 (Levine). *Suppose Γ is a p -group, and C_* is a chain complex consisting of free $\mathbb{Z}\Gamma$ -modules such that $\bigoplus_{i \leq n} C_i$ is finitely generated. If $H_i(C_* \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}_p)$ vanishes for $i \leq n$ (where \mathbb{Z}_p is regarded as a $\mathbb{Z}\Gamma$ -module with trivial Γ -action), then $H_i(C_* \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ vanishes for $i \leq n$.*

We remark that in [36], $\mathbb{Z}_{(p)}$ (the localization of \mathbb{Z} away from p) is used instead of \mathbb{Z}_p . Since $H_i(C_* \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ vanishes for $i \leq n$ if and only if so does $H_i(C_* \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$ for any finitely generated free chain complex C_* over \mathbb{Z} , Lemma 3.2 is equivalent to its $\mathbb{Z}_{(p)}$ -analogue.

For an abelian group R , we say that a map $f: Y \rightarrow X$ is n -connected with respect to R -coefficients if $H_i(X, Y; R) = 0$ for $i \leq n$. If X and Y have finite n -skeletons, then $f: X \rightarrow Y$ is n -connected with respect to \mathbb{Z}_p -coefficients if and only if f is n -connected with respect to $\mathbb{Z}_{(p)}$ -coefficients. The following is an immediate consequence of Levine’s lemma:

Lemma 3.3. *Suppose X and Y are CW-complexes with finite n -skeletons, $f: Y \rightarrow X$ is n -connected with respect to \mathbb{Z}_p -coefficients, and $\pi_1(X) \rightarrow \Gamma$ is a map into a p -group Γ . Then $H_i(X, Y; \mathbb{Z}_{(p)}\Gamma) = 0$ for $i \leq n$. In other words, denoting the associated Γ -covers of X and Y by X_Γ and Y_Γ , respectively, the lift $Y_\Gamma \rightarrow X_\Gamma$ of f is n -connected with respect to \mathbb{Z}_p -coefficients.*

Proof of Proposition 3.1. (1) If (M, ϕ) is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$, then there is a null-cobordism, say V , of rM for some $r \in S$. Attaching rW to V along rM , we obtain a null-cobordism of rM' over Γ . It follows that (M', ϕ') is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$. The converse is proved similarly.

By the definition of $\lambda(-, -)$ and a Novikov additivity argument, it can be seen that

$$\lambda(M, \phi) - \lambda(M', \phi') = [\lambda_{\mathcal{K}}(W)] - i_{\mathcal{K}}^* \sigma(W).$$

Since $H_*(W, M; \mathbb{Z}_p) = 0$, $\sigma(W)$ vanishes. Also, by Lemma 3.3, it follows that $H_*(W, M; \mathbb{Z}_{(p)}\Gamma) = 0$. Since \mathcal{K} has characteristic zero, $\mathbb{Z} \rightarrow \mathbb{Z}\Gamma \rightarrow \mathcal{K}$ is injective. Since \mathcal{K} is a division ring, the map $\mathbb{Z}\Gamma \rightarrow \mathcal{K}$ factors through $\mathbb{Z}_{(p)}\Gamma$. It follows that $H_*(W, M; \mathcal{K}) = 0$ so that $[\lambda_{\mathcal{K}}(W)] = 0$.

(2) Since $H_*(W, M; \mathbb{Z}_p) = 0$, we have $H_*(W_\Gamma, M_\Gamma; \mathbb{Z}_p) = 0$ by Lemma 3.3. Similarly $H_*(W_\Gamma, M'_\Gamma; \mathbb{Z}_p) = 0$. \square

In general, determining whether a given ϕ factors through $\pi_1(W)$ for some homology cobordism W is known to be a difficult problem. As an easy special case, if Γ is abelian, then ϕ factors through $\pi_1(W)$ for any homology cobordism W , since any map into an abelian group factors through $H_1(-)$. In order to investigate further sophistication beyond the abelianization, we consider certain towers of covers of M in the next subsection.

3.1. p -towers

Let X be a CW-complex. As in the introduction, a tower

$$X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X$$

of covering maps is called a p -tower of height n if each $X_{i+1} \rightarrow X_i$ is a regular cover whose covering transformation group is an abelian p -group Γ_i . Such a tower is determined by inductively choosing maps $\phi_i: \pi_1(X_i) \rightarrow \Gamma_i$. In this paper, we always assume that a p -tower consists of connected spaces, unless stated otherwise. In other words, (X is connected and) the maps ϕ_i are surjective.

Obviously, given a map $f: Y \rightarrow X$, a p -tower for X induces one for Y via pullback. We will frequently consider f which gives rise to a 1-1 correspondence between p -towers for X and Y . We give a formal definition below.

Definition 3.4. A map $f_0: Y_0 \rightarrow X_0$ is called a p -tower map of height n if for any abelian p -groups $\Gamma_0, \dots, \Gamma_n$, the following holds:

- (0) f_0 induces a bijection $f_0^*: \text{Hom}(\pi_1(X_0), \Gamma_0) \rightarrow \text{Hom}(\pi_1(Y_0), \Gamma_0)$.
- (1) Let X_1 and Y_1 be the covers of X_0 and Y_0 determined by a map $\phi_0: \pi_1(X_0) \rightarrow \Gamma_0$ and $\psi_0 = f_0^*(\phi_0): \pi_1(Y_0) \rightarrow \Gamma_0$, respectively. Then the lift $f_1: Y_1 \rightarrow X_1$ of f_0 induces a bijection

$$f_1^*: \text{Hom}(\pi_1(X_1), \Gamma_1) \rightarrow \text{Hom}(\pi_1(Y_1), \Gamma_1).$$

...

- (n) Let X_n and Y_n be the covers of X_{n-1} and Y_{n-1} determined by a map $\phi_{n-1}: \pi_1(X_{n-1}) \rightarrow \Gamma_{n-1}$ and $\psi_{n-1} = f_{n-1}^*(\phi_{n-1}): \pi_1(Y_{n-1}) \rightarrow \Gamma_{n-1}$, respectively. Then the lift $f_n: Y_n \rightarrow X_n$ of f_{n-1} induces a bijection

$$f_n^*: \text{Hom}(\pi_1(X_n), \Gamma_n) \rightarrow \text{Hom}(\pi_1(Y_n), \Gamma_n).$$

If the above condition is satisfied for all n , then f_0 is called a p -tower map.

Remark 3.5. For $f: Y \rightarrow X$ and $g: Z \rightarrow Y$, if any two of f , g , and $f \circ g$ are p -tower maps, then so is the third.

The following simple observation says that in our case pullback preserves the connectedness condition.

Lemma 3.6. Suppose G and π are finitely presented groups and $f: G \rightarrow \pi$ is a homomorphism inducing an injection $f^*: \text{Hom}(\pi, \mathbb{Z}_p) \rightarrow \text{Hom}(G, \mathbb{Z}_p)$. Then for any abelian p -group Γ , a map $\phi: \pi \rightarrow \Gamma$ is surjective if and only if so is $\phi f: G \rightarrow \Gamma$.

Proof. The “if” part is obvious. For the “only if” part, observe that for any abelian p -group A and a power r of p such that $r \geq |A|$, $\text{Hom}(-, A) \cong \text{Hom}(H_1(-) \otimes \mathbb{Z}_r, A)$ and a group homomorphism $B \rightarrow A$ is surjective if and only if so is the induced map $H_1(B) \otimes \mathbb{Z}_r \rightarrow A$. So we may assume that both G and π are abelian p -groups by applying $H_1(-) \otimes \mathbb{Z}_r$ to G and π where $r = |\Gamma|$.

Let C be the cokernel of f . Since f^* is injective and $\text{Hom}(-, \mathbb{Z}_p)$ is left exact, $\text{Hom}(C, \mathbb{Z}_p) = 0$. Thus C is p -torsion free, that is, there is no nontrivial element in C whose order is a power of p . However, being a quotient of a p -group, C is a p -group. It follows that $C = 0$, that is, f is surjective. So ϕf is surjective whenever so is ϕ . \square

Lemma 3.7. *If X and Y are CW-complexes with finite 2-skeletons and $f : Y \rightarrow X$ is 2-connected with respect to \mathbb{Z}_p -coefficients, then f is a p -tower map.*

Proof. We claim the following: for any CW-complexes X and Y with finite 2-skeletons and any abelian p -group Γ , if $f : Y \rightarrow X$ is 2-connected with respect to \mathbb{Z}_p -coefficients, then

$$f^* : \text{Hom}(\pi_1(X), \Gamma) \rightarrow \text{Hom}(\pi_1(Y), \Gamma)$$

is a one-to-one correspondence. For, let $r = |\Gamma|$. Then

$$\text{Hom}(\pi_1(X), \Gamma) \cong \text{Hom}(H_1(X) \otimes \mathbb{Z}_r, \Gamma) = \text{Hom}(H_1(X; \mathbb{Z}_r), \Gamma)$$

and similarly for Y . Since $H_i(X, Y; \mathbb{Z}_p) = 0$ for $i \leq 2$, we have $H_i(X, Y; \mathbb{Z}_r) = 0$ for $i \leq 2$. From the long exact sequence for (X, Y) , it follows that $Y \rightarrow X$ induces an isomorphism on $H_1(-; \mathbb{Z}_r)$. This proves the claim.

Now, we use induction on n to show that Definition 3.4(n) holds, and in addition, that the map $f_n : Y_n \rightarrow X_n$ is 2-connected with respect to \mathbb{Z}_p -coefficients. By the above claim, Definition 3.4(0) holds. Also, f is 2-connected with respect to \mathbb{Z}_p -coefficients by the hypothesis. Suppose Definition 3.4($n - 1$) holds and f_{n-1} is 2-connected with respect to \mathbb{Z}_p -coefficients. Then by Lemma 3.3, the lift $f_n : Y_n \rightarrow X_n$ is 2-connected with respect to \mathbb{Z}_p -coefficients. By the above claim, it follows that f_n induces a bijection on $\text{Hom}(\pi_1(-), \Gamma_n)$, that is, Definition 3.4(n) holds. \square

Lemma 3.7 enables us to apply Proposition 3.1 inductively to p -towers. As an immediate consequence, one obtains a sequence of homology cobordism invariants, as stated below. From now on, a cyclic group $\Gamma = \mathbb{Z}_d$ is always endowed with the map $\mathbb{Z}\Gamma \rightarrow \mathcal{K} = \mathbb{Q}(\zeta_d)$ sending $1 \in \mathbb{Z}_d$ to $\zeta_d = \exp(2\pi\sqrt{-1}/d)$. Note that $H_4(\mathbb{Z}_d) = 0$.

Theorem 3.8. *Suppose W is a \mathbb{Z}_p -homology cobordism between 3-manifolds M and M' . Then the following holds:*

- (1) $M \rightarrow W$ and $M' \rightarrow W$ are p -tower maps.
- (2) For a given p -tower $M_n \rightarrow \dots \rightarrow M_1 \rightarrow M_0 = M$, let

$$W_n \rightarrow \dots \rightarrow W_1 \rightarrow W_0 = W, \quad M'_n \rightarrow \dots \rightarrow M'_1 \rightarrow M'_0 = M'$$

be the p -towers of W and M' which correspond to the M_i via pullback along the p -tower maps $M \rightarrow W \leftarrow M'$. Then W_i is a \mathbb{Z}_p -homology cobordism between M_i and M'_i for each i .

(3) Let d be a power of p and

$$\text{Hom}(\pi_1(M_n), \mathbb{Z}_d) \approx \text{Hom}(\pi_1(M'_n), \mathbb{Z}_d)$$

be the bijection induced by the p -tower maps $M \rightarrow W \leftarrow M'$. For any $\phi_n: \pi_1(M_n) \rightarrow \mathbb{Z}_d$ and the corresponding $\phi'_n: \pi_1(M'_n) \rightarrow \mathbb{Z}_d$, (M_n, ϕ_n) is S -torsion in $\Omega_3^{\text{top}}(B\mathbb{Z}_d)$ if and only if so is (M'_n, ϕ'_n) , and in that case

$$\lambda(M_n, \phi_n) = \lambda(M'_n, \phi'_n) \quad \text{in } S^{-1}\mathbb{Z} \otimes L^0(\mathbb{Q}(\zeta_d)).$$

Since a homology cobordism is a \mathbb{Z}_p -homology cobordism, Theorem 1.1 follows immediately from Theorem 3.8.

3.2. Algebraic closures and p -towers

For later use, we generalize Lemma 3.7 by weakening the 2-connectedness condition. For a group G , let \widehat{G} be the algebraic closure with respect to $\mathbb{Z}_{(p)}$ -coefficients in the sense of [8]. It is known that the assignment $G \mapsto \widehat{G}$ is a functor on the category of groups and there is a natural transformation $p_G: G \rightarrow \widehat{G}$ [8]. The most essential property of \widehat{G} is the following: as in case of spaces, we say that a group homomorphism $f: \pi \rightarrow G$ is n -connected with respect to R -coefficients if $H_i(f; R) = 0$ for $i \leq n$. Then, whenever π, G are finitely presented and $f: \pi \rightarrow G$ is 2-connected with respect to \mathbb{Z}_p (or equivalently $\mathbb{Z}_{(p)}$)-coefficients, f induces an isomorphism $\widehat{\pi} \rightarrow \widehat{G}$. For proofs of these facts and more details, see [8]. In fact the functor $G \mapsto \widehat{G}$ is initial among those satisfying this inverting property whenever f is a homomorphism between finitely presented groups which is 2-connected with respect to \mathbb{Z}_p -coefficients.

Proposition 3.9. *If X and Y are CW-complexes with finite 2-skeletons and $f: X \rightarrow Y$ induces an isomorphism $\widehat{\pi_1(X)} \rightarrow \widehat{\pi_1(Y)}$, then f is a p -tower map.*

Proof. By [8], there is a sequence of 2-connected (with respect to \mathbb{Z}_p -coefficients) maps

$$\pi_1(X) = G_0 \rightarrow G_1 \rightarrow \dots$$

on finitely presented groups G_k such that $\widehat{\pi_1(X)} \cong \varinjlim G_k$ and the natural map $p_{\pi_1(X)}: \pi_1(X) \rightarrow \widehat{\pi_1(X)}$ is the limit map $G_0 \rightarrow \varinjlim G_k$. Since $\pi_1(Y)$ is finitely presented, the composition

$$\pi_1(Y) \rightarrow \widehat{\pi_1(Y)} \xrightarrow{\cong} \widehat{\pi_1(X)}$$

factors through some G_k , so that we have the following commutative diagram:

$$\begin{array}{ccccc} \pi_1(X) & \longrightarrow & G_k & \longrightarrow & \widehat{\pi_1(X)} \\ \downarrow & \nearrow & & & \downarrow \cong \\ \pi_1(Y) & \longrightarrow & & & \widehat{\pi_1(Y)} \end{array}$$

Since the horizontal maps induce isomorphisms on $H_1(-; \mathbb{Z}_p)$, the map $\pi_1(Y) \rightarrow G_k$ induces an isomorphism on $H_1(-; \mathbb{Z}_p)$. Since $H_2(\pi_1(X); \mathbb{Z}_p) \rightarrow H_2(G_k; \mathbb{Z}_p)$ is surjective, $H_2(\pi_1(Y); \mathbb{Z}_p) \rightarrow H_2(G_k; \mathbb{Z}_p)$ is surjective. It follows that $\pi_1(Y) \rightarrow G_k$ is 2-connected with respect to \mathbb{Z}_p -coefficients.

Let $Z = K(G_k, 1)$ be the Eilenberg–MacLane space. Since G_k is finitely presented, we may assume that Z has finite 2-skeleton. Consider maps $X \rightarrow Z$ and $Y \rightarrow Z$ which induce our $\pi_1(X) \rightarrow G_k$ and $\pi_1(Y) \rightarrow G_k$. Since $\pi_1(X) \rightarrow G_k$ and $\pi_1(Y) \rightarrow G_k$ are 2-connected with respect to \mathbb{Z}_p -coefficients, so are $X \rightarrow Z$ and $Y \rightarrow Z$, and therefore they are p -tower maps by Lemma 3.7. By Remark 3.5, it follows that $f: X \rightarrow Y$ is a p -tower map. \square

Remark 3.10. In [35] Levine defined another algebraic closure of a group G (much earlier than [8]). Levine’s algebraic closure has the following property which is appropriate for studying manifold embeddings into a simply connected ambient space: if π, G are finitely presented, $f: \pi \rightarrow G$ is integrally 2-connected, and $f(\pi)$ normally generates G , then f induces an isomorphism on Levine’s algebraic closures. Comparing this with the universal property of the algebraic closure \widehat{G} of [8], it follows easily that there is a natural transformation from Levine’s algebraic closure to \widehat{G} . An immediate consequence is that if $\pi \rightarrow G$ induces an isomorphism on Levine’s algebraic closures, then $\widehat{\pi} \rightarrow \widehat{G}$ is also an isomorphism. Therefore Proposition 3.9 applies for $f: X \rightarrow Y$ which induces an isomorphism on Levine’s algebraic closures.

Remark 3.11. The 2-connectedness assumption in Lemma 3.7 is stronger than the assumption of Proposition 3.9. For example, a p -tower map considered in Proposition 6.3 induces an isomorphism on $\widehat{\pi_1(-)}$ while it is not H_2 -onto.

4. Computation of intersection form defects

In this section we suppose that Γ is endowed with $\mathbb{Z}\Gamma \rightarrow \mathcal{K}$ and $H_4(\Gamma) = 0$.

4.1. Connected sum

Suppose M and M' are closed 3-manifolds M endowed with $\phi: \pi_1(M) \rightarrow \Gamma$ and $\phi': \pi_1(M') \rightarrow \Gamma$, respectively. Let $\psi: \pi_1(M\#M') \rightarrow \Gamma$ be the map induced by ϕ and ϕ' , regarding $\pi_1(M\#M')$ as the free product $\pi_1(M) * \pi_1(M')$. Note that any map $\pi_1(M\#M') \rightarrow \Gamma$ is of this form.

Lemma 4.1. *If (M, ϕ) and (M', ϕ') are S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$, then $(M\#M', \psi)$ is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$ and*

$$\lambda(M\#M', \psi) = \lambda(M, \phi) + \lambda(M', \phi')$$

in $S^{-1}\mathbb{Z} \otimes L^0(\mathcal{K})$.

Proof. Choose W and W' such that $\partial W = rM$ and $\partial W' = r'M'$ over Γ for some r and r' in S . Consider $r'W$ and rW' which have boundaries $rr'M$ and $rr'M'$, respectively. Let

M_i be the i th copy of M in $\partial(r'W)$ for $i = 1, \dots, rr'$. Choose a 4-ball B_i in $r'W$ which is disjoint from M_j for $j \neq i$ and intersects M_i in a 3-ball contained in ∂B_i . Choose a 4-ball B'_i in rW' for each $i = 1, \dots, rr'$ in a similar way. Let $V = (r'W \cup rW')/\sim$ where $B_i \subset r'W$ and $B'_i \subset rW'$ are identified for each i . It can be seen that $\partial V = rr'(M\#M')$ over Γ . Therefore $(M\#M', \psi)$ is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$ and we can compute $\lambda(M\#M', \psi)$ using the intersection form of V . Since each B_i is contractible, a standard Mayer–Vietoris argument shows that

$$[\lambda_{\mathcal{K}}(V)] = r'[\lambda_{\mathcal{K}}(W)] + r[\lambda_{\mathcal{K}}(W')],$$

and similarly for the ordinary signature $\sigma(V)$. From this the desired additivity of $\lambda(-, -)$ follows. \square

The following example will be used later to show that our invariant vanishes for some manifolds:

Lemma 4.2. *Let M be the connected sum of disjoint copies of $S^1 \times S^2$. Then for any $\phi: \pi_1(M) \rightarrow \Gamma$, $\lambda(M, \phi)$ is well-defined as an element in $L^0(\mathcal{K})$ and vanishes.*

Proof. By Lemma 4.1, we may assume that $M = S^1 \times S^2$. Then, letting $W = S^1 \times D^3$, we have $\partial W = M$ over $\pi_1(M)$. So $(M, \phi) = 0$ in $\Omega_3^{\text{top}}(B\Gamma)$, and $\lambda(M, \phi)$ can be computed from the intersection form of W . Since W has the homotopy type of a 1-complex, $H_2(W; \mathbb{Q}) = 0 = H_2(W; \mathcal{K})$. It follows that $\lambda(W, \phi) = 0$ for any ϕ . \square

4.2. Toral sum

We consider a special case of toral sum described below. Let M and M' be closed 3-manifolds, and T and T' be solid tori embedded in M and M' , respectively. Choose an orientation reversing homeomorphism $h: T' \rightarrow T$, and let N be the manifold obtained by gluing the boundaries of $M - \text{int } T$ and $M' - \text{int } T'$ along $h|_{\partial}$. Suppose there is a retraction $s: M' \rightarrow T'$ onto T' . For a homomorphism $\phi: \pi_1(M) \rightarrow \Gamma$, let ϕ' be the composition

$$\pi_1(M') \xrightarrow{s_*} \pi_1(T') \xrightarrow{h_*} \pi_1(T) \rightarrow \pi_1(M) \xrightarrow{\phi} \Gamma.$$

Then ϕ and ϕ' induce a map $\psi: \pi_1(N) \rightarrow \Gamma$.

Lemma 4.3. (1) *(M', ϕ') is null-bordant over Γ . Consequently, (M', ϕ') is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$.*

(2) *If (M, ϕ) is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$, then so is (N, ψ) , and*

$$\lambda(N, \psi) = \lambda(M, \phi) + \lambda(M', \phi') \quad \text{in } S^{-1}\mathbb{Z} \otimes L^0(\mathcal{K}).$$

Proof. It can be seen that $\Omega_3^{\text{top}}(\mathbb{Z}) = 0$ from the Atiyah–Hirzebruch spectral sequence; indeed, each E^2 term $E_{p,q}^2 = H_p(\mathbb{Z}; \Omega_q^{\text{top}})$ vanishes for $p + q = 3$ since $B\mathbb{Z}$ has the homotopy type a 1-complex, namely S^1 , and $\Omega_q^{\text{top}} = 0$ for $1 \leq q \leq 3$. Therefore M' endowed with $s_*: \pi_1(M') \rightarrow \pi_1(T') = \mathbb{Z}$ has a null-bordism W' , i.e., $\partial W' = M'$ over \mathbb{Z} . By the definition of ϕ' , $\partial W' = M'$ over Γ as well. This proves the first assertion.

Now suppose that (M, ϕ) is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$. Choose W such that $\partial W = rM$ over Γ for some $r \in S$. Let V be the manifold obtained by taking the disjoint union of W and rW' and then attaching the j th copy of T in $\partial W = rM$ to the j th copy of T' in $\partial(rW') = rM'$. It can be seen that $\partial V = rN$ over Γ . Thus (N, ψ) is S -torsion in $\Omega_3^{\text{top}}(B\Gamma)$.

By Mayer–Vietoris, we have an exact sequence

$$\begin{aligned} H_2(S^1 \times D^2; \mathcal{K})^r &\rightarrow H_2(W; \mathcal{K}) \oplus H_2(W'; \mathcal{K})^r \rightarrow H_2(V; \mathcal{K}) \\ &\rightarrow H_1(S^1 \times D^2; \mathcal{K})^r \rightarrow H_1(W; \mathcal{K}) \oplus H_1(W'; \mathcal{K})^r. \end{aligned}$$

Obviously $H_2(S^1 \times D^2; \mathcal{K}) = 0$. The map

$$H_1(S^1 \times D^2; \mathcal{K}) = H_1(T'; \mathcal{K}) \rightarrow H_1(M'; \mathcal{K}) \rightarrow H_1(W'; \mathcal{K})$$

is injective since it has a left inverse. Therefore it follows that

$$H_2(V; \mathcal{K}) \cong H_2(W; \mathcal{K}) \oplus H_2(W'; \mathcal{K})^r.$$

From this we obtain an orthogonal decomposition of the intersection form, namely

$$[\lambda_{\mathcal{K}}(V)] = [\lambda_{\mathcal{K}}(W)] + r[\lambda_{\mathcal{K}}(W')].$$

An analogous formula for the ordinary signature is proved by a similar argument. From this the desired additivity follows. \square

For later use, we state the following lemma which was proved in the proof of Lemma 4.3(1).

Lemma 4.4. *If $\phi: \pi_1(M) \rightarrow \Gamma$ factors through \mathbb{Z} , then (M, ϕ) is null-bordant over Γ so that $\lambda(M, \phi)$ is well-defined as an element in $L^0(\mathcal{K})$.*

4.3. p -Towers of a toral sum

Suppose M, M', T, T', s , and N are as above. Note that $M - \text{int } T$ can be viewed as a subspace of both M and N . From the existence of the retraction $s: M' \rightarrow T'$, it follows that the inclusion $M - \text{int } T \rightarrow M$ extends to a map $f: N \rightarrow M$; we give a proof below. Since s is a retraction, we have the following commutative diagram, where the vertical map is induced by the inclusion:

$$\begin{array}{ccc} \pi_1(\partial T') & & \\ \downarrow & \searrow & \\ \pi_1(M' - \text{int } T') & \xrightarrow{(s)_*} & \pi_1(T') \end{array}$$

It follows that the inclusion $\partial T' \rightarrow T'$ extends to $r: M' - \text{int } T' \rightarrow T'$. Now, define $f: N \rightarrow M$ by

$$f: N = (M - \text{int } T) \cup_{\partial} (M' - \text{int } T') \xrightarrow{\text{id} \cup r} (M - \text{int } T) \cup_{\partial} T' \cong M. \quad \square$$

Suppose

$$M_n \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 = M$$

is a p -tower determined by $\{\phi_i : \pi_1(M_i) \rightarrow \Gamma_i\}$. We consider the pullback p -tower

$$N_n \rightarrow \cdots \rightarrow N_1 \rightarrow N_0 = N$$

of N via f . By “lifting” the toral sum structure of N , we can obtain N_n from M_n via toral sum. Note that T may not be lifted to M_n ; in general, the pre-image of $T \subset M$ under $M_n \rightarrow M$ is a disjoint union of solid tori, say $\tilde{T}_1, \tilde{T}_2, \dots$, and the restriction $\tilde{T}_j \rightarrow T$ is a r_j -fold cyclic cover for some divisor r_j of $\prod_{i=0}^{n-1} |\Gamma_i|$. (In general, r_j depends on j since $M_n \rightarrow M$ may not be a regular cover.) Therefore, we need to consider a corresponding r_j -fold covers of M' to construct N_n from M_n . Details are as follows. Let \tilde{M}'_j be the r_j -fold cyclic cover of M' determined by

$$\pi_1(M') \xrightarrow{s_*} \pi_1(T') = \mathbb{Z} \xrightarrow{\text{proj.}} \mathbb{Z}_{r_j}$$

and $\tilde{T}'_j \subset \tilde{M}'_j$ be the pre-image of T' . Obviously each \tilde{T}'_j is a solid torus and $\tilde{T}'_j \rightarrow T'$ is a r_j -fold cyclic cover which can be identified with $\tilde{T}_j \rightarrow T$. Let

$$N_n = \left[M_n - \bigcup_j \text{int } \tilde{T}_j \right] \cup \left[\bigcup_j (\tilde{M}'_j - \text{int } \tilde{T}'_j) \right] / \{ \partial \tilde{T}_j \sim \partial \tilde{T}'_j \text{ for } j = 1, 2, \dots \}.$$

Also, s lifts to $\tilde{s}_j : \tilde{M}'_j \rightarrow \tilde{T}'_j$. For any $\phi_n : \pi_1(M_n) \rightarrow \mathbb{Z}_d$ with d a power of p , ϕ_n and the map $\tilde{\phi}'_j$ defined to be

$$\tilde{\phi}'_j : \pi_1(\tilde{M}'_j) \xrightarrow{(\tilde{s}_j)^*} \pi_1(\tilde{T}'_j) = \pi_1(\tilde{T}_j) \rightarrow \pi_1(M_n) \xrightarrow{\phi_n} \mathbb{Z}_d$$

induce a map $\psi_n : \pi_1(N_n) \rightarrow \mathbb{Z}_d$. The following lemma follows immediately by applying Lemma 4.3 to the toral decomposition of N_n .

Lemma 4.5. (N_n, ψ_n) is S -torsion in $\Omega_3^{\text{top}}(B\mathbb{Z}_d)$ if and only if so is (M_n, ϕ_n) , and in that case

$$\lambda(N_n, \psi_n) = \lambda(M_n, \phi_n) + \sum_j \lambda(\tilde{M}'_j, \tilde{\phi}'_j).$$

4.4. Infection by a knot

The following toral sum construction will be used as a main tool to construct examples in our applications. Let K be a knot in S^3 . The *zero-surgery manifold* M_K of K is defined to be the manifold obtained by filling in the exterior of K with a solid torus, say T' , in such a way that the preferred longitude of K bounds a disk in T' . Note that there is a retraction $s : M_K \rightarrow T'$ as assumed in the previous subsection. Let M be a closed 3-manifold and α be a simple closed curve in M with tubular neighborhood T . Forming a toral sum of M and M_K via a homeomorphism $h : T \cong T'$, we obtain a new manifold N . We say that N is obtained from M by *infection by K along α* . Note that a meridian of K is identified with a parallel of α , and a preferred longitude of K is identified with a meridian of α .

As in the above subsection, let

$$M_n \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 = M, \quad N_n \rightarrow \cdots \rightarrow N_1 \rightarrow N_0 = N$$

be a p -tower of M and the p -tower of N induced by it, respectively. Let X_r be the r -fold cyclic cover of M_K . Then by Lemma 4.5, $\lambda(N_n, -)$ is determined by $\lambda(M_n, -)$ and $\lambda(X_{r_j}, -)$ where r_j is as above. The relevant characters of X_{r_j} are of the following form: the canonical surjection $\pi_1(M_K) \rightarrow \mathbb{Z}$ sending the (positive) meridian μ of K to 1 restricts to a surjection $\pi_1(X_r) \rightarrow r\mathbb{Z}$, viewing $\pi_1(X_r)$ as a subgroup of $\pi_1(M_K)$. Composing it with an appropriate map $r\mathbb{Z} \rightarrow \mathbb{Z}_d$, we define a map $\phi_r^{s,d}: \pi_1(X_r) \rightarrow \mathbb{Z}_d$ sending μ^r to $s \in \mathbb{Z}_d$. Note that $\lambda(X_r, \phi_r^{s,d})$ is always well-defined as an element in $L^0(\mathbb{Q}(\zeta_d))$ by Lemma 4.4.

Lemma 4.6. *Let A be a Seifert matrix of K . Then*

$$\lambda(X_r, \phi_r^{s,d}) = [\lambda_r(A, \zeta_d^s)] - [\lambda_r(A, 1)] \quad \text{in } L^0(\mathbb{Q}(\zeta_d))$$

where $[\lambda_r(A, \omega)]$ is the Witt class of (the nonsingular part of) the hermitian form represented by the following $r \times r$ block matrix:

$$\lambda_r(A, \omega) = \begin{bmatrix} A + A^T & -A & & & -\omega^{-1}A^T \\ -A^T & A + A^T & -A & & \\ & -A^T & A + A^T & \ddots & \\ & & & \ddots & \ddots & -A \\ -\omega A & & & & -A^T & A + A^T \end{bmatrix}_{r \times r}.$$

For $r = 1, 2$, $\lambda_r(A, \omega)$ should be understood as

$$[(1 - \omega)A + (1 - \omega^{-1})A^T] \quad \text{and} \quad \begin{bmatrix} A + A^T & -A - \omega^{-1}A^T \\ -A^T - \omega A & A + A^T \end{bmatrix}.$$

We postpone the proof of Lemma 4.6 to the Appendix.

Now, from Lemmas 4.5 and 4.6, we obtain a formula for the intersection form defect invariants of manifolds infected by knots:

Corollary 4.7. *Let $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots \subset M_n$ be the components of the pre-image of $\alpha \subset M$, and r_j be the degree of the covering map $\tilde{\alpha}_j \rightarrow \alpha$. Let $\phi_n: \pi_1(M_n) \rightarrow \mathbb{Z}_d$ be a character with d a power of p and $\psi_n: \pi_1(N_n) \rightarrow \mathbb{Z}_d$ be the character induced by ϕ_n . Then*

$$\lambda(N_n, \psi_n) = \lambda(M_n, \phi_n) + \sum_j ([\lambda_{r_j}(A, \zeta_d^{\phi_n([\tilde{\alpha}_j]_1)})] - [\lambda_{r_j}(A, 1)]).$$

In the previous subsection, we defined a map $f: N \rightarrow M$ for N obtained from M by a toral sum with M' ; in general, it can easily be seen that f is not necessarily a p -tower map in the sense of Definition 3.4. However, for knot infection, we have the following result:

Proposition 4.8. *Suppose N is obtained from M by knot infection along a solid torus $T \subset M$. Then the map $f: N \rightarrow M$ is a p -tower map for any prime p .*

Proof. Let E be the exterior of K . Using the fact that $H_*(E) \cong H_*(S^1 \times D^2)$, it can be easily shown that there is a homology equivalence $h: E \rightarrow S^1 \times D^2$ which restricts to a homeomorphism on the boundary. Using h , the inclusion $M - \text{int } T \rightarrow M$ extends to

$$f: N = (M - \text{int } T) \cup_{\partial} E \xrightarrow{\text{id} \cup h} (M - \text{int } T) \cup_{\partial} (S^1 \times D^2) = M.$$

By a Mayer–Vietoris argument, it follows that f is a homology equivalence. By Lemma 3.7, f is a p -tower map. \square

4.5. Invariants of $L^0(\mathbb{Q}(\zeta_d))$ and norm residue symbols

We give a quick review of known invariants of $L^0(\mathbb{Q}(\zeta_d))$. For more details, the reader is referred to Milnor–Husemoller’s book [38]. See also [7, Chapter 3].

For a nonsingular hermitian form λ on a finite-dimensional $\mathbb{Q}(\zeta_d)$ -space V , the following invariants are defined:

Signature: The natural inclusion $\mathbb{Q}(\zeta_d) \rightarrow \mathbb{C}$ gives rise to

$$\text{sign}: L^0(\mathbb{Q}(\zeta_d)) \rightarrow L^0(\mathbb{C}) \cong \mathbb{Z}.$$

In other words, $\text{sign } \lambda$ is the signature of λ viewed as a hermitian form over \mathbb{C} .

Rank modulo 2: Let r be the $\mathbb{Q}(\zeta_d)$ -dimension of the underlying space V . Since every hyperbolic form has even dimension,

$$\text{rank } \lambda = \text{modulo } 2 \text{ residue class of } r \in \mathbb{Z}_2$$

is an invariant of the Witt class of λ .

Discriminant: Since λ is nonsingular and hermitian, the determinant of the matrix representing λ is nonzero and fixed under the involution, i.e., $\det \lambda \in \mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times$. The discriminant of λ is defined by

$$\text{dis } \lambda = (-1)^{r(r+1)/2} \det \lambda \in \frac{\mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times}{\{z \cdot \bar{z} \mid z \in \mathbb{Q}(\zeta_d)^\times\}}.$$

Here $z \mapsto \bar{z}$ denotes the involution on $\mathbb{Q}(\zeta_d)$ induced by $\zeta_d \mapsto \zeta_d^{-1}$. It can be verified easily that $\text{dis } \lambda$ is an invariant of the Witt class of λ .

We remark that $\text{dis } \lambda$ is regarded as an element of a multiplicative group, while $\text{sign } \lambda$ and $\text{rank } \lambda$ are in additive groups. It is known that $\{\text{sign}, \text{rank}, \text{dis}\}$ is a complete set of invariants of $L^0(\mathbb{Q}(\zeta_d))$ if $d > 2$, i.e., $\zeta_d \neq \pm 1$ [38].

Since $\text{dis } \lambda$ is defined up to multiplication by $z \cdot \bar{z}$ where $z \in \mathbb{Q}(\zeta_d)$, detecting a nonvanishing value of $\text{dis } \lambda$ is a nontrivial problem. For this purpose, we will employ some algebraic number theory. The remaining part of this subsection is devoted to a quick summary of necessary results.

We call $x \in \mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times$ a *norm* if $x = z \cdot \bar{z}$ for some $z \in \mathbb{Q}(\zeta_d)^\times$. In fact, $L = \mathbb{Q}(\zeta_d)$ is a quadratic extension over $K = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$, and the norm map $N_K^L: L^\times \rightarrow K^\times$ is given by $N_K^L(z) = z\bar{z}$. Note that $L = K(\sqrt{D})$ where

$$D = (\zeta_d + \zeta_d^{-1})^2 - 4 \in K.$$

The problem of deciding whether $x \in K^\times$ is a norm or not can be reduced to the computation of *norm residue symbols*. For a prime \mathfrak{p} of K (that is, a prime ideal in the ring of algebraic integers of K) and $a, b \in K^\times$, there is defined the norm residue symbol $(a, b)_\mathfrak{p}$ satisfying the following properties:

- (1) $(a, b)_\mathfrak{p} = +1$ or -1 .
- (2) $(a, b)_\mathfrak{p}$ is symmetric and bilinear, i.e., $(a, b)_\mathfrak{p} = (b, a)_\mathfrak{p}$ and $(aa', b)_\mathfrak{p} = (a, b)_\mathfrak{p}(a', b)_\mathfrak{p}$.
Consequently, $(a, b)_\mathfrak{p} = (a^{-1}, b)_\mathfrak{p}$.
- (3) If x is a norm, then $(x, D)_\mathfrak{p} = 1$ for all \mathfrak{p} .

For our purpose, (3) is essential. From (3) it easily follows that the induced homomorphism

$$(-, D)_\mathfrak{p}: \frac{\mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times}{\{z \cdot \bar{z} \mid z \in \mathbb{Q}(\zeta_d)^\times\}} \rightarrow \{\pm 1\}$$

is well-defined for each prime \mathfrak{p} . We remark that the converse of (3) holds if we think of the norm residue symbols for archimedean valuations as well as those for primes. For more detailed discussions on the norm residue symbol, the reader is referred to [43, 5].

For the case that $K = \mathbb{Q}$ and \mathfrak{p} is (the ideal generated by) an odd prime p , which will be used in our applications discussed in later sections, the following lemma provides a formula for the computation of the norm residue symbols:

Lemma 4.9. *Suppose p is an odd prime and $a, b \in \mathbb{Q}$. Then*

$$(a, b)_p = \left((-1)^{v_p(a)v_p(b)} \cdot \frac{a^{v_p(b)}}{b^{v_p(a)}} \right)^{(p-1)/2} \text{ in } \mathbb{Z}_p,$$

where $v_p(x)$ is the valuation associated to p , i.e., if we write $x = p^r(s/t)$ where s, t are integers relatively prime to p , then $v_p(x)$ is defined to be r .

Note that $x^{(p-1)/2} \equiv \pm 1 \pmod{p}$ for any $x \not\equiv 0 \pmod{p}$. For a proof of Lemma 4.9, refer to [43]. Also, [7, Section 3.4] provides a summary of results including the general case where K is not necessarily \mathbb{Q} , for nonexperts of algebraic number theory.

5. Exotic homology cobordism types of rational homology 3-spheres with vanishing signature invariants

In this section we construct rational homology spheres Σ_i satisfying the conclusion of Theorem 1.3. Indeed, we will show that Σ_i and Σ_j are not \mathbb{Z}_2 -homology cobordant for $i \neq j$, while each Σ_i has vanishing known signature invariants.

5.1. Construction of rational homology spheres by infection

We construct the manifolds Σ_i as follows. First we describe a “seed” manifold M . Choose two lens spaces $L_1 = L(r_1, s_1)$ and $L_2 = L(r_2, s_2)$ such that r_i is a power of 2 and all the multisignatures of L_i vanish for $i = 1, 2$. (For example, if we choose (r_i, s_i) such that L_i bounds a rational 4-ball, then L_i has vanishing multisignatures.) Let $M = L_1 \# L_2$. Figure 4 illustrates a Kirby diagram of M , together with a simple closed curve α in M . (r_i/s_i represents the surgery slope.) It can be easily seen that $\pi_1(M) = \mathbb{Z}_{r_1} * \mathbb{Z}_{r_2}$, and if we denote the generators of the \mathbb{Z}_{r_1} and \mathbb{Z}_{r_2} factors by x and y , then α represents the commutator $(x, y) = xyx^{-1}y^{-1} \in \pi_1(M)$.

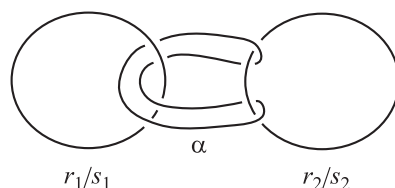


Fig. 4.

For a given integer a , consider the knot K_a shown in Figure 3. Note that K_a is (negatively) amphichiral, that is, isotopic to its (concordance) inverse $-K_a$. The knot K_a has an obvious Seifert surface of genus one, on which the following Seifert matrix is defined:

$$A = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix}.$$

Let $\Sigma_0 = M$. For $i > 0$ the manifolds Σ_i are obtained by infection on M along α by K_{a_i} , where a_1, a_2, \dots are integers which will be specified later. By (the proof of) Proposition 4.8, there is a homology equivalence $\Sigma_i \rightarrow \Sigma_0$ for all i . Before discussing how to choose the a_i , we investigate the Witt class defect invariants of the infected manifolds. Let $K = K_a$ and N be M infected along α by K .

Let

$$\Gamma_0 = \mathbb{Z}_{r_1} \oplus \mathbb{Z}_{r_2}, \quad \Gamma_1 = \mathbb{Z}_2, \quad \Gamma_2 = \mathbb{Z}_4.$$

We will consider 2-towers of height two with deck transformation groups Γ_0 and Γ_1 , and intersection form defect invariants associated to Γ_2 -valued characters. In this section we always assume that a p -tower consists of connected spaces.

For the special case of $a = 0$, that is, when K is unknotted and $N = M$, the invariant vanishes for any such tower and character:

Lemma 5.1. *For any 2-tower $M_2 \rightarrow M_1 \rightarrow M_0 = M$ with deck transformation groups Γ_0 and Γ_1 and for any $\phi_2: \pi_1(M_2) \rightarrow \Gamma_2$, $(M, \phi_2) = 0$ in $\Omega_3^{\text{top}}(B\Gamma_2)$, and $\lambda(M_2, \phi_2) = 0$ in $L^0(\mathbb{Q}(\sqrt{-1}))$.*

Proof. The map $\phi_0: \pi_1(M) \rightarrow \Gamma_0$ inducing $M_1 \rightarrow M$ factors through $H_1(M)$, which has the same cardinality as Γ_0 . It follows that ϕ_1 is identical with the abelianization map, since ϕ_1 is surjective. (Note that the M_i are assumed to be connected.) Therefore M_1 is

the universal abelian cover of M . Note that $M = L_1 \# L_2$ and the universal (abelian) cover of each L_i is S^3 . From this it follows that M_1 is a connected sum of disjoint copies of $S^1 \times S^2$, and therefore so is M_2 . By Lemma 4.2, the conclusion follows. \square

Next, we investigate the general case:

Proposition 5.2. (1) For any 2-tower $N_2 \rightarrow N_1 \rightarrow N_0 = N$ with deck transformation groups Γ_0 and Γ_1 and for any $\psi_2: \pi_1(N_2) \rightarrow \Gamma_2$, $\lambda(N_2, \psi_2)$ is always well-defined as an element in $L^0(\mathbb{Q}(\sqrt{-1}))$, and

$$\text{dis } \lambda(N_2, \psi_2) = (2a^2 + 1)^{n_1} (2a^4 + 4a^2 + 1)^{n_2}$$

for some integers n_1 and n_2 .

(2) For some 2-tower $N_2 \rightarrow N_1 \rightarrow N_0 = N$ with deck transformation groups Γ_0 and Γ_1 and for some $\psi_2: \pi_1(N_2) \rightarrow \Gamma_2$,

$$\text{dis } \lambda(N_2, \psi_2) = (2a^2 + 1)(2a^4 + 4a^2 + 1).$$

Here, $\text{dis } \lambda(N_2, \psi_2)$ is understood as an element in $\mathbb{Q}^\times / \{z\bar{z} \mid z \in \mathbb{Q}(\sqrt{-1})^\times\}$.

In the proof of Proposition 5.2, we need the following lemma. We define the (symmetrized) Alexander polynomial $\Delta_V(t)$ of a $2g \times 2g$ matrix V by

$$\Delta_V(t) = t^{-g} \cdot \det(tV - V^T).$$

We remark that, in contrast to the case of knots, there is no $(\pm t^r)$ -factor ambiguity in the definition of the Alexander polynomial of a matrix. $\Delta_V(1)$ is always $+1$ for any Seifert matrix V . For our Seifert matrix A of the knot K , we have

$$\Delta_A(t) = -a^2t + (2a^2 + 1) - a^2t^{-1}.$$

Lemma 5.3. (1) If $\omega = \zeta_d^k$ and $\Delta_A(\omega) \neq 0$, then

$$\text{dis}[\lambda_1(A, \omega)] = \Delta_A(\omega) \quad \text{in } \mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times / \{z\bar{z} \mid z \in \mathbb{Q}(\zeta_d)^\times\}.$$

(2) If $\omega = \zeta_d^k$ and $\Delta_A(\sqrt{\omega}) \neq 0 \neq \Delta_A(-\sqrt{\omega})$, then

$$\text{dis}[\lambda_2(A, \omega)] = \Delta_A(\sqrt{\omega})\Delta_A(-\sqrt{\omega}) \quad \text{in } \mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times / \{z\bar{z} \mid z \in \mathbb{Q}(\zeta_d)^\times\},$$

where $\sqrt{\omega}$ and $-\sqrt{\omega}$ denote the zeros of $x^2 = \omega$.

Indeed, it turns out that Lemma 5.3 generalizes to the case of $\lambda_r(A, \omega)$ with $r > 2$ in an obvious form. Since we do not need it, we do not address the case that $r > 2$; below we prove it for $r = 1, 2$ by a straightforward computation.

Proof. (1) For $\omega = 1$,

$$\lambda_1(A, \omega) = (1 - \omega)A + (1 - \omega^{-1})A^T$$

is a zero matrix, and so $\text{dis}[\lambda_1(A, \omega)] = 1 = \Delta_A(1)$ as claimed. Suppose $\omega \neq 1$ and A is $2g \times 2g$. It can be easily verified that

$$(-1)^{g(2g+1)} \det \lambda_1(A, \omega) = (\omega - 1)^g (\omega^{-1} - 1)^g \Delta_A(\omega).$$

Since $\Delta_A(\omega) \neq 0$, the matrix $\lambda_1(A, \omega)$ is nonsingular. So $\text{dis}[\lambda_1(A, \omega)] = \Delta_A(\omega)$.

(2) For $\omega = 1$, the matrix

$$\lambda_2(A, \omega) = \begin{bmatrix} A + A^T & -A - \omega^{-1}A^T \\ -A^T - \omega A & A + A^T \end{bmatrix}$$

is singular. By a simple basis change it can be seen that the nonsingular part of $\lambda_2(A, 1)$ is given by $A + A^T$. (Note that $A + A^T$ is nonsingular for any Seifert matrix A .) Therefore

$$\text{dis} \lambda_2(A, 1) = (-1)^{g(2g+1)} \det(A + A^T) = \Delta_A(-1) = \Delta_A(1)\Delta_A(-1)$$

as claimed. Suppose $\omega \neq 1$. To compute $\det \lambda_2(A, \omega)$, we make a variable change: let $G = (A - A^T)^{-1}A$. Then $(A - A^T)^{-1}A^T = G - 1$. (We denote the identity matrix by 1.) Since $\det(A - A^T) = 1$, we have

$$\det \lambda_2(A, \omega) = \det \begin{bmatrix} 2G - 1 & -G - \omega^{-1}(G - 1) \\ -(G - 1) - \omega G & 2G - 1 \end{bmatrix}.$$

Now, viewing the above matrix as one over the commutative domain $\mathbb{Q}(G)$, we can compute the determinant in a straightforward way; this gives us

$$\begin{aligned} \det \lambda_2(A, \omega) &= \det \left[(1 - \omega)(1 - \omega^{-1}) \left(G - \frac{1}{1 - \sqrt{\omega}} \right) \left(G - \frac{1}{1 + \sqrt{\omega}} \right) \right] \\ &= \det \left[A - \frac{1}{1 + \sqrt{\omega}}(A - A^T) \right] \det \left[A - \frac{1}{1 - \sqrt{\omega}}(A - A^T) \right] \\ &= \left(\frac{1}{1 - \omega} \right)^g \left(\frac{1}{1 - \omega^{-1}} \right)^g \Delta_A(\sqrt{\omega}) \Delta_A(-\sqrt{\omega}). \end{aligned}$$

It follows that $\lambda_2(A, \omega)$ is nonsingular and $\text{dis}[\lambda_2(A, \omega)] = \Delta_A(\sqrt{\omega})\Delta_A(-\sqrt{\omega})$ for $\omega \neq 1$. □

Proof of Proposition 5.2(1). Since N is obtained from M by knot infection, there is a 2-tower $M_2 \rightarrow M_1 \rightarrow M_0 = M$ which gives rise to $N_2 \rightarrow N_1 \rightarrow N_0 = N$ and there is $\phi_2: \pi_1(M_2) \rightarrow \Gamma_2$ inducing the given ψ_2 via pullback by Proposition 4.8. From this it follows that $\lambda(N_2, \psi_2)$ is well-defined as an element of $L^0(\mathbb{Q}(\sqrt{-1}))$, by Lemmas 4.3 and 5.1.

Let $\tilde{\alpha}_1, \dots, \tilde{\alpha}_k \subset M_2$ be the components of the pre-image of $\alpha \subset M_0$. We claim that the degree r_j of $\tilde{\alpha}_j \rightarrow \alpha$ is either 1 or 2. For, since Γ_0 is abelian and $\alpha = (x, y)$ is in the commutator subgroup $[\pi_1(M), \pi_1(M)]$, the preimage of α in M_1 consists of simple closed curves which are lifts of α . Since $M_2 \rightarrow M_1$ is a double covering, its restriction on each $\tilde{\alpha}_j$ is either one-to-one or two-to-one. The claim follows from this.

Now, from Corollary 4.7, it follows that $\lambda(N_2, \psi_2)$ is a linear combination of terms of the form $[\lambda_r(A, \sqrt{-1}^s)]$ where $r \in \{1, 2\}$ and $s \in \{0, 1, 2, 3\}$. By a straightforward computation using Lemma 5.3, we immediately obtain the following, which we state as a lemma for later use.

Lemma 5.4. *For the Seifert matrix A of our K_a ,*

$$\begin{aligned} \text{dis}[\lambda_1(A, \pm 1)] &= 1, \\ \text{dis}[\lambda_1(A, \pm\sqrt{-1})] &= 2a^2 + 1, \\ \text{dis}[\lambda_2(A, \pm 1)] &= 1, \\ \text{dis}[\lambda_2(A, \pm\sqrt{-1})] &= 2a^4 + 4a^2 + 1 \quad \text{in } \mathbb{Q}^\times / \{z\bar{z} \mid z \in \mathbb{Q}(\sqrt{-1})^\times\}. \end{aligned}$$

This completes the proof of Proposition 5.2(1). □

5.2. *Combinatorial computation of intersection form defects*

In this subsection, we show Proposition 5.2(2) by computing the invariant for a specific tower and character. In order to compute the invariant using Corollary 4.7, we need to understand explicitly the behavior of the pre-image of the infection curve $\alpha \subset M$. As a general technique for this, we will consider a p -tower map $X \rightarrow M$ of a 2-complex X , from which the pre-images of the infection curve α can be read off algorithmically. This provides a combinatorial method to compute the intersection form defect invariants. Since we will use the same technique again in later sections, we illustrate how this method proceeds in detail.

For convenience, for a 1-complex X and a group G , we will describe a map $\pi_1(X) \rightarrow G$ as an assignment of elements in G to 1-cells of X ; such an assignment defines (the homotopy class of) a map $X \rightarrow K(G, 1)$ sending the 0-skeleton of X to a basepoint and sending 1-cells of X to paths in $K(G, 1)$ representing the associated elements in G .

We construct the complex X and a tower of height two with deck transformation groups $\Gamma_0 = \mathbb{Z}_{r_1} \oplus \mathbb{Z}_{r_2}$ and $\Gamma_1 = \mathbb{Z}_2$ as follows:

- (0) We start with $S^1 \vee S^1$, which is a 1-complex with one 0-cell $*$ and two 1-cells that we denote by c_1 and c_2 . Let $X = X_0$ be the complex obtained by attaching two 2-cells to $S^1 \vee S^1$ along c_1^2 and c_2^2 respectively.
- (1) Let X_1 be the universal abelian cover of X_0 . Then X_1 is the union of $r_1 + r_2$ 2-disks which are lifts of the 2-cells of X_0 , and has the homotopy type of its 1-dimensional subcomplex (which is indeed a strong deformation retract of X_1), as illustrated in Figure 5; \tilde{c}_1, \tilde{c}_2 are the lifts of c_1 and c_2 based at the basepoint $* \in X_2$.
- (2) Assigning $1 \in \Gamma_1 = \mathbb{Z}_2$ to the 1-cell denoted by e_1 in Figure 5 and $0 \in \Gamma_1$ to all the other 1-cells, we obtain a map $\pi_1(X) \rightarrow \Gamma_1 = \mathbb{Z}_2$ which gives rise to a double cover X_2 of X_1 . Then X_2 has the homotopy type of the 1-complex (which is a strong deformation retract of X_2) illustrated in Figure 5; $*$ is the basepoint.

Finally, assigning $1 \in \Gamma_2$ to the 1-cell denoted by e_2 in Figure 5 and $0 \in \Gamma_2$ to the other 1-cells, we define a map $\phi_2 : \pi_1(X_2) \rightarrow \Gamma_2 = \mathbb{Z}_4$.

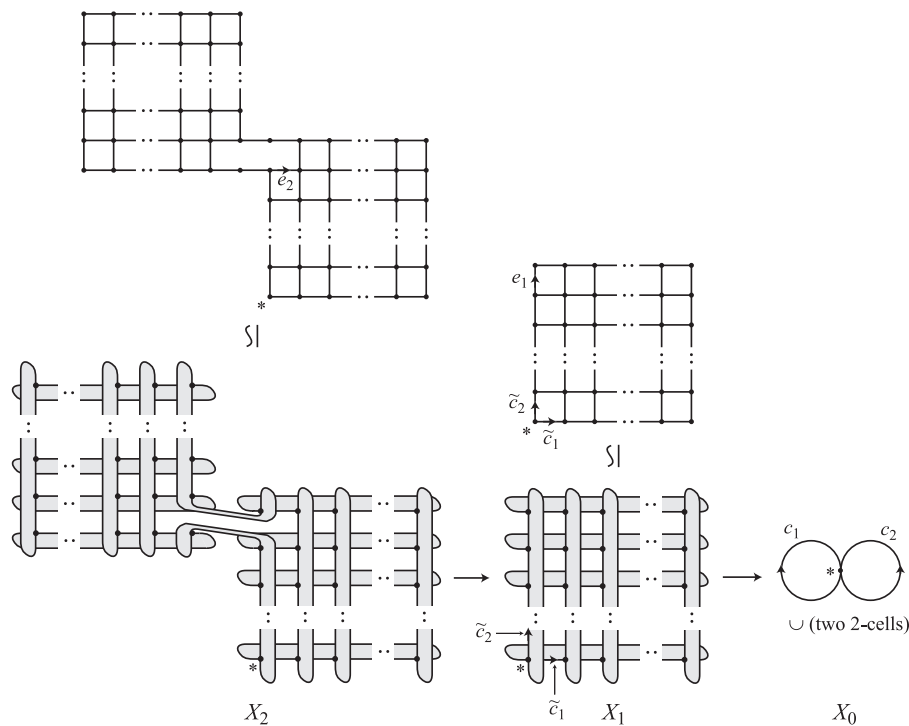


Fig. 5.

Obviously there is a map $X \rightarrow M$ which induces an isomorphism $\pi_1(X) \rightarrow \pi_1(M)$ sending $[c_1]$ and $[c_2]$ to x and y , respectively. $X \rightarrow M$ is a p -tower map for any prime p since $\pi_1(X) \cong \pi_1(M)$ (or alternatively by Lemma 3.7). Therefore, there is a unique 2-tower $M_2 \rightarrow M_1 \rightarrow M_0 = M$ which gives rise to our tower of X via pullback. Since N is obtained by infection from M , there is a map $N \rightarrow M$ which is a p -tower map for any prime p , by Proposition 4.8. Pullback via $N \rightarrow M$ gives rise to an associated 2-tower $N_2 \rightarrow N_1 \rightarrow N_0 = N$. Also, $\phi_2: \pi_1(X_2) \rightarrow \Gamma_2$ gives rise to $\psi_2: \pi_1(N_2) \rightarrow \Gamma_2$.

Now we compute $\lambda(N_2, \psi_2)$, using Corollary 4.7. Recall that the infection is performed along the curve α in M . Let $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots \subset M_2$ be the components of the pre-image of α . To apply Corollary 4.7 we need to compute the degree r_j of the covering $\tilde{\alpha}_i \rightarrow \alpha$, and $\phi_2([\tilde{\alpha}_j]) \in \Gamma_2 = \mathbb{Z}_4$.

We can read off the necessary data r_j and $\phi_2([\alpha_j])$ algorithmically from the 1-complexes in Figure 5. As a general case, suppose that $p: \tilde{X} \rightarrow X$ is a finite covering map and α is a loop in X based at $* \in X$. For a path γ in X based at $*$, we denote by $\tilde{\gamma}_v$ the lift of γ in \tilde{X} based at $v \in V = p^{-1}(*)$. We construct a collection S of loops in \tilde{X} as follows: initially let S be the empty set and mark all $v \in V$ as “white”. While there is a “white” v in V , we repeat the following: as a new element, insert into S the loop $(\tilde{\alpha}^r)_v$ where r is the minimal positive integer such that $(\tilde{\alpha}^r)_v$ is a loop, and mark the endpoints of $(\tilde{\alpha}^k)_v$ as “black” for all $1 \leq k \leq r$. We denote the result by the following notation.

Definition 5.5. We define $\mathcal{L}(\alpha, \tilde{X}|X)$ to be the collection S constructed above.

We apply the above algorithm to our cover $X_2 \rightarrow X_0$. As an abuse of notation, let $\alpha = (c_1, c_2) = c_1 c_2 c_1^{-1} c_2^{-1}$ be the loop in X_0 which represents the class $[\alpha] \in \pi_1(M_0) = \pi_1(X_0)$ of the infection curve, and write $\mathcal{L}(\alpha, X_2|X_0) = \{\tilde{\alpha}_j\}$. For each $\tilde{\alpha}_j$, r_j is the integer such that $\tilde{\alpha}_j$ is a lift of α^{r_j} .

As examples, in Figure 6 we illustrate four elements of the collection $\mathcal{L}(\alpha, X_2|X_0)$, say $\tilde{\alpha}_1, \dots, \tilde{\alpha}_4$, as curves in (the 1-complex with the homotopy type of) X_2 . The black dot(s) on each $\tilde{\alpha}_j$ represents the basepoint(s) giving $\tilde{\alpha}_j$. Note that $r_1 = 2$ and $r_2 = r_3 = r_4 = 1$.

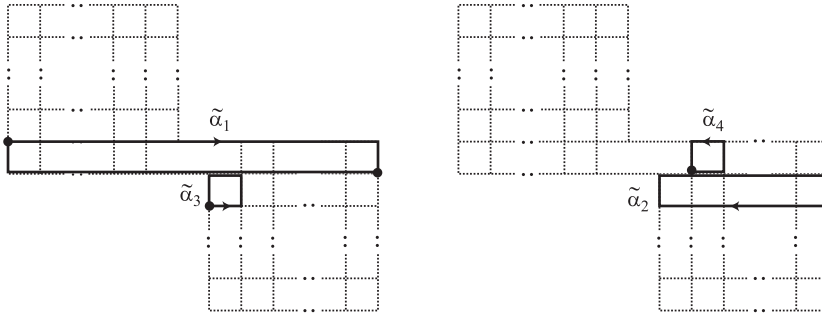


Fig. 6.

We observe the following: among the loops in $\mathcal{L}(\alpha, X_2|X_0)$, only $\tilde{\alpha}_1, \dots, \tilde{\alpha}_4$ pass through the 1-cell e_2 . (In fact, for each 1-cell, exactly four loops in $\mathcal{L}(\alpha, X_2|X_0)$ pass through it.) From our definition of ϕ_2 , it follows immediately that $\tilde{\alpha}_1, \dots, \tilde{\alpha}_4$ are sent by ϕ_2 respectively to $3, 1, 1, 3 \in \mathbb{Z}_4$, and all other loops in $\mathcal{L}(\alpha, X_2|X_0)$ are in the kernel of ϕ_2 .

Now we can compute the intersection form defect invariant from the data obtained above. Let A be a Seifert matrix of K . Then, by Corollary 4.7, we have

$$\begin{aligned} \lambda(N_2, \psi_2) &= [\lambda_2(A, -\sqrt{-1})] - [\lambda_2(A, 1)] + [\lambda_1(A, -\sqrt{-1})] - [\lambda_1(A, 1)] \\ &\quad + 2([\lambda_1(A, \sqrt{-1})] - [\lambda_1(A, 1)]) \in L^0(\mathbb{Q}(\sqrt{-1})). \end{aligned}$$

From the discriminant computation in Lemma 5.4, it follows that

$$\text{dis } \lambda(N_2, \psi_2) = (2a^2 + 1)(2a^4 + 4a^2 + 1).$$

This proves Proposition 5.2(2).

5.3. Realization of independent discriminants

Recall that our Σ_i is M infected by the knot K_{a_i} . By the invariance of the intersection form defects under homology cobordism (Theorem 3.8) and by the computation done in Proposition 5.2, if the a_i are chosen in such a way that

$$(2a_i^2 + 1)(2a_i^4 + 4a_i^2 + 1) \neq (2a_j^2 + 1)^{n_1} (2a_j^4 + 4a_j^2 + 1)^{n_2} \quad \text{in } \mathbb{Q}^\times / \{z\bar{z} \mid z \in \mathbb{Q}(\sqrt{-1})\}$$

for any $i \neq j$ and any n_1, n_2 , then Σ_i is not (\mathbb{Z}_2) -homology cobordant to Σ_j for any $i \neq j$. This subsection is devoted to the proof of the existence of such a sequence $\{a_i\}$.

In order to distinguish the elements in \mathbb{Q}^\times modulo norms, we appeal to some number theory. Recall from Section 4 that for each prime p the norm residue symbol induces a homomorphism

$$(-, D)_p: \mathbb{Q}^\times / \{z\bar{z} \mid z \in \mathbb{Q}(\sqrt{-1})\} \rightarrow \{\pm 1\}$$

where $D = -4$. Since $-4 = 2^2(-1)$, we may assume that $D = -1$. In the following proposition, we will construct a sequence $\{a_i\}$ together with a sequence of primes $\{p_i\}$ which are “dual” to the values of the invariants of the Σ_i in the following sense:

$$\begin{aligned} ((2a_i^2 + 1)(2a_i^4 + 4a_i^2 + 1), -1)_{p_i} &= -1 \quad \text{for any } i, \\ ((2a_j^2 + 1)^{n_1}(2a_j^4 + 4a_j^2 + 1)^{n_2}, -1)_{p_i} &= 1 \quad \text{for any } i \neq j \text{ and } n_1, n_2. \end{aligned}$$

The existence of such p_i completes the proof that the manifolds Σ_i obtained from the a_i (including Σ_0) are not (\mathbb{Z}_2) -homology cobordant to each other.

Proposition 5.6. *There are sequences a_1, a_2, \dots of positive integers and p_1, p_2, \dots of primes such that*

- (1) $(2a_i^2 + 1, -1)_{p_i} = -1$ and $(2a_i^4 + 4a_i^2 + 1, -1)_{p_i} = 1$.
- (2) $(2a_j^2 + 1, -1)_{p_i} = 1$ and $(2a_j^4 + 4a_j^2 + 1, -1)_{p_i} = 1$ for $i \neq j$.

Proof. Note that for an odd prime p , by Lemma 4.9,

$$\begin{aligned} (x, -1)_p &= \left((-1)^{v_p(x)v_p(-1)} \frac{(-1)^{v_p(x)}}{x^{v_p(-1)}} \right)^{(p-1)/2} = (-1)^{v_p(x) \cdot (p-1)/2} \\ &= \begin{cases} -1 & \text{if } v_p(x) \text{ is odd and } p \equiv -1 \pmod{4}, \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{*}$$

We will inductively choose a_n and a prime factor p_n of $2a_n^2 + 1$ in such a way that (1) and (2) are satisfied whenever $i, j \leq n$. Let $a_1 = 1$ and $p_1 = 3$. From the formula (*) it follows that a_1 and p_1 satisfy (1).

Suppose a_1, \dots, a_{n-1} and p_1, \dots, p_{n-1} have been chosen. To choose a_n , we need the following:

Lemma 5.7. *For any odd integer q , there is an odd multiple a of q such that $2a^2 + 1$ is not a square.*

We postpone the proof of Lemma 5.7 and continue the proof of Proposition 5.6. By Lemma 5.7, there is a positive odd multiple a_n of

$$q = \prod_{j < n} (2a_j^2 + 1)(2a_j^4 + 4a_j^2 + 1)$$

such that $2a_n^2 + 1$ is not a square; then there is a prime factor p_n of $2a_n^2 + 1$ such that $v_{p_n}(2a_n^2 + 1)$ is odd. We may assume that $p_n \equiv -1 \pmod{4}$; for, if $p \equiv 1 \pmod{4}$ for all

prime p such that $v_p(2a_n^2 + 1)$ is odd, then $2a_n^2 + 1 \equiv 1 \pmod{4}$, but this is impossible since a_n is odd.

From the formula (*), it follows that $(2a_n^2 + 1, -1)_{p_n} = -1$. Also, $v_{p_n}(2a_n^4 + 4a_n^2 + 1) = 0$ since $2a_n^2 + 1$ and $2a_n^4 + 4a_n^2 + 1$ are relatively prime. So by (*) we have $(2a_n^4 + 4a_n^2 + 1, -1)_{p_n} = 1$.

Suppose $j < i$. Then p_j divides neither $2a_i^2 + 1$ nor $2a_i^4 + 4a_i^2 + 1$ since p_j divides a_i . From (*), it follows that

$$(2a_i^2 + 1, -1)_{p_j} = 1 = (2a_i^4 + 4a_i^2 + 1, -1)_{p_j}.$$

If p_i divides $2a_j^2 + 1$, then p_i divides a_j by our choice of a_j . However, since p_i is a prime factor of $2a_i^2 + 1$, this implies $p_i = 1$, which is a contradiction. So $v_{p_i}(2a_j^2 + 1) = 0$. Similarly $v_{p_i}(2a_j^4 + 4a_j^2 + 1) = 0$. By (*), it follows that

$$(2a_j^2 + 1, -1)_{p_i} = 1 = (2a_j^4 + 4a_j^2 + 1, -1)_{p_i}. \quad \square$$

Proof of Lemma 5.7. We may assume that q is positive. Consider a Diophantine equation $x^2 = 2y^2 + 1$. From the theory of Pell's equation and continued fractions (for example, refer to [27]), it follows that all positive solutions (x_n, y_n) are given by the following recurrence relation: $x_1 = 3, y_1 = 2$, and

$$x_{n+1} = 3x_n + 4y_n, \quad y_{n+1} = 2x_n + 3y_n.$$

Choose any positive odd integer k . If $kq \neq y_n$ for all n , then for $a = kq, 2a^2 + 1$ is not a square. Suppose $kq = y_n$ for some n . Then $x_n > y_n > q$. So $y_{n+1} > 5y_n$, and $(k + 2)q = y_n + 2q < 3y_n < y_{n+1}$. Since $\{y_n\}$ is strictly increasing, it follows that $(k + 2)q \neq y_i$ for all i , that is, for $a = (k + 2)q, 2a^2 + 1$ is not a square. \square

5.4. Vanishing of signature invariants

In this subsection we prove that the multisignatures, η -invariants, and Harvey's L^2 -invariants vanish for our Σ_i .

Recall that the η -invariant $\tilde{\eta}(M, \phi)$ is defined for a closed 3-manifold M endowed with a finite-dimensional unitary representation $\phi: \pi_1(M) \rightarrow U(k)$ as a signature defect, as in [1, 2]. We need the following formula:

Lemma 5.8. *Suppose M is a closed 3-manifold and $\phi: \pi_1(M) \rightarrow U(k)$ is a unitary representation. Let K be a knot with zero-surgery manifold M_K , and N be the manifold obtained from M by infection by K along a simple closed curve $\alpha \subset M$. Let ψ be the composition*

$$\psi: \pi_1(N) \rightarrow \pi_1(M) \xrightarrow{\phi} U(k),$$

where the first map is induced by the p -tower map $N \rightarrow M$ given by Proposition 4.8. Let ϕ_K be the composition

$$\pi_1(M_K) \rightarrow H_1(M_K) = \mathbb{Z} \rightarrow U(k),$$

where the last map sends a (positive) meridian of K to $\phi([\alpha]) \in U(k)$. Then

$$\tilde{\eta}(N, \psi) = \tilde{\eta}(M, \phi) + \tilde{\eta}(M_K, \phi_K).$$

Since Lemma 5.8 can be proved using a standard folklore argument using the Atiyah–Patodi–Singer index theorem, we just give a sketch of a proof: by Lemma 4.4, there is a 4-manifold W_K bounded by M_K over \mathbb{Z} . View M_K as the exterior of K filled in with a solid torus T . Attaching W_K and $M \times [0, 1]$ along T and a regular neighborhood of α in $M = M \times 1$, we obtain a 4-manifold V with $\partial V = (-M) \cup N$ over $U(k)$. Using the fact that $T \hookrightarrow W_K$ has a left homotopy inverse, one can show that the signature defect of V is the sum of those of $M \times [0, 1]$ and W_K , which are equal to zero and $\tilde{\eta}(M_K, \phi_K)$, respectively. By the index theorem, this is exactly $\tilde{\eta}(N, \psi) - \tilde{\eta}(M, \phi)$.

In this subsection, \widehat{G} denotes the algebraic closure of a group G with respect to \mathbb{Z} -coefficients in the sense of [8], or Levine’s algebraic closure [35]. We denote by $p_G: G \rightarrow \widehat{G}$ the natural map into the algebraic closure. Following [36], we consider invariants of M of the form $\tilde{\eta}(M, \theta \circ p_{\pi_1(M)})$, where θ is a representation of $\widehat{\pi_1(M)}$.

Lemma 5.9. *Suppose M is a 3-manifold such that $\tilde{\eta}(M, \theta \circ p_{\pi_1(M)}) = 0$ for any $\theta: \widehat{\pi_1(M)} \rightarrow U(k)$. Then all multisignatures of M vanish.*

Proof. Since the multisignatures associated to \mathbb{Z}_d -valued characters are known to be equivalent to the η -invariants associated to representations ρ of $\pi_1(M)$ factoring through \mathbb{Z}_d , it suffices to check that every such representation ρ factors through $\widehat{\pi_1(M)}$. Since ρ factors through $H_1(M)$, the proof is finished by applying the following property of the algebraic closure to the case of $G = \pi_1(M)$: for any group G , the map $p_G: G \rightarrow \widehat{G}$ induces an isomorphism on $H_1(-)$. \square

Now suppose $M = L_1 \# L_2$ is our seed manifold used in the previous subsections, and N is obtained from M by infection along α by a knot K . Recall that L_i is a lens space with vanishing multisignatures.

Lemma 5.10. *If the Alexander polynomial of K has no zero in the unit circle, then for any $\theta: \widehat{\pi_1(N)} \rightarrow U(k)$, $\tilde{\eta}(N, \theta \circ p_{\pi_1(N)})$ vanishes.*

Proof. By (the proof of) Proposition 4.8, the p -tower map $N \rightarrow M$ is 2-connected. It follows that the induced map $\widehat{\pi_1(N)} \rightarrow \widehat{\pi_1(M)}$ is an isomorphism, by results of [35, 8]. (To obtain this when \widehat{G} designates Levine’s algebraic closure, we need an additional condition that $\pi_1(N) \rightarrow \pi_1(M)$ is normally surjective, which can be verified easily by a van Kampen argument.) So, for any $\theta: \widehat{\pi_1(N)} \rightarrow U(k)$, there is $\theta': \widehat{\pi_1(M)} \rightarrow U(k)$ making the following diagram commute:

$$\begin{array}{ccccc}
 \pi_1(N) & \xrightarrow{p_{\pi_1(N)}} & \widehat{\pi_1(N)} & \xrightarrow{\theta} & U(k) \\
 \downarrow & & \downarrow & \nearrow \theta' & \\
 \pi_1(M) & \xrightarrow{p_{\pi_1(M)}} & \widehat{\pi_1(M)} & &
 \end{array}$$

So, $\psi = \theta \circ p_{\pi_1(N)}$ is induced by $\phi = \theta' \circ p_{\pi_1(M)}$ as in Lemma 5.8. It follows that $\tilde{\eta}(N, \psi)$ is the sum of $\tilde{\eta}(M, \phi)$ and $\tilde{\eta}(M_K, \phi_K)$ for some ϕ_K . We will show that both

$\tilde{\eta}(M, \phi)$ and $\tilde{\eta}(M_K, \phi_K)$ are zero. Since $M = L_1 \# L_2$,

$$\tilde{\eta}(M, \phi) = \tilde{\eta}(L_1, \phi_1) + \tilde{\eta}(L_2, \phi_2)$$

for some ϕ_i . Since $\pi_1(L_i)$ is a (finite) cyclic group, ϕ_i is a sum of 1-dimensional representations. So we may assume that ϕ_i is 1-dimensional, that is, $\tilde{\eta}(L_i, \phi_i)$ is a multisignature, which vanishes by our choice of L_i . Let A be a Seifert matrix of K . It is known that $\tilde{\eta}(M_K, \phi_K)$ is determined by $\{\text{sign}(1 - \omega)A + (1 - \bar{\omega})A^T\}_{\omega \in S^1}$ (e.g., see [23]). Since $\Delta_A(t)$ has no zero on S^1 , $\text{sign}(1 - \omega)A + (1 - \bar{\omega})A^T = 0$ whenever $\omega \in S^1$, and therefore, $\tilde{\eta}(M_K, \rho_K)$ vanishes. \square

From Lemmas 5.10 and 5.9, it follows that the manifolds Σ_i constructed in Section 5.1 have vanishing multisignatures and Levine's $\tilde{\eta}$ -invariants, as claimed in Theorem 1.3(2) and (3).

For each integer $n \geq 0$, Harvey defined the L^2 -signature invariants $\rho_n(N)$ [26]. It is the von Neumann–Cheeger–Gromov L^2 -signature defect associated to the natural map $\pi_1(N) \rightarrow \pi_1(N)/\pi_1(N)_H^{(n)}$, where $G_H^{(n)}$ denotes the n th term of the torsion-free derived series [15] for a group G .

Lemma 5.11. *If N is a rational homology sphere, then $\rho_n(N) = 0$ for any $n \geq 0$.*

Proof. From the definition of the torsion-free derived series in [15], it follows that $\pi_1(N)_H^{(n)} = \pi_1(N)$ for all n , since the first Betti number $b_1(N)$ is zero. Being the signature defect for untwisted coefficients, $\rho_n(N)$ vanishes. \square

From this it follows that the manifolds Σ_i have vanishing Harvey's invariants, as claimed in Theorem 1.3(4). This completes the proof of Theorem 1.3.

6. Intersection form defects of links

Two links L and L' in S^3 are said to be (topologically) *concordant* if there is a locally flat s -cobordism C embedded in $S^3 \times [0, 1]$ from $L \times \{0\} \subset S^3 \times \{0\}$ to $L' \times \{1\} \subset S^3 \times \{1\}$; C is called a *concordance*. (When C is a smooth submanifold in $S^3 \times [0, 1]$, L and L' are said to be *smoothly concordant*.) A link concordant to a trivial link is called a *slice link*. Or equivalently, a slice link is a link which bounds a disjoint union of locally flat 2-disks in D^4 , regarding S^3 as the boundary of D^4 .

For a link L in S^3 , the closed 3-manifold obtained from S^3 by performing surgery along the zero-linking framing of each component of L is called the *zero-surgery manifold*. The following fact is well-known:

Lemma 6.1. *If two links are concordant, their zero-surgery manifolds are homology cobordant.*

By Lemma 6.1, one can apply Theorem 3.8 to extract link concordance invariants from the intersection form defects of (p -towers of) zero-surgery manifolds. In particular, for a slice link, we obtain the following vanishing theorem.

Theorem 6.2. *Suppose that L is a slice link with zero-surgery manifold M ,*

$$M_n \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 = M$$

is a p -tower, and $\phi_n : \pi_1(M_n) \rightarrow \mathbb{Z}_d$ is a character with $d = p^a$. Then

- (1) (M_n, ϕ_n) is trivial in $\Omega_3^{\text{top}}(B\mathbb{Z}_d)$,
- (2) $\lambda(M_n, \phi_n) = 0$ in $L^0(\mathbb{Q}(\zeta_d))$.

Proof. By Lemma 6.1 and Theorem 3.8, we may assume that L is a trivial link. Then M is the connected sum of m disjoint copies of $S^1 \times S^2$, where m is the number of components of L . Therefore, being a cover of M , M_n is a connected sum of disjoint copies of $S^1 \times S^2$. Appealing to Lemma 4.2 completes the proof. \square

6.1. p -towers of surgery manifolds of \widehat{F} -links

In this subsection we will show that there are many highly nontrivial p -towers of the zero-surgery manifold for a large class of links in S^3 . We start with a description of the class of links we think of.

In this section, we denote by \widehat{G} the algebraic closure of a group G with respect to $\mathbb{Z}_{(p)}$ -coefficients, in the sense of [8]. Suppose L is an m -component link with zero surgery manifold M , and let $X = \bigvee^m S^1$, the wedge of m circles. We say that a map of X into $S^3 - L$ or M is a *meridian map* if the image of the i th circle is an i th meridian of L . Let $F = \pi_1(X)$ be the free group of rank m , and $\pi = \pi_1(S^3 - L)$. We say that L is a $\mathbb{Z}_{(p)}$ -coefficient \widehat{F} -link if there is a meridian map $X \rightarrow S^3 - L$ inducing an isomorphism $\widehat{F} \rightarrow \widehat{\pi}$ and the preferred longitudes of L are in the kernel of $\pi \rightarrow \widehat{\pi}$. We note that this is a $\mathbb{Z}_{(p)}$ -analogue of the notion of an \widehat{F} -link due to Levine [35]; the definition of an \widehat{F} -link in [35] is identical with ours except that Levine’s algebraic closure is used in place of our \widehat{G} . Henceforth, an \widehat{F} -link always designates a $\mathbb{Z}_{(p)}$ -coefficient \widehat{F} -link in our sense. (An \widehat{F} -link in the sense of [35] is a $\mathbb{Z}_{(p)}$ -coefficient \widehat{F} -link; it might be an interesting question whether the converse is true, i.e., whether the two notions are equivalent.)

By arguments of [35], we have the following facts: the property that $\widehat{F} \rightarrow \widehat{\pi}$ is an isomorphism is independent of the choice of a meridian map, and a link concordant to an \widehat{F} -link is an \widehat{F} -link. We remark that it is a long-standing conjecture that any link with vanishing Milnor $\bar{\mu}$ -invariants is an \widehat{F} -link (in the sense of [35]).

Proposition 6.3. *Suppose L is an \widehat{F} -link with zero-surgery manifold M . Then any meridian map $X = \bigvee^m S^1 \rightarrow M$ is a p -tower map, in the sense of Definition 3.4.*

Proof. Since L is an \widehat{F} -link, a meridian map $X \rightarrow S^3 - L$ is a p -tower map. So it suffices to show the inclusion $S^3 - L \rightarrow M$ is a p -tower map. To prove this, we need the following two properties of the algebraic closure functor $E(G) = \widehat{G}$:

- (1) The algebraic closure functor preserves direct limits, that is, $E(\varinjlim G_i)$ is the direct limit of the system $\{E(G_i)\}$ in the full subcategory of algebraically closed groups.
- (2) The algebraic closure functor is an idempotent, that is, $E(E(G)) = E(G)$.

(1) holds since $E: \{\text{groups}\} \rightarrow \{\text{algebraically closed groups}\}$ is a left adjoint of the inclusion functor $\{\text{algebraically closed groups}\} \rightarrow \{\text{groups}\}$. For a proof of (2), refer to [35].

Let $\pi = \pi_1(S^3 - L)$, $G = \pi_1(M)$, F be a free group of rank m , and $\ell: F \rightarrow \pi$ be a map sending the i th generator of F to an i th preferred longitude of L . (Here m is the number of components of L as before.) Then $G = \text{Coker}\{\ell\}$. By our hypothesis that L is an \widehat{F} -link, the composition $F \xrightarrow{\ell} \pi \rightarrow \widehat{\pi}$ is the zero map. So it induces the zero map $\widehat{F} \rightarrow \widehat{\pi}$. By (2) above, it follows that $\widehat{\ell}: \widehat{F} \rightarrow \widehat{\pi}$ is the zero map. Therefore $\widehat{\pi} \cong \text{Coker}\{\widehat{\ell}\} \cong \widehat{G}$ by (1) above (recall that the cokernel is a direct limit). Now, by Proposition 3.9, the inclusion $S^3 - L \rightarrow M$ is a p -tower map. \square

Remark 6.4. An interesting question related to Proposition 6.3 is the following: if L is an m -component link with vanishing $\bar{\mu}$ -invariants, then is a meridian map of $\bigvee^m S^1$ into the surgery manifold of L a p -tower map? An affirmative answer may be viewed as an evidence supporting the conjecture that a link with vanishing $\bar{\mu}$ -invariant is an \widehat{F} -link.

As stated in the corollary below, it follows that the character groups of iterated p -covers of the surgery manifold of L are highly nontrivial, provided that L is an \widehat{F} -link which is not a knot.

Corollary 6.5. *If M is the zero-surgery manifold of an m -component \widehat{F} -link L and $M_n \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 = M$ is a p -tower consisting of connected covers $M_{i+1} \rightarrow M_i$ with deck transformation groups Γ_i , then for any abelian group Γ_n , we have $\text{Hom}(\pi_1(M_n), \Gamma_n) \cong (\Gamma_n)^{r_n}$, where the rank r_n is given by*

$$r_n = \left(\prod_{i=0}^{n-1} |\Gamma_i| \right) (m - 1) + 1.$$

Proof. By Proposition 6.3, we may assume that $M = \bigvee^m S^1$. Since M is a 1-complex, $H_1(M_n)$ is a free abelian group. Therefore it suffices to show that the first Betti number $b_1(M_n)$ is the number r_n given above. Let $d = |\Gamma_0| \cdots |\Gamma_{n-1}|$. Since M_n is a d -fold cover of M , the Euler characteristic $\chi(M_n) = 1 - b_1(M_n)$ can also be computed as follows:

$$\chi(M_n) = d \cdot \chi(M) = d(1 - m).$$

From this it follows that $b_1(M_n) = r_n$. \square

7. Computation for iterated Bing doubles

For a link L in S^3 , let $BD(L)$ be the (untwisted) Bing double of L . It is obtained from L as follows: Let V be an unknotted solid torus in S^3 , and L_{orbit} be the 2-component link in V illustrated in Figure 7. For a link L with m components, let h_i be a homeomorphism of V onto a tubular neighborhood of the i th component of L which sends a preferred longitude and meridian of V to those of the i th component of L respectively ($1 \leq i \leq m$). Then $BD(L) = \bigcup_{i=1}^m h_i(L_{\text{orbit}}) \subset S^3$.

We define the n th iterated Bing double $BD_n(L)$ of L inductively by $BD_0(L) = L$ and $BD_n(L) = BD_{n-1}(BD(L))$ for $n > 0$.

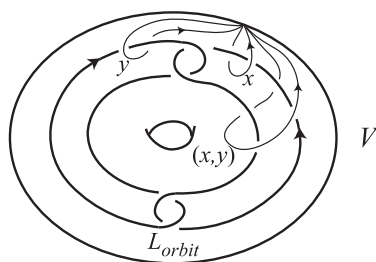


Fig. 7. The link L_{orbit} in an unknotted solid torus $V \subset S^3$.

For a knot K in S^3 , $BD_n(K)$ is obtained from a trivial link by infection as follows: Let α be a meridional curve of a tubular neighborhood U of a trivial knot O in S^3 , and take an iterated Bing double $BD_n = BD_n(O)$ contained in U . Then BD_n is a trivial link in S^3 , and by performing infection on BD_n by K along α , we obtain $BD_n(K)$. Figure 8 illustrates BD_n and α for $n = 2$.

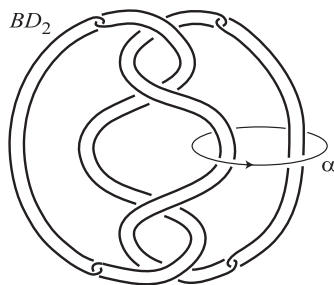


Fig. 8. The infection curve α for $n = 2$.

We denote the mirror image of a link L with reversed orientation by $-L$. Recall from the introduction that a link L is said to be *2-torsion* if a connected sum of L and itself is slice. Note that a connected sum of two links is defined by choosing a “disk basing” for each link in the sense of Lin and Habegger [25]. The defining condition of a 2-torsion link L should be understood as that $L\#L$ is slice for some choice of disk basings. (We remark that $L\#L$ may not be slice for some other disk basings even when L is 2-torsion.) If L is 2-torsion, then L is concordant to $-L$, and when L is a knot, the converse is also true. An amphichiral knot is 2-torsion.

Observe that $BD_n \cup \alpha$ is isotopic to $-(BD_n \cup \alpha)$; applying a π -rotation to Figure 8 about a horizontal axis, we obtain one from the other. From this, it follows that $BD_n(K)$ is isotopic to $-(BD(-K))$. As a consequence, we have

Lemma 7.1. *If K is amphichiral, then $BD_n(K)$ is isotopic to $-BD_n(K)$. If K is 2-torsion, then $BD_n(K)$ is 2-torsion.*

We prove that certain iterated Bing doubles are 2-torsion but not slice, as stated below:

Theorem 7.2. *Let $\{a_i\}$ be the sequence of integers given by Proposition 5.6 and K_a be the amphichiral knot shown in Figure 3. Then $BD_n(K_{a_i})$ is not slice for any i and n .*

Theorem 1.6 stated in the introduction is an immediate consequence of Theorem 7.2.

Remark 7.3. It is known that many known invariants vanish for (iterated) Bing doubles. (Refer to Cimasoni’s paper [12] for an excellent discussion on this.) Harvey’s invariant ρ_k [26] and Levine’s $\tilde{\eta}$ -invariants [36] detect some examples of non-slice Bing doubles; however, both invariant vanish for our $BD_n(K_{a_i})$, as explained below:

- (1) The 2-torsion property of $BD_n(K_{a_i})$ implies the vanishing of ρ_k ; since $\rho_k(L) = -\rho_k(-L)$ for any link L [26], we have $\rho_k(BD_n(K_{a_i})) = -\rho_k(-BD_n(K_{a_i})) = -\rho_k(BD_n(K_{a_i}))$, and so $\rho_k(BD_n(K_{a_i})) = 0$. Or alternatively, one may appeal to a formula for ρ_k of infected manifold: it is known that $\rho_k(BD_n(K)) = \rho_k(BD_n) + \epsilon\rho_0(K) = \epsilon\rho_0(K)$ for some $\epsilon \in \{0, 1\}$ [26], and $\rho_0(K)$ is equal to the integral of the Levine–Tristram signature function σ_K over the unit circle [19, 20]. So, if the integral of σ_K vanishes (in particular if K is torsion in the knot concordance group), then $\rho_k(BD_n(K))$ vanishes.
- (2) Using a similar argument (see also the proof of Lemma 5.10), it is shown that if $\sigma_K = 0$, or equivalently if K is torsion in the algebraic knot concordance group, then Levine’s invariant $\tilde{\eta}(M, \theta \circ p_{\pi_1(M)})$ of the zero-surgery manifold M of $BD_n(K)$ vanishes for any unitary representation θ of the algebraic closure of $\pi_1(M)$.

7.1. p -towers for iterated Bing doubles

The proof of Theorem 7.2 proceeds similarly to that of Theorem 1.3 discussed in Section 5. The outline is as follows: denote the zero-surgery manifolds of BD_n and $BD_n(K)$ by M and N , respectively, and let $X = \sqrt[2^n]{S^1}$. By Proposition 4.8 there is a p -tower map $N \rightarrow M$ since N is obtained from M by infection along α . Also, the meridian map $X \rightarrow M$ sending the i th circle to the i th meridian of BD_n is a p -tower map by Proposition 6.3 (or more directly, since it is a π_1 -isomorphism). We consider p -towers of M and N which are induced by a p -tower of X that will be constructed combinatorially. We show the nontriviality of the intersection form defects extracted from the p -tower of N by a computation using Corollary 4.7 as in Section 5.2. This proves Theorem 7.2.

Our construction of a p -tower of $X_0 = X$ is as follows. $p = 2$ will be used throughout this section. We define inductively covers $X_k \rightarrow X_{k-1}$ for $1 \leq k \leq n$, some 1-cells $c_i^{(k)}$ of X_k for $1 \leq i \leq 2^{n-k}$, and a map $\phi_k: \pi_1(X_k) \rightarrow \Gamma_k = (\mathbb{Z}_2)^{2^{n-k}}$. Initially, viewing X_0 as a 1-complex with one 0-cell $*$ and 2^n 1-cells, let $c_i^{(0)}$ be the i th (oriented) 1-cell of X_0 and define $\phi_0: \pi_1(X_0) \rightarrow \Gamma_0 = (\mathbb{Z}_2)^{2^n}$ by assigning to each 1-cell $c_i^{(0)}$ the i th standard basis element $e_i \in \Gamma_0$. Suppose $k < n$ and $X_k, c_i^{(k)}$, and $\phi_k: \pi_1(X_k) \rightarrow \Gamma_k$ have been defined. We define X_{k+1} to be the Γ_k -cover of X_k determined by ϕ_k , and choose a basepoint $*$ $\in X_{k+1}$ from the preimage of $*$ $\in X_k$. Let $c_i^{(k+1)} \subset X_{k+1}$ be the lift of $c_{2i-1}^{(k)} \subset X_k$ based at $*$ $\in X_{k+1}$. Then $\phi_{k+1}: \pi_1(X_{k+1}) \rightarrow \Gamma_{k+1} = (\mathbb{Z}_2)^{2^{n-k-1}}$ is defined to be the map induced by the assignment $c_i^{(k+1)} \mapsto e_i$, (other cells) $\mapsto 0$. Finally, let X_{n+1} be the double cover of X_n determined by $\phi_n: \pi_1(X_n) \rightarrow \Gamma_n = \mathbb{Z}_2$.

As an abuse of notation, we denote by α a loop in X_0 representing the class of $[\alpha] \in \pi_1(M_0) = \pi_1(X_0)$ of the infection curve $\alpha \subset M_0$. Then α can be described as an iterated commutator of the loops $c_i^{(0)}$. In order to give an explicit commutator expression, we define inductively loops $x_i^{(k)}$ in X_0 based at $*$ for $0 \leq k \leq n$ and $1 \leq i \leq 2^{n-k}$ as follows: $x_i^{(0)} = c_i^{(0)}$ and

$$x_i^{(k+1)} = (x_{2i-1}^{(k)}, x_{2i}^{(k)}) = x_{2i-1}^{(k)} x_{2i}^{(k)} (x_{2i-1}^{(k)})^{-1} (x_{2i}^{(k)})^{-1}.$$

Then $\alpha = x_1^{(n)}$. For, in Figure 7, the meridian of the solid torus V containing L_{orbit} is the commutator of the meridians of the two components of L_{orbit} , and applying this relation inductively, it follows that $\alpha = x_1^{(n)}$. Note that $[x_i^{(k)}] \in \pi_1(X_0)^{(k)} \subset \pi_1(X_k) \subset \pi_1(X_0)$. From this it follows that any lift of α in X_n is a loop.

In order to compute intersection form defects, we need to investigate the collection $\mathcal{L}(\alpha, X_{n+1}|X_0)$ described in Definition 5.5, as we did in Section 5.2. Since any lift of α in X_n is a loop and since X_{n+1} is a double cover of X_n , some lifts of α in X_{n+1} may not be loops but any lift of α^2 in X_{n+1} is a loop. In other words, if we write $\mathcal{L}(\alpha, X_{n+1}|X_0) = \{\tilde{\alpha}_j\}$, then each $\tilde{\alpha}_j$ is a lift of α^{r_j} in X_{n+1} for some $r_j \in \{1, 2\}$.

The essential property of our 2-tower is the following:

Lemma 7.4. *For any $d > 0$ and $0 \leq s < d$, there is a character $\varphi_{n+1}: \pi_1(X_{n+1}) \rightarrow \mathbb{Z}_d$ such that all the $\tilde{\alpha}_j \in \mathcal{L}(\alpha, X_{n+1}|X)$ are in the kernel of φ_{n+1} except two, say $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, with the following properties: $r_1 = 2, r_2 = 1$, and $\tilde{\alpha}_1, \tilde{\alpha}_2$ are sent by φ_{n+1} respectively to $s, -s \in \mathbb{Z}_d$.*

Before proving Lemma 7.4, we give a proof of Theorem 7.2 using our invariants associated to the characters given by Lemma 7.4.

Proof of Theorem 7.2. As before, let M and N be the zero-surgery manifolds of BD_n and $BD_n(K_{a_i})$, and $X \rightarrow M$ be the meridian map. Let

$$M_{n+1} \rightarrow M_n \rightarrow \dots \rightarrow M_0 = M \quad \text{and} \quad N_{n+1} \rightarrow N_n \rightarrow \dots \rightarrow N_0 = N$$

be the 2-towers which correspond, via the 2-tower maps $X \rightarrow M \leftarrow N$, to the 2-tower of X constructed above.

Let $\varphi_{n+1}: \pi_1(X_{n+1}) \rightarrow \mathbb{Z}_4$ be the map given by Lemma 7.4 applied to the case of $d = 4$ and $s = 1$. Let $\psi_{n+1}: \pi_1(N_{n+1}) \rightarrow \mathbb{Z}_4$ be the map induced by φ_{n+1} . By Lemma 7.4(1) and Corollary 4.7 we have

$$\lambda(N_{n+1}, \psi_{n+1}) = [\lambda_2(A, \sqrt{-1})] + [\lambda_1(A, -\sqrt{-1})] \in L^0(\mathbb{Q}(\sqrt{-1})).$$

where A is a Seifert matrix of K_{a_i} . Applying the discriminant map

$$\text{dis}: L^0(\mathbb{Q}(\sqrt{-1})) \rightarrow \mathbb{Q}^\times / \{z \cdot \bar{z} \mid z \in \mathbb{Q}(\sqrt{-1})^\times\}$$

defined in Section 4.5, we obtain

$$\text{dis } \lambda(N_{n+1}, \psi_{n+1}) = (2a_i^2 + 1)(2a_i^4 + 4a_i^2 + 1)$$

by Lemma 5.4. From the norm residue symbol computation in Proposition 5.6, it follows that $\text{dis } \lambda(N_{n+1}, \psi_{n+1})$ is nontrivial. Hence $BD_n(K_{a_i})$ is not slice by Theorem 6.2. \square

7.2. Lifts of the infection curve α

In this subsection we complete the proof of Theorem 7.2 by showing Lemma 7.4. In order to give a precise description of lifts of $x_i^{(k)}$ in X_k , we consider the following cut-paste construction of our tower. For $0 \leq k \leq n$, let Y_k be X_k with the 1-cells $c_i^{(k)}$ removed ($1 \leq i \leq 2^{n-k}$). For each $g \in \Gamma_k$, let $Y_k(g)$ be a copy of Y_k . We obtain X_{k+1} by taking the disjoint union $\bigcup_{g \in \Gamma_k} Y_k(g)$ and then attaching $2^{2(n-k)}$ 1-cells $c_i^{(k)}(g)$ which goes from (starting point of $c_i^{(k)} \in Y_k(g)$) to (endpoint of $c_i^{(k)} \in Y_k(g + e_i)$) for $g \in \Gamma_k$ and $1 \leq i \leq 2^{n-k}$. We regard $* \in Y_k \subset X_k$ as the basepoint of Y_k , and $* \in Y_k = Y_k(0) \subset X_{k+1}$ as the basepoint of X_{k+1} , where $0 \in \Gamma_k$ is the (additive) identity. Then $c_i^{(k+1)}$ is the 1-cell $c_{2i-1}^{(k)}(0) \subset X_{k+1}$. Figure 9 is a schematic diagram of Y_k and (part of) X_{k+1} .

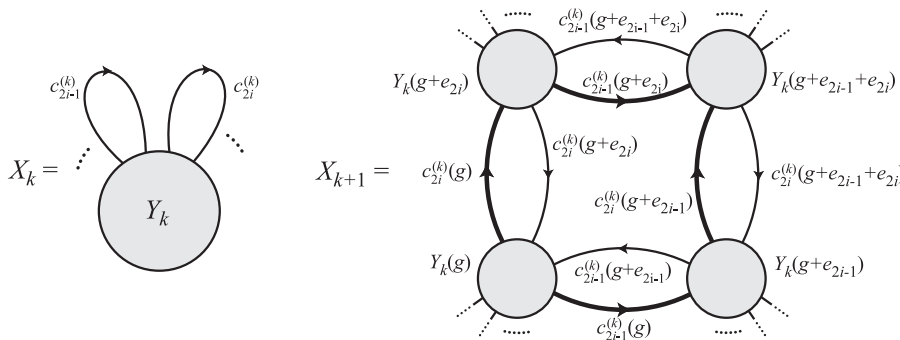


Fig. 9. A schematic diagram of Y_k and X_{k+1} .

Let \bar{X}_{k+1} be the 1-complex obtained by collapsing each $Y_k(g) \subset X_{k+1}$ to a point for $g \in \Gamma_k$. For a path γ in X_{k+1} , denote by $\bar{\gamma}$ its composition with $X_{k+1} \rightarrow \bar{X}_{k+1}$. In particular $\bar{c}_i^{(k+1)}(g)$ is the image of $c_i^{(k+1)}(g) \subset X_{k+1}$ in \bar{X}_{k+1} . Note that the map $\phi_{k+1} : \pi_1(X_{k+1}) \rightarrow \Gamma_{k+1}$ factors through $\pi_1(\bar{X}_{k+1})$.

Lemma 7.5. For a 0-cell v in X_{k+1} , let γ_v be the lift of the path $x_i^{(k+1)}$ in X_{k+1} based at v . Note that v lies in $Y_k(g)$ for some $g \in \Gamma_k$. Then the following holds:

- (1) For $j \neq i$, $\bar{\gamma}_v$ never meets (the interior of) $\bar{c}_j^{(k+1)} \subset \bar{X}_{k+1}$, and passes through $\bar{c}_i^{(k+1)}$ algebraically

$$\begin{Bmatrix} +1 \\ -1 \\ 0 \end{Bmatrix} \text{ times } \begin{Bmatrix} \text{if } v = * \in Y_k(0) \subset X_{k+1}, \\ \text{if } v = * \in Y_k(e_{2i}) \subset X_{k+1}, \\ \text{otherwise.} \end{Bmatrix}$$

- (2) If $v \neq * \in Y_k(g)$, then $\bar{\gamma}_v$ is null-homotopic (rel ∂) in \bar{X}_{k+1} . If $v = * \in Y_k(g)$, then $\bar{\gamma}_v$ is homotopic (rel ∂) to a 4-gon

$$\bar{c}_{2i-1}^{(k)}(g) \cdot \bar{c}_{2i}^{(k)}(g + e_{2i-1}) \cdot (\bar{c}_{2i-1}^{(k)}(g + e_{2i}))^{-1} \cdot (\bar{c}_{2i}^{(k)}(g))^{-1}$$

in \bar{X}_{k+1} . (See Figure 9, where the 4-gon is illustrated as bold edges.)

Proof. Denote (1) and (2) stated above by (1_{k+1}) and (2_{k+1}) . We show the lemma by proving the following implications: $(1_k) \Rightarrow (2_{k+1}) \Rightarrow (1_{k+1})$. Note that for $k + 1 = 0$, the initial condition (1_0) holds obviously. \square

Proof of $(2_{k+1}) \Rightarrow (1_{k+1})$. Indeed, the first statement of (1_{k+1}) is proved without using (2_{k+1}) . For, observe that $c_j^{(k+1)}$ is a lift of the 1-cell $c_\ell^{(0)} \subset X_0$ representing $x_\ell \in \pi_1(X_0)$, where $\ell = (j - 1) \cdot 2^{k+1} + 1$. If $i \neq j$, then since $x_i^{(k+1)}$ is a word in $x_{(i-1) \cdot 2^{k+1} + 1}, \dots, x_{i \cdot 2^{k+1}}, x_i^{(k+1)}$ does not contain $x_\ell^{\pm 1}$. Therefore any lift of $x_i^{(k+1)}$ in X_{k+1} never passes through $c_j^{(k+1)}$. From this the first conclusion of (1_{k+1}) follows.

To prove the second statement of (1_{k+1}) , suppose (2_{k+1}) holds and suppose $\tilde{\gamma}_v$ passes through $\tilde{c}_i^{(k+1)} = \tilde{c}_{2i-1}^{(k)}(0)$ algebraically nonzero times. Then by the first statement of (2_{k+1}) , v should be $* \in Y_k(g) \subset X_{k+1}$ for some $g \in \Gamma_{k-1}$. By the second statement of (2_{k+1}) , it follows that $g = 0$ or e_{2i} and in each case $\tilde{\gamma}_v$ passes through $\tilde{c}_{2i-1}^{(k)}(0)$ algebraically $+1$ and -1 times, respectively. \square

Proof of $(1_k) \Rightarrow (2_{k+1})$. Suppose (1_k) holds. Let a_ℓ be the lift of $x_\ell^{(k)}$ in X_k which is based at $v \in Y_k(g) = Y_k \subset X_k$. Note that a_ℓ is a loop in X_k . Since $x_i^{(k+1)} = x_{2i-1}^{(k)} x_{2i}^{(k)} (x_{2i-1}^{(k)})^{-1} (x_{2i}^{(k)})^{-1}$, γ_v is obtained by concatenating some lifts of a_{2i-1} , a_{2i} , a_{2i-1}^{-1} , and a_{2i}^{-1} in X_{k+1} .

We claim that if \bar{a}_ℓ passes through $\bar{c}_\ell^{(k)}$ algebraically 0 times in \bar{X}_k , then for any lift \bar{a}'_ℓ in X_{k+1} of a_ℓ , \bar{a}'_ℓ is a loop null-homotopic (rel ∂) in \bar{X}_{k+1} . For, observe the following facts: first, by the first statement of (1_k) , \bar{a}_ℓ never meets $\bar{c}_j^{(k)}$ in \bar{X}_k for $j \neq \ell$. So, from the construction of X_{k+1} from X_k , it follows that \bar{a}'_ℓ is contained in the circle $\bar{c}_\ell^{(k)}(g) \cup \bar{c}_\ell^{(k)}(g + e_\ell) \subset \bar{X}_{k+1}$ (see Figure 9). Second, from the hypothesis of the claim it follows that \bar{a}'_ℓ is a loop, and furthermore, has degree zero as a map into the circle $\bar{c}_\ell^{(k)}(g) \cup \bar{c}_\ell^{(k)}(g + e_\ell)$. Thus \bar{a}'_ℓ is null-homotopic.

From the claim, it follows that $\tilde{\gamma}_v$ is not null-homotopic in \bar{X}_{k+1} only if \bar{a}_ℓ passes through $\bar{c}_\ell^{(k)}$ algebraically nonzero times for both $\ell = 2i - 1$ and $2i$. This is equivalent to the condition that $v = * \in Y_k(g)$ by the second statement of (1_k) . In this case, looking at the lifts of a_{2i-1} , a_{2i} , a_{2i-1}^{-1} , and a_{2i}^{-1} in X_{k+1} , it is easily verified that $\tilde{\gamma}_v$ is of the desired form. \square

Proof of Lemma 7.4. From the construction discussed at the beginning of this subsection, one can see that the 1-complex X_n is as in Figure 10. Recall that X_{n+1} is the double cover of X_n determined by the assignment $c^{(n)} \mapsto 1$, (other 1-cells) $\mapsto 0$. So, taking two disjoint copies of the 1-complex R obtained from X_n by cutting $c_1^{(n)} = c_1^{(n-1)}(0) \subset X_n$ and then attaching them appropriately, one obtains X_{n+1} , as illustrated in Figure 10. Let e be the one of the copies $c_2^{(n-1)}(0)$ in X_{n+1} as shown in Figure 10. Define a character $\phi: \pi_1(X_{n+1}) \rightarrow \mathbb{Z}_d$ by assigning $-s \in \mathbb{Z}_d$ to e and 0 to other 1-cells.

For our purpose, it suffices to investigate loops $\tilde{\alpha}_j \in \mathcal{L}(\alpha, X_{n+1}|X_0)$ which are not null-homotopic in

$$X'_{n+1} = (X_{n+1} \text{ with each } Y_{n-1}(-) \text{ in Figure 10 collapsed}),$$

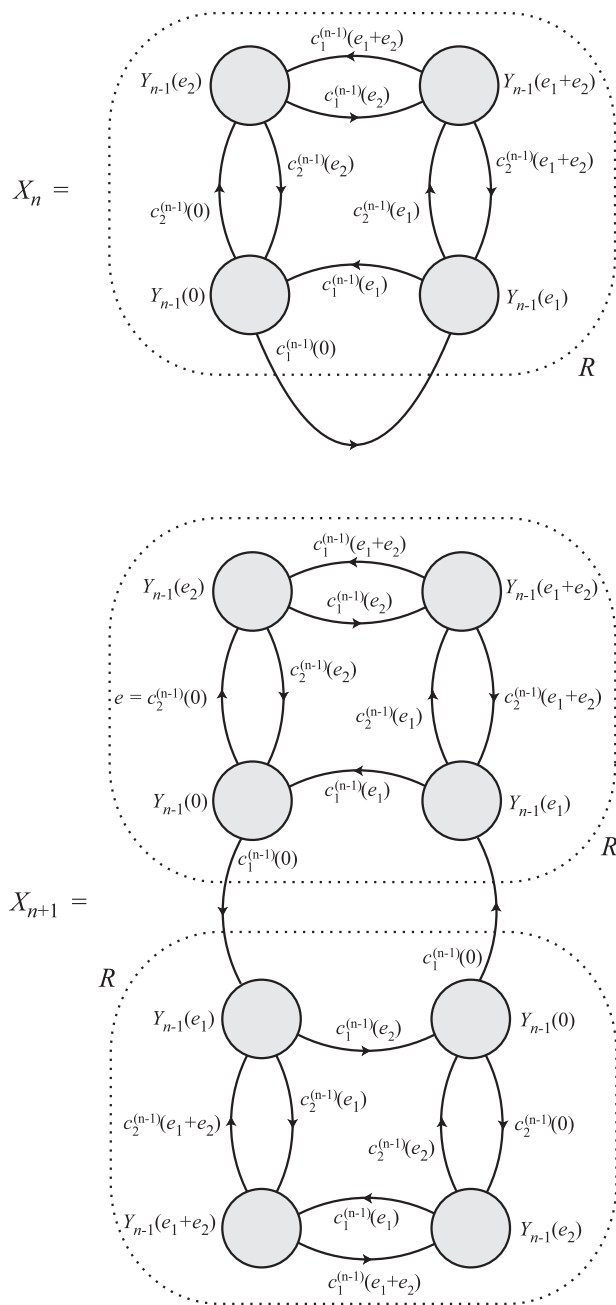


Fig. 10. The covers X_n and X_{n+1} .

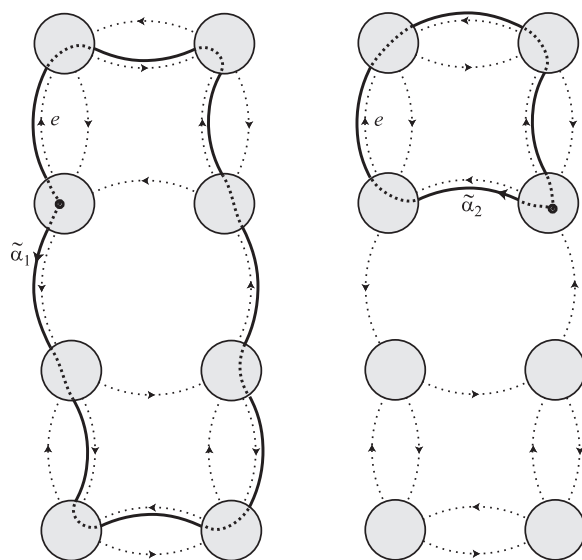


Fig. 11. Some lifts of α^2 and α in X_{n+1} .

since those null-homotopic in X'_{n+1} are in the kernel of ϕ defined above. Recall that, in Lemma 7.5, we gave a description of loops in $\mathcal{L}(\alpha, X_n|X_0)$ which are not null-homotopic in \bar{X}_n . Lifting those loops (or their squares), one can easily sketch the $\tilde{\alpha}_j$ as curves in X'_{n+1} . From this it can be seen that two of the $\tilde{\alpha}_j$, say $\tilde{\alpha}_1$ and $\tilde{\alpha}_3$, are lifts of α^2 , i.e., $r_1 = r_3 = 2$, and there are four other loops that are non-null-homotopic in X'_{n+1} , say $\tilde{\alpha}_2, \tilde{\alpha}_4, \tilde{\alpha}_5, \tilde{\alpha}_6$, with $r_2 = r_4 = r_5 = r_6 = 1$.

Also, it can be verified that exactly two of the $\tilde{\alpha}_j$ pass through the 1-cell $e \subset X_{n+1}$; $\tilde{\alpha}_1$ passes through e algebraically -1 times, and one of $\tilde{\alpha}_2, \tilde{\alpha}_4, \tilde{\alpha}_5, \tilde{\alpha}_6$, say $\tilde{\alpha}_2$, passes through e algebraically $+1$ times. We illustrate $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ in Figure 11. Therefore, it follows that $\phi(\tilde{\alpha}_1) = s, \phi(\tilde{\alpha}_3) = -s$, and all other elements in $\mathcal{L}(\alpha, X_{n+1}|X_0)$ are in the kernel of ϕ . \square

7.3. Bing doubles and the Levine–Tristram signature

In this subsection, as a by-product of the proof of Theorem 7.2, we prove the following generalization of the result of Harvey [26] and Teichner:

Theorem 7.6. *For any knot K and any positive integer n , the Levine–Tristram signature function σ_K is determined by $BD_n(K)$. In particular, if σ_K is nontrivial, then $BD_n(K)$ is not slice.*

For concreteness, we recall a precise definition of the Levine–Tristram signature function. For a Seifert matrix A of a knot K and $\omega \in S^1 \subset \mathbb{C}$,

$$\text{sign } \lambda_1(A, \omega) = \text{sign}((1 - \omega)A + (1 - \bar{\omega})A^T)$$

is often called the ω -signature of K . It is known that if K is algebraically slice and ω is not a zero of the Alexander polynomial of K , then the ω -signature vanishes. Note that the zero set of the Alexander polynomial is not invariant under concordance. This leads us to think of the average

$$\sigma_K(\omega) = \lim \frac{\text{sign } \lambda_1(A, \omega_+) + \text{sign } \lambda_1(A, \omega_-)}{2}$$

as a concordance invariant, where $\omega_+, \omega_- \in S^1$ approach ω from different sides. The function $\sigma_K : S^1 \rightarrow \mathbb{Z}$ is referred to as the *Levine–Tristram signature function of K* .

Proof of Theorem 7.6. We use the notations of the previous subsections: let N be the zero-surgery manifold of $BD_n(K)$, which is obtained from the zero-surgery manifold M of BD_n by infection by K along α , and $X \rightarrow M$ be the meridian map such that $[\alpha] \in \pi_1(M) = \pi_1(X)$ is represented by the loop $x_1^{(n)}$ in X . We consider the 2-tower $X_n \rightarrow \dots \rightarrow X_0 = X$ constructed above. We need the following analogue of Lemma 7.4:

Lemma 7.7. *For any $d > 0$ and $0 \leq s < d$, there is a character $\varphi_n : \pi_1(X_n) \rightarrow \mathbb{Z}_d$ such that all $\tilde{\alpha}_j \in \mathcal{L}(\alpha, X_n|X)$ are in the kernel of φ_n except two, which are sent by φ_n respectively to $s, -s \in \mathbb{Z}_d$.*

Proof. Recall that the 1-complex X_n is as in Figure 10. Define $\phi : \pi_1(X_n) \rightarrow \mathbb{Z}_d$ by assigning $s \in \mathbb{Z}_d$ to the 1-cell $c_1^{(n-1)}(0)$, and $0 \in \mathbb{Z}_d$ to other 1-cells of X_n .

By Lemma 7.5(1), all lifts of α in X_n are killed by ϕ except those based at $*$ $\in Y_{k-1}(0)$ or $*$ $\in Y_{k-1}(e_2)$, which we denote by $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. We illustrate $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ in Figure 12 (a dot on each $\tilde{\alpha}_i$ represents the point that $\tilde{\alpha}_i$ is based at). Note that $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ meet the 1-cell $c_1^{(n-1)}(0) + 1$ and -1 times algebraically, respectively. Since s is assigned to $c_1^{(n-1)}(0)$, we have $\phi(\tilde{\alpha}_1) = s$ and $\phi(\tilde{\alpha}_2) = -s$. □

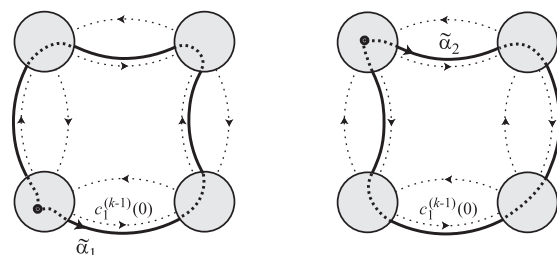


Fig. 12. Some lifts of α in X_n .

We continue the proof of Theorem 7.6. For a given $d = 2^r$ and $0 \leq s < d$, let $\varphi_n : \pi_1(X_n) = \pi_1(M_n) \rightarrow \mathbb{Z}_d$ be a map given by the above lemma. Let $\psi_n : \pi_1(N_n) \rightarrow \mathbb{Z}_d$ be the map induced by φ_n . Since BD_n is a trivial link, the invariant $\lambda(M_n, -)$ always vanishes by Theorem 6.2. Therefore, by the above lemma and Corollary 4.7, we have

$$\lambda(N_n, \psi_n) = [\lambda_1(A, \zeta_d^s)] + [\lambda_1(A, \zeta_d^{-s})] \in L^0(\mathbb{Q}(\zeta_d))$$

where A is a Seifert matrix of K . We apply the signature map $\text{sign}: L^0(\mathbb{Q}(\zeta_d)) \rightarrow \mathbb{Z}$ discussed in Section 3. Since $\lambda_1(A, \omega)$ and $\lambda_1(A, \omega^{-1})$ are the transposes of each other, they have the same signature. So we have

$$\text{sign } \lambda(N_n, \psi_n) = 2 \text{ sign } \lambda_1(A, \zeta_d^s).$$

It is known that the ω -signature has nontrivial jumps only at finitely many points on S^1 . Since $\{\zeta_{2^r}^s \mid r, s \in \mathbb{Z}\}$ is a dense subset of S^1 , it follows that σ_K is determined by $\{\lambda(N_n, \psi_n) \mid r, s \in \mathbb{Z}\}$. In particular, if $BD_n(K)$ is slice, then σ_K is trivial by Theorem 6.2. \square

8. Intersection form defects of (n) -solvable manifolds and links

Following [20], we say that a closed 3-manifold M is (n) -solvable if there is a spin 4-manifold W with $\pi = \pi_1(W)$ satisfying the following: $\partial W = M$, the inclusion induces an isomorphism $H_1(M) \cong H_1(W)$, and there are elements

$$u_1, \dots, u_r, v_1, \dots, v_r \in H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$$

such that $2r = b_2(W)$ and $\lambda_n(v_i, v_j) = 0$, $\mu_n(v_i) = 0$, $\lambda_n(v_i, u_j) = \delta_{ij}$ for any i, j , where λ_n is the $\mathbb{Z}[\pi/\pi^{(n)}]$ -valued intersection pairing on $H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$ and μ_n is the $(\mathbb{Z}[\pi/\pi^{(n)}]/\text{involution})$ -valued self-intersection quadratic form on $H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$. In this case W is called an (n) -solution of M . A link L is (n) -solvable if the zero-surgery manifold of L is (n) -solvable.

Remark 8.1. In this paper, all results on solvability are proved without using that a solution W is spin or that the self-intersection μ_n vanishes on the v_i .

8.1. Obstructions to being (n) -solvable

In this subsection we show that our invariants from p -towers of height $< n$ vanish for (n) -solvable manifolds and links:

Theorem 8.2. *Suppose W is an (n) -solution of M and $H_1(M)$ is p -torsion free. Then the following holds:*

- (1) *The inclusion $M \rightarrow W$ is a p -tower map of height n , in the sense of Definition 3.4.*
- (2) *For any p -tower $M_{n-1} \rightarrow \dots \rightarrow M_1 \rightarrow M_0 = M$ and for any $\phi: \pi_1(M_{n-1}) \rightarrow \mathbb{Z}_d$ with d a power of p , $\lambda(M_{n-1}, \phi) = 0$ in $L^0(\mathbb{Q}(\zeta_d))$.*

Theorem 1.7 is an immediate consequence of Theorem 8.2.

Proof of Theorem 8.2(1). We use induction on n . Fix n , and if $n > 0$, then suppose that $W_n \rightarrow W_{n-1} \rightarrow \dots \rightarrow W_0 = W$ is a p -tower of height n for W , and let $M_i = \partial W_i$ for $i \leq n$. As we did in Section 3, it suffices to show that $H_1(M_n; \mathbb{Z}_r) \rightarrow H_1(W_n; \mathbb{Z}_r)$ is an isomorphism for any power r of p . Since $H_1(M; \mathbb{Z}_p) \cong H_1(W; \mathbb{Z}_p)$, it follows that $H_1(W, M; \mathbb{Z}_p) = 0$. By applying Levine’s Lemma 3.2 inductively, we have $H_1(W_n, M_n; \mathbb{Z}_p) = 0$, and so $H_1(W_n, M_n; \mathbb{Z}_r) = 0$. It follows that $H_1(M_n; \mathbb{Z}_r) \rightarrow H_1(W_n; \mathbb{Z}_r)$ is surjective. So, it suffices to prove:

Assertion 1. $H_2(W_n; \mathbb{Z}_r) \rightarrow H_2(W_n, M_n; \mathbb{Z}_r)$ is an isomorphism.

To prove Assertion 1, we investigate the intersection form of W_n . Let $W^{(n)}$ be the cover of W corresponding to the n th derived subgroup $\pi^{(n)}$ of π . Since W is an (n) -solution, there are

$$u_1, \dots, u_r, v_1, \dots, v_r \in H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) = H_2(W^{(n)})$$

such that $\lambda_n(u_i, u_j) = 0$ and $\lambda_n(u_i, v_j) = \delta_{ij}$ where $r = \frac{1}{2}b_2(W)$. Note that (the $\pi^{(n)}$ -coset of) an element $g \in \pi$ acts on $H_2(W^{(n)})$ via the covering transformation on $W^{(n)}$. (By convention we assume that it is a right action.)

Fixing basepoints of the covers W_i , we can regard $\pi_1(W_i)$ as a (possibly non-normal) subgroup of π . Since each $W_i \rightarrow W_{i-1}$ is an abelian cover, we have $\pi^{(n)} \subset \pi_1(W_n)$, that is, $W^{(n)}$ is a cover of W_n . So there is an induced map $H_2(W^{(n)}) \rightarrow H_2(W_n)$. Note that the index $s = [\pi : \pi_1(W_n)]$ is finite. Choosing coset representatives $g_k \in \pi$, we write the right cosets of the subgroup $\pi_1(W_n) \subset \pi$ as $\pi_1(W_n)g_k$ for $1 \leq k \leq s$. Let $u_{ik}, v_{ik} \in H_2(W_n)$ be the images of $u_i \cdot g_k, v_i \cdot g_k \in H_2(W^{(n)})$, respectively, where $\cdot g_k$ denotes the action of g_k on $H_2(W^{(n)})$.

For a 4-manifold V , denote the untwisted \mathbb{Z} -valued intersection pairing on $H_2(V)$ by $I_V(-, -)$.

Assertion 2. $I_{W_n}(u_{ik}, u_{jl}) = 0$ and $I_{W_n}(u_{ik}, v_{jl}) = \delta_{ij}\delta_{kl}$ for any i, j, k, l .

Proof. Note that

$$0 = \lambda_n(u_i, u_j) = \sum_{g\pi^{(n)} \in \pi/\pi^{(n)}} I_{W^{(n)}}(u_i, u_j \cdot g) \cdot (g\pi^{(n)}) \text{ in } \mathbb{Z}_p[\pi/\pi^{(n)}].$$

It follows that $I_{W^{(n)}}(u_i, u_j \cdot g) = 0$ for any $g \in \pi$. Therefore

$$\begin{aligned} I_{W_n}(u_{ik}, u_{jl}) &= \sum_{g\pi^{(n)} \in \pi_1(W_n)/\pi^{(n)}} I_{W^{(n)}}(u_i \cdot g_k, u_j \cdot g_l g) \\ &= \sum_{g\pi^{(n)} \in \pi_1(W_n)/\pi^{(n)}} I_{W^{(n)}}(u_i, u_j \cdot g_l g g_k^{-1}) = 0. \end{aligned}$$

Similarly, for $g \in \pi$,

$$I_{W^{(n)}}(u_i, v_j \cdot g) = \begin{cases} 1 & \text{if } i = j \text{ and } g \in \pi^{(n)}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$I_{W_n}(u_{ik}, v_{jl}) = \sum_{g\pi^{(n)} \in \pi_1(W_n)/\pi^{(n)}} I_{W^{(n)}}(u_i, v_j \cdot g_l g g_k^{-1}).$$

Note that $g_l g g_k^{-1} \in \pi^{(n)}$ if and only if $g g_k^{-1} g_l \in \pi^{(n)}$ since $\pi^{(n)}$ is a normal subgroup. If $g g_k^{-1} g_l \in \pi^{(n)}$ for $g \in \pi_1(W_n)$, then $k = l$ since $\pi^{(n)} \subset \pi_1(W_n)$, and consequently $g \in \pi^{(n)}$. It follows that $I_{W_n}(u_{ik}, v_{jl}) = \delta_{ij}\delta_{kl}$. \square

We denote the i th Betti number of a space or a pair by $b_i(-)$, and the \mathbb{Z}_p -coefficient Betti number by $b_i(-; \mathbb{Z}_p) = \text{rank}_{\mathbb{Z}_p} H_i(-; \mathbb{Z}_p)$.

Assertion 3. $b_2(W_n) = 2rs = b_2(W_n; \mathbb{Z}_p)$.

Proof. First we consider the case of $n = 0$. Recall the assumption that $H_1(M)$ is p -torsion free. Since $H_1(M) \cong H_1(W)$, we have $b_1(W; \mathbb{Z}_p) = b_1(W)$. Since $M \rightarrow W$ induces an H_1 -isomorphism, it follows that $b_3(W; \mathbb{Z}_p) = b_1(W, M; \mathbb{Z}_p) = 0$ and $b_3(W) = b_1(W, M) = 0$. Since the \mathbb{Z}_p -coefficient Euler characteristic of W is equal to the integral coefficient Euler characteristic, we have $b_2(W; \mathbb{Z}_p) = b_2(W) = 2r$.

For the case of $n \geq 1$, we need the following lemma, whose proof is postponed:

Lemma 8.3. *Suppose V is a compact 4-manifold. Then for any connected regular p^a -fold cover \tilde{V} of V , $b_2(\tilde{V}; \mathbb{Z}_p) \leq p^a \cdot b_2(V; \mathbb{Z}_p)$.*

Applying this lemma to W_0, \dots, W_n inductively, we have

$$b_2(W_n) \leq b_2(W_n; \mathbb{Z}_p) \leq s \cdot b_2(W; \mathbb{Z}_p) = s \cdot b_2(W) = 2rs.$$

By Assertion 2, $b_2(W_n) \geq 2rs$. From this Assertion 3 follows. □

Now we continue the proof of Assertion 1. From Assertion 3 and the universal coefficient theorem

$$H_2(W_n; \mathbb{Z}_p) = (H_2(W_n) \otimes \mathbb{Z}_p) \oplus \text{Tor}(H_1(W_n), \mathbb{Z}_p),$$

it follows that $H_2(W_n)$ is p -torsion free and $H_1(W_n)$ is torsion and p -torsion free. So, for any power r of p , we have $H_2(W_n; \mathbb{Z}_r) = (H_2(W_n)/\text{torsion}) \otimes \mathbb{Z}_r$ by the universal coefficient theorem. By a similar argument, we have $H_2(W_n, M_n; \mathbb{Z}_r) = (H_2(W_n, M_n)/\text{torsion}) \otimes \mathbb{Z}_r$. By Assertions 2 and 3, the map

$$H_2(W_n)/\text{torsion} \rightarrow H_2(W_n, M_n)/\text{torsion}$$

is an isomorphism. From this Assertion 1 follows. This completes the proof of Theorem 8.2(1). □

Proof of Theorem 8.2(2). By Theorem 8.2(1), there is a p -tower

$$W_{n-1} \rightarrow \dots \rightarrow W_1 \rightarrow W_0 = W$$

such that $\partial W_i = M_i$ and $\text{Hom}(\pi_1(W_{n-1}), \mathbb{Z}_d) \approx \text{Hom}(\pi_1(M_{n-1}), \mathbb{Z}_d)$. Thus the given $\phi: \pi_1(M_{n-1}) \rightarrow \mathbb{Z}_d$ extends to $\psi: \pi_1(W_{n-1}) \rightarrow \mathbb{Z}_d$. From this it follows that $\lambda(M_{n-1}, \phi)$ is well-defined as an element in $L^0(\mathbb{Q}(\zeta_d))$.

We will show that $\lambda(M_{n-1}, \phi)$ vanishes by investigating the intersection form of the bounding 4-manifold W_{n-1} . By applying Assertions 2 and 3 of the proof of Theorem 8.2(1) to $n - 1$, it follows that the untwisted intersection form on $H_2(W_{n-1}; \mathbb{Q})$ is metabolic and so the ordinary signature $\sigma(W_{n-1})$ vanishes.

It remains to show that the intersection form $\lambda_{W_{n-1}}^{\mathbb{Q}(\zeta_d)}$ on $H_2(W_{n-1}; \mathbb{Q}(\zeta_d))$ is Witt trivial. Indeed, we can construct a ‘‘Lagrangian’’ and its dual for $\lambda_{W_{n-1}}^{\mathbb{Q}(\zeta_d)}$, as we did for the untwisted intersection form I_{W_n} in Assertion 2 of the proof of Theorem 8.2(1). Details are as follows. Let $u_i, v_i \in H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) = H_2(W^{(n)})$, where $1 \leq i \leq r = \frac{1}{2}b_2(W)$,

be the elements such that $\lambda_n(u_i, u_j) = 0$ and $\lambda_n(u_i, v_j) = \delta_{ij}$ as before. Let $\pi_1(W_{n-1})g_k$ be the right cosets of $\pi_1(W_{n-1}) \subset \pi = \pi_1(W)$, where $1 \leq k \leq s = [\pi : \pi_1(W_{n-1})]$. Let $u_{ik}, v_{jl} \in H_2(W_n)$ be the images of $u_i \cdot g_k, v_j \cdot g_l$. Similarly to Assertion 2 of the proof of Theorem 8.2(1), for any $h \in \pi_1(W_{n-1})$ we have

$$\begin{aligned} I_{W_n}(u_{ik}, u_{jl} \cdot h) &= \sum_{g\pi^{(n)} \in \pi_1(W_n)/\pi^{(n)}} I_{W^{(n)}}(u_i, u_j \cdot g_\ell h g g_k^{-1}) = 0, \\ I_{W_n}(u_{ik}, v_{jl} \cdot h) &= \sum_{g\pi^{(n)} \in \pi_1(W_n)/\pi^{(n)}} I_{W^{(n)}}(u_i, v_j \cdot g_\ell h g g_k^{-1}) \\ &= \begin{cases} 1 & \text{if } i = j, k = l, \text{ and } h \in \pi_1(W_n), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

To obtain the last equality, observe that the relevant summand can be nonzero if and only if $h g g_k^{-1} g_l \in \pi^{(n)}$; if this is the case, then k should be equal to l since $h, g \in \pi_1(W_{n-1})$, and consequently g should be in the coset $h^{-1}\pi^{(n)}$. Since $g \in \pi_1(W_n)$, this can occur only when $h \in \pi_1(W_n)$.

Let $\lambda_{W_{n-1}}^{\mathbb{Z}[\mathbb{Z}_d]}$ be the $\mathbb{Z}[\mathbb{Z}_d]$ -valued intersection form on $H_2(W_{n-1}; \mathbb{Z}[\mathbb{Z}_d]) = H_2(W_n)$, where \mathbb{Z}_d is identified with $\pi_1(W_{n-1})/\pi_1(W_n) =$ the covering transformation group of $W_n \rightarrow W_{n-1}$. From the above computation, it follows that

$$\begin{aligned} \lambda_{W_{n-1}}^{\mathbb{Z}[\mathbb{Z}_d]}(u_{ik}, u_{jl}) &= 0, \\ \lambda_{W_{n-1}}^{\mathbb{Z}[\mathbb{Z}_d]}(u_{ik}, v_{jl}) &= \sum_{h\pi_1(W_n) \in \pi_1(W_{n-1})/\pi_1(W_n)} I_{W_n}(u_{ik}, v_{ik} \cdot h) \cdot h\pi_1(W_n) = \delta_{ij}\delta_{kl}. \end{aligned}$$

Therefore, by naturality, the values of $\lambda_{W_{n-1}}^{\mathbb{Q}(\zeta_d)}$ evaluated at the images of (u_{ik}, u_{jl}) and (u_{ik}, v_{jl}) are 0 and $\delta_{ij}\delta_{kl}$, respectively, for $1 \leq i \leq r, 1 \leq k \leq s$.

Now, in order to conclude that $\lambda_{W_{n-1}}^{\mathbb{Q}(\zeta_d)}$ is Witt trivial, it suffices to show that $b_2(W_{n-1}; \mathbb{Q}(\zeta_d)) = 2rs$. From the properties of the u_{ik}, v_{jl} proved above, it follows that $b_2(W_{n-1}; \mathbb{Q}(\zeta_d)) \geq 2rs$. For the opposite inequality, we appeal to the following analogue of Lemma 8.3, which will be proved later:

Lemma 8.4. *Suppose V is a 4-manifold and $\pi_1(V) \rightarrow \Gamma$ is a map, where Γ is a p -group endowed with a map $\mathbb{Z}\Gamma \rightarrow \mathcal{K}$ into a (skew-)field with characteristic zero. If the induced map $\mathbb{Z}\pi_1(V) \rightarrow \mathbb{Z}\Gamma \rightarrow \mathcal{K}$ is nontrivial, then $b_2(V; \mathcal{K}) \leq b_2(V; \mathbb{Z}_p)$.*

If $\phi: \pi_1(M_{n-1}) \rightarrow \mathbb{Z}_d$ is a trivial map, then $\lambda(M_{n-1}, \phi) = 0$. So we may assume that ϕ is nontrivial. Then $\mathbb{Z}\pi_1(W_{n-1}) \rightarrow \mathbb{Z}[\mathbb{Z}_d] \rightarrow \mathbb{Q}(\zeta_d)$ is nontrivial since its composition with $\mathbb{Z}\pi_1(M_{n-1}) \rightarrow \mathbb{Z}\pi_1(W_{n-1})$ is nontrivial. From this and Assertion 3 of the proof of Theorem 8.2(1) for $n - 1$, it follows that

$$b_2(W_{n-1}; \mathbb{Q}(\zeta_d)) \leq b_2(W_{n-1}; \mathbb{Z}_p) = 2rs$$

by Lemma 8.4. □

Proof of Lemma 8.3. This is a p -covering analogue of previously known results for quotient (skew-)field coefficient homology modules of poly-torsion-free-abelian covers, which were dealt with in [20] for the special case that $H_1(\partial V) \rightarrow H_1(V)$ is an isomorphism, and in Proposition 2.1 of [9] for the general case. Since our statement can also be proved along the same lines, we will only indicate how the proof of Proposition 2.1 of [9] has to be modified. As an analogue of Lemma 2.5 of [9], we need the following: Suppose (X, A) is a finite CW-pair with X connected, \tilde{X} is the cover of X induced by $\phi: \pi_1(X) \rightarrow \Gamma$ where Γ is a p -group, and $\tilde{A} \subset \tilde{X}$ is the pre-image of A .

- (1) If A is nonempty, then $b_1(\tilde{X}, \tilde{A}; \mathbb{Z}_p) \leq |\Gamma| \cdot b_1(X, A; \mathbb{Z}_p)$.
- (2) If ϕ is surjective, then $b_1(\tilde{X}; \mathbb{Z}_p) \leq |\Gamma| \cdot (b_1(X; \mathbb{Z}_p) - 1) + 1$.

The proof of (1), (2) is exactly the same as that of Lemma 2.5 of [9], except that we should use Levine’s Lemma 3.2 in place of Lemma 2.3(2) of [9]. Using the above (1), (2) in place of Lemma 2.5 of [9], the argument of the proof of [9, Proposition 2.1] proves our Lemma 8.3. □

Proof of Lemma 8.4. We proceed similarly to the proof of Lemma 8.3, along the same lines as the proof of Proposition 2.1 of [9]. In this case we need the following analogue of Lemma 2.5 of [9]:

- (1) If A is nonempty, then $b_1(X, A; \mathcal{K}) \leq b_1(X, A; \mathbb{Z}_p)$.
- (2) If ϕ is nontrivial, then $b_1(X; \mathcal{K}) \leq b_1(X; \mathbb{Z}_p) - 1$.

The proof of (1) proceeds as follows. As in the proof of Lemma 2.5 of [9], we consider an inclusion

$$(Y, B) = \bigcup^{b_1(X, A; \mathbb{Z}_p)} ([0, 1], \{0, 1\}) \hookrightarrow (X, A)$$

such that $Y \cap A = B$ and $H_1(X, Y \cup A; \mathbb{Z}_p) = 0$. By Levine’s Lemma 3.2, we have $H_1(X, Y \cup A; \mathbb{Z}_{(p)}\Gamma) = 0$. It follows that $H_1(X, Y \cup A; \mathcal{K}) = 0$, and so $b_1(X, A; \mathcal{K}) \leq b_1(Y, B; \mathcal{K}) \leq b_1(X, A; \mathbb{Z}_p)$. (2) is proved exactly as in the proof of Lemma 2.5 of [9]. Now, the arguments of the proof of Proposition 2.1 of [9] can be applied to prove Lemma 8.4. □

8.2. Iterated Bing doubles and solvable and grope filtrations

In this subsection we show that there are nontrivial torsion elements (as well as infinite order elements) in an arbitrary depth of the solvable filtration and grope filtration of link concordance. Recall that in Theorems 7.2 and 7.6 we proved that the iterated Bing doubles $BD_n(K)$ of certain knots K are not slice. Note that we used 2-towers of height $n + 1$ and n to prove Theorems 7.2 and 7.6, respectively. Therefore, by Theorem 8.2, we immediately obtain the following nonsolvability results: (1) For the amphichiral knots K_{a_i} considered in Theorem 7.2, $BD_n(K_{a_i})$ is 2-torsion but $BD_n(K_{a_i})$ is not $(n + 2)$ -solvable for any n . (2) If σ_K is nontrivial, then $BD_n(K)$ is not $(n + 1)$ -solvable for any n .

Modifying the construction slightly, we can easily obtain (n) -solvable examples:

- Theorem 8.5.** (1) *There are infinitely many amphichiral knots K such that $BD_n(K)$ is 2-torsion and (n) -solvable but not $(n + 2)$ -solvable for any n .*
 (2) *If σ_K is nontrivial and $\text{Arf}(K) = 0$, then $BD_n(K)$ is (n) -solvable but not $(n + 1)$ -solvable for any n .*

Here $\text{Arf}(K)$ is the Arf invariant of a knot K .

Proof. (1) For the solvability part, we need the following result of Cochran–Orr–Teichner [20, Proposition 3.1]: if K is a knot with $\text{Arf}(K) = 0$ and L is obtained by infection on an (n) -solvable link L_0 by K along a curve α such that $[\alpha] \in \pi_1(S^3 - L_0)^{(n)}$, then L is (n) -solvable. (In [20] they stated the result when L_0 is a knot, but their argument works for links as well.)

Choose any $i \neq j$ and let $K = K_{a_i} \# K_{a_j}$ where the K_{a_i} are as in Theorem 7.2. Since both K_{a_i} and K_{a_j} are amphichiral, so is K . So, by Lemma 7.1, $BD_n(K)$ has the desired 2-torsion property. Due to Levine [33], $\text{Arf}(K) = 1$ if and only if $\Delta_K(-1) = \pm 3 \pmod{8}$. Since

$$\Delta_{K_{a_i}}(-1) = -(4a_i^2 + 1) \equiv 3 \pmod{8}$$

(recall that the a_i are all odd), $\text{Arf}(K_{a_i}) = 1$ for all i . Since Arf is additive under connected sum, $\text{Arf}(K) = 1 + 1 = 0$. From this it follows that $BD_n(K)$ is (n) -solvable.

In order to show the non-solvability of $BD_n(K)$, recall that in the proof of Theorem 7.2 we considered the discriminant of the invariant $\lambda(N_{n+1}, \psi_{n+1})$ where N_{n+1} is the $(n + 1)$ st term of a height $(n + 1)$ 2-tower of the zero-surgery manifold of $BD_n(K)$ and ψ_{n+1} is a \mathbb{Z}_4 -valued character of $\pi_1(N_{n+1})$. Since $\text{dis } \lambda(N_{n+1}, \psi_{n+1})$ is determined by the Alexander polynomial of K as in the proof of Theorem 7.2 (see also Lemma 5.3) and the Alexander polynomial is multiplicative under connected sum, $\text{dis } \lambda(N_{n+1}, \psi_{n+1})$ for our K is equal to the product of those for K_{a_i} and K_{a_j} . So, for our K we have

$$\lambda(N_{n+1}, \psi_n) = (2a_i^2 + 1)(2a_i^4 + 4a_i^2 + 1) \cdot (2a_j^2 + 1)(2a_j^4 + 4a_j^2 + 1)$$

by the computation for K_{a_i} done in the proof of Theorem 7.2. Since $i \neq j$, by applying the norm residue symbol $(\cdot, -1)_{p_i}$ or $(\cdot, -1)_{p_j}$ which was computed in Proposition 5.6, it follows that $\lambda(N_{n+1}, \psi_n)$ is nontrivial.

(2) From the Arf invariant condition, it follows immediately that $BD_n(K)$ is (n) -solvable by [20, Proposition 3.1]. The nonsolvability has already been discussed. \square

For the iterated Bing doubles $BD(K_{a_i})$ in Theorem 8.5, Harvey’s invariant ρ_k vanishes (for all k) as mentioned in Remark 7.3. Also, there are infinitely many knots K such that σ_K is nontrivial but the integral of σ_K over the unit circle is zero. So, we have the following consequence:

- Corollary 8.6.** (1) *For any n , there are infinitely many (n) -solvable 2-torsion links which are not $(n + 2)$ -solvable but have vanishing ρ_k -invariants.*
 (2) *For any n , there are infinitely many (n) -solvable links which are not $(n + 1)$ -solvable but have vanishing ρ_k -invariants.*

From this it follows that the invariant ρ_n , which is viewed as a homomorphism of “ (n) -solvable boundary string links modulo $(n + 1)$ -solvable boundary string links” as in [26, 17], has nontrivial kernel for any n . We remark that Cochran, Harvey, and Leidy have announced a (different) proof of the nontriviality of the kernel of ρ_n .

In a subsequent paper [6], it will be shown that the links in Corollary 8.6(1) and (2) (can be chosen so that they) are independent modulo $\mathcal{F}_{(n+2)}$ and $\mathcal{F}_{(n+1)}$, respectively, in an appropriate sense. In fact, considering the subgroup generated by these links, it can be proved that the kernel of ρ_n contains a subgroup whose abelianization is isomorphic to \mathbb{Z}^∞ .

In [19, 20, 21, 26], another filtration $\{\mathcal{G}_n\}$ of the set of link concordance classes is defined in terms of *grope*s, instead of the solvability; $\mathcal{G}_{(n)}$ is defined to be the subset of concordance classes of links in S^3 which bound an embedded symmetric grope of height n in the 4-ball. (As an abuse of notation, we will write $L \in \mathcal{G}_{(n)}$ when the concordance class of L is in $\mathcal{G}_{(n)}$.) $\{\mathcal{G}_{(n)}\}$ is called the *grope filtration*. For a precise definition of a grope and related discussions, the reader is referred to [19, 21]. The following result on the existence of torsion elements in the grope filtration is a consequence of Theorem 8.5:

Corollary 8.7. (1) *There are infinitely many knots K such that $BD_n(K)$ is 2-torsion and $BD_n(K) \in \mathcal{G}_{(n+1)}$ but $BD_n(K) \notin \mathcal{G}_{(n+4)}$ for any n .*
 (2) *If σ_K is nontrivial and $\text{Arf}(K) = 0$, then $BD_n(K) \in \mathcal{G}_{(n+1)}$ but $BD_n(K) \notin \mathcal{G}_{(n+3)}$ for any n .*

As in Corollary 8.6, the iterated Bing doubles in Corollary 8.7 can be chosen so that the invariant ρ_k vanishes.

Proof. In [19], it was shown that $\mathcal{G}_{(n+2)} \subset \mathcal{F}_{(n)}$. So, the iterated Bing doubles considered in Theorem 8.5(1) and (2) are not in $\mathcal{G}_{(n+4)}$ and $\mathcal{G}_{(n+3)}$, respectively.

Recall that $BD_n(K)$ is obtained by performing infection on the trivial link BD_n along a curve α shown in Figure 8. It is well known that α bounds an embedded symmetric grope of height n in $S^3 - BD_n$. So, by the argument of [26, proof of Theorem 6.13], it follows that $BD_n(K) \in \mathcal{G}_{(n+1)}$ if $\text{Arf}(K) = 0$. This completes the proof. \square

Finally we remark that $BD_n(K)$ is a boundary link for any knot K , so that our results hold in the solvable and grope filtrations of boundary links.

Appendix: Computation for zero-surgery manifolds of knots

In this appendix we compute the intersection form defect invariants of cyclic covers of the zero-surgery manifold M of a knot K . We use the same notation as in Section 4.4: let X_r be the r -fold cyclic cover of M which is determined by the canonical map $\pi_1(M) \rightarrow \mathbb{Z}_r$ sending a (positive) meridian of K to $1 \in \mathbb{Z}_r$. The image of the natural map $\pi_1(X_r) \rightarrow \pi_1(M) \rightarrow H_1(M) = \mathbb{Z}$ is $r\mathbb{Z}$. Composing it with $r\mathbb{Z} \rightarrow \mathbb{Z}_d$ sending $r \in r\mathbb{Z}$ to $s \in \mathbb{Z}_d$, we obtain a character $\phi_r^{s,d}: \pi_1(X_r) \rightarrow \mathbb{Z}_d$ sending the lift of the r th power of a meridian of K to $s \in \mathbb{Z}_d$. The following result was stated as Lemma 4.6 in Section 4.4:

Lemma. $(X_r, \phi_r^{s,d})$ is null-bordant over \mathbb{Z}_d , and

$$\lambda(X_r, \phi_r^{s,d}) = [\lambda_r(A, \zeta_d^s)] - [\lambda_r(A, 1)] \text{ in } L^0(\mathbb{Q}(\zeta_d))$$

where A is a Seifert matrix of K and $[\lambda_r(A, \omega)]$ is the Witt class of (the nonsingular part of) the hermitian form represented by the following $r \times r$ block matrix:

$$\lambda_r(A, \omega) = \begin{bmatrix} A + A^T & -A & & & -\omega^{-1}A^T \\ -A^T & A + A^T & -A & & \\ & -A^T & A + A^T & \ddots & \\ & & & \ddots & \ddots & -A \\ -\omega A & & & & -A^T & A + A^T \end{bmatrix}_{r \times r}.$$

For $r = 1, 2$, $\lambda_r(A, \omega)$ should be understood as

$$[(1 - \omega)A + (1 - \omega^{-1})A^T] \text{ and } \begin{bmatrix} A + A^T & -A - \omega^{-1}A^T \\ -A^T - \omega A & A + A^T \end{bmatrix}.$$

Proof. Let V be the 4-manifold obtained by attaching a 2-handle to D^4 along the zero-framing of $K \subset S^3$. Then $\partial V = M$. Consider a Seifert surface of K from which the Seifert matrix A is defined. By capping it off using a disk in the boundary of the 2-handle of V , we obtain a closed surface in M , which is usually called a ‘‘capped-off Seifert surface’’. Pushing it slightly into the interior of V , we obtain a surface F in V with trivial normal bundle. In fact the trace of pushing induces a framing of the normal bundle of F . We identify a tubular neighborhood of F with $F \times D^2$ using this framing. Let $W = V - \text{int}(F \times D^2)$. Obviously $\partial W = M \cup (F \times D^2)$. By a Thom–Pontryagin construction along the trace of pushing $\cong F \times [0, 1] \subset W$, W can be viewed as a space over \mathbb{Z} . Let W_r be the r -fold cyclic cover associated to $\pi_1(W) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_r$.

$H_1(W) \cong \mathbb{Z}$ and is generated by a meridian of F . Therefore the canonical map $\pi_1(M) \rightarrow H_1(M) = \mathbb{Z}$ is the restriction of $\pi_1(W) \rightarrow H_1(W) = \mathbb{Z}$. This enables us to define $\psi_r^{s,d}: \pi_1(W_r) \rightarrow \mathbb{Z}_d$ exactly in the same way as $\phi_r^{s,d}: \pi_1(X_r) \rightarrow \mathbb{Z}_d$, so that $\phi_r^{s,d}$ is identical with

$$\pi_1(X_r) \rightarrow \pi_1(W_r) \xrightarrow{\psi_r^{s,d}} \mathbb{Z}_d.$$

Obviously $\partial W_r = X_r \cup (F \times S^1)$ over \mathbb{Z}_d . Here $F \times S^1$ is endowed with

$$\phi': \pi_1(F \times S^1) \xrightarrow{\text{proj.}} \pi_1(S^1) = \mathbb{Z} \xrightarrow{\text{proj.}} \mathbb{Z}_d.$$

We claim that $\lambda(F \times S^1, \phi') = 0$. For, choosing a handlebody H bounded by F , it can be seen that $\partial(H \times S^1) = F \times S^1$ over \mathbb{Z}_d . Since $H_2(F \times S^1; R) \rightarrow H_2(H \times S^1; R)$ is surjective for $R = \mathbb{Q}$ and $\mathbb{Q}(\zeta_d)$, both $[\lambda_{\mathbb{Q}(\zeta_d)}(H \times S^1)]$ and $\sigma(H \times S^1)$ vanish. This proves the claim.

From the claim it follows that

$$\lambda(X_r, \phi_r^{s,d}) = [\lambda_{\mathbb{Q}(\zeta_d)}(W_r)] - [t_{\mathbb{Q}(\zeta_d)}^* \sigma(W_r)].$$

To compute the intersection form of W_r from the Seifert matrix A of K , we use a known cut-paste construction of W_r (e.g., similar arguments were used in [29] and [10] to compute some signature invariants). Details are as follows. Let Y be the manifold obtained by cutting W along the trace of pushing the capped-off Seifert surface F . Obviously there are inclusion maps $i_{\pm}: F \times [0, 1] \rightarrow \partial Y$ corresponding to the positive and negative normal directions of F such that $i_+(F \times [0, 1])$ and $i_-(F \times [0, 1])$ are disjoint and

$$W = Y / \{i_+(z) \sim i_-(z) \text{ for } z \in F \times [0, 1]\}.$$

The covering W_r is obtained by gluing r disjoint copies $t^0 Y, tY, \dots, t^{r-1} Y$ of Y :

$$W_r = \left(\bigcup_{k=0}^{r-1} t^k Y \right) / \{t^{k+1} i_+(z) \sim t^k i_-(z) \text{ for } z \in F \times [0, 1], k = 0, \dots, r-1\}$$

where t^r is understood as t^0 . From this we have a Mayer–Vietoris long exact sequence

$$\rightarrow \bigoplus^r H_2(Y) \rightarrow H_2(W_r) \rightarrow \bigoplus^r H_1(F) \rightarrow \bigoplus^r H_1(Y) \rightarrow .$$

Since Y can also be obtained by cutting D^4 along the trace of pushing F , it can be seen that Y is homeomorphic to D^4 . It follows that

$$H_2(W_r) \cong \bigoplus^r H_1(F).$$

For a 1-cycle x on F , the corresponding element in $H_2(W_r)$ is described as follows. Since Y is contractible, there are 2-chains u_+ and u_- in Y such that ∂u_{\pm} is equal to $i_{\pm}(x \times *)$, which is a pushoff of x along the \pm -normal direction of the Seifert surface. Then the homology class x^k of the 2-chain $t^{k+1} u_+ \cup t^k u_-$ in W_r corresponds to the class of x in the k th $H_1(F)$ factor. For another 1-cycle y , choosing v_+ and v_- in X such that $\partial v_{\pm} = i_{\pm}(y \times 1)$, the intersection number $x_k \cdot y_{\ell}$ in W_r is given by

$$\begin{cases} (u_+ \cdot v_+) + (u_- \cdot v_-) & \text{if } k = \ell, \\ u_+ \cdot v_- & \text{if } k = \ell - 1, \\ u_- \cdot v_+ & \text{if } k = \ell + 1, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition, $u_+ \cdot v_-$ is exactly the value of the Seifert form on (x, y) , and the other terms in the above formula can also be interpreted similarly. (A technical issue is that in computing the expression for $k = \ell$, one needs to push one of u_{\pm} and v_{\pm} further along the \pm -direction to remove intersection points on the boundary of the 2-chains; pushing u_{\pm} , one can see that $u_+ \cdot v_+$ and $u_- \cdot v_-$ are the Seifert form evaluated at (x, y) and (y, x) .)

From this it follows that the intersection form on $H_2(W_r)$ is given by the block matrix $\lambda_r(A, 1)$. So $\sigma(W_r)$ is the Witt class of $\lambda_r(A, 1)$.

To compute the intersection form $\lambda_{\mathbb{Q}(\zeta_d)}(W_r)$, we consider W_{dr} ; indeed,

$$H_2(W_r; \mathbb{Z}[\mathbb{Z}_d]) = H_2(W_{dr}).$$

The above argument shows that $H_2(W_{dr})$ is the direct sum of dr copies of $H_1(F)$. It can be easily seen that the covering transformation action of a generator of \mathbb{Z}_d , say g , is exactly sending the k th $H_1(F)$ factor of $H_2(W_{dr})$ to the $(k+r)$ th factor. Therefore,

$$H_2(W_r; \mathbb{Z}[\mathbb{Z}_d]) \cong H_2(W_r) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}_d].$$

The above computation also shows that the $\mathbb{Z}[\mathbb{Z}_d]$ -intersection on $H_2(W_r; \mathbb{Z}[\mathbb{Z}_d])$ is represented by the matrix $\lambda_r(A, g)$. Note that $\mathbb{Q}(\zeta_d)$ is $\mathbb{Q}[\mathbb{Z}_d]$ -projective, being an irreducible factor of the regular representation $\mathbb{Q}[\mathbb{Z}_d]$ of \mathbb{Z}_d . Therefore the universal coefficient theorem gives

$$H_2(W_r; \mathbb{Q}(\zeta_d)) = H_2(W_r; \mathbb{Z}[\mathbb{Z}_d]) \otimes_{\mathbb{Z}[\mathbb{Z}_d]} \mathbb{Q}(\zeta_d).$$

It follows that the intersection form on $H_2(W_r; \mathbb{Q}(\zeta_d))$ is represented by $\lambda_r(A, \zeta_d^s)$. This completes the computation of $\lambda(X_r, \phi_r^{s,d})$. \square

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