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Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$

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Abstract. Several Liouville-type theorems are presented for stable solutions of the equation $-\Delta u = f(u)$ in $\mathbb{R}^N$, where $f > 0$ is a general convex, nondecreasing function. Extensions to solutions which are merely stable outside a compact set are discussed.

1. Introduction

For $N \geq 1$ and $f \in C^1(\mathbb{R})$ consider the equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1}$$

The aim of this paper is to classify solutions $u \in C^2(\mathbb{R}^N)$ which are *stable*, i.e. such that for all $\varphi \in C_c^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f'(u)\varphi^2\,dx \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx. \tag{2}$$

For some of our results, we shall assume in addition $u > 0$ in $\mathbb{R}^N$ and/or $u \in L^\infty(\mathbb{R}^N)$. We shall also discuss extensions to solutions which are merely stable outside a compact set (i.e. (2) holds for test functions supported in the complement of a given compact set $K \subset\subset \mathbb{R}^N$).

Stable *radial* solutions of (1) are by now well-understood: by the work of Cabré and Capella [4], refined by Villegas in [21], every bounded radial stable solution of (1) must be constant if $N \leq 10$. The result holds for any nonlinearity $f \in C^1(\mathbb{R})$. Conversely, there exist unbounded radial stable solutions in any dimension. Take, for example, $u(x) = |x|^2/2N$ solving (1) with $f(u) = -1$. Also, there are examples of bounded radial stable solutions when $N \geq 11$. See e.g. [21], [14]. When dealing with nonradial solutions, much less is known. In the case $N = 2$, any stable solution of (1) with bounded gradient is one-dimensional (i.e. up to a rotation of space, u depends only on one variable) under the sole assumption that f is locally Lipschitz continuous (see [15]). For $N = 3$, this result remains valid under the additional assumptions that $f \geq 0$ and u is bounded, as demonstrated by N. Dancer in [7].

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In arbitrary dimension, a complete analysis of stable solutions and solutions which are stable outside a compact set is provided for two important nonlinearities $f(u) = |u|^{p-1}u$, $p > 1$ and $f(u) = e^u$ in [12], [14], [13] and [8] (see also [6], [11]). After completing this work, we were informed by B. Sciunzi about the recent work [5], where nonlinearities with a vertical asymptote are also considered. To our knowledge, the most general classification result for bounded stable solutions of (1) in arbitrary dimension is now given in [10]. Regarding the class of solutions which are stable outside a compact set, we also mention the recent paper [16] on symmetry of solutions of low Morse index.

Under a mere nonnegativity assumption on the nonlinearity, we begin this paper by stating that up to space dimension $N = 4$, bounded stable solutions of (1) are trivial:

Theorem 1.1. *Assume $f \in C^1(\mathbb{R})$, $f \geq 0$ and $1 \leq N \leq 4$. Assume $u \in C^2(\mathbb{R}^N)$ is a bounded, stable solution of* (1). *Then u is constant.*

Remark 1.2. It would be interesting to know whether Theorem 1.1 still holds if one assumes that u is unbounded but $|\nabla u|$ is bounded.

1.1. Power-type nonlinearities

For our next set of results, we restrict to the following class of nonlinearities:

$$f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+_*),\ f > 0 \text{ is nondecreasing and convex in } \mathbb{R}^+_*. \tag{3}$$

As demonstrated in [14] for the particular case of the power nonlinearities $f(u) = |u|^{p-1}u$, two critical exponents play an important role, namely the classical Sobolev exponent

$$p_S(N) = \frac{N+2}{N-2} \quad \text{for } N \geq 3 \tag{4}$$

and the Joseph–Lundgren exponent

$$p_c(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} \quad \text{for } N \geq 11. \tag{5}$$

In order to relate the nonlinearity f and the above exponents, we introduce a quantity q defined for $u \in \mathbb{R}^+_*$ by

$$q(u) = \frac{f'^2}{ff''}(u) = \frac{(\ln f)'}{(\ln f')'}(u) \tag{6}$$

whenever $ff''(u) \neq 0$, and $q(u) = +\infty$ otherwise. If $f(u) = |u|^{p-1}u$, $p \geq 1$, then q is independent of u and coincides with the conjugate exponent of p, i.e. $1/p + 1/q = 1$. In this section, we assume that $q(u)$ converges as $u \to 0^+$ and denote its limit by

$$q_0 = \lim_{u \to 0^+} q(u) \in \overline{\mathbb{R}}. \tag{7}$$

Remark 1.3. If $u \in C^2(\mathbb{R}^N)$ with $u \geq 0$ solves (1), and (3) holds, then $f(0) = 0$.

In dimension $N = 1, 2$, this follows directly from the classical Liouville theorem for superharmonic nonnegative functions. For a proof in dimension $N \geq 3$, see Step 6. in Section 6. We then observe that

Lemma 1.4. *If $f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+_*)$ is convex nondecreasing, $f(0) = 0$ and (7) holds, then in fact $q_0 \in [1, +\infty]$.*

Proof. Indeed, assume by contradiction that there exists $\theta > 1$ such that $0 \le q(u) \le 1/\theta$ in a neighbourhood of 0. Consequently, near 0,

$$\frac{f''}{f'} - \theta \frac{f'}{f} \ge 0.$$

So, f'/f^θ is nondecreasing, hence bounded above near 0. Integrating again, we deduce that $f^{1-\theta}(u) \le Cu + C'$ near 0, which is not possible if $f(0) = 0$. □

Define now $p_0 \in \overline{\mathbb{R}}$, the conjugate exponent of q_0, by

$$1/p_0 + 1/q_0 = 1. \tag{8}$$

The exponent p_0 must be understood as a measure of the "flatness" of f at 0. All nonlinearities f such that (3) holds and which either are analytic at the origin, or have at least one nonzero derivative at the origin, or are merely of the form $f(u) = u^p g(u)$, where $p \ge 1$ and $g(0) \ne 0$, satisfy (7). Exponentially flat functions such as $f(u) = e^{-1/u^2}$ also qualify (with $p_0 = +\infty$). However, although we have not tried to prove it, there most probably exist (convex increasing) nonlinearities failing (7). This being said, we establish the following theorem.

Theorem 1.5. *Assume $f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+_*)$ is nondecreasing, convex, $f > 0$ in $\mathbb{R}^+_*$ and (7) holds. Assume $u \in C^2(\mathbb{R}^N)$ is a bounded, nonnegative, stable solution of (1). Then $u \equiv 0$ if any one of the following conditions holds:*

1. $1 \le N \le 9$,
2. $N = 10$ *and* $p_0 < +\infty$, *where* p_0 *is given by* (8),
3. $N \ge 11$ *and* $p_0 < p_c(N)$, *where* p_0 *is given by* (8) *and* $p_c(N)$ *by* (5).

Remark 1.6. Theorem 1.5 was first proved by A. Farina when $f(u) = |u|^{p-1}u$. See [14]. As observed e.g. in [14], for $N \ge 11$, there exists a nonconstant bounded positive stable solution for $f(u) = |u|^{p-1}u$ as soon as $p \ge p_c(N)$. So our result is sharp in the class of power-type nonlinearities for $N \ge 11$. We do not know whether Theorem 1.5 remains true when $N = 10$ and $p_0 = +\infty$. Nor do we know if for $N \le 10$, assumption (7) can be completely removed. See Theorem 1.11 in Section 1.2 for partial results in this direction. See also [21] for a positive answer in the radial case.

1.2. Some generalizations: unbounded and sign-changing solutions, beyond power-type nonlinearities

First, we discuss the case of unbounded solutions. When $f(u) = |u|^{p-1}u$, the assumption $u \in L^\infty(\mathbb{R}^N)$ is unnecessary (see [14]). For general power-type nonlinearities, Theorem 1.5 remains true for unbounded solutions under an additional assumption on the behaviour of f at $+\infty$:

Corollary 1.7. *Assume as before that $f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+_*)$ is nondecreasing, convex, $f > 0$ in $\mathbb{R}^+_*$ and* (7) *holds. Let $\underline{p_\infty} \in \overline{\mathbb{R}}$ be defined by*

$$\overline{q_\infty} := \limsup_{u \to +\infty} q(u), \qquad 1/\underline{p_\infty} + 1/\overline{q_\infty} = 1. \tag{9}$$

Let $u \in C^2(\mathbb{R}^N)$ denote a nonnegative, stable solution of (1). *Then $u \equiv 0$ if any one of the following conditions holds:*

1. $1 \leq N \leq 9$ *and* $1 < \underline{p_\infty}$,
2. $N = 10$, $p_0 < +\infty$ *and* $1 < \underline{p_\infty} < +\infty$,
3. $N \geq 11$, $p_0 < p_c(N)$ *and* $1 < \underline{p_\infty} < p_c(N)$.

Next, we look at solutions which may change sign. When $f(u) = |u|^{p-1}u$, the assumption $u \geq 0$ is also unnecessary (see [14]). For power-type nonlinearities, Theorem 1.5 can be extended to the case of solutions of arbitrary sign if f is odd:

Corollary 1.8. *Assume that $f \in C^0(\mathbb{R}) \cap C^2(\mathbb{R}^+_*)$ is nondecreasing and that when restricted to $\mathbb{R}^+_*$, f is convex and $f > 0$. Assume* (7) *holds. Assume in addition that f is odd. Let $u \in C^2(\mathbb{R}^N)$ denote a bounded, stable solution of* (1). *Then $u \equiv 0$ if any one of the following conditions holds:*

1. $1 \leq N \leq 9$ *and* $1 < p_0$,
2. $N = 10$ *and* $1 < p_0 < +\infty$,
3. $N \geq 11$ *and* $1 < p_0 < p_c(N)$.

Remark 1.9. The above corollary remains true if f is not odd but simply $f(0) = 0$ and the assumptions made on f also hold for $\tilde{f}$ defined for $u \in \mathbb{R}^+$ by $\tilde{f}(u) = -f(-u)$.

Corollary 1.10. *Assuming in addition $1 < \underline{p_\infty}$ if $N \leq 9$ (respectively $1 < \underline{p_\infty} < +\infty$ if $N = 10$ and $1 < \underline{p_\infty} < p_c(N)$ when $N \geq 11$), Corollary* 1.8 *remains valid for any stable solution. That is, one can drop from Corollaries* 1.7 *and* 1.8 *both assumptions $u \geq 0$ and $u \in L^\infty(\mathbb{R}^N)$.*

Finally, we study nonlinearities for which (7) fails. To do so, we introduce $\overline{q_0}, \underline{q_0} \in \overline{\mathbb{R}}$ defined by

$$\overline{q_0} = \limsup_{u \to 0^+} q(u), \qquad \underline{q_0} = \liminf_{u \to 0^+} q(u). \tag{10}$$

Theorem 1.11. *Assume $f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+_*)$ is nondecreasing, convex, $f > 0$ in $\mathbb{R}^+_*$ and let $\overline{q_0}, \underline{q_0}$ be defined by* (10). *Assume $u \in C^2(\mathbb{R}^N)$ is a bounded, nonnegative, stable solution of* (1). *Then $u \equiv 0$ if any one of the following conditions holds:*

1. $N \leq 4$,
2. $5 \leq N$ *and* $\underline{q_0} > N/2$,
3. $5 \leq N \leq 6$ *and* $\overline{q_0} < \infty$,
4. $5 \leq N$, $\overline{q_0} < \infty$ *and* $\frac{4}{N-2}(1 + 1/\sqrt{\overline{q_0}}) > 1/\underline{q_0}$.

Remark 1.12. The above theorem is of particular interest when f' is convex or concave near the origin. Assume $f(0) = f'(0) = 0$ (this is not restrictive, see Remark 3.3). Apply Cauchy's mean value theorem: given $u_n \in \mathbb{R}^+_*$, there exists $v_n \in (0, u_n)$ such that

$$q(u_n) = \left.\frac{f'^2}{ff''}\right|_{u=u_n} = \left.\frac{2f'f''}{f'f'' + ff'''}\right|_{u=v_n}.$$

If $f''' \geq 0$ near 0, we deduce that $\overline{q_0} \leq 2$. By case 3 of Theorem 1.11, we conclude that if f' is convex near 0 and $N \leq 6$, then $u \equiv 0$. Similarly, if f' is concave near 0, then $\underline{q_0} \geq 2$. By case 4 of the theorem, if f' is concave near 0, $N \leq 10$ and $\overline{q_0} < +\infty$, then $u \equiv 0$.

Remark 1.13. Our methods yield absolutely no result under the assumption $10 \geq N \geq 5$ and $\underline{q_0} \leq N/2 < \overline{q_0} = \infty$.

1.3. Solutions which are stable outside a compact set

Stability outside a compact set is a much weaker requirement than stability throughout $\mathbb{R}^N$. For example, the well-known bubble solutions of (1) for the critical nonlinearity $f(u) = u^{(N+2)/(N-2)}$ are stable only outside a compact set. The same observation can be made for any solution of (1) having finite Morse index (see for instance [12], [14]). Setting aside the case where f is a power or an exponential nonlinearity, little is known about the classification of solutions of (1) which are stable outside a compact set, even in the radial case. Now, recall the definition of the critical exponents given in (4) and (5). As demonstrated in [14], the nonlinearities $f(u) = |u|^{p-1}u$, $p = p_S(N)$, $N \geq 3$ and $p \geq p_c(N)$, $N \geq 11$, must be singled out. For such values of p, radial solutions which are stable outside a compact set are nontrivial and completely classified, while for other values of $p > 1$, all solutions which are stable outside a compact set (whether radial or not) must be constant. See [14]. When dealing with more general nonlinearities, the first basic step consists in determining the behaviour of a solution u at infinity. This can be done by exploiting the classification of stable solutions obtained in Theorem 1.5 and Corollary 1.8:

Proposition 1.14. *Assume $f \in C^0(\mathbb{R})$. Assume $u = 0$ is the only bounded stable C^2 solution of* (1). *If $u \in C^2(\mathbb{R}^N)$ is a bounded solution of* (1) *which is stable outside a compact set, then*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Remark 1.15. As follows from the proof, the same result is valid for bounded *positive* solutions which are stable outside a compact set, under the weaker assumption that all bounded *positive* stable solutions of the equation are constant.

Remark 1.16. If $f'(0) > 0$, then in fact there exists no bounded solution of (1) which is stable outside a compact set. See the proof of Proposition 1.14 below.

Remark 1.17. Clearly, if we assume instead that f vanishes only at $u_0 \neq 0$ then $\lim_{|x|\to\infty} u(x) = u_0$. Similarly, we leave it to the reader to check that if the set of zeros of f is totally disconnected and the only bounded stable solutions of the equation are constant, then $\lim_{|x|\to\infty} u(x) = u_0$, where u_0 is a zero of f.

Remark 1.18. We do not know if a version of Proposition 1.14 holds if one assumes that f vanishes only at $-\infty$ or $+\infty$. If $f(u) = e^u$ and $N = 2$ (see e.g. [13]), there exist (infinitely many) solutions of (1) which are stable outside a compact set and such that

$$\lim_{|x|\to\infty} u(x) = -\infty.$$

Proof of Proposition 1.14. For $k \geq 1$, let $\tau_k \in \mathbb{R}^N$ be such that $\lim_{k\to\infty} |\tau_k| = +\infty$ and let $u_k(x) = u(x + \tau_k)$ for $x \in \mathbb{R}^N$. Standard elliptic regularity implies that a subsequence of (u_k) converges in the topology of $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ to a solution v of (1). In addition, since u is stable outside a compact set, v is stable. Therefore, v is constant and $f(v) = 0$, so $v = 0$. If $f'(0) > 0$, then $v = 0$ is clearly unstable, which is absurd. This proves Remark 1.16. In addition, since $v = 0$ is the unique cluster point of (u_k), the whole sequence must converge to 0, and Proposition 1.14 follows. □

In light of Proposition 1.14, it is natural to try to characterize the speed of decay of our solutions as $|x| \to \infty$. When f is power-type, we have the following:

Theorem 1.19. *Assume $f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+_*)$ is nondecreasing, convex, $f > 0$ in $\mathbb{R}^+_*$, $f(0) = 0$ and* (7) *holds: Assume $u \in C^2(\mathbb{R}^N)$ is a bounded positive solution of* (1) *which is stable outside a compact set. If any one of the following conditions holds:*

1. $1 \leq N \leq 9$,
2. $N = 10$ *and* $p_0 < +\infty$,
3. $N \geq 11$ *and* $p_0 < p_c(N)$,

then there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^N$ sufficiently large,

$$u(x) \leq Cs(|x|). \tag{11}$$

In the above inequality, the speed of decay $s(R)$ is defined for $R > 0$ as the unique solution $s = s(R)$ of

$$f(A_1 R^2 f(s)) = A_2 f(s), \tag{12}$$

where A_1, A_2 are two positive constants depending on N only. In other words, s is given by $s(R) = f^{-1}(C_1 R^{-2} g(C_2 R^{-2}))$ where C_1, C_2 are two positive constants depending on N only and g is the inverse function of $t \mapsto f(t)/t$.

Remark 1.20. In the above theorem, we have implicitly assumed that the functions f and $t \mapsto f(t)/t$ are invertible in a neighborhood of 0. This is indeed true: by convexity of f, $t \mapsto f(t)/t$ is nondecreasing. By Step 6 in Section 6, we must have $f(0) = 0$ and $\lim_{t\to 0^+} f(t)/t = 0$. If there existed two values $0 < t_1 < t_2$ such that $f(t_1)/t_1 = f(t_2)/t_2$, then, by convexity, f would be linear on (t_1, t_2), hence on $(0, t_2)$ by convexity. This contradicts $\lim_{t\to 0^+} f(t)/t = 0$. So, $t \mapsto f(t)/t$ is invertible for $t > 0$ small and so too must be f.

Remark 1.21. Equation (12) looks somewhat complicated at first glance. For many nonlinearities (including $f(u) = |u|^{p-1}u$), one can actually set the constants A_1, A_2 equal to 1. Then (12) takes the simplified form

$$f(s)/s = R^{-2}.$$

In particular, when $f(u) = |u|^{p-1}u$, we recover the familiar speed $s(R) = R^{-2/(p-1)}$.

Remark 1.22. If $p_0 < \infty$, then for each $\epsilon > 0$ there exists $C > 0$ such that

$$s(R) \le CR^{-2/(p_0-1)+\epsilon} \quad \text{for } R \ge 1.$$

However, even when $p_0 < \infty$, there do exist nonlinearities f failing the estimate $s(R) \le CR^{-2/(p_0-1)}$.

Proof of Remark 1.22. An easy calculation shows that for all $\delta > 0$ small, there exist $C, \varepsilon > 0$ such that $C^{-1}u^{p_0+\delta} \le f(u) \le Cu^{p_0-\delta}$ and $C^{-1}u^{p_0+\delta-1} \le f'(u) \le Cu^{p_0-\delta-1}$ for $u \in (0, \varepsilon)$ provided (7) holds and $p_0 < +\infty$. Plugging this information into the definition of $s(R)$ yields the desired conclusion. □

From now on, our aim is to prove a Liouville-type result for solutions which are stable outside a compact set. As follows from the analysis in [14], we must distinguish the subcritical and supercritical cases. We first consider the case where p_0 is *subcritical*, i.e.

$$p_0 < \infty,\ N \le 2 \quad \text{or} \quad p_0 < p_S(N),\ N \ge 3. \tag{13}$$

In this case, we make the following extra global assumption on f:

$$(p_0 + 1)F(s) \ge sf(s) \quad \text{for all } s \in \mathbb{R}, \tag{14}$$

where F denotes the antiderivative of f vanishing at 0. Then we have

Theorem 1.23. *Assume $f \in C^0(\mathbb{R}^+) \cap C^2(\mathbb{R}^+_*)$ is nondecreasing, convex, $f > 0$ in $\mathbb{R}^*_+$ and* (7) *holds. Assume $u \in C^2(\mathbb{R}^N)$ is a bounded, nonnegative solution of* (1)*, which is stable outside a compact set. Assume p_0 is subcritical (i.e.* (13) *holds) and f satisfies the global inequality* (14)*. Then $u = 0$.*

We turn next to the supercritical case. We say that p_0 is in the *supercritical* range if

$$p_S(N) < p_0 < +\infty,\ 3 \le N \le 10 \quad \text{or} \quad p_S(N) < p_0 < p_c(N),\ N \ge 11. \tag{15}$$

In this case, we begin by showing that the asymptotic decay estimate (11) can be further improved. Namely, we show that not only $u(x) = O(s(|x|))$ but in fact $u(x) = o(s(|x|))$. The price we pay is the following set of assumptions *near the origin*: we require that there exist constants $\varepsilon, c_1, c_2 > 0$ such that

$$f(u) \ge c_1 u^{p_0} \qquad \text{for } u \in (0, \varepsilon), \tag{16}$$

$$f'(u) \le c_2 u^{p_0-1} \qquad \text{for } u \in (0, \varepsilon). \tag{17}$$

By convexity of f, the above inequalities reduce to one when $f(0) = 0$:

$$c_2 u^{p_0} \geq u f'(u) \geq f(u) \geq c_1 u^{p_0} \quad \text{for } u \in (0, \varepsilon). \tag{18}$$

Compare this assumption with the already known estimate given in the proof of Remark 1.22.

Corollary 1.24. *Make the same assumptions as in Theorem* 1.19. *Assume in addition that f satisfies the local estimates* (16), (17). *For p_0 in the supercritical range* (15), *any bounded positive solution $u \in C^2(\mathbb{R}^N)$ of* (1) *which is stable outside a compact set satisfies*

$$u(x) = o(|x|^{-2/(p_0-1)}) \quad \textit{and} \quad |\nabla u(x)| = o(|x|^{-2/(p_0-1)-1}) \quad \textit{as } |x| \to \infty. \tag{19}$$

Finally, to obtain the Liouville theorem in the supercritical range, we assume in addition that

$$(p_0 + 1)F(s) \leq s f(s) \quad \text{for all } s \in \mathbb{R}. \tag{20}$$

Note that the inequality is reversed compared to (14). Also note that since f is nondecreasing, we automatically have $F(s) \leq s f(s)$. (20) can thus be seen as an improved global convexity assumption on F. We have

Theorem 1.25. *Assume $f \in C^2(\mathbb{R}^+)$ is nondecreasing, convex, $f > 0$ in $\mathbb{R}_+^*$ and* (7) *holds. Assume $u \in C^2(\mathbb{R}^N)$ is a bounded, nonnegative solution of* (1) *which is stable outside a compact set. Assume p_0 is in the supercritical range* (15) *and f satisfies the local bounds* (16), (17) *as well as the global inequality* (20). *Then $u \equiv 0$.*

Remark 1.26. As mentioned in Remark 1.6, the above theorem is false for exponents $p_0 \geq p_c(N)$, $N \geq 11$ and for $p_0 = p_S(N)$, $N \geq 3$.

Remark 1.27. For the nonlinearity $f(u) = |u|^{p-1}u$, all the extra assumptions (14), (20), (16), (17) are automatically satisfied.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.1. Theorem 1.5 is the object of Section 3. In Section 4, we discuss the extensions given in Corollaries 1.7, 1.8 and 1.10. Theorem 1.11, which deals with nonlinearities which are not of power-type, is proved in Section 5. Section 6 is devoted to the proof of Theorem 1.19, pertaining to the rate of decay of solutions which are stable outside a compact set. The refined asymptotics obtained in Corollary 1.24 is also derived in that section. Section 7 covers Theorem 1.23, dealing with subcritical nonlinearities, while the supercritical case is addressed in Section 8.

2. The case of low dimensions $1 \leq N \leq 4$: proof of Theorem 1.1

The proof bears resemblance to an argument found in [2] (see also Theorem 6.2 of [1]). It relies on two simple arguments: a growth estimate of the Dirichlet energy on balls and a Liouville-type result for certain divergence-form equations (due to Berestycki, Caffarelli and Nirenberg [3]), which applies to solutions with controlled energy. The specific form of the aforementioned equation is obtained by linearizing (1) and taking advantage of the stability assumption. The limitation $N \leq 4$ arises from the energy estimate on balls.

Proof of Theorem 1.1. For $R > 0$, let B_R denote the ball of radius R centred at the origin. We begin by proving that there exists a constant $C > 0$ independent of $R > 0$ such that

$$\int_{B_R} |\nabla u|^2 \, dx \leq C R^{N-2}. \tag{21}$$

Let $M \geq \|u\|_\infty$, $\varphi \in C_c^2(\mathbb{R}^N)$ and multiply (1) by $(u - M)\varphi$:

$$\int_{\mathbb{R}^N} -\Delta u (u - M)\varphi \, dx = \int_{\mathbb{R}^N} f(u)(u - M)\varphi \, dx.$$

Integrating by parts and recalling that $f \geq 0$, it follows that

$$\int_{\mathbb{R}^N} |\nabla u|^2 \varphi \, dx + \int_{\mathbb{R}^N} (u - M)\nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^N} f(u)(u - M)\varphi \, dx \leq 0,$$

whence

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2 \varphi \, dx &\leq -\int_{\mathbb{R}^N} \frac{1}{2}\nabla (u - M)^2 \nabla \varphi \, dx = \int_{\mathbb{R}^N} \frac{(u - M)^2}{2} \Delta \varphi \, dx \\ &\leq 2M^2 \int_{\mathbb{R}^N} |\Delta \varphi| \, dx.\end{aligned}$$

Let φ_0 denote any nonnegative test function such that $\varphi_0 = 1$ on B_1 and apply the above inequality with $\varphi(x) = \varphi_0(x/R)$. We obtain (21).

Since u is stable, there exists a solution $v > 0$ of the linearized equation

$$-\Delta v = f'(u) v \quad \text{in } \mathbb{R}^N. \tag{22}$$

Let $\sigma_j = \frac{1}{v}\frac{\partial u}{\partial x_j}$ for $j = 1, \ldots, N$. Then since v and $\partial u / \partial x_j$ both solve the linearized equation (22), it follows that

$$-\nabla \cdot (v^2 \nabla \sigma_j) = 0 \quad \text{in } \mathbb{R}^N. \tag{23}$$

It is known that any solution $\sigma \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ of (23) such that

$$\int_{B_R} v^2 \sigma^2 \leq C R^2$$

must be constant (see Proposition 2.1 in [2]). By (21), we deduce that if $N \leq 4$, then σ_j is constant, i.e. there exists a constant C_j such that

$$\frac{\partial u}{\partial x_j} = C_j v.$$

In particular, the gradient of u points in a fixed direction, i.e. u is one-dimensional and solves

$$-u'' = f(u) \quad \text{in } \mathbb{R}.$$

Since $f \geq 0$ and u is bounded, this is possible only if u is constant and $f(u) = 0$. □

3. The Liouville theorem for stable solutions: proof of Theorem 1.5

The proof is split into two separate cases, according to the value of q_0. We first consider the case $q_0 > N/2$. It suffices to prove the following lemma.

Lemma 3.1. *Assume $f \in C^2(\mathbb{R}^+)$, $f > 0$ is nondecreasing, convex and*

$$\underline{q_0} := \liminf_{u \to 0^+} q(u) > N/2.$$

Assume $u \in C^2(\mathbb{R}^N)$, $u \geq 0$ and

$$-\Delta u \geq f(u) \quad \text{in } \mathbb{R}^N. \tag{24}$$

Then $u \equiv 0$.

Remark 3.2. The above lemma is standard (see the earlier work [19], as well as [9] for the most general result in this direction). We provide a proof of Lemma 3.1 for the convenience of the reader.

Proof. Assume by contradiction that $u \not\equiv 0$. By the Strong Maximum Principle, $u > 0$.

Step 1. Since $\underline{q_0} > N/2$, there exists $q > N/2$ such that

$$\frac{f'' f}{f'^2} < \frac{1}{q}$$

in a neighborhood of 0. Equivalently, $\frac{f''}{f'} - \frac{1}{q}\frac{f'}{f} < 0$. Hence, the function $f'/f^{1/q}$ is decreasing near 0. In particular, there exists a constant $C > 0$ such that $f'/f^{1/q} \geq C$ near 0, which implies that for some $p < N/(N-2)$ and $c_1 > 0$,

$$f(u) \geq c_1 u^p. \tag{25}$$

The above inequality holds in a neighborhood of 0.

Step 2. Since $p < p_S(N)$, there exists $\varphi > 0$ solving

$$\begin{cases} -\Delta\varphi = c_1\varphi^p & \text{in } B_1, \\ \varphi = 0 & \text{on } \partial B_1. \end{cases} \tag{26}$$

We are going to prove that a rescaled version of φ must lie below u. Let indeed $R > 0$ and $\varphi_R(x) = R^{-2/(p-1)}\varphi(x/R)$ for $x \in B_R$, $\varphi_R(x) = 0$ for $|x| \geq R$. Then

$$\begin{cases} -\Delta\varphi_R = c_1(\varphi_R)^p & \text{in } B_R, \\ \varphi_R = 0 & \text{on } \partial B_R. \end{cases}$$

Furthermore, since $p < N/(N-2)$,

$$\frac{\|\varphi_R\|_{L^\infty(B_R)}}{R^{2-N}} \leq \frac{R^{-2/(p-1)}}{R^{2-N}}\|\varphi\|_{L^\infty(B_1)} \to 0 \quad \text{as } R \to +\infty. \tag{27}$$

Step 3. Since $u > 0$ is superharmonic, there exists a constant $c > 0$ such that

$$u(x) \geq c|x|^{2-N} \quad \text{for } |x| \geq 1. \tag{28}$$

Indeed, the above inequality clearly holds for $|x| = 1$, with $c = \min_{[|x|=1]} u$. In addition, the function $z = u - c|x|^{2-N}$ is superharmonic in $[1 \leq |x| \leq M]$. By the Maximum Principle we have $z \geq \min(0, \min_{[|x|=M]} z(x))$ in $[1 \leq |x| \leq M]$. Hence, $z \geq \liminf_{M\to\infty} \min(0, \min_{[|x|=M]} z(x)) = 0$, and (28) is established.

Step 4. Collecting (27) and (28), we obtain for $R > 0$ sufficiently large

$$u \geq \varphi_R.$$

We conclude using the celebrated sliding method: first, by (27), $\|\varphi_R\|_\infty \to 0$ as $R \to \infty$, so that by (25), $f(\varphi_R) \geq c_1(\varphi_R)^p$, provided R is sufficiently large. In particular,

$$-\Delta(u - \varphi_R) \geq f(u) - f(\varphi_R) \geq 0.$$

By the Strong Maximum Principle, $u > \varphi_R$. Next, we slide φ_R in a given direction, say $\tilde{\varphi}_{R,t}(x) = \varphi_R(x + te_1)$, where $e_1 = (1, 0, \ldots, 0)$. We want to prove that $u \geq \tilde{\varphi}_{R,t}$ for all $t \geq 0$. If not, there exists $t_0 \in (0, +\infty)$ such that $u \geq \tilde{\varphi}_{R,t_0}$ and $u(x_0) = \tilde{\varphi}_{R,t_0}(x_0)$ at some point $x_0 \in \mathbb{R}^N$. But again we have

$$-\Delta(u - \tilde{\varphi}_{R,t_0}) \geq f(u) - f(\tilde{\varphi}_{R,t_0}) \geq 0,$$

and the Strong Maximum Principle would imply that $u \equiv \tilde{\varphi}_{R,t_0}$. This is not possible since $\tilde{\varphi}_{R,t_0}$ is compactly supported while u is not. The above argument holds if e_1 is replaced by any other direction $e \in S^{N-1}$. In particular, $u \geq \max \varphi_R > 0$, which is possible, since u is superharmonic, only if u is constant. Since $f > 0$, we obtain a contradiction. Hence, $u \equiv 0$. □

Remark 3.3. If $f(0) \neq 0$ or $f'(0) \neq 0$, then (25) clearly holds in a neighborhood of 0 and we may work as above to conclude that u is constant. We may therefore assume for the rest of the proof that $f(0) = f'(0) = 0$.

We turn next to the case $q_0 \leq N/2$, which is a consequence of the following theorem.

Theorem 3.4. *Assume $f \in C^2(\mathbb{R}^+)$ is nondecreasing, convex, $f > 0$ in $\mathbb{R}^*_+$, (7) holds and $q_0 < +\infty$. Then the differential inequality*

$$-\Delta u \leq f(u) \quad \text{in } \mathbb{R}^N \tag{29}$$

does not admit any solution $u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $u > 0$ such that (2) holds if any one of the following conditions holds:

1. $1 \leq N \leq 9$,
2. $N = 10$ *and* $p_0 < +\infty$,
3. $N \geq 11$ *and* $p_0 < p_c(N)$,

Remark 3.5. With no change to the proof, Theorem 3.4 remains true if u is only assumed to be locally Lipschitz continuous. The differential inequality (29) must then be understood in the weak sense, i.e.

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi \, dx \leq \int_{\mathbb{R}^N} f(u)\varphi \, dx$$

for all Lipschitz functions $\varphi \geq 0$ with compact support.

It remains to prove Theorem 3.4. We begin with the following weighted Poincaré inequality.

Lemma 3.6. *Assume Ω is an arbitrary open set in $\mathbb{R}^N$. Let $u \in C^2(\Omega)$ with $u \geq 0$ satisfy*

$$-\Delta u \leq f(u) \quad \text{in } \Omega.$$

Assume in addition that for all $\varphi \in C^1_c(\Omega)$,

$$\int_\Omega f'(u)\varphi^2 \, dx \leq \int_\Omega |\nabla \varphi|^2 \, dx. \tag{30}$$

Let $\phi \in W^{1,\infty}_{\text{loc}}(\mathbb{R}; \mathbb{R})$ denote a convex function and $\eta \in C^1_c(\mathbb{R}^N)$. Let

$$\psi(u) = \int_0^u \phi'^2(t) \, dt.$$

Then

$$\int_\Omega [(f'\phi^2 - f\psi) \circ u]\eta^2 \, dx \leq \int_\Omega [\phi^2 \circ u]|\nabla \eta|^2. \tag{31}$$

Remark 3.7. If ϕ is not convex, then the following variant of (31) holds:

$$\int_\Omega [(f'\phi^2 - f\psi)\circ u]\eta^2\,dx \le \int_\Omega [K\circ u]\Delta(\eta^2)\,dx - \int_\Omega [\phi^2\circ u]\eta\Delta\eta\,dx, \qquad (32)$$

where $K(u) = \int_0^u \psi(s)\,ds$.

Proof of Lemma 3.6. Multiply (29) by $\psi(u)\eta^2$ and integrate by parts:

$$\int_\Omega \nabla u\nabla(\psi(u)\eta^2)\,dx \le \int_\Omega f(u)\psi(u)\eta^2\,dx,$$

$$\int_\Omega \phi'(u)^2|\nabla u|^2\eta^2\,dx + \int_\Omega \psi(u)\nabla u\nabla\eta^2\,dx \le \int_\Omega f(u)\psi(u)\eta^2\,dx,$$

$$\int_\Omega \phi'(u)^2|\nabla u|^2\eta^2\,dx - \int_\Omega K(u)\Delta\eta^2\,dx \le \int_\Omega f(u)\psi(u)\eta^2\,dx,$$

where $K(u) = \int_0^u \psi(s)\,ds$. Hence,

$$\int_\Omega \phi'(u)^2|\nabla u|^2\eta^2\,dx \le \int_\Omega K(u)\Delta\eta^2\,dx + \int_\Omega f(u)\psi(u)\eta^2\,dx. \qquad (33)$$

Next, we apply (30) with $\varphi = \phi(u)\eta$ to obtain

$$\begin{aligned}\int_\Omega f'(u)\phi(u)^2\eta^2\,dx &\le \int_\Omega |\nabla(\phi(u)\eta)|^2\,dx = \int_\Omega |\phi'(u)\eta\nabla u + \phi(u)\nabla\eta|^2\,dx,\\ &\le \int_\Omega \phi'(u)^2\eta^2|\nabla u|^2\,dx + \int_\Omega \phi(u)^2|\nabla\eta|^2\,dx + 2\int_\Omega \phi(u)\phi'(u)\eta\nabla\eta\nabla u\,dx\\ &\le \int_\Omega \phi'(u)^2\eta^2|\nabla u|^2\,dx + \int_\Omega \phi(u)^2|\nabla\eta|^2\,dx + \frac{1}{2}\int_\Omega \nabla\eta^2\nabla\phi(u)^2\,dx\\ &\le \int_\Omega \phi'(u)^2\eta^2|\nabla u|^2\,dx + \int_\Omega \phi(u)^2\left(|\nabla\eta|^2 - \frac{1}{2}\Delta\eta^2\right)dx.\end{aligned}$$

Plug (33) in the above. Then

$$\begin{aligned}&\int_\Omega (f'(u)\phi(u)^2 - f(u)\psi(u)^2)\eta^2\,dx\\ &\qquad \le \int_\Omega K(u)\Delta\eta^2\,dx + \int_\Omega \phi(u)^2\left(|\nabla\eta|^2 - \frac{1}{2}\Delta\eta^2\right)dx.\end{aligned}$$

This proves Remark 3.7. Finally, when ϕ is convex,

$$\psi(u) = \int_0^u \phi'^2(s)\,ds \le \phi'(u)\phi(u).$$

Integrating, we obtain $K \le \frac{1}{2}\phi^2$ and (31) follows. □

Proof of Theorem 3.4. Take $\alpha \geq 1$ and $\phi = f^{\alpha}$. In order to take advantage of Lemma 3.6, we need to make sure that the quantity $(f'\phi^2 - f\psi) \circ u$ remains nonnegative and better, bounded below by some positive function of u. Clearly, the best one can hope for is an inequality of the form

$$(f'\phi^2 - f\psi) \circ u \geq c f'\phi^2 \circ u.$$

To obtain such an inequality, we apply L'Hôpital's Rule:

$$\begin{aligned}\lim_{0^+} \frac{f'\phi^2}{f\psi} &= \lim_{0^+} \frac{f' f^{2\alpha-1}}{\psi} = \lim_{0^+} \frac{f'' f^{2\alpha-1} + (2\alpha - 1) f^{2\alpha-2} f'^2}{\alpha^2 f^{2\alpha-2} f'^2} \\ &= \frac{1}{\alpha^2}(1/q_0 + 2\alpha - 1) > 1,\end{aligned}$$

where the last inequality holds if $\alpha \in [1, 1 + 1/\sqrt{q_0})$. Note that this interval is nonempty since we assumed $q_0 < +\infty$. Hence, for some constant $c > 0$,

$$f'\phi^2 - f\psi \geq c f'\phi^2 \tag{34}$$

in a neighbourhood $[0, \epsilon]$ of the origin. Modifying ϕ, we can extend the above inequality to a given compact interval $[0, M]$ as follows. Take $\phi \in W^{1,\infty}_{\text{loc}}(\mathbb{R}; \mathbb{R})$ defined by

$$\phi(u) = \begin{cases} f(u)^{\alpha} & \text{if } 0 \leq u \leq \varepsilon, \\ f(\varepsilon)^{\alpha-1} f(u) \exp\left(\displaystyle\int_{\varepsilon}^{u} \sqrt{\frac{f''}{f}}\, ds\right) & \text{if } u > \varepsilon, \end{cases} \tag{35}$$

where ε, α are chosen as before. Then $\phi \in W^{1,\infty}_{\text{loc}}(\mathbb{R}; \mathbb{R})$. For $u > \varepsilon$, we claim that the quantity $\frac{f'}{f}\phi^2 - \psi$ is constant. Indeed,

$$\begin{aligned}\left(\frac{f'}{f}\phi^2 - \psi\right)' &= \left(\frac{f'}{f}\right)'\phi^2 + 2\frac{f'}{f}\phi\phi' - \phi'^2 = \left(\frac{f''}{f} - \frac{f'^2}{f^2}\right)\phi^2 + 2\frac{f'}{f}\phi\phi' - \phi'^2 \\ &= \frac{f''}{f}\phi^2 - \left(\frac{f'}{f}\phi - \phi'\right)^2 = \phi^2\left(\frac{f''}{f} - \left(\frac{f'}{f} - \frac{\phi'}{\phi}\right)^2\right) = 0,\end{aligned}$$

where we used the definition of ϕ in the last equality. So for $u > \varepsilon$,

$$\begin{aligned}f'\phi^2 - f\psi &= f\left(\frac{f'}{f}\phi^2 - \psi\right) = f\left(\frac{f'(\varepsilon)}{f(\varepsilon)}\phi^2(\varepsilon) - \psi(\varepsilon)\right) \geq f'(\varepsilon)\phi^2(\varepsilon) - f(\varepsilon)\psi(\varepsilon) \\ &\geq c_{\varepsilon} > 0,\end{aligned}$$

where we used (34) at $u = \varepsilon$. Since $f'\phi^2$ is bounded above by a constant on any compact interval of the form $[\varepsilon, M]$, we conclude that (34) holds throughout $[0, M]$ for a constant $c > 0$ perhaps smaller. We have just proved that given $\alpha \in [1, 1 + 1/\sqrt{q_0})$ and a bounded positive function u, there exists $c > 0$ such that

$$[f'\phi^2 - f\psi] \circ u \geq c[f'\phi^2] \circ u. \tag{36}$$

Recall that we established the above inequality in order to apply Lemma 3.6. Unfortunately, since the function ϕ we introduced in (35) may not be convex, we cannot apply Lemma 3.6 directly. We make use of (32) instead. In order to obtain a meaningful result, we need to understand how the different functions of u introduced in (32) compare. By definition of ϕ, we easily deduce the following set of inequalities:

$$\begin{cases} [f'\phi^2 - f\psi] \circ u \geq cf'f^{2\alpha} \circ u, \\ \phi^2 \circ u \leq Cf^{2\alpha} \circ u, \\ K \circ u \leq Cf^{2\alpha} \circ u. \end{cases} \tag{37}$$

So, we just need to relate f and f' to be able to compare all quantities involved in the estimate. Fix $q_1 < q_0$. By definition of q_0, there exists a neighbourhood of zero where

$$\frac{ff''}{f'^2} \leq 1/q_1.$$

In particular, $f'/f^{1/q_1}$ is nonincreasing and in a neighbourhood of zero we have

$$f' \geq cf^{1/q_1}. \tag{38}$$

By continuity, up to choosing $c > 0$ smaller, the above inequality holds in the whole range of a given bounded positive function u. Recall now (37), (38) and apply (32). The estimate reduces to

$$\int_{\mathbb{R}^N} [f^{1/q_1+2\alpha} \circ u]\eta^2\, dx \leq C \int_{\mathbb{R}^N} [f^{2\alpha} \circ u](|\nabla\eta|^2 + |\eta\Delta\eta|)\, dx.$$

Choose $\eta = \zeta^m$, $m \geq 1$, $\zeta \in C_c^2(\mathbb{R}^N)$, $1 \geq \zeta \geq 0$:

$$\begin{aligned} \int_{\mathbb{R}^N} [f^{1/q_1+2\alpha} \circ u]\zeta^{2m}\, dx &\leq C \int_{\mathbb{R}^N} [f^{2\alpha} \circ u](\zeta^{2m-2}|\nabla\zeta|^2 + \zeta^{2m-1}|\Delta\zeta|)\, dx \\ &\leq C \int_{\mathbb{R}^N} [f^{2\alpha} \circ u]\zeta^{2m-2}(|\nabla\zeta|^2 + |\Delta\zeta|)\, dx. \end{aligned}$$

Using Hölder's inequality, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^N} [f^{1/q_1+2\alpha} \circ u]\zeta^{2m}\, dx \\ &\qquad \leq C\left(\int_{\mathbb{R}^N} [f^{2\alpha m'} \circ u]\zeta^{2m}\, dx\right)^{1/m'} \left(\int_{\mathbb{R}^N} (|\nabla\zeta|^2 + |\Delta\zeta|)^m\, dx\right)^{1/m}. \end{aligned}$$

Assume temporarily that

$$f^{1/q_1+2\alpha} \circ u \geq cf^{2\alpha m'} \circ u. \tag{39}$$

Then the inequality simplifies to

$$\int_{\mathbb{R}^N} [f^{2\alpha m'} \circ u]\zeta^{2m}\, dx \leq C \int_{\mathbb{R}^N} (|\nabla\zeta|^2 + |\Delta\zeta|)^m\, dx.$$

Choose now ζ such that $\zeta \equiv 1$ in B_R and $|\nabla\zeta| \le C/R$, $|\Delta\zeta| \le C/R^2$:

$$\int_{\mathbb{R}^N} [f^{2\alpha m'} \circ u]\zeta^{2m}\, dx \le C R^{N-2m}. \tag{40}$$

The above inequality is true as soon as (39) holds, which itself reduces to choosing the exponents such that

$$2\alpha m' \ge 1/q_1 + 2\alpha.$$

This holds for some $q_1 < q_0$ provided $2\alpha(m'-1) > 1/q_0$. For $N \le 2$, fix such an $m' \in (1, +\infty)$ and let $R \to +\infty$ in (40). We conclude that $f \circ u = u = 0$. In dimension $N \ge 3$, since α can be chosen arbitrarily close to $1 + 1/\sqrt{q_0}$ and restricting to m' less than but as close as we wish to $N/(N-2)$, we finally need only assume

$$\frac{4}{N-2}(1 + 1/\sqrt{q_0}) > 1/q_0 \quad \text{if } N \ge 3. \tag{41}$$

Since $m' < N/(N-2)$, we have $N - 2m < 0$. So, the right-hand side of (40) converges to 0 as $R \to \infty$, whence $f \circ u = 0$ and $u = 0$, as desired. Solving (41) for q_0 yields the conditions stated in Theorem 3.4. □

4. Extensions to unbounded and sign-changing solutions

We deal first with possibly unbounded solutions.

Proof of Corollary 1.7. Note that by Lemma 3.1, we need only consider the case $q_0 < +\infty$. We modify the rest of the proof of Theorem 3.4 as follows: take $\phi \in W^{1,\infty}_{\text{loc}}(\mathbb{R};\mathbb{R})$ defined by

$$\phi(u) = \begin{cases} f(u)^\alpha & \text{if } 0 \le u \le \varepsilon, \\ f(\varepsilon)^{\alpha-1} f(u) \exp\left(\displaystyle\int_\varepsilon^u \sqrt{\frac{f''}{f}}\, ds\right) & \text{if } \varepsilon < u \le 1/\varepsilon, \\ f(u)^\beta + A & \text{if } u > 1/\varepsilon, \end{cases}$$

where α is chosen in $[1, 1 + 1/\sqrt{q_0})$ as previously, β in $[1, 1 + 1/\sqrt{q_\infty})$ and A such that ϕ is $W^{1,\infty}_{\text{loc}}(\mathbb{R};\mathbb{R})$. Then (36) holds if in addition

$$\liminf_{u \to +\infty} \frac{f'\phi^2}{f\psi}(u) > 1.$$

We leave to the reader the check that this is true under assumption (9), for $\beta \in [1, 1 + 1/\sqrt{q_\infty})$. Apply (32) with $\eta = \zeta^m$, $m \ge 1$, $\zeta \in C^2_c(\mathbb{R}^N)$, $0 \le \zeta \le 1$:

$$\begin{aligned} \int_{\mathbb{R}^N} [f'\phi^2 \circ u]\zeta^{2m}\, dx &\le C \int_{\mathbb{R}^N} [(\phi^2 + K) \circ u]\zeta^{2m-2}(|\nabla\zeta|^2 + |\Delta\zeta|)\, dx \\ &\le C\left(\int_{\mathbb{R}^N} [(\phi^2 + K)^{m'} \circ u]\zeta^{2m}\, dx\right)^{1/m'} \left(\int_{\mathbb{R}^N} (|\nabla\zeta|^2 + |\Delta\zeta|)^m\, dx\right)^{1/m}. \end{aligned} \tag{42}$$

By definition of ϕ and (38), there exist constants $c, c' > 0$ such that for $u \in [0, 1]$, $f'\phi^2(u) \geq cf' f^{2\alpha}(u) \geq c' f^{2\alpha+q_1}(u)$, where $q_1 < q_0$. We also clearly have the estimate $(\phi^2 + K)^{m'}(u) \leq Cf^{2\alpha m'}$ for $u \in [0, 1]$. So,

$$f'\phi^2 \geq c(\phi^2 + K)^{m'} \qquad \text{on } [0, 1],$$

provided that $2\alpha m' \geq 1/q_1 + 2\alpha$. Similarly, the reader will easily check using (9) that given $q_2 < \overline{q_\infty}$, there exists $c > 0$ such that

$$f' \geq cf^{1/q_2} \qquad \text{in } [1, +\infty),$$

whence $f'\phi^2 \geq c(\phi^2 + K)^{m'}$ in $[1, +\infty)$ provided that $2\beta m' \geq 1/q_2 + 2\alpha$. We conclude that

$$f'\phi^2 \circ u \geq c(\phi^2 + K)^{m'} \circ u, \tag{43}$$

provided that $2\alpha m' \geq 1/q_1 + 2\alpha$ and $2\beta m' \geq 1/q_2 + 2\alpha$. Since α can be chosen arbitrarily close to $1 + 1/\sqrt{q_0}$, β to $1 + 1/\sqrt{\overline{q_\infty}}$, q_1 to q_0, q_2 to $\overline{q_\infty}$ and m' to $N/(N-2)$ for $N \geq 3$ (respectively $m' \in (1, +\infty)$ for $N \leq 2$), we conclude that suitable parameters can be chosen provided (41) holds and either $N \leq 2$ or

$$\frac{4}{N-2}(1 + 1/\sqrt{\overline{q_\infty}}) > 1/\overline{q_\infty} \qquad \text{if } N \geq 3.$$

These inequalities are true under the assumptions of Corollary 1.7. So, collecting (42) and (43), we obtain for some $m > N/2$,

$$\int_{\mathbb{R}^N} [(\phi^2 + K)^{m'} \circ u]\zeta^{2m}\, dx \leq C \int_{\mathbb{R}^N} (|\nabla\zeta|^2 + |\Delta\zeta|)^{m'}\, dx. \tag{44}$$

Finally, choose ζ such that $\zeta \equiv 1$ in B_R and $|\nabla\zeta| \leq C/R$, $|\Delta\zeta| \leq C/R^2$: the right-hand side of (44) converges to 0 as $R \to \infty$ and the conclusion follows. □

We work next with sign-changing solutions.

Proof of Corollaries 1.8 and 1.10. We simply remark that if $u \in C^2(\mathbb{R}^N)$ is a solution of (1), then u^+ (respectively u^-) is locally Lipschitz continuous and solves the differential inequality (29) (respectively $-\Delta u^- \leq \tilde{f}(u^-)$ in $\mathbb{R}^N$, where $\tilde{f}(t) := -f(-t)$ for $t \in \mathbb{R}^-$). Since we assumed $q_0 < +\infty$, we may then apply Theorem 3.4 and Remark 3.5, and Corollary 1.8 follows. For Corollary 1.10, we replace Theorem 3.4 by the adaptation presented in the proof of Corollary 1.7. □

5. Beyond power-type nonlinearities

Proof of Theorem 1.11. Case 1 of the theorem was proved in Theorem 1.1, while case 2 was proved in Lemma 3.1. For cases 3 and 4 take $\alpha \geq 1$ and $\phi = f^\alpha$. Let $L = \liminf_{0^+} f'\phi^2/f\psi$ and let (u_n) denote a sequence along which $f'\phi^2/f\psi$ converges to L.

By Remark 3.3, we may always assume that $f(0) = 0$. So, applying Cauchy's mean value theorem, there exists $v_n \in (0, u_n)$ such that

$$\frac{f'\phi^2}{f\psi}(u_n) = \frac{f' f^{2\alpha-1}}{\psi}(u_n) = \frac{f'' f^{2\alpha-1} + (2\alpha-1) f^{2\alpha-2} f'^2}{\alpha^2 f^{2\alpha-2} f'^2}\bigg|_{u=v_n}.$$

Passing to the limit, we obtain

$$L = \liminf_{0^+} \frac{f'\phi^2}{f\psi} \geq \frac{1}{\alpha^2}\left(\frac{1}{\overline{q_0}} + 2\alpha - 1\right) > 1, \tag{45}$$

where the last inequality holds if $\alpha \in [1, 1 + 1/\sqrt{\overline{q_0}})$. Note that this interval is nonempty since we assumed $\overline{q_0} < \infty$. At this point, we repeat the steps performed in the proof of Theorem 3.4: from (45), we deduce that (34) holds in a neighborhood $[0, \varepsilon]$ of the origin. Modifying ϕ as in (35), the *verbatim* arguments lead to (36) and (37). For the rest of the proof, we argue slightly differently according to the case considered.

Case 2 of Theorem 1.11. In place of (38), we simply use the convexity of f. Since u is bounded, there exists a constant $c > 0$ such that

$$f'(u) \geq f(u)/u \geq cf(u).$$

So, (40) holds for some $m > N/2$ whenever $\frac{4}{N-2}(1 + 1/\sqrt{\overline{q_0}}) > 1$, which is true for $N \leq 6$, provided $\overline{q_0} < \infty$. Following the proof of Theorem 3.4, we obtain case 2 of Theorem 1.11.

Case 3 of Theorem 1.11. By definition of $\underline{q_0}$, (38) now holds for $q_1 < \underline{q_0}$. Resuming our inspection of the proof of Theorem 3.4, we see that (40) holds under assumption 3 of Theorem 1.11, and the desired conclusion follows. □

6. Speed of decay: proof of Theorem 1.19

In this section, we characterize the speed of decay of solutions which are stable outside a compact set. To do so, we shall again take advantage of Lemma 3.6 or actually its general form (32), with a different choice of test function $\phi \circ u$. We divide the proof into several steps.

Step 1. We begin by proving the usual estimate

$$[f'\phi^2 - f\psi](u) \geq c[f'\phi^2](u)$$

where this time $\phi(u) = (f(u)/u)^\alpha$ and α is chosen in a suitable range.

First, by Lemma 3.1 and Remark 3.3, we may restrict to the case where $q_0 < +\infty$, whence $p_0 > 1$, and we may also assume $f(0) = f'(0) = 0$. By Proposition 1.14, $\lim_{|x|\to\infty} u(x) = 0$. For $u \in \mathbb{R}_+^*$, take $\phi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ defined by

$$\phi(u) = (f(u)/u)^\alpha, \tag{46}$$

where $\alpha > 1/2$. We begin by computing

$$L = \liminf_{u \to 0^+} \frac{f'\phi^2}{f\psi}(u). \tag{47}$$

Let (u_n) denote a sequence along which $f'\phi^2/f\psi$ converges to L. Observe that since $f(0) = f'(0) = 0$, we have $\psi(0) = 0$ and

$$\lim_{u \to 0} f' f^{2\alpha-1} u^{-2\alpha} = \lim_{u \to 0} \frac{f'(u)}{u} \left(\frac{f(u)}{u} \right)^{2\alpha-1} = 0 \tag{48}$$

if $\alpha > 1/2$. So, by Cauchy's mean value theorem, there exists $v_n \in (0, u_n)$ such that

$$\begin{aligned}
\frac{f'\phi^2}{f\psi}(u_n) &= \left.\frac{f' f^{2\alpha-1} u^{-2\alpha}}{\psi}\right|_{u=u_n} \\
&= \left.\frac{f'' f^{2\alpha-1} u^{-2\alpha} + (2\alpha-1) f'^2 f^{2\alpha-2} u^{-2\alpha} - 2\alpha f' f^{2\alpha-1} u^{-2\alpha-1}}{\alpha^2 u^{-2\alpha-2} f^{2\alpha} (-1 + uf'/f)^2}\right|_{u=v_n} \\
&= \left.\frac{f'' u^2/f + (2\alpha-1) f'^2 u^2/f^2 - 2\alpha u f'/f}{\alpha^2(-1 + uf'/f)^2}\right|_{u=v_n} \\
&= \left.\frac{ff''/f'^2 + (2\alpha-1) - 2\alpha f/(uf')}{\alpha^2(1 - f/(uf'))^2}\right|_{u=v_n}.
\end{aligned}$$

For $u \in \mathbb{R}_+^*$, let

$$p(u) = \frac{uf'(u)}{f(u)}. \tag{49}$$

It follows that

$$\frac{f'\phi^2}{f\psi}(u_n) = \left.\frac{1/q + 2\alpha - 1 - 2\alpha/p}{\alpha^2(1 - 1/p)^2}\right|_{u=v_n} = 1 + \left.\frac{1/q - (\alpha(1 - 1/p) - 1)^2}{\alpha^2(1 - 1/p)^2}\right|_{u=v_n}. \tag{50}$$

We claim that (7) implies

$$p_0 = \lim_{u \to 0^+} p(u), \tag{51}$$

where p_0 is the conjugate exponent of q_0, i.e. (8) holds. Take indeed any cluster point p_1 of p and a sequence (u_n) such that p converges to p_1 along (u_n). By Cauchy's mean value theorem, there exists $v_n \in (0, u_n)$ such that

$$p(u_n) = \left.\frac{f' + uf''}{f'}\right|_{u=v_n} = 1 + p/q(v_n).$$

Let $\underline{p_0} = \liminf_{u \to 0^+} p(u)$ and $\overline{p_0} = \limsup_{u \to 0^+} p(u)$. Pass to the limit as $n \to +\infty$:

$$1 + \underline{p_0}/q_0 \le p_1 \le 1 + \overline{p_0}/q_0.$$

Applying the above inequality to $p_1 = \underline{p_0}, \overline{p_0}$, we obtain

$$\overline{p_0}(1 - 1/q_0) \leq 1 \leq \underline{p_0}(1 - 1/q_0)$$

and (51) follows. Next, we apply (51) in (50). Thus,

$$L = 1 + \frac{1/q_0 - (\alpha/q_0 - 1)^2}{\alpha^2/q_0^2}.$$

So, $L > 1$ if

$$\alpha \in (q_0 - \sqrt{q_0}, q_0 + \sqrt{q_0}). \tag{52}$$

We conclude that given $\alpha > 1/2$ in the range (52), there exists $c > 0$ such that for u small enough,

$$[f'\phi^2 - f\psi](u) \geq c[f'\phi^2](u) \geq c(f(u)/u)^{2\alpha+1}, \tag{53}$$

where we used the convexity of f in the last inequality. Note that since $u(x) \to 0$ as $|x| \to +\infty$, the above inequality holds for $u = u(x)$ and x in the complement of a ball of large radius.

Step 2. Next, we need to estimate the other functions of u appearing in (32). We claim that for small values of u,

$$K(u) \leq C(f(u)/u)^{2\alpha}. \tag{54}$$

To see this, it suffices to prove that $\limsup_{u\to 0^+} K(u)/\phi^2(u) < \infty$. Take a sequence (u_n) converging to zero and apply Cauchy's mean value theorem: there exists $v_n \in (0, u_n)$ such that

$$\frac{K}{\phi^2}(u_n) = \frac{\psi}{2\phi\phi'}(v_n).$$

It follows from (53) that $f'\phi^2 - f\psi \geq 0$ for small u. So, $\psi(v_n) \leq [f'\phi^2/f](v_n)$ for large n, so that

$$\frac{K}{\phi^2}(u_n) \leq \frac{f'\phi}{2f\phi'}(v_n) = \frac{1}{2\alpha(1 - 1/p(v_n))}.$$

Recalling (51) and since we assumed that $p_0 > 1$, (54) follows.

Step 3. In this step, we prove an estimate of the form

$$\int_{B_R(x_0)} \left(\frac{f(u)}{u}\right)^{2\alpha+1} dx \leq CR^{N-2m},$$

where $m = 2\alpha + 1$ and $B_R(x_0)$ is a suitably chosen ball shifted towards infinity.

Choose $\zeta \in C_c^2(\mathbb{R}^N)$, $0 \leq \zeta \leq 1$, supported outside a ball $B_{R_0}(0)$ of large radius, so that (2) holds for functions supported outside $B_{R_0}(0)$ and that (53) and (54) hold for

$u = u(x)$, $x \in \operatorname{supp} \zeta$. By Lemma 3.6, we may apply (32) with $\eta = \zeta^m$, $m \geq 1$. Using (53), (54) and the convexity of f, we obtain, for $\alpha > 1/2$ in the range (52),

$$\int_{\mathbb{R}^N} \left(\frac{f(u)}{u}\right)^{2\alpha+1} \zeta^{2m}\, dx \leq \int_{\mathbb{R}^N} f'(u) \left(\frac{f(u)}{u}\right)^{2\alpha} \zeta^{2m}\, dx$$
$$\leq C \int_{\mathbb{R}^N} \left(\frac{f(u)}{u}\right)^{2\alpha} \zeta^{2m-2}(|\nabla \zeta|^2 + |\Delta \zeta|)\, dx.$$

Fix $m = 2\alpha + 1$ and apply Hölder's inequality. It follows that

$$\int_{\mathbb{R}^N} \left(\frac{f(u)}{u}\right)^{2\alpha+1} \zeta^{2m}\, dx \leq C \int_{\mathbb{R}^N} (|\nabla \zeta|^2 + |\Delta \zeta|)^{2m}\, dx. \tag{55}$$

We work on balls shifted towards infinity. More precisely, we take a point $x_0 \in \mathbb{R}^N$ such that $|x_0| > 10R_0$ and set $R = |x_0|/4$. Then $B_{2R}(x_0) \subset \{x \in \mathbb{R}^N : |x| \geq R_0\}$ and we may apply (55) with $\zeta = \varphi(|x - x_0|/R)$ and $\varphi \in C_c^2(\mathbb{R})$ given by

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \geq 2. \end{cases}$$

We get

$$\int_{B_R(x_0)} \left(\frac{f(u)}{u}\right)^{2\alpha+1} dx \leq C_3 R^{N-2m}. \tag{56}$$

Step 4. In this step, we prove the estimate

$$R^\epsilon \|f(u)/u\|_{L^{N/(2-\varepsilon)}(B_R(x_0))} \leq C.$$

By Lemma 1.4, $q_0 \geq 1$. Under the assumptions of Theorem 1.19, we can choose the exponent m so large that for small $\varepsilon > 0$, $m > N/(2 - \varepsilon)$ (recall that $m = 2\alpha + 1$ and $\alpha > 1/2$ can be chosen freely in the range (52)). Furthermore, by Hölder's inequality and (56), we obtain

$$R^\epsilon \|f(u)/u\|_{L^{N/(2-\varepsilon)}(B_R(x_0))} \leq R^\epsilon \|f(u)/u\|_{L^m(B_R(x_0))} |B_R|^{(2-\varepsilon)/N-1/m}$$
$$\leq C R^\varepsilon (R^{N-2m})^{1/m} R^{2-\varepsilon-N/m} = C. \tag{57}$$

Step 5. Now, we think of u as a solution of a linear problem, namely

$$-\Delta u = \frac{f(u)}{u} u =: V(x)u \quad \text{in } \mathbb{R}^N. \tag{58}$$

According to classical results of J. Serrin [18] and N. Trudinger [20] (see also Theorem 7.1.1 on page 154 of [17]), for any $p \in (1, +\infty)$ and any $x_0 \in \mathbb{R}^N$, there exists a constant

$$C_S = C_S(R^\varepsilon \|V\|_{L^{N/(2-\varepsilon)}(B_{2R}(x_0))}, N, p) > 0$$

such that

$$\|u\|_{L^\infty(B_R(x_0))} \leq C_S R^{-N/p} \|u\|_{L^p(B_{2R}(x_0))}. \tag{59}$$

Note that for our choice of x_0, (57) holds and so C_S is a true constant, independent of R and x_0.

Step 6. The inequality (59) gives a pointwise estimate in terms of an integral average of u. In order to control the latter, we consider the average $\tilde{u}$ of u over the sphere $\partial B_r(x_0)$, defined for $r > 0$ by $\tilde{u}(r) = \fint_{\partial B_r(x_0)} u \, d\sigma$. We claim that there exists $C = C(N) > 0$ such that

$$\frac{f(\tilde{u}(r))}{\tilde{u}(r)} \leq \frac{C}{r^2}. \tag{60}$$

To prove this, we first observe that since f is convex, $\tilde{u}$ satisfies the differential inequality

$$-\tilde{u}'' - \frac{N-1}{r}\tilde{u}' \geq f(\tilde{u}).$$

Now, since $f \geq 0$, we have $\tilde{u}' \leq 0$. In particular $r \mapsto f(\tilde{u}(r))$ is nonincreasing. Fix $\lambda \in (0, 1)$ and integrate the differential inequality between 0 and r:

$$-\tilde{u}'(r) \geq r^{1-N} \int_0^r s^{N-1} f(\tilde{u}(s)) \, ds \geq r^{1-N} \int_0^{\lambda r} s^{N-1} f(\tilde{u}(s)) \, ds \geq \frac{\lambda^N r f(\tilde{u}(\lambda r))}{N}.$$

Integrate a second time between r and r/λ to obtain

$$\tilde{u}(r) \geq \tilde{u}(r/\lambda) + \frac{\lambda^N}{N} \int_r^{r/\lambda} s f(\tilde{u}(\lambda s)) \, ds \geq r^2 f(\tilde{u}(r)) \frac{\lambda^N}{2N} \left(\frac{1}{\lambda^2} - 1 \right).$$

If we take $\lambda = (N-2)/N$, (60) follows with $C = \frac{1}{2N}\left(\frac{N-2}{N}\right)^N \left(\left(\frac{N}{N-2}\right)^2 - 1\right)$.

Step 7. Recall that we are trying to establish an L^p estimate, $p > 1$, in order to use (59). To start, we use (60) to obtain an L^1 estimate of $f(u)$. Namely, we prove that there exist constants $C_1, C_2 > 0$, depending on N only, such that

$$\fint_{B_R(x_0)} f(u) \, dx \leq C_1 R^{-2} g(C_2/R^2), \tag{61}$$

where g is the inverse function of $t \mapsto f(t)/t$, which exists for small values of t by Remark 1.20. For simplicity, we write B_R in place of $B_R(x_0)$ in what follows. To prove (61), observe that for $r \in (R, 2R)$,

$$\begin{aligned}
\int_{B_R} f(u) \, dx &= c_N R^{N-2} \int_R^{2R} r^{1-N} \, dr \int_{B_R} f(u) \, dx \\
&\leq c_N R^{N-2} \int_R^{2R} r^{1-N} \, dr \int_{B_r} f(u) \, dx = c_N R^{N-2} \int_R^{2R} r^{1-N} \, dr \int_{B_r} -\Delta u \, dx \\
&\leq -c_N R^{N-2} \int_R^{2R} r^{1-N} \, dr \int_{\partial B_r} \frac{\partial u}{\partial n} \, d\sigma = -c_N R^{N-2} \int_R^{2R} \tilde{u}' \, dr \leq c_N R^{N-2} \tilde{u}(R).
\end{aligned}$$

Estimate (61) follows, using (60).

Step 8. The assumptions on f allow us to convert (61) into an L^p estimate. Indeed, since $q_0 < \infty$ (in fact, one only needs $\overline{q_0} < \infty$), one can easily check that there exists $p > 1$ such that the function $h(t) = f(t^{1/p})$ is convex for small t. By Jensen's inequality,

$$h\left(\fint_{B_R} u^p \, dx \right) \le \fint_{B_R} f(u) \, dx \le C_1 R^{-2} g(C_2/R^2).$$

By Remark 1.20, f is invertible and so too is h. Composing with h^{-1}, we obtain

$$\int_{B_R} u^p \, dx \le C R^N h^{-1}(C_1 R^{-2} g(C_2/R^2)).$$

Combining this with (59), we finally obtain

$$\begin{aligned} \|u\|_{L^\infty(B_R)} &\le C R^{-N/p} (R^N h^{-1}(C_1 R^{-2} g(C_2/R^2)))^{1/p} \\ &= C f^{-1}(C_1 R^{-2} g(C_2/R^2)) = C s(R). \end{aligned}$$ □

We conclude this section by proving Corollary 1.24. Namely, we improve the rate of decay from $O(s(|x|))$ to $o(s(|x|))$ when additional information on the nonlinearity is available.

Proof of Corollary 1.24. To start, observe that under assumption (18), there exists a constant $C > 0$ such that

$$s(R) \le C R^{-2/(p_0 - 1)}. \tag{62}$$

Recall now (55). We choose a suitable cut-off function $\zeta \in C_c^2(\mathbb{R}^N)$ as follows. Let $\varphi \in C_c^2(\mathbb{R})$ satisfy $0 \le \varphi \le 1$ everywhere on $\mathbb{R}$ and

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \le 1, \\ 0 & \text{if } |t| \ge 2. \end{cases}$$

For $s > 0$, let $\theta_s \in C_c^2(\mathbb{R})$ satisfy $0 \le \theta_s \le 1$ everywhere on $\mathbb{R}$ and

$$\theta_s(t) = \begin{cases} 0 & \text{if } |t| \le s + 1, \\ 1 & \text{if } |t| \ge s + 2. \end{cases}$$

Given $R > R_0 + 3$, we define ζ at last by

$$\zeta(x) = \begin{cases} \theta_{R_0}(|x|) & \text{if } |x| \le R_0 + 3, \\ \varphi(|x|/R) & \text{if } |x| \ge R_0 + 3. \end{cases}$$

Applying (55) with ζ as above, we deduce that for some constants $C_1, C_2 > 0$,

$$\int_{B_R \setminus B_{R_0+2}} \left(\frac{f(u)}{u} \right)^{2\alpha+1} dx \le C_1 + C_2 R^{N-2m}. \tag{63}$$

Recall that (63) holds for $m = 2\alpha + 1$ and any $\alpha > 1/2$ such that $q_0 - \sqrt{q_0} < \alpha < q_0 + \sqrt{q_0}$. In fact, the restriction $\alpha > 1/2$ can be lifted and replaced by $\alpha > 0$. Indeed, the restriction $\alpha > 1/2$ was used for the sole purpose of proving (48). But (48) clearly holds under the finer assumption (18) for any $\alpha > 0$.

We would like to choose α such that $m := 2\alpha + 1 = N/2$. Since p_0 is in the supercritical range (15), straightforward algebraic computations show that such a choice is indeed possible in the range $q_0 - \sqrt{q_0} < \alpha < q_0 + \sqrt{q_0}$. By (63), we deduce that

$$\int_{\mathbb{R}^N} u^{(p_0-1)N/2} < \infty.$$

In particular, given $\eta > 0$ small, there exists $R > 0$ so large that given any point $x_0 \in \mathbb{R}^N$ such that $|x_0| = 4R$,

$$\int_{B_R(x_0)} u^{(p_0-1)N/2} < \eta.$$

We apply again (59), this time with $p = (p_0 - 1)N/2$, to obtain

$$\|u\|_{L^\infty(B_R(x_0))} \leq C_S R^{-N/p} \|u\|_{L^p(B_{2R}(x_0))} \leq C_S \eta R^{-2/(p_0-1)}. \tag{64}$$

This shows that $u(x) = o(|x|^{-2/(p_0-1)})$. It remains to prove the estimate on $|\nabla u|$. Observe that any partial derivative $v = \partial u/\partial x_i$ solves the linearized equation

$$-\Delta v = f'(u)\, v \quad \text{in } \mathbb{R}^N.$$

Apply again the Serrin inequality (59), this time with potential $\tilde{V}(x) = f'(u)$ and solution v. Since $0 \leq f'(u) \leq C u^{p_0-1}$, the potential $\tilde{V}$ is equivalent to $V(x) = f(u)/u$ and so the Serrin constant C_S is again independent of R and x_0 under our assumptions. We get

$$\|v\|_{L^\infty(B_R(x_0))} \leq C_S R^{-N/p} \|v\|_{L^p(B_{2R}(x_0))}.$$

Serrin's Theorem (cf. Theorem 1 on page 256 of [18]) also gives the estimate

$$\|\nabla u\|_{L^p(B_R(x_0))} \leq C_S R^{-1} \|u\|_{L^p(B_{2R}(x_0))}$$

for solutions of (58). Collecting these inequalities, we obtain

$$\|\nabla u\|_{L^\infty(B_R(x_0))} \leq C_S R^{-N/p-1} \|u\|_{L^p(B_{2R}(x_0))}.$$

Using that $u(x) = o(|x|^{-2/(p_0-1)})$, we obtain the desired estimate. □

7. Proof of Theorem 1.23: the subcritical case

By Remark 1.22, since p_0 is subcritical, we have

$$\int_{\mathbb{R}^N} f(u)u\, dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^N} F(u)\, dx < +\infty. \tag{65}$$

Multiply equation (1) by $u\zeta$, where ζ is a standard cut-off, i.e. $\zeta \equiv 1$ in B_R, $\zeta \equiv 0$ in B_{2R} and $|\nabla\zeta| \le C/R$, $|\Delta\zeta| \le C/R^2$. Then integrate:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \zeta\, dx + \int_{\mathbb{R}^N} u \nabla u \nabla \zeta\, dx = \int_{\mathbb{R}^N} f(u) u \zeta\, dx,$$
$$\int_{\mathbb{R}^N} |\nabla u|^2 \zeta\, dx - \frac{1}{2}\int_{\mathbb{R}^N} u^2 \Delta \zeta\, dx = \int_{\mathbb{R}^N} f(u) u \zeta\, dx.$$

By Remark 1.22, the second term on the left-hand side of the above equality converges to 0 as $R \to +\infty$. Hence, by monotone convergence we have

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx = \int_{\mathbb{R}^N} u f(u)\, dx < +\infty. \tag{66}$$

As in the classical Pokhozhaev identity, we may now multiply the equation by $x \cdot \nabla u \zeta$ to obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx = \frac{2N}{N-2}\int_{\mathbb{R}^N} F(u)\, dx. \tag{67}$$

We now collect (66) and (67). By assumption (14), if u is not identically zero, then

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2\, dx &= \int_{\mathbb{R}^N} u f(u)\, dx \le (p_0+1)\int_{\mathbb{R}^N} F(u)\, dx < \frac{2N}{N-2}\int_{\mathbb{R}^N} F(u)\, dx\\ &= \int_{\mathbb{R}^N} |\nabla u|^2\, dx,\end{aligned}$$

a contradiction. □

8. Proof of Theorem 1.25: the supercritical case

In what follows, we prove Theorem 1.25 in the supercritical case, i.e. when p_0 is in the range (15) and f satisfies (16), (17) and (20). In polar coordinates, a function u takes the form $u = u(r, \sigma)$, where $r \in \mathbb{R}_+^*$, $\sigma \in S^{N-1}$, $N \ge 2$, while its Laplacian is given by

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2}\Delta_{S^{N-1}} u.$$

Recall the classical Emden change of variables and unknowns $t = \ln r$ and $u(r,\sigma) = r^{-\alpha} v(t,\sigma)$, where $\alpha = 2/(p_0 - 1)$. Then

$$\begin{aligned} v(t,\sigma) &= e^{\alpha t} u(e^t, \sigma),\\ v_t(t,\sigma) &= e^{\alpha t}(e^t u_r + \alpha u) = e^{(\alpha+1)t} u_r + e^{\alpha t}\alpha u,\\ v_{tt}(t,\sigma) &= e^{\alpha t}(e^{2t} u_{rr} + (2\alpha+1) e^t u_r + \alpha^2 u),\\ \Delta_{S^{N-1}} v &= e^{\alpha t}\Delta_{S^{N-1}} u. \end{aligned} \tag{68}$$

Writing

$$\alpha = \frac{2}{p_0 - 1}, \quad A = (N - 2 - 2\alpha), \quad B = \alpha^2 + \alpha A, \tag{69}$$

we obtain

$$\begin{aligned} v_{tt} + A v_t &= e^{(\alpha+2)t}\left(u_{rr} + \frac{N-1}{r} u_r\right) + B e^{\alpha t} u \\ &= e^{(\alpha+2)t}(-e^{-2t}\Delta_{S^{N-1}} u - f(u)) + B e^{\alpha t} u \\ &= -e^{(\alpha+2)t} f(e^{-\alpha t} v) - e^{\alpha t}\Delta_{S^{N-1}} v + B v. \end{aligned}$$

To summarize, v solves

$$v_{tt} + A v_t + B v + \Delta_{S^{N-1}} v + f(e^{-\alpha t} v) e^{(\alpha+2)t} = 0 \quad \text{for } t \in \mathbb{R}, \sigma \in S^{N-1}. \tag{70}$$

Multiply the above equation by v_t and integrate over S^{N-1}. For $t \in \mathbb{R}$, we find

$$\begin{aligned} &\int_{S^{N-1}} \left(\frac{v_t^2}{2}\right)_t d\sigma + A \int_{S^{N-1}} v_t^2 \, d\sigma + B \int_{S^{N-1}} \left(\frac{v^2}{2}\right)_t d\sigma \\ &\qquad - \int_{S^{N-1}} \left(\frac{|\nabla_{S^{N-1}} v|^2}{2}\right)_t d\sigma + \int_{S^{N-1}} f(v e^{-\alpha t}) v_t e^{(\alpha+2)t} \, d\sigma = 0. \end{aligned} \tag{71}$$

Let F denote the antiderivative of f such that $F(0) = 0$. Then

$$\begin{aligned} &\frac{d}{dt}[F(v e^{-\alpha t}) e^{(p_0+1)\alpha t}] \\ &\qquad = f(v e^{-\alpha t})(v_t e^{-\alpha t} - \alpha v e^{-\alpha t}) e^{(p_0+1)\alpha t} + F(v e^{-\alpha t}) \alpha (p_0 + 1) e^{(p_0+1)\alpha t}. \end{aligned}$$

So,

$$\begin{aligned} &f(v e^{-\alpha t}) v_t e^{p_0 \alpha t} \\ &\quad = \frac{d}{dt}[F(v e^{-\alpha t}) e^{(p_0+1)\alpha t}] + \alpha f(v e^{-\alpha t}) v e^{\alpha p_0 t} - \alpha F(v e^{-\alpha t})(p_0 + 1) e^{(p_0+1)\alpha t}. \end{aligned}$$

Applying (20), we conclude that

$$f(v e^{-\alpha t}) v_t e^{p_0 \alpha t} \geq \frac{d}{dt}[F(v e^{-\alpha t}) e^{(p_0+1)\alpha t}].$$

Using this inequality in (71), we obtain

$$\begin{aligned} &\int_{S^{N-1}} \left(\frac{v_t^2}{2}\right)_t d\sigma + A \int_{S^{N-1}} v_t^2 \, d\sigma + B \int_{S^{N-1}} \left(\frac{v^2}{2}\right)_t d\sigma \\ &\qquad - \int_{S^{N-1}} \left(\frac{|\nabla_{S^{N-1}} v|^2}{2}\right)_t d\sigma + \int_{S^{N-1}} \frac{d}{dt}[F(v e^{-\alpha t}) e^{(p_0+1)\alpha t}] \, d\sigma \leq 0. \end{aligned}$$

Integrating for $t \in (-s, s)$, $s > 0$, we then derive

$$\frac{1}{2}\left[\int_{S^{N-1}} v_t^2\, d\sigma\right]_{t=-s}^{t=s} + A\int_{t=-s}^{t=s}\int_{S^{N-1}} v_t^2\, d\sigma\, dt + \frac{B}{2}\left[\int_{S^{N-1}} v^2\, d\sigma\right]_{t=-s}^{t=s} - \frac{1}{2}\left[\int_{S^{N-1}} |\nabla_{S^{N-1}} v|^2\, d\sigma\right]_{t=-s}^{t=s} + \left[\int_{S^{N-1}} F(ve^{-\alpha t})e^{(p_0+1)\alpha t}\, d\sigma\right]_{t=-s}^{t=s} \le 0. \quad (72)$$

Recall the definition of v given in (68) and use the improved decay estimates (19): we see that $v(t,\cdot)$, $v_t(t,\cdot)$, $|\nabla_{S^{N-1}} v(t,\cdot)|$ converge to 0 as $t \to \pm\infty$, uniformly in $\sigma \in S^{N-1}$. Passing to the limit as $s \to +\infty$ in (72), we finally obtain

$$A\int_{\mathbb{R}}\int_{S^{N-1}} v_t^2\, d\sigma\, dt + \limsup_{s\to+\infty}\int_{S^{N-1}} F(ve^{-\alpha s})e^{(p_0+1)\alpha s}\, d\sigma \le 0. \quad (73)$$

Since $p_0 > (N+2)/(N-2)$, it follows from (69) that $A > 0$. So, both terms in (73) are nonnegative. In particular, $v_t \equiv 0$ and v is a function depending only on σ. Since $\lim_{t\to+\infty} v(t,\sigma) = 0$ by (19), we deduce that $v \equiv 0$ and $u \equiv 0$ as claimed.

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