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Variational problems with free boundaries for the fractional Laplacian

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Abstract. We discuss properties (optimal regularity, nondegeneracy, smoothness of the free boundary etc.) of a variational interface problem involving the fractional Laplacian; due to the nonlocality of the Dirichlet problem, the task is nontrivial. This difficulty is bypassed by an extension formula, discovered by the first author and Silvestre, which reduces the study to that of a codimension 2 (degenerate) free boundary.

1. Introduction

The goal of this paper is to derive local properties—optimal regularity, nondegeneracy, smoothness—of a free boundary problem involving the fractional Laplacian, generalising the classical phase transition problem for the standard Laplacian with prescribed gradient jump [11]. Let us recall that the fractional Laplacian $(-\Delta)^\alpha$ is given by

$$(-\Delta)^\alpha u(x) = c_{N,\alpha}\, \mathrm{PV} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}}\, dy, \tag{1.1}$$

where PV is the Cauchy principal value and $c_{N,\alpha}$ a normalisation constant. Let us also say that a function $u \in C^{1,\gamma}(\mathbb{R}^N)$ (with $\gamma > \alpha$) is *α-harmonic* in a domain Ω of $\mathbb{R}^N$ if it satisfies $(-\Delta)^\alpha u(x) = 0$ for all $x \in \Omega$.

The strong form—i.e. the one that assumes that the unknowns have at least as many derivatives as those appearing in the formulation—of our problem is the following: given $\alpha \in (0, 1)$ and $A > 0$, consider a function $u \in C(\mathbb{R}^N)$ solving, in a domain D of $\mathbb{R}^N$:

$$\begin{aligned} (-\Delta)^\alpha u(x) &= 0 && \text{if } x \in D \cap \{u > 0\}, \\ \lim_{y \to x} \frac{u(y)}{((y - x)\, .\, \nu(x))^\alpha} &= A && \text{if } x \in D \cap \partial(\{u = 0\}). \end{aligned} \tag{1.2}$$

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with prescribed value $f(x)$ outside D. Also recall that the strong form of (1.2) for the standard Laplacian is to study a function $u \in C(\mathbb{R}^N)$ such that, in D, one has

$$\begin{aligned} -\Delta u(x) &= 0 \quad \text{if } u(x) > 0, \\ u_\nu(x) &= A \quad \text{if } x \in \partial(\{u = 0\}). \end{aligned} \tag{1.3}$$

Let us immediately notice the following explicit solution to (1.2). On the line $\mathbb{R}$, the function $(x^+)^\alpha$ is a solution of (1.2) with $A = 1$. To see it, a quick argument for $\alpha = 1/2$ is that, in the complex plane cut along the negative axis, the function $z \mapsto z^{1/2}$ is analytic, hence its real part is harmonic. Moreover, because it is even in y, its y-derivative on the positive axis vanishes; this means that the half-Laplacian of $\mathcal{R}(z^{1/2}) = \sqrt{x}$ is zero on $\mathbb{R}_+$. To prove the validity of the statement for $\alpha \neq 1/2$, a possible way goes once again through elementary complex analysis, by (after scaling in x) noticing that (for instance if $\alpha < 1/2$)

$$\int_{\{x \pm i\varepsilon \,:\, x \in \mathbb{R}\}} \frac{1 - (1+z)^\alpha}{z^{1+2\alpha}}\, dz = 0,$$

and letting $\varepsilon \to 0$. Of course, regularity of the free boundary is rather easy to study in this example, but (i) it proves that our problem is not void, (ii) this solution will follow us in the whole paper. Note, moreover, that this boundary behaviour is typical of α-harmonic functions at regular boundary points which are minima (see the generalised Hopf lemma in Section 2 below). Once again this parallels exactly the classical Laplacian case: the classical Hopf lemma indeed states that, at a minimum which is a regular boundary point, a harmonic function grows linearly away from the boundary.

The motivation for studying problems of the form (1.2) for the classical Laplacian comes from reaction-diffusion problems in plasma physics, semi-conductor theory, flame propagation etc. When turbulence or long-range interactions are present, it is relevant to replace the Laplacian by nonlocal operators, such as $(-\Delta)^\alpha$. For further information on the modelling, see the review papers [5] and [21]. The particular problem we will discuss appears in flame propagation and also in the propagation of surfaces of discontinuities, like planar crack expansion. In this context, (1.2) is related to reaction-diffusion equations: in a companion paper [8] we will interpret (1.2) as the singular limit of a singularly perturbed elliptic reaction-diffusion model.

Potential theory for the fractional Laplacian is well developed: see for instance [4], [19], [20], especially from the point of view of the boundary Harnack principle. Studying local properties of the free boundary requires, however, rescaling: we sometimes want to forget about what happens far away from the point under consideration. This does not agree well with an operator which precisely takes information from the whole space. Even more basically, we are not able, in general, to prove existence theorems for (1.2) in such a strong sense.

Let us devise a weak form for (1.2). A possible way to do it (see [1]) is to try to minimise the energy

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2\alpha}}\, dx\, dy + \mathcal{L}_N(\{u > 0\}), \tag{1.4}$$

where $\mathcal{L}_N$ is the N-dimensional Lebesgue measure. When the first term in (1.4) is replaced by the Dirichlet integral $\int_{\mathbb{R}^N} |\nabla u|^2$, sufficiently smooth local minimisers can be proved to satisfy (1.2)—with $\alpha = 1$—in the strong sense. This does not however suppress the nonlocality of the Dirichlet integral appearing in (1.4); we would really wish to use only local information on our unknown. To bypass the inconvenience, let us make use of the extension property presented in [11], which generalises the Poisson formula. Consider the upper half-plane $\mathbb{R}^{N+1}_+ = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}_+\}$, and set $\beta = 1 - 2\alpha$. For $u \in C^2(\mathbb{R}^N)$ solve the Dirichlet problem

$$\begin{aligned} -\text{div}(y^\beta \nabla v) &= 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\ v(x, 0) &= u(x). \end{aligned} \tag{1.5}$$

This can be done by convolution with the the Poisson kernel $P_{N,\alpha}(x, y)$ of the operator $-\text{div}(y^\beta \nabla)$ in $\mathbb{R}^{N+1}_+$; we have (see [11])

$$P_{N,\alpha}(x, y) = q_{N,\alpha} \frac{y^{2\alpha}}{(x^2 + y^2)^{(N+2\alpha)/2}}, \tag{1.6}$$

where $q_{N,\alpha}$ ensures that $\int P_{N,\alpha}(x, 1)\, dx = 1$.

Theorem 0.1 ([11]). *We have* $(-\Delta)^\alpha u(x) = -\lim_{y\to 0} y^\beta v_y(x, y)$.

Because of the divergence form of the elliptic operator at stake in (1.5), a Dirichlet integral is available and we may introduce an energy to minimise. Notice also that if u solves (1.5), we may extend it evenly across the hyperplane $\{y = 0\}$, and the new equation satisfied by u is

$$\begin{aligned} -\text{div}(|y|^\beta \nabla v) &= 0 \quad \text{in } \mathbb{R}^{N+1}, \\ v(x, 0) &= u(x). \end{aligned} \tag{1.7}$$

For any open subset Ω of $\mathbb{R}^{N+1}$, let us introduce the weighted Hilbert space

$$H^1(\beta, \Omega) = \{u \in L^2(\Omega_+) : |y|^{\beta/2} \nabla u \in L^2(\Omega)\}, \tag{1.8}$$

where $\Omega_+ = \Omega \cap \mathbb{R}^{N+1}_+$. We then set

$$\forall v \in H^1(\beta, \Omega), \qquad \mathcal{J}(v, \Omega) = \int_\Omega |y|^\beta |\nabla v|^2\, dx\, dy + \mathcal{L}_N(\{v > 0\} \cap \mathbb{R}^N \cap \Omega), \tag{1.9}$$

where $\mathcal{L}_N$ still denotes the N-dimensional Lebesgue measure on the hyperplane $\mathbb{R}^N$. For any $r > 0$ and $(x, y) \in \mathbb{R}^{N+1}$ let $B_r(x, y)$ be the ball in $\mathbb{R}^{N+1}$ centred at (x, y) and of radius r, and let $B_{r+}(x, y)$ be its intersection with the upper half-plane. When $x = 0$ we will simply use the notations B_r and B_{r+}. Finally, if $x \in \mathbb{R}^N$ we denote by $B^N_r(x)$ the ball in $\mathbb{R}^N$ centred at x with radius r.

The study of (1.2) is now replaced by the study of *local minimisers* of $\mathcal{J}$, i.e. functions u that are in $H^1(\beta, B_1)$ and satisfy

$$\forall B \subset B_1,\ \forall v \in H^1(\beta, B) \text{ with } v = u \text{ on } \partial B, \qquad \mathcal{J}(u, B) \le \mathcal{J}(v, B). \tag{1.10}$$

We take the opportunity to define what a *global minimiser* is: it is a function $u \in H^1_{\text{loc}}(\beta, \mathbb{R}^{N+1})$ which minimises $\mathcal{J}(\cdot, B)$ in every ball B in $\mathbb{R}^{N+1}$. It is a simple task to prove that a local minimiser u satisfies $\text{div}(|y|^\beta \nabla u)(x, y) = 0$ for (x, y) in any open subset of its set of positivity. If u is in $C(\mathbb{R}^{N+1})$—the continuity is not obvious and will have to be established—we will prove (this is not trivial) in Section 3 below that $(-\Delta)^\alpha u = 0$ on $\mathbb{R}^N \cap \{u > 0\}$.

The free boundary condition comes of course from the area integral, but deriving it precisely is more delicate than in the classical ($\alpha = 1$) case, and a special section will be devoted to it.

Notice once again the analogy with the classical Laplacian. The weak form of (1.3) is to study local minimisers of the functional

$$\forall v \in H^1(B), \quad \mathcal{J}(v, B) = \int_B |\nabla v|^2 \, dx + \mathcal{L}_N(\{v > 0\} \cap B), \tag{1.11}$$

where this time B is a ball in $\mathbb{R}^N$. Here are the main results that we will prove in this paper.

Basic properties of local minimisers

Consider a local minimiser u of Problem (1.10), posed in B_1.

Theorem 1.1 (Optimal regularity). *We have $u \in C^{0,\alpha}(K)$ for all compact sets $K \subset B_1$.*

Theorem 1.2 (Nondegeneracy). *There exists a constant $c_0 > 0$ such that for all $x \in B^N_{1/2}(0) \cap \{u > 0\}$,*

$$u(x, 0) \geq c_0 d(x, \partial \{u > 0\})^\alpha.$$

Theorem 1.3 (Positive density). *Suppose that $(0, 0)$ is a free boundary point. There is $\delta > 0$ such that, for every $r > 0$,*

$$\mathcal{L}_N(\{u = 0\} \cap B^N_r) \geq \delta r^N, \quad \mathcal{L}_N(\{u > 0\} \cap B^N_r) \geq \delta r^N. \tag{1.12}$$

The free boundary condition

Here we assume that u has an actual free boundary, i.e. the set $\partial\{x \in \mathbb{R}^N : u(x) > 0\}$ is non-void. We will see that this is the case if the data are not too large.

Theorem 1.4. *Let u be a solution of (1.10). Define the constant A_α by*

$$A_\alpha = \left(c_{1,\alpha} \int_{-1}^0 \frac{(1+x)^\alpha}{(-x)^\alpha} \, dx \int_1^{+\infty} \frac{(1+x)^\alpha}{x^{1+2\alpha}} \, dx \right)^{-1}, \tag{1.13}$$

where $c_{1,\alpha}$ is the constant in (1.1) with $N = 1$. Let x_0 be a free boundary point having a measure-theoretic normal $\nu(x_0)$. Then

$$\lim_{x \to x_0} \frac{u(x) - u(x_0)}{((x - x_0) \, . \, \nu(x_0))^\alpha_+} = A_\alpha. \tag{1.14}$$

Regularity of the free boundary

Finally, we are interested in proving conditional regularity properties for the free boundary, i.e. regularity away from possible singularities—similarly to what happens in minimal surface theory [15]. Recall that singularities may occur in the minimisation problem for the classical Laplacian case (see [12]). The following theorem is in the spirit of [1].

Theorem 1.5. *Assume* $N = 2$ *and let* u *solve* (1.10) *in* B_1. *Assume that the free boundary is a Lipschitz graph in* $B_1 \cap \mathbb{R}^2$*:*

$$\partial\{u > 0\} = \{(x_1, x_2) : x_2 > f(x_1)\},$$

where f *is a Lipschitz function. Assume also that* 0 *is on the free boundary. Then the free boundary is a* C^1 *graph in* $B_{1/2} \cap \mathbb{R}^2$.

The previous theorems are the main results of this paper. We will also provide a classification of global solutions. As for the assumption in Theorem 1.5, Lipschitz regularity of the free boundary can be attained from scratch from some special geometric configurations in cylinders or star-shaped domains. It has been shown (in the Laplacian case) for some particular models of conical flames (see [16]).

Theorem 1.5 generalises to nonlocal operators the main theorem of [6], which proves in the case of the Laplacian that if one starts with a Lipschitz free boundary (as a graph) then the free boundary is locally $C^{1,\gamma}$ for some $0 < \gamma < 1$. However, our theorem gives a weaker result since we just obtain C^1 with a nonexplicit modulus of continuity. Whether this modulus is actually Hölder remains an open problem. Notice that such a result might also be accessible via "flatness of the free boundary implies regularity"—as in [1]—but this needs some measure-theoretic properties on the free boundary we do not know yet.

The paper is organised as follows. In Section 2, we give some (sometimes well-known, sometimes new) properties of the fractional Laplacian, which will be useful in the sequel. In Section 3, we start the study of (1.10) and prove Theorems 1.1 to 1.3. Section 4 is devoted to the classification of global solutions to (1.2), resulting in the derivation of the free boundary condition. Finally, we prove Theorem 1.5 in Section 5.

2. Properties of $(-\Delta)^\alpha$ and its extension

The Dirichlet integral appearing in (1.10) comes from the degenerate elliptic operator $-\text{div}(|y|^\beta\nabla)$. Because $\beta = 1 - 2\alpha$ we have $\beta \in (-1, 1)$, and the weight $|y|^\beta$ is, with the notable exception of $\alpha = 1/2$, singular at 0 (for $\alpha > 1/2$) or degenerate at 0 (for $\alpha < 1/2$). One has to make sure that important properties like the Poincaré inequality or the Harnack principle hold, and this is what the next subsection is devoted to. In the second subsection, we turn to the particular case of the fractional Laplacian and prove a monotonicity formula for the Dirichlet energy.

In the particular case $\alpha = 1/2$, the function u is harmonic in the (x, y) variables and $(-\Delta)^{1/2}u$ coincides with $-u_y$, the normal derivative of u at $y = 0$. The reader should always keep this example in mind.

2.1. Degenerate elliptic equations with A_2 weights

Set $L_\beta = -\text{div}(|y|^\beta \nabla)$ in $\mathbb{R}^{N+1}$. Its weight $|y|^\beta$ belongs to the second Muckenhoupt class A_2:

Definition 2.1. A function $w \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ belongs to A_2 if for every ball B in $\mathbb{R}^{N+1}$ we have

$$\int_B w \int_B w^{-1} < \infty. \tag{2.1}$$

Clearly, $|y|^\beta$ falls into this class for $\beta \in (-1, 1)$. Another interesting property of this weight is its independence from the tangential variable x. This allows us to consider translations in x. The series of papers ([13]–[14]) develops a theory for this kind of operator: Sobolev embeddings, Poincaré inequality, Harnack inequality, local solvability in Hölder spaces, estimates of the Green function. In the following, we recall some of their results which will be useful later. In the next three results we denote $w(E) = \int_E w$.

Theorem 2.2 (Weighted embedding theorem). *Given $w \in A_2$, there exist constants C and $\delta > 0$ such that for all balls B_R, all $u \in C_0^\infty(B_R)$ and all numbers k satisfying $1 \le k \le N/(N-1) + \delta$,*

$$\left(\frac{1}{w(B_R)} \int_{B_R} |u|^{2k} w\right)^{1/2k} \le CR\left(\frac{1}{w(B_R)} \int_{B_R} |\nabla u|^2 w\right)^{1/2}.$$

Theorem 2.3 (Poincaré inequality). *Given $w \in A_2$, there exist constants C and $\delta > 0$ such that for all balls B_R, all u Lipschitz continuous in B_R and all numbers k satisfying $1 \le k \le N/(N-1) + \delta$,*

$$\left(\frac{1}{w(B_R)} \int_{B_R} |u - A_R|^{2k} w\right)^{1/2k} \le CR\left(\frac{1}{w(B_R)} \int_{B_R} |\nabla u|^2 w\right)^{1/2},$$

where either $A_R = (1/w(B_R)) \int_{B_R} uw$ or $A_R = (1/w(B_R)) \int_{B_R} u$.

In the next two results, we set $L_w = -\text{div}(w(x, y)\nabla)$. If $(x_0, y_0) \in \mathbb{R}^{N+1}$ and $r > 0$, we let $Q_r(x_0, y_0)$ be the cube with centre (x_0, y_0) and sidelength r. The set $Q_r^N(x_0)$ is the cube of $\mathbb{R}^N$ with centre x_0 and sidelength r.

Theorem 2.4 (Harnack inequality). *Let $u \ge 0$ solve $L_w u = 0$ in $Q_1(x_0, y_0)$.*

[i] (Interior Harnack) *For each compact set $K \subset Q_1(x_0, y_0)$, there exists a constant M, independent of u, such that*

$$\max_K u \le M \min_K u.$$

[ii] (Boundary Harnack) *Assume that $u = 0$ on the face $\Sigma := Q^N_{1/2}(x_0) \times \{y_0 + 1/2\}$. Let $v \ge 0$ solve $L_w v = 0$ and be such that $u(x_0, y_0) = v(x_0, y_0)$. Assume also that $v = 0$ on Σ. For any compact subset K of $Q_1(x_0, y_0)$ containing a neighbourhood of x_0, there exists a constant $M_K > 1$, independent of u and v, such that*

$$\max_K \frac{u}{v} \le M_K \min_K \frac{u}{v}.$$

Theorem 2.4 classically implies

Theorem 2.5 (Oscillation lemma). [i] *Let u be a solution of $L_w u = 0$ in a domain Ω. Then the oscillation of u decays geometrically in concentric balls inside Ω: if $(x_0, y_0) \in \Omega$, there exists $\lambda \in (0, 1)$, depending only of w and the distance of (x_0, y_0) to $\partial\Omega$, such that, for small enough $r > 0$,*

$$\mathrm{osc}_{B_r(x_0,y_0)}\, u \le \lambda\, \mathrm{osc}_{B_{2r}(x_0,y_0)}\, u.$$

[ii] *In the situation of Theorem 2.4[ii], there exists $\lambda \in (0, 1)$, depending only of w, such that, for small enough $r > 0$,*

$$\mathrm{osc}_{B_r(x_0,y_0+1/2)\cap Q_1(x_0,y_0)}\, \frac{u}{v} \le \lambda\, \mathrm{osc}_{B_{2r}(x_0,y_0+1/2)\cap Q_1(x_0,y_0)}\, \frac{u}{v}.$$

2.2. The particular case of $(-\Delta)^\alpha$

We specialise here the weights to those of the fractional Laplacian, i.e. we study solutions of

$$-\mathrm{div}(|y|^\beta \nabla u) = 0 \quad ((x, y) \in B), \tag{2.2}$$

where B is some ball in $\mathbb{R}^{N+1}$. We wish to prove a monotonicity formula in the spirit of the well-known one for the Laplacian, as well as results of the type: if u is harmonic in, say, $B_1 \subset \mathbb{R}^N$, then

$$\forall 0 < r \le R < 1, \qquad \int_{B_r} |\nabla u|^2 \le \left(\frac{r}{R}\right)^N \int_{B_R} |\nabla u|^2.$$

This just comes from the fact that $-\Delta|\nabla u|^2 \le 0$. Coming back to (2.2), the precise result is the following.

Theorem 2.6. *Let u be a solution of (2.2) in B_1. Then, for $0 < r < R < 1$,*

$$\int_{B_r} |y|^\beta |\nabla u|^2\, dx\, dy \le \left(\frac{r}{R}\right)^{N+1+\beta} \int_{B_R} |y|^\beta |\nabla u|^2\, dx\, dy. \tag{2.3}$$

Proof. Denote, for all $r > 0$ and all $v \in H^1(\beta, B_r)$

$$E_r[v] = \int_{B_r} |y|^\beta |\nabla v|^2\, dx\, dy.$$

If u is as described above, then for all $r \in (0, 1)$,

$$\forall v \in H^1(\beta, B_r) \text{ with } v = u \text{ on } \partial B_r, \qquad E_r[u] \le E_r[v]. \tag{2.4}$$

For a small $\varepsilon > 0$, let us take the test function

$$v_\varepsilon(x, y) = \begin{cases} \dfrac{1}{1+\varepsilon} u((1+\varepsilon)(x, y)) & \text{if } |(x, y)| \le \dfrac{r}{1+\varepsilon}, \\[2mm] \dfrac{1 + r^{-1}|(x, y)|\varepsilon}{1+\varepsilon} u\left(\dfrac{(x, y)}{r|(x, y)|}\right) & \text{if } |(x, y)| \in \left(\dfrac{r}{1+\varepsilon}, r\right]. \end{cases}$$

In other words, v_ε is an $(1+\varepsilon)^{-1}$ Lipschitz dilation of u, extended in a radially linear fashion. We claim that

$$\lim_{\varepsilon\to 0}\frac{E_r[v_\varepsilon]-E_r[u]}{\varepsilon}=r\int_{\partial B_r}|y|^\beta|\nabla u|^2\,d\sigma(x,y)-(N+1+\beta)E_r[u].$$

Indeed, we have

$$E_r[v_\varepsilon]=E_{r(1+\varepsilon)^{-1}}[v_\varepsilon]+\int_{B_r\backslash B_{r/(1+\varepsilon)}}|y|^\beta|\nabla v_\varepsilon|^2\,dx\,dy=:I_\varepsilon+II_\varepsilon.$$

Now,

$$I_\varepsilon=(1+\varepsilon)^{-N-1-\beta}\int_{B_r}|y|^\beta|\nabla u|^2\,dx\,dy=(1-(N+1+\beta)\varepsilon)E_r[u]+O(\varepsilon^2).$$

Because v_ε is radially linear on the annulus $B_r\backslash B_{r/(1+\varepsilon)}$ the term II_ε is computed as follows:

$$II_\varepsilon=\varepsilon r\int_{\partial B_r}|y|^\beta|\nabla u|^2\,dx\,dy+O(\varepsilon^2)=\varepsilon\frac{dE_r[u]}{dr}+O(\varepsilon^2).$$

This computation needs C^1 regularity for u inside B_1, which is provided in [10]. Hence

$$r\frac{dE_r[u]}{dr}-(N+1+\beta)E_r[u]\geq 0,$$

which proves our theorem. □

We end up this section by quoting the strong maximum principle for α-harmonic functions in domains. It could be derived from the Harnack inequality, but admits simpler proofs—either by inspection or from Riesz potentials (see [4]). Quite often it will be sufficient to use it, therefore it is justified to present it separately.

Proposition 2.7. [i] (Strong maximum principle) *If a smooth function $v(x)$ satisfies $(-\Delta)^\alpha v=0$ in some domain Ω of $\mathbb{R}^N$, and if v is nonnegative and nonzero in $\mathbb{R}^N$, then $v>0$ in Ω.*

[ii] (Generalised Hopf Lemma) *If a smooth function $v(x)$ satisfies $(-\Delta)^\alpha v=0$ in some smooth domain Ω of $\mathbb{R}^N$, if v is nonnegative and nonzero in $\mathbb{R}^N$, and if there is a point $X_0\in\partial\Omega$ for which $v(X_0)=0$, then there exists $\lambda>0$ such that $v(x)\geq\lambda((x-X_0)\,.\,\nu(X_0))^\alpha$, where $\nu(X_0)$ is the inner normal to $\partial\Omega$ at X_0.*

3. Existence and general properties of local minimisers

After proving that Theorems 1.1 to 1.3 are not void—by explaining why the minimisation (1.10) has solutions with nontrivial free boundaries—we give a proof of these results. In passing we deduce a positive density consequence, which is almost enough, but not completely, to infer that the free boundary has finite perimeter.

3.1. Behaviour of a minimiser in its positivity set

As is well-known, a local minimiser in the $\alpha = 1$ case—i.e. a minimiser of the functional given in (1.11)—is harmonic in its positivity set. An analogous property is true for minimisers of (1.10). Here is the statement.

Proposition 3.1. *Let u be a local minimiser in* (1.10)*. Assume moreover that it is continuous in $\overline{B}_1$, and let $x_0 \in \mathbb{R}^N$ be such that $u(x_0, 0) > 0$. Then*

$$\lim_{y\to 0} |y|^{\beta} u(x_0, y) = 0.$$

If moreover u is defined in $\mathbb{R}^{N+1}$, is positive outside the hyperplane $\{y = 0\}$ and satisfies $-\mathrm{div}(|y|^{\beta}\nabla u) = 0$ in its positivity set, together with the estimate $u(x, y) = O(|(x, y)|^{\alpha})$, then $(-\Delta)^{\alpha} u(\cdot, 0) = 0$ on $\mathbb{R}^N \cap \{u > 0\}$.

Proof. If $u \in C(B_1)$, then $\{u > 0\}$ is open, and because u is a local minimiser in $H^1(\beta, B_1)$, it solves $-\mathrm{div}(|y|^{\beta}\nabla u) = 0$ inside $\{u > 0\}$. Assume now that u satisfies $-\mathrm{div}(|y|^{\beta}\nabla u) = 0$ in its positivity set, together with the estimate $u(x, y) = O(|(x, y)|^{\alpha})$. Then

$$u(x, y) = \int_{\mathbb{R}^N} P_{N,\alpha}(x - x', y) u(x', 0)\, dx'. \tag{3.1}$$

To see this, it is enough to prove that any solution $v(x, y)$ of

$$-\mathrm{div}(y^{\beta}\nabla v) = 0 \quad \text{in } \mathbb{R}^{N+1}_+, \qquad v(x, 0) = 0, \qquad v(x, y) = O(|(x, y)|^{\alpha}) \tag{3.2}$$

is zero. Let therefore $v(x, y)$ be such a solution. By scaling we have, for all integers p,

$$|\Delta_x^p v(x, y)| = O(|(x, y)|^{\alpha - 2p}).$$

Choose $p \geq 2N$ and set $\Delta_x^p v =: v_p$. Its Fourier transform in x, denoted by $\hat{v}_p(\xi, y)$, solves

$$-\left(\partial_{yy} + \frac{\beta}{y}\partial_y - |\xi|^2\right)\hat{v}_p = 0 \quad (y > 0), \qquad \hat{v}_p(\xi, 0) = 0, \qquad v_p(x, y) = O(y^{2p-\alpha}).$$

This implies $\hat{v}_p \equiv 0$, thus $|\xi|^{2p}\hat{v} \equiv 0$, where $\hat{v}$ is the Fourier transform of v in x. Thus there exists a set of tempered distributions $(a_{\gamma}(y))_{\gamma}$, the multi-index $\gamma = (\gamma_1, \ldots, \gamma_N)$ being of length less than $2Np$, such that

$$\hat{v} = \sum_{\gamma} a_{\gamma}(y) \otimes \partial^{\gamma}\delta_{\xi=0}.$$

And thus, denoting (as usual) $x^{\gamma} = x_1^{\gamma_1} \ldots x_N^{\gamma_N}$ we obtain

$$v(x, y) = \sum_{\gamma} a_{\gamma}(y) x^{\gamma}.$$

However, the growth condition on v imposes that only a_0 is zero; hence

$$-a_0'' - \frac{\beta}{y} a_0' = 0,$$

and the growth condition once again imposes that $a_0 \equiv 0$.

Thus u is even in y and we have (3.1). Take x_0 such that $u(x_0) > 0$. We have $-\mathrm{div}(|y|^\beta \nabla u) = 0$ in a small neighbourhood of x_0 and thus, from Lemma 4.2 in [11], the limit $\lim_{y\to 0^+} y^\beta u(x_0, y)$ exists. Because u is even, we have

$$-\mathrm{div}(|y|^\beta \nabla u) = 2 \lim_{y\to 0^+} y^\beta u(x_0, y)\delta_{y=0},$$

therefore the RHS of the equality vanishes. By Theorem 0.1, $(-\Delta)^\alpha u(x_0) = 0$. □

For any bounded subset Ω of $\mathbb{R}^{N+1}$, set $\Omega_+ = \mathbb{R}^{N+1}_+ \cap \Omega$. With this notation, let $\mathcal{J}_+$ be defined by

$$\forall v \in H^1(\beta, B_+), \qquad \mathcal{J}(v, B_+) = \int_{B_+} y^\beta |\nabla v|^2\, dx\, dy + \mathcal{L}_N(\{v > 0\} \cap \mathbb{R}^N \cap B), \quad (3.3)$$

where B is any ball centred on the hyperplane $\{y = 0\}$. If u is a local minimiser in B_1, its restriction to B_{1+} is a local minimum of $\mathcal{J}_+$ in $H^1(\beta, B_1)$. This fact will be used freely in what follows.

3.2. Existence of minimisers with nontrivial free boundaries

We will not show here the existence of nontrivial local minimisers *defined on the whole space* $\mathbb{R}^{N+1}$. This is a hard challenge, and a way to get a low-cost result would be to add first order derivatives in the operator $-\mathrm{div}(|y|^\beta \nabla)$. We will not dwell on this aspect here, leaving it to [8]. Let $f(x, y) \in C^\infty(B_1)$.

Proposition 3.2. *Problem* (1.10) *has an absolute minimum u coinciding with f on* $\partial B_1 \cap \mathbb{R}^{N+1}_+$. *Moreover, we may choose f such that u has a nontrivial free boundary.*

Proof. Since the functional $\mathcal{J}$ is nonnegative, there exists a minimising sequence $(u_k)_{k\in\mathbb{N}}$. The sequence is bounded in $H^\alpha(B_1)$ and, thanks to the compactness of the embedding $H^\alpha \hookrightarrow L^{2N/(N-2\alpha)}$, the sequence $(u_k)_k$ converges—up to a subsequence—to a function u strongly in $L^{2N/(N-2\alpha)}$ and almost everywhere in $\mathbb{R}^N$.

Moreover, there exists a function $0 \le \gamma \le 1$ such that

$$\mathcal{L}_N(u_k > 0) \to \gamma$$

weak* in $L^\infty(\mathbb{R}^N)$. Using the fact that $\gamma = 1$ a.e. in $\{u > 0\}$ we deduce that

$$\mathcal{J}(u) \le \liminf_{k\to+\infty} \mathcal{J}(u_k).$$

This yields the existence of an absolute minimiser. As is classical, it is also a local minimiser.

Let us prove that, for some choices of $f \ge 0$, u has a free boundary. We use an argument that will be encountered in the next section. Set $\varepsilon = \|f\|_{C(B_1)}$, where we have extended f by symmetry. Assume u has no free boundary. Then $-\mathrm{div}(|y|^\beta \nabla u) = 0$ in B_1. This, by the maximum principle and Theorem 2.5, implies $\|u\|_{L^\infty(B_1)} \le C_0\varepsilon$ for some constant $C_0 > 1$ independent of ε. If ϕ_0 is a C^∞ function equal to 2 on ∂B_1 and 0 in $B_{1/2}$, set $\underline{u} = \min(u, \varepsilon C_0\phi_0)$. Then $\underline{u} = f$ on ∂B_1, therefore

$$\mathcal{J}(u, B_1) \le \mathcal{J}(\underline{u}, B_1).$$

However,

$$\int_{B_1} |y|^\beta |\nabla \underline{u}|^2 \le \int_{B_1} |y|^\beta |\nabla u|^2 + O(\varepsilon),$$

and

$$\mathcal{L}_N(\{\underline{u} > 0\}) \le \mathcal{L}_N((B_1 \cap \mathbb{R}^N) \setminus (B_{1/2} \cap \mathbb{R}^N)) = \mathcal{L}_N(\{u > 0\} \cap \mathbb{R}^N) - \mathcal{L}_N(B_{1/2} \cap \mathbb{R}^N).$$

This contradicts the minimality of u, as soon as $\varepsilon > 0$ is small enough. □

3.3. *Optimal regularity*

We use here the characterisation of Hölder functions (Morrey [17]): given $0 < \alpha < 1$, if B is a ball in $\mathbb{R}^{N+1}$, and if there are $C > 0$ and $p \in (1, N+1)$ such that

$$\forall x \in B,\ \forall r < d(x, \partial B), \qquad \int_{B_r(x)} |\nabla u|^p \le C r^{N+1-p+p\alpha}, \tag{3.4}$$

then $u \in C^\alpha(B)$.

Proof of Theorem 1.1. Let u be a local minimiser in B_1. For every $r \in (0, 1)$ and (x_0, y_0) in B_1, let us consider the harmonic replacement of u in $B_r(x_0, y_0)$ (we have chosen $r < 1 - |x_0|$), i.e. the solution of

$$-\mathrm{div}(|y|^\beta \nabla h_r^{x_0,y_0}) = 0 \quad \text{in } B_r(x_0, y_0), \qquad h_r^{x_0,y_0}|_{\partial B_r(x_0,y_0)} = u. \tag{3.5}$$

From the translation invariance in x we may assume $x_0 = 0$. We simply denote by h_r the solution of (3.5). Notice that, thanks to Theorems 2.2 and 2.3, u is an admissible Dirichlet datum. For all $r > 0$ note that $\mathcal{J}(u, B_r) \le \mathcal{J}(h_r, B_r)$; this implies

$$\int_{B_r} |y|^\beta |\nabla u|^2 \le \int_{B_r} |y|^\beta |\nabla h_r|^2 + C r^N.$$

This, due to the identity $\int_{B_r} |y|^\beta \nabla h_r . \nabla(u - h_r) = 0$, translates into

$$\int_{B_r} |y|^\beta |\nabla(u - h_r)|^2 \le C r^N.$$

Therefore, if $r < \rho < 1$ we have

$$\begin{aligned}\int_{B_r} |y|^\beta |\nabla u|^2 &= \int_{B_r} |y|^\beta |\nabla(u - h_\rho + h_\rho)|^2 \\ &\le 2\left(\int_{B_\rho} |y|^\beta |\nabla(u - h_\rho)|^2 + \int_{B_r} |y|^\beta |\nabla h_\rho|^2\right) \\ &\le C\rho^N + 2\int_{B_r} |y|^\beta |\nabla h_\rho|^2 \\ &\le C\rho^N + C\left(\frac{r}{\rho}\right)^{N+1+\beta} \int_{B_\rho} |y|^\beta |\nabla h_\rho|^2 \quad \text{by Theorem 2.6} \\ &\le C\rho^N + C\left(\frac{r}{\rho}\right)^{N+1+\beta} \int_{B_\rho} |y|^\beta |\nabla u|^2. \end{aligned} \tag{3.6}$$

Take now any $\delta < 1/2$. The last line of (3.6) with

$$\rho = \delta^n, \quad r = \delta^{n+1}, \quad \mu := \delta^N \tag{3.7}$$

yields

$$\int_{B_{\delta^{n+1}}} |y|^\beta |\nabla u|^2 \le C\mu^n + C\mu\delta^{2(1-\alpha)} \int_{B_{\delta^n}} |y|^\beta |\nabla u|^2.$$

Choosing δ such that $q := C\delta^{2(1-\alpha)} < 1$, we infer from the above (and an elementary induction) that

$$\int_{B_{\delta^n}} |y|^\beta |\nabla u|^2 \le \frac{C^2}{1-q}\mu^{n-1}.$$

This implies in turn, for all $r < 1/2$, and for a possibly different constant,

$$\int_{B_r} |y|^\beta |\nabla u|^2 \le Cr^N. \tag{3.8}$$

Case 1: $\alpha \le 1/2$. Then $\beta \ge 0$ and we write

$$\int_{B_r} |\nabla u| \le \left(\int_{B_r} |y|^{-\beta}\right)^{1/2} \left(\int_{B_r} |y|^\beta |\nabla u|^2\right)^{1/2} \le Cr^{N+\alpha}.$$

This is (3.4) with $p = 1$. We have $u \in C^\alpha(B_{1/2})$.

Case 2: $\alpha > 1/2$. This time we have

$$\int_{B_r} |\nabla u|^2 \le r^{-\beta} \int_{B_r} |y|^\beta |\nabla u|^2 \le Cr^{N-\beta} = Cr^{N-1+2\alpha}.$$

This is once again (3.4) with $p = 2$, which ends the proof of Theorem 1.1. □

3.4. Nondegeneracy

At this point, it is convenient to define the blow-up of a local minimiser around a free boundary point. If $x_0 \in \mathbb{R}^N$ is a free boundary point for u, let us define the *blow-up* of u at x_0 as

$$u_r(x, y) = \frac{1}{r^\alpha} u(x_0 + rx, ry). \tag{3.9}$$

For every $r > 0$, $\lambda > 0$ and $u \in H^1(\beta, B_r)$ we have

$$\mathcal{J}(u, B_\lambda) = r^N \mathcal{J}(u_r, B_{\lambda/r}). \tag{3.10}$$

Consequently, u is a local minimiser of $\mathcal{J}(\cdot, B_\lambda)$ if and only if u_r is a local minimiser of $r^N \mathcal{J}(\cdot, B_{\lambda/r})$. Moreover, the family (u_r) is equicontinuous, because each of its elements is a dilation of a unique function.

Proof of Theorem 1.2. We do not lose any generality if we prove the following: for u satisfying the assumptions of Theorem 1.3, if $(x_0, 0)$ is at distance 1 from the free boundary, then $\varepsilon := u(x_0, 0)$ is not too small.

From the Harnack inequality in Theorem 2.4 there is $C_0 > 0$ such that, since $u(x_0, 0) = \varepsilon$, we have $u \leq C_0\varepsilon$ in $B_1(x_0, 0)$. Let γ be a smooth nonnegative function such that

$$\gamma(x, y) = 0 \quad \text{in } B_{1/2}(x_0, 0), \qquad \gamma(x, y) = 2C_0 \quad \text{in } B_{7/8}(x_0, 0) \setminus B_{3/4}(x_0, 0).$$

The function

$$v(x, y) = \min(u(x, y), \varepsilon\gamma(x, y))$$

is an admissible test function for (1.10) in $B_1(x_0, 0)$: indeed, it belongs to $H^1(\beta, B_1(x_0, 0))$ and satisfies $v = u$ at the boundary of the ball. We should therefore have

$$\mathcal{J}(u, B_1(x_0, 0)) \leq \mathcal{J}(v, B_1(x_0, 0)). \tag{3.11}$$

However, from the very definition of v,

$$\int_{B_1(x_0,0)} |y|^\beta |\nabla v|^2 \leq \int_{B_1(x_0,0)} |y|^\beta |\nabla u|^2 + O(\varepsilon),$$

and because $v \equiv 0$ on $B_{1/2}(x_0, 0)$, we have

$$\mathcal{L}_N(\{v > 0\}) \leq \mathcal{L}_N(\{u > 0\}) - \mathcal{L}_N(B_{1/2}(x_0, 0) \cap \mathbb{R}^N).$$

Consequently, $\mathcal{J}(u, B_1(x_0, 0)) > \mathcal{J}(v, B_1(x_0, 0))$, contradicting (3.11). □

The next step to Theorem 1.3 is an improvement of Theorem 1.2, which says that u grows like r^α away from a free boundary point. From Theorem 1.3, the set $\{u > 0\}$ could have a narrow cusp going into a free boundary. Here is the precise statement, showing that the scenario is not possible.

Proposition 3.3. *If u is a local minimiser defined in B_1 and $(0,0)$ is a free boundary point, then there is $C > 0$ such that, for $0 < r < 1/2$,*

$$\sup_{B_r^N} u \geq Cr^{\alpha}. \tag{3.12}$$

Proof. The proof is divided into two steps.

Step 1. Let u be a local minimiser in $B_M^N(0)$ such that

- the origin is a free boundary point,
- $B_1(e_1, 0) \subset \{u > 0\}$,
- $u(e_1, 0) = \tau > 0$.

From Theorem 1.3, the constant τ is universally bounded and bounded away from 0. We claim the existence of $\lambda > 0$ and $M > 0$ universal, the latter being large, such that

$$\sup_{B_M^N(0)} u \geq (1+\lambda)\tau. \tag{3.13}$$

Suppose not. This implies the existence of a sequence $(u_k)_{k\in\mathbb{N}}$ of solutions such that

$$\lim_{k\to+\infty} \sup_{B_k^N(0)} u = \tau.$$

From optimal regularity, the family $(u_k)_k$ is equicontinuous in $B_2(0)$, hence it may be assumed to converge uniformly on every compact subset of $\mathbb{R}^{N+1}$ to a function u_∞ which, by Proposition 3.1, is α-harmonic on its positivity set restricted to the hyperplane $\{y = 0\}$. Moreover $u_\infty(\cdot, 0)$ has a maximum at e_1, thus it is constant from the strong maximum principle (Proposition 2.7). Hence $u_\infty \equiv \tau$, a contradiction because 0 is a free boundary point.

Step 2. Assume that 0 is a free boundary point. The argument now follows as in [3]: starting at the origin, we construct inductively a sequence $(x_n)_n$ of points such that

- $u(x_{n+1}, 0) \geq (1+\lambda)u(x_n, 0)$,
- if $r_n := d(x_n, \{u = 0\})$ and $\tilde{x}_n$ is a free boundary point realising the distance, we have $x_{n+1} \in B_{Mr_n}^N(\tilde{x}_n)$. This is allowed by the construction of Step 1, applied to the blow-up $(1/r_n^{\alpha})u(\tilde{x}_n + r_n x, r_n y)$.

In particular, we have

$$|x_{n+1} - x_n| \leq (M+1)r_n.$$

We end the induction at the first index n_0 such that x_n leaves B_1^N. This is indeed possible, since the sequence $(u(x_n, 0))_n$ grows geometrically, and is controlled by $|x_n|^{\alpha}$. Let n_0 be therefore the first x_n leaving B_1^N. Then

$$
\begin{aligned}
u(x_{n_0+1},0) &= \sum_{n\le n_0}(u(x_{n+1},0)-u(x_n,0)) \ge \lambda \sum_{n\le n_0} u(x_n,0)\\
&\ge C\lambda \sum_{n\le n_0} d(x_n,\{u=0\}\cap B_1^N)^\alpha \quad \text{by nondegeneracy}\\
&\ge C'\lambda \sum_{n\le n_0} |x_{n+1}-x_n|^\alpha\\
&\ge C''\lambda \sum_{n=0}^{n_0} |x_{n+1}-x_n| \quad \text{because } |x_{n+1}-x_n|\le 1\\
&\ge C'''|x_{n_0}|.
\end{aligned}
$$

The constants C to C''' do not depend on n. Set $q := C'''|x_{n_0}|$; it is universal from the above considerations. Our argument proves that for all $r > 0$,

$$\sup_{B_{Mr}^N} u(\cdot,0) \ge qr^\alpha,$$

which is the sought-for estimate just by replacing r by r/M. □

Proof of Theorem 1.3. With the aid of the blow-up u_r, the problem is now to prove that if 0 is a free boundary point, there is $\delta \in (0,1)$ such that

$$\mathcal{L}_N(\{u>0\}\cap B_1^N) \ge \delta \tag{3.14}$$

and

$$\mathcal{L}_N(\{u=0\}\cap B_1^N) \ge \delta. \tag{3.15}$$

Property (3.14) is readily proved by combination of Theorem 1.1 (optimal regularity) and Proposition 3.3 just proved: indeed, the latter implies the existence of a ball with radius comparable to unity, contained in $\{u>0\}\cap B_1^N$. Let us prove (3.15); for this we assume the contrary, i.e. there is a sequence $(u_n)_n$ of minimisers, defined in B_1, such that

$$\lim_{n\to+\infty} \mathcal{L}_N(\{u_n=0\}) = 0.$$

Also assume, without loss of generality, that 0 is a common free boundary point to all the u_n. The sequence $(u_n)_n$ may be assumed to converge to u_∞; moreover we have

$$\int_{B_1} |y|^\beta |\nabla u_\infty|^2\,dx\,dy \le \liminf_{n\to+\infty} \int_{B_1} |y|^\beta |\nabla u_n|^2\,dx\,dy.$$

For every v agreeing with u_n on ∂B_1 we have $\mathcal{J}(u_n,B_1) \le \mathcal{J}(v,B_1)$. Because the measure of the zero set of u_n goes to 0 as $n\to+\infty$, the above inequality implies, for every v in $H^1(\beta,B_1)$ and agreeing with u_∞ on ∂B_1,

$$\mathcal{L}_N(B_1^N) + \int_{B_1} |y|^\beta |\nabla u_\infty|^2\,dx\,dy \le \mathcal{J}(v,B_1) \le \mathcal{L}_N(B_1^N) + \int_{B_1} |y|^\beta |\nabla v|^2\,dx\,dy.$$

Consequently, u_∞ minimises the Dirichlet integral over the unit ball of $\mathbb{R}^{N+1}$ and, as such, satisfies $\operatorname{div}(|y|^\beta \nabla u_\infty) = 0$ in B_1. By nondegeneracy, it cannot be uniformly 0 (recall that 0 is a free boundary point). But the interior Harnack inequality implies $u_\infty > 0$ in B_1, a contradiction. □

We notice that we have proved in fact that the set $\{u_\infty = 0\}$ is the a.e.-limit of $\{u_n = 0\}$. Theorems 1.1 and 1.3 imply the following important corollary.

Corollary 3.4 (Sequences of minimisers converge to minimisers). [i] *Let* $(u_n)_n$ *be a sequence of minimisers of* $\mathcal{J}$*, bounded in* $H^1(\beta, B_1)$*. Then any (weakly) converging subsequence of* $(u_n)_n$ *converges to a minimiser of* $\mathcal{J}$ *in* B_1.

[ii] (The particular case of blow-ups) *Let* u *solve* (1.10)*, and let* $x_0 \in \mathbb{R}^N$ *be a free boundary point. For* $r \in (0, 1)$ *consider the blow-up* u_r *given by* (3.9)*. Then* u_r *is a local minimiser of* $\mathcal{J}$ *in* $B_{1/r}$*, and any uniform limit of the family* $(u_r)_r$ *is a global minimiser of* $\mathcal{J}$.

Proof. Part [ii] is just a consequence of [i] and the fact that all the blow-ups u_r are rescalings of the same function. As for [i], consider an $H^1(\beta, B_1)$-bounded sequence $(u_n)_n$ of local minimisers. From optimal regularity there is a subsequence uniformly (and also $H^1(\beta, B_1)$-weakly) converging to some $u_\infty \in C^\alpha(B_1) \cap H^1(\beta, B_1)$. From the lower semicontinuity of the Dirichlet integral in the weak H^1_β topology, we have

$$\int_{B_1} |y|^\beta \nabla u_\infty \, dx\, dy + \limsup_{n \to +\infty} \mathcal{L}_N(\{u_n = 0\}) \le \mathcal{J}(v, B_1)$$

for all $v \in H^1(\beta, B_1)$ coinciding with u_∞ on ∂B_1. The issue is now to prove that

$$\{u_\infty = 0\} \subset \bigcap_p \bigcup_{n \ge p} \{u_n = 0\} =: \limsup_{n \to +\infty} \{u_n = 0\},$$

with the possible exception of a set with zero measure. Now, by Lebesgue's differentiability theorem, almost every point of B_1 is a differentiability point of $\mathbf{1}_{\{u_\infty = 0\}}$, which implies

$$\lim_{r \to 0} \frac{\mathcal{L}_N(\{u_\infty = 0\} \cap B_r^N(x_0))}{\mathcal{L}_N(B_r^N(x_0))} = 1$$

if x_0 is such a point. But, from Theorem 1.3, x_0 has to be an interior point of $\{u = 0\}$: otherwise, the quantity

$$\frac{\mathcal{L}_N(\{u_\infty > 0\} \cap B_r^N(x_0))}{\mathcal{L}_N(B_r^N(x_0))}$$

would be bounded from below. In other words there is $\delta > 0$ such that $B_{2\delta}(x_0) \subset \{u_\infty = 0\}$, and thus, by uniform convergence, $B_\delta(x_0) \subset \{u_n = 0\}$ for large n. □

4. Regular points, free boundary relation

In this section we start the study of the free boundary of a local minimiser, i.e. a solution of (1.10). Let u be such a minimiser; denote by $\Gamma(u) \subset \mathbb{R}^N$ its free boundary, $\Omega_-(u) \subset \mathbb{R}^N$ the set where it is 0, and $\Omega_+(u) \subset \mathbb{R}^N$ its positivity set. Let x_0 be a free boundary point. We now know from the preceding section that u is C^α and nondegenerate. Hence any blow-up limit of u centred at x_0—i.e. any limit of blow-ups u_r defined by (3.9)—is a nontrivial C^α function. We want to prove Theorem 1.4, i.e. the existence of $A_* > 0$ such that, for each regular point x_0 of $\Gamma(u)$, each blow-up limit of u around x_0 satisfies, in some coordinate system, $u(x', x_N, 0) = A_*(x_N)_+^\alpha$. By regularity we mean the existence of a measure-theoretical normal or, as we shall see later, a tangent ball from inside or outside.

Definition 4.1. The *reduced part* $\Gamma^*(u)$ of the free boundary $\Gamma(u)$ is the set of points x_0 at which the following holds: given the half ball $(B_r^N)^+(x_0) := \{(x-x_0).\nu \geq 0\} \cap B_r^N(x_0)$, we have

$$\lim_{r\to 0} \frac{\mathcal{L}_N((B_r^N)^+(x_0) \triangle \Omega_+(u))}{\mathcal{L}_N(B_r^N(x_0))} = 0.$$

The definition means (see [15]) that the vector measure $\nabla \mathbf{1}_\Omega(B_r^N(x_0))$ has a density at the point, in other words there is $\nu(x_0)$ (with $|\nu(x_0)| = 1$) such that the quantity

$$\lim_{r\to 0} \frac{\nabla \mathbf{1}_\Omega(B_r^N(x_0))}{|\nabla \mathbf{1}_\Omega(B_r^N(x_0))|} \tag{4.1}$$

exists and is equal to $\nu(x_0)$. Note that from the uniform density of $\Omega^\pm$ we have, as $r \to 0$ and at the free boundary point x_0,

$$B_r^N(x_0) \cap \Gamma^*(u) \subset \{|(x - x_0) \,.\, \nu(x_0)| \leq o(r)\}. \tag{4.2}$$

Indeed, if $u(x) = 0$ for $(x-x_0).\nu(x_0) \geq \delta r$, there is $q > 0$ such that $\mathcal{L}_N(B_{\delta r}^N(x) \cap \{u = 0\}) \geq q\delta r^N$, implying

$$\liminf_{r\to 0} \frac{\mathcal{L}_N((B_r^N)^+(x_0) \triangle \Omega_+(u))}{\mathcal{L}_N(B_r(x_0))} \geq q\delta,$$

a contradiction to the definition. The same argument is valid if $x \in \Omega_-$ is such that $(x - x_0) \,.\, \nu(x_0) \leq -\delta r$.

In the first subsection, we prove that blow-up limits at regular points are one-dimensional. In the second one, we prove the free boundary relation at different kinds of regular points.

4.1. Blow-up limits

The main result of the subsection is the following.

Proposition 4.2. *Consider $x_0 \in \Gamma^*(u)$. Then, for any blow-up limit $u_\infty(x, y)$ of u about x_0, there exist $A > 0$ and a coordinate system $(x', x_N) \in \mathbb{R}^N$ centred at 0 such that $u(x, 0) = A(x_N)_+^\alpha$.*

Proof. Let u_∞ be such a blow-up limit. There exists a coordinate system (x', x_N) centred at 0 such that:

- $\Omega_+(u_\infty) = \mathbb{R}^N_+$ (this is due to (4.2)),
- $(-\Delta)^\alpha u_\infty = 0$ in $\Omega_+(u)$.

Set $u_0(x) = (x_N)^\alpha_+$; by optimal regularity and nondegeneracy there are constants $0 < C_1 u_0 \le u_\infty \le C_2 u_0$. On the other hand, the Harnack constants are invariant under the scaling (3.9). Thus, the oscillation lemma (Theorem 2.5[ii]) holds at every scale, the solutions being global. Thus we may apply it all the way down from a ball of radius $2^n r$ (n arbitrarily large) to a ball of radius r. And so, u_∞/u_0 is constant. □

4.2. The free boundary condition

Since the blow-up profile depends on taking a subsequence, the constant A exhibited in the first step is *a priori* not universal, and this is what we are going to fix now. Let $P_{N,\alpha}(x, y)$ be the Poisson kernel of the operator $-\mathrm{div}(|y|^\beta \nabla)$ in $\mathbb{R}^{N+1}$. We have

$$P_{1,\alpha}(x, y) = q_{1,\alpha} \frac{y^{2\alpha}}{(x^2 + y^2)^{(1+2\alpha)/2}}. \tag{4.3}$$

Set $u_0(x) = (x^+)^\alpha$. By Corollary 3.4, the function

$$U_0(x, y) = A \int_{x\in\mathbb{R},\, y>0} P_{1,\alpha}(\bar{x}, y) u_0(x - \bar{x})\, d\bar{x} \tag{4.4}$$

is a global minimiser in $\mathbb{R}^2_+$. As a preliminary step we want to see which A allow the function U_0 given by (4.4) to be a local minimum; a suitable choice of the test function in the general space $\mathbb{R}^{N+1}$ will conclude the proof. The argument as a whole is classical: it consists in perturbing the free boundary of u_0 along its normal, but the calculations are more involved than in the classical case due to the nonlocality of the fractional Laplacian.

Proposition 4.3. *If AU_0 is a global minimiser in $\mathbb{R}^2_+$, then $A = A_\alpha$, given in* (1.13).

Proof. For all small ε (no sign condition on ε) let us define u_ε as

$$u_\varepsilon(x) = \frac{(x + \varepsilon)^\alpha_+}{(1 + \varepsilon)^\alpha}, \tag{4.5}$$

and $\tilde{u}_\varepsilon$ as

$$\tilde{u}_\varepsilon(x) = u_\varepsilon(x) \quad \text{if } |x| \le 1, \qquad \tilde{u}_\varepsilon(x) = x^\alpha_+ \quad \text{if } |x| \ge 1. \tag{4.6}$$

In particular, we may take $\varepsilon = 0$ and have

$$u_0(x) = \tilde{u}_0(x) = (x_+)^\alpha. \tag{4.7}$$

This time we use the fact that a minimiser in B_1 can be viewed as a minimiser in B_{1+}.

Define U_ε by

$$\begin{aligned} -\mathrm{div}(y^\beta \nabla U_\varepsilon) &= 0 && \text{in } B_{1+},\\ U_\varepsilon(x,0) &= u_\varepsilon(x) && \text{in } (-1,1),\\ U_\varepsilon(x) &= U_0(x) && \text{in } \{|(x,y)|=1,\ y>0\} \end{aligned} \tag{4.8}$$

and note that

$$E[AU_0] + \mathcal{L}_1(\{U_0>0\}) \le E[AU_\varepsilon] + \mathcal{L}_1(\{U_\varepsilon>0\}). \tag{4.9}$$

Here we have denoted by $E[\cdot]$ the Dirichlet integral

$$\forall v \in H^1(\beta, B_{1+}), \qquad E[v] = \int_{B_{1+}} y^\beta |\nabla v|^2\,dx\,dy.$$

Obviously we have

$$\mathcal{L}_1(\{U_\varepsilon>0\}) - \mathcal{L}_1(\{U_0>0\}) = \mathcal{L}_1(\{(x+\varepsilon)_+^\alpha>0\}) - \mathcal{L}_1(\{x_+^\alpha>0\}) = \varepsilon.$$

The difference of the Dirichlet integrals is

$$\begin{aligned} E[U_\varepsilon] - E[U_0] &= -2\int_{B_{1+}} y^\beta \nabla U_0 \,.\, \nabla(U_\varepsilon - U_0)\,dx\,dy\\ &\quad + \int_{B_{1+}} y^\beta |\nabla(U_\varepsilon - U_0)|^2\,dx\,dy\\ &:= -2\cdot I + II. \end{aligned}$$

Integrating by parts, we compute the term I as

$$\begin{aligned} I &= -\int_{-1}^{1} (u_\varepsilon(x) - x_+^\alpha) \lim_{y\to 0}(y^\beta \partial_y U_0(x,y))\,dx\\ &= -\int_{-\varepsilon}^{0} u_\varepsilon(x)(-\partial_{xx})^\alpha x_+^\alpha\,dx \qquad \text{because } (-\partial_{xx})^\alpha x_+^\alpha = 0 \text{ if } x>0\\ &= \frac{c_{1,\alpha}}{(1+\varepsilon)^\alpha}\int_{-\varepsilon}^{0} \frac{(x+\varepsilon)^\alpha}{(-x)^\alpha}\left(\int_x^{+\infty} \frac{(x+y)_+^\alpha}{y^{1+2\alpha}}\,dy\right)dx\\ &= \varepsilon c_{1,\alpha}\int_{-1}^{0} \frac{(x+1)_+^\alpha}{(-x)^\alpha}\,dx \int_1^{+\infty} \frac{(1+y)_+^\alpha}{y^{1+2\alpha}}\,dy = \varepsilon A_\alpha + O(\varepsilon^2). \end{aligned}$$

Arguing in a similar fashion we have

$$\begin{aligned} II &= -\int_{-\varepsilon}^{0} \tilde u_\varepsilon(x).(-\partial_{xx})^\alpha(\tilde u_\varepsilon - x_+^\alpha)\,dx\\ &= -\int_{-\varepsilon}^{0} \tilde u_\varepsilon(x).(-\partial_{xx})^\alpha(\tilde u_\varepsilon - u_\varepsilon(x))\,dx - \int_{-\varepsilon}^{0} \tilde u_\varepsilon(x).(-\partial_{xx})^\alpha(u_\varepsilon - u_\varepsilon(x))\,dx\\ &= -\int_{-\varepsilon}^{0} \tilde u_\varepsilon(x).(-\partial_{xx})^\alpha(u_\varepsilon - u_\varepsilon(x))\,dx + O(|\varepsilon|^{1+\alpha})\\ &\qquad\qquad \text{because } (-\partial_{xx})^\alpha(\tilde u_\varepsilon - u_\varepsilon(x)) = O(1) \text{ on } (-1,1)\\ &= I + O(|\varepsilon|^{1+\alpha}). \end{aligned}$$

Gathering everything, we obtain $-\varepsilon A_\alpha A + \varepsilon + O(|\varepsilon|^{1+\alpha}) \geq 0$, which, by letting ε go to 0^+ or 0^-, yields the sought-for value of A. □

Now, we may complete this section by giving the

Proof of Theorem 1.4. It remains to prove that if AU_0 is a solution of the minimisation problem (1.10) (i.e. this time in $N+1$ space dimensions), defined in the whole space $\mathbb{R}^{N+1}$, then we still have $A = A_\alpha$. In this proof only, let $\tilde{B}_r^2$ be the two-dimensional ball having one direction in the plane and one in the extension:

$$\tilde{B}_r^2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |(x, y)| \leq r\} \text{ (and } x \in \mathbb{R}^N\text{, the reference hyperplane).}$$

A p-dimensional ball with radius r, included in the reference hyperplane $\mathbb{R}^N$ and centred at 0, will be denoted by B_r^p.

With these notations, consider a smooth, nonnegative function $\varphi(x')$ such that φ is identically equal to 1 in B_R^{N-1}, and to 0 outside B_{R+1}^{N-1}; we may require $\|\nabla\varphi\|_\infty \leq 1$. Let $w \in H^1(\beta, \tilde{B}_{1+}^2)$ be such that $w = AU_0$ on $\{y > 0,\ |(x, y)| = 1\}$; consider

$$v(x', x_N, y) = \varphi(x')w(x_N, y) + A(1 - \varphi(x'))U_0(x_N, y).$$

This is an admissible test function on $B_{R+1}^{N-1} \times \tilde{B}_{1+}^2$, coinciding with AU_0 on the boundary $\partial(B_{R+1}^{N-1} \times \tilde{B}_{1+}^2)$. Hence

$$\mathcal{J}(AU_0, B_{R+1}^{N-1} \times \tilde{B}_{1+}^2) \leq \mathcal{J}(v, B_{R+1}^{N-1} \times \tilde{B}_{1+}^2).$$

We have

$$\mathcal{J}(AU_0, B_{R+1}^{N-1} \times \tilde{B}_{1+}^2) = \mathcal{L}_{N-1}(B_{R+1}^{N-1})\mathcal{J}(AU_0, \tilde{B}_{1+}^2),$$
$$\mathcal{J}(v, B_{R+1}^{N-1} \times \tilde{B}_{1+}^2) = \mathcal{L}_{N-1}(B_{R+1}^{N-1})(\mathcal{J}(w, \tilde{B}_{1+}^2) + O(1/R)).$$

Letting $R \to +\infty$ yields

$$\mathcal{J}(AU_0, \tilde{B}_{1+}^2) \leq \mathcal{J}(w, \tilde{B}_{1+}^2).$$

Because w is an arbitrary admissible test function, AU_0 is a 2D minimiser, and we may apply Proposition 4.3. □

4.3. *Tangent balls from one side*

We show that points at which the free boundary has a tangent ball are regular points. First, recall the definition.

Definition 4.4. A point $x_0 \in \Gamma(u)$ has a *tangent ball from outside* if there is a ball $B \subset \Omega_-(u)$ such that $x_0 \in B \cap \Gamma(u)$. A point $x_0 \in \Gamma(u)$ has a *tangent ball from inside* if there is a ball $B \subset \Omega_+(u)$ such that $x_0 \in B \cap \Gamma(u)$. A point $x_0 \in \Gamma(u)$ is *regular* if $\Gamma(u)$ has a tangent hyperplane at x_0.

The additional information is the following:

Proposition 4.5. *A point $x_0 \in \Gamma(u)$ which has a tangent ball from outside or from inside is regular.*

Proof. It is enough to prove that if $\Gamma(u)$ has a tangent ball from one side at a point x_0, it has a tangent plane from the other side at x_0. The proof follows the lines of Lemma 11.17 of [9], and we will only stress what modifications need to be done. If $B_1^N(x_1)$ is tangent to $\Gamma(u)$ at x_0, we use as a lower barrier the fundamental solution u^* with pole at x_1 (see [4] for instance) vanishing at $\partial B_1^N(x_1)$, and work with its extension in $B_1(x_1, 0)$ (the $(N+1)$-dimensional ball). From nondegeneracy, some small multiple $q_0 u^*$ is a lower barrier of u in $B_1(x_1, 0)$. Let $q_r > 0$ be the supremum of all q's such that $u \geq q u^*$ in $B_r(x_1, 0)$; clearly q_r increases with r and, by optimal regularity, converges to some constant q_∞ as $r \to 0$. As in [9], this forces the asymptotic behaviour

$$u(x, 0) = q_\infty((x - x_0) \,.\, \nu(x_0))^\alpha + o(((x - x_0) \,.\, \nu(x_0))^\alpha)$$

with $\nu(x_0) = x_1 - x_0$. Thus the plane orthogonal to $\nu(x_0)$ is tangent to $\Gamma(u)$.

If instead $B_1^N(y_1)$ is tangent from the $\{u = 0\} \cap B_1^N$ side, we use as an exterior barrier the inversion of the fundamental solution—see [11]. □

5. The planar case: Lipschitz implies C^1

In this final section we assume that $N = 2$; in this section only a point in the plane $\mathbb{R}^2$ will be denoted by $X = (x_1, x_2)$ and the ball in $\mathbb{R}^2$ with centre X and radius r will be denoted by $B_r^2(X)$. For every $\theta \in (0, \pi/2]$ and every unit vector ν, the planar cone of centre 0, direction ν and opening θ will be denoted by $\mathcal{C}(\nu, \theta)$. The situation is the following: we are given

- a function $u(X, y) \in C^\alpha(B_1)$, nondegenerate, i.e. satisfying the conclusion of Theorem 1.3,
- a Lipschitz graph in $B_1 \cap \mathbb{R}^N$: $\Omega_+(u) \cap B_1^2(0) = \{(x_1, x_2) : x_2 > f(x_1)\}$ where f is a Lipschitz function and $f(0) = 0$,

such that

$$\begin{aligned}
-\mathrm{div}(|y|^\beta \nabla u) &= 0 \quad \text{in } B_1, \\
u(x_1, x_2, 0) &= 0 \quad \text{in } \{x_2 < f(x_1)\}, \\
\lim_{y \to 0} (y^\beta u_y)(x_1, x_2, y) &= 0 \quad \text{in } \{x_2 < f(x_1)\}, \qquad (5.1)\\
u(X, 0) \sim_{X \to \bar{X}} A_\alpha((X - \bar{X}) \,.\, \nu(\bar{X}))^\alpha \quad &\text{if } \bar{X} \in \Gamma(u) \text{ is regular and } X \in \Omega_+(u).
\end{aligned}$$

In (5.1), a regular point of $\Gamma(u)$ is a point $\bar{X} = (\bar{x}_1, \bar{x}_2)$ such that $f'(\bar{x}_1)$ exists. The vector $\nu(\bar{X})$ is the normal to $\Gamma(u)$ at $\bar{X}$ pointing into $\Omega_+(u)$:

$$\nu(\bar{X}) = \frac{1}{\sqrt{1 + f'(\bar{x}_1)^2}}(f'(\bar{x}_1), -1).$$

The constant A_α is given by (1.13) in Theorem 1.4.

We will prove that f is necessarily C^1 in a neighbourhood of 0, and the strategy is in the spirit of [6], [9]. We prove that, in a nested sequence of balls centred at 0, the Lipschitz constant of the graph (modulo rotations) goes to 0. What we will not be able to retrieve is a control on how the Lipschitz constant of f goes to 0—were it the case, we would infer that the free boundary is $C^{1,\gamma}$ in the vicinity of 0. The argument is inductive, and the idea is to substitute the 'iterative' hypothesis: 'the free boundary is a Lipschitz graph with smaller and smaller Lipschitz constant' by the richer hypothesis: 'the function u is, in smaller and smaller balls, monotone in a larger and larger cone of directions'. In other words, all level sets of u—and not only the zero level set—are Lipschitz with smaller and smaller constants.

5.1. More on regular points

The 1D solution $(x_+)^\alpha$ is from now on (as in (4.7)) denoted by $u_0(x)$. The following two corollaries that follow from Proposition 4.5 quantify how fast u converges to the global profile at a boundary point.

Corollary 5.1. *Assume $u(x_1, x_2, y)$ satisfies the assumptions of this section, is defined in B_{2M} ($M > 0$ large), and $B^2_M(0)$ is tangent to $\Gamma(u)$ from one side. For every $\varepsilon > 0$, there is $M_\varepsilon > 0$ such that if $M \geq M_\varepsilon$, then up to a rotation of the coordinates,*

$$|u(x_1, x_2, 0) - A_\alpha u_0(x_2)| \leq \varepsilon \qquad \text{in } B^2_1(0).$$

Proof. Assume the contrary. Let u_M be the blow-down

$$u_M(X, y) = \frac{1}{M^\alpha} u\left(\frac{X}{M}, \frac{y}{M}\right).$$

Once again it is an equicontinuous family of local minimisers, which therefore may be assumed to converge to $u_\infty \in C^\alpha(B_1)$, a local minimiser in B_1, having 0 as a free boundary point, and such that $B^2_1(0)$ is tangent to the free boundary from one side. From Proposition 4.5, 0 is a regular point, and so

$$u_\infty(x_1, x_2, 0) \sim A_\alpha (x_2)^\alpha_+ \qquad \text{as } |(x_1, x_2)| \to 0.$$

However, (a subsequence of) the sequence $(u_M)_M$ converges uniformly to u_∞ in $B_{1/2}$, and this entails a contradiction with the assumption. □

Corollary 5.2. *Let $X_0 \in \Gamma(u)$ have a tangent ball from one side, of radius 1. There exist $r_0 > 0$, independent of X_0, and a function $\omega(\rho)$ defined in $[0, r_0]$, such that*

$$\lim_{\rho \to 0^+} \frac{\omega(\rho)}{\rho^\alpha} = 0, \tag{5.2}$$

and, for every $r \in (0, r_0)$,

$$\begin{aligned} |u(X_0 + r\nu(X_0), 0) - A_\alpha u_0(r)| &\leq \omega(r), \\ |\nabla u(X_0 + r\nu(X_0), 0) - A_\alpha u_0'(r)\nu(X_0)| &\leq \omega(r)/r, \end{aligned} \tag{5.3}$$

where $\nu(X_0)$ is the inner normal to $\Gamma(u)$ at X_0. The function ω can be chosen independently of X_0. More generally, for every $\delta \in [0, 1)$, there exist $\omega_\delta(\rho)$ and r_δ such that:

- *for every* $\delta' \in [0, 1-\delta]$, $\omega_{\delta'}$ *satisfies* (5.2) *uniformly with respect to* $\delta' \in [0, 1-\delta]$,
- *for every* $r \in (0, r_0)$ *and every* e *on the unit sphere such that* $e.\nu(X_0) \geq 1-\delta$ *we have*

$$\begin{aligned} |u(X_0+re,0) - A_\alpha u_0(re\,.\,\nu(X_0))| &\leq \omega_\delta(r), \\ |\nabla u(X_0+re,0) - A_\alpha u_0'(re\,.\,\nu(X_0))\nu(X_0)| &\leq \omega_\delta(r)/r. \end{aligned} \tag{5.4}$$

This is just Corollary 5.1, made uniform in a small ball around $(X_0, 0)$. Therefore a standard compactness argument works. Notice that, in the notations of the corollary, we have $\omega = \omega_0$. The final corollary of this section then shows how monotonicity in a cone of directions at the free boundary propagates to the neighbouring level lines of u.

Corollary 5.3. *Let* $X_0 \in \Gamma(u)$ *have a tangent ball from one side, and let* $\nu(X_0)$ *be the normal to* $\Gamma(u)$ *at* X_0. *For every* $\theta \in (0, \pi/2]$, *there exist* $r_\theta, \varepsilon_\theta > 0$ *such that for all* $\varepsilon \in [0, \varepsilon_\theta]$, $X \in B^2_{r_\theta}(X_0)$, $\varepsilon \in [0, \varepsilon_\theta]$ *and* $e \in \mathcal{C}(\pm\nu(X_0), \theta)$ *we have*

$$u(X+\varepsilon e, 0) - u(X, 0) \geq 0 \quad (\textit{resp.} \leq 0). \tag{5.5}$$

Proof. Let $(X_n)_n$, $(e_n)_n$ and ε_n be sequences contradicting (5.5) with, for instance, the plus sign. If $\lim_{n\to+\infty} \varepsilon_n/x_{2n} = +\infty$, then, from nondegeneracy,

$$u(X_n+\varepsilon_n e_n, 0) \geq Ce_n\,.\,\nu(X_0)\varepsilon_n^\alpha,$$

and the constant C is universal. By optimal regularity,

$$u(X_n, 0) \leq C'(x_{2n})_+^\alpha.$$

Thus

$$u(X_n+\varepsilon_n e_n, 0) \geq Ce_n\,.\,\nu(X_0)\varepsilon_n^\alpha - O((x_{2n})^\alpha) \geq 0 \quad \text{for } n \text{ large},$$

a contradiction. If the sequence $(\varepsilon_n/x_{2n})_n$ is bounded, then we contradict Corollary 5.2. The minus sign case is treated similarly. □

5.2. Initial configuration (monotonicity in a cone of directions)

We start by showing that the free boundary being Lipschitz implies that all level surfaces of u nearby are Lipschitz in the X variables. The function f being Lipschitz implies that, for each $(x_1, f(x_1))$ which is a differentiability point of $\Gamma(u)$, we have

$$\text{angle}(\nu((x_1, f(x_1)), Ox_1) \subset [\pi/2 - \arctan L^0, \pi/2]. \tag{5.6}$$

Lemma 5.4. *Set* $\theta_\mu = \pi/2 - (1+\mu)\arctan L^0$. *For every* $\mu \in (0, \pi/(2\arctan L^0))$, *there are* $\rho_\mu, \delta_\mu > 0$ *such that, in the vertical cylinder* $B^2_{\rho_\mu}(0) \times (-\delta_\mu, \delta_\mu)$, *the function* u *is increasing in every direction of* $\mathcal{C}(e_2, \theta_\mu)$, *and decreasing in every direction of* $\mathcal{C}(-e_2, \theta_\mu)$.

Proof. Consider such a μ, and take $r_0 = 1/10$. Choose a direction e on the unit sphere such that

$$\text{angle}(e, e_2) \leq \pi/2 - (1+\mu)\arctan L^0.$$

In other words, the direction ε lies within the complementary cone of all possible directions of the normals to $\Gamma(u)$.

For all X in $B^2_1(0) \cap \{d(X, \Gamma(u)) = r_0\}$, denote by $\pi(X)$ its projection on $\Gamma(u)$; we have $|X - \pi(X)| = r_0$. Moreover there is $\delta_0 > 0$ such that

$$e \, . \, \nu(\pi(X)) \geq \delta_0 \mu. \tag{5.7}$$

From Corollary 5.2 and, in particular, property (5.3), we may find a universal $t_0 > 0$ such that

$$\partial_e u(\pi(X) + t_0 \nu(\pi(X)), 0) \geq 0.$$

This property, applied to all the blow-ups u_{r/r_0} with $r \leq r_0$, implies that for all $r < r_0$ and all X such that $d(X, \Gamma(u)) = t_0 r$, we have

$$\text{for all directions } e \text{ in } \mathcal{C}(e_2, \theta_\mu), \qquad \partial_e u(X, 0) \geq 0. \tag{5.8}$$

In particular, (5.8) is true in $B^2_{t_0 r_0}(0)$. In the same fashion we have

$$\text{for all directions } e \text{ in } \mathcal{C}(-e_2, \theta_\mu), \qquad \partial_e u(X, 0) \leq 0. \tag{5.9}$$

Inequalities (5.8) and (5.9) are trivially true in $\Omega_-(u) \cap B^2_{r_0}(0)$.

Let us now go to the extension, and more precisely look at the restriction of u to the (narrow) box $B^2_{r_0}(0) \times [-d, d]$. We want to prove that (5.8) and (5.9) are true for all $e \in \mathcal{C}(e_2, \theta_{\mu_0}) \cup \mathcal{C}(-e_2, \theta_{\mu_0})$. For this once again consider the sequences of blow-ups

$$u_d(X, y) = \frac{1}{d^\alpha} u(dX, dy), \qquad \text{defined in } B^2_{r_0/d}(0) \times [1, 1].$$

By optimal regularity and nondegeneracy, a subsequence of $(u_d)_d$ converges, as $d \to 0^+$, to a solution u_∞ of (5.1), but this time posed in $\mathbb{R}^3$. Moreover let $P_{2,\alpha}(X, y)$ be the Poisson kernel of the operator $-\text{div}(y^\beta \nabla)$ in $\mathbb{R}^3_+$. We have

$$\partial_e u_\infty(X, 1) = \int_{\mathbb{R}^2} P_{2,\alpha}(X - \bar{X}, y) \partial_e u_\infty(\bar{X}, 0) \, d\bar{X}.$$

By nondegeneracy this quantity (or its opposite) is uniformly controlled from below, independently of the limit u_∞. This proves

$$\text{for all directions } e \text{ in } \mathcal{C}(e_2, \theta_\mu) \text{ and } (X, y) \in B^2_{r_0}(0) \times [-d, d], \qquad \partial_e u(X, y) \geq 0.$$

In particular, (5.8) is true in $B^2_{t_0 r_0}(0)$. In the same fashion we have

$$\text{for all directions } e \text{ in } \mathcal{C}(-e_2, \theta_\mu) \text{ and } (X, y) \in B^2_{r_0}(0) \times [-d, d], \qquad \partial_e u(X, 0) \leq 0.$$

This is exactly what is claimed by the lemma. □

Choose now $\mu_0 \le \theta_0/100$ and rename $\theta_0 = \pi/2 - (1+\mu_0)\arctan L^0$. Rescale the picture of Lemma 5.4 into the unit ball by setting, for instance,

$$u(X, y) := \frac{1}{(r_0/10)^\alpha} u\left(\frac{r_0}{10}X, \frac{r_0}{10}y\right),$$

and considering only what happens in the new cylinder $B_1^2(0) \times [-d_{\mu_0}/10, d_{\mu_0}/10]$. Set $d_0 = d_{\mu_0}/10$. The situation is now as follows: in the cylinder $B_1^2(0) \times [-d_0, d_0]$, $\Gamma(u)$ is a Lipschitz planar graph, still denoted by $\{x_2 = f(x_1)\}$. Moreover, u is monotone in every direction $e \in \mathcal{C}(e_2, \theta_0) \cup \mathcal{C}(-e_2, \theta_0)$. This is our starting point.

5.3. Improvement of monotonicity at two points

The idea comes from [1]. We start by finding two free boundary points of $\mathbb{R}^2$, on each side of and at distance of order one from the origin, in such a way that, at these two points (i) we have tangent discs of radius of order one, (ii) the corresponding normal vectors form with each other an angle better than what the Lipschitz constant of f would dictate. The argument is an estimate on how the free boundary is separated both from the cone and its opposite.

Lemma 5.5. *There exist $M > 0$ and $\delta_M \in (0, \pi/2 - \theta_0)$, depending on θ_0 and M, such that if $u(\cdot, 0)$ is defined in $B_M^2(0)$, then for every unit vector ν,*

$$\sup_{X \in \Gamma(u) \cap B_1^2(0)} d(X, \mathcal{C}(\pm\nu, \theta_0)) \ge \delta_M.$$

In other words, as soon as we are close enough to the origin, the free boundary is δ_M-away from every cone.

Proof. Assume the lemma is false: there is a sequence $(M_n)_n$ going to infinity, a sequence $(\nu_n)_n$ of unit vectors with $\pm\nu_n \in \mathcal{C}(e_2, \pi/2 - \theta_0)$, as well as a sequence $(u_n)_n$ of solutions having 0 as a free boundary point, such that $u_n(\cdot, 0)$ is defined in $B_{M_n}^2(0)$ and

$$\lim_{n \to +\infty} d(\Gamma(u_n) \cap B_1^2(0), \mathcal{C}(\nu_n, \theta_0)) = 0.$$

In the limit (along a subsequence) $n \to +\infty$, there is a unit vector ν_∞ and a solution u_∞ whose free boundary coincides with $\mathcal{C}(\nu_\infty, \theta_0)$ in $B_1^2(0)$. There is obviously a tangent ball at 0 from one side, but then 0 has to be a regular point of $\Gamma(u_\infty)$; a contradiction. □

Corollary 5.6. *There are $x_- < 0 < x_+$, three real numbers: $\gamma > 0$, $r_1 > 0$, $d_1 \in (0, d_0]$, and a direction ν_1, all depending on θ_0, such that:*

- *the points $X_\pm = (x_\pm, f(x_\pm))$ are in $B_{r_0}^2$ and are regular points of $\Gamma(u)$,*
- *for all $y \in [-\delta_1, \delta_1]$, the function $(X, y) \in B_{r_1}^2(X_\pm) \mapsto u(X, y)$ is increasing in every direction of $\mathcal{C}(\nu_1, \theta_0 + \gamma)$ and decreasing in every direction of $\mathcal{C}(-\nu_1, \theta_0 + \gamma)$.*

Proof. First, rescale the picture at the end of Section 5.2 so that our function u is now defined in $B^2_M(0) \times [-d_0, d_0]$, with a value of M to which we may apply Lemma 5.5. Let $\delta_M := \delta$ (recall that $\delta < \pi/2 - \theta_0$) be such that $\Gamma(u)$ is δ-away from every cone with vertex 0 and opening θ_0. Then consider a point on $\mathcal{C}(e_2, \theta_0)$ at distance exactly δ from $\Gamma(u)$. We always assume that its projection on e_1 is negative; call this point $\tilde{X}_- = (\tilde{x}_{1-}, \tilde{x}_{2-})$. We wish to find a point on the other side of the origin, at a controlled distance from the free boundary. Let $q_0 \in (0, 1)$ be small enough so that $\arctan(q\delta) \sim q\delta$ for every $\delta \in (0, \pi/2)$ and $q \le q_0$. Now, either

- there is $\tilde{X}_+ = (\tilde{x}_{1+}, \tilde{x}_{2+} > 0) \in \mathcal{C}(e_2, \theta_0)$ at distance $q_0\delta/1000$ from $\Gamma(u)$, and we are done, or
- every point of $\mathcal{C}(e_2, \theta_0) \cap B^2_1(0)$ is at distance less than $q_0\delta/1000$ from $\Gamma(u) \cap B^2_1(0)$.

Assume the second case holds. Denote by ν_δ the image of e_2 under the rotation of angle $-\arctan(q_0\delta/10)$; then

- $d(\tilde{X}_-, \Gamma(u)) \ge \delta/2$,
- there is $\tilde{X}_+ \in X_0 + \mathcal{C}(\nu_\delta, \theta_0)$ such that $\tilde{x}_{1+} > 0$ and $d(\tilde{X}_+, \Gamma(u)) \ge \delta/100$.

If the first case holds, set $\nu_1 = e_2$; if the second case holds, set $\nu_1 := \nu_\delta$. In both cases, set $2\gamma = q_0\delta/1000$.

Let now $X_\pm = (x_\pm, f(x_\pm))$ be the projections of $\tilde{X}_\pm$ onto $\Gamma(u)$. We have $\nu(X_\pm) \in \mathcal{C}(\pm\nu_1, \pi/2 - \theta_0 - \delta)$; consequently, by Corollary 5.3 with this time $\theta = \arctan(\delta/10^6)$, there exists $r_1 < r_0$ such that, in $B^2_{r_1}(X_\pm)$,

$$\text{for all directions } e \text{ in } \mathcal{C}(\pm\nu_1, \theta_0 + \gamma), \qquad \partial_e u(X, 0) \ge 0 \ \text{ (resp. } \le 0). \tag{5.10}$$

This ends the proof of the corollary. □

5.4. Improvement of monotonicity in a whole ball, iteration

These two points having been found, we prove that the Lipschitz constant improvement propagates inwards, thus implying a better monotonicity cone in a smaller ball. Here is the main lemma.

Lemma 5.7. *There are $r_1 > 0$, $d_1 > 0$ and $\theta_1 > \theta_0$ (possibly smaller than the ones of Corollary 5.6) and a unit vector ν_1 (once again possibly different from the one of Corollary 5.6) such that:*

- *θ_1 is bounded away from 0 if θ_0 is bounded away from $\pi/2$,*
- *for all $y \in [-d_1, d_1]$ and $e \in \mathcal{C}(\nu_1, \theta_1)$, the function $X \in B^2_{r_1}(0) \mapsto u(X, y)$ is increasing in the direction e. For every $e \in \mathcal{C}(-\nu_1, \theta_1)$ the function is decreasing.*

Proof. If $X_\pm = (x_{i\pm}, f(x_{i\pm}))$ are given by Corollary 5.6, set

$$\Omega = \Big(\bigcup_{X \in \Gamma(u),\, x_{1-} < x_1 < x_{1+}} B^2_{r_1}(X) \Big) \times (-d_1, d_1). \tag{5.11}$$

We are going to propagate the monotonicity inside this cylinder. We first deal with the directions inside $\mathcal{C}(e_2, \theta_0)$; the whole argument is then repeated in the new, smaller ball in order to get the monotonicity improvement in the negative directions.

1. On the line $\Omega_+(u) \cap \{d(X, \Gamma(u)) = r_1\}$, we use a (by now classical: see for instance [6], [9]) Harnack inequality argument. Let c_0 be the nondegeneracy constant of Theorem 1.2. There is $q_0 > 0$, universal, and a point $\bar{X} \in \Omega_+(u) \cap \{d(X, \Gamma(u)) = (r_1/10c_0)^{1/\alpha}\}$ such that $|\nabla u(\bar{X}, 0)| \leq q_0$. Now, recall that $\nabla u(\bar{X}, 0) \subset \mathcal{C}(e_2, \pi/2 - \theta_0)$ and set $\nu_2 = \nabla u(\bar{X}, 0)/|\nabla u(\bar{X}, 0)|$. Let R_θ be the rotation of angle θ. Then either $\nu_2 \,.\, R_{\theta_0} e_2 \neq 0$ or $\nu_2 \,.\, R_{-\theta_0} e_2 \neq 0$. Assume the former. For convenience let here $d(X, B)$ be the *signed* distance from the point X to the set B. By the Harnack inequality we have

$$\forall (X, y) \in \partial\Omega \cap \big(\{d(X, \Gamma(u)) = r_1\} \times [-d_1, d_1] \cup \{|d(X, \Gamma(u))| \leq r_1\} \times \{-d_1, d_1\}\big),$$
$$\partial_{R_{\theta_0}} u(X, y) \geq C q_0 \nu_2 \,.\, R_{-\theta_0} e_2. \qquad (5.12)$$

We also have to treat the part of $\partial\Omega$ hitting $\Omega_-(u)$. Recall that the function $y \mapsto y^{2\alpha}$ solves $-\mathrm{div}(y^\beta \nabla u) = 0$ in $\mathbb{R}^3_+$, is positive and vanishes for $y = 0$. From the boundary Harnack inequality we have

$$\forall (X, y) \in \partial\Omega \cap \{d(X, \Gamma(u)) = -r_1\} \times [-d_1, d_1],$$
$$\partial_{R_{\theta_0}} u(X, y) \geq C q_0 \nu_2 \,.\, R_{-\theta_0} e_2 |y|^{2\alpha}. \qquad (5.13)$$

Inequalities (5.12) and (5.13) imply the existence of $\bar{\gamma}$ such that, on all $\partial\Omega$ except the lateral sides, i.e. the rectangles $[X_\pm - r_1\nu(X_\pm), X_\pm + r_1\nu(X_\pm)] \times [-d_1, d_1]$, and for all e such that $-\theta_0 \leq \mathrm{angle}(e, e_2) \leq \theta_0 + \bar{\gamma}$, we have $\partial_e u \geq 0$. To retrieve the lateral sides, we just have to apply Corollary 5.6 at $X_\pm$ and use the same argument as in Lemma 5.4 to propagate the extra monotonicity into the extension.

As a conclusion, there is $d_1 > 0$, an angle (renamed θ_1) strictly larger than θ_0 and a unit vector $\nu_1 \in \mathcal{C}(e_2, \theta_0)$ such that for all $y \in [-d_1, d_1]$ and $e \in \mathcal{C}(\nu_1, \theta_1)$, the function $X \in B^2_{r_1}(X_\pm) \mapsto u(X, y)$ is increasing in the direction e.

2. Let us finally prove that u is increasing in every direction of $\mathcal{C}(\nu_1, \theta_1)$ in the whole Ω. For this we consider, for every $\varepsilon > 0$ small enough and every $\theta \in [\theta_0, \theta_1]$ (following once again [6], [9]), the function

$$\underline{u}(X, y) = \sup_{e \in \mathcal{C}(0,\theta)} u(X - \varepsilon e, y) = \sup_{X' \in B_{\sin\theta}} u(X - \varepsilon X', y).$$

The family $(\underline{u}^\theta)_\theta$ is a continuous family of subsolutions of $-\mathrm{div}(y^\beta \nabla u) = 0$ in $\Omega \backslash \Omega_-(u)$. If we prove that $\underline{u}^{\theta_1} \leq u$ we are done; to do so let $\bar{\theta}$ be the last $\theta \geq \theta_0$ (possibly equal to θ_0) such that $\underline{u}^\theta \leq u$. The only possibility is a contact point between $\underline{u}^\theta$ and u. By the strong maximum principle this point—denote it by $(\bar{X}, 0)$—can only be on $\Gamma(u)$, and strictly between X_- and X_+. By the definition of $\underline{u}^\theta$, there is (provided $\varepsilon > 0$ is small enough) $X_\theta \in \Gamma(u)$ such that:

- there is an outside ball of radius $\varepsilon \sin\theta$ touching $\Gamma(u)$ at X_θ,
- there is an inside ball of radius $(\varepsilon/2)\sin\theta$ touching $\Gamma(u)$ at $\bar{X}$, and with $\nu(\bar{X}) = \nu(X_\theta)$.

From Theorem 1.4, we have $u(X,0) \sim A_\alpha((X-\bar{X}) \,.\, \nu(\bar{X}))^\alpha$ in a neighbourhood of $\bar{X}$. In the same fashion we have $u(X,0) \sim A_\alpha((X-X_\theta) \,.\, \nu(X_\theta))^\alpha$ in a neighbourhood of X_θ. Hence $\underline{u}^\theta(X,0) \geq A_\alpha((X-\bar{X}) \,.\, \nu(\bar{X}))^\alpha$ in a neighbourhood of $\bar{X}$. From the generalised Hopf lemma there is $\delta > 0$ such that $u(X,0) \sim (A_\alpha+\delta)((X-\bar{X}) \,.\, \nu(\bar{X}))^\alpha$ in a vicinity of $\bar{X}$. This is a contradiction, hence $\bar{\theta} = \theta_1$. □

Remark. We have not been very careful here in making explicit the respective dependences of $r_1, \theta_1, \ldots$ on θ_0. The only useful information is that θ_1 is bounded away from 0 if θ_0 is bounded away from $\pi/2$. It would, on the other hand, be crucial to have a more explicit control on θ_1 as $\theta_0 \to \pi/2$ in order to prove a $C^{1,\gamma}$ property—which we do not have at the moment.

Proof of Theorem 1.5. Iterate Lemma 5.7: at each step we obtain a ball of radius r_n with $\lim_{n\to+\infty} r_n = 0$, an angle θ_n with $\lim_{n\to+\infty}(\pi/2-\theta_n) = 0$ and a unit vector ν_n such that u is increasing (resp. decreasing) in $\mathcal{C}(\nu_n, \theta_n)$ (resp. $\mathcal{C}(-\nu_n, \theta_n)$. This implies the differentiability of the free boundary at 0, and because the estimates in Lemmas 5.4 to 5.7 only depend on the initial Lipschitz constant of the free boundary, the differentiability of the free boundary at every point in $B^2_{1/2}(0)$. The C^1 character also follows, because the iteration process implies that normal vectors at neighbouring points are close to each other. □

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