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Fredholm theory and transversality for the parametrized and for the S^1-invariant symplectic action

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Abstract. We study the parametrized Hamiltonian action functional for finite-dimensional families of Hamiltonians. We show that the linearized operator for the L^2-gradient lines is Fredholm and surjective, for a generic choice of Hamiltonian and almost complex structure. We also establish the Fredholm property and transversality for generic S^1-invariant families of Hamiltonians and almost complex structures, parametrized by odd-dimensional spheres. This is a foundational result used to define S^1-equivariant Floer homology. As an intermediate result of independent interest, we generalize Aronszajn's unique continuation theorem to a class of elliptic integro-differential inequalities of order two.

Contents

1. Introduction

Motivation I. Hamiltonian Floer homology is commonly referred to as Morse homology for the symplectic action functional on the free loop space of a symplectic manifold. One of the most important features of the free loop space is that it carries an S^1-action by reparametrization at the source

$$(\tau \cdot \gamma)(\theta) := \gamma(\theta - \tau), \qquad \tau \in S^1,\ \gamma : S^1 \to W,$$

where (W, ω) is the target symplectic manifold. It was realized at an early stage of the theory that Floer homology should admit an S^1-equivariant version. In the last section

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of the foundational article [10], Floer, Hofer, and Salamon explicitly set the goal of constructing it.

Such an S^1-equivariant theory was first defined by Viterbo [19], in the context of symplectic homology. Viterbo's paper contains a wealth of structural properties with rich applications, but it does not give any kind of technical details for the definition. The present paper grew out of our efforts to understand S^1-equivariant Floer homology and put it on firm grounds.

The topological motivation of the definition is the following. Let X be a topological space endowed with an S^1-action, and ES^1 be a contractible space on which S^1 acts freely. The *Borel construction of* X, denoted X_{S^1}, is defined to be the quotient of $X \times ES^1$ by the free diagonal action. The *S^1-equivariant homology of* X is defined to be

$$H_*^{S^1}(X) := H_*(X_{S^1}).$$

Taking as a model for ES^1 the inductive limit $\varinjlim S^{2N+1}$ of the unit spheres $S^{2N+1} \subset \mathbb{C}^{N+1}$, one sees that $X_{S^1} = \varinjlim X \times_{S^1} S^{2N+1}$. Moreover, we have

$$H_*^{S^1}(X) = \varinjlim H_*(X \times_{S^1} S^{2N+1}).$$

Assume now that X is a finite-dimensional manifold. Morse theory on the finite-dimensional approximation $X \times_{S^1} S^{2N+1}$ of the Borel construction is the same as S^1-invariant Morse theory on $X \times S^{2N+1}$. Viterbo's idea is to define S^1-equivariant Floer homology as the direct limit of S^1-invariant Floer homology groups for S^1-invariant action functionals defined on $C^\infty(S^1, W) \times S^{2N+1}$. The latter space carries the diagonal S^1-action

$$\tau \cdot (\gamma, \lambda) \mapsto (\gamma(\cdot - \tau), \tau \cdot \lambda).$$

The equation. Let $H : S^1 \times W \times S^{2N+1} \to \mathbb{R}$, $H = H(\theta, x, \lambda)$, be a smooth function, which we view as an S^{2N+1}-family of Hamiltonians $H_\lambda : S^1 \times W \to \mathbb{R}$. Let J_λ^θ, $\theta \in S^1$, $\lambda \in S^{2N+1}$, be an S^{2N+1}-family of time-dependent almost complex structures which are compatible with ω. Let g be a Riemannian metric on S^{2N+1}. The *parametrized Floer equation* for a pair of maps $u : \mathbb{R} \times S^1 \to W$ and $\lambda : \mathbb{R} \to S^{2N+1}$ is the integro-differential system

$$\partial_s u + J^\theta_{\lambda(s)}(\partial_\theta u - X^\theta_{H_{\lambda(s)}}(u)) = 0, \tag{1.1}$$

$$\dot\lambda(s) - \int_{S^1} \vec\nabla_\lambda H(\theta, u(s,\theta), \lambda(s))\, d\theta = 0, \tag{1.2}$$

subject to the asymptotic conditions

$$\lim_{s\to-\infty}(u(s,\cdot), \lambda(s)) = (\overline{\gamma}, \overline{\lambda}), \qquad \lim_{s\to+\infty}(u(s,\cdot), \lambda(s)) = (\underline{\gamma}, \underline{\lambda}), \tag{1.3}$$

where $(\overline{\gamma}, \overline{\lambda})$, $(\underline{\gamma}, \underline{\lambda})$ are elements of

$$\mathcal{P}(H) := \left\{(\gamma, \lambda) : \dot\gamma - X_{H_\lambda}(\gamma) = 0,\ \int_{S^1} \frac{\partial H}{\partial \lambda}(\theta, \gamma(\theta), \lambda)\, d\theta = 0\right\}. \tag{1.4}$$

Here and in what follows we use the notation $\vec{\nabla}$ for a gradient vector field, whereas ∇ will denote a covariant derivative. Our convention for the Hamiltonian vector field is that $\omega(X_H, \cdot) = dH$.

The Fredholm and transversality analysis contained in this paper apply to any symplectic manifold and any component of the free loop space of W. However, in order to interpret (1.1–1.2) as a (negative) gradient equation, it is convenient to restrict to the component $C^\infty_{\mathrm{contr}}(S^1, W)$ of contractible loops and to assume that (W, ω) is symplectically aspherical, i.e. $\langle[\omega], \pi_2(W)\rangle = 0$. The equations (1.1–1.2) are in this case the negative gradient equations of the *parametrized action functional*

$$\mathcal{A} : C^\infty_{\mathrm{contr}}(S^1, \widehat{W}) \times S^{2N+1} \to \mathbb{R},$$

defined by

$$\mathcal{A}(\gamma, \lambda) := -\int_{D^2} \overline{\gamma}^*\omega - \int_{S^1} H_\lambda(\theta, \gamma(\theta))\, d\theta. \tag{1.5}$$

Here $\overline{\gamma} : D^2 \to W$ is a smooth extension of γ to the disc. The metric on $C^\infty_{\mathrm{contr}}(S^1, W) \times S^{2N+1}$ is the product of the (λ-dependent) L^2-metric determined by $(J^\theta_\lambda)_{\theta\in S^1}$ with the metric g. The elements of $\mathcal{P}(H)$ are the critical points of $\mathcal{A}$.

S^1-invariance. Let us now assume that H and J are S^1-invariant with respect to the diagonal S^1-action on $S^1 \times S^{2N+1}$, meaning that

$$H_{\tau\lambda}(\theta + \tau, \cdot) = H_\lambda(\theta, \cdot), \qquad J^{\theta+\tau}_{\tau\lambda} = J^\theta_\lambda \tag{1.6}$$

for all $\theta \in S^1$, $\tau \in S^1$, $\lambda \in S^{2N+1}$. Let us also assume that the metric g on S^{2N+1} is S^1-invariant. Then equations (1.1–1.2) are invariant under the diagonal S^1-action on $C^\infty_{\mathrm{contr}}(S^1, W) \times S^{2N+1}$.

Equation (1.3) is not, since S^1 acts freely on the asymptotes $\overline{p} = (\overline{\gamma}, \overline{\lambda})$, $\underline{p} = (\underline{\gamma}, \underline{\lambda})$. To fix this, we introduce the S^1-orbits

$$S_p := S^1 \cdot p, \qquad p \in \mathcal{P}(H),$$

and the condition

$$\lim_{s\to-\infty} (u(s, \cdot), \lambda(s)) \in S_{\overline{p}}, \qquad \lim_{s\to+\infty} (u(s, \cdot), \lambda(s)) \in S_{\underline{p}}. \tag{1.7}$$

The results of Sections 5 and 7 are summarized in the following statement.

Theorem A. (a) *For a generic choice of the S^1-invariant Hamiltonian H, and for any choice of S^1-invariant (J, g), the operator which linearizes* (1.1–1.2) *is Fredholm between Sobolev spaces with suitable exponential weights.*

(b) *There exists an explicit class consisting of S^1-invariant triples (H, J, g) with H as above such that, for a generic choice of (H, J, g) inside this class, the Fredholm operator which linearizes* (1.1–1.2) *is surjective for all solutions of* (1.1–1.3) *and all $\overline{p}, \underline{p} \in \mathcal{P}(H)$.*

Part (b) is proved as Theorem 7.4. As a matter of fact, that theorem is more precise. It states that we can achieve transversality within a special class of almost complex structures (called *adapted*, Definition 7.2), after possibly perturbing a Hamiltonian which is either generic (in the sense that it belongs to the class $\mathcal{H}_{\rm gen}$ defined in Section 7), or split (in the sense that it belongs to the class $\mathcal{H}_{\rm split}$, loc. cit.). In the case of split Hamiltonians, our proof works under the assumption that W is symplectically aspherical. For generic Hamiltonians, this assumption is not used.

The Hamiltonians satisfying (a) are those for which the Hessian of $\mathcal{A}$ at a critical point is degenerate only along the infinitesimal generator of the S^1-action. As a consequence of (b), for a generic choice of (H, J, g) inside the given class, the spaces of trajectories

$$\widehat{\mathcal{M}}(\overline{p}, \underline{p}; H, J, g) := \{(u, \lambda) \text{ solving } (1.1, 1.2, 1.3)\}$$

and

$$\widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; H, J, g) := \{(u, \lambda) \text{ solving } (1.1, 1.2, 1.7)\}$$

are smooth manifolds, for all $\overline{p}, \underline{p} \in \mathcal{P}(H)$. Viterbo's definition of S^1-equivariant Floer homology relies on counting modulo the S^1-action the elements of the moduli spaces

$$\mathcal{M}(S_{\overline{p}}, S_{\underline{p}}; H, J, g) := \widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)/\mathbb{R}.$$

The parameter space. In part (a) of the above theorem we need to consider the linearized operator acting between weighted Sobolev spaces. This is necessary since, for a generic choice of the S^1-invariant Hamiltonian H, the elements of $\mathcal{P}(H)$ come in Morse–Bott nondegenerate families of dimension 1 given by the free S^1-action. In order to prove the Fredholm property, one first has to establish it for operators of the same form and having nondegenerate asymptotics. This corresponds to considering the linearization of equations (1.1–1.2) for a generic and *non*-invariant H.

The point is that equations (1.1–1.3) and the action functional (1.5) still make sense if one replaces the parameter space S^{2N+1} by some arbitrary manifold Λ, and so do the spaces of trajectories $\widehat{\mathcal{M}}(\overline{p}, \underline{p}; H, J, g)$.

We summarize the results of Sections 2 and 4 in the following statement.

Theorem B. *Let Λ be an arbitrary finite-dimensional parameter space.*

(a) *For a generic choice of H and for any choice of (J, g), the operator which linearizes* (1.1–1.2) *is Fredholm between suitable Sobolev spaces.*
(b) *For a generic choice of the triple (H, J, g), the Fredholm operator which linearizes* (1.1–1.2) *is surjective for all solutions of* (1.1–1.3) *and all $\overline{p}, \underline{p} \in \mathcal{P}(H)$.*

The Hamiltonians satisfying (a) are those for which the Hessian of $\mathcal{A}$ at a critical point is nondegenerate. As a consequence of (b), for a generic choice of (H, J, g) the moduli spaces of parametrized Floer trajectories

$$\mathcal{M}(\overline{p}, \underline{p}; H, J, g) := \widehat{\mathcal{M}}(\overline{p}, \underline{p}; H, J, g)/\mathbb{R}$$

are smooth manifolds, for all $\overline{p}, \underline{p} \in \mathcal{P}(H)$. We use these moduli spaces in [3] to define parametrized symplectic homology groups and establish a Gysin long exact sequence for symplectic homology.

Motivation II. Our initial motivation was the desire to interpret the long exact sequence in [2] as a Gysin exact sequence. To this end, we prove in [4] that, given an aspherical symplectic manifold W with contact type boundary $M = \partial W$, the (positive part of) the S^1-equivariant symplectic homology of W is isomorphic to the linearized contact homology of M, provided the latter is well defined. Via this isomorphism, the long exact sequence of [2] is isomorphic to the Gysin exact sequence of [4].

However, we believe that the present paper has ramifications going well beyond S^1-equivariant symplectic homology.

- *Transversality in linearized contact homology.* The second author is currently developing with Cieliebak a version of "nonequivariant" contact homology [9]. The Borel construction can be applied to it in order to define an invariant which is isomorphic to linearized contact homology. The results of the present paper will be instrumental in proving that transversality can be achieved for this theory, modulo having it for finite energy holomorphic planes or, alternatively, modulo the data of a linearization for the contact complex. Transversality can currently be achieved for linearized contact homology only for homotopy classes of loops which contain only simple Reeb orbits.
- *Lagrange multiplier problems.* Equations (1.1–1.2) can be viewed as a Floer type Lagrange multiplier problem. To prove unique continuation for this integro-differential system, we were led to prove a generalization of Aronszajn's theorem for integro-differential inequalities (see below). This is relevant for any Floer-type problem involving an additional parameter space. Examples are Rabinowitz–Floer homology [8], or G-equivariant Floer homology [14].
- *Floer homology for families.* Our methods can be extended to define parametrized Floer homology groups for a symplectic fibration. We expect these to coincide with the target of the Hutchings spectral sequence [12].
- *Relation to Givental's point of view.* Given a closed symplectic manifold X, Givental defined in [11] a D-module structure on $H^*(X;\mathbb{C}) \otimes \Lambda_{\mathrm{Nov}} \otimes \mathbb{C}[\hbar]$, where Λ_{Nov} is a suitable Novikov ring and $\hbar$ is the generator of $H^*(BS^1)$. He interprets this as being the S^1-equivariant Floer cohomology of X. Our construction of S^1-equivariant Floer homology in [3] provides an interpretation of the underlying homology group as the homology of a Floer-type complex. We expect that the D-module structure can also be defined within our setup. Note that, in Givental's setup, the quantum product is typically nontrivial so that we cannot assume that X is symplectically aspherical. Therefore, we have to restrict to the class $\mathcal{H}_{\mathrm{gen}}$ of generic Hamiltonians defined in Section 7.

Aronszajn's theorem. We prove in Section 3 the following unique continuation result for solutions of integro-differential inequalities, as Theorem 3.2. This generalizes a celebrated theorem of Aronszajn [1]. It allows one to prove unique continuation for solutions of the system (1.1–1.2).

Theorem C. *Let $h > 0$ and denote $Z_h :=]-h, h[\times S^1$. Assume $u \in C^\infty(Z_h, \mathbb{C}^n)$ satisfies*

$$|\Delta u(s,\theta)|^2 \leq M\left[|u(s,\theta)|^2 + |\nabla u(s,\theta)|^2 + \int_{S^1} |u(s,\tau)|^2\, d\tau\right]$$

for all $(s,\theta) \in Z_h$, where $M > 0$ is a positive constant. If u vanishes together with all its derivatives on $\{0\} \times S^1$, then $u \equiv 0$ on Z_h.

Noncompact setup. We use the setup of symplectic homology, since this was our initial motivation. The consequences are merely cosmetic, and the adaptation to the setup of closed manifolds is straightforward.

Structure of the paper. In §2 we prove Theorem B(a) as Proposition 2.4 and Theorem 2.5. In §3 we prove several results on unique continuation, and in particular Theorem C as Theorem 3.2. In §4 we prove Theorem B(b) as Theorem 4.1. In §5 we prove Theorem A(a) as Propositions 5.1 and 5.2. In §6 we prove a unique continuation result needed for the S^1-invariant theory. Finally, in §7 we prove Theorem A(b) as Theorem 7.4.

2. Fredholm theory for the parametrized Floer equation

In this section, we prove Theorem B(a). The setup is that of symplectic homology. Our ambient symplectic manifold, denoted $(\widehat{W}, \widehat{\omega})$, is the symplectic completion of a compact symplectic manifold (W, ω) with contact type boundary. This means that there exists a vector field X defined in a neighbourhood of ∂W, transverse and pointing outwards along ∂W, such that $\mathcal{L}_X\omega = \omega$. The 1-form $\alpha := (\iota_X\omega)|_{\partial W}$ is a contact form, and the flow of X determines a symplectic trivialization of a neighbourhood of ∂W as $([-\delta, 0] \times \partial W, d(e^t\alpha))$. The symplectic completion is

$$\widehat{W} = W \cup_{\partial W} [0, \infty[\times \partial W.$$

Moreover, we assume that $\widehat{W}$ (or, equivalently, W) is symplectically aspherical, i.e. $\langle\widehat{\omega}, \pi_2(\widehat{W})\rangle = 0$. The Reeb vector field R_α on $M := \partial W$ is defined by the conditions $\ker \omega|_M = \langle R_\alpha\rangle$ and $\alpha(R_\alpha) = 1$. The contact distribution on M is defined by $\xi = \ker \alpha$. Finally, we define the *action spectrum* of (M, α) by

$$\mathrm{Spec}(M, \alpha) := \{T \in \mathbb{R}^+ : \text{there is a closed } R_\alpha\text{-orbit of period } T\}.$$

Let Λ denote a finite-dimensional closed manifold of dimension m, which we call "parameter space". The elements of Λ are denoted by λ.

We define the set $\mathcal{H}_\Lambda$ of *admissible Hamiltonian families* to consist of elements $H \in C^\infty(S^1 \times \widehat{W} \times \Lambda, \mathbb{R})$ which satisfy the following conditions:

- $H < 0$ on $S^1 \times W \times \Lambda$;
- there exists $t_0 \geq 0$ such that $H(\theta, p, t, \lambda) = \beta e^t + \beta'(\lambda)$ for $t \geq t_0$, with $0 < \beta \notin \mathrm{Spec}(M, \alpha)$ and $\beta' \in C^\infty(\Lambda, \mathbb{R})$.

Let $H : S^1 \times \widehat{W} \times \Lambda \to \mathbb{R}$ be an admissible Hamiltonian family denoted by $H(\theta, x, \lambda) = H_\lambda(\theta, x)$. The differential of the corresponding action functional $\mathcal{A}$ defined by (1.5) is given by

$$d\mathcal{A}(\gamma,\lambda)\cdot(\zeta,\ell) = \int_{S^1} \omega(\dot\gamma(\theta) - X_{H_\lambda}(\gamma(\theta)), \zeta(\theta))\, d\theta - \int_{S^1} \frac{\partial H}{\partial \lambda}(\theta,\gamma(\theta),\lambda)\, d\theta \cdot \ell \quad (2.1)$$

and therefore (γ, λ) is a critical point of $\mathcal{A}$ if and only if

$$\gamma \in \mathcal{P}(H_\lambda) \quad \text{and} \quad \int_{S^1} \frac{\partial H}{\partial \lambda}(\theta,\gamma(\theta),\lambda)\, d\theta = 0. \quad (2.2)$$

In (1.4) we denoted the set of critical points of $\mathcal{A}$ by $\mathcal{P}(H)$.

Remark 2.1. Equation (2.2) can be interpreted as follows. Every loop $\gamma : S^1 \to \widehat{W}$ determines a function

$$F_\gamma : \Lambda \to \mathbb{R}, \qquad \lambda \mapsto \int_{S^1} H(\theta,\gamma(\theta),\lambda)\, d\theta. \quad (2.3)$$

A pair (γ, λ) belongs therefore to $\mathcal{P}(H)$ if and only if

$$\gamma \in \mathcal{P}(H_\lambda) \quad \text{and} \quad \lambda \in \mathrm{Crit}(F_\gamma).$$

Let $J = (J^\theta_\lambda)$, $\lambda \in \Lambda$, $\theta \in S^1$, be a family of θ-dependent compatible almost complex structures on $\widehat{W}$ which, at infinity, are invariant under translations in the t-variable and satisfy the relations

$$J^\theta_\lambda \xi = \xi, \qquad J^\theta_\lambda(\partial/\partial t) = R_\alpha. \quad (2.4)$$

Such an *admissible family of almost complex structures* J induces a family of L^2-metrics on the space $C^\infty(S^1, \widehat{W})$, parametrized by Λ and defined by

$$\langle \zeta, \eta\rangle_\lambda := \int_{S^1} \omega(\zeta(\theta), J^\theta_\lambda \eta(\theta))\, d\theta, \qquad \zeta,\eta \in T_\gamma C^\infty(S^1,\widehat{W}) = \Gamma(\gamma^* T\widehat{W}).$$

Such a metric can be coupled with any metric g on Λ and gives rise to a metric on $C^\infty(S^1, \widehat{W}) \times \Lambda$ acting at a point (γ, λ) by

$$\langle (\zeta,\ell),(\eta,k)\rangle_{J,g} := \langle \zeta,\eta\rangle_\lambda + g(\ell,k), \qquad (\zeta,\ell),(\eta,k) \in \Gamma(\gamma^*T\widehat{W}) \oplus T_\lambda\Lambda.$$

We denote by $\mathcal{J}_\Lambda$ the set of pairs (J, g) consisting of an admissible family of almost complex structure J on $\widehat{W}$ and of a Riemannian metric g on Λ. The *parametrized Floer equations* (1.1–1.2) are the gradient equation for $\mathcal{A}$ with respect to such a metric $\langle\cdot,\cdot\rangle_{J,g}$. For the reader's convenience, we rewrite them:

$$\partial_s u + J^\theta_{\lambda(s)}(\partial_\theta u - X^\theta_{H_{\lambda(s)}}(u)) = 0, \quad (2.5)$$

$$\dot\lambda(s) - \int_{S^1} \vec\nabla_\lambda H(\theta, u(s,\theta), \lambda(s))\, d\theta = 0, \quad (2.6)$$

and, for $(\overline{\gamma}, \overline{\lambda}), (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$,

$$\lim_{s\to-\infty}(u(s,\cdot),\lambda(s)) = (\overline{\gamma},\overline{\lambda}), \qquad \lim_{s\to+\infty}(u(s,\cdot),\lambda(s)) = (\underline{\gamma},\underline{\lambda}). \quad (2.7)$$

Remark 2.2. Equation (2.6) is equivalent to

$$\dot{\lambda}(s) - \vec{\nabla} F_{u(s,\cdot)}(\lambda(s)) = 0, \tag{2.8}$$

where $F_{u(s,\cdot)}$ is defined by (2.3). Thus, the parametrized Floer equation is a system involving a Floer equation and a finite-dimensional gradient equation.

Let us fix $p \geq 2$. The linearization of the equations (2.5–2.6) gives rise to the operator

$$D_{(u,\lambda)} : W^{1,p}(u^*T\widehat{W}) \oplus W^{1,p}(\lambda^*T\Lambda) \to L^p(u^*T\widehat{W}) \oplus L^p(\lambda^*T\Lambda),$$

$$D_{(u,\lambda)}(\zeta,\ell) := \begin{pmatrix} D_u\zeta + (D_\lambda J \cdot \ell)(\partial_\theta u - X_{H_\lambda}(u)) - J_\lambda(D_\lambda X_{H_\lambda} \cdot \ell) \\ \nabla_s \ell - \nabla_\ell \int_{S^1} \vec{\nabla}_\lambda H(\theta,u,\lambda)\, d\theta - \int_{S^1} \nabla_\zeta \vec{\nabla}_\lambda H(\theta,u,\lambda)\, d\theta \end{pmatrix},$$

where

$$D_u : W^{1,p}(u^*T\widehat{W}) \to L^p(u^*T\widehat{W})$$

is the usual Floer operator given by

$$D_u\zeta := \nabla_s\zeta + J_\lambda\nabla_\theta\zeta - J_\lambda\nabla_\zeta X_{H_\lambda} + \nabla_\zeta J_\lambda(\partial_\theta u - X_{H_\lambda}).$$

The Hessian of $\mathcal{A}$ at a critical point $p = (\gamma,\lambda)$ is given by the formula

$$\begin{aligned} d^2\mathcal{A}(\gamma,\lambda)((\zeta,\ell),(\eta,k)) &= \int_{S^1} \omega(\nabla_\theta\eta - \nabla_\eta X_{H_\lambda}, \zeta)\, d\theta - \int_{S^1} \eta\left(\frac{\partial H}{\partial\lambda}\cdot\ell\right) d\theta \\ &\quad - \int_{S^1} k(dH_\lambda\cdot\zeta)\, d\theta - \int_{S^1} \frac{\partial^2 H}{\partial\lambda^2}(\ell,k)\, d\theta \\ &= d^2\mathcal{A}_{H_\lambda}(\gamma)(\zeta,\eta) - \int_{S^1} \eta\left(\frac{\partial H}{\partial\lambda}\cdot\ell\right) d\theta \\ &\quad - \int_{S^1} k(dH_\lambda\cdot\zeta)\, d\theta - d^2F_\gamma(\lambda)(\ell,k). \end{aligned} \tag{2.9}$$

We define the asymptotic operator at a critical point (γ,λ) by

$$\begin{aligned} &D_{(\gamma,\lambda)} : H^1(S^1,\gamma^*T\widehat{W}) \times T_\lambda\Lambda \to L^2(S^1,\gamma^*T\widehat{W}) \times T_\lambda\Lambda, \\ &D_{(\gamma,\lambda)}(\zeta,\ell) = \begin{pmatrix} J_\lambda(\nabla_\theta\zeta - \nabla_\zeta X_{H_\lambda} - (D_\lambda X_{H_\lambda})\cdot\ell) \\ -\int_{S^1}\nabla_\zeta \frac{\partial H}{\partial\lambda}\, d\theta - \int_{S^1}\nabla_\ell\frac{\partial H}{\partial\lambda}\, d\theta \end{pmatrix}. \end{aligned} \tag{2.10}$$

Note that $D_{(\gamma,\lambda)}$ is obtained from $D_{(u,\lambda)}$ for $(u(s,\theta),\lambda(s)) \equiv (\gamma(\theta),\lambda)$ and $(\zeta(s,\theta),\ell(s)) \equiv (\zeta(\theta),\ell)$.

Lemma 2.3. *The Hessian $d^2\mathcal{A}(\gamma,\lambda)$ has trivial kernel if and only if the asymptotic operator $D_{(\gamma,\lambda)}$ is injective.*

Proof. The conclusion follows readily from the identity

$$d^2\mathcal{A}(\gamma,\lambda)((\zeta,\ell),(\eta,k)) = \langle D_{(\gamma,\lambda)}(\zeta,\ell),(\eta,k)\rangle. \qquad \square$$

We say that a critical point (γ,λ) is *nondegenerate* if the Hessian $d^2\mathcal{A}(\gamma,\lambda)$ has trivial kernel. Since the operator $D_{(\gamma,\lambda)}$ is self-adjoint, this is equivalent to its surjectivity by Lemma 2.3.

An admissible Hamiltonian family H is called *nondegenerate* if $\mathcal{P}(H)$ consists of nondegenerate elements. We denote the set of nondegenerate and admissible Hamiltonian families by $\mathcal{H}_{\Lambda,\mathrm{reg}} \subset \mathcal{H}_\Lambda$.

Proposition 2.4. *The set $\mathcal{H}_{\Lambda,\mathrm{reg}}$ is of the second Baire category in $\mathcal{H}_\Lambda$. Moreover, if $H \in \mathcal{H}_{\Lambda,\mathrm{reg}}$ the set $\mathcal{P}(H)$ is discrete.*

Proof. Given an integer $r \geq 2$, we denote by $\mathcal{H}^r_\Lambda$ the set of functions $H : S^1 \times \widehat{W} \times \Lambda \to \mathbb{R}$ of class C^r which satisfy the defining conditions for an admissible Hamiltonian family. This is a Banach manifold with respect to the C^r-norm. As a matter of fact, it is an open subset of the Banach space of C^r-functions $h : S^1 \times \widehat{W} \times \Lambda \to \mathbb{R}$ which, outside a compact set, have the form $\beta e^t + \beta'(\lambda)$ with $\beta \in \mathbb{R}$ and $\beta' : \Lambda \to \mathbb{R}$ of class C^r. Hence the tangent space $T_H\mathcal{H}^r_\Lambda$ is identified with this Banach space. We denote by $\mathcal{H}^r_{\Lambda,\mathrm{reg}} \subset \mathcal{H}^r_\Lambda$ the set of Hamiltonians H such that $\mathcal{P}(H)$ consists of nondegenerate elements as defined above. For $t_0 \geq 0$, we denote $\{t \leq t_0\} := W \cup M \times [0,t_0]$, and let $\mathcal{H}^r_{\Lambda,\mathrm{reg},t_0} \subset \mathcal{H}^r_\Lambda$ be the set of Hamiltonians H such that the elements $(\gamma,\lambda) \in \mathcal{P}(H)$ with $\mathrm{im}(\gamma) \subset \{t \leq t_0\}$ are nondegenerate. Then

$$\mathcal{H}^r_{\Lambda,\mathrm{reg}} = \bigcap_{t_0\geq 0} \mathcal{H}^r_{\Lambda,\mathrm{reg},t_0}.$$

Our first claim is that each $\mathcal{H}^r_{\Lambda,\mathrm{reg},t_0}$ is open and dense in $\mathcal{H}^r_\Lambda$, so that $\mathcal{H}^r_{\Lambda,\mathrm{reg}}$ is of the second Baire category. To prove that $\mathcal{H}^r_{\Lambda,\mathrm{reg},t_0}$ is dense, we consider the Banach bundle $\mathcal{E} \to \mathcal{H}^r_\Lambda \times C^r(S^1,\{t \leq t_0\}) \times \Lambda$ whose fibre at (H,γ,λ) is $\mathcal{E}_{(H,\gamma,\lambda)} := C^{r-1}(S^1,\gamma^*T\widehat{W}) \times T_\lambda\Lambda$, and the section f given by

$$f(H,\gamma,\lambda) := \Big(\dot\gamma - X_H \circ \gamma, -\int_{S^1} \vec\nabla_\lambda H\Big).$$

The main step is to prove that $\mathcal{P} := f^{-1}(0)$ is a Banach submanifold of $\mathcal{H}^r_\Lambda \times C^r(S^1,\{t \leq t_0\}) \times \Lambda$. Indeed, the vertical differential of f at a point $(H,\gamma,\lambda) \in \mathcal{P}$ is given by

$$df(H,\gamma,\lambda)\cdot(h,\zeta,\ell) = \begin{pmatrix} \nabla_\theta\zeta - \nabla_\zeta X_H - (D_\lambda X_H)\cdot\ell - X_h \\ -\int_{S^1}\nabla_\zeta\vec\nabla_\lambda H - \int_{S^1}\nabla_\ell\vec\nabla_\lambda H - \int_{S^1}\vec\nabla_\lambda h \end{pmatrix},$$

where $h \in T_H\mathcal{H}^r_\Lambda$ and X_h is its Hamiltonian vector field. That $df(H,\gamma,\lambda)$ is surjective is seen as follows. Given $k \in T_\lambda\Lambda$, we have $(0,k) = df(H,\gamma,\lambda)\cdot(h,0,0)$, with $h(\cdot,\cdot,\lambda) =$ const in some neighbourhood of $\mathrm{im}(\gamma)$ and $\vec\nabla_\lambda h = k$. Given $\eta \in C^{r-1}(S^1,\gamma^*T\widehat{W})$, we have $(\eta,0) = df(H,\gamma,\lambda)\cdot(h,0,0)$, with h independent of λ and such that $X_h = -\eta$

along γ. This proves that $\mathcal{P}$ is a Banach submanifold as desired. Since $\mathcal{H}^r_{\Lambda,\mathrm{reg},t_0}$ coincides with the set of regular values of the natural projection $\mathcal{P} \to \mathcal{H}^r_\Lambda$, we conclude by the Sard–Smale theorem that it is dense.

To prove that $\mathcal{H}^r_{\Lambda,\mathrm{reg},t_0}$ is open in $\mathcal{H}^r_\Lambda$, we prove that its complement is closed. Let therefore $H^\nu \in \mathcal{H}^r_\Lambda \setminus \mathcal{H}^r_{\Lambda,\mathrm{reg},t_0}$ be a sequence such that $H^\nu \to H \in \mathcal{H}^r_\Lambda$ as $\nu \to \infty$. Let $(\gamma^\nu, \lambda^\nu) \in \mathcal{P}(H^\nu)$ be such that $D_{(\gamma^\nu,\lambda^\nu)}$ is not surjective and $\mathrm{im}(\gamma^\nu) \subset \{t \le t_0\}$. Since Λ is compact, it follows from the Arzelà–Ascoli theorem that, up to a subsequence, $(\gamma^\nu, \lambda^\nu)$ converges to some $(\gamma, \lambda) \in \mathcal{P}(H)$, with $\mathrm{im}(\gamma) \subset \{t \le t_0\}$. Since the sequence $D_{(\gamma^\nu,\lambda^\nu)}$ converges to $D_{(\gamma,\lambda)}$, the latter cannot be surjective, so that $H \in \mathcal{H}^r_\Lambda \setminus \mathcal{H}^r_{\Lambda,\mathrm{reg},t_0}$ as desired.

Let $\mathcal{H}_{\Lambda,\mathrm{reg},t_0} := \bigcap_{r\ge 2} \mathcal{H}^r_{\Lambda,\mathrm{reg},t_0} \subset \mathcal{H}_\Lambda$. The same argument as above shows that $\mathcal{H}_{\Lambda,\mathrm{reg},t_0}$ is open. We claim that it is also dense, so that $\mathcal{H}_{\Lambda,\mathrm{reg}} = \bigcap_{t_0\ge 0} \mathcal{H}_{\Lambda,\mathrm{reg},t_0}$ is of the second Baire category in $\mathcal{H}_\Lambda$. To see this, let $H \in \mathcal{H}_\Lambda$ be fixed and consider a sequence $H^r \in \mathcal{H}^r_{\Lambda,\mathrm{reg},t_0}$ such that $H^r \to H$ in any fixed norm C^{r_0}, i.e. in the C^∞-topology as $r \to \infty$. Since $\mathcal{H}^r_{\Lambda,\mathrm{reg},t_0}$ is open in $\mathcal{H}^r_\Lambda$ and $\mathcal{H}_\Lambda$ is dense in $\mathcal{H}^r_\Lambda$, there exists $\widetilde{H}^r \in \mathcal{H}^r_{\Lambda,\mathrm{reg},t_0} \cap \mathcal{H}_\Lambda = \mathcal{H}_{\Lambda,\mathrm{reg},t_0}$ such that $\|H^r - \widetilde{H}^r\|_{C^r} \le \varepsilon_r$, with $\varepsilon_r \to 0$ as $r \to \infty$. Then $\widetilde{H}^r \to H$ in the C^∞-topology, which shows that $\mathcal{H}_{\Lambda,\mathrm{reg},t_0} \subset \mathcal{H}_\Lambda$ is dense.

It remains to prove that, given $H \in \mathcal{H}_{\Lambda,\mathrm{reg}}$, the elements of $\mathcal{P}(H)$ are isolated. This follows from the nondegeneracy of the Hessian $d^2\mathcal{A}$, as can be easily seen using a Taylor expansion at first order for $d\mathcal{A}$. □

Let $I \subset \mathbb{R}$ be any interval. We denote

$$\mathcal{W}^{1,p}(I) := W^{1,p}(I \times S^1, u^*T\widehat{W}) \oplus W^{1,p}(I, \lambda^*T\Lambda),$$
$$\mathcal{L}^p(I) := L^p(I \times S^1, u^*T\widehat{W}) \oplus L^p(I, \lambda^*T\Lambda),$$

and we abbreviate $\mathcal{W}^{1,p} := \mathcal{W}^{1,p}(\mathbb{R})$ and $\mathcal{L}^p := \mathcal{L}^p(\mathbb{R})$.

Given $(\overline{\gamma}, \overline{\lambda}), (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$ and $(u, \lambda) \in \widehat{\mathcal{M}}((\overline{\gamma}, \overline{\lambda}), (\underline{\gamma}, \underline{\lambda}); H, J, g)$, we denote $D := D_{(u,\lambda)}$. We can choose a unitary trivialization of $u^*T\widehat{W}$ and a trivialization of $\lambda^*T\Lambda$ in which D has the form

$$D\begin{pmatrix}\zeta\\ \ell\end{pmatrix} := \left[\begin{pmatrix}\partial_s + J_0\partial_\theta & 0\\ 0 & d/ds\end{pmatrix} + N\right]\begin{pmatrix}\zeta\\ \ell\end{pmatrix}, \tag{2.11}$$

with $N : \mathbb{R} \times S^1 \to \mathrm{Mat}_{2n+m}(\mathbb{R})$ pointwise bounded and $\lim_{s\to\pm\infty} N(s, \theta)$ symmetric.

Theorem 2.5. *Assume $(\overline{\gamma}, \overline{\lambda}), (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$ are nondegenerate. For any (u, λ) in $\widehat{\mathcal{M}}((\overline{\gamma}, \overline{\lambda}), (\underline{\gamma}, \underline{\lambda}); H, J, g)$ the operator*

$$D := D_{(u,\lambda)} : \mathcal{W}^{1,p} \to \mathcal{L}^p$$

is Fredholm for $1 < p < \infty$.

Remark 2.6. The nonlinear theory only requires the case $p > 2$, so that our $W^{1,p}$-maps to $\widehat{W} \times \Lambda$ are continuous.

Remark 2.7 (Structure of the proof). There are two main ingredients in the proof of Theorem 2.5. The first is that D is an elliptic operator, so that it satisfies the estimates in Lemma 2.8 below. The second ingredient is that the constant operators at the asymptotes are bijective, due to our standing nondegeneracy assumption. This is proved in Lemma 2.9 below, and allows us to refine the elliptic estimate by introducing a compact operator (Lemma 2.10).

Proof of Theorem 2.5. By Lemma 2.10 below, the operator D satisfies an estimate of the form

$$\|x\|_{\mathcal{W}^{1,p}} \le C(\|Dx\|_{\mathcal{L}^p} + \|Kx\|_{\mathcal{L}^p([-T,T])}), \tag{2.12}$$

where $K : \mathcal{W}^{1,p} \to \mathcal{L}^p([-T, T])$ is the restriction operator and $T > 0$ is large enough. The embedding $W^{1,p} \hookrightarrow C^0$ with $p > 2$ is compact if the domain is bounded and has dimension at most 2, so that K is a compact operator. By [13, Lemma A.1.1] it follows that D has a finite-dimensional kernel and a closed image.

To show that D has a finite-dimensional cokernel, we introduce its formal adjoint $D^* : \mathcal{W}^{1,q} \to \mathcal{L}^q$, $1/p + 1/q = 1$, defined by

$$D^*\begin{pmatrix}\zeta\\ \ell\end{pmatrix} := \left[\begin{pmatrix}-\partial_s + J_0\partial_\theta & 0\\ 0 & -d/ds\end{pmatrix} + N^T\right]\begin{pmatrix}\zeta\\ \ell\end{pmatrix}, \tag{2.13}$$

where N^T denotes the transpose of N. Lemma 2.10 applies also to the operator D^*, which therefore satisfies an estimate of the form

$$\|x\|_{\mathcal{W}^{1,q}} \le C(\|D^*x\|_{\mathcal{L}^q} + \|Kx\|_{\mathcal{L}^q([-T,T])}), \tag{2.14}$$

with $K : \mathcal{W}^{1,q} \to \mathcal{L}^q([-T, T])$ the restriction operator and $T > 0$ large enough. The embedding $W^{1,q} \hookrightarrow L^q$ is compact for a bounded domain of dimension at most 2, so that K is compact and we infer that D^* has a finite-dimensional kernel.

Given an element $y \in \mathcal{L}^q$ which annihilates the image of D, we have $D^*y = 0$. On the other hand, by elliptic regularity for D^*, we have $y \in \mathcal{W}^{1,q}$. The cokernel of D therefore coincides with the kernel of D^* and is finite-dimensional. This proves the Fredholm property for D. □

Lemma 2.8. *Under the hypotheses of Theorem* 2.5, *and for* $p > 1$, *there exists a constant* $C > 0$ *such that, for any* $k \in \mathbb{Z}$ *and* $x \in \mathcal{W}^{1,p}([k-1, k+2])$, *we have*

$$\|x\|_{\mathcal{W}^{1,p}([k,k+1])} \le C(\|Dx\|_{\mathcal{L}^p([k-1,k+2])} + \|x\|_{\mathcal{L}^p([k-1,k+2])}). \tag{2.15}$$

Similarly, there exists a constant $C_1 > 0$ *such that, for* $x \in \mathcal{W}^{1,p}$, *we have*

$$\|x\|_{\mathcal{W}^{1,p}} \le C_1(\|Dx\|_{\mathcal{L}^p} + \|x\|_{\mathcal{L}^p}). \tag{2.16}$$

Proof. Let us denote

$$D_1 := \begin{pmatrix}\partial_s + J_0\partial_\theta & 0\\ 0 & d/ds\end{pmatrix}$$

and let D_0 be the operator given by multiplication with N, so that $D = D_1 + D_0$. The crucial point is that D_1 is diagonal and each of its components satisfies an estimate of the form (2.15). For the component $\partial_s + J_0\partial_\theta$, this follows immediately from [13, Lemma B.4.6(ii)] (with the notations therein, one has to take $q = r$, $p = \infty$, $\Omega' =]k, k+1[\times S^1$, $\Omega =]k-1, k+2[\times S^1$). For the component d/ds, the estimate follows from the fact that the right hand side of (2.15) defines a norm which is equivalent to the Sobolev norm on $W^{1,p}([k-1, k+2], T_\lambda\Lambda)$.

We have $\|D_0 x\|_{\mathcal{L}^p([k-1,k+2])} \le C_1\|x\|_{\mathcal{L}^p([k-1,k+2])}$ since N is pointwise bounded, so that

$$\begin{aligned}\|x\|_{\mathcal{W}^{1,p}([k,k+1])} &\le C(\|D_1 x\|_{\mathcal{L}^p([k-1,k+2])} + \|x\|_{\mathcal{L}^p([k-1,k+2])})\\ &\le C(\|Dx\|_{\mathcal{L}^p([k-1,k+2])} + \|D_0 x\|_{\mathcal{L}^p([k-1,k+2])} + \|x\|_{\mathcal{L}^p([k-1,k+2])})\\ &\le C_2(\|Dx\|_{\mathcal{L}^p([k-1,k+2])} + \|x\|_{\mathcal{L}^p([k-1,k+2])}).\end{aligned}$$

The estimate (2.16) follows from (2.15) by summing over $k \in \mathbb{Z}$. □

Lemma 2.9. *Let $(\gamma, \lambda_0) \in \mathcal{P}(H)$ and (u, λ) be the constant trajectory at (γ, λ_0), defined by $u(s, \theta) := \gamma(\theta)$ and $\lambda(s) = \lambda_0$. If (γ, λ_0) is nondegenerate, then the operator $D := D_{(u,\lambda)} : \mathcal{W}^{1,p} \to \mathcal{L}^p$ is bijective for $p > 1$.*

Proof. We follow [16, Lemma 2.4] and [16, Exercise 2.5].

Step 1. *The claim holds for $p = 2$.*

Let

$$A = D_{(\gamma,\lambda)} : H^1(S^1, \gamma^*T\widehat{W}) \oplus T_\lambda\Lambda \to L^2(S^1, \gamma^*T\widehat{W}) \oplus T_\lambda\Lambda$$

be the asymptotic operator at (γ, λ), defined by (2.10). Our nondegeneracy assumption on (γ, λ) ensures that A is bijective. We view A as an unbounded self-adjoint operator on $H := L^2(S^1, \gamma^*T\widehat{W}) \oplus T_\lambda\Lambda$ with domain $W := H^1(S^1, \gamma^*T\widehat{W}) \oplus T_\lambda\Lambda$. The Hilbert space H admits an orthogonal decomposition into negative and positive eigenspaces as $H = E^+ \oplus E^-$. Let $P^\pm : H \to E^\pm$ be the corresponding orthogonal projections, and denote $A^\pm := A|_{E^\pm}$. These operators generate strongly continuous semigroups $s \mapsto e^{-A^+s}$ and $s \mapsto e^{A^-s}$ defined for $s \ge 0$ and acting on $E^\pm$ respectively. We define $K : \mathbb{R} \to \mathcal{L}(H)$ by

$$K(s) := \begin{cases} e^{-A^+s}P^+, & s \ge 0,\\ -e^{-A^-s}P^-, & s < 0.\end{cases}$$

This function is discontinuous at $s = 0$, and strongly continuous for $s \ne 0$. Moreover, it satisfies

$$\|K(s)\|_{\mathcal{L}(H)} \le e^{-\delta s}$$

for a suitable constant $\delta > 0$, because A is bijective and therefore its eigenvalues are bounded away from 0. We define the operator $Q : L^2(\mathbb{R}, H) \to W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, W)$ by

$$Qy(s) = \int_{-\infty}^{\infty} K(s-\tau)y(\tau)\,d\tau.$$

We note that $W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, W) = \mathcal{W}^{1,2}$ and $L^2(\mathbb{R}, H) = \mathcal{L}^2$, and we claim that Q is the inverse of D. Indeed, given $y \in \mathcal{L}^2$, the orthogonal decomposition of $x = Qy = x^+ + x^-$ is given by

$$x^+(s) = \int_{-\infty}^{s} e^{-A^+(s-\tau)} y^+(\tau)\, d\tau, \qquad x^-(s) = -\int_{s}^{\infty} e^{-A^-(s-\tau)} y^-(\tau)\, d\tau.$$

One computes directly that $\dot{x}^\pm + A^\pm x^\pm = y^\pm$, so that $\dot{x} + Ax = Dx = y$. This proves Step 1.

Step 2. *Let $p \geq 2$. There exists a constant $C > 0$ such that, for all $k \in \mathbb{Z}$ and $x \in \mathcal{W}^{1,p}([k-1, k+2])$, we have*

$$\|x\|_{\mathcal{W}^{1,p}([k,k+1])} \leq C(\|Dx\|_{\mathcal{L}^p([k-1,k+2])} + \|x\|_{\mathcal{L}^2([k-1,k+2])}).$$

We have

$$\begin{aligned}
\|x\|_{\mathcal{W}^{1,p}([k,k+1])} &\leq C_1(\|Dx\|_{\mathcal{L}^p([k-1/2,k+3/2])} + \|x\|_{\mathcal{L}^p([k-1/2,k+3/2])}) \\
&\leq C_2(\|Dx\|_{\mathcal{L}^p([k-1/2,k+3/2])} + \|x\|_{\mathcal{W}^{1,2}([k-1/2,k+3/2])}) \\
&\leq C_3(\|Dx\|_{\mathcal{L}^p([k-1/2,k+3/2])} + \|Dx\|_{\mathcal{L}^2([k-1,k+2])} + \|x\|_{\mathcal{L}^2([k-1,k+2])}) \\
&\leq C_4(\|Dx\|_{\mathcal{L}^p([k-1,k+2])} + \|x\|_{\mathcal{L}^2([k-1,k+2])}).
\end{aligned}$$

The first and third inequalities follow from Lemma 2.8. The second inequality follows from the Sobolev embedding $\mathcal{W}^{1,2}([k-1/2, k+3/2]) \hookrightarrow \mathcal{L}^p([k-1/2, k+3/2])$ (see [13, Theorem B.1.12] and the subsequent discussion for the summands defined on $[k-1/2, k+3/2] \times S^1$, and [13, Theorem B.1.11] for the summands defined on $[k-1/2, k+3/2]$). The last inequality holds because $p \geq 2$, so that $\mathcal{L}^p(I) \hookrightarrow \mathcal{L}^2(I)$ for any bounded interval I.

Step 3. *Let $p \geq 2$. There exists a constant $C > 0$ such that, if $x \in \mathcal{W}^{1,2}$ and $Dx \in \mathcal{L}^p$, then $x \in \mathcal{W}^{1,p}$ and*

$$\|x\|_{\mathcal{W}^{1,p}} \leq C\|Dx\|_{\mathcal{L}^p}. \tag{2.17}$$

We first remark that if $x \in \mathcal{W}^{1,2}$ and $Dx \in \mathcal{L}^p_{\mathrm{loc}}$, then $x \in \mathcal{W}^{1,p}_{\mathrm{loc}}$. Indeed, as seen in Step 2, we have an embedding $\mathcal{W}^{1,2}(I) \hookrightarrow \mathcal{L}^p(I)$ for any bounded interval I. The remark then follows from elliptic regularity for D (see [18, Proposition 1.2.1] and the references therein).

Let $H := L^2(S^1, \gamma^* T\widehat{W}) \oplus T_\lambda \Lambda$ and, for an interval $I \subset \mathbb{R}$, denote the natural norm on $L^p(I, H)$ by $\|\cdot\|_{L^p(I,H)}$. It follows from Step 2 and the inequality $(a+b)^p \leq 2^p(a^p + b^p)$ that

$$\begin{aligned}
\|x\|^p_{\mathcal{W}^{1,p}([k,k+1])} &\leq C(\|Dx\|_{\mathcal{L}^p([k-1,k+2])} + \|x\|_{\mathcal{L}^2([k-1,k+2])})^p \\
&\leq 2^p C(\|Dx\|^p_{\mathcal{L}^p([k-1,k+2])} + \|x\|^p_{\mathcal{L}^2([k-1,k+2])}) \\
&= 2^p C(\|Dx\|^p_{\mathcal{L}^p([k-1,k+2])} + \|x\|^p_{L^2([k-1,k+2],H)}) \\
&\leq 2^p C(\|Dx\|^p_{\mathcal{L}^p([k-1,k+2])} + 3^{p/2-1}\|x\|^p_{L^p([k-1,k+2],H)}) \\
&\leq 3^{p/2-1} 2^p C(\|Dx\|^p_{\mathcal{L}^p([k-1,k+2])} + \|x\|^p_{L^p([k-1,k+2],H)}).
\end{aligned}$$

The third inequality is Hölder's. By summing over $k \in \mathbb{Z}$ we obtain

$$\|x\|^p_{\mathcal{W}^{1,p}} \leq C_1(\|Dx\|^p_{\mathcal{L}^p} + \|x\|^p_{L^p(\mathbb{R},H)}). \tag{2.18}$$

Let $Q : \mathcal{L}^2 \to \mathcal{W}^{1,2}$ be the inverse of $D : \mathcal{W}^{1,2} \to \mathcal{L}^2$ as in Step 1. Then

$$\begin{aligned}\|x\|_{L^p(\mathbb{R},H)} &= \|QDx\|_{L^p(\mathbb{R},H)} = \|K * (Dx)\|_{L^p(\mathbb{R},H)} \\ &\leq \|K\|_{L^1(\mathbb{R},\mathcal{L}(H))}\|Dx\|_{L^p(\mathbb{R},H)} \leq C_2\|Dx\|_{L^p(\mathbb{R},H)} \leq C_3\|Dx\|_{\mathcal{L}^p}.\end{aligned}$$

The first inequality is Young's inequality for a convolution [6, Théorème 4.30], and the last inequality follows from the fact that $\|\cdot\|_{L^2(S^1)} \leq \|\cdot\|_{L^p(S^1)}$, while any two norms are equivalent on the finite-dimensional space $T_\lambda\Lambda$. Combining the above inequality with (2.18), we obtain (2.17). This proves Step 3.

Step 4. *We prove the lemma for $p \geq 2$.*

The estimate (2.17) holds in particular for $x \in C_0^\infty(\mathbb{R} \times S^1, u^*T\widehat{W}) \oplus C_0^\infty(\mathbb{R}, T_\lambda\Lambda)$ and, by density, for all $x \in \mathcal{W}^{1,p}$. We infer that $D : \mathcal{W}^{1,p} \to \mathcal{L}^p$ is injective and has a closed image. To prove that it is surjective, it is therefore enough to show that its image is dense in $\mathcal{L}^p$. Indeed, it follows from Steps 1 and 3 that its image contains the dense subspace $\mathcal{L}^p \cap \mathcal{L}^2$.

Step 5. *We prove the lemma for $1 < p < 2$.*

Let $q > 2$ be such that $1/p + 1/q = 1$. Define $\mathcal{W}^{-1,p} := W^{-1,p}(\mathbb{R} \times S^1, u^*T\widehat{W}) \oplus W^{-1,p}(\mathbb{R}, T_\lambda\Lambda)$, where $W^{-1,p}$ is the dual space of $W^{1,q}$, so that $\mathcal{W}^{-1,p}$ is the dual space of $\mathcal{W}^{1,q}$. Note also that $\mathcal{L}^q$ is the dual of $\mathcal{L}^p$. The formal adjoint D^* defined in (2.13) is canonically identified with the functional analytic adjoint $D^* : \mathcal{W}^{1,q} \to \mathcal{L}^q$ of $D : \mathcal{L}^p \to \mathcal{W}^{-1,p}$. By Step 4, there exists a constant $C > 0$ such that, for any $x \in \mathcal{W}^{1,q}$, we have

$$\|x\|_{\mathcal{W}^{1,q}} \leq C\|D^*x\|_{\mathcal{L}^q}. \tag{2.19}$$

Using that D^* is bijective and duality, we obtain, for $y \in \mathcal{L}^p$,

$$\begin{aligned}\|y\|_{\mathcal{L}^p} &= \sup_{\|z\|_{\mathcal{L}^q}=1} |\langle z, y\rangle| = \sup_{\|D^*x\|_{\mathcal{L}^q}=1} |\langle D^*x, y\rangle| = \sup_{\|D^*x\|_{\mathcal{L}^q}=1} |\langle x, Dy\rangle| \\ &\leq \sup_{\|D^*x\|_{\mathcal{L}^q}=1} \|x\|_{\mathcal{W}^{1,q}}\|Dy\|_{\mathcal{W}^{-1,p}} \leq \sup_{\|D^*x\|_{\mathcal{L}^q}=1} C\|D^*x\|_{\mathcal{L}^q}\|Dy\|_{\mathcal{W}^{-1,p}} \\ &= C\|Dy\|_{\mathcal{W}^{-1,p}}.\end{aligned} \tag{2.20}$$

The last inequality uses (2.19).

We now prove that there exists a constant $C > 0$ such that, for any $x \in \mathcal{W}^{1,p}$, we have

$$\|x\|_{\mathcal{W}^{1,p}} \leq C\|Dx\|_{\mathcal{L}^p}. \tag{2.21}$$

For $x = (\zeta, \ell) \in \mathcal{W}^{1,p}$, we have

$$\|x\|_{\mathcal{W}^{1,p}} \leq C_1(\|x\|_{\mathcal{L}^p} + \|\partial_s x\|_{\mathcal{L}^p} + \|\partial_\theta\zeta\|_{L^p}). \tag{2.22}$$

Using (2.20) and the inclusion $\mathcal{L}^p \hookrightarrow \mathcal{W}^{-1,p}$ we obtain

$$\|x\|_{\mathcal{L}^p} \le C\|Dx\|_{\mathcal{W}^{-1,p}} \le C\|Dx\|_{\mathcal{L}^p}.$$

On the other hand we have

$$\begin{aligned}\|\partial_s x\|_{\mathcal{L}^p} &\le C\|D(\partial_s x)\|_{\mathcal{W}^{-1,p}} = C\|\partial_s(Dx) - (\partial_s N)x\|_{\mathcal{W}^{-1,p}} \\ &\le C_1(\|\partial_s(Dx)\|_{\mathcal{W}^{-1,p}} + \|x\|_{\mathcal{W}^{-1,p}}) \le C_1(\|Dx\|_{\mathcal{L}^p} + \|x\|_{\mathcal{L}^p}) \le C_2\|Dx\|_{\mathcal{L}^p}.\end{aligned}$$

The first and last inequalities use (2.20) for $\partial_s x$ and x, the second inequality uses the fact that $\partial_s N$ and its derivatives are pointwise bounded, and the third inequality uses that $\partial_s : \mathcal{L}^p \to \mathcal{W}^{-1,p}$ is bounded (and of norm 1). Similarly, we have $\|\partial_\theta \zeta\|_{L^p} \le C_3\|Dx\|_{\mathcal{L}^p}$. Using (2.22) we obtain (2.21).

It follows from (2.21) that $D : \mathcal{W}^{1,p} \to \mathcal{L}^p$ is injective and has a closed image. To prove that it is surjective, it is enough to show that its image is dense in $\mathcal{L}^p$. Consider therefore $y \in \mathcal{L}^q$ such that $\langle Dx, y\rangle = 0$ for all $x \in \mathcal{W}^{1,p}$. We obtain $D^*y = 0$ in $\mathcal{W}^{-1,q}$. By elliptic regularity for D^*, we infer $y \in \mathcal{W}^{1,q}$. Since $D^* : \mathcal{W}^{1,q} \to \mathcal{L}^q$ is injective by Step 4, we obtain $y = 0$. □

Lemma 2.10. *Let $p > 1$. Under the hypotheses of Theorem 2.5, there exists $T > 0$ and a constant $C > 0$ such that*

$$\|x\|_{\mathcal{W}^{1,p}} \le C(\|Dx\|_{\mathcal{L}^p} + \|Kx\|_{\mathcal{L}^p([-T,T])}), \tag{2.23}$$

where $K : \mathcal{W}^{1,p} \to \mathcal{L}^p([-T,T])$ is the restriction operator.

Proof. Let $(\overline{u}, \widetilde{\overline{\lambda}})$ and $(\underline{u}, \widetilde{\underline{\lambda}})$ be the constant trajectories at $(\overline{\gamma}, \overline{\lambda})$ and $(\underline{\gamma}, \underline{\lambda})$ respectively. Denote by $\overline{D} := D_{(\overline{u},\widetilde{\overline{\lambda}})}$ and $\underline{D} := D_{(\underline{u},\widetilde{\underline{\lambda}})}$ the corresponding operators which, by Lemma 2.9, are isomorphisms. Since invertibility is an open property in the space of operators, and because the order 0 part of D converges as $s \to \pm\infty$ to the order 0 part of $\underline{D}$ and $\overline{D}$ respectively, we infer the existence of constants $T > 0$ and $C > 0$ such that, for every $x \in \mathcal{W}^{1,p}$ such that $x(s) = 0$ for $|s| \le T - 1$, we have

$$\|x\|_{\mathcal{W}^{1,p}} \le C\|Dx\|_{\mathcal{L}^p}. \tag{2.24}$$

Let $\beta : \mathbb{R} \to [0,1]$ be a smooth cutoff function such that $\beta(s) = 0$ for $|s| \ge T$ and $\beta(s) = 1$ for $|s| \le T - 1$. We obtain

$$\begin{aligned}\|x\|_{\mathcal{W}^{1,p}} &\le \|\beta x\|_{\mathcal{W}^{1,p}} + \|(1-\beta)x\|_{\mathcal{W}^{1,p}} \\ &\le C_1(\|D(\beta x)\|_{\mathcal{L}^p} + \|\beta x\|_{\mathcal{L}^p} + \|D((1-\beta)x)\|_{\mathcal{L}^p}) \\ &\le C_2(\|Dx\|_{\mathcal{L}^p} + \|Kx\|_{\mathcal{L}^p([-T,T])}).\end{aligned}$$

The first and the third inequalities are straightforward, whereas the second uses (2.16) and (2.24). This proves the lemma. □

3. Unique continuation for the parametrized Floer equation

The fundamental property on which rest transversality results in Floer theory [10] is the unique continuation principle for Floer trajectories. We know of two ways to prove it. The first one is the Carleman similarity principle [10, Theorem 2.2], which cannot hold in our setup due to the integral term, which makes the system of equations (2.5–2.6) nonlocal. The second one is Aronszajn's theorem, stating that a solution of a *pointwise* differential inequality involving an elliptic operator of order 2 satisfies the unique continuation property [1]. Again, one cannot apply it to our setup because of the integral term.

Aronszajn's theorem relies on a local estimate [1, (2.4)] which is nowadays called a *Carleman-type inequality*. We will extend Aronszajn's theorem to a class of integro-differential elliptic inequalities by proving a semi-local Carleman-type inequality. This generalization of Aronszajn's theorem will apply to the solutions of our system of equations. Our arguments closely follow the ones of Aronszajn [1].

For $r > 0$, we denote $Z_r :=]-r, r[\times S^1$, with coordinates (s, θ).

Proposition 3.1 (Semi-local Carleman inequality). *Let $h > 0$. There exist $c > 0$ and $\alpha_0 > 0$ such that, for any $0 < r \le h$, $\alpha \ge \alpha_0$ and $u \in C_0^\infty(Z_r, \mathbb{C}^n)$ which vanishes together with all its derivatives along $\{0\} \times S^1$, we have*

$$cr^2 \int_{Z_r} |s|^{-2\alpha} |\Delta u|^2 \, ds \, d\theta \ge \int_{Z_r} |s|^{-2\alpha} (|\nabla u|^2 + |u|^2) \, ds \, d\theta. \tag{3.1}$$

First proof. In our first proof, we use a change of variables inspired by the original paper of Aronszajn [1]. It is enough to prove the inequality on $Z_r^+ :=]0, r[\times S^1$. We make the change of variables $s = e^{-\rho}$, $\chi < \rho < \infty$, $\chi = -\log r$ and define

$$w :]\chi, \infty[\times S^1 \to \mathbb{C}^n, \quad w(\rho, \theta) = e^{\beta\rho} u(e^{-\rho}, \theta),$$

with $\beta = \alpha + 3/2$. Our assumption on u guarantees that w vanishes together with all its derivatives as $\rho \to \infty$. We denote $w' = \partial w / \partial \rho$ and $w'' = \partial^2 w / \partial \rho^2$. A straightforward computation shows that

$$\frac{\partial u}{\partial s}(e^{-\rho}, \theta) = -e^{(1-\beta)\rho}(w'(\rho, \theta) - \beta w(\rho, \theta)),$$
$$\frac{\partial^2 u}{\partial s^2}(e^{-\rho}, \theta) = e^{(2-\beta)\rho}(w''(\rho, \theta) - (2\beta - 1) w'(\rho, \theta) + (\beta^2 - \beta) w(\rho, \theta)).$$

We therefore obtain

$$\int_{Z_r^+} |s|^{-2\alpha} |\Delta u|^2 \, ds \, d\theta = \int_\chi^\infty \int_{S^1} |w'' - (2\beta - 1) w' + (\beta^2 - \beta) w + \Delta_\theta w|^2 \, d\rho \, d\theta,$$

with $\Delta_\theta w := e^{-2\rho}\partial^2 w/\partial\theta^2$. We denote the last integral by I. Expanding the integrand in I we obtain

$$\begin{aligned} I = &\int_\chi^\infty \int_{S^1} (|w''|^2 + |(\beta^2-\beta)w + \Delta_\theta w|^2) \\ &+ \int_\chi^\infty \int_{S^1} ((2\beta-1)^2|w'|^2 + w''[(\beta^2-\beta)\bar{w} + \Delta_\theta \bar{w}] + \bar{w}''[(\beta^2-\beta)w + \Delta_\theta w]) \\ &+ \int_\chi^\infty \int_{S^1} (1-2\beta)(w'\Delta_\theta \bar{w} + \bar{w}'\Delta_\theta w). \end{aligned}$$

We have used that $\iint (w''\bar{w}' + \bar{w}''w') = 0$ and that $\iint (w'\bar{w} + \bar{w}'w) = 0$, which follow from the fact that w and w' vanish for $\rho \to \infty$ and ρ near χ. We denote the above three integrals by J^1, J^2, and J^3 respectively. Since $J^1 \geq 0$ we have $I \geq J^2 + J^3$.

We treat J^2. Using integration by parts with respect to ρ we obtain $\iint (w''\bar{w} + \bar{w}''w) = -2\iint |w'|^2$. Using integration by parts with respect to θ and ρ we obtain

$$\begin{aligned} \int_\chi^\infty \int_{S^1} w''\Delta_\theta \bar{w} &= \int_\chi^\infty \int_{S^1} \left(\left| \frac{\partial w'}{\partial \theta} \right|^2 - 2\frac{\partial w'}{\partial \theta}\frac{\partial \bar{w}}{\partial \theta} \right) e^{-2\rho} \\ &= \int_\chi^\infty \int_{S^1} \left(\left| \frac{\partial w'}{\partial \theta} \right|^2 + 2\frac{\partial w}{\partial \theta}\frac{\partial \bar{w}'}{\partial \theta} - 4\left|\frac{\partial w}{\partial \theta}\right|^2 \right) e^{-2\rho}. \end{aligned}$$

For the second equality we have used another integration by parts with respect to ρ. Thus

$$\int_\chi^\infty \int_{S^1} (w''\Delta_\theta \bar{w} + \bar{w}''\Delta_\theta w) = \int_\chi^\infty \int_{S^1} \left(2\left| \frac{\partial w'}{\partial \theta} \right|^2 - 4\left|\frac{\partial w}{\partial \theta}\right|^2 \right) e^{-2\rho}.$$

Finally

$$J^2 = \int_\chi^\infty \int_{S^1} \left(\left(2\left(\beta - \frac{1}{2}\right)^2 + \frac{1}{2} \right) |w'|^2 + 2\left|\frac{\partial w'}{\partial \theta}\right|^2 e^{-2\rho} \right) - \int_\chi^\infty \int_{S^1} 4\left|\frac{\partial w}{\partial \theta}\right|^2 e^{-2\rho}. \tag{3.2}$$

We treat J^3. Integrating by parts with respect to θ and ρ we obtain

$$\int_\chi^\infty \int_{S^1} w'\Delta_\theta \bar{w} = -\int_\chi^\infty \int_{S^1} \frac{\partial w'}{\partial \theta}\frac{\partial \bar{w}}{\partial \theta} e^{-2\rho} = \int_\chi^\infty \int_{S^1} \left(\frac{\partial w}{\partial \theta}\frac{\partial \bar{w}'}{\partial \theta} - 2\left|\frac{\partial w}{\partial \theta}\right|^2 \right) e^{-2\rho},$$

so that

$$\int_\chi^\infty \int_{S^1} (w'\Delta_\theta \bar{w} + \bar{w}'\Delta_\theta w) = -\int_\chi^\infty \int_{S^1} 2\left|\frac{\partial w}{\partial \theta}\right|^2 e^{-2\rho}.$$

We denote by I^2 the first integral in the expression (3.2) for J^2, and set

$$I^3 := J^3 - \int_\chi^\infty \int_{S^1} 4\left|\frac{\partial w}{\partial \theta}\right|^2 e^{-2\rho} = \int_\chi^\infty \int_{S^1} (4\beta - 6)\left|\frac{\partial w}{\partial \theta}\right|^2 e^{-2\rho}$$

so that $J^2 + J^3 = I^2 + I^3$.

We now treat the right hand side in (3.1). Using the same change of variables as above we obtain

$$\int_{Z_r^+} |s|^{-2\alpha}(|\nabla u|^2 + |u|^2)\, ds\, d\theta$$
$$= \int_\chi^\infty \int_{S^1} \left(|w' - \beta w|^2 e^{-2\rho} + \left|\frac{\partial w}{\partial \theta}\right|^2 e^{-4\rho} + |w|^2 e^{-4\rho} \right) d\rho\, d\theta.$$

The first term in the integrand is $|w' - \beta w|^2 e^{-2\rho} = |w'|^2 e^{-2\rho} + \beta^2 |w|^2 e^{-2\rho} - \beta(w'\bar{w} + \bar{w}'w)e^{-2\rho}$, and we have

$$\int_\chi^\infty \int_{S^1} (w'\bar{w} + \bar{w}'w)e^{-2\rho} = \int_\chi^\infty \int_{S^1} (w'\bar{w} - \bar{w}(w' - 2w))e^{-2\rho} = \int_\chi^\infty \int_{S^1} 2|w|^2 e^{-2\rho}.$$

The right hand side in (3.1) is therefore equal to

$$\int_\chi^\infty \int_{S^1} \left(|w'|^2 e^{-2\rho} + (\beta^2 - 2\beta)|w|^2 e^{-2\rho} + \left|\frac{\partial w}{\partial \theta}\right|^2 e^{-4\rho} + |w|^2 e^{-4\rho} \right).$$

We now recall the inequality (4.10) in [1] which reads in our case, for $\theta \in S^1$ fixed,

$$\int_\chi^\infty |w|^2 e^{-\tau\rho}\, d\rho \le \frac{e^{-\tau\chi}}{\tau^2} \int_\chi^\infty |w'|^2\, d\rho, \qquad \tau > 0. \tag{3.3}$$

To prove (3.3) we write $w(\rho) = \int_\chi^\rho w'$, and use the Cauchy–Schwarz inequality to obtain $|w(\rho)|^2 \le (\rho - \chi)\int_\chi^\infty |w'|^2$. On the other hand, $\int_\chi^\infty e^{-\tau\rho}(\rho - \chi)\, d\rho = e^{-\tau\chi}/\tau^2$.

Using (3.3) with $\tau = 2$ and the relations $4\beta - 6 = 4\alpha \ge 4$ and $I \ge I^2 + I^3$, we obtain the desired conclusion with the constants $c = 1$ and $\alpha_0 = 1$. □

Second proof, by Luc Robbiano. We again work on Z_r^+. We define $v := s^{-\alpha}u$, so that v vanishes with all its derivatives along $\{0\} \times S^1$. We have

$$\partial_s u = s^\alpha \partial_s v + \alpha s^{\alpha-1} v,$$
$$\partial_s^2 u = s^\alpha \partial_s^2 v + 2\alpha s^{\alpha-1}\partial_s v + \alpha(\alpha - 1)s^{\alpha-2} v.$$

We obtain

$$s^{-\alpha}\Delta u = \partial_s^2 v + \partial_\theta^2 v + 2\alpha s^{-1}\partial_s v + \alpha(\alpha - 1)s^{-2} v.$$

In order to estimate $A := \int_{Z_r^+} |s^{-\alpha}\Delta u|^2$, we separate self-adjoint and anti-adjoint terms in the previous expression. Denoting by $\langle \cdot, \cdot \rangle$ the L^2-scalar product for functions defined on Z_r^+, and by $\|\cdot\|$ the corresponding L^2-norm, we obtain

$$A = \|\partial_s^2 v + \partial_\theta^2 v + \alpha(\alpha - 1)s^{-2} v\|^2 + 4\alpha^2 \|s^{-1}\partial_s v\|^2$$
$$+ 2\,\mathrm{Re}\,\langle \partial_s^2 v + \partial_\theta^2 v + \alpha(\alpha - 1)s^{-2} v, 2\alpha s^{-1}\partial_s v \rangle.$$

Let us further compute the last term. We have

$$\langle \partial_s^2 v, 2\alpha s^{-1}\partial_s v \rangle = -\langle \partial_s v, 2\alpha \partial_s(s^{-1}\partial_s v) \rangle = -\langle \partial_s v, -2\alpha s^{-2}\partial_s v \rangle - \langle \partial_s v, 2\alpha s^{-1}\partial_s^2 v \rangle,$$

so that

$$2\,\mathrm{Re}\,\langle \partial_s^2 v, 2\alpha s^{-1}\partial_s v\rangle = 2\alpha\|s^{-1}\partial_s v\|^2.$$

Similarly,

$$\begin{aligned}\langle \partial_\theta^2 v, 2\alpha s^{-1}\partial_s v\rangle &= -\langle 2\alpha\partial_s(s^{-1}\partial_\theta^2 v), v\rangle\\ &= -2\alpha\langle s^{-1}\partial_s\partial_\theta^2 v, v\rangle + 2\alpha\langle s^{-2}\partial_\theta^2 v, v\rangle\\ &= -2\alpha\langle s^{-1}\partial_s v, \partial_\theta^2 v\rangle + 2\alpha\langle s^{-2}\partial_\theta^2 v, v\rangle,\end{aligned}$$

so that

$$2\,\mathrm{Re}\,\langle \partial_\theta^2 v, 2\alpha s^{-1}\partial_s v\rangle = 2\alpha\langle s^{-2}\partial_\theta^2 v, v\rangle = -2\alpha\|s^{-1}\partial_\theta v\|^2.$$

Finally, we have

$$\begin{aligned}\langle \alpha(\alpha-1)s^{-2}v, 2\alpha s^{-1}\partial_s v\rangle &= -\langle 2\alpha^2(\alpha-1)\partial_s(s^{-3}v), v\rangle\\ &= -\langle 2\alpha^2(\alpha-1)s^{-3}\partial_s v, v\rangle + \langle 6\alpha^2(\alpha-1)s^{-4}v, v\rangle,\end{aligned}$$

so that

$$2\,\mathrm{Re}\,\langle \alpha(\alpha-1)s^{-2}v, 2\alpha s^{-1}\partial_s v\rangle = 6\alpha^2(\alpha-1)\|s^{-2}v\|^2.$$

Let us now denote $Bv := \partial_s^2 v + \partial_\theta^2 v + \alpha(\alpha-1)s^{-2}v$, so that

$$\begin{aligned}A = \|Bv\|^2 &+ (4\alpha^2+2\alpha)\|s^{-1}\partial_s v\|^2 + 6\alpha^2(\alpha-1)\|s^{-2}v\|^2 + 2\alpha\|s^{-1}\partial_\theta v\|^2\\ &+ 4\alpha\langle s^{-2}(Bv - \partial_s^2 v - \alpha(\alpha-1)s^{-2}v), v\rangle.\end{aligned}$$

We again further compute the last term. We have

$$|\langle 4\alpha s^{-2}Bv, v\rangle| \le \|Bv\|^2 + 4\alpha^2\|s^{-2}v\|^2.$$

We also have

$$-4\alpha\langle s^{-2}\partial_s^2 v, v\rangle = 4\alpha\langle \partial_s v, \partial_s(s^{-2}v)\rangle = 4\alpha\langle \partial_s v, s^{-2}\partial_s v\rangle - 8\alpha\langle \partial_s v, s^{-3}v\rangle,$$

and

$$|8\alpha\langle \partial_s v, s^{-3}v\rangle| \le 4\|s^{-1}\partial_s v\|^2 + 4\alpha^2\|s^{-2}v\|^2.$$

We obtain

$$\begin{aligned}A &\ge (4\alpha^2+6\alpha-4)\|s^{-1}\partial_s v\|^2 + 2\alpha^2(\alpha-5)\|s^{-2}v\|^2 + 2\alpha\|s^{-1}\partial_\theta v\|^2\\ &\ge 2\|s^{-1}\partial_s v\|^2 + 4\alpha^2\|s^{-2}v\|^2 + 2\|s^{-1}\partial_\theta v\|^2.\end{aligned}$$

The last inequality holds if $\alpha \ge 7$. Now since $v := s^{-\alpha}u$, we have

$$\begin{aligned}\|s^{-\alpha-1}\partial_s u\| &\le \|s^{-1}\partial_s v\| + \alpha\|s^{-2}v\|,\\ \|s^{-\alpha-2}u\| &= \|s^{-2}v\|,\\ \|s^{-\alpha-1}\partial_\theta u\| &= \|s^{-1}\partial_\theta v\|.\end{aligned}$$

Substituting these in the above estimate, we obtain

$$A \ge \|s^{-\alpha-1}\partial_s u\|^2 + \|s^{-\alpha-1}\partial_\theta u\|^2 + \|s^{-\alpha-2}u\|^2.$$

Since $s < r \le h$, we deduce the desired inequality with the constants $c = \max(1, h^2)$ and $\alpha_0 = 7$. □

Theorem 3.2 (Unique continuation for integro-differential inequalities). *Let $h > 0$. Assume $u \in C^\infty(Z_h, \mathbb{C}^n)$ satisfies*

$$|\Delta u(s,\theta)|^2 \le M\left[|u(s,\theta)|^2 + |\nabla u(s,\theta)|^2 + \int_{S^1} |u(s,\tau)|^2\, d\tau\right] \tag{3.4}$$

for all $(s,\theta) \in Z_h$, where $M > 0$ is a positive constant. If u vanishes together with all its derivatives on $\{0\} \times S^1$, then $u \equiv 0$ on Z_h.

Proof. It is enough to prove that u vanishes in a neighbourhood of $\{0\} \times S^1$. The conclusion then follows by a connectedness argument on $]-h, h[$. Let $0 < r < 1/\sqrt{(2\pi+1)cM}$ be fixed, where $c > 0$ is the constant in (3.1). Let $\varphi :]-r, r[\to [0,1]$ be a smooth function equal to 1 for $|s| \le r/3$ and to 0 for $|s| \ge 2r/3$. Let $u_1(s,\theta) := \varphi(s)u(s,\theta)$. Then

$$\begin{aligned}
&\int_{Z_{r/3}} |s|^{-2\alpha}\left[|u(s,\theta)|^2 + |\nabla u(s,\theta)|^2 + \int_{S^1} |u(s,\tau)|^2\, d\tau\right] ds\, d\theta \\
&\quad \le (2\pi+1) \int_{Z_{r/3}} |s|^{-2\alpha}[|u(s,\theta)|^2 + |\nabla u(s,\theta)|^2]\, ds\, d\theta \\
&\quad \le (2\pi+1) \int_{Z_r} |s|^{-2\alpha}(|u_1|^2 + |\nabla u_1|^2) \le (2\pi+1)cr^2 \int_{Z_r} |s|^{-2\alpha}|\Delta u_1|^2 \\
&\quad = (2\pi+1)cr^2 \int_{Z_{r/3}} |s|^{-2\alpha}|\Delta u|^2 + (2\pi+1)cr^2 \int_{Z_r \setminus Z_{r/3}} |s|^{-2\alpha}|\Delta u_1|^2 \\
&\quad \le (2\pi+1)cr^2 M \int_{Z_{r/3}} |s|^{-2\alpha}\left[|u|^2 + |\nabla u|^2 + \int_{S^1} |u|^2\right] \\
&\qquad + (2\pi+1)cr^2 \int_{Z_r \setminus Z_{r/3}} |s|^{-2\alpha}|\Delta u_1|^2.
\end{aligned}$$

The third inequality follows from Proposition 3.1, for $\alpha \ge \alpha_0$. It follows that

$$\begin{aligned}
\int_{Z_{r/3}} |s|^{-2\alpha}\left[|u|^2 + |\nabla u|^2 + \int_{S^1} |u|^2\right] &\le C \int_{Z_r \setminus Z_{r/3}} |s|^{-2\alpha}|\Delta u_1|^2 \\
&\le \frac{C}{(r/3)^{2\alpha}} \int_{Z_r \setminus Z_{r/3}} |\Delta u_1|^2,
\end{aligned}$$

with $C = (2\pi+1)cr^2/(1 - (2\pi+1)cr^2 M)$. We claim that $u \equiv 0$ on $Z_{r/3}$. Following Carleman [7], we assume this is false: there is $(s_0, \theta_0) \in Z_{r/3}$ such that $u(s_0, \theta_0) \ne 0$. Hence there exists a constant $k > 0$ (depending on u, but not on α) such that

$$\frac{k}{|s_0|^{2\alpha}} \le \int_{Z_{r/3}} |s|^{-2\alpha}|u|^2$$

for all $\alpha \ge \alpha_0$. In view of the above, we obtain

$$0 < k \le C \frac{|s_0|^{2\alpha}}{(r/3)^{2\alpha}} \int_{Z_r \setminus Z_{r/3}} |\Delta u_1|^2.$$

Since $|s_0| < r/3$, we obtain a contradiction as $\alpha \to \infty$. □

In the next statement we denote $I_h :=]-h, h[$ for $h > 0$.

Proposition 3.3. *Let $h > 0$ and $u : Z_h \to \mathbb{C}^n$, $\lambda : I_h \to \mathbb{R}^m$ be C^∞-functions satisfying*

$$\begin{aligned} \partial_s u + J(s,\theta)\partial_\theta u + C(s,\theta)u + D(s,\theta)\lambda = 0, \\ \partial_s \lambda + \int_{S^1} E(s,\theta)u(s,\theta)\,d\theta + F(s)\lambda = 0, \end{aligned} \tag{3.5}$$

with C, D, E, F of class C^1, J of class C^∞ and $J^2 = -\mathbb{1}$. Assume there exists a nonempty open set $\mathcal{U} \subset Z_h$ such that $(u(s,\theta), \lambda(s)) = (0,0)$ for all $(s,\theta) \in \mathcal{U}$. Then $u \equiv 0$ on Z_h and $\lambda \equiv 0$ on I_h.

Proof. We first notice that, for any $(s,\theta) \in \mathcal{U}$, there exists $\varepsilon > 0$ such that $u \equiv 0$ on $]s-\varepsilon, s+\varepsilon[\times S^1$ and $\lambda \equiv 0$ on $]s-\varepsilon, s+\varepsilon[$. Indeed, choose $\varepsilon > 0$ small enough such that $]s-\varepsilon, s+\varepsilon[\times]\theta-\varepsilon, \theta+\varepsilon[\subset \mathcal{U}$. The condition on λ then follows from the hypothesis. On the other hand, u satisfies $\partial_s u + J\partial_\theta u + Cu = 0$ on $]s-\varepsilon, s+\varepsilon[\times S^1$ and vanishes on this domain by the standard unique continuation property [10, Theorem 2.2, Proposition 3.1]. Let us assume without loss of generality that $(0,\theta) \in \mathcal{U}$ for some $\theta \in S^1$. The previous discussion shows that the pair (u,λ) vanishes together with all its derivatives along $\{0\} \times S^1$.

Let i denote the standard complex structure on $\mathbb{C}^n$. We choose a C^∞ function $\Psi : Z_h \to \mathrm{GL}_{\mathbb{R}}(\mathbb{C}^n)$ such that $J\Psi = \Psi i$, and we define $v : Z_h \to \mathbb{C}^n$ by $u = \Psi v$. Then v is C^∞ and satisfies

$$\partial_s v + i\partial_\theta v + \widetilde{C}(s,\theta)v + \widetilde{D}(s,\theta)\lambda = 0,$$

with $\widetilde{C} = \Psi^{-1}(\partial_s \Psi + J\partial_\theta \Psi + C\Psi)$ and $\widetilde{D} = \Psi^{-1}D$. Moreover, λ satisfies

$$\partial_s \lambda + \int_{S^1} \widetilde{E}(s,\theta)v(s,\theta)\,d\theta + F(s)\lambda = 0,$$

with $\widetilde{E} = E\Psi$. Thus $\widetilde{C}$, $\widetilde{D}$, and $\widetilde{E}$ are C^1. We assume in what follows without loss of generality that $J = i$.

Denote $U(s,\theta) := (u(s,\theta), \lambda(s), 0) \in \mathbb{C}^n \times \mathbb{C}^m$, so that $U : Z_h \to \mathbb{C}^{n+m}$ satisfies an equation of the form

$$\partial_s U + i\partial_\theta U + A(s,\theta)U + \int_{S^1} B(s,\tau)U(s,\tau)\,d\tau = 0$$

for some A, B of class C^1. Applying ∂_s and $-i\partial_\theta$ to this equation, summing, substituting $\partial_s U$ from the equation, and integrating once by parts with respect to θ, we obtain

$$\Delta U + A_1 U + A_2 \partial_s U + A_3 \partial_\theta U + \int_{S^1} A_4(s,\tau)U(s,\tau)\,d\tau = 0.$$

Here A_j, $j = 1, \ldots, 4$, are C^0 and given by $A_1 = \partial_s A - i\partial_\theta A$, $A_2 = A$, $A_3 = -iA$, $A_4 = \partial_s B - BA + \partial_\theta B i - B\int_{S^1} B$. By restricting to a smaller cylinder $Z_{h'}$, $h' < h$ so that the A_j are pointwise bounded by some constant $K > 0$, we obtain

$$|\Delta U|^2 \le 4K\left[|U|^2 + |\nabla U|^2 + \left(\int_{S^1} |U|\right)^2\right] \le 8\pi K\left[|U|^2 + |\nabla U|^2 + \int_{S^1} |U|^2\right].$$

The conclusion follows from Theorem 3.2. □

Remark 3.4. Assuming that the coefficients C, D, E, F in (3.5) are C^∞, the conclusion of Proposition 3.3 holds under the assumption that $\lambda(0) = 0$ and $u(0, \cdot) \equiv 0$. Indeed, by successive differentiation in (3.5), we find that the pair (u, λ) vanishes together with all its derivatives along $\{0\} \times S^1$.

Proposition 3.5 (Unique continuation). *Let $h > 0$ and $u_i : Z_h \to \widehat{W}$, $\lambda_i : I_h \to \Lambda$, $i = 0, 1$, be smooth functions satisfying equations* (2.5–2.6)*, i.e.*

$$\partial_s u + J^\theta_{\lambda(s)}(\partial_\theta u - X^\theta_{H_{\lambda(s)}}(u)) = 0,$$

$$\dot{\lambda}(s) - \int_{S^1} \vec{\nabla}_\lambda H(\theta, u(s, \theta), \lambda(s))\, d\theta = 0.$$

If (u_0, λ_0) and (u_1, λ_1) coincide on some nonempty open set $\mathcal{U} \subset Z_h$, then they coincide on Z_h.

Proof. We can assume without loss of generality that $\mathcal{U} = I_\delta \times I_\varepsilon$ for some $\delta, \varepsilon > 0$. Since $\lambda_0 = \lambda_1$ on I_δ, it follows that u_0 and u_1 satisfy the same Floer equation $\partial_s u + J^\theta_s(\partial_\theta u - X^\theta_s) = 0$ on $I_\delta \times S^1$. Since u_0 and u_1 coincide on $\mathcal{U}$, it follows by the unique continuation property for the Floer equation [10, Proposition 3.1] that $u_0 = u_1$ on $I_\delta \times S^1$.

Let $I \subset I_h$ be the set of points s such that $u_0 = u_1$ on $\{s\} \times S^1$, and $\lambda_0(s) = \lambda_1(s)$. Then $I \supset I_\delta$ and hence is nonempty. Moreover, it is closed. To prove the proposition, it is enough to show that I is open. Let $s_0 \in I$ be a point on the boundary of a connected component of I with nonempty interior, and denote $\gamma := u_0(s_0, \cdot) = u_1(s_0, \cdot)$. We consider a trivialization of $\gamma^* T\widehat{W}$ of the form $S^1 \times \mathbb{C}^n$, and a local chart in Λ around $\lambda_0(s_0) = \lambda_1(s_0)$, which we identify with $\mathbb{R}^m$. Then, for s close to s_0, we can view $u_0(s, \cdot)$ and $u_1(s, \cdot)$ as taking values in $\mathbb{C}^n$, and similarly $\lambda_0(s)$ and $\lambda_1(s)$ as taking values in $\mathbb{R}^m$. The difference $(u, \lambda) := (u_0 - u_1, \lambda_0 - \lambda_1)$ then satisfies an equation of the form (3.5) with smooth coefficients (the computation is similar to the one in the proof of [10, Proposition 3.1]). Moreover, (u, λ) vanishes to infinite order along $\{s_0\} \times S^1$. By Proposition 3.3, we conclude that $(u, \lambda) \equiv 0$ on a small strip around $\{s_0\} \times S^1$, so that s_0 belongs to the interior of I. □

Remark 3.6. The conclusion of Proposition 3.5 holds under the assumption that $u_0(s_0, \cdot) = u_1(s_0, \cdot)$ and $\lambda_0(s_0) = \lambda_1(s_0)$ for some $s_0 \in \mathbb{R}$ (use Remark 3.4). By successive differentiation, this hypothesis implies that (u_0, λ_0) and (u_1, λ_1) coincide together with all their derivatives along $\{s_0\} \times S^1$.

4. Transversality for the parametrized Floer equation

Let $H \in \mathcal{H}_{\Lambda,\mathrm{reg}}$. A pair $(J, g) \in \mathcal{J}_\Lambda$ is called *regular for H* if the operator $D_{(u,\lambda)}$ is surjective for any solution (u, λ) of (2.5–2.7). We denote the space of such pairs by $\mathcal{J}_{\Lambda,\mathrm{reg}}(H)$. In this section we prove Theorem B(b) as the following statement.

Theorem 4.1. *There exists a subset $\mathcal{HJ}_{\Lambda,\mathrm{reg}} \subset \mathcal{H}_{\Lambda,\mathrm{reg}} \times \mathcal{J}_\Lambda$ of second Baire category such that $(J, g) \in \mathcal{J}_{\Lambda,\mathrm{reg}}(H)$ whenever $(H, J, g) \in \mathcal{HJ}_{\Lambda,\mathrm{reg}}$.*

Remark 4.2. In general, it is not possible to first fix $H \in \mathcal{H}_{\Lambda,\mathrm{reg}}$ and then prove that $\mathcal{J}_{\Lambda,\mathrm{reg}}(H)$ is of the second Baire category in $\mathcal{J}_\Lambda$, as the following example shows. Consider a Hamiltonian of the form $H(\theta, x, \lambda) = K(\theta, x) + g(x) f(\lambda)$. Assume K has non-degenerate 1-periodic orbits with disjoint geometric images, fix a regular almost complex structure J for K, consider a Floer trajectory u for (K, J) with asymptotes $\overline{\gamma}, \underline{\gamma}$, let $g \equiv 1$ near $\overline{\gamma}$, $g \equiv -1$ near $\underline{\gamma}$, and let λ_0 be a minimum of f. If $\dim \Lambda > \mathrm{ind}(u)$, then (u, λ_0) is a parametrized Floer trajectory of negative index, independently of the choice of Riemannian metric g on Λ. Moreover, since u survives under small perturbations of J, the parametrized trajectory (u, λ_0) will survive under small perturbations of the pair (J, g). This shows that the latter cannot be chosen generically to be regular.

This phenomenon is similar to the one arising in the construction of the continuation morphism in Morse homology from a regular pair (f_-, g_-) to a regular pair (f_+, g_+). In that situation, we again *cannot* first fix the homotopy (f_t) and then choose the homotopy (g_t) generically. One has to choose the *pair* (f_t, g_t) generically. An explicit example is provided by the homotopy $f_t : \mathbb{R} \to \mathbb{R}$ given by $f_t(x) = -\frac{1}{2} t x^2$ for $t \in [-1, 1]$, $x \in \mathbb{R}$. In this case the constant trajectory at $x = 0$ has index -1 and exists for any choice of metric.

Let $(u, \lambda) \in \mathcal{M}(\overline{p}, \underline{p})$, $\overline{p} = (\overline{\gamma}, \overline{\lambda})$, $\underline{p} = (\underline{\gamma}, \underline{\lambda})$. We define the set of *regular points for* (u, λ) by

$$R(u,\lambda) := \left\{ (s,\theta) \in \mathbb{R} \times S^1 : \begin{array}{c} (\partial_s u(s,\theta), \partial_s \lambda(s)) \neq (0,0), \\ (u(s,\theta), \lambda(s)) \neq (\overline{\gamma}(\theta), \overline{\lambda}), (\underline{\gamma}(\theta), \underline{\lambda}), \\ (u(s,\theta), \lambda(s)) \notin (u(\cdot,\theta), \lambda(\cdot))(\mathbb{R} \setminus \{s\}) \end{array} \right\}.$$

Notation. In the following we denote $U(s, \theta) = (u(s, \theta), \lambda(s))$ and assume that U satisfies equations (2.5–2.6). We also denote $R(U) := R(u, \lambda)$.

Proposition 4.3 (Regular points). *Assume* $\partial_s U \not\equiv (0, 0)$. *Then:*

(i) *The set* $\{(s, \theta) : \partial_s U(s, \theta) \neq (0, 0)\}$ *is open and dense in* $\mathbb{R} \times S^1$. *Moreover, given* $s \in \mathbb{R}$, *there exists* $\theta \in S^1$ *such that* $\partial_s U(s, \theta) \neq (0, 0)$.

(ii) *The set* $R(u, \lambda)$ *is open.*

(iii) *If* $\partial_s u \equiv 0$, *then* $R(u, \lambda)$ *is equal to* $\mathbb{R} \times S^1$. *If* $\partial_s u \not\equiv 0$, *then* $R(u, \lambda)$ *is dense in the open set* $\{(s, \theta) : \partial_s u(s, \theta) \neq 0\}$.

Remark 4.4. If $\partial_s u \not\equiv 0$ and $\partial_s \lambda \equiv 0$, the proposition implies that $R(u, \lambda)$ is dense in $\mathbb{R} \times S^1$. Indeed, u satisfies a Floer equation which is independent of s and the open set $\{(s, \theta) : \partial_s u(s, \theta) \neq 0\}$ is dense in $\mathbb{R} \times S^1$ [10, Lemma 4.1].

To prove Proposition 4.3, we need the following enhancement of Proposition 3.5 (this is the analogue of Lemma 4.2 in [10]). In the next statement we denote $V_h(s, \theta) :=]s - h, s + h[\times]\theta - h, \theta + h[\subset \mathbb{R} \times S^1$ and $I_h(s) :=]s - h, s + h[\subset \mathbb{R}$ for $h > 0$.

Lemma 4.5. *Let* $U_i = (u_i, \lambda_i)$, $i = 0, 1$, *be smooth functions defined on a strip* $I_{h_0} \times S^1$, $h_0 > 0$, *and satisfying equations* (2.5–2.6) *in Proposition* 3.5. *Assume that*

$$U_0(s_0, \theta_0) = U_1(s_0, \theta_0), \quad \partial_s u_0(s_0, \theta_0) \neq 0, \quad \partial_s U_1(s_0, \theta_0) \neq (0, 0)$$

for some $(s_0, \theta_0) \in \mathbb{R}\times S^1$. *Assume also that, for any* $0 < h' \le h_0$*, there exists* $0 < h \le h_0$ *with the following property: for any* $(s, \theta) \in V_h(s_0, \theta_0)$*, there exists* $(s', \theta) \in V_{h'}(s_0, \theta_0)$ *such that*

$$U_0(s, \theta) = U_1(s', \theta).$$

Then $U_0 = U_1$.

Remark 4.6. We could not prove Lemma 4.5 under the more general assumption $\partial_s U_0(s_0, \theta_0) \neq (0, 0)$ (instead of $\partial_s u_0(s_0, \theta_0) \neq 0$). This in turn influences the conclusion of (iii) in Proposition 4.3: we only show that $R(u, \lambda)$ is dense in the set $\{(s, \theta) : \partial_s u(s, \theta) \neq 0\}$.

Proof of Lemma 4.5. By Proposition 3.5, it is enough to prove that $U_0 = U_1$ on some open neighbourhood of (s_0, θ_0). Let us choose $h' > 0$ small enough so that $I_{h'}(s_0) \to \widehat{W} \times \Lambda$, $s \mapsto U_1(s, \theta)$, is an embedding for all $\theta \in I_{h'}(\theta_0)$. By further diminishing the corresponding $h > 0$ we can also assume that $I_h(s_0) \to \widehat{W}$, $s \mapsto u_0(s, \theta)$, is an embedding for all $\theta \in I_h(\theta_0)$.

For each $\theta \in I_h(\theta_0)$, we have by assumption $U_0(I_h(s_0), \theta) \subset U_1(I_{h'}(s_0), \theta)$. We can therefore define smooth embeddings $G_\theta := (U_1(\cdot, \theta))^{-1} \circ U_0(\cdot, \theta) : I_h(s_0) \to I_{h'}(s_0)$. Moreover, for h small enough, we have $s_0 \in \mathrm{im}(G_\theta)$. Let us choose $0 < h'' < h'$ small enough such that $I_{h''}(s_0) \subset \mathrm{im}(G_\theta)$ for all $\theta \in I_h(\theta_0)$. By the implicit function theorem, we obtain a smooth embedding $F_\theta := (G_\theta)^{-1} : I_{h''}(s_0) \to I_h(s_0)$. The collection $\{F_\theta\}$ of maps gives rise to the smooth map $F : V_{h''}(s_0, \theta_0) \to V_h(s_0, \theta_0)$ defined by $F(s, \theta) := (F_\theta(s), \theta) := (\phi(s, \theta), \theta)$. We have

$$U_1(s, \theta) = U_0(\phi(s, \theta), \theta)$$

for all $(s, \theta) \in V_{h''}(s_0, \theta_0)$. Substituting in the Floer equation for u_1, we obtain

$$\begin{aligned}
0 &= \partial_s u_1 + J^\theta_{\lambda_1}(u_1)(\partial_\theta u_1 - X^\theta_{H_{\lambda_1}}(u_1)) \\
&= \partial_s u_0(F) \cdot \partial_s \phi + J^\theta_{\lambda_0(F)}(u_0(F))(\partial_s u_0(F) \cdot \partial_\theta \phi + \partial_\theta u_0(F) - X^\theta_{H_{\lambda_0(F)}}(u_0(F))) \\
&= \partial_s u_0(F) \cdot (\partial_s \phi - 1) + J^\theta_{\lambda_0(F)}(u_0(F))\partial_s u_0(F) \cdot \partial_\theta \phi.
\end{aligned}$$

The last equality follows from the Floer equation for u_0. Since $\partial_s u_0 \neq 0$ on $V_h(s_0, \theta_0)$ we see that the vectors $\partial_s u_0(F)$ and $J^\theta_{\lambda_0(F)} \partial_s u_0(F)$ are linearly independent, so that $\partial_s \phi \equiv 1$ and $\partial_\theta \phi \equiv 0$. Since $\phi(s_0, \theta_0) = s_0$, we obtain $\phi(s, \theta) = s$ for all $(s, \theta) \in V_{h''}(s_0, \theta_0)$ and the conclusion follows. □

Proof of Proposition 4.3. (i) A straightforward computation shows that, for any $s \in \mathbb{R}$, the pair $\partial_s U = (\partial_s u, \partial_s \lambda)$ satisfies an equation of the form (3.5) with smooth coefficients in a local trivialization along the loop $u(s, \cdot)$ and in a local chart around $\lambda(s)$. Assume by contradiction that $\partial_s U \equiv (0, 0)$ on some nonempty open set $\mathcal{U}$. By Proposition 3.3, we see that $\partial_s U \equiv (0, 0)$ on some open strip around $\mathcal{U}$. By the standard open-closed argument we get $\partial_s U \equiv 0$ on $\mathbb{R} \times S^1$, which contradicts the hypothesis.

We now prove the second statement. Assuming by contradiction the existence of a point $s_0 \in \mathbb{R}$ such that $\partial_s U \equiv (0, 0)$ along $\{s_0\} \times S^1$, we deduce by Remark 3.4 that $\partial_s U \equiv (0, 0)$ on a strip around $\{s_0\} \times S^1$. We then conclude as above.

(ii) The first two conditions defining the elements of $R(u,\lambda)$ are clearly open. We need to show that the third one is open as well. Arguing by contradiction, we find a point $(s_0,\theta_0)\in R(u,\lambda)$, a sequence $(s^\nu,\theta^\nu)\to(s_0,\theta_0)$, and a sequence $s'^\nu\neq s^\nu$ such that $U(s'^\nu,\theta^\nu)=U(s^\nu,\theta^\nu)$. Since $\partial_s U(s_0,\theta_0)\neq(0,0)$, we can find $h>0$ such that $U(\cdot,\theta_0)$ is an embedding on $I_h(s_0)$ and $U(\cdot,\theta^\nu)$ is an embedding on $I_h(s^\nu)$ for ν large enough. Thus, we can assume without loss of generality that s'^ν is bounded away from s_0 (otherwise $s'^\nu\in I_h(s^\nu)$ for ν large enough, a contradiction). Since U converges at $\pm\infty$ to its asymptotes, and $U(s_0,\theta_0)$ does not lie on those asymptotes by assumption, we infer the existence of some $T>0$ such that $s'^\nu\in[-T,T]$ for all ν. We can therefore find a convergent subsequence, still denoted s'^ν, such that $s'^\nu\to s'_0\neq s_0$. Then $U(s'_0,\theta_0)=U(s_0,\theta_0)$, which contradicts the assumption that $(s_0,\theta_0)\in R(u,\lambda)$.

(iii) If $\partial_s u\equiv 0$, then λ satisfies an ordinary differential equation independent of s, and since $\partial_s\lambda\not\equiv 0$, we infer that $\partial_s\lambda\neq 0$ on $\mathbb{R}$ and every point $(s,\theta)\in\mathbb{R}\times S^1$ is regular.

Let us now assume $\partial_s u\not\equiv 0$. It is enough to show that for any (s,θ) such that $\partial_s u(s,\theta)\neq 0$, there exists a neighbourhood $\mathcal{U}$ such that $R(u,\lambda)\cap\mathcal{U}$ is dense in $\mathcal{U}$. Let $(s_0,\theta_0)\in\mathbb{R}\times S^1$ be such that $\partial_s u(s_0,\theta_0)\neq 0$. We choose $h>0$ small enough such that $\partial_s u\neq 0$ on $V_h(s_0,\theta_0)$ and $I_h(s_0)\to\widehat{W}$, $s\mapsto u(s,\theta)$, is an embedding for all $\theta\in I_h(\theta_0)$. Then $I_h(s_0)\to\widehat{W}\times\Lambda$, $s\mapsto U(s,\theta)$, is a fortiori also an embedding for all $\theta\in I_h(\theta_0)$. Since every point $(s,\theta)\in V_h(s_0,\theta_0)$ can be approximated by a sequence (s^ν,θ^ν) satisfying $U(s^\nu,\theta^\nu)\neq\overline{p}(\theta^\nu),\underline{p}(\theta^\nu)$, we can assume without loss of generality that

$$\forall\,(s,\theta)\in V_h(s_0,\theta_0),\qquad U(s,\theta)\neq\overline{p}(\theta),\underline{p}(\theta). \tag{4.1}$$

The conclusion of the proposition now reduces to showing that (s_0,θ_0) can be approximated by a sequence $(s^\nu,\theta^\nu)\in R(U)$. Assuming this is false, there exists $0<\varepsilon<h$ such that $V_\varepsilon(s_0,\theta_0)\cap R(U)=\emptyset$, i.e.

$$\forall(s,\theta)\in V_\varepsilon(s_0,\theta_0),\ \exists\,s'\neq s,\qquad U(s',\theta)=U(s,\theta). \tag{4.2}$$

Since $\lim_{s\to-\infty}U(s,\theta)=\overline{p}(s,\theta)$ and $\lim_{s\to\infty}U(s,\theta)=\underline{p}(s,\theta)$ uniformly in θ, we infer from (4.1) the existence of a constant $T>0$ such that $|s'|\leq T$ in (4.2).

Let us denote $C(U):=\{(s,\theta)\in\mathbb{R}\times S^1:\partial_s U(s,\theta)=(0,0)\}$. Note that, by the proof of (i), the set $C(U)$ has empty interior. We now claim that (s_0,θ_0) can be approximated by a sequence (s^ν,θ_0) such that, for all ν and all $s'\in\mathbb{R}$ with $U(s',\theta_0)=U(s^\nu,\theta_0)$, we have $(s',\theta_0)\notin C(U)$. Assuming the claim, we can suppose without loss of generality that, for each $s'\in\mathbb{R}$ such that $U(s',\theta_0)=U(s_0,\theta_0)$, we have $(s',\theta)\notin C(U)$. Moreover, after further diminishing $\varepsilon>0$, we can assume without loss of generality that

$$\forall(s,\theta)\in V_\varepsilon(s_0,\theta_0),\ \forall s'\in\mathbb{R},\qquad U(s,\theta)=U(s',\theta)\ \Rightarrow\ (s',\theta)\notin C(U). \tag{4.3}$$

Indeed, if this failed for all $\varepsilon>0$, we could find a sequence $(s^\nu,\theta^\nu)\to(s_0,\theta_0)$ and a sequence s'^ν such that $(s'^\nu,\theta^\nu)\in C(U)$, $U(s'^\nu,\theta^\nu)=U(s^\nu,\theta^\nu)$, and $|s'^\nu-s_0|\geq\varepsilon_0>0$. Up to a subsequence, we have $s'^\nu\to s'\in[-T,T]$, $\theta^\nu\to\theta_0$, and $U(s',\theta_0)=U(s_0,\theta_0)$ with $(s',\theta_0)\in C(U)$. This contradicts our last assumption on (s_0,θ_0), obtained via the claim.

To prove the claim, let us choose a neighbourhood $\mathcal{V}$ of $U(I_\varepsilon(s_0),\theta_0)$ in $\widehat{W}\times\Lambda$, of the form $I_\varepsilon(s_0)\times\mathbb{R}^{2n+m-1}$, and denote pr_1 the projection to the first coordinate interval $I_\varepsilon(s_0)$. Let $f := \mathrm{pr}_1\circ U(\cdot,\theta_0)$, with $f:\mathrm{dom}(f) := U(\cdot,\theta_0)^{-1}(\mathcal{V})\to I_\varepsilon(s_0)$. Let $C(U)_{\theta_0} := \{s\in\mathbb{R} : (s,\theta_0)\in C(U)\}$. Then $f(C(U)_{\theta_0}\cap\mathrm{dom}(f))$ is contained in the set of critical values of f. By Sard's theorem, this is a nowhere dense set in $I_\varepsilon(s_0)$, and the claim follows.

We now closely follow the proof of Theorem 4.3 in [10]. We first remark that, for any $(s,\theta)\in V_\varepsilon(s_0,\theta_0)$, there are only a finite number of values $s'\in\mathbb{R}$ such that $U(s',\theta)=U(s,\theta)$. If not, we could find an accumulation point $s'\in[-T,T]$ such that $\partial_s U(s',\theta)=(0,0)$ and $U(s',\theta)=U(s,\theta)$, in contradiction with (4.3). Let $s_1,\dots,s_N\in[-T,T]$ be the points such that $U(s_j,\theta_0)=U(s_0,\theta_0)$, $j=1,\dots,N$.

We now claim that, for any $r>0$ there exists $\delta>0$ such that

$$\forall(s,\theta)\in V_{2\delta}(s_0,\theta_0),\ \exists(s',\theta)\in\bigcup_{j=1}^N V_r(s_j,\theta_0),\qquad U(s,\theta)=U(s',\theta).$$

If this failed, we could find $r>0$ and a sequence $(s^\nu,\theta^\nu)\to(s_0,\theta_0)$ such that, for all ν and for all $(s',\theta^\nu)\in\bigcup_{j=1}^N V_r(s_j,\theta_0)$, we have $U(s^\nu,\theta^\nu)\neq U(s',\theta^\nu)$. On the other hand, by (4.2) there exists $s'^\nu\in[-T,T]$ such that $U(s'^\nu,\theta^\nu)=U(s^\nu,\theta^\nu)$, and in particular $|s'^\nu-s_j|\geq r$ for all j. Up to a subsequence we have $s'^\nu\to s'$ and $\theta^\nu\to\theta_0$, so that $U(s',\theta_0)=U(s_0,\theta_0)$ and $s'\neq s_j$, $j=1,\dots,N$, a contradiction.

Following [10], we define

$$\Sigma_j := \{(s,\theta)\in\overline{V_\delta}(s_0,\theta_0) : \exists(s',\theta)\in\overline{V_r}(s_j,\theta_0),\ U(s',\theta)=U(s,\theta)\}.$$

Then Σ_j is closed and $\overline{V_\delta}(s_0,\theta_0)=\Sigma_1\cup\dots\cup\Sigma_N$. It follows from Baire's theorem that one of the Σ_j, say Σ_1, has nonempty interior.

Let $(\overline{s},\overline{\theta})\in\mathrm{int}(\Sigma_1)$ and denote by $(\overline{s}_1,\overline{\theta})$ the unique preimage of $U(\overline{s},\overline{\theta})$ in $V_r(s_1,\theta_0)$. Let $0<r_1<r$ be such that $V_{r_1}(\overline{s}_1,\overline{\theta})\subset V_r(s_1,\theta_0)$, and $0<\delta_1<\delta$ be such that $V_{\delta_1}(\overline{s},\overline{\theta})\subset\Sigma_1$, and such that for all $(s,\theta)\in V_{\delta_1}(\overline{s},\overline{\theta})$, there exists $(s',\theta)\in V_{r_1}(\overline{s}_1,\overline{\theta})$ such that $U(s,\theta)=U(s',\theta)$. It follows from our construction that, for all $0<h'\leq r_1$, there exists $0<h\leq\delta_1$ such that for all $(s,\theta)\in V_h(\overline{s},\overline{\theta})$ there exists $(s',\theta)\in V_{h'}(\overline{s}_1,\overline{\theta})$, such that $U(s,\theta)=U(s',\theta)$. We can therefore apply Lemma 4.5 with $(s_0,\theta_0):=(\overline{s},\overline{\theta})$, $U_0:=U$, $U_1:=U(\cdot+\overline{s}_1-\overline{s},\cdot)$, and $h_0=r_1$ to obtain $U_0=U_1$. This implies

$$U(s,\theta)=\lim_{k\to\pm\infty}U(s+k(\overline{s}_1-\overline{s}),\theta)=\overline{p}(\theta)=\underline{p}(\theta).$$

This contradicts our standing assumption $\partial_s U\not\equiv(0,0)$. Proposition 4.3 is proved. □

Proof of Theorem 4.1. Let $\mathcal{J}^r_\Lambda$, $r\geq 1$, denote the space of pairs (J,g) of class C^r such that J is an admissible almost complex structure on $\widehat{W}$. Let $\mathcal{H}^r_{\Lambda,\mathrm{reg}}$, $r\geq 1$, denote the space of regular admissible Hamiltonians of class C^r. Let $\mathcal{HJ}^r_{\Lambda,\mathrm{reg}}\subset\mathcal{H}^r_{\Lambda,\mathrm{reg}}\times\mathcal{J}^r_\Lambda$ denote the space of triples (H,J,g) such that (J,g) is regular for H. By a standard argument due to Taubes [13, p. 52], it is enough to prove that $\mathcal{HJ}^r_{\Lambda,\mathrm{reg}}$ is of the second Baire category in $\mathcal{H}^r_{\Lambda,\mathrm{reg}}\times\mathcal{J}^r_\Lambda$.

Given $p > 2$ and $\overline{p}, \underline{p} \in \mathcal{P}(H)$, we denote by $\mathcal{B}$ the space of pairs (u, λ) consisting of maps $u : \mathbb{R} \times S^1 \to \widehat{W}$ and $\lambda : \mathbb{R} \to \Lambda$ which are locally of class $W^{1,p}$, which satisfy (2.7), and which are of class $W^{1,p}$ in local charts near the asymptotes. Then $\mathcal{B} \times \mathcal{H}^r_{\Lambda,\mathrm{reg}} \times \mathcal{J}^r_\Lambda$ is a Banach manifold. There is a Banach bundle $\mathcal{E} \to \mathcal{B} \times \mathcal{H}^r_{\Lambda,\mathrm{reg}} \times \mathcal{J}^r_\Lambda$ whose fibre at (u, λ, H, J, g) is $\mathcal{L}^p := L^p(\mathbb{R}\times S^1, u^*T\widehat{W}) \oplus L^p(\mathbb{R}, \lambda^*T\Lambda)$. The solutions of the parametrized Floer equations (2.5–2.6) for (H, J, g) are the zeroes of the section $f : \mathcal{B} \times \mathcal{H}^r_{\Lambda,\mathrm{reg}} \times \mathcal{J}^r_\Lambda \to \mathcal{E}$ given by

$$f(u, \lambda, H, J, g) := \begin{pmatrix} \partial_s u + J^\theta_{\lambda(s)}(\partial_\theta u - X^\theta_{H_{\lambda(s)}}(u)) \\ \dot\lambda(s) - \int_{S^1} \vec\nabla_\lambda H(\theta, u(s,\theta), \lambda(s))\, d\theta \end{pmatrix}.$$

The crucial step is to prove that the universal moduli space $\mathcal{M} := f^{-1}(0)$ is a Banach submanifold of $\mathcal{B} \times \mathcal{H}^r_{\Lambda,\mathrm{reg}} \times \mathcal{J}^r_\Lambda$. Then the claim follows easily from the Sard–Smale theorem as in [13, proof of Theorem 3.1.5(ii)].

The vertical differential of f at a point $(u, \lambda, H, J, g) \in \mathcal{M}$ is given by

$$\begin{aligned} df(u, \lambda, H, J, g) \cdot (\zeta, \ell, h, Y, A) &:= D_{(u,\lambda)}(\zeta, \ell) \\ &+ \begin{pmatrix} -J^\theta_{\lambda(s)} X^\theta_{h_{\lambda(s)}}(u) + Y^\theta_{\lambda(s)}(\partial_\theta u - X^\theta_{H_{\lambda(s)}}(u)) \\ -\int_{S^1} \vec\nabla_\lambda h(\theta, u(s,\theta), \lambda(s))\, d\theta + A \cdot \int_{S^1} \vec\nabla_\lambda H(\theta, u(s,\theta), \lambda(s))\, d\theta \end{pmatrix}, \end{aligned}$$

where $h \in T_H \mathcal{H}^r_{\Lambda,\mathrm{reg}}$ and $(Y, A) \in T_{(J,g)}\mathcal{J}^r_\Lambda$. For a description of h we refer to the proof of Proposition 2.4. We view Y as a family $Y = (Y^\theta_\lambda)$, $\lambda \in \Lambda$, $\theta \in S^1$, with $Y^\theta_\lambda \in \mathrm{End}(T\widehat{W})$, such that $Y^\theta_\lambda J^\theta_\lambda + J^\theta_\lambda Y^\theta_\lambda = 0$ and $\widehat\omega(Y^\theta_\lambda \cdot, \cdot) + \widehat\omega(\cdot, Y^\theta_\lambda \cdot) = 0$. Moreover, for $t \geq 0$ large enough, Y^θ_λ is independent of t, it preserves ξ and vanishes on $\langle \partial/\partial t, R_\alpha \rangle$. The element A is a tangent vector at g to the space $\mathrm{Met}^r(\Lambda)$ of Riemannian metrics of class C^r on Λ. Considering a 1-parameter family $g^\varepsilon \in \mathrm{Met}^r(\Lambda)$ such that $g^0 = g$, we define A by $g(A\cdot, \cdot) = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} g^\varepsilon$, so that A is an element of $\mathrm{End}(T\Lambda)$ which is symmetric with respect to g. Denoting by $\vec\nabla^\varepsilon_\lambda H$ the λ-gradient of H with respect to g^ε, we then have

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \vec\nabla^\varepsilon_\lambda H = -A \cdot \vec\nabla_\lambda H,$$

hence the formula for the vertical differential of f.

We need to show that df is surjective. Since the image of $D_{(u,\lambda)}$ is closed and has finite codimension, it follows that the image of df has the same property. Thus, it suffices to show that it is dense. Let $(\eta, k) \in \mathcal{L}^q = (\mathcal{L}^p)^*$, $1/p + 1/q = 1$, be an element annihilating $\mathrm{Im}(df)$. Using that $(u, \lambda, H, J, g) \in \mathcal{M}$, this means that

$$\begin{aligned} &\int_{\mathbb{R}\times S^1} \langle \eta, D_u\zeta + (D_\lambda J \cdot \ell) J \partial_s u - J(D_\lambda X_{H_\lambda} \cdot \ell) - J X_{h_\lambda} + Y^\theta_\lambda J \partial_s u \rangle \, ds\, d\theta \\ &\quad + \int_{\mathbb{R}} \left\langle k, \nabla_s \ell - \nabla_\ell \int_{S^1} \vec\nabla_\lambda H - \int_{S^1} \nabla_\zeta \vec\nabla_\lambda H - \int_{S^1} \vec\nabla_\lambda h + A \cdot \dot\lambda \right\rangle ds = 0 \end{aligned} \tag{4.4}$$

for any $(\zeta, \ell, h, Y, A) \in T_{(u,\lambda)}\mathcal{B} \oplus T_H\mathcal{H}^r_{\Lambda,\mathrm{reg}} \oplus T_{(J,g)}\mathcal{J}^r_\Lambda$. We claim that $(\eta, k) = (0, 0)$. Taking $h = 0$, $Y = 0$, $A = 0$ we find that (η, k) lies in the kernel of the formal adjoint $D^*_{(u,\lambda)}$. The latter has the same form as $D_{(u,\lambda)}$ and is therefore elliptic with smooth coefficients. By elliptic regularity, it follows that η and k are smooth. We distinguish now three cases.

Case 1: $\partial_s u \equiv 0$ and $\partial_s\lambda \equiv 0$. By Lemma 2.9 the operator $D_{(u,\lambda)}$ is bijective, so that df is surjective.

Case 2: $\partial_s u \equiv 0$ and $\dot\lambda \not\equiv 0$. In this case λ satisfies an ordinary differential equation independent of s and therefore $\dot\lambda \neq 0$ on $\mathbb{R}$ and every point $(s, \theta) \in \mathbb{R} \times S^1$ is regular. We claim $k \equiv 0$. Indeed, if there existed $s_0 \in \mathbb{R}$ with $k(s_0) \neq 0$, we could take $\zeta = 0$, $\ell = 0$, $h = 0$, $Y = 0$ and A supported in a small neighbourhood of $\lambda(s_0)$ such that $A(\lambda(s_0))\dot\lambda(s_0) = k(s_0)$, so that the sum of the integrals in (4.4) would be > 0. We claim $\eta \equiv 0$. Indeed, if there existed (s_0, θ_0) such that $\eta(s_0, \theta_0) \neq 0$, we could take $\zeta = 0$, $\ell = 0$, $Y = 0$, $A = 0$ and h supported near $\lambda(s_0)$, and satisfying $J^\theta_{\lambda(s_0)} X_h(\theta, \overline{\gamma}(\theta), \lambda(s_0)) = -\eta(s_0, \theta)$ for all $\theta \in S^1$. Then the sum of the integrals in (4.4) would be > 0.

Case 3: $\partial_s u \not\equiv 0$. By Proposition 4.3, there exists a nonempty open set $\Omega \subset \mathbb{R} \times S^1$ consisting of regular points (s, θ) such that $\partial_s u(s, \theta) \neq 0$.

We first claim that $\eta \equiv 0$ on Ω. Arguing by contradiction, we find $(s_0, \theta_0) \in \Omega$ such that $\eta(s_0, \theta_0) \neq 0$. We then take $\zeta = 0$, $\ell = 0$, $h = 0$, $A = 0$ and Y supported near $(\theta_0, u(s_0, \theta_0), \lambda(s_0))$ such that $Y^{\theta_0}_{\lambda(s_0)} J^{\theta_0}_{\lambda(s_0)} \partial_s u(s_0, \theta_0) = \eta(s_0, \theta_0)$. The second integral in (4.4) is zero, whereas the first one localizes near (s_0, θ_0) and is positive. This contradicts (4.4).

We now claim that $k(s) = 0$ for all $(s, \theta) \in \Omega$. Arguing again by contradiction, we find $(s_0, \theta_0) \in \Omega$ such that $k(s_0) \neq 0$. We consider a function h of the form $h(\theta, x, \lambda) = \phi(\theta)\psi(x)h_1(\lambda)$ such that ϕ is a cutoff function supported near θ_0, ψ is a cutoff function supported near $u(s_0, \theta_0)$, and h_1 is supported in a neighbourhood of $\lambda(s_0)$ and satisfies $\vec\nabla_\lambda h_1(\lambda(s_0)) = -k(s_0)$. The crucial observation is that if the support of ψ is small enough (depending on the choice of h_1), then

$$\langle k(s), \vec\nabla_\lambda h(\theta, u(s, \theta), \lambda(s))\rangle \geq 0$$

on $\mathbb{R} \times S^1$, and vanishes outside a small neighbourhood of (s_0, θ_0). Here we use that (s_0, θ_0) is a regular point and $\partial_s u(s_0, \theta_0) \neq 0$. We now take $\zeta = 0$, $\ell = 0$, $Y = 0$, $A = 0$, and h as above, so that the first integral in (4.4) vanishes, and the second integral is positive, a contradiction. □

Remark 4.7. We needed the possibility to deform the metric g in the proof of Theorem 4.1 only to treat Case 2.

5. Fredholm theory in the S^1-invariant case

In this section we take the parameter space to be $\Lambda = S^{2N+1}$, $N \geq 1$.

We denote by $\mathcal{H}^{S^1}_N \subset \mathcal{H}_{S^{2N+1}}$ the set of admissible Hamiltonian families $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ which are invariant with respect to the diagonal S^1-action on $S^1 \times S^{2N+1}$, meaning that $H(\theta+\tau, x, \tau\lambda) = H(\theta, x, \lambda)$ for all $\tau \in S^1$. It follows from the definitions that there exists $t_0 \geq 0$ such that, for $t \geq t_0$, we have $H(\theta, p, t, \lambda) = \beta e^t + \beta'(\lambda)$ with $0 < \beta \notin \mathrm{Spec}(M, \alpha)$, and $\beta' \in C^\infty(S^{2N+1}, \mathbb{R})$ being S^1-invariant.

Given $H \in \mathcal{H}_N^{S^1}$, the parametrized action functional $\mathcal{A}$ is S^1-invariant, and so is the set $\mathcal{P}(H)$ of its critical points. For $p = (\gamma, \lambda) \in \mathcal{P}(H)$, we denote

$$S_p = S_{(\gamma,\lambda)} := \{(\tau\gamma, \tau\lambda) : \tau \in S^1\} \subset \mathcal{P}(H),$$

so that $S_p = S_{\tau\cdot p}$, $\tau \in S^1$. We refer to S_p as an *S^1-orbit of critical points*.

We denote by $\mathcal{J}_N^{S^1}$ the set of pairs (J, g) consisting of an S^1-invariant admissible S^{2N+1}-family J of almost complex structures on $\widehat{W}$, and of an S^1-invariant Riemannian metric g on S^{2N+1}. The S^1-invariance condition on J means that $J_{\tau\lambda}^{\theta+\tau} = J_\lambda^\theta$ for all $\tau \in S^1$.

An S^1-orbit of critical points $S_p \subset \mathcal{P}(H)$ is called *nondegenerate* if the Hessian $d^2\mathcal{A}(\gamma, \lambda)$ has a 1-dimensional kernel V_p for some (and hence any) $(\gamma, \lambda) \in S_p$. It follows from Lemma 2.3 that nondegeneracy is equivalent to the fact that the kernel of the asymptotic operator D_p is also 1-dimensional and equal to V_p. In both cases, a generator of V_p is given by the infinitesimal generator of the S^1-action.

We define the set $\mathcal{H}_{N,\mathrm{reg}}^{S^1} \subset \mathcal{H}_N^{S^1}$ to consist of all elements H such that, for any $p \in \mathcal{P}(H)$, the S^1-orbit S_p is nondegenerate.

Proposition 5.1. *The set $\mathcal{H}_{N,\mathrm{reg}}^{S^1}$ is of the second Baire category in $\mathcal{H}_N^{S^1}$. Moreover, if $H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1}$, each S^1-orbit $S_p \subset C^\infty(S^1, \widehat{W}) \times S^{2N+1}$ is isolated.*

Proof. The proof is similar to that of Proposition 2.4. Given an integer $r \geq 2$, we denote by $\mathcal{H}_N^{r,S^1}$ the space of S^1-invariant admissible Hamiltonian families of class C^r. We denote by $\mathcal{H}_{N,\mathrm{reg}}^{r,S^1} \subset \mathcal{H}_{S^{2N+1}}^{r,S^1}$ the set of Hamiltonians H such that, for any $p \in \mathcal{P}(H)$, the S^1-orbit S_p is nondegenerate. For $t_0 \geq 0$, we denote $\{t \leq t_0\} := W \cup M \times [0, t_0]$ and let $\mathcal{H}_{N,\mathrm{reg},t_0}^{r,S^1} \subset \mathcal{H}_N^{r,S^1}$ be the set of Hamiltonians H such that for any $p = (\gamma, \lambda) \in \mathcal{P}(H)$ with $\mathrm{im}(\gamma) \subset \{t \leq t_0\}$, the S^1-orbit S_p is nondegenerate. Then

$$\mathcal{H}_{N,\mathrm{reg}}^{r,S^1} = \bigcap_{t_0 \geq 0} \mathcal{H}_{N,\mathrm{reg},t_0}^{r,S^1}.$$

As in Proposition 2.4, it is enough to prove that $\mathcal{H}_{N,\mathrm{reg},t_0}^{r,S^1}$ is open and dense, so that $\mathcal{H}_{N,\mathrm{reg}}^{r,S^1}$ is of the second Baire category. The proof that $\mathcal{H}_{N,\mathrm{reg},t_0}^{r,S^1}$ is open is similar to the proof that $\mathcal{H}_{\Lambda,\mathrm{reg},t_0}^{r}$ is open in Proposition 2.4. We now prove that $\mathcal{H}_{N,\mathrm{reg},t_0}^{r,S^1}$ is dense. We consider the Banach bundle $\mathcal{E} \to \mathcal{H}_N^{r,S^1} \times C^r(S^1, \{t \leq t_0\}) \times S^{2N+1} \times \mathbb{R}$ whose fibre at (H, γ, λ, a) is $\mathcal{E}_{(H,\gamma,\lambda,a)} := C^{r-1}(S^1, \gamma^*T\widehat{W}) \times T_\lambda S^{2N+1}$, and the section $\bar{f}$ given by

$$\bar{f}(H, \gamma, \lambda, a) := \left(\dot{\gamma} - X_H \circ \gamma + a\dot{\gamma}, -\int_{S^1} \vec{\nabla}_\lambda H + aX\right),$$

where the vector field X denotes the infinitesimal generator of the S^1-action on S^{2N+1}. We first prove that $\overline{\mathcal{P}} := \bar{f}^{-1}(0)$ is a Banach submanifold of $\mathcal{H}_N^{r,S^1} \times C^r(S^1, \{t \le t_0\}) \times S^{2N+1} \times \mathbb{R}$. Indeed, the vertical differential of $\bar{f}$ at a point $(H, \gamma, \lambda, a) \in \overline{\mathcal{P}}$ is given by

$$d\bar{f}(H,\gamma,\lambda,a)\cdot(h,\zeta,\ell,b) = \begin{pmatrix} \nabla_\theta \zeta - \nabla_\zeta X_H - (D_\lambda X_H)\cdot \ell - X_h + b\dot{\gamma} \\ -\int_{S^1} \nabla_\zeta \vec{\nabla}_\lambda H - \int_{S^1} \nabla_\ell \vec{\nabla}_\lambda H - \int_{S^1} \vec{\nabla}_\lambda h + bX \end{pmatrix}$$
$$= df(H,\gamma,\lambda)\cdot(h,\zeta,\ell) + b\begin{pmatrix}\dot{\gamma} \\ X\end{pmatrix},$$

where $f(H, \gamma, \lambda)$ is the restriction of $\bar{f}$ to $\{a = 0\}$. That $d\bar{f}(H, \gamma, \lambda, a)$ is surjective is seen as follows. First, $d\bar{f}(H, \gamma, \lambda, a) \cdot (h, 0, 0, a + 1) = (0, X)$ for $h = H$ near $\mathrm{im}(\gamma)$. Given $k \in T_\lambda S^{2N+1}$ such that $g(k, X) = 0$, we have $(0, k) = d\bar{f}(H, \gamma, \lambda, a)\cdot(h, 0, 0, 0)$, with $h(\cdot, \cdot, \lambda) = \text{const}$ in some neighbourhood of $\mathrm{im}(\gamma)$, h is S^1-invariant and $\vec{\nabla}_\lambda h = k$. Given $\eta \in C^{r-1}(S^1, \gamma^* T\widehat{W})$, let us choose $h \in T_H \mathcal{H}_N^{r,S^1}$ such that $X_h = -\eta$ along γ. Then the first component of $d\bar{f}(H, \gamma, \lambda, a) \cdot (h, 0, 0, 0)$ is equal to η. This proves that $d\bar{f}(H, \gamma, \lambda, a)$ is surjective and that $\overline{\mathcal{P}}$ is a Banach submanifold as desired.

We now claim that the set of regular values of the natural projection $\mathrm{pr} : \overline{\mathcal{P}} \to \mathcal{H}_N^{r,S^1}$ is contained in $\mathcal{H}_{N,\mathrm{reg},t_0}^{r,S^1}$. It then follows from the Sard–Smale theorem that the latter is dense. Given such a regular value H, for any $(H, \gamma, \lambda, a) \in \mathrm{pr}^{-1}(H)$ we see that

$$\forall h \in T_H\mathcal{H}_N^{r,S^1},\ \exists(\zeta,\ell,b),\qquad \begin{pmatrix} -X_h \\ -\int_{S^1}\vec{\nabla}_\lambda h\end{pmatrix} + D_{(\gamma,\lambda)}(\zeta,\ell) + b\begin{pmatrix}\dot{\gamma}\\ X\end{pmatrix} = 0.$$

Since the restriction of $d\bar{f}$ to $T_H\mathcal{H}_N^{r,S^1} \oplus 0 \oplus 0 \oplus \mathbb{R}$ is surjective, we deduce that the cokernel of $D_{(\gamma,\lambda)}$ has dimension at most 1 for any $(H, \gamma, \lambda, a) \in \mathrm{pr}^{-1}(H)$ and in particular for any $(\gamma, \lambda) \in \mathcal{P}(H)$. On the other hand, since $D_{(\gamma,\lambda)}$ is self-adjoint, the same holds for $\dim \ker D_{(\gamma,\lambda)}$. But the latter is at least 1 by S^1-symmetry, which proves the claim. □

Let $d > 0$ be small enough (for a fixed $H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1}$, one can take $d > 0$ to be smaller than the minimal spectral gap of the asymptotic operators D_p, $p \in \mathcal{P}(H)$), and fix $1 < p < \infty$. Given $\overline{p}, \underline{p} \in \mathcal{P}(H)$ and $(u, \lambda) \in \widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$, we define

$$\mathcal{W}^{1,p,d} := W^{1,p}(u^*T\widehat{W}; e^{d|s|}ds\,d\theta) \oplus W^{1,p}(\lambda^*TS^{2N+1}; e^{d|s|}ds) \oplus V_{\overline{p}} \oplus V_{\underline{p}},$$
$$\mathcal{L}^{p,d} := L^p(u^*T\widehat{W}; e^{d|s|}ds\,d\theta) \oplus L^p(\lambda^*TS^{2N+1}; e^{d|s|}ds).$$

Here we identify $V_{\overline{p}}$, $V_{\underline{p}}$ with the 1-dimensional spaces generated by the sections $\beta(s)(\dot{\overline{\gamma}}, X_{\overline{\lambda}})$, respectively $\beta(-s)(\dot{\underline{\gamma}}, X_{\underline{\lambda}})$ of $u^*T\widehat{W} \oplus \lambda^*TS^{2N+1}$. For this identification, we denote by $X_{\overline{\lambda}}$, $X_{\underline{\lambda}}$ the values of the infinitesimal generator of the S^1-action on S^{2N+1} at the points $\overline{\lambda}$, respectively $\underline{\lambda}$, and choose a cut-off function $\beta : \mathbb{R} \to [0, 1]$ which is equal to 1 near $-\infty$, and vanishes near $+\infty$.

Proposition 5.2. *Assume that $S_{\overline{p}}, S_{\underline{p}} \subset \mathcal{P}(H)$ are nondegenerate. For any (u,λ) in $\widehat{\mathcal{M}}(S_{\overline{p}}, S_{\underline{p}}; H, J, g)$ the operator*

$$D_{(u,\lambda)} : \mathcal{W}^{1,p,d} \to \mathcal{L}^{p,d}$$

is Fredholm.

Proof. Let $\mathcal{W}^{1,p}$ and $\mathcal{L}^p$ be defined as $\mathcal{W}^{1,p,d}$ and $\mathcal{L}^{p,d}$ above, with $d = 0$ and without taking into account the direct summands $V_{\overline{p}}$, $V_{\underline{p}}$. Let $\widetilde{D}_{(u,\lambda)} : \mathcal{W}^{1,p} \to \mathcal{L}^p$ be the operator obtained by conjugating with $e^{(d/p)|s|}$ the restriction of $D_{(u,\lambda)}$ to $W^{1,p}(u^*T\widehat{W}; e^{d|s|}ds\,d\theta) \oplus W^{1,p}(\lambda^*TS^{2N+1}; e^{d|s|}ds)$. Then $\widetilde{D}_{(u,\lambda)}$ has nondegenerate asymptotics, hence it is Fredholm by Theorem 2.5 (the asymptotic operator at $-\infty$ is $\widetilde{D}_{\overline{p}} = D_{\overline{p}} + (d/p)\mathbb{1}$, and the asymptotic operator at $+\infty$ is $\widetilde{D}_{\underline{p}} = D_{\underline{p}} - (d/p)\mathbb{1}$). It follows that the operator $D_{(u,\lambda)}$ is also Fredholm. □

6. Unique continuation in the S^1-invariant case

The purpose of this section is to prove a unique continuation result which is slightly more general than the one in Section 3. This is needed in the proof of Theorem A(b).

Notation. We denote by X the infinitesimal generator of the S^1-action on the parameter space $\Lambda = S^{2N+1}$. We denote by $\mathcal{H}_N^{S^1} \subset \mathcal{H}_{S^{2N+1}}$ the set of admissible Hamiltonian families $H : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$ which are invariant with respect to the diagonal S^1-action on $S^1 \times S^{2N+1}$. We denote by $\mathcal{J}_N^{S^1}$ the set of pairs (J, g) consisting of an S^1-invariant admissible S^{2N+1}-family of almost complex structures J on $\widehat{W}$ and of an S^1-invariant Riemannian metric g on S^{2N+1}.

Definition 6.1. Given $H \in \mathcal{H}_N^{S^1}$, we define $\widetilde{H} : \widehat{W} \times S^{2N+1} \to \mathbb{R}$ by

$$\widetilde{H}(x,\lambda) := H(0,x,\lambda).$$

Given an S^1-invariant almost complex structure $J = (J_\lambda^\theta)$, we define

$$\widetilde{J}_\lambda(x) := J_\lambda^0(x).$$

Given maps $u : \mathbb{R} \times S^1 \to \widehat{W}$, $\lambda : \mathbb{R} \to S^{2N+1}$, we define maps $\widetilde{\lambda} : \mathbb{R} \times S^1 \to S^{2N+1}$ and $\widetilde{U} : \mathbb{R} \times S^1 \to \widehat{W} \times S^{2N+1}$ by

$$\widetilde{\lambda}(s,\theta) := (-\theta)\cdot\lambda(s), \qquad \widetilde{U}(s,\theta) := (u(s,\theta), \widetilde{\lambda}(s,\theta)).$$

It follows from the definitions that the pair (u,λ) satisfies equations (2.5–2.7) if and only if $\widetilde{U} = (u, \widetilde{\lambda})$ satisfies the equations

$$\partial_s u + \widetilde{J}_{\widetilde{\lambda}}(\partial_\theta u - X_{\widetilde{H}_{\widetilde{\lambda}}}(u)) = 0, \tag{6.1}$$

$$\partial_s \widetilde{\lambda} - \int_{S^1} \tau_* \vec{\nabla}_\lambda \widetilde{H}(\widetilde{U}(s,\theta+\tau))\,d\tau = 0, \tag{6.2}$$

$$\partial_\theta \widetilde{\lambda} + X_{\widetilde{\lambda}} = 0, \tag{6.3}$$

and

$$\lim_{s\to-\infty}\widetilde{U}(s,\theta)=(\overline{\gamma}(\theta),(-\theta)\cdot\overline{\lambda}),\qquad \lim_{s\to+\infty}\widetilde{U}(s,\theta)=(\underline{\gamma}(\theta),(-\theta)\cdot\underline{\lambda}) \tag{6.4}$$

for all $\theta\in S^1$. We note that equations (6.1–6.3) are independent of the variables s and θ.

Proposition 6.2 (Unique continuation). *Let $h>0$ and $\widetilde{U}_i=(u_i,\widetilde{\lambda}_i):Z_h\to\widehat{W}\times S^{2N+1}$, $i=0,1$, be smooth functions satisfying* (6.1–6.3). *If $\widetilde{U}_0=\widetilde{U}_1$ on some nonempty open set $\mathcal{U}\subset Z_h$, then $\widetilde{U}_0=\widetilde{U}_1$ on Z_h.*

To prove Proposition 6.2, we need the following enhancement of Proposition 3.3.

Proposition 6.3. *Let $h>0$ and $\widetilde{U}=(u,\widetilde{\lambda}):Z_h\to\mathbb{C}^n\times\mathbb{R}^{2N+1}$ be C^∞-functions satisfying*

$$\begin{aligned}\partial_s u+J(s,\theta)\partial_\theta u+C(s,\theta)u+D(s,\theta)\lambda&=0,\\ \partial_s\widetilde{\lambda}+\int_{S^1}E(s,\theta,\tau)\widetilde{U}(s,\tau)\,d\tau&=0,\\ \partial_\theta\widetilde{\lambda}+F(s,\theta)\widetilde{\lambda}&=0,\end{aligned} \tag{6.5}$$

with C,D,E,F of class C^1, J of class C^∞ and $J^2=-\mathbb{1}$. Assume there exists a nonempty open set $\mathcal{U}\subset Z_h$ such that $\widetilde{U}(s,\theta)=(0,0)$ for all $(s,\theta)\in\mathcal{U}$. Then $\widetilde{U}\equiv(0,0)$ on Z_h.

Proof. We first remark that $\widetilde{U}$ must vanish on some strip $]s_0-\varepsilon,s_0+\varepsilon[\times S^1\subset Z_h$. More precisely, let us choose $(s_0,\theta_0)\in\mathcal{U}$ and $\varepsilon>0$ such that $]s_0-\varepsilon,s_0+\varepsilon[\times]\theta_0-\varepsilon,\theta_0+\varepsilon[\subset\mathcal{U}$. Then, for $s\in]s_0-\varepsilon,s_0+\varepsilon[$, we see that $\widetilde{\lambda}(s,\cdot)$ solves a linear ODE on S^1 and vanishes at θ_0, hence vanishes identically. Thus u solves $\partial_s u+J\partial_\theta u+Cu=0$ on $]s_0-\varepsilon,s_0+\varepsilon[\times S^1$, and therefore must also vanish identically by the standard unique continuation property [10, Theorem 2.2, Proposition 3.1]. In particular, $\widetilde{U}$ vanishes with all its derivatives along $\{s_0\}\times S^1$.

As in the proof of Proposition 3.3, we can assume without loss of generality that $J=i$. Let us denote

$$\widetilde{V}(s,\theta):=(u(s,\theta),\widetilde{\lambda}(s,\theta),0)\in\mathbb{C}^n\times\mathbb{C}^{2N+1}.$$

Then $\widetilde{V}$ satisfies an equation of the form

$$\partial_s\widetilde{V}+i\partial_\theta\widetilde{V}+A(s,\theta)\widetilde{V}+\int_{S^1}B(s,\theta,\tau)\widetilde{V}(s,\tau)\,d\tau=0,$$

with A,B of class C^1. As in Proposition 3.3, we infer an inequality

$$|\Delta\widetilde{V}|^2\le 8\pi K\left[|\widetilde{V}|^2+|\nabla\widetilde{V}|^2+\int_{S^1}|\widetilde{V}|^2\right].$$

The only difference with respect to Proposition 3.3 is that the function A_4 therein depends now on θ. However, it is still pointwise bounded and the same argument carries through.

The conclusion follows from Theorem 3.2. □

Proof of Proposition 6.2. The proof follows the same pattern as that of Proposition 3.5, and makes use of Proposition 6.3.

Let us assume without loss of generality that $\mathcal{U} = I_\delta \times I_\varepsilon$ for some $\delta, \varepsilon > 0$. Since $\widetilde{\lambda}_0(s,\cdot)$ and $\widetilde{\lambda}_1(s,\cdot)$ solve the same ODE on S^1 and coincide on I_ε, we infer that they coincide on S^1 for all $s \in I_\delta$. By the unique continuation property for the Floer equation [10, Proposition 3.1], we infer that $u_0 = u_1$ on the strip $I_\delta \times S^1$.

Let $I \subset I_h$ be the set of points s such that $\widetilde{U}_0 = \widetilde{U}_1$ on $\{s\} \times S^1$. Then I is nonempty (it contains I_δ), closed, and we must prove that it is open. Let $s_0 \in I$ be a point on the boundary of a connected component of I with nonempty interior, and denote $\gamma := \widetilde{U}_0(s_0,\cdot) = \widetilde{U}_1(s_0,\cdot)$. We consider a trivialization of $\gamma^*(T\widehat{W} \times TS^{2N+1})$ of the form $S^1 \times \mathbb{C}^n \times \mathbb{R}^{2N+1}$. Then, for s close to s_0, we can view $\widetilde{U}_0(s,\cdot)$ and $\widetilde{U}_1(s,\cdot)$ as taking values in $\mathbb{C}^n \times \mathbb{R}^{2N+1}$. The difference $\widetilde{U} := (u, \widetilde{\lambda}) := (u_0 - u_1, \widetilde{\lambda}_0 - \widetilde{\lambda}_1)$ satisfies an equation of the form (6.5) with smooth coefficients. The computation is similar to the one in [10, Proposition 3.1], and we just establish the second equation in (6.5). We have, for suitable matrices $\widehat{E}$ and E,

$$\begin{aligned}\partial_s \widetilde{\lambda}(s,\theta) &= \int_{S^1} (\tau-\theta)_*[\vec{\nabla}_\lambda \widetilde{H}(\widetilde{U}_0(s,\tau)) - \vec{\nabla}_\lambda \widetilde{H}(\widetilde{U}_1(s,\tau))]\,d\tau \\ &= \int_{S^1} (\tau-\theta)_* \widehat{E}(s,\tau)\widetilde{U}(s,\tau)\,d\tau = \int_{S^1} E(s,\theta,\tau)\widetilde{U}(s,\tau)\,d\tau.\end{aligned}$$

The conclusion follows from Proposition 6.3. □

7. Transversality in the S^1-invariant case

We prove in this section that transversality for the S^1-invariant Floer equations can be achieved within the following two classes of Hamiltonians.

A. **Generic Hamiltonians.** We require such Hamiltonians H to be admissible, regular, and to satisfy the following two conditions:
- for all $(\gamma,\lambda) \in \mathcal{P}(H)$, γ is a simple embedded curve;
- for all distinct elements $(\gamma_1,\lambda_1), (\gamma_2,\lambda_2) \in \mathcal{P}(H)$ we have $\gamma_1 \neq \gamma_2$.

We denote the class of generic Hamiltonians by $\mathcal{H}_{\text{gen}}$.

B. **Split Hamiltonians.** We require such Hamiltonians to be admissible and of the form $K(x) + f(\lambda)$, with K being C^2-small on W. Here f is S^1-invariant and K has either constant and nondegenerate 1-periodic orbits, or nonconstant and transversally nondegenerate ones.

We denote the class of split Hamiltonians by $\mathcal{H}_{\text{split}}$. We denote

$$\mathcal{H}_* := \mathcal{H}_{\text{gen}} \cup \mathcal{H}_{\text{split}}.$$

Definition 7.1. An admissible Hamiltonian $H \in \mathcal{H}_N^{S^1}$ is called *strongly admissible* if the following two conditions hold:

1. For every $(\gamma,\lambda) \in \mathcal{P}(H)$ such that γ is not constant, we have

$$X^\theta_{H_\lambda}(\gamma(\theta)) \neq 0, \qquad \forall \theta \in S^1.$$

2. For every $(\gamma, \lambda) \in \mathcal{P}(H)$ such that γ is constant (equal to $x \in \widehat{W}$), there exists a neighbourhood $\mathcal{U}$ of $\{x\} \times (S^1 \cdot \lambda)$ in $\widehat{W} \times S^{2N+1}$ such that $H(\theta, x', \lambda') = K(x') + f(\lambda')$ for all $\theta \in S^1$ and $(x', \lambda') \in \mathcal{U}$. Moreover, x is an isolated critical point of K.

We denote by $\mathcal{H}'$ the class of strongly admissible Hamiltonians.

We clearly have $\mathcal{H}_* \subset \mathcal{H}'$.

Definition 7.2. Given $H \in \mathcal{H}'$, an almost complex structure $J \in \mathcal{J}_N^{S^1}$ is called *adapted to* H if the following hold:

1. For every $(\gamma, \lambda_0) \in \mathcal{P}(H)$, we have
$$[J_\lambda^\theta X_{H_\lambda}^\theta, X_{H_\lambda}^\theta](\gamma(\theta)) \notin \mathrm{Span}(J_\lambda^\theta X_{H_\lambda}^\theta, X_{H_\lambda}^\theta), \qquad \forall \theta \in S^1,\ \lambda \in S^1 \cdot \lambda_0.$$
2. For every $(\gamma, \lambda_0) \in \mathcal{P}(H)$ such that γ is constant (equal to $x \in \widehat{W}$), there exists a neighbourhood $\mathcal{U}$ of $\{x\} \times (S^1 \cdot \lambda)$ in $\widehat{W} \times S^{2N+1}$ such that J_λ^θ is independent of θ and λ on $\mathcal{U}$, i.e. $J_\lambda^\theta(x') = J(x')$ for all $(x', \lambda) \in \mathcal{U}$ and $\theta \in S^1$.

We denote by $\mathcal{J}'(H) \subset \mathcal{J}_N^{S^1}$ the set of almost complex structures adapted to H.

Remark 7.3. The set $\mathcal{J}'(H)$ is nonempty for every choice of strongly admissible Hamiltonian H. This is proved by a genericity argument: given a nonzero vector field X along a curve, one can choose generically a nonzero vector field Y which is linearly independent of X along the same curve, and such that the distribution spanned by X and Y is noninvolutive and symplectic.

We denote
$$\mathcal{H}_*\mathcal{J}' := \left\{ (H, J, g) : \begin{array}{c} H \in \mathcal{H}_*,\ (J, g) \in \mathcal{J}'(H), \\ J \text{ admissible, cylindrical for } t \geq 1, \\ \text{independent of } (\theta, \lambda) \text{ if } H \in \mathcal{H}_{\mathrm{split}} \end{array} \right\},$$
and
$$\mathcal{H}\mathcal{J}' := \{(H, J, g) : H \in \mathcal{H}',\ (J, g) \in \mathcal{J}'(H)\},$$
so that $\mathcal{H}_*\mathcal{J}' \subset \mathcal{H}\mathcal{J}'$.

Let $H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1}$. A pair $(J, g) \in \mathcal{J}_N^{S^1}$ is called *regular for* H if the operator $D_{(u,\lambda)}$ is surjective for any $\overline{p}, \underline{p} \in \mathcal{P}(H)$ and any $(u, \lambda) \in \widehat{\mathcal{M}}(\overline{p}, \underline{p}; H, J, g)$. We denote the set of such regular pairs by $\mathcal{J}_{N,\mathrm{reg}}^{S^1}(H)$.

The next result proves Theorem A(b).

Theorem 7.4. *There exists an open subset $\mathcal{H}\mathcal{J}'_{\mathrm{reg}} \subset \mathcal{H}\mathcal{J}'$ which is dense in a neighbourhood of $\mathcal{H}_*\mathcal{J}' \subset \mathcal{H}\mathcal{J}'$ and consisting of triples (H, J, g) such that*
$$H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1}, \qquad (J, g) \in \mathcal{J}_{N,\mathrm{reg}}^{S^1}(H).$$

Remark 7.5 (on symplectic asphericity). The previous theorem can be rephrased by saying that we can achieve transversality within the special class of adapted almost complex structures, after possibly perturbing a Hamiltonian which is either generic (in the sense that it belongs to $\mathcal{H}_{\mathrm{gen}}$), or split (in the sense that it belongs to $\mathcal{H}_{\mathrm{split}}$). We would like to draw the reader's attention to the fact that, in the case of split Hamiltonians, our proof uses the assumption that W is symplectically aspherical and the Hamiltonian is C^2-small on W. For generic Hamiltonians, these assumptions are not used.

In this section we denote by X the infinitesimal generator of the S^1-action on S^{2N+1}. Also, we make extensive use of the notation $\widetilde{H}$, $\widetilde{J}$, $\widetilde{U}$ introduced in Definition 6.1, and of the fact that $(\widetilde{H}, \widetilde{J}, \widetilde{U}=(u, \widetilde{\lambda}))$ solve (6.1–6.4) if and only if $(H, J, U=(u, \lambda))$ solve (2.5–2.7).

Our first result is an analogue of Lemma 4.5. We recall the notation $V_h(s_0, \theta_0) :=]s_0 - h, s_0 + h[\times]\theta_0 - h, \theta_0 + h[$.

Lemma 7.6. *Let $H \in \mathcal{H}^{S^1}_{N,\mathrm{reg}} \cap \mathcal{H}'$ and $(J, g) \in \mathcal{J}'(H)$. Let $\overline{p} = (\overline{\gamma}, \overline{\lambda})$, $\underline{p} = (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$ and $\widetilde{U}_i = (u_i, \widetilde{\lambda}_i) : \mathbb{R} \times S^1 \to \widehat{W} \times S^{2N+1}$, $i = 0, 1$, be solutions of (6.1–6.4). Assume that, for some $(s_0, \theta_0) \in \mathbb{R} \times S^1$,*

$$\widetilde{U}_0(s_0, \theta_0) = \widetilde{U}_1(s_0, \theta_0) \quad \text{and} \quad du_0(s_0, \theta_0),\ d\widetilde{U}_1(s_0, \theta_0) \text{ are injective.}$$

Also assume there exists $h_0 > 0$ such that, for all $0 < h' \le h_0$, there exists $h > 0$ with the following property: for any $(s, \theta) \in V_h(s_0, \theta_0)$, there exists $(s', \theta') \in V_{h'}(s_0, \theta_0)$ such that

$$\widetilde{U}_0(s, \theta) = \widetilde{U}_1(s', \theta').$$

Then there exists a neighbourhood $\mathcal{U} \subset \widehat{W} \times S^{2N+1}$ of $\underline{\gamma}(S^1) \times (S^1 \cdot \underline{\lambda})$, independent of $\widetilde{U}_0$ and $\widetilde{U}_1$, such that, if $\widetilde{U}_0(s_0, \theta_0) \in \mathcal{U}$, the above assumptions imply $\widetilde{U}_0 = \widetilde{U}_1$ (the same holds for the asymptote at $-\infty$).

Proof. By Proposition 6.2, it is enough to prove that $\widetilde{U}_0$ and $\widetilde{U}_1$ coincide on some open neighbourhood of (s_0, θ_0). Let us choose $0 < h' \le h_0$ small enough so that $\widetilde{U}_1 : V_{h'}(s_0, \theta_0) \to \widehat{W} \times S^{2N+1}$ is an embedding. Upon further diminishing the corresponding $h > 0$, we can assume that $u_0 : V_h(s_0, \theta_0) \to \widehat{W}$ is also an embedding.

By assumption we have $\widetilde{U}_0(V_h(s_0, \theta_0)) \subset \widetilde{U}_1(V_{h'}(s_0, \theta_0))$. We can therefore define a smooth embedding $G := (\widetilde{U}_1)^{-1} \circ \widetilde{U}_0 : V_h(s_0, \theta_0) \to V_{h'}(s_0, \theta_0)$. Moreover, we know by assumption that $(s_0, \theta_0) = G(s_0, \theta_0) \in \mathrm{im}(G)$. There exists therefore $0 < h'' < h'$ such that $V_{h''}(s_0, \theta_0) \subset \mathrm{im}(G)$. By the implicit function theorem, we obtain a smooth embedding $F := G^{-1} := (\phi, \psi) : V_{h''}(s_0, \theta_0) \to V_h(s_0, \theta_0)$. It follows from the definition that

$$\widetilde{U}_1(s, \theta) = \widetilde{U}_0(\phi(s, \theta), \psi(s, \theta))$$

for all $(s,\theta) \in V_{h''}(s_0,\theta_0)$. Substituting this relation into (6.1) for u_1 we obtain

$$\begin{aligned}
0 &= \partial_s u_1(s,\theta) + \widetilde{J}_{\widetilde{\lambda}_1}(u_1)(\partial_\theta u_1(s,\theta) - X_{\widetilde{H}_{\widetilde{\lambda}_1}}(u_1)) \\
&= \partial_s u_0(F)\partial_s\phi + \partial_\theta u_0(F)\partial_s\psi \\
&\quad + \widetilde{J}_{\widetilde{\lambda}_0(F)}(u_0(F))[\partial_s u_0(F)\partial_\theta\phi + \partial_\theta u_0(F)\partial_\theta\psi - X_{\widetilde{H}_{\widetilde{\lambda}_0(F)}}(u_0(F))] \\
&= \partial_s u_0(F)\partial_s\phi + \partial_\theta u_0(F)\partial_s\psi + \widetilde{J}_{\widetilde{\lambda}_0(F)}[\widetilde{J}_{\widetilde{\lambda}_0(F)}(-\partial_\theta u_0 + X_{\widetilde{H}_{\widetilde{\lambda}_0(F)}})\partial_\theta\phi \\
&\quad + (\widetilde{J}_{\widetilde{\lambda}_0(F)}\partial_s u_0 + X_{\widetilde{H}_{\widetilde{\lambda}_0(F)}})\partial_\theta\psi - X_{\widetilde{H}_{\widetilde{\lambda}_0(F)}}] \\
&= (\partial_s\phi - \partial_\theta\psi)\partial_s u_0(F) + (\partial_s\psi + \partial_\theta\phi)\partial_\theta u_0(F) - \partial_\theta\phi\, X_{\widetilde{H}_{\widetilde{\lambda}_0(F)}}(u_0(F)) \\
&\quad - (1-\partial_\theta\psi)\,\widetilde{J}_{\widetilde{\lambda}_0(F)}(u_0(F))X_{\widetilde{H}_{\widetilde{\lambda}_0(F)}}(u_0(F)). \qquad (7.1)
\end{aligned}$$

The third equality uses the Floer equation (6.1) for $(u_0, \widetilde{\lambda}_0)$.

By Definition 7.2, we can choose a neighbourhood $\mathcal{U} \subset \widehat{W} \times S^{2N+1}$ of $\underline{\gamma}(S^1) \times (S^1 \cdot \underline{\lambda})$ such that

$$[\widetilde{J}_\lambda X_{\widetilde{H}_\lambda}, X_{\widetilde{H}_\lambda}](x) \notin \operatorname{Span}(\widetilde{J}_\lambda(x)X_{\widetilde{H}_\lambda}(x), X_{\widetilde{H}_\lambda}(x))$$

for all $(x,\lambda) \in \mathcal{U}$.

Up to further diminishing $h > 0$, we can assume that $\widetilde{U}_0(V_h(s_0,\theta_0)) \subset \mathcal{U}$. We now claim that the four vectors $\partial_s u_0$, $\partial_\theta u_0$, $X_{\widetilde{H}_{\widetilde{\lambda}_0}}(u_0)$, $\widetilde{J}_{\widetilde{\lambda}_0}(u_0)X_{\widetilde{H}_{\widetilde{\lambda}_0}}(u_0)$ are linearly independent on an open dense subset of $V_h(s_0,\theta_0)$. This follows from the argument in [10, Lemma 7.7]. More precisely, assume by contradiction the existence of a nonempty open subset $\Omega \subset V_h(s_0,\theta_0)$ such that these four vectors are linearly dependent on Ω.

Let us first use the assumption of strong admissibility on H. Since u_0 is an embedding on $V_h(s_0,\theta_0)$, we can further assume, after slightly moving the base point (s_0,θ_0), that $u_0(V_h(s_0,\theta_0))$ does not intersect the geometric image of $\underline{\gamma}$. Also, by assumption, $\widetilde{\lambda}_0$ is close to $S^1 \cdot \underline{\lambda}$, and therefore $X_{\widetilde{H}_{\widetilde{\lambda}_0}}(u_0) \neq 0$ on $V_h(s_0,\theta_0)$.

On the other hand, by assumption the vectors $\partial_s u_0$ and $\partial_\theta u_0$ are linearly independent on $V_h(s_0,\theta_0)$. Let us use the shorthand notation $\widetilde{J} = \widetilde{J}_{\widetilde{\lambda}_0}$ and $X_{\widetilde{H}} = X_{\widetilde{H}_{\widetilde{\lambda}_0}}$. Since $\partial_\theta u_0 = \widetilde{J}\partial_s u_0 + X_{\widetilde{H}}(u_0)$, the linear dependence of the above four vectors on Ω is equivalent to the linear dependence of $\partial_s u_0, \widetilde{J}\partial_s u_0, X_{\widetilde{H}}, \widetilde{J}X_{\widetilde{H}}$. This in turn implies that $\partial_s u_0 \in \operatorname{Span}(\widetilde{J}X_{\widetilde{H}}, X_{\widetilde{H}})$, i.e. there exist smooth functions $a, b : \Omega \to \mathbb{R}$ such that

$$\partial_s u_0 = a\widetilde{J}X_{\widetilde{H}}(u_0) + bX_{\widetilde{H}}(u_0).$$

From $\partial_\theta u_0 = \widetilde{J}\partial_s u_0 + X_{\widetilde{H}}$ we also obtain

$$\partial_\theta u_0 = b\widetilde{J}X_{\widetilde{H}}(u_0) + (1-a)X_{\widetilde{H}}(u_0).$$

We now use the fact that $[\partial_s u_0, \partial_\theta u_0] = 0$ on Ω to obtain

$$(a^2+b^2-a)[\widetilde{J}X_{\widetilde{H}}, X_{\widetilde{H}}] = (\partial_\theta a - \partial_s b)\widetilde{J}X_{\widetilde{H}} + (\partial_s a + \partial_\theta b)X_{\widetilde{H}}. \qquad (7.2)$$

Note that the linear independence of $\partial_s u_0$ and $\partial_\theta u_0$ is equivalent to the condition $a^2 + b^2 - a \neq 0$. We infer that, for all $(s, \theta) \in \Omega$,

$$[\widetilde{J}_{\widetilde{\lambda}_0(s,\theta)} X_{\widetilde{H}_{\widetilde{\lambda}_0(s,\theta)}}, X_{\widetilde{H}_{\widetilde{\lambda}_0(s,\theta)}}](u_0(s,\theta)) \in \mathrm{Span}(\widetilde{J}_{\widetilde{\lambda}_0(s,\theta)} X_{\widetilde{H}_{\widetilde{\lambda}_0(s,\theta)}}, X_{\widetilde{H}_{\widetilde{\lambda}_0(s,\theta)}}).$$

This contradicts our choice of $\mathcal{U}$. We have thus proved that the four vectors $\partial_s u_0$, $\partial_\theta u_0$, $X_{\widetilde{H}_{\widetilde{\lambda}_0}}(u_0)$, $\widetilde{J}_{\widetilde{\lambda}_0}(u_0) X_{\widetilde{H}_{\widetilde{\lambda}_0}}(u_0)$ are linearly independent on an open dense subset $\mathcal{V} \subset V_h(s_0, \theta_0)$.

Since $F : V_{h''}(s_0, \theta_0) \to V_h(s_0, \theta_0)$ is an embedding, we infer that $F^{-1}(\mathcal{V})$ is open and nonempty. Equation (7.1) now implies that $\partial_s \phi - \partial_\theta \psi = 0$, $\partial_s \psi + \partial_\theta \phi = 0$, $\partial_\theta \phi = 0$, and $1 - \partial_\theta \psi = 0$, so that $F(s, \theta) = (s + \overline{s}, \theta + \overline{\theta})$ on $F^{-1}(\mathcal{V})$ for suitable constants $\overline{s}$ and $\overline{\theta}$. Since $F(s_0, \theta_0) = (s_0, \theta_0)$, we must have $\overline{s} = 0$ and $\overline{\theta} = 0$, and therefore $\widetilde{U}_1 = \widetilde{U}_0$ on $F^{-1}(\mathcal{V})$. This concludes the proof. □

The next lemma is the analogue of Lemma 7.6 in [10].

Lemma 7.7. *Let $\widetilde{U} = (u, \widetilde{\lambda})$ be a solution of* (6.1–6.4)*, and assume $\partial_s \widetilde{U} \not\equiv (0, 0)$.*

(i) *If $du \not\equiv 0$, the set of points $(s, \theta) \in \mathbb{R} \times S^1$ such that the vectors*

$$\partial_s \widetilde{U}(s, \theta) \quad \textit{and} \quad \partial_\theta \widetilde{U}(s, \theta) \tag{7.3}$$

are linearly independent is open and dense in $\{(s, \theta) : du(s, \theta) \neq 0\}$.

(ii) *If $du \equiv 0$, the above vectors are linearly independent on $\mathbb{R} \times S^1$.*

Remark 7.8. The key point in the statement is that $\partial_s \widetilde{U}$ and $\partial_\theta \widetilde{U}$ lie in the kernel of the operator which linearizes equations (6.1–6.3), since the latter are independent of s and θ. Equivalently, $(\partial_s u, \dot{\lambda})$ and $(\partial_\theta u, -X)$ lie in the kernel of the linearized operator $D_{(u,\lambda)}$.

Proof of Lemma 7.7. Openness is clear by continuity of $\partial_s \widetilde{U}$ and $\partial_\theta \widetilde{U}$. To prove density, we argue by contradiction and assume that $\partial_s \widetilde{U}$ and $\partial_\theta \widetilde{U}$ are linearly dependent on some open set $\Omega \subset \mathbb{R} \times S^1$. By Proposition 4.3(i), the set of points where $\partial_s \widetilde{U} \neq (0, 0)$ is open and dense in $\mathbb{R} \times S^1$, so that we can assume without loss of generality that $\partial_s \widetilde{U} \neq (0, 0)$ on Ω. We thus find a smooth function $\mu : \Omega \to \mathbb{R}$ such that $\partial_\theta \widetilde{U}(s, \theta) = \mu(s, \theta) \partial_s \widetilde{U}(s, \theta)$ for all $(s, \theta) \in \Omega$. More explicitly,

$$(\partial_\theta u(s, \theta), (-\theta)_* X_{\lambda(s)}) = \mu(s, \theta)(\partial_s u(s, \theta), (-\theta)_* \partial_s \lambda(s)).$$

Since $X \neq 0$, we infer $\mu \neq 0$. Since $X_{\lambda(s)}$ and $\partial_s \lambda(s)$ do not depend on θ, we infer that the same holds for μ, so that

$$\mu(s, \theta) = \mu(s).$$

In the computations that follow we denote total derivatives by d, and partial derivatives by ∂. We compute

$$\begin{aligned} d_s(\widetilde{H} \circ \widetilde{U}) &= \partial_x \widetilde{H} \cdot \partial_s u + \partial_\lambda \widetilde{H} \cdot \partial_s \widetilde{\lambda} = \omega(X_{\widetilde{H}}, \partial_s u) + \partial_\lambda \widetilde{H} \cdot \partial_s \widetilde{\lambda} \\ &= \omega(-\widetilde{J} \partial_s u + \partial_\theta u, \partial_s u) + \partial_\lambda \widetilde{H} \cdot \partial_s \widetilde{\lambda} = |\partial_s u|^2 + \partial_\lambda \widetilde{H} \cdot \partial_s \widetilde{\lambda}. \end{aligned}$$

For the third equality we used the Floer equation (6.1) for u. We also have

$$d_\theta(\widetilde{H}\circ\widetilde{U}) = d\widetilde{H}\cdot\partial_\theta\widetilde{U} = \mu\, d\widetilde{H}\cdot\partial_s\widetilde{U} = \frac{1}{\mu}|\partial_\theta u|^2 + \partial_\lambda\widetilde{H}\cdot\partial_\theta\widetilde{\lambda}.$$

We now compute mixed second derivatives:

$$\begin{aligned} d_\theta d_s(\widetilde{H}\circ\widetilde{U}) &= 2\langle\nabla_\theta\partial_s u, \partial_s u\rangle + \omega(\partial_s u, (\partial_\lambda\widetilde{J}\cdot\partial_\theta\widetilde{\lambda})\cdot\partial_s u) + \partial_\theta(\partial_\lambda\widetilde{H}\cdot\partial_s\widetilde{\lambda})\\ d_s d_\theta(\widetilde{H}\circ\widetilde{U}) &= -\frac{\mu'}{\mu^2}|\partial_\theta u|^2 + \frac{2}{\mu}\langle\nabla_s\partial_\theta u, \partial_\theta u\rangle + \frac{1}{\mu}\omega(\partial_\theta u, (\partial_\lambda\widetilde{J}\cdot\partial_s\widetilde{\lambda})\cdot\partial_\theta u)\\ &\quad + \partial_s(\partial_\lambda\widetilde{H}\cdot\partial_\theta\widetilde{\lambda}). \end{aligned}$$

The equality $d_\theta d_s(\widetilde{H}\circ\widetilde{U}) = d_s d_\theta(\widetilde{H}\circ\widetilde{U})$ implies $(\mu'/\mu^2)|\partial_\theta u|^2 = 0$. Indeed,

$$2\langle\nabla_\theta\partial_s u, \partial_s u\rangle = 2\langle\nabla_s\partial_\theta u, \partial_s u\rangle = \frac{2}{\mu}\langle\nabla_s\partial_\theta u, \partial_\theta u\rangle$$

because $\nabla_\theta\partial_s u = \nabla_s\partial_\theta u$, and we have

$$\omega(\partial_s u, (\partial_\lambda\widetilde{J}\cdot\partial_\theta\widetilde{\lambda})\cdot\partial_s u) = \frac{1}{\mu}\omega(\partial_\theta u, (\partial_\lambda\widetilde{J}\cdot\partial_s\widetilde{\lambda})\cdot\partial_\theta u)$$

because $\partial_\theta u = \mu\partial_s u$, $\partial_\theta\widetilde{\lambda} = \mu\partial_s\widetilde{\lambda}$; similarly,

$$\begin{aligned} \partial_\theta(\partial_\lambda\widetilde{H}\cdot\partial_s\widetilde{\lambda}) &= \nabla_{\partial_\theta\widetilde{\lambda}}(\partial_\lambda\widetilde{H})\cdot\partial_s\widetilde{\lambda} + (\partial_\lambda\widetilde{H})\cdot\nabla_\theta\partial_s\widetilde{\lambda} + \partial_x\partial_\lambda\widetilde{H}\cdot(\partial_\theta u, \partial_s\widetilde{\lambda})\\ &= \nabla_{\partial_s\widetilde{\lambda}}(\partial_\lambda\widetilde{H})\cdot\partial_\theta\widetilde{\lambda} + (\partial_\lambda\widetilde{H})\cdot\nabla_s\partial_\theta\widetilde{\lambda} + \partial_x\partial_\lambda\widetilde{H}\cdot(\partial_s u, \partial_\theta\widetilde{\lambda})\\ &= \partial_s(\partial_\lambda\widetilde{H}\cdot\partial_\theta\widetilde{\lambda}). \end{aligned}$$

Thus

$$(\mu'/\mu^2)|\partial_\theta u|^2 = \mu'|\partial_s u|^2 = 0. \tag{7.4}$$

We now prove (i). In this case we have $\partial_s u \neq 0$ or $\partial_\theta u \neq 0$ on Ω. Then (7.4) implies $\mu' = 0$, so that μ is constant on Ω. We now claim that

$$\widetilde{U}(s-\mu\tau, \theta+\tau) = \widetilde{U}(s,\theta) \tag{7.5}$$

for τ sufficiently close to 0 and (s,θ) in some nonempty open subset of Ω. Indeed, this clearly holds for $\tau = 0$ and the derivative of the left hand side with respect to τ is given by

$$-\mu\partial_s\widetilde{U} + \partial_\theta\widetilde{U} = 0.$$

Both sides in (7.5) define solutions of (6.1–6.4), and they must coincide by the unique continuation property (Proposition 6.2). In particular, their asymptotes must also coincide. This leads to a contradiction, since the asymptote (say at $-\infty$) of the left term in (7.5) is $(-\tau)\cdot\overline{p}$, which is different from the asymptote $\overline{p}$ for small $\tau \neq 0$, due to the fact that S^1 acts freely on S^{2N+1}.

We now prove (ii). In this case we have $\partial_s u = \partial_\theta u = 0$ on $\mathbb{R}\times S^1$, so that $u(s,\theta) \equiv x$. We have

$$0 = \frac{d}{d\theta}\int_{S^1} \widetilde{H}(x, \widetilde{\lambda}(s,\theta+\tau))\,d\tau = \int_{S^1} \vec{\nabla}_\lambda \widetilde{H}(x, \widetilde{\lambda}(s,\theta+\tau)) \cdot X_{\widetilde{\lambda}(s,\theta+\tau)}\,d\tau$$
$$= \int_{S^1} \tau_* \vec{\nabla}_\lambda \widetilde{H}(x, \widetilde{\lambda}(s,\theta+\tau))\,d\tau \cdot X_{\widetilde{\lambda}(s,\theta)}.$$

For the last equality, we used that $X_{\widetilde{\lambda}(s,\theta)} = \tau_* X_{\widetilde{\lambda}(s,\theta+\tau)}$. Assuming by contradiction that $-X_{\widetilde{\lambda}(s,\theta)} = \mu(s)\partial_s\widetilde{\lambda}(s,\theta)$ at some point $(s,\theta) \in \mathbb{R}\times S^1$, we obtain $0 = \partial_s\widetilde{\lambda}(s,\theta)\cdot X_{\widetilde{\lambda}(s,\theta)} = -(1/\mu(s))\|X_{\widetilde{\lambda}(s,\theta)}\|^2$, which is impossible. □

Definition 7.9. Let $H \in \mathcal{H}_N^{S^1}$. Given maps $u : \mathbb{R}\times S^1 \to \widehat{W}$ and $\lambda : \mathbb{R} \to S^{2N+1}$, denote $\widetilde{U} := (u, \widetilde{\lambda})$ as in Definition 6.1, and assume that $\widetilde{U}$ satisfies the asymptotic conditions (6.4) for $(\overline{\gamma}, \overline{\lambda}), (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$. A point $(s_0,\theta_0) \in \mathbb{R}\times S^1$ is called *injective* if

$$\widetilde{U}^{-1}(\widetilde{U}(s_0,\theta_0)) = \{(s_0,\theta_0)\}, \quad d\widetilde{U}(s_0,\theta_0) \text{ is injective}$$

and

$$\widetilde{U}(s_0,\theta_0) \neq (\overline{\gamma}(\theta), (-\theta)\cdot\overline{\lambda}), \quad \widetilde{U}(s_0,\theta_0) \neq (\underline{\gamma}(\theta), (-\theta)\cdot\underline{\lambda}), \quad \forall \theta \in S^1. \tag{7.6}$$

We denote the set of injective points by $R(\widetilde{U})$.

Proposition 7.10. *Let $H \in \mathcal{H}_{N,\mathrm{reg}}^{S^1} \cap \mathcal{H}'$ and $(J,g) \in \mathcal{J}'(H)$. Let $\overline{p}, \underline{p} \in \mathcal{P}(H)$ and $\widetilde{U} = (u,\widetilde{\lambda}) : \mathbb{R}\times S^1 \to \widehat{W}\times S^{2N+1}$ be a solution of* (6.1–6.4) *satisfying $\partial_s\widetilde{U} \not\equiv (0,0)$. For every $R > 0$, there exists a nonempty open set $\Omega \subset [R,\infty[\,\times S^1$ consisting of injective points.*

Proof. We proceed in several steps.

Step 1. *We prove that $R(\widetilde{U})$ is open.*

The second and third conditions in the definition of an injective point are clearly open, and we must prove that the first one is open as well. Arguing by contradiction, there exists $(s_0,\theta_0) \in R(\widetilde{U})$, a sequence $(s^\nu,\theta^\nu) \to (s_0,\theta_0)$, and a sequence $(s'^\nu,\theta'^\nu) \neq (s^\nu,\theta^\nu)$ such that $\widetilde{U}(s'^\nu,\theta'^\nu) = \widetilde{U}(s^\nu,\theta^\nu)$. Since $d\widetilde{U}(s_0,\theta_0)$ is injective, the sequence (s'^ν,θ'^ν) is bounded away from (s_0,θ_0). On the other hand, since $\widetilde{U}(s_0,\theta_0)$ does not belong to any of the asymptotes, it follows that the sequence s'^ν is bounded. Therefore (s'^ν,θ'^ν) has a subsequence converging to $(s'_0,\theta'_0) \neq (s_0,\theta_0)$. On the other hand, we must have $\widetilde{U}(s'_0,\theta'_0) = \widetilde{U}(s_0,\theta_0)$, which contradicts the assumption that $(s_0,\theta_0) \in R(\widetilde{U})$.

Step 2. *The set $R(\widetilde{U})$ is dense in*

$$\widetilde{U}^{-1}(\mathcal{U}) \cap \{(s,\theta) \in \mathbb{R}\times S^1 : du(s,\theta) \text{ injective}\},$$

where $\mathcal{U}$ is chosen as in Lemma 7.6.

Arguing by contradiction, we find a nonempty open set $\Omega \subset \mathbb{R} \times S^1$ consisting of noninjective points, and such that $du(s,\theta)$ is injective for all $(s,\theta) \in \Omega$. By Lemma 7.7(i) we can assume without loss of generality that $d\widetilde{U}$ is injective on Ω. This implies that the set of points (s,θ) such that condition (7.6) is satisfied is open and dense in Ω, so that we can assume without loss of generality that it is satisfied on $\overline{\Omega}$. Thus, the fact that points in Ω are noninjective is equivalent to

$$\forall (s,\theta) \in \Omega,\ \exists (s',\theta') \neq (s,\theta), \qquad \widetilde{U}(s',\theta') = \widetilde{U}(s,\theta).$$

By further shrinking Ω, we can assume that $\widetilde{U}|_\Omega$ is an embedding. Following [10, proof of Lemma 7.8] we denote

$$\Omega' := \{(s',\theta') \in \mathbb{R} \times S^1 \setminus \Omega : \widetilde{U}(s',\theta') \in \widetilde{U}(\Omega)\}.$$

Since condition (7.6) is satisfied on Ω, we infer the existence of some $T > 0$ such that $\Omega' \subset [-T,T] \times S^1$. We claim now that Ω' must contain a nonempty open set. To prove this, consider the map $\Phi : \Omega' \to \Omega$ defined by the commutative diagram

$$\begin{array}{ccc} \Omega' & \xrightarrow{\Phi} & \Omega \\ & \searrow^{\widetilde{U}} \quad \nearrow_{\widetilde{U}^{-1}} & \\ & \widetilde{U}(\Omega) & \end{array} \tag{7.7}$$

This extends to a smooth map on an open neighbourhood of Ω' (compose $\widetilde{U}$ in the target with a projection onto the submanifold $\widetilde{U}(\Omega)$, then apply $\widetilde{U}^{-1}$). If a point $(s,\theta) \in \Omega$ is a regular value of Φ, then $d\widetilde{U}(s',\theta')$ is injective for all $(s',\theta') \in \Omega'$ such that $\widetilde{U}(s',\theta') = \widetilde{U}(s,\theta)$. This implies that a regular value of Φ has only a finite number of preimages in Ω' (otherwise we could find an accumulation point of preimages, which would be a preimage at which the condition of injectivity of $d\widetilde{U}$ would be violated). By Sard's theorem, we can choose such a regular value (s_0,θ_0). Let $(s_1,\theta_1),\ldots,(s_N,\theta_N)$ be the other preimages of $\widetilde{U}(s_0,\theta_0)$. As in the proof of Proposition 4.3, one sees that for any $r > 0$ there exists $\delta > 0$ such that

$$\forall (s,\theta) \in V_{2\delta}(s_0,\theta_0),\ \exists (s',\theta') \in \bigcup_{j=1}^{N} V_r(s_j,\theta_j), \qquad \widetilde{U}(s,\theta) = \widetilde{U}(s',\theta')$$

(if this were not true, one would produce by a compactness argument in $[-T,T] \times S^1$ a preimage of $\widetilde{U}(s_0,\theta_0)$ distinct from (s_j,θ_j), $j = 0,\ldots,N$). Let us define

$$\Sigma_j := \{(s,\theta) \in \overline{V}_\delta(s_0,\theta_0) : \exists (s',\theta') \in \overline{V}_r(s_j,\theta_j),\ \widetilde{U}(s',\theta') = \widetilde{U}(s,\theta)\}.$$

Then Σ_j is closed and $\overline{V}_\delta(s_0,\theta_0) = \Sigma_1 \cup \cdots \cup \Sigma_N$. It follows from Baire's theorem that some Σ_j, say Σ_1, has nonempty interior. Then $\widetilde{U}(\mathrm{int}(\Sigma_1))$ is nonempty and open in $\widetilde{U}(\Omega)$, so that $\Sigma_1' := (\widetilde{U}|_{V_r(s_1,\theta_1)})^{-1}(\widetilde{U}(\mathrm{int}(\Sigma_1)))$ is nonempty and open in Ω', which proves our claim.

Let $(\overline{s}, \overline{\theta}) \in \mathrm{int}(\Sigma_1)$ and denote by $(\overline{s}_1, \overline{\theta}_1) \in \Sigma_1'$ the unique preimage of $\widetilde{U}(\overline{s}, \overline{\theta})$. Let $0 < r_1 < r$ be such that $V_{r_1}(\overline{s}_1, \overline{\theta}_1) \subset \Sigma_1'$, and $0 < \delta_1 < \delta$ be such that $V_{\delta_1}(\overline{s}, \overline{\theta}) \subset \Sigma_1$ and $\widetilde{U}(V_{\delta_1}(\overline{s}, \overline{\theta})) \subset \widetilde{U}(V_{r_1}(\overline{s}_1, \overline{\theta}_1))$. It follows from our construction that, for all $0 < h' \le r_1$, there exists $0 < h \le \delta_1$ such that $\widetilde{U}(V_h(\overline{s}, \overline{\theta})) \subset \widetilde{U}(V_{h'}(\overline{s}_1, \overline{\theta}_1))$. We can therefore apply Lemma 7.6 with $(s_0, \theta_0) := (\overline{s}, \overline{\theta})$, $\widetilde{U}_0 := \widetilde{U}$, $\widetilde{U}_1 := \widetilde{U}(\cdot + \overline{s}_1 - \overline{s}, \cdot + \overline{\theta}_1 - \overline{\theta})$, and $h_0 = r_1$. Since $\widetilde{U}(s_0, \theta_0) \in \mathcal{U}$, we obtain $\widetilde{U}_0 = \widetilde{U}_1$.

We can now get the desired contradiction as follows. We first note that, by construction, we have $(\overline{s}_1, \overline{\theta}_1) \neq (\overline{s}, \overline{\theta})$. Assume first that $\overline{\theta}_1 \neq \overline{\theta}$. Since $\lim_{s \to -\infty} \widetilde{U}(s, \theta) = (\overline{\gamma}(\theta), (-\theta) \cdot \overline{\lambda})$, we deduce from $\widetilde{U}_0 = \widetilde{U}_1$ that $\overline{\lambda} = (\overline{\theta} - \overline{\theta}_1) \cdot \overline{\lambda}$, a contradiction. Thus $\overline{\theta}_1 = \overline{\theta}$, so that $\overline{s}_1 \neq \overline{s}$. Then

$$\widetilde{U}(s, \theta) = \lim_{k \to \pm\infty} \widetilde{U}(s + k(\overline{s}_1 - \overline{s}), \theta) = (\overline{\gamma}(\theta), (-\theta) \cdot \overline{\lambda}),$$

so that $\partial_s \widetilde{U} \equiv (0, 0)$, a contradiction again. The proof of Step 2 is complete.

Step 3. *Assume there exists $R_0 > 0$ such that $du(s, \theta)$ is noninjective for all $s \ge R_0$ and $\theta \in S^1$. Assume that $\lim_{s \to \infty}(u(s, \theta), \lambda(s)) = (\underline{\gamma}(\theta), \underline{\lambda})$ and $\underline{\gamma}$ is nonconstant. Then $\partial_s u \equiv 0$ on $[R_0, \infty[\times S^1$ and $R(\widetilde{U})$ is dense in $[R_0, \infty[\times S^1$.*

Let us choose $R \ge R_0$ large enough so that $X^{\theta}_{H_{\lambda(s)}}(u(s, \theta)) \neq 0$ for all $s \ge R$ and $\theta \in S^1$. This is possible since, by assumption, the Hamiltonian H is strongly admissible (see Definition 7.1). As a consequence of the Floer equation for u, the vectors $\partial_s u$ and $\partial_\theta u$ cannot vanish simultaneously for $s \ge R$.

We first show that there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \partial_s u(s, \theta) + \beta \partial_\theta u(s, \theta) = 0 \tag{7.8}$$

for all $(s, \theta) \in]R, \infty[\times S^1$. By our assumption on du, this relation holds for some choice of smooth functions $\alpha, \beta :]R, \infty[\times S^1 \to \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$. Let us use the shorthand notation $\widetilde{J} := \widetilde{J}_{\widetilde{\lambda}}$, $X_{\widetilde{H}} := X_{\widetilde{H}_{\widetilde{\lambda}}}$, so that the Floer equation for u reads $\partial_s u + \widetilde{J} \partial_\theta u = \widetilde{J} X_{\widetilde{H}}$. We obtain

$$\partial_s u = \beta^2 \widetilde{J} X_{\widetilde{H}} - \alpha\beta X_{\widetilde{H}} \quad \text{and} \quad \partial_\theta u = -\alpha\beta \widetilde{J} X_{\widetilde{H}} + \alpha^2 X_{\widetilde{H}}.$$

Let us denote $a := \beta^2$, $b := -\alpha\beta$, so that $\partial_s u = a\widetilde{J} X_{\widetilde{H}} + b X_{\widetilde{H}}$, $\partial_\theta u = b\widetilde{J} X_{\widetilde{H}} + (1 - a) X_{\widetilde{H}}$, and $a^2 + b^2 - a = 0$. From $[\partial_s u, \partial_\theta u] = 0$ we obtain (see also (7.2))

$$0 = (\partial_\theta a - \partial_s b) \widetilde{J} X_{\widetilde{H}} + (\partial_s a + \partial_\theta b) X_{\widetilde{H}}.$$

By our choice of $R > 0$ we have $X_{\widetilde{H}} \neq 0$, hence the linear combination above must be trivial. The map $(b, a) :]R, \infty[\times S^1 \to \mathbb{C}$ is therefore holomorphic. On the other hand, its image lies on the circle $a^2 + b^2 - a = 0$, and this map must be constant. It then follows that α and β are constant as well.

By assumption, the asymptote $\underline{\gamma}$ is nonconstant. This implies that $\beta = 0$, as seen by passing to the limit $s \to \infty$ in (7.8). Thus $\partial_s u \equiv 0$ on $[R, \infty[\times S^1$.

We now prove that $\partial_s u \equiv 0$ on $[R_0, \infty[\times S^1$. Let $\mathcal{I} := \{R \in [R_0, \infty[\,:\, \partial_s u \equiv 0$ on $[R, \infty[\times S^1\}$. We have just showed $\mathcal{I} \neq \emptyset$, and clearly $\mathcal{I}$ is closed. On the other hand, the previous proof also shows that $\mathcal{I}$ is open (if $R \in \mathcal{I}$, then $u([R, \infty[\times S^1) = \underline{\gamma}(S^1)$, hence $u([R-\varepsilon, \infty[\times S^1)$ is close to $\gamma(S^1)$ for $\varepsilon > 0$ small enough, so that X_H is nonzero along the image of u and one can apply the previous argument). Thus $\mathcal{I} = [R_0, \infty[$.

We claim that $d\widetilde{U}(s, \theta)$ is injective for all $(s, \theta) \in [R_0, \infty[\times S^1$. Indeed, the component λ of (u, λ) solves a time-independent ODE on $[R_0, \infty[$, and since $\partial_s \widetilde{U} \not\equiv 0$, we infer that $\dot{\lambda}(s) \neq 0$ for all $s \geq R_0$. Since $\partial_\theta u \neq 0$ on $[R_0, \infty[\times S^1$, the claim follows.

We finally claim that the set $R(\widetilde{U})$ of injective points is open and dense in $[R_0, \infty[\times S^1$. Arguing by contradiction, we find a nonempty open set $\Omega \subset [R_0, \infty] \times S^1$ consisting of noninjective points. Since $\partial_\theta u \neq 0$ on Ω and $d\widetilde{U}$ is injective, it follows that the second and third conditions in the definition of an injective point are satisfied. Thus, the fact that points in Ω are noninjective is equivalent to

$$\forall (s, \theta) \in \Omega,\ \exists (s', \theta') \neq (s, \theta), \qquad \widetilde{U}(s', \theta') = \widetilde{U}(s, \theta).$$

Arguing verbatim as in Step 2, we find (after possibly shrinking Ω) an open set $\Omega' \subset \,]-\infty, R_0[\times S^1$, disjoint from Ω, and a diffeomorphism $\Phi := (\phi, \psi) : \Omega' \to \Omega$ such that $\widetilde{U}|_{\Omega'} = \widetilde{U}|_{\Omega} \circ \Phi$ (see diagram (7.7)). Substituting the relation $u(s, \theta) = u(\phi(s, \theta), \psi(s, \theta))$ for all $(s, \theta) \in \Omega'$ in the Floer equation (6.1) for u, we obtain, as in (7.1),

$$\begin{aligned} 0 = (\partial_s \phi - \partial_\theta \psi) \partial_s u(\Phi) + (\partial_s \psi + \partial_\theta \phi) \partial_\theta u(\Phi) - \partial_\theta \phi\, X_{\widetilde{H}_{\widetilde{\lambda}(\Phi)}}(u(\Phi)) \\ - (1 - \partial_\theta \psi)\, \widetilde{J}_{\widetilde{\lambda}(\Phi)}(u(\Phi)) X_{\widetilde{H}_{\widetilde{\lambda}(\Phi)}}(u(\Phi)). \end{aligned}$$

Using that $\partial_s u = 0$ and $\partial_\theta u = X_{\widetilde{H}_{\widetilde{\lambda}}} \neq 0$ on Ω, we obtain

$$\partial_s \psi = 0, \qquad \partial_\theta \psi = 1.$$

The same substitution in (6.3) for $\widetilde{\lambda}$ yields

$$\begin{aligned} 0 = \partial_\theta \widetilde{\lambda} + X_{\widetilde{\lambda}} &= \partial_s \widetilde{\lambda}(\Phi) \partial_\theta \phi + \partial_\theta \widetilde{\lambda}(\Phi) \partial_\theta \psi + X_{\widetilde{\lambda}(\Phi)} \\ &= \partial_s \widetilde{\lambda}(\Phi) \partial_\theta \phi + (\partial_\theta \psi - 1)(-X_{\widetilde{\lambda}(\Phi)}). \end{aligned} \tag{7.9}$$

The third equality uses equation (6.3) for $\widetilde{\lambda}$. Since $\partial_\theta \psi = 1$ and $\partial_s \widetilde{\lambda} \neq 0$ on Ω, we obtain

$$\partial_\theta \phi = 0.$$

Thus $\phi(s, \theta) = \phi(s)$ and $\psi(s, \theta) = \theta + \theta_0$, $\theta_0 \in S^1$ are actually defined on some open strip $I' \times S^1$ which intersects Ω'. Let us denote $I := \phi(I')$, so that we have a diffeomorphism

$$\overline{\Phi} = (\phi, \psi) : I' \times S^1 \to I \times S^1.$$

We first observe that

$$\widetilde{\lambda}(\overline{\Phi}(s, \theta)) = \widetilde{\lambda}(s, \theta), \qquad \forall (s, \theta) \in I' \times S^1.$$

This follows from the fact that both $\widetilde{\lambda} \circ \overline{\Phi}$ and $\widetilde{\lambda}$ solve the same ODE (6.3), due to the special form of $\overline{\Phi}$. We now claim that $u \circ \overline{\Phi}$ and u coincide on $I' \times S^1$. This follows from the unique continuation property for the standard Floer equation, since $u \circ \overline{\Phi}(s, \theta) = \underline{\gamma}(\theta + \theta_0)$ and therefore

$$\partial_s(u \circ \Phi) + \widetilde{J}_{\widetilde{\lambda}}(u \circ \Phi)(\partial_\theta(u \circ \overline{\Phi}) - X_{\widetilde{H}_{\widetilde{\lambda}}}) = 0$$

on $I' \times S^1$. We have thus obtained

$$\widetilde{U} \circ \overline{\Phi} = \widetilde{U}$$

on $I' \times S^1$. Let now $s_0' \in I'$ and denote $s_0 := \phi(s_0')$. The maps $\widetilde{U}$ and $\widetilde{U}(\cdot + s_0 - s_0', \cdot + \theta_0)$ coincide along $\{s_0'\} \times S^1$ and solve (6.1–6.3), hence by unique continuation (Proposition 6.2) they coincide on $\mathbb{R} \times S^1$. Arguing as in the last paragraph of Step 2, we obtain a contradiction with our standing assumption $\partial_s \widetilde{U} \not\equiv (0, 0)$. This proves Step 3.

Step 4. *Assume there exists $R_0 > 0$ such that $du(s, \theta)$ is noninjective for all $s \geq R_0$ and $\theta \in S^1$. Assume that $\lim_{s \to \infty}(u(s, \theta), \lambda(s)) = (x, \underline{\lambda})$ for some $x \in \widehat{W}$ (recall that, in this case, we have $h = K + f$ near $(x, \underline{\lambda})$). Then either*

- *$du \equiv 0$ on $[R_0, \infty[\times S^1$ and $R(\widetilde{U})$ is dense in $[R_0, \infty[\times S^1$, or*
- *there exists $R \geq R_0$ such that $\partial_\theta u \equiv 0$ and u is a nonconstant gradient trajectory of K on $[R, \infty[\times S^1$. In this case, $R(\widetilde{U})$ is dense in $[R, \infty[\times S^1$.*

By condition (2) in Definition 7.1, there exists $R \geq R_0$ such that, for $s \geq R$, the components u and λ solve the decoupled equations

$$\partial_s u + J(u)(\partial_\theta u - X_K(u)) = 0,$$
$$\dot{\lambda} - \nabla f(\lambda) = 0.$$

The first equation implies that $C_R(u) := \{(s, \theta) \in]R, \infty[\times S^1 : \partial_s u(s, \theta) = 0\}$ either coincides with $]R, \infty[\times S^1$ or is discrete [10, Lemma 4.1]. In the first case, we obtain $du \equiv 0$ on $]R, \infty[\times S^1$. In the second case, the complement of $C_R(u)$ is connected. As in Step 3, one then shows that there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$ and

$$\alpha \partial_s u(s, \theta) + \beta \partial_\theta u(s, \theta) = 0$$

for all $(s, \theta) \in]R, \infty[\times S^1$.

If $\alpha \neq 0$, let us assume without loss of generality that $\alpha > 0$. Then $u(s, \theta) = u(s + \alpha t, \theta + \beta t)$ for all $t \geq 0$ and $(s, \theta) \in]R, \infty[\times S^1$. Letting $t \to \infty$ we see that $u(s, \theta) = x$ and we again obtain $du \equiv 0$. If $\alpha = 0$, then $\partial_\theta u \equiv 0$ and u is a gradient trajectory of K.

We now prove the following: if $du \equiv 0$ on $[R, \infty[\times S^1$ for some $R \geq R_0$, then the same holds on $[R_0, \infty[\times S^1$. Arguing as in Step 3, we consider the set $\mathcal{I} := \{R \in [R_0, \infty[: du \equiv 0 \text{ on } [R, \infty[\times S^1\}$. Then $\mathcal{I} \neq \emptyset$ and $\mathcal{I}$ is closed. On the other hand, $\mathcal{I}$ is open: if $R \in \mathcal{I}$, then $u([R, \infty[\times S^1) = x$ and therefore $u([R - \varepsilon, \infty[\times S^1)$ belongs to a neighbourhood of x where H has the form $K + f$, provided $\varepsilon > 0$ is small enough. The above argument shows that either $du \equiv 0$ on $[R - \varepsilon, \infty[\times S^1$, or u is a nonconstant gradient trajectory of K on the same domain. The latter is impossible since $du \equiv 0$ on $[R, \infty[\times S^1$. This shows $\mathcal{I} = [R_0, \infty[$.

Let us refer to the case $du \equiv 0$ as *Case 1*, and to the case when u is a nonconstant gradient trajectory of K as *Case 2*. We denote $\widetilde{R} := R_0$ in Case 1, and $\widetilde{R} := R$ in Case 2. We now prove that injective points are dense in $[\widetilde{R}, \infty[\times S^1$.

Let us first assume that $\dot{\lambda} \not\equiv 0$ on $[\widetilde{R}, \infty[$. Since X and $\dot{\lambda}$ are orthogonal (by S^1-invariance of f), we infer that $d\widetilde{U}$ is injective on $[\widetilde{R}, \infty[\times S^1$. Moreover, since $\lambda(s) \notin S^1 \cdot \underline{\lambda}$ for $s \geq \widetilde{R}$, the third condition in the definition of an injective point is satisfied. Arguing by contradiction as in Step 2 (using Baire's theorem), we find open sets $\Omega' \subset]-\infty, \widetilde{R}[\times S^1$ and $\Omega \subset [\widetilde{R}, \infty[\times S^1$ and a diffeomorphism $\Phi = (\phi, \psi) : \Omega' \to \Omega$ such that $\widetilde{U}(s, \theta) = \widetilde{U}(\Phi(s, \theta))$ for all $(s, \theta) \in \Omega'$. Using (7.9) of Step 3 we obtain $\partial_\theta \phi = 0$ and $\partial_\theta \psi = 1$. Thus $\phi = \phi(s)$ and $\psi = \theta + \overline{\psi}(s)$, and Φ admits an extension $\overline{\Phi} = (\phi, \psi) : I' \times S^1 \to I \times S^1$ as in Step 3. The same arguments as in Step 3 show that $\widetilde{U} \circ \overline{\Phi} = \widetilde{U}$ on $I' \times S^1$. We then fix $s_0' \in I'$ and denote $s_0 := \phi(s_0')$. The maps $\widetilde{U}$ and $\widetilde{U}(\cdot + s_0 - s_0', \cdot + \overline{\psi}(s_0'))$ coincide along $\{s_0'\} \times S^1$ and solve (6.1–6.3), hence by unique continuation (Proposition 6.2) they coincide on $\mathbb{R} \times S^1$. As in the final paragraph of Step 2, this contradicts $\partial_s \widetilde{U} \not\equiv (0, 0)$.

We now assume that $\dot{\lambda} \equiv 0$ on $[R, \infty[$. Then u is a nonconstant gradient trajectory of K (Case 2). Under our assumptions $d\widetilde{U}$ is injective on $[R, \infty[\times S^1$. Moreover, u is not equal to x on this domain and hence the third condition in the definition of an injective point is satisfied. Arguing as above and using the same notation, we find that $\widetilde{U}(s, \theta) = \widetilde{U}(\Phi(s, \theta))$ for all $(s, \theta) \in \Omega'$, with $\Phi = (\phi, \psi)$. Using (7.1) we obtain $\partial_s \phi = 1$ and $\partial_\theta \phi = 0$, so that $\phi(s, \theta) = s + \overline{s}$ for some $\overline{s} \in \mathbb{R}$. Using (7.9) we obtain $\partial_\theta \psi = 1$, so $\psi(s, \theta) = \theta + \overline{\psi}(s)$. We conclude exactly as above. This proves Step 4. □

Proposition 7.11. *Let $H \in \mathcal{H}^{S^1}_{N,\mathrm{reg}} \cap \mathcal{H}'$ and $(J, g) \in \mathcal{J}'(H)$. Let $(u, \widetilde{\lambda}) : \mathbb{R} \times S^1 \to \widehat{W} \times S^{2N+1}$ be a solution of* (6.1–6.4) *satisfying* $\partial_s u \not\equiv (0, 0)$. *Assume one of the following holds:*

- *one of the asymptotes of u has a nonconstant first component,*
- *both asymptotes have a constant first component and u differs from a nonconstant gradient trajectory in the neighbourhood of $-\infty$ or $+\infty$.*

Then there exists a nonempty open set $\Omega \subset \mathbb{R} \times S^1$ consisting of injective points and such that du is injective on Ω.

Proof. We distinguish several cases.

Assume there exists a sequence (s^ν, θ^ν) such that $s^\nu \to \infty$, $\nu \to \infty$ and $du(s^\nu, \theta^\nu)$ is injective. In this case, the claim follows from Step 2 in the proof of Proposition 7.10.

Now assume there exists R_0 such that du is noninjective on $[R_0, \infty[\times S^1$, and the asymptote $\underline{\gamma} := \lim_{s \to \infty} u(s, \cdot)$ is nonconstant. Let $R_- := \inf\{R : \partial_s u \equiv 0$ on $[R, \infty[\times S^1\}$. By Step 3 in Proposition 7.10, we have $R_- \leq R_0$. Moreover, the assumption $\partial_s u \not\equiv 0$ ensures that $R_- > -\infty$. Applying Step 3 again, we find a sequence (s^ν, θ^ν) such that $s^\nu \to R_-$ as $\nu \to \infty$ (with $s^\nu < R_-$), and $du(s^\nu, \theta^\nu)$ is injective. Then the claim follows from Step 2.

Finally, assume there exists R_0 such that du is noninjective on the domain $[R_0, \infty[\, \times S^1$, and the asymptote $\underline{\gamma} := \lim_{s\to\infty} u(s,\cdot)$ is constant. By Step 4 in the proof of Proposition 7.10 and our assumption above, we must have $du \equiv 0$ on $[R_0, \infty[\, \times S^1$. Let $R_- := \inf\{R : du \equiv 0 \text{ on } [R, \infty[\, \times S^1\}$. Then $R_- \leq R_0$ and, because $\partial_s u \not\equiv 0$, we have $R_- > -\infty$. Applying Step 4 again, we find a sequence (s^ν, θ^ν) such that $s^\nu \to R_-$ as $\nu \to \infty$ (with $s^\nu < R_-$), and $du(s^\nu, \theta^\nu)$ is injective (if such a sequence did not exist, we could find $\varepsilon > 0$ such that du is noninjective on $[R_- - \varepsilon, \infty[\, \times S^1$, so that, by Step 4, we either have $du \equiv 0$ on this domain and get a contradiction with the definition of R_-, or u is a nonconstant gradient trajectory and we get a contradiction with our assumption). The claim then follows from Step 2. □

Proof of Theorem 7.4. We start by defining the neighbourhood of $\mathcal{H}_*\mathcal{J}' \subset \mathcal{H}\mathcal{J}'$ for which we will prove the theorem. Let us fix $(H_0, J_0, g_0) \in \mathcal{H}_*\mathcal{J}'$ and define which perturbations (H, J, g) of (H_0, J_0, g_0) are allowed. If $H_0 \in \mathcal{H}_{\text{gen}}$, then we allow any $H \in \mathcal{H}_{\text{gen}}$ and any $(J, g) \in \mathcal{J}'(H)$. If $H_0 \in \mathcal{H}_{\text{split}}$, then the S^1-invariant metric g on S^{2N+1} is allowed to be arbitrary. The pair (H, J) is required to be a perturbation of (H_0, J_0) supported away from the constant orbits of H, and close enough to (H_0, J_0) so that the following two conditions hold:

- For all $(\gamma_1, \lambda_1), (\gamma_2, \lambda_2) \in \mathcal{P}(H)$ such that $\gamma_1 = \gamma_2$ and $\lambda_1 \neq \lambda_2$, and for every solution $\lambda : \mathbb{R} \to S^{2N+1}$ of the equation
$$\dot{\lambda} = \int_{S^1} \vec{\nabla}_\lambda H(\theta, \gamma_1(\theta), \lambda)\, d\theta \tag{7.10}$$
with $\lim_{s\to-\infty} \lambda(s) = \lambda_1$ and $\lim_{s\to\infty} \lambda(s) = \lambda_2$, there exists a nonempty open interval $\mathcal{I} \subset \mathbb{R}$ such that, for any $s \in \mathcal{I}$ and $s' \in \mathbb{R} \setminus \{s\}$, we have $\lambda(s') \notin S^1 \cdot \lambda(s)$.
- For any $\overline{p} = (\overline{\gamma}, \overline{\lambda})$, $\underline{p} = (\underline{\gamma}, \underline{\lambda}) \in \mathcal{P}(H)$ such that $\overline{\gamma} \equiv \overline{x}$, $\underline{\gamma} \equiv \underline{x}$ are constant, and any $(u, \lambda) \in \widehat{\mathcal{M}}(\overline{p}, \underline{p}; H, J, g)$ such that, near $\overline{p}$ and $\underline{p}$, the components u, λ are nonconstant gradient trajectories of $\overline{K}$, $\bar{f}$, respectively $\underline{K}$, $\underline{f}$, we have
$$\lambda(\mathbb{R}) \cap (S^1 \cdot \overline{\lambda}) = \emptyset \quad \text{or} \quad \lambda(\mathbb{R}) \cap (S^1 \cdot \underline{\lambda}) = \emptyset. \tag{7.11}$$

The condition involving (7.10) is clearly satisfied for (H_0, g) with g arbitrary. Hence it will still be satisfied for small enough perturbations of H_0.

The condition involving (7.11) is also satisfied for the pair (H_0, J_0). Let us write $H_0 = K_0 + f_0$. By the maximum principle and taking into account that constant orbits of K_0 are situated in W, the trajectories involved in condition (7.11) are contained in W. Since K_0 is C^2-small on W, and W is symplectically aspherical, these must be gradient trajectories of K_0 [17]. Similarly, the λ-components are gradient trajectories of f_0. Hence (7.11) is satisfied due to S^1-invariance of f_0. As a consequence, it will still be satisfied after a small perturbation of the pair (H_0, J_0).

Once the above neighbourhood of $\mathcal{H}_*\mathcal{J}'$ has been defined, the proof is set up as for Theorem 4.1, with obvious modifications dictated by S^1-invariance and the fact that, in the split case, we only allow perturbations supported away from the constant orbits. The main equation is (4.4), namely

$$\int_{\mathbb{R}\times S^1} \langle \eta, D_u\zeta + (D_\lambda J\cdot\ell)J\partial_s u - J(D_\lambda X_{H_\lambda}\cdot\ell) - JX_{h_\lambda} + Y_\lambda^\theta J\partial_s u\rangle\, ds\, d\theta$$
$$+ \int_{\mathbb{R}} \left\langle k, \nabla_s\ell - \nabla_\ell \int_{S^1}\vec{\nabla}_\lambda H - \int_{S^1}\nabla_\zeta\vec{\nabla}_\lambda H - \int_{S^1}\vec{\nabla}_\lambda h + A\cdot\dot{\lambda}\right\rangle ds = 0. \qquad (7.12)$$

We must show that if (7.12) is satisfied for all $(\zeta,\ell,h,Y,A)\in T_{(u,\lambda)}\mathcal{B}\oplus T_h\mathcal{H}^{r,S^1}_{N,\mathrm{reg}}\oplus T_{(J,g)}\mathcal{J}^{r,S^1}_N$, then $(\eta,k)\equiv 0$. Taking $h=0$, $Y=0$, $A=0$ we find that (η,k) lies in the kernel of the formal adjoint $D^*_{(u,\lambda)}$. The latter has the same form as $D_{(u,\lambda)}$ and is therefore elliptic with smooth coefficients. By elliptic regularity, it follows that η and k are smooth and the pair (η,k) satisfies the unique continuation property. It is therefore enough to show that (η,k) vanishes on a nonempty open set. We now distinguish three cases.

Case 1: $\partial_s u\equiv 0$ and $\partial_s\lambda\equiv 0$. In this case $(u,\lambda)\equiv(\gamma,\lambda_0)\in\mathcal{P}(H)$. The operator $D_{(u,\lambda)}$ is Fredholm of index 1 (using the notation in the proof of Proposition 5.2, the index is easily seen to differ by 1 from the index of the operator $\widetilde{D}_{(u,\lambda)}$, which is equal to 0 since $\widetilde{D}_{(u,\lambda)}$ is bijective). We must therefore show that $D_{(u,\lambda)}$ has a 1-dimensional kernel. Let $V\in\ker D_{(u,\lambda)}$, and denote $V(s):=V(s,\cdot)\in H^1(S^1,\gamma^*T\widehat{W})\oplus T_{\lambda_0}S^{2N+1}$. Let $V(s)^\perp$ be the L^2-orthogonal complement of $(\dot\gamma,-X)$ and consider the asymptotic operator $D_{(\gamma,\lambda)}: H^1(S^1,\gamma^*T\widehat{W})\oplus T_{\lambda_0}S^{2N+1}\to L^2(S^1,\gamma^*T\widehat{W})\oplus T_{\lambda_0}S^{2N+1}$. In suitable coordinates, we can write $D_{(u,\lambda)}=\partial_s+D_{(\gamma,\lambda_0)}$. Since $V\in\ker D_{(u,\lambda)}$, we have

$$(\partial_s - D_{(\gamma,\lambda_0)})(\partial_s + D_{(\gamma,\lambda_0)})V = \partial_s^2 V - D^2_{(\gamma,\lambda_0)}V = 0.$$

Taking the L^2-scalar product with $V(s)^\perp$, using that $D_{(\gamma,\lambda_0)}$ is self-adjoint, and that $(\partial_s V)^\perp=\partial_s(V^\perp)$, we obtain $\langle\partial_s^2V^\perp,V^\perp\rangle-\|D_{(\gamma,\lambda_0)}V^\perp\|^2=0$. By assumption, the kernel of $D_{(\gamma,\lambda_0)}$ has dimension 1 and is generated by $(\dot\gamma,-X)$. Hence there exists a constant $c>0$ such that

$$\|D_{(\gamma,\lambda_0)}V(s)^\perp\|^2_{L^2}\ge c\|V(s)^\perp\|^2_{L^2},\qquad \forall s\in\mathbb{R}.$$

As a consequence $\partial_s^2\|V^\perp\|^2\ge 2\langle\partial_s^2V^\perp,V^\perp\rangle\ge 2c\|V^\perp\|^2$. Since $\|V^\perp\|\to 0$ as $s\to\pm\infty$, we infer by the maximum principle that $V^\perp\equiv 0$. Thus $V(s)=a(s)(\dot\gamma,-X)$ and we obtain

$$0=D_{(u,\lambda)}V=a'(s)(\dot\gamma,-X)+a(s)D_{(\gamma,\lambda_0)}(\dot\gamma,-X)=a'(s)(\dot\gamma,-X),$$

so that a is constant. This proves that $\ker D_{(u,\lambda)}$ is generated by $(\dot\gamma,-X)$, as desired.

Case 2: $\partial_s u\equiv 0$ and $\partial_s\lambda\not\equiv 0$. By Steps 3 and 4 in the proof of Proposition 7.10, the set of injective points is open and dense in $\mathbb{R}\times S^1$. By condition (7.10), there exists a nonempty open set $\Omega\subset\mathbb{R}\times S^1$ consisting of injective points such that $\lambda(s')\notin S^1\cdot\lambda(s)$ for all $s'\neq s$ and all $(s,\theta)\in\Omega$. Note that $\dot\lambda\neq 0$ and, up to further shrinking Ω, we can assume without loss of generality that $\widetilde{\lambda}$ is an embedding on Ω.

We claim that $k(s) = 0$ for all $(s, \theta) \in \Omega$. Arguing by contradiction, we find $(s_0, \theta_0) \in \Omega$ such that $k(s_0) \neq 0$. We take $\zeta = 0$, $\ell = 0$, $Y = 0$, $h = 0$, and A supported near $S^1 \cdot \lambda(s_0)$ and satisfying $A(\lambda(s_0)) \cdot \dot{\lambda}(s_0) = k(s_0)$. The first integral in (7.12) vanishes, and the second integral is localized near s_0 and is positive. This contradicts (7.12).

We now claim that $\eta \equiv 0$ on Ω. If not, let $(s_0, \theta_0) \in \Omega$ be such that $\eta(s_0, \theta_0) \neq 0$. Let us consider a function $\widetilde{h}$ of the form $\widetilde{h}(x, \lambda) = \phi(x)\psi(\lambda)$ such that ψ is a cutoff function supported near $\widetilde{\lambda}(s_0, \theta_0) = (-\theta_0) \cdot \lambda(s_0)$, ϕ is supported near $u(s_0, \theta_0)$ and satisfies $-\widetilde{J}_{\widetilde{\lambda}(s_0,\theta_0)} X_{\widetilde{h}_{\widetilde{\lambda}(s_0,\theta_0)}}(u(s_0, \theta_0)) = \eta(s_0, \theta_0)$. This determines uniquely an S^1-invariant function h via $h(\theta, x, \lambda) = \widetilde{h}(x, (-\theta) \cdot \lambda)$. We now remark that if the support of ψ is small enough (depending on the choice of ϕ), we have

$$\langle \eta(s, \theta), -\widetilde{J}_{\widetilde{\lambda}(s,\theta)} X_{\widetilde{h}_{\widetilde{\lambda}(s,\theta)}}(u(s, \theta)) \rangle \geq 0$$

on $\mathbb{R} \times S^1$, and vanishes outside a small neighbourhood of (s_0, θ_0). To see this, one uses that (s_0, θ_0) is an injective point and that $\widetilde{\lambda}$ is an embedding on Ω. We now take ζ, ℓ, Y, A to be zero, and h as above. Then both integrals in (7.12) are localized near (s_0, θ_0). Since k vanishes on Ω, the second integral vanishes, whereas the first one is positive. This contradicts (7.12).

Remark. The perturbation h is admissible even if $u \equiv x$ is a constant orbit. Indeed, in this case λ is a gradient trajectory of an S^1-invariant function on S^{2N+1}, so that $\lambda(s_0) \notin S^1 \cdot \underline{\lambda}$, with $\underline{\lambda} := \lim_{s \to \infty} \lambda(s)$. Thus, the Hamiltonian H remains "split" in a neighbourhood of $\{x\} \times (S^1 \cdot \underline{\lambda})$ under perturbations that are supported away from this set.

Case 3: $\partial_s u \not\equiv 0$. Let us first assume that u satisfies the assumptions of Proposition 7.11, and let $\Omega \subset \mathbb{R} \times S^1$ be a nonempty open set consisting of injective points and such that du is injective on Ω.

We claim that $\eta \equiv 0$ on Ω. If not, we can find $(s_0, \theta_0) \in \Omega$ such that $\eta(s_0, \theta_0) \neq 0$. Moreover, we have $\partial_s u(s_0, \theta_0) \neq 0$ by the definition of Ω. Let $\widetilde{Y} : \widehat{W} \times S^{2N+1} \to \mathrm{End}(T\widehat{W})$ be a function supported near $p_0 := (u(s_0, \theta_0), \widetilde{\lambda}(s_0, \theta_0)) = (u(s_0, \theta_0), (-\theta_0) \cdot \lambda_0)$, and which satisfies the relation $\widetilde{Y}(p_0) J_\lambda^{\theta_0}(u(s_0, \theta_0)) \partial_s u(s_0, \theta_0) = \eta(s_0, \theta_0)$. This uniquely determines an S^1-invariant function Y via $Y_\lambda^\theta(x) := \widetilde{Y}(x, (-\theta) \cdot \lambda)$. Taking $\zeta = 0$, $\ell = 0$, $h = 0$, $A = 0$, and Y as above, the first integral in (7.12) is localized near the injective point (s_0, θ_0) and hence is positive, whereas the second integral in (7.12) is zero. This contradicts (7.12) and proves that $\eta \equiv 0$ on Ω.

We now claim that $k(s) = 0$ for all $(s, \theta) \in \Omega$. Arguing by contradiction, we find $(s_0, \theta_0) \in \Omega$ such that $k(s_0) \neq 0$. Let $\widetilde{h} : \widehat{W} \times S^{2N+1} \to \mathbb{R}$ be a function of the form $\widetilde{h}(x, \lambda) := \phi(x)\psi(\lambda)$ such that ϕ is a cutoff function near $u(s_0, \theta_0)$, and ψ is supported near $\widetilde{\lambda}(s_0, \theta_0) = (-\theta_0) \cdot \lambda(s_0)$ and satisfies $\vec{\nabla}_\lambda \psi(\widetilde{\lambda}(s_0, \theta_0)) = -k(s_0)$. This uniquely determines an S^1-invariant function h via $h(\theta, x, \lambda) := \widetilde{h}(x, (-\theta) \cdot \lambda)$. The main observation is that if the support of ϕ is small enough, then

$$\left\langle k(s), \int_{S^1} \vec{\nabla}_\lambda h(\theta, u(s, \theta), \lambda(s))\, d\theta \right\rangle \geq 0$$

and vanishes outside a small neighbourhood of (s_0, θ_0). This follows from the fact that (s_0, θ_0) is injective and the assumption that $du(s_0, \theta_0)$ is injective, so that u is an embedding near (s_0, θ_0). Taking $\zeta = 0$, $\ell = 0$, $Y = 0$, $A = 0$, and h as above, we see that both integrals in (7.12) are localized near (s_0, θ_0). Since η was shown to vanish on Ω, the first integral vanishes, whereas the second integral is positive. This contradicts (7.12) and proves the claim.

We are now left with the case when both asymptotes of u are constant and u is a nonconstant gradient trajectory near $\pm\infty$. It follows from Step 4 in Proposition 7.10 that there exists $R > 0$ large enough such that $\partial_s u \neq 0$ on $\Omega := (]-\infty, -R] \cup [R, \infty[) \times S^1$ and the set of injective points is open and dense in this domain. The same argument as above shows that $\eta \equiv 0$ on Ω.

We claim that $k(s) = 0$ for all $(s, \theta) \in \Omega$. If λ is constant near $-\infty$ or $+\infty$, the same construction as above proves the claim. Let us therefore assume that λ is a nonconstant gradient trajectory near both $\pm\infty$.

We now use that (u, λ) satisfies (7.11), say at $+\infty$. This implies that, for $s > 0$ large enough, we have $\lambda(\mathbb{R} \setminus \{s\}) \cap S^1 \cdot \lambda(s) = \emptyset$. Let us choose an injective point (s_0, θ_0) with s_0 large enough such that $k(s_0) \neq 0$. Since $\dot\lambda(s_0) \neq 0$, we can choose an S^1-invariant function A supported in a neighbourhood of $S^1 \cdot \lambda(s_0)$ and satisfying $A(\lambda(s_0)) \cdot \dot\lambda(s_0) = k(s_0)$. The first integral in (7.12) vanishes since η was shown to be zero near $+\infty$, and the second integral is localized near s_0 and is positive. This contradicts (7.12) and finishes the proof. □

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