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Bubbling along boundary geodesics near the second critical exponent

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Abstract. The role of the *second critical exponent* $p = (n+1)/(n-3)$, the Sobolev critical exponent in one dimension less, is investigated for the classical Lane–Emden–Fowler problem $\Delta u + u^p = 0$, $u > 0$ under zero Dirichlet boundary conditions, in a domain Ω in $\mathbb{R}^n$ with bounded, smooth boundary. Given Γ, a geodesic of the boundary with negative inner normal curvature we find that for $p = (n+1)/(n-3) - \varepsilon$, there exists a solution u_ε such that $|\nabla u_\varepsilon|^2$ converges weakly to a Dirac measure on Γ as $\varepsilon \to 0^+$, provided that Γ is nondegenerate in the sense of second variations of length and ε remains away from a certain explicit discrete set of values for which a resonance phenomenon takes place.

Keywords. Critical Sobolev exponent, blowing-up solution, nondegenerate geodesic

Contents

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1. Introduction and statement of main results

A basic model of nonlinear elliptic PDE is the classical Lane–Emden–Fowler problem [20],

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$ and $p > 1$. Though it looks simple, the structure of the solution set of this problem is in general very complex and a number of basic questions remain mostly unsolved. Among those, solvability for powers p above the critical exponent $(n+2)/(n-2)$ is especially difficult. When $1 < p < (n+2)/(n-2)$, compactness of Sobolev's embedding yields a solution as a minimizer of the variational problem

$$S(p) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{(\int_\Omega |u|^{p+1})^{2/(p+1)}}. \tag{1.2}$$

For $p \geq (n+2)/(n-2)$ this approach fails and essential obstructions to existence arise: Pokhozhaev [25] found that no solution to (1.1) exists if the domain is star-shaped. In contrast, Kazdan and Warner [22] observed that if Ω is a symmetric annulus then compactness holds for any $p > 1$ within the class of radial functions, and a solution can again always be found by the above minimizing procedure. Compactness in the minimization is also restored, without symmetries, by the addition of suitable linear perturbations exactly at the critical exponent $p = (n+2)/(n-2)$, as established by Brezis and Nirenberg [6].

Topology and geometry of the domain are crucial factors for solvability: when $p = (n+2)/(n-2)$ it was proven by Bahri and Coron [2] that solutions to (1.1) exist whenever the topology of Ω is nontrivial in a suitable sense. For powers larger than critical direct use of variational arguments seems hopeless, and finding general conditions for solvability is a notoriously open issue.

A question raised by Rabinowitz, stated by Brezis in [5], is whether the presence of nontrivial topology in the domain suffices for solvability in the supercritical case $p > (n+2)/(n-2)$. Strikingly enough, the answer was found to be negative in dimension $n \geq 4$: Passaseo [23] discovered that for the domain being a thin tubular neighborhood of a copy of the sphere S^{n-2} embedded in $\mathbb{R}^n$, a Pokhozhaev-type identity implies that no solution exists if $p \geq (n+1)/(n-3)$. We call the latter number, which is strictly greater than $(n+2)/(n-2)$, the *second critical exponent*.

The purpose of this paper is to construct solutions of (1.1) when p is below but sufficiently close to the (supercritical) second critical exponent. Assuming that $\partial\Omega$ contains a nondegenerate, closed geodesic Γ with strictly negative curvature, we find a solution to (1.1) with a concentration behavior as p approaches $(n+1)/(n-3)$ in the form of a *bubbling line*, eventually collapsing onto Γ. One should generically expect that this geometric condition holds if for instance Ω has a convex hole or it is a deformation of a torus-like solid of revolution like Passaseo's domain.

We next recall the familiar notion of “point bubbling” in the slightly subcritical case for problem (1.1),

$$\begin{cases} \Delta u + u^{\frac{n+2}{n-2}-\varepsilon} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for small $\varepsilon > 0$. The loss of compactness of Sobolev’s embedding as $\varepsilon \to 0$ triggers the presence of *bubbling solutions* around special points of the domain, which resemble a sharp extremal of the best Sobolev constant in $\mathbb{R}^n$,

$$S_n := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2}{(\int_{\mathbb{R}^n} |u|^{2n/(n-2)})^{(n-2)/n}},$$

a type of point-concentration behavior extensively considered in the literature. This is precisely the behavior of a solution u_ε of (1.3) which minimizes $S(p)$ in (1.2) for

$$p = p_\varepsilon = \frac{n+2}{n-2} - \varepsilon$$

(see [7, 14, 26, 19]). We have $S(p_\varepsilon) \to S_n$ and

$$u_\varepsilon(x) = \mu_\varepsilon^{-\frac{n-2}{2}} w_n(\mu_\varepsilon^{-1}(x - x_\varepsilon)) + o(1), \qquad \mu_\varepsilon \sim \varepsilon^{\frac{1}{n-2}},$$

as $\varepsilon \to 0^+$, where w_n is the *standard bubble*,

$$w_n(x) = \left(\frac{c_n}{1+|x|^2}\right)^{\frac{n-2}{2}}, \qquad c_n = (n(n-2))^{\frac{1}{n-2}}, \tag{1.4}$$

a radial solution of

$$\Delta w + w^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n$$

corresponding to an extremal for S_n [1, 28]. The blow-up point x_ε approaches (up to a subsequence) a harmonic center x_0 of Ω, a minimizer for Robin’s function of the domain, the diagonal of the regular part of Green’s function. The solution concentrates as a Dirac mass at x_0, namely

$$|\nabla u_\varepsilon|^2 \rightharpoonup S_n^{n/2} \delta_{x_0} \quad \text{as } \varepsilon \to 0 \tag{1.5}$$

in the sense of measures. It is found in [26] that actually solutions of (1.3) with this behavior exist, concentrating at any given nondegenerate critical point x_0 of Robin’s function. We refer the reader to the works [3, 10, 21] and to the survey [13] for related results on construction of point-bubbling solutions for problems near the critical exponent.

Now, we are interested in problem (1.1) for powers slightly below the second critical exponent, namely

$$\begin{cases} \Delta u + u^{\frac{n+1}{n-3}-\varepsilon} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

We want to find a solution u_ε with a behavior analogous to that just described for (1.3), now concentrating *along a curve*, with a sectional profile given by a scaled standard bubble in one dimension less. This problem is substantially harder than (1.3), in particular because a global variational characterization of the solution does not seem possible in view of its supercritical character. In addition, this solution has formally a large ε-dependent Morse index, and the construction requires us to avoid special values of ε where a change of topological type occurs.

We shall assume that $\partial\Omega$ contains a closed geodesic Γ, nondegenerate, which has globally *negative curvature*, and in addition a nonresonance condition of the form

$$|k^2\varepsilon^{2\frac{n-2}{n-3}} - \kappa^2| > \delta\varepsilon^{\frac{n-2}{n-3}} \quad \text{for all } k = 1, 2, \dots, \tag{1.7}$$

where $\kappa > 0$ is given explicitly in terms of Γ by formula (8.10).

Theorem 1.1. *Let $n \geq 8$ and $\Omega \subset \mathbb{R}^n$ be a domain with smooth, bounded boundary $\partial\Omega$, which contains a closed geodesic Γ, nondegenerate with negative inner normal curvature. Then, given $\delta > 0$, for all $\varepsilon > 0$ sufficiently small satisfying condition* (1.7)*, problem* (1.6) *has a solution u_ε that satisfies*

$$|\nabla u_\varepsilon|^2 \rightharpoonup S_{n-1}^{\frac{n-1}{2}}\delta_\Gamma$$

as $\varepsilon \to 0$ in the sense of measures, where δ_Γ is the Dirac measure supported on the curve Γ. Moreover, u_ε can be described according to formula (1.9) *below.*

Much more precise information on the solution can indeed be gathered as we shall explain later. The condition $n \geq 8$ seems essential for the method used, while we believe the phenomenon described should also be true for lower dimensions.

Theorem 1.1 includes the case of an *exterior domain*, $\Omega \setminus \Lambda$, with Λ bounded. It is worth mentioning that for this case it was established in [8, 9] that problem (1.1) is actually *always* solvable if $p > (n+2)/(n-2)$. In fact a continuum of solutions exist *but* they are of slow decay (infinite energy). Finding finite-energy (fast decay) solutions for supercritical powers is a much harder question, which is only answered in [9] for p very close from above to $(n+2)/(n-2)$. In turns out that a dramatic change of structure in the set of slow decay solutions takes place *precisely* when $p = (n+1)/(n-3)$, the second critical exponent.

The line-bubbling phenomenon here discovered is conceptually quite different from point bubbling. In spite of zero boundary data, concentration eventually collapses on the boundary. On the other hand, point concentration is determined by global information on the domain encoded in Green's function, while only local structure of the domain near the curve Γ is relevant to the line bubbling. In order to describe the solution more precisely, we introduce a local system of coordinates near Γ.

For notational simplicity we will write $N = n - 1$ in the remainder of this paper, so that the problem is embedded in $\mathbb{R}^{N+1}$.

We consider the metric induced by the Euclidean one on $\partial\Omega$ and denote by $\bar{\nabla}$ the associated connection. We introduce Fermi coordinates in a neighborhood of Γ in $\partial\Omega$.

Given $q \in \Gamma$, there is a natural splitting

$$T_q \partial\Omega = T_q \Gamma \oplus N_q \Gamma$$

into the normal and tangent bundle over Γ. We assume that Γ is parameterized by arc-length x_0, $x_0 \mapsto \gamma(x_0)$, and denote by E_0 a unit tangent vector to Γ. In a neighborhood of a point q of Γ, assume we are given an orthonormal basis E_i, $i = 1, \ldots, N-1$, of $N_q\Gamma$. We can assume that the E_i are parallel along Γ, which means that

$$\bar{\nabla}_{E_0} E_i = 0$$

for $i = 1, \ldots, N-1$. The geodesic condition for Γ translates precisely into

$$\bar{\nabla}_{E_0} E_0 = 0.$$

To parameterize a neighborhood of a point of Γ in $\partial\Omega$ we define

$$F(x_0, \bar{x}) := \mathrm{Exp}^{\partial\Omega}_{\gamma(x_0)}(x_i E_i), \qquad \bar{x} := (x_1, \ldots, x_N),$$

where $\mathrm{Exp}^{\partial\Omega}$ is the exponential map on $\partial\Omega$ and summation over $i = 1, \ldots, N-1$ is understood. To parameterize a neighborhood of Γ in $\bar{\Omega}$, we consider the system of coordinates $(x_0, x) \in \mathbb{R}^{N+1}$ given by

$$G(x_0, x) = F(x_0, \bar{x}) - x_N \mathbf{n}(F(x_0, \bar{x})), \qquad x = (\bar{x}, x_N) \in \mathbb{R}^N, \tag{1.8}$$

where x is close to 0 and $\mathbf{n}$ designates the outward unit normal.

In terms of $\mathbf{n}$, we assume that Γ has globally negative curvature in the sense that

$$\partial^2_{x_0} \gamma = \bar{h}_{00} \mathbf{n},$$

with $\bar{h}_{00}$ a strictly positive function along Γ.

The solution u_ε predicted by the theorem can be described in these coordinates at main order as follows:

$$u_\varepsilon(x_0, x) = \mu_\varepsilon^{-\frac{N-2}{2}} w_N(\mu_\varepsilon^{-1}(x - d_\varepsilon)) + o(1), \tag{1.9}$$

where

$$d_{\varepsilon j}(x_0) \sim \varepsilon \tilde{d}_j(x_0), \qquad j = 1, \ldots, N, \qquad \mu_\varepsilon(x_0) \sim \varepsilon^{\frac{N-1}{N-2}} \tilde{\mu}(x_0),$$

where $\tilde{d}_j$ and $\tilde{\mu}$ are smooth functions of x_0 with $\tilde{d}_N$ and $\tilde{\mu}$ strictly positive, and w_N is given by (1.4).

Finally, let us make explicit the meaning of nondegeneracy of the geodesic Γ. Let us denote by $\bar{R}$ the Ricci tensor on $\partial\Omega$. Then nondegeneracy of Γ translates exactly into the fact that the linear system of equations

$$-\ddot{\bar{d}}_k + \sum_{j=1}^{N-1} (\bar{R}(E_0, E_j)E_0 \cdot E_k)\bar{d}_j = 0, \qquad x_0 \in [-\ell, \ell],\ k = 1, \ldots, N-1, \tag{1.10}$$

has only the trivial 2ℓ-periodic solution $\bar{d} \equiv 0$.

The rest of this paper will be devoted to the proof of Theorem 1.1. We point out that the resonance phenomenon has already been found to arise in the analysis of higher dimensional concentration in other elliptic boundary value problems, in particular for a Neumann singular perturbation problem in [17, 18, 16, 15] and in Schrödinger equations in the plane in [12]. Theorem 1.1 seems to be the first result on higher dimensional concentration phenomena associated to critical exponents. The question of whether one can find concentration results for larger critical exponents, say k-dimensional concentration slightly below $(n+2-k)/(n-2-k)$ arises naturally but we will not treat it in this paper.

2. Scheme of the proof of Theorem 1.1

Let us write problem (1.6) as

$$\begin{cases} \Delta u + u^{p-\varepsilon} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{2.1}$$

here and in what follows we set $p = (N+2)/(N-2)$. A key element of the proof of Theorem 1.1 is the construction of a first approximation of the solution to our problem. The main part of the construction is performed close to the geodesic. Let us consider the system of coordinates $(x_0, \bar{x}, x_N)$ introduced in (1.8), which straightens the boundary of Ω in a neighborhood of the geodesic to the hyperplane $x_N = 0$. In this language the geodesic is represented by the x_0-axis. We recall that x_0 designates arclength of the curve and $x_N > 0$ is the normal coordinate to the boundary. Then for a function u defined on this neighborhood we write

$$\tilde{u}(x_0, x) = u(G(x_0, x)). \tag{2.2}$$

Let 2ℓ represent the total length of the geodesic. Extending $\tilde{u}$ in a 2ℓ-periodic manner with respect to x_0, it is convenient to regard it as a function defined on the infinite half cylinder

$$D = \{(x_0, \bar{x}, x_N) : |\bar{x}|^2 + |x_N|^2 < a,\ x_N > 0\},$$

where $a > 0$ is a fixed small number. Equation (2.1) for u reads in terms of $\tilde{u}$ in D as

$$\begin{cases} \Delta\tilde{u} + B(\tilde{u}) + \tilde{u}^{p-\varepsilon} = 0, \quad u > 0 \quad \text{in } D, \\ \tilde{u}(x_0, \bar{x}, 0) = 0 \quad \text{for all } (x_0, \bar{x}), \\ \tilde{u}(x_0 + 2\ell, \bar{x}, x_N) = \tilde{u}(x_0, \bar{x}, x_N) \quad \text{for all } (x_0, \bar{x}, x_N), \end{cases} \tag{2.3}$$

where B is a second order linear operator of the form

$$B = b_{lk}(x_0, x)\partial_{lk} + b_l(x_0, x)\partial_l$$

with smooth coefficients, 2ℓ-periodic in x_0, $b_{lk}(x_0, 0) \equiv 0$, which we explicitly find in terms of geometric quantities in §4. If a is sufficiently small, the differential operator involved in (2.3) can be regarded as a small perturbation of the Laplacian inside D. To construct an approximation to a solution of (2.3) with the desired properties, the main observation we make is that if

$$\omega(x) := \left(\frac{c_N}{1+|x|^2}\right)^{\frac{N-2}{2}}, \tag{2.4}$$

then for small $\mu > 0$ and $d = (\bar{d}, d_N) \in \mathbb{R}^N$ the function

$$u_0 = \mu^{-\frac{N-2}{2}}\omega(\mu^{-1}(x-d)) = \left(\frac{c_N\mu}{\mu^2 + |\bar{x}-\bar{d}|^2 + |x_N - d_N|^2}\right)^{\frac{N-2}{2}}$$

satisfies

$$\begin{cases} \Delta u + u^p = 0, \quad u > 0 \quad \text{in } D, \\ u(x_0 + 2\ell, \bar{x}, x_N) = u(x_0, \bar{x}, x_N) \quad \text{for all } (x_0, \bar{x}, x_N), \end{cases} \tag{2.5}$$

and can therefore be considered as an approximation of a solution to (2.3). We assume $d_N > 0$ so that the maximum set of u_0 is inside the domain, with value $\sim \mu^{-(N-2)/2}$. In addition, we want the boundary values to be small compared with this order, which is achieved if $\mu \ll d_N$. In this case the boundary values are bounded by $\sim \mu^{-(N-2)/2}(\mu/d_N)^{(N-2)/2}$. Unfortunately, to obtain a good approximation it does not suffice to choose μ and d just to be constants. We assume instead that they define smooth functions of x_0. As we will see later, a sound choice is to take

$$d_\varepsilon(x_0) = \varepsilon\tilde{d}_\varepsilon(x_0), \qquad \mu_\varepsilon(x_0) = \rho\tilde{\mu}_\varepsilon(x_0), \qquad \rho = \varepsilon^{\frac{N-1}{N-2}}, \tag{2.6}$$

where $\tilde{\mu}_\varepsilon$ and $\tilde{d}_\varepsilon$ are uniformly bounded 2ℓ-periodic smooth functions so that also $\tilde{\mu}_\varepsilon$ an $\tilde{d}_{\varepsilon N}$ are positive and uniformly bounded below away from zero. In particular, observe that $\mu_\varepsilon \sim \varepsilon^{1/(N-2)} d_{\varepsilon N}$, and we set as an approximation to a solution of (2.3),

$$\tilde{u}_0(x_0, x) = \mu_\varepsilon^{-\frac{N-2}{2}}\omega(\mu_\varepsilon^{-1}(x - d_\varepsilon)).$$

It is natural to consider the further change of variables

$$\tilde{u}(x_0, x) = \mu_\varepsilon^{-\frac{N-2}{2}} v(\rho^{-1}x_0, \mu_\varepsilon^{-1}(x - d_\varepsilon)), \qquad v = v(y_0, y), \tag{2.7}$$

under which $\tilde{u}_0$ reads simply $\omega(y)$. Equation (2.3) is transformed in terms of v into

$$\begin{cases} S(v) := a_0(\rho y_0)\partial_{00}v + \Delta_y v + \tilde{\mathcal{A}}(v) + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon} v^{p-\varepsilon} = 0 \quad \text{in } \mathcal{D}, \\ v\left(y_0, \bar{y}, -\dfrac{d_{\varepsilon N}}{\mu_\varepsilon}(\rho y_0)\right) = 0, \\ v(y_0 + 2\ell\rho^{-1}, y) = v(y_0, y), \end{cases} \tag{2.8}$$

where

$$\tilde{\mathcal{A}} = a_{ij}(y_0, y)\partial_{ij} + a_i(y_0, y)\partial_i + c(y_0, y)$$

is again a small operator and now we reduce the original cylinder to take $\mathcal{D}$ as a region of the form

$$\mathcal{D} = \left\{(y_0, \bar{y}, y_N) : -\frac{d_{\varepsilon N}}{\mu_\varepsilon}(\rho y_0) < y_N < \frac{\hat{\delta}}{\rho},\ |\bar{y}| < \frac{\hat{\delta}}{\rho}\right\}, \tag{2.9}$$

where $\hat{\delta} > 0$ is a small number which will be further reduced if necessary. Here

$$a_0(x_0) = \rho^{-2}\mu_\varepsilon(x_0)^2 = \tilde{\mu}_\varepsilon(x_0)^2, \tag{2.10}$$

and $\tilde{\mathcal{A}}$ is a differential operator with coefficients becoming small with ε, which we will fully identify later. Noting that $\mu_\varepsilon^{-(N-2)\varepsilon/2} \to 1$ and that the domain $\mathcal{D}$ is expanding into entire $\mathbb{R}^{N+1}$, we see that $\omega(y)$ indeed approximates a solution to the equation. We will actually take an approximation $\mathtt{w}$ and differs little from ω which in particular satisfies the boundary condition.

Now, if we set $v = \mathtt{w} + \phi$ with ϕ small, the equation takes the form

$$L(\phi) := a_0\partial_{00}\phi + \Delta_y\phi + p\omega^{p-1}\phi + \tilde{\mathcal{A}}(\phi) = -S_\varepsilon(\mathtt{w}) - N(\phi)$$

where the operator $N(\phi)$ is of order smaller than linear in ϕ. More precisely

$$N(\phi) = \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}(\mathtt{w} + \phi)^{p-\varepsilon} - \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}\mathtt{w}^{p-\varepsilon} - p\omega^{p-1}\phi.$$

It is therefore important to understand bounded solvability of a linear equation involving the operator L. This is a rather subtle issue since the limiting L does have a kernel in the space of bounded functions in $\mathbb{R}^{N+1}$. Indeed, the equation

$$\partial_{00}\phi + \Delta_y\phi + p\omega^{p-1}\phi = 0$$

has the bounded solutions $Z_i, i = 1, \dots, N+1$, and $Z_0(x)\cos(\sqrt{\lambda_1}x_0)$, $Z_0(x)\sin(\sqrt{\lambda_1}x_0)$, where

$$Z_i = \partial_i w, \quad i = 1, \dots, N, \quad Z_{N+1} = x \cdot \nabla w + \frac{N-2}{2}w, \tag{2.11}$$

and we denote by Z_0, $\lambda_1 > 0$ the first eigenfunction and eigenvalue in $L^2(\mathbb{R}^N)$ of the problem

$$\Delta_y\phi + p\omega(y)^{p-1}\phi = \lambda\phi \quad \text{in } \mathbb{R}^N. \tag{2.12}$$

As we shall show these are *all* the bounded solutions of the equation.

Let us consider a bounded function $h(y_0, y)$ 2ℓ-periodic in y_0 and the following *projected problem* in which we mod out the above functions, and look for bounded functions $c_i(y_0)$ and ϕ such that

$$\begin{cases} L(\phi) := a_0\partial_{00}\phi + \Delta_y\phi + p\omega^{p-1}\phi + \tilde{\mathcal{A}}(\phi) = h + \sum_{i=0}^{N+1} c_i(y_0)Z_i \quad \text{in } \mathcal{D}, \\ \phi = 0 \quad \text{on } \partial\mathcal{D}, \\ \phi(y_0 + 2\ell\rho^{-1}, y) = \phi(y_0, y), \\ \int_{\mathcal{D}_{y_0}} \phi(y_0, y)Z_i(y)\,dy = 0 \quad \text{for all } y_0 \in \mathbb{R},\ i = 0, \dots, N. \end{cases} \tag{2.13}$$

As we will see, this problem has a unique solution whenever ε is small enough provided that certain uniform estimates hold for the parameters involved and their derivatives. In addition ϕ satisfies a uniform a priori estimate in L^∞-weighted norms. We develop this theory in fact in larger generality in §3. Then we consider the projected nonlinear problem

$$\begin{cases} L(\phi) = -S_\varepsilon(\mathtt{w}) - N(\phi) + \displaystyle\sum_{i=0}^{N+1} c_i(y_0) Z_i \quad \text{in } \mathcal{D}, \\ \phi = 0 \quad \text{on } \partial\mathcal{D}, \\ \phi(y_0 + 2\ell\rho^{-1}, y) = \phi(y_0, y), \\ \displaystyle\int_{\mathcal{D}_{y_0}} \phi(y_0, y) Z_i(y)\, dy = 0 \quad \text{for all } y_0 \in \mathbb{R},\ i = 0, \dots, N+1, \end{cases} \tag{2.14}$$

where $\mathcal{D}_{y_0} = \{y : (y_0, y) \in \mathcal{D}\}$, to which we can apply the linear solvability theory and contraction mapping principle to find a unique small solution. Moreover,

$$c_i(y_0) \int_{\mathbb{R}^N} Z_i^2 \sim \int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_i\, dy$$

and therefore to have a solution of the original problem (with $c_i \equiv 0$) we need a set of relations that look (approximately!) like

$$\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_i\, dy = 0 \quad \text{for all } y_0,\ i = 0, \dots, N+1. \tag{2.15}$$

At this point we mention that the approximation $\mathtt{w}$ carries as an additive term a function of the form $e_\varepsilon(\rho y_0) Z_0(y)$ where e_ε is another parameter of the form $e_\varepsilon(x_0) = \varepsilon \tilde{e}_\varepsilon(x_0)$. It turns out that adjusting conveniently the $(N+2)$ parameters $\mu_\varepsilon, d_\varepsilon, e_\varepsilon$ we can achieve that the above $N+2$ relations hold as a system of differential equations for these quantities, which turns out to be solvable because of the nondegeneracy assumptions made. The story is however more involved since the parameters enter the nonlinear relations at different orders so that a further improvement of the approximation $\mathtt{w}$ of the form $\mathtt{W} = \mathtt{w} + \Pi$ is needed. This is the main purpose of the work in §5. Π is built upon solving the linear problem (2.13) for $h = -S_\varepsilon(\mathtt{w})$, after identifying the right main order values of the parameters in the solvability conditions (2.15), which turns out to reduce substantially the size of the approximation error $S_\varepsilon(\mathtt{W})$. Another crucial step is a gluing procedure carried out in §6, where the full problem (2.1), for which a global approximation is built by just multiplying $\mathtt{W}$ by a cut-off function, is reduced to solving an equation similar to (2.14) for $c_i \equiv 0$, just in a neighborhood of the geodesic, but where the operator $N(\phi)$ is replaced by a similar one which includes nonlocal terms in ϕ encoding the information on the rest of the domain. This is what tells us that the influence of geometry of the remaining part of the domain is basically negligible. The corresponding projected version of the nonlinear problem is solved in §7 and the final adjustment of the remaining parts of the parameters is done in §8, thus completing the proof of Theorem 1.1. We devote the rest of this paper to carrying out the program outlined above.

3. The linear theory

In this section we will develop a linear theory suitable to solve problem (2.13). Our main result is contained in Proposition 3.2 below, for which we need some preliminaries. Let

$\omega(x)$ be the function defined in (2.4) as

$$\omega(x) := \left(\frac{c_N}{1+|x|^2}\right)^{\frac{N-2}{2}},$$

where $x \in \mathbb{R}^N$ and $c_N = (N(N-2))^{1/2}$, which is, we recall, an entire solution of the problem

$$\Delta_{\mathbb{R}^N}\omega + \omega^p = 0 \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

where $p = (N+2)/(N-2)$. Let us consider the operator

$$L_0 := \Delta_{\mathbb{R}^N} + p\omega^{p-1},$$

which corresponds to the nonlinear operator in (3.1) linearized at ω.

To analyze the point spectrum of this operator, we use the conformal invariance of (3.1). Let us consider on $\mathbb{R}^N$ the metric

$$g_{S^N} := \left(\frac{2}{1+|x|^2}\right)^2 dx^2,$$

which is conformal to the euclidean metric dx^2 and corresponds to the standard metric on S^N when parameterized by the inverse of the stereographic projection

$$x \in \mathbb{R}^N \mapsto \left(\frac{2}{1+|x|^2}x, \frac{1-|x|^2}{1+|x|^2}\right) \in S^N.$$

In polar coordinates, we have the expression of the Laplace–Beltrami operator on S^N given by

$$\Delta_{S^N} = \left(\frac{2}{1+r^2}\right)^{-n} r^{1-n}\partial_r\left(\left(\frac{2}{1+r^2}\right)^{n-2} r^{n-1}\partial_r\right) + \left(\frac{2}{1+r^2}\right)^{-2} r^{-2}\Delta_{S^{N-1}},$$

where $r = |x|$. The following identity follows from the conformal invariance of the so called *conformal Laplacian* or can be obtained by direct computation:

$$L = \left(\frac{2}{1+|x|^2}\right)^{\frac{N+2}{2}} (\Delta_{S^N} + N)\left(\frac{2}{1+|x|^2}\right)^{\frac{2-N}{2}}.$$

We also have

$$\int_{S^N} Z(\Delta + N)Z\, d\mathrm{vol}_{S^N} = \int_{\mathbb{R}^N} \tilde{Z}L\tilde{Z}\, d\mathrm{vol}_{\mathbb{R}^N},$$

where $\tilde{Z}$ and Z are related by

$$\tilde{Z} = \left(\frac{2}{1+r^2}\right)^{\frac{N-2}{2}} Z.$$

Now, the operator $\Delta_{S^N} + N$ has an $N+1$-dimensional kernel corresponding to the coordinate functions on S^N (since N is an eigenvalue of $-\Delta_{S^N}$). This implies that the L^2-null space of the operator L is $N+1$-dimensional and spanned by the functions

$$Z_j := \partial_{x_j}\omega, \quad j = 1, \ldots, N, \quad \text{and} \quad Z_{N+1} := x \cdot \nabla\omega + \frac{N-2}{2}\omega$$

(see (2.11)). The fact that $LZ_j = 0$ can also be checked directly or can be proved using the fact that (3.1) enjoys some translation and dilation invariance in the sense that, for all $\lambda > 0$ and $a \in \mathbb{R}^N$, the function

$$x \mapsto \lambda^{\frac{n-2}{2}} u(\lambda x + a)$$

is a solution of (3.1) whenever u is. Differentiation with respect to λ or with respect to a, at $\lambda = 1$ and $a = 0$, directly shows that Z_j is a solution of $LZ_j = 0$.

Moreover, the space where the quadratic form

$$\tilde{Z} \mapsto -\int_{S^N} \tilde{Z}(\Delta + N)\tilde{Z}\, d\mathrm{vol}_{S^N}$$

is negative definite is one-dimensional, and coincides with the space of constant functions, which implies that the space where

$$Z \mapsto -\int_{\mathbb{R}^N} ZLZ\, d\mathrm{vol}_{\mathbb{R}^N}$$

is negative is also one-dimensional. Hence, the operator L_0 has one negative eigenvalue $-\lambda_1 < 0$, and we denote by Z_0 the corresponding eigenfunction (normalized to have L^2-norm equal to 1). See (2.12). We observe that this eigenfunction decays exponentially at infinity with exponential order $O(e^{-\sqrt{\lambda_1}|x|})$.

Having understood the point spectrum of the operator L we have

Lemma 3.1. *Assume that $\xi \notin \{0, \pm\sqrt{\lambda_1}\}$. Then given $h \in L^\infty(\mathbb{R}^N)$, there exists a unique bounded solution of*

$$(L_0 - |\xi|^2)\psi = h$$

in $\mathbb{R}^N$. Moreover

$$\|\psi\|_{L^\infty} \leq c_\xi \|h\|_{L^\infty}$$

for some constant $c_\xi > 0$ only depending on ξ.

Proof. For all $r > 0$, we denote B_r the ball of radius r in $\mathbb{R}^N$ centered at the origin. We assume that $\xi \notin \{0, \pm\sqrt{\lambda_1}\}$ is fixed. We first prove that there exists $r_\xi > 0$ (depending on ξ) such that, for all $r \geq r_\xi$, the a priori estimate

$$\|\psi\|_{L^\infty(B_r)} \leq c_\xi \|(L - |\xi|^2)\psi\|_{L^\infty(B_r)} \tag{3.2}$$

holds for any bounded function ψ vanishing on ∂B_r.

Assume for the time being that this estimate is already proven. Then, for $r \geq r_\xi$, the operator $L_0 - |\xi|^2$ is injective on the ball of radius r (it being understood that we consider the zero Dirichlet boundary conditions). The Fredholm alternative implies that, for all $r \geq r_\xi$, we can find a unique solution of

$$(L_0 - |\xi|^2)\psi_r = h$$

on B_r with $\psi_r = 0$ on ∂B_r. Given a sequence r_j tending to ∞, the a priori estimate (3.2), elliptic estimates and Ascoli–Arzelà's Theorem allow one to extract from $(\psi_{r_j})_j$ a subsequence which converges (uniformly on compact sets) to a function ψ satisfying

$$(L_0 - |\xi|^2)\psi = h$$

in $\mathbb{R}^N$. Moreover, passing to the limit in (3.2), we find that $\|\psi\|_{L^\infty} \leq c_\xi \|h\|_{L^\infty}$. This completes the proof of the existence of ψ. Uniqueness follows at once from the fact that (3.2) extends to the case where the functions are defined on $\mathbb{R}^N$.

It remains to prove (3.2). First observe that, since $\xi \neq 0$, there exists $\bar{r}_\xi > 0$ such that

$$p\omega^{p-1} - |\xi|^2 \leq -\tfrac{1}{2}|\xi|^2$$

in $\mathbb{R}^N \setminus B_{\bar{r}_\xi}$. Given $r > \bar{r}_\xi$ and using the constant function as a barrier, we immediately find that

$$\|\psi\|_{L^\infty(B_r \setminus B_{\bar{r}_\xi})} \leq c_\xi (\|(L_0 - |\xi|^2)\psi\|_{L^\infty(B_r \setminus B_{\bar{r}_\xi})} + \|\psi\|_{L^\infty(\partial B_{\bar{r}_\xi})}) \tag{3.3}$$

for any bounded function ψ vanishing on ∂B_r.

We now assume that (3.2) does not hold. Then there exists a sequence of radii r_j tending to ∞, and functions ψ_j vanishing on ∂B_{r_j}, such that

$$\|\psi\|_{L^\infty(B_{r_j})} = 1 \quad \text{while} \quad \lim_{j \to \infty} \|(L_0 - |\xi|^2)\psi_j\|_{L^\infty(B_{r_j})} = 0.$$

Observe that, without loss of generality, we can assume that $r_j \geq \bar{r}_\xi$, and (3.3) implies that $\|\psi_j\|_{L^\infty(B_{\bar{r}_\xi})}$ remains bounded away from 0 as j tends to ∞.

Elliptic estimates and Ascoli–Arzelà's Theorem allow us to extract from $(\psi_j)_j$ a subsequence which converges (uniformly on compact sets) to a function ψ satisfying

$$(L_0 - |\xi|^2)\psi = 0$$

in $\mathbb{R}^N$. Moreover, ψ is bounded and not identically equal to 0 (since $\|\psi_j\|_{L^\infty(B_{\bar{r}_\xi})}$ remains bounded away from 0). But, since $\xi \notin \{0, \pm\sqrt{\lambda_1}\}$, this contradicts the classification of the point spectrum of L. The proof of the a priori estimate is therefore complete. □

We shall use the previous result in order to obtain a priori estimates and a solvability theory for problem (2.13). We consider here a slightly more general problem that involves the essential features needed. For a positive smooth function $R(y_0)$ and a constant $M > 0$ we consider the domain $\mathcal{D}$ defined as

$$\mathcal{D} = \{(y_0, \bar{y}, y_N) \in \mathbb{R}^{N+1} : -R(y_0) < y_N < M,\ |\bar{y}| < M\}$$

and for functions ϕ defined on $\mathcal{D}$, an operator of the form

$$L(\phi) := b(y_0)\partial_{00}\phi + \Delta_y \phi + p\omega^{p-1}\phi + b_{ij}(y_0, y)\partial_{ij}\phi + b_i(y_0, y)\partial_i \phi + d(y_0, y)\phi$$

where $b_{00} \equiv 0$. Then for a given function h we want to solve the following projected problem:

$$\begin{cases} L(\phi) = h + \displaystyle\sum_{i=0}^{N+1} c_i(y_0) Z_i(y) \quad \text{in } \mathcal{D}, \\ \phi = 0 \quad \text{on } \partial\mathcal{D}, \\ \displaystyle\int_{\mathcal{D}_{y_0}} \phi(y_0, y) Z_i(y)\, dy = 0 \quad \text{for all } y_0 \in \mathbb{R},\ i = 0, \dots, N, \end{cases} \tag{3.4}$$

where

$$\mathcal{D}_{y_0} = \{ y \in \mathbb{R}^N : (y_0, y) \in \mathcal{D} \}.$$

We fix a number $2 \le \nu < N$ and consider the L^∞-weighted norms

$$\|\phi\|_* = \sup_{\mathcal{D}} (1 + |y|^{\nu-2}) |\phi(y_0, y)| + \sup_{\mathcal{D}} (1 + |x|^{\nu-1}) |D\phi(x_0, x)|,$$
$$\|h\|_{**} = \sup_{\mathcal{D}} (1 + |y|^{\nu}) |h(y_0, y)|.$$

We assume that all functions involved are smooth. We will establish existence and uniform a priori estimates for problem (3.4) in the above norms, provided that appropriate bounds for the coefficients hold.

Proposition 3.2. *Assume that $N \ge 7$ and $N - 2 \le \nu < N$. Assume that for a number $m > 0$ we have*

$$m \le b(y_0) \le m^{-1} \quad \textit{for all } y_0 \in \mathbb{R}.$$

Then there exist positive numbers δ, C such that if, for all i, j,

$$\|\partial_0 R\|_\infty + M\|\partial_{00} R\|_\infty + M\|\partial_0 b\|_\infty + \|b_{ij}\|_\infty + \|Db_{ij}\|_\infty + \|(1+|y|)b_i\|_\infty + \|(1+|y|^2)d\|_\infty < \delta, \tag{3.5}$$

and

$$\delta^{-1} < R(y_0), \quad M^{-1}R(y_0) < \delta \quad \textit{for all } y_0 \in \mathbb{R}, \tag{3.6}$$

then for any h with $\|h\|_{**} < +\infty$ there exists a unique solution $\phi = T(h)$ of problem (3.4) *with $\|\phi\|_* < +\infty$, and we have*

$$\|\phi\|_* \le C\|h\|_{**}.$$

Proof. The proof of this result will be carried out in three steps.

Step 1. Let us assume that in problem (3.4) the coefficients b_i, d, and the functions c_i are identically zero. We will prove that δ, C as in the above statement can then be chosen so that for any h with $\|h\|_{**} < +\infty$ and any solution ϕ of problem (3.4) with $\|\phi\|_* < +\infty$ we have

$$\|\phi\|_* \le C\|h\|_{**}.$$

Arguing towards a contradiction, we assume the existence of b^n, ϕ_n, h_n, b^n_{ij}, R_n, M_n such that

$$m \le b^n(y_0) \le m^{-1} \quad \text{for all } x_0 \in \mathbb{R},$$
$$\|\phi_n\|_* = 1, \quad \|h_n\|_{**} \to 0,$$
$$M_n\|\partial_0 b^n\|_\infty + M_n^{-1}\|R_n\|_\infty + \|\partial_0 R_n\|_\infty + M_n\|\partial_{00} R_n\|_\infty + \|b^n_{ij}\|_\infty \to 0, \quad \inf_{x_0} R_n \to +\infty$$

and

$$b^n(y_0)\partial_{00}\phi_n + \Delta_y\phi_n + b^n_{ij}\partial_{ij}\phi_n + pw(y)^{p-1}\phi_n = h_n \quad \text{in } \mathcal{D}, \tag{3.7}$$

together with the orthogonality and boundary conditions.

To achieve a contradiction we will first show that

$$\|\phi_n\|_\infty \to 0. \tag{3.8}$$

If this is not the case then we may assume that there is a positive number γ for which $\|\phi_n\|_\infty > \gamma$. Since we also know that

$$|\phi(y_0, y)| \le C/(1+|y|)^{\nu-2},$$

we conclude that for some $A > 0$,

$$\|\phi_n\|_{L^\infty(|x|\le A)} \ge \gamma.$$

Let us fix a y_{0n} such that

$$\|\phi_n(y_{0n}, \cdot)\|_{L^\infty(|y|\le A)} \ge \gamma/2.$$

By elliptic estimates and compactness of Sobolev embeddings, we see that we may assume that the sequence of functions $\phi(y_0 + y_{0n}, y)$ converges uniformly over compact subsets of $\mathbb{R}^{N+1}$ to a nontrivial, bounded solution of

$$\Delta_y\tilde{\phi} + a_0^\infty\partial_{00}\tilde{\phi} + p\omega(y)^{p-1}\tilde{\phi} = 0 \quad \text{in } \mathbb{R}^{N+1},$$

where a_0^∞ is a positive constant, which with no loss of generality, via scaling, we may assume to be equal to one. By virtue of Lemma 3.1 and the orthogonality conditions assumed, which pass to the limit thanks to dominated convergence, and the assumptions $N \ge 7$, $N - 2 < \alpha$, we then find that $\tilde{\phi} \equiv 0$. This is a contradiction that shows (3.8).

Let us now deduce the result of Step 1. Since $\|\phi_n\|_* = 1$, there exists (y_{0n}, y_n) with $r_n := |y_n| \to +\infty$ such that

$$r_n^{\nu-2}|\phi_n(y_{0n}, y_n)| + r_n^{\nu-1}|D\phi_n(y_{0n}, y_n)| \ge \gamma > 0.$$

Let us now consider the scaled function

$$\tilde{\phi}_n(z, x) = r_n^{\nu-2}\phi_n(y_{0n} + r_n z_0, r_n z)$$

defined on

$$\tilde{\mathcal{D}} = \{(z_0, \bar{z}, z_N) : -\tilde{R}_n(z_0) < z_N < M_n r_n^{-1},\ |\bar{z}| < M_n r_n^{-1}\}$$

with $\tilde{R}_n(z_0) = r_n^{-1} R_n(y_{0n} + r_n z_0)$. Note that $M_n r_n^{-1} \geq 1/\sqrt{2}$. Then we have

$$|\tilde{\phi}_n(z_0, z)| + |z|\, |D\tilde{\phi}(z_0, z)| \leq |z|^{2-\nu} \quad \text{in } \tilde{\mathcal{D}},$$

and for some z_n with $|z_n| = 1$,

$$|\tilde{\phi}_n(0, z_n)| + |D\tilde{\phi}(0, z_n)| \geq \gamma > 0.$$

The function $\tilde{\phi}_n$ satisfies

$$\tilde{a}_{0n}\partial_{00}\tilde{\phi}_n + \Delta_z\tilde{\phi}_n + o(1)\partial_{ij}\tilde{\phi}_n + O(r_n^{-2})|z|^{-4}\tilde{\phi}_n = \tilde{h}_n \quad \text{in } \tilde{\mathcal{D}},$$

where

$$\tilde{h}_n(z_0, z) = r_n^{\nu} h_n(y_{0n} + r_n z_0, r_n z), \qquad \tilde{b}^n(z_0) = b^n(y_{0n} + r_n z_0).$$

Let us observe that from the assumptions made we get

$$\|\partial_0 \tilde{b}^n\|_\infty + \|\partial_0 \tilde{R}_n\|_\infty + \|\partial_{00}\tilde{R}_n\|_\infty \to 0.$$

Then we may assume that

$$\tilde{b}^n(z_0) \to b_* > 0,$$

and that the function $\tilde{\phi}_n$ converges uniformly, in the C^1 sense over compact subsets of $\mathcal{D}_* \setminus \{z = 0\}$, to $\tilde{\phi}$ which satisfies

$$b_*\partial_{00}\tilde{\phi} + \Delta_z\tilde{\phi} = 0 \quad \text{in } \mathcal{D}_* \setminus \{z = 0\},$$

where either

$$\mathcal{D}_* = \{(z_0, \bar{z}, z_N) : 0 < z_N < d_*,\ |\bar{z}| < d_*\}$$

with $1 < d_* < +\infty$, or

$$\mathcal{D}_* = \{(z_0, \bar{z}, z_N) : a_* < z_N\}$$

with $a_* \geq 0$, or

$$\mathcal{D}_* = \mathbb{R}^{N+1},$$

and where $\tilde{\phi}$ satisfies

$$|\tilde{\phi}(z_0, z)| + |z|\, |\tilde{\phi}(z_0, z)| \leq |z|^{2-\nu} \quad \text{in } \mathbb{R}^{N+1}_{d_*} \setminus \{z = 0\}$$

with the value $\tilde{\phi} = 0$ assumed continuously on the boundary of $\partial\mathcal{D}_* \setminus \{z = 0\}$. Moreover, since $\partial_{00}\tilde{R}_n$ is uniformly bounded, standard elliptic estimates at the boundary yield a uniform $C^{1,\alpha}$ bound for $\tilde{\phi}_n$, which thus implies that the limit of the derivative is uniform, therefore $\tilde{\phi} \not\equiv 0$. With no loss of generality we may assume that $b_* = 1$. If the singular line $z = 0$ lies inside $\mathcal{D}_*$, the fact that $\nu < N$ makes it removable. Indeed, the limit $\tilde{\phi}$ is easily seen to be weakly harmonic in $\mathcal{D}_*$. This plus boundedness of the boundary value zero yields $\tilde{\phi} \equiv 0$ in all cases. If the singularity lies on the boundary, this happens on the hyperplane $z_N = 0$. In such a case, an odd reflection reduces us to the case of an interior singularity, so that in any event, $\tilde{\phi} \equiv 0$. We have obtained a contradiction which concludes Step 1.

Step 2. We claim that the a priori estimate estimate obtained in Step 1 is in reality valid for the full problem (3.4), maybe on reducing the value of δ. Let δ be a small number so that the conclusion of Step 1 holds. Now we additionally assume

$$\|Db_{ij}\|_\infty + \|(1+|y|)b_i\|_\infty + \|(1+|y|^2)d\|_\infty \le \delta, \tag{3.9}$$

where δ will be taken smaller if necessary. Then there exist positive numbers δ, C such that if the conditions of Proposition 3.2 and estimate (3.9) hold for all i, j, then for any h with $\|h\|_{**} < +\infty$ and any solution ϕ of problem (3.4) with $\|\phi\|_* < +\infty$ we have, for all i,

$$|c_i|_\infty + \|\phi\|_* \le C\|h\|_{**}.$$

Moreover

$$c_l(y_0)\int_{\mathcal{D}_{y_0}} Z_l^2 = -\int_{\mathcal{D}_{y_0}} h(y_0, y)Z_l(y)\,dy + o(1)\|h\|_{**},$$

where $o(1) \to 0$ as $\delta \to 0$.

Testing the equation against $Z_l(y)$ and integrating only in y we find

$$\begin{aligned} c_l(y_0)\int_{\mathcal{D}_{y_0}} Z_l^2 &= b(y_0)\int_{\mathcal{D}_{y_0}} \partial_{00}\phi Z_l - \int_{\mathcal{D}_{y_0}} hZ_l + \int_{\mathcal{D}_{y_0}} b_{ij}\partial_{ij}\phi Z_l \\ &\quad + \int_{\mathcal{D}_{y_0}} (b_i\partial_i\phi + d\phi)Z_l + \int_{\mathbb{R}^{N-1}} Z_l(\bar{y}, R(y_0))\partial_{y_N}\phi(y_0, \bar{y}, R(y_0))\,d\bar{y}. \end{aligned} \tag{3.10}$$

Now, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^{N-1}} Z(y', R(x_0))\partial_{y_N}\phi(x_0, y', R(x_0))\,dy'\right| \\ &\qquad\qquad \le \|\phi\|_* \int_{\mathbb{R}^{N-1}} (|y'| + R(x_0))^{2-N+1-\alpha}\,dy' \le \delta^\sigma\|\phi\|_* \end{aligned}$$

for some $\sigma > 0$ depending on α and N. We immediately find that also

$$\left|\int_{\mathcal{D}_{y_0}} (b_i\partial_i\phi + c\phi)Z_l\right| \le C\delta\|\phi\|_*,$$

while integrating by parts in indices carrying the y' variables gives

$$\left|\int_{\mathcal{D}_{y_0}} a_{ij}\partial_{ij}\phi Z_l\right| = \left|\int_{\mathcal{D}_{y_0}} \partial_i(a_{ij}Z_l)\partial_j\phi\right| \le C\delta\|\phi\|_*$$

and

$$\left|\int_{\mathcal{D}_{y_0}} hZ_l\right| \le C\|h\|_{**}.$$

Now, we know that

$$\int_{\mathcal{D}_{y_0}} \phi(y_0, y)Z_l(y)\,dy = 0$$

and hence, using the boundary value zero,

$$\int_{\mathcal{D}_{y_0}} \partial_0\phi(y_0, y) Z_l(y)\, dy = 0,$$

or

$$\int_{\mathbb{R}^{N-1}} dy' \int_{-\infty}^{R(y_0)} \partial_0\phi(y_0, \bar{y}, t) Z_l(y', t)\, dt = 0,$$

so that differentiating once more we find

$$0 = \int_{\mathcal{D}_{y_0}} \partial_{00}\phi Z_l\, dx + \partial_0 R(x_0) \int_{\mathbb{R}^{N-1}} \partial_0\phi(y_0, \bar{y}, R(y_0)) Z_l(y', R(y_0))\, dy',$$

which implies that

$$\left| \int_{\mathcal{D}_{y_0}} \partial_{00}\phi Z_l\, dy \right| \le C\delta^\sigma \|\phi\|_*.$$

Combining the above inequalities into (3.10) we then find the estimate

$$|c_l(y_0)| \le C(\|h\|_{**} + \delta^\sigma \|\phi\|_*)\ . \tag{3.11}$$

On the other hand, Lemma 3.2 implies that

$$\|\phi\|_* \le C\Big[\|h\|_{**} + \sum_i \|c_i Z_i\|_{**}\Big] \le C\Big[\|h\|_{**} + \sum_i \|c_i\|_\infty + \delta\|\phi\|_*\Big].$$

Combining this last inequality and (3.11), and reducing the value of δ if necessary, we find that the c_is are controlled by h,

$$\|c_i\|_\infty \le C\|h\|_{**},$$

and the result of Step 2 readily follows.

Step 3. We shall next discuss the existence for problem (3.4), under the assumptions such that the result of Step 2 holds true. We consider first the case of the right hand sides $h(y_0, y)$ which are T-periodic in y_0, for an arbitrarily large but fixed T, the same property being valid for the coefficients.

We then look for a weak solution ϕ to (3.4) in the space H_T defined as the subspace of functions ψ which are in $H^1(B)$ for any bounded subset B of $\mathcal{D}$, which are T-periodic in y_0, such that in addition $\psi = 0$ on $\partial\mathcal{D}$ in the trace sense, and

$$\int_{\mathcal{D}_{y_0}} \psi(y_0, y) Z_j(y)\, dy = 0 \quad \text{for all } y_0 \in \mathbb{R},\ j = 0, \dots, N+1.$$

Let $\mathcal{D}_T = \{y \in \mathcal{D} : y_0 \in (-T, T)\}$ and define a bilinear form in H_T (after one integration by parts) by

$$B(\phi, \psi) := \int_{\mathcal{D}_T} \psi L\phi.$$

Then problem (3.4) gets weakly formulated as that of finding $\phi \in H_T$ such that

$$B(\phi, \psi) = \int_{\mathcal{D}_T} h\psi \quad \text{for all } \psi \in H_T.$$

If h is smooth, elliptic regularity implies that a weak solution is a classical one. The weak formulation can be readily put in the form

$$\phi + K(\phi) = \hat{h}$$

in H_T, where $\hat{h}$ is a linear operator of h and K is compact. The a priori estimate of Step 2 shows that for $h = 0$ there is only the trivial solution. The Fredholm alternative thus applies, proving that problem (3.4) is solvable in the periodic setting. While this is enough for our purposes, it is worth observing that approximating a general h by periodic functions of increasing period, and using the uniform estimate provided by Step 2, we obtain in the limit a solution to the problem with the desired property. This completes the proof of the proposition. □

4. Geometric setting

We consider the metric induced by the Euclidean one on $\partial\Omega$ and denote by $\bar{\nabla}$ the associated connection. We introduce Fermi coordinates in a neighborhood of Γ in

$$\Sigma := \partial\Omega.$$

Given $q \in \Gamma$, there is a natural splitting

$$T_q\Sigma = T_q\Gamma \oplus N_q\Gamma$$

into the normal and tangent bundle over Γ. We assume that Γ is parameterized by arc-length $x_0 \in (-\ell, \ell)$,

$$x_0 \mapsto \gamma(x_0),$$

and denote by E_0 a unit tangent vector to Γ. In a neighborhood of a point q of Γ, assume that we are given an orthonormal basis E_i, $i = 1, \ldots, N-1$, of $N_q\Gamma$. We can assume that the E_i are parallel along Γ, which means that

$$\bar{\nabla}_{E_0} E_i = 0$$

for $i = 1, \ldots, N-1$. The geodesic condition for Γ translates precisely into

$$\bar{\nabla}_{E_0} E_0 \equiv 0.$$

To parameterize a neighborhood of $q \in \Gamma$ in Σ we define

$$F(x_0, \bar{x}) := \mathrm{Exp}^{\Sigma}_{\gamma(x_0)}\Big(\sum_i x_i E_i\Big), \qquad \bar{x} := (x_1, \ldots, x_{N-1}),$$

where Exp^Σ is the exponential map on Σ and summation over $i = 1, \ldots, N-1$ is understood. This parameterization induces coordinate vector fields

$$X_a := F_*(\partial_{x_a}),$$

for $a = 0, \ldots, N-1$. By construction $X_a = E_a$ along Γ and

$$\bar{\nabla}_{E_a} E_b = 0. \tag{4.1}$$

Let $\bar{g}$ denote the metric on Σ which is induced by the Euclidean metric. The Fermi coordinates above are defined in such a way that the coefficients of $\bar{g}$,

$$\bar{g}_{ab} = X_a \cdot X_b,$$

are equal to δ_{ab} along Γ. We now compute higher order terms in the Taylor expansions of the functions g_{ab}. The metric coefficients at $q := F(x_0, \bar{x})$ are given in terms of geometric data at $p := F(x_0, 0)$ and $\bar{x}$.

Notation. The symbol $\mathcal{O}(|\bar{x}|^r)$ indicates a smooth function whose Taylor expansion does not involve any term up to order r in the variables x_i, $i = 1, \ldots, N-1$.

We now give the expansion of the metric coefficients. The expansion of the $\bar{g}_{ij}$, $i, j = 1, \ldots, N-1$, agrees with the well known expansion for the metric in normal coordinates but we briefly recall the proof here for completeness. We agree that indices $a, b, c, \ldots$ run from 0 to $N-1$ while $i, j, k, \ldots$ run from 1 to $N-1$.

Proposition 4.1. *At the point $q = F(x_0, \bar{x})$, the following expansions hold:*

$$\begin{aligned} \bar{g}_{ij} &= \delta_{ij} + \tfrac{1}{3}(\bar{R}(E_i, E_k)E_j \cdot E_l)x_k x_l + \mathcal{O}(|\bar{x}|^3), \\ \bar{g}_{0i} &= \mathcal{O}(|\bar{x}|^2), \\ \bar{g}_{00} &= 1 + (\bar{R}(E_0, E_k)E_0 \cdot E_l)x_k x_l + \mathcal{O}(|\bar{x}|^3). \end{aligned} \tag{4.2}$$

where $i, j, k, l = 1, \ldots, N-1$ and summation over repeated indices is understood. Here $\bar{R}$ denotes the curvature tensor on $(\Sigma, \bar{g})$.

Proof. We compute

$$X_i \bar{g}_{ab} = \bar{\nabla}_{X_i} X_a \cdot X_b + X_a \cdot \bar{\nabla}_{X_i} X_b,$$

Using (4.1) we get $X_i \bar{g}_{ab} = 0$ along Γ. This yields the first order Taylor expansion

$$\bar{g}_{ab} = \mathcal{O}(|\bar{x}|^2).$$

To compute the second order terms, it is enough to compute $X_k X_k \bar{g}_{ab}$ at a point of Γ and then to polarize (i.e. replace X_k by $X_i + X_j, \ldots$). We compute

$$X_k X_k \bar{g}_{ab} = \bar{\nabla}^2_{X_k} X_a \cdot X_b + X_a \cdot \bar{\nabla}^2_{X_k} X_b + 2\bar{\nabla}_{X_k} X_a \cdot \bar{\nabla}_{X_k} X_b. \tag{4.3}$$

Recall that, since X_a are coordinate vector fields, we have

$$\bar{\nabla}^2_{X_k} X_a = \bar{\nabla}_{X_k} \bar{\nabla}_{X_a} X_k = \bar{\nabla}_{X_a} \bar{\nabla}_{X_k} X_k + \bar{R}(X_k, X_a) X_k. \tag{4.4}$$

Therefore, we get

$$X_k X_k \bar{g}_{ab} = 2\bar{R}(X_k, X_a)X_k \cdot X_b + 2\bar{\nabla}_{X_k} X_a \cdot \bar{\nabla}_{X_k} X_b + \bar{\nabla}_{X_a} \bar{\nabla}_{X_k} X_k \cdot X_b + X_a \cdot \bar{\nabla}_{X_b} \bar{\nabla}_{X_k} X_k. \tag{4.5}$$

Using this, together with (4.1) we get

$$E_k E_k \bar{g}_{ij} = 2\bar{R}(E_k, E_i)E_k \cdot E_i + \bar{\nabla}_{E_i} \bar{\nabla}_{E_k} E_k \cdot E_j + E_i \cdot \bar{\nabla}_{E_j} \bar{\nabla}_{E_k} E_k \tag{4.6}$$

along Γ. To proceed, first observe that

$$\bar{\nabla}_X X_{|p} = \bar{\nabla}^2_X X = 0$$

along Γ, for any $X \in N_p\Gamma$. Indeed, for all $p \in \Gamma$, $X \in N_p\Gamma$ is tangent to the geodesic $s \mapsto \exp^{\Sigma}_p(sX)$, and so $\bar{\nabla}_X X = \bar{\nabla}^2_X X = 0$ at p. In particular, taking $X = X_k + \varepsilon X_j$, we obtain

$$0 = \bar{\nabla}_{E_k+\varepsilon E_j} \bar{\nabla}_{E_k+\varepsilon E_j} (E_k + \varepsilon E_j).$$

Equating the coefficient of ε to 0 gives $\bar{\nabla}_{E_j} \bar{\nabla}_{E_k} E_k = -2\bar{\nabla}_{E_k} \bar{\nabla}_{E_k} E_j$, and hence

$$3\bar{\nabla}^2_{E_k} E_j = \bar{R}(E_k, E_j)E_k.$$

So finally, using (4.3) together with (4.6), we get

$$E_k E_k \bar{g}_{ij} = \tfrac{2}{3}\bar{R}(E_k, E_i)E_k \cdot E_j$$

along Γ. The formula for the second order Taylor coefficient for $\bar{g}_{ij}$ now follows at once.

Finally, it follows from (4.5) together with (4.1) that

$$E_k E_k \bar{g}_{00} = 2\bar{R}(E_k, E_0)E_k \cdot E_0 + 2\bar{\nabla}_{E_0} \bar{\nabla}_{E_k} E_k \cdot E_0$$

along Γ. Since $\bar{\nabla}_{E_k} E_k = 0$ along K, we also get $\bar{\nabla}_{E_0} \bar{\nabla}_{E_k} E_k = 0$ along Γ. We conclude that

$$E_k E_k \bar{g}_{00} = 2\bar{R}(E_k, E_0)E_k \cdot E_0$$

along Γ and this gives the formula for the second order Taylor expansion for $\bar{g}_{00}$. □

Notation. In what follows, we will use the notation

$$R_{ijlm} = \bar{R}(E_i, E_j)E_l \cdot E_m. \tag{4.7}$$

To parameterize a neighborhood of a point $q \in \Gamma$ in $\bar{\Omega}$, we consider the system of coordinates $(x_0, x) \in \mathbb{R}^{N+1}$ introduced in (1.8) given by

$$G(x_0, x) = F(x_0, \bar{x}) - x_N \mathbf{n}(F(x_0, \bar{x})), \qquad x = (\bar{x}, x_N) \in \mathbb{R}^N,$$

where $x \in \mathbb{R}^N$ is close to 0 and $\mathbf{n}$ designates the outward unit normal to Σ.

In these coordinates, the coefficients of the Euclidean metric read

$$g_{NN} = 1 \quad \text{and} \quad g_{aN} = g_{Na} = 0 \tag{4.8}$$

for all $a = 0, \dots, N-1$. Finally, for $a, b = 0, \dots, N-1$, the coefficients g_{ab} can be expanded in powers of x_N as

$$g_{ab} = \bar{g}_{ab} + 2\bar{h}_{ab} x_N + \bar{k}_{ab} x_N^2 + O(x_N^3),$$

where $\bar{g}$ is the metric on Σ whose expansion has been given in the last section,

$$\bar{h}_{ab} := -E_a \cdot \nabla_{E_b} \mathbf{n} = -E_b \cdot \nabla_{E_a} \mathbf{n} \tag{4.9}$$

are the coefficients of the second fundamental form $\bar{h}$ of Σ and

$$\bar{k}_{ab} := (\bar{h} \otimes \bar{h})_{ab} = \sum_{c,d} \bar{h}_{ac} \bar{g}^{cd} \bar{h}_{db} \tag{4.10}$$

are the coefficients of the square of the second fundamental form. An important remark is that $\bar{h}_{00}$, computed along Γ, is a smooth function of the arclength which represents the normal curvature along the geodesic in the sense that

$$\partial_{x_0}^2 \gamma = \nabla_{E_0} E_0 = \bar{h}_{00} \mathbf{n} \tag{4.11}$$

along Γ.

Building on the expansion of the metric which has been obtained above, we give the expansion of the Laplace operator in the above defined coordinates. Recall that the Laplacian is given, in terms of the coefficients of the metric, by

$$\Delta = \frac{1}{\sqrt{|g|}} \partial_{x_\alpha} (\sqrt{|g|} g^{\alpha\beta} \partial_{x_\beta}) = g^{pq} \partial_{x_\alpha} \partial_{x_\beta} + \partial_p g^{\alpha\beta} \partial_{x_\beta} + \tfrac{1}{2} \operatorname{Tr}_g (\partial_{x_\alpha} g) g^{\alpha\beta} \partial_{x_\beta},$$

where the indices α, β run from 0 to N and where $|g|$ denotes the determinant of the metric. Since (4.8) holds, the above formula simplifies to

$$\Delta = \partial_{x_N}^2 + \tfrac{1}{2} \operatorname{Tr}_g (\partial_{x_N} g) \partial_{x_N} + g^{ab} \partial_{x_a} \partial_{x_b} + \partial_{x_a} g^{ab} \partial_{x_b} + \tfrac{1}{2} \operatorname{Tr}_g (\partial_{x_a} g) g^{ab} \partial_{x_b},$$

where the indices a, b run from 0 to $N-1$.

We have the following decomposition (recall that $i, j, k, l, m, \dots$ run from 1 to $N-1$):

$$\begin{aligned}
\Delta = {} & \partial_{x_0}^2 + \sum_j \partial_{x_j}^2 + \partial_{x_N}^2 + A^{00} \partial_{x_0}^2 + \sum_j A^{0j} \partial_{x_0} \partial_{x_j} \\
& + \sum_{i,j} \Big(-\tfrac{1}{3} \sum_{k,l} (\bar{R}(E_i, E_k) E_j \cdot E_l) x_k x_l - 2\bar{h}_{ij} x_N + A^{ij} \Big) \partial_{x_i} \partial_{x_j} \\
& + B^0 \partial_{x_0} + \sum_j \Big(\sum_k \big(\tfrac{2}{3} \bar{R}(E_i, E_j) E_i \cdot E_k + \bar{R}(E_0, E_j) E_0 \cdot E_k \big) x_k + B^j \Big) \partial_{x_j} \\
& + (\operatorname{Tr}_{\bar{g}} \bar{h} - \operatorname{Tr}_{\bar{g}} \bar{k} x_N + B^N) \partial_{x_N},
\end{aligned} \tag{4.12}$$

where the curvature tensor $\bar{R}$, the metric $\bar{g}$ and the tensors $\bar{h}$ and $\bar{k}$ are computed along Γ, and hence only depend on x_0, while the functions $A^{\alpha\beta}$ and B^{α} do depend on $x_0, x_1, \ldots, x_N$ and enjoy the following decompositions:

$$
\begin{aligned}
A^{00} &= A^{00}_N x_N + \sum_{k,l} A^{00}_{kl} x_k x_l, \\
A^{ij} &= A^{ij}_N x_N^2 + \Big(\sum_k A^{ij}_{Nk} x_k\Big) x_N + \sum_{k,l,m} A^{ij}_{kl} x_k x_l x_m, \\
A^{0j} &= A^{0j}_N x_N + \sum_{k,l} A^{0j}_{kl} x_k x_l, \\
B^{0} &= B^{0}_N x_N + \sum_{k} B^{0}_{k} x_k, \\
B^{j} &= B^{j}_N x_N + \sum_{k,l} B^{j}_{kl} x_k x_l, \\
B^{N} &= B^{N}_N x_N^2 + \Big(\sum_k B^{N}_{k} x_k\Big) x_N + \sum_{j} B^{N}_{j} x_j.
\end{aligned}
\tag{4.13}
$$

Here $A^{00}_N, A^{00}_{kl}, A^{ij}_N, \ldots$ and $B^0_N, B^0_k, B^j_N, \ldots$ are smooth functions depending on $x_0, \ldots, x_N$, hence they can be further decomposed using Taylor's expansion. More precise expansions can be given in terms of the geometric data defined above but they will not appear in the final result so we have chosen to leave the expansion as it is. For example A^{0j}_N can be further expanded in powers of x_N and we have

$$A^{0j}_N = -4\bar{h}_{0j} x_N + \tilde{A}^{0j}_N x_N^2,$$

where $\tilde{A}^{0j}_N$ is a smooth function depending on $x_0, \ldots, x_N$.

5. Construction of a first approximation

This section is devoted to the construction of an approximation for a solution to our problem

$$\Delta u + u^{\frac{N+2}{N-2}-\varepsilon} = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

As explained in Section 2, the idea is to build the approximation using the standard bubble ω in $\mathbb{R}^N$ satisfying

$$\Delta u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}^N,$$

centered and translated along a curve which is located inside the domain Ω and, at the same time, very close to the geodesic Γ in $\partial\Omega$. We will thus first introduce a precise description of the approximation in a region extremely close to the geodesic, without taking into account the outer region. Since the solution turns out to be very concentrated, this description is accurate enough and a gluing procedure we perform in Section 6 is the key instrument to gather together this thin region close to the geodesic with the outer region.

Let $(x_0, x) \in \mathbb{R}^{N+1}$ be the local coordinates along the geodesic introduced in (1.8). We perform the change of variables introduced in Section 2 (formula (2.7)),

$$u(G(x_0, x)) = \mu_\varepsilon^{-\frac{N-2}{2}} v(\rho^{-1} x_0, \mu_\varepsilon^{-1}(x - d_\varepsilon)), \quad v = v(y_0, y), \quad \rho = \varepsilon^{\frac{N-1}{N-2}}, \tag{5.2}$$

where

$$\mu_\varepsilon(x_0) = \rho \tilde{\mu}_\varepsilon(x_0), \quad d_\varepsilon(x_0) = \varepsilon \tilde{d}_\varepsilon(x_0) \tag{5.3}$$

are functions of the arclength $x_0 \in (-\ell, \ell)$ (see (2.6)). We now need to be more precise in the description of μ_ε and d_ε. We assume that

$$\tilde{\mu}_\varepsilon(x_0) = \mu_\varepsilon^0(x_0) + \varepsilon \mu(x_0), \quad \tilde{d}_{\varepsilon N}(x_0) = d_{\varepsilon N}(x_0) + \varepsilon d_N(x_0), \tag{5.4}$$

and

$$\tilde{d}_{\varepsilon j}(x_0) = \varepsilon d_j(x_0) \quad \text{for all } j = 1, \dots, N-1. \tag{5.5}$$

In (5.4), μ_ε^0 and $d_{\varepsilon N}(x_0)$ are explicit smooth functions of x_0 of the form

$$\mu_\varepsilon^0 = \mu_0(x_0) + \varepsilon^{\frac{1}{N-2}} \mu_1(x_0), \quad d_{\varepsilon N}(x_0) = d_{0N}(x_0) + \varepsilon^{\frac{1}{N-2}} d_{1N}(x_0), \tag{5.6}$$

with

$$\mu_0(x_0) = \frac{\alpha}{\bar{h}_{00}(x_0)}, \quad d_{0N}(x_0) = \frac{\beta}{\bar{h}_{00}(x_0)}, \tag{5.7}$$

where α and β are positive constants depending only on the dimension N, and $\bar{h}_{00}$ is the normal curvature along the geodesic Γ, which is assumed to be smooth and strictly positive (see (4.11)). The functions μ_1, d_{1N} in (5.6) are smooth functions of x_0, uniformly bounded in ε together with their derivatives, whose precise definition we give later in Section 5 (see (5.37)).

Finally in (5.4) and (5.5), we assume that μ, $d = (d_1, \dots, d_{N-1}, d_N)$ are parameter functions defined in $(-\ell, \ell)$ to be adjusted only in the final finite-dimensional reduction. For now, we assume they are smooth functions of x_0 and that they have the following norms bounded:

$$\|\mu\|_a = \|\varepsilon^{\frac{N}{N-2}} \ddot{\mu}\|_\infty + \|\varepsilon^{\frac{N}{2(N-2)}} \dot{\mu}\|_\infty + \|\mu\|_\infty \tag{5.8}$$

and

$$\|d\|_d = \|d_N\|_b + \sum_{j=1}^{N-1} \|d_j\|_c, \tag{5.9}$$

where

$$\|d_N\|_b = \|\varepsilon \ddot{d}_N\|_\infty + \|\varepsilon^{1/2} \dot{d}_N\|_\infty + \|d_N\|_\infty, \tag{5.10}$$

$$\|d_j\|_c = \|\ddot{d}_j\|_\infty + \|\dot{d}_j\|_\infty + \|d_j\|_\infty \quad \text{for } j = 1, \dots, N-1. \tag{5.11}$$

In the previous expressions and in the rest of the paper, the dot denotes the derivative with respect to x_0.

The (y_0, y) variables belong to the set $\mathcal{D}$ defined in (2.9). We recall the definition

$$\mathcal{D} = \left\{ (y_0, \bar{y}, y_N) : -\frac{d_{\varepsilon N}}{\mu_\varepsilon}(\rho y_0) < y_N < \frac{\hat{\delta}}{\rho},\ |\bar{y}| < \frac{\hat{\delta}}{\rho} \right\},$$

for some fixed positive number $\hat{\delta}$ we will choose later. The domain $\mathcal{D}$ expands as $\varepsilon \to 0$ to the whole space $\mathbb{R}^N$. Observe that, with our choice of μ_ε and $d_{\varepsilon N}$ in (5.4)–(5.6), we have

$$-d_{\varepsilon N}/\mu_\varepsilon = -\varepsilon^{-\frac{1}{N-2}}[\gamma + \varepsilon^{\frac{1}{N-2}} O(1)], \tag{5.12}$$

where γ is a positive constant, depending only on N, and where $O(1)$ denotes a smooth function of x_0, which is uniformly bounded in ε, together with its derivative, for μ and d with $\|\mu\|_a + \|d\|_d \le c$ (see (5.8)–(5.9)). In particular, the function $R = d_{\varepsilon N}/\mu_\varepsilon$ satisfies assumption (3.6). Not only this. We have

$$\|\partial_0(d_{\varepsilon N}/\mu_\varepsilon)\|_\infty \le c\rho\varepsilon^{-\frac{1}{N-2}}(\varepsilon\|\dot{\mu}\|_\infty + \varepsilon\|\dot{d}_N\|_\infty) \le c\varepsilon^{3/2},$$

and

$$\rho^{-1}\|\partial_{00}(d_{\varepsilon N}/\mu_\varepsilon)\|_\infty \le c\rho\varepsilon^{-\frac{1}{N-2}}(\varepsilon\|\ddot{\mu}\|_\infty + \varepsilon\|\ddot{d}_N\|_\infty) \le c\varepsilon^{\frac{3N-8}{2(N-2)}}.$$

Thus the function $d_{\varepsilon N}/\mu_\varepsilon$ satisfies (3.5).

As we rigorously prove in Lemma 5.1 below, the Laplace operator, whose expansion is described in (4.12), after the change of variable (2.7) gets transformed by the following relation:

$$\mu_\varepsilon^{\frac{N+2}{2}} \Delta u = \mathcal{A}(v), \tag{5.13}$$

where, in $\mathcal{D}$, the differential operator $\mathcal{A}$ can be written in the compact form

$$\mathcal{A}v = a_0 \partial_0^2 v + \Delta_y v + \tilde{\mathcal{A}}v. \tag{5.14}$$

In (5.14), a_0 is given by

$$a_0 = (\mu_0 + \varepsilon^{\frac{1}{N-2}}\mu_1 + \varepsilon\mu)^2$$

(see (2.10)). Observe that

$$\rho^{-1}\|\partial_0 a_0\|_\infty \le c\varepsilon\|\dot{\mu}\|_\infty \le c\varepsilon^{\frac{N-4}{2(N-2)}},$$

thus the function a_0 satisfies (3.5).

Furthermore, in $\mathcal{D}$ the differential operator $\tilde{\mathcal{A}}$ can be described as follows:

$$\tilde{\mathcal{A}}v = \sum_{(\alpha,\beta)} a_{\alpha,\beta}\partial_{\alpha,\beta} v + \sum_\alpha b_\alpha \partial_\alpha v + cv, \tag{5.15}$$

where $a_{\alpha,\beta}$, d_α and c are functions of the variable $(\rho y_0, y)$, depending in an algebraic way on the parameter functions μ_ε and d_ε. More precisely, given the choice in (5.3)–(5.5), one has, in the region under consideration,

$$a_{\alpha,\beta} = O(\varepsilon + \rho^2|y|^2) \quad \text{if } \alpha \ne 0,\ \beta \ne 0, \qquad a_{0,\beta} = O(\varepsilon), \quad \text{and} \quad a_{0,0} = 0,$$

while

$$b_\alpha = \rho O(\varepsilon + \rho|y|) \quad \text{and} \quad c = \rho^2 O(1).$$

Condition (3.5) is thus satisfied by the differential operator $\mathcal{A}$. This fact, together with the estimates on $d_{\varepsilon N}/\mu_\varepsilon$ in the definition of $\mathcal{D}$ in (2.9), gives that the linear theory developed in Section 3 for the linear operator $\mathcal{A} + p\omega^{p-1}$ in the domain $\mathcal{D}$ can be applied.

The next lemma gives the detailed computation of the differential operator $\mathcal{A}$ in terms of the geometry of the problem.

Lemma 5.1. *After the change of variable* (2.7)*, the following holds true:*

$$\mu_\varepsilon^{\frac{N+2}{2}} \Delta u = \mathcal{A}(v) := a_0 \partial_0^2 v + \Delta_y v + \sum_{k=0}^{5} \mathcal{A}_k v + B(v), \tag{5.16}$$

where a_0 is defined in (2.10)*. In the previous expression $\mathcal{A}_k$ denotes the following differential operators:*

$$\begin{aligned}
\mathcal{A}_0 v = {} & \dot{\mu}_\varepsilon^2 \big[D_{yy} v[y]^2 + 2(1+\gamma) D_y v[y] + \gamma(1+\gamma) v \big] \\
& + \dot{\mu}_\varepsilon [D_{yy} v[y] + \gamma D_y v][\dot{d}_\varepsilon] + D_{yy} v[\dot{d}_\varepsilon]^2 \\
& - 2\mu_\varepsilon \big[\varepsilon^{-\frac{N-1}{N-2}} D_y(\partial_0 v)[\dot{\mu}_\varepsilon y + \dot{d}_\varepsilon] + \gamma \dot{\mu}_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_0 v \big] \\
& - \mu_\varepsilon D_y v[\ddot{d}_\varepsilon] - \mu_\varepsilon \ddot{\mu}_\varepsilon (\gamma v + D_y v[y]),
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\mathcal{A}_1 v = {} & \sum_{i,j} \Big[-\tfrac{1}{3} R_{ikjl} (\mu_\varepsilon y_k + d_{\varepsilon k})(\mu_\varepsilon y_l + d_{\varepsilon l}) - 2\bar{h}_{ij} (\mu_\varepsilon y_N + d_{\varepsilon N}) \\
& + \sum_k a_{Nk}^{ij} (\mu_\varepsilon y_k + d_{\varepsilon k})(\mu_\varepsilon y_N + d_{\varepsilon N}) \Big] \partial_{ij} v,
\end{aligned} \tag{5.18}$$

where R_{ikjl} is defined in (4.7)*, $\bar{h}_{ij}$ is given in* (4.9) *and the functions $a_{Nk}^{ij} = a_{Nk}^{ij}(\varepsilon^{\frac{N-1}{N-2}} y_0)$ are given by*

$$A_{Nk}^{ij} = a_{Nk}^{ij} x_N + O(x_N^2),$$

with A_{Nk}^{ij} defined in (4.13)*. Furthermore,*

$$\begin{aligned}
\mathcal{A}_2 v = {} & \sum_j \big[-4\bar{h}_{0j} (\mu_\varepsilon y_N + d_{\varepsilon N}) \\
& \times \big(-D_y(\partial_j v)[d] + \mu_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_{0j} v - (\gamma \partial_j v + D_y(\partial_j v)[y]) \dot{\mu}_\varepsilon \big) \big]
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
\mathcal{A}_3 v = {} & \Big(\sum_k b_k^0 [\mu_\varepsilon y_k + d_{\varepsilon k}] + b_N^0 (\mu_\varepsilon y_N + d_{\varepsilon N}) \Big) \\
& \times \big\{ \mu_\varepsilon \big[-D_y v[\dot{d}_\varepsilon] + \mu_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_0 v - \dot{\mu}_\varepsilon (\gamma v + D_y v[y]) \big] \big\},
\end{aligned} \tag{5.20}$$

where b_k^0 are smooth functions of $\varepsilon^{\frac{N-1}{N-2}} y_0$ given by

$$B_k^0 = b_k^0 x_N + O(x_N^2)$$

(see (4.13) *for B_k^0). Finally,*

$$\mathcal{A}_4 v = \sum_j \Big[\sum_k (\tfrac{2}{3} R_{ijik} + R_{0j0k})(\mu_\varepsilon y_k + d_{\varepsilon k}) + b_N^j(\mu_\varepsilon y_N + d_{\varepsilon N})\Big]\mu_\varepsilon \partial_j v, \tag{5.21}$$

where b_N^j are smooth functions of $\varepsilon^{\frac{N-1}{N-2}} y_0$ given by

$$B_N^j = b_N^j x_N + O(x_N^2)$$

(see (4.13) *for B_N^j), and*

$$\mathcal{A}_5 v = (\mathrm{Tr}_{\bar g}\, \bar h - \mathrm{Tr}_{\bar g}\, \bar k(\mu_\varepsilon y_N + d_{\varepsilon N}))\mu_\varepsilon \partial_N v, \tag{5.22}$$

where $\bar h$ is given by (4.9) *and $\bar k$ by* (4.10). *The operator $B(v)$ can be described as follows:*

$$\begin{aligned} B(v) = {} & O\big(|\mu_\varepsilon \bar y + \bar d_\varepsilon|^2 + (\mu_\varepsilon y_N + d_{\varepsilon N}) + (\mu_\varepsilon y_N + d_{\varepsilon N})(\mu_\varepsilon \bar y + \bar d_\varepsilon)\big)\mathcal{A}_0(v) \\ & + O\big(|\mu_\varepsilon \bar y + \bar d_\varepsilon|^3 + (\mu_\varepsilon y_N + d_{\varepsilon N})|\mu_\varepsilon \bar y + \bar d_\varepsilon|^2 + (\mu_\varepsilon y_N + d_{\varepsilon N})^2\big)\partial_{ij} v \\ & + O\big(|\mu_\varepsilon \bar y + \bar d_\varepsilon|^2 + (\mu_\varepsilon y_N + d_{\varepsilon N})|\mu_\varepsilon \bar y + \bar d_\varepsilon| + (\mu_\varepsilon y_N + d_{\varepsilon N})^2\big) \\ & \times \big[\mu_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_{0j} v + \mu_\varepsilon \varepsilon^{-\frac{N-1}{N-2}} \partial_0 v - D_y(\partial_j v)[d_\varepsilon] \\ & - (\gamma \partial_j v + D_y(\partial_j v)[y])\dot\mu_\varepsilon - D_y v \dot d_\varepsilon - \dot\mu_\varepsilon(\gamma v + D_y v[y]) + \mu_\varepsilon \partial_j v\big] \\ & + O\big((\mu_\varepsilon \bar y + \bar d_\varepsilon)^2 + (\mu_\varepsilon \bar y + \bar d_\varepsilon)(\mu_\varepsilon y_N + d_{\varepsilon N}) + (\mu_\varepsilon y_N + d_{\varepsilon N})^2\big)\mu_\varepsilon \partial_N v. \end{aligned}$$

Proof. We will show first that

$$\mu_\varepsilon^{\gamma+2} \partial_0^2 u(x_0, x) = \rho^{-2} \mu_\varepsilon^2 \partial_0^2 v(y_0, y) + \mathcal{A}_0(v(y_0, y)). \tag{5.23}$$

If $v = v(y_0, y)$, we define

$$\tilde v(z_0, z, \mu_\varepsilon) := \mu_\varepsilon^{-\gamma} v(z_0, \mu_\varepsilon^{-1} z).$$

We have $u(x_0, x) = \tilde v(\rho^{-1} x_0, x - d, \mu_\varepsilon)$. Then we compute

$$\partial_0 u = D_z \tilde v[-\dot d_\varepsilon] + \rho^{-1} \partial_0 \tilde v + \dot\mu_\varepsilon \partial_{\mu_\varepsilon} \tilde v,$$

and

$$\begin{aligned} \partial_0^2 u = {} & D_{zz} \tilde v[\dot d_\varepsilon]^2 + \rho^{-2} \partial_0^2 \tilde v + \dot\mu_\varepsilon^2 \partial_{\mu_\varepsilon}^2 \tilde v - 2\rho^{-1} D_z(\partial_0 \tilde v)[\dot d_\varepsilon] \\ & + 2\rho^{-1} \dot\mu_\varepsilon \partial_{0\mu_\varepsilon} \tilde v - 2\dot\mu_\varepsilon D_z(\partial_{\mu_\varepsilon} \tilde v)[\dot d_\varepsilon] - D_z \tilde v[\ddot d_\varepsilon] - \ddot\mu_\varepsilon \partial_{\mu_\varepsilon} \tilde v. \end{aligned}$$

Thus formula (5.23) follows by expressing the previous computations in terms of v. To get the rest of (5.16), one argues in a similar way. □

With respect to the local coordinates along the geodesic Γ previously introduced and after scaling the variables as in (2.7), the original equation reduces locally close to the geodesic to

$$\mathcal{A}v + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon} v^{p-\varepsilon} = 0, \tag{5.24}$$

where $\mathcal{A}$ is defined in (5.14) and $p = (N+2)/(N-2)$. We denote by S_ε the operator given by (5.24),

$$S_\varepsilon(v) := \mathcal{A}v + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon} v^{p-\varepsilon}. \tag{5.25}$$

In the rest of this section we study equation (5.24) in the set $(y_0, y) \in \mathcal{D}$ and we build an approximate solution to (5.24) which furthermore satisfies the zero Dirichlet boundary condition in the region $y_N = -d_{\varepsilon N}/\mu_\varepsilon$. Indeed, our approximation close to the geodesic is

$$\mathtt{W} = \mathtt{w} + \Pi. \tag{5.26}$$

We start with the description of $\mathtt{w}$. The definition of Π will be given at the end of this section.

We define $\mathtt{w}$ to be

$$\mathtt{w} = \tilde{\omega} + e_\varepsilon(\rho y_0)\chi_\varepsilon(y) Z_0. \tag{5.27}$$

The first term in (5.27) is $\tilde{\omega}$ defined as follows:

$$\tilde{\omega}(y) := (1+\alpha_\varepsilon)(\omega(y) - \bar{\omega}(y)), \tag{5.28}$$

with ω given in (2.4), $\alpha_\varepsilon := \mu_\varepsilon^{(N-2)^2\varepsilon/8} - 1$ and

$$\bar{\omega}(y) = \omega(\bar{y}, y_N + 2d_{\varepsilon N}/\mu_\varepsilon).$$

Observe that

$$\Delta((1+\alpha_\varepsilon)\omega) + \mu_\varepsilon^{-\frac{N-2}{2}\mu_\varepsilon}((1+\alpha_\varepsilon)\omega)^p = 0 \quad \text{in } \mathbb{R}^N.$$

In the second term in (5.27), Z_0 denotes the first eigenfunction in $L^2(\mathbb{R}^N)$ of the problem

$$\Delta\phi + pw(x)^{p-1}\phi = \lambda\phi \quad \text{in } \mathbb{R}^N, \quad \lambda_1 > 0$$

with $\int Z_0^2 = 1$ and χ_ε is a cut-off function defined as follows. Let $\chi = \chi(s)$ for $s \in \mathbb{R}$, with $\chi(s) = 1$ if $s < \hat{\delta}$, $\chi(s) = 0$ if $s > 2\hat{\delta}$, for some fixed $\hat{\delta} > 0$ chosen in such a way that $\chi_\varepsilon(\bar{y}, -d_{\varepsilon N}/\mu_\varepsilon) = 0$, where $\chi_\varepsilon(y) = \chi(\varepsilon^{1/(N-2)}|y|)$. Observe that the function $\mathtt{w}$ satisfies the Dirichlet boundary condition for $y_N = -d_{\varepsilon N}/\mu_\varepsilon$.

Finally, in (5.27) the function $e_\varepsilon(\rho y_0)$ is defined as follows:

$$e_\varepsilon = \varepsilon\tilde{e}_\varepsilon \quad \text{with} \quad \tilde{e}_\varepsilon = e_\varepsilon^0 + \varepsilon e \quad \text{and} \quad e_\varepsilon^0 = e_0 + \varepsilon^{\frac{1}{N-2}} e_1, \tag{5.29}$$

where e_1 is an explicit smooth function, uniformly bounded in ε, whose expression we give in (5.37), and

$$e_0 = \frac{2\int_{\mathbb{R}^N} \partial_{ii}\omega Z_0}{\lambda_1}(\mathrm{Tr}_{\bar{g}}\,\bar{h} - \bar{h}_{00})d_{0N}. \tag{5.30}$$

Finally, in (5.29), the function e is unknown and, for now, it plays the role of a parameter. It will be chosen later on, together with $\mu, d_1, \dots, d_N$ in (5.4) and (5.5), to be a solution of a system of $N+2$ ordinary differential equations. For the moment, we assume that e is a smooth function with the norm

$$\|e\|_e = \|\varepsilon^{2+\frac{2}{N-2}}\ddot{e}\|_\infty + \|\varepsilon^{1+\frac{1}{N-2}}\dot{e}\|_\infty + \|e\|_\infty \tag{5.31}$$

uniformly bounded by a positive constant independent of ε.

The error one commits by considering $\mathtt{w}$ a real solution to (5.24) is given by the size of $S_\varepsilon(\mathtt{w})$, which is itself a function of the parameter functions μ, d and e. Assume that μ, d and e, defined respectively in (5.4), (5.5) and (5.29), satisfy the assumption

$$\|(\mu, d, e)\| := \|\mu\|_a + \|d\|_d + \|e\|_e \le c \tag{5.32}$$

for some constant $c > 0$, independent of ε.

Then for all ε small enough and $(y_0, y) \in \mathcal{D}$, we have the expansion

$$\begin{aligned}
S_\varepsilon(\mathtt{w}) &= -p\omega^{p-1}\bar{\omega} - \varepsilon\omega^p \log\omega + \varepsilon[-2\bar{h}_{ij}d^0_{\varepsilon N}\partial_{ij}\omega + \lambda_1 e^0_\varepsilon Z_0] \\
&+ \varepsilon^{1+\frac{1}{N-2}}\mu^0_\varepsilon[-2\bar{h}_{ij}y_N\partial_{ij}\omega + \mathrm{Tr}_{\bar{g}}\,\bar{h}\partial_N\omega] \\
&+ \varepsilon^2\Big[(\rho^2 a_0\ddot{e} + \lambda_1 e)Z_0 - 2\bar{h}_{ij}d_N\partial_{ij}\omega \\
&+ \sum_{ij}(\dot{d}_i\dot{d}_j - \tfrac{1}{3}R_{ijkl}d_k d_l + a^{ij}_{Nk}d_k d^0_{\varepsilon N} + 4\bar{h}_{0j}d_i d^0_{\varepsilon N})\partial_{ij}\omega + \Upsilon_\varepsilon\Big] \\
&+ \varepsilon^{2+\frac{1}{N-2}}\mu^0_\varepsilon\Big[-\sum_j \partial_j\omega\cdot\ddot{d}_j + \Big(-\sum_{ij}\tfrac{1}{3}R_{ijkl}y_k d_l\partial_{ij}\omega + 2a^{ij}_{Nk}y_k d^0_{\varepsilon N}\partial_{ij}\omega\Big) \\
&+ (\tfrac{2}{3}R_{ijik} + R_{0j0k})d_k\partial_j\omega + 4\bar{h}_{0j}\dot{d}_i y_N\partial_{ij}\omega\Big] \\
&+ \varepsilon^{3+\frac{1}{N-2}}\big[-\mu^0_\varepsilon\partial_N\omega\cdot\ddot{d}_N - \tfrac{1}{3}\mu^0_\varepsilon R_{ijkl}y_k d_l\partial_{ij}\omega + \mu(\tfrac{2}{3}R_{ijik} + R_{0j0k})d_k\partial_j\omega \\
&+ (\mu^0_\varepsilon d_N + \mu d^0_{\varepsilon N})(2a^{ij}_{Nk}y_k\partial_{ij}\omega + b^j_N\partial_j\omega - \mathrm{Tr}_{\bar{g}}\,\bar{h}\partial_N\omega) \\
&+ (\mu^0_\varepsilon e + \mu e^0_\varepsilon)(-2\bar{h}_{ij}y_N\partial_{ij}Z_0 + \mathrm{Tr}_{\bar{g}}\,\bar{h}\partial_N Z_0)\big] \\
&+ \varepsilon^{3+\frac{2}{N-2}}\big[-\ddot{\mu}\mu Z_{N+1} \\
&+ 2\mu\mu^0_\varepsilon\big(-\tfrac{1}{3}R_{ikjl}y_k y_l\partial_{ij}\omega + (\tfrac{2}{3}R_{ijik} + R_{0j0k})y_k\partial_j\omega + b^j_N y_N\partial_j\omega - \mathrm{Tr}_{\bar{g}}\,\bar{k}y_N\partial_N\omega\big)\big] \\
&+ \varepsilon^4(\log\varepsilon)r,
\end{aligned} \tag{5.33}$$

where

$$\Upsilon_\varepsilon = \Upsilon_0 + \varepsilon^{\frac{1}{N-2}}\Upsilon^1_\varepsilon \tag{5.34}$$

with

$$\Upsilon_0 = -2\bar{h}_{ij}d_{0N}e_0\partial_{ij}Z_0 + p(p-1)e_0^2\omega^{p-2}Z_0^2 + pe_0\omega^{p-1}\log\omega\, Z_0,$$

and Υ^1_ε a sum of functions of the form

$$f_1(\varepsilon^{1+\frac{1}{N-2}}y_0)f_2(\mu, d, e)f_3(y)$$

with f_1 a smooth explicit function of the variable $\varepsilon^{1+\frac{1}{N-2}} y_0$, uniformly bounded in ε, f_2 a smooth function of μ, d and e, uniformly bounded in ε for μ, d and e satisfying (5.32), and f_3 a smooth function of the variable y, with $\sup(1+|y|^{N-2})|f_3(y)| < +\infty$.

In the previous expansion, $\bar{h}$ is the second fundamental form on Σ defined in (4.9), $\bar{k}$ is the square of the second fundamental form defined in (4.10), and R_{ijkl} are the components of the curvature tensor $\bar{R}$ on $(\Sigma, \bar{g})$ as defined in (4.7). Here indices i, j, k, l run from 1 to $N-1$ and summation over repeated indices is understood. Finally a_{Nk}^{ij} is defined as $A_{Nk}^{ij} = a_{Nk}^{ij} x_N + O(x_N^2)$ (see (4.13)).

Finally the term r in the expansion (5.33) is a sum of functions of the form

$$h_0(\varepsilon^{1+\frac{1}{N-2}} y_0)[f_1(\mu, d, \dot{\mu}, \dot{d}) + o(1) f_2(\mu, d, e, \dot{\mu}, \dot{d}, \dot{e}, \ddot{\mu}, \ddot{d}, \ddot{e})] f_3(y)$$

with h_0 a smooth function uniformly bounded in ε, and f_1 and f_2 smooth functions of their arguments, uniformly bounded in ε when μ, d and e satisfy (5.32). An important remark is that the function f_2 depends linearly on the argument $(\ddot{\mu}, \ddot{d}, \ddot{e})$. Concerning f_3, we have

$$\sup\,(1+|y|^{N-2})|f_3(y)| < +\infty.$$

We postpone the proof of the expansion (5.33) to the Appendix, Section 9 and we continue the description of $\mathtt{w}$ in (5.27).

We now use formula (5.33) to compute, for each y_0, the $L^2(\mathcal{D}_{y_0})$ projection of the error $S_\varepsilon(\mathtt{w})$ (see (5.25) and (5.27)) along the functions Z_i, $i = 0, 1, \ldots, N+1$ (see (2.11) and (2.12)). Here $\mathcal{D}_{y_0}$ denotes the y_0 section of the domain $\mathcal{D}$, defined in (2.9),

$$\mathcal{D}_{y_0} = \{y : (y_0, y) \in \mathcal{D}\}.$$

Denote

$$C_1 := \int_{\mathbb{R}^N} Z_i^2, \quad C_2 := \int_{\mathbb{R}^N} Z_{N+1}^2, \quad C_3 := \int_{\mathbb{R}^N} Z_0^2.$$

We start with the projections in the tangential directions Z_i, for $i = 1, \ldots, N-1$. Assume μ, d and e satisfy (5.32). Then for ε small enough, and any $k = 1, \ldots, N-1$,

$$\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_k = \varepsilon^{2+\frac{1}{N-2}} C_1[\mu_0(-\ddot{d}_k + R_{0j0k} d_j) + \alpha_k(\rho y_0) + \varepsilon \beta_k(\rho y_0; \mu, d, e)] + \varepsilon^3 r. \tag{5.35}$$

In (5.35), R_{0j0k} are the components as defined in (4.7) of the curvature tensor $\bar{R}$ on $(\Sigma, \bar{g})$ as in Proposition 4.1, and the functions α_k are explicit, smooth and uniformly bounded in ε. The functions β_k are smooth functions of their arguments, they are bounded in ε when μ, d and e satisfy (5.32), and they do not depend on the derivatives of μ, d and e. Finally the term r denotes a sum of functions of the form

$$h_0(\rho y_0)[h_1(\mu, d, e, \dot{\mu}, \dot{d}, \dot{e}) + o(1) h_2(\mu, d, e, \dot{\mu}, \dot{d}, \dot{e}, \ddot{\mu}, \ddot{d}, \ddot{e})], \tag{5.36}$$

where h_0 is a smooth function uniformly bounded in ε, h_1 and h_2 are smooth functions of their arguments, uniformly bounded in ε when μ, d and e satisfy (5.32), and $o(1) \to 0$

as $\varepsilon \to 0$. An important remark is that h_2 depends linearly on the argument $(\ddot{\mu}, \ddot{d}, \ddot{e})$. We postpone the proof of (5.35) to the Appendix, Section 9.

Concerning the projection of $S_\varepsilon(\mathtt{w})$ in the remaining directions Z_{N+1}, Z_N and Z_0, they turn out to be much bigger than the projections along Z_i, for $i = 1, \dots, N-1$. Indeed, roughly speaking, they are at main order of size ε. To reduce this size, we expand of $\tilde{\mu}_\varepsilon$, $\tilde{d}_{\varepsilon N}$ and $\tilde{e}_\varepsilon$ in terms of the functions μ_0, d_{0N}, μ_1, d_{1N} in (5.6) and of e_0, e_1 in (5.30).

Indeed, if we assume μ, d and e satisfy (5.32), then we can prove that there exist a constant $\varpi > 0$ depending on N and smooth functions

$$\mu_0, d_{0N}, e_0, \mu_1, d_{1N}, e_1 : (-\ell, \ell) \to \mathbb{R}, \tag{5.37}$$

in the definitions (5.6), (5.29), (5.30) such that, as $\varepsilon \to 0$, for all $y_0 \in (-\rho^{-1}\ell, \rho^{-1}\ell)$, we have

$$\begin{aligned}\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_{N+1} = \varepsilon^2 [A\bar{h}_{00}\mu + B\bar{h}_{00} d_N + \alpha_{N+1}(\rho y_0) + \varepsilon\beta_{N+1}(\rho y_0; \mu, d, e)] \\ + \varepsilon^{3+\frac{2}{N-2}}[-C_2\mu_0\ddot{\mu}] + \varepsilon^4 r\end{aligned} \tag{5.38}$$

and

$$\begin{aligned}\varpi \int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_N = \varepsilon^{2+\frac{1}{N-2}}[B\bar{h}_{00}\mu + C\bar{h}_{00} d_N + \alpha_N(\rho y_0) + \varepsilon\beta_N(\rho y_0; \mu, d, e)] \\ + \varepsilon^{3+\frac{1}{N-2}}[-C_1\mu_0\ddot{d}_N] + \varepsilon^4 r.\end{aligned} \tag{5.39}$$

In (5.38) and (5.39), A, B and C are explicit constants which depend only on the dimension N, with $A, C > 0$ and $AC - B^2 > 0$. The function $\bar{h}_{00}$ is the curvature of the geodesic Γ on the boundary Σ as defined in (4.11). The functions α_{N+1}, α_N are explicit, smooth and uniformly bounded in ε. The functions β_{N+1}, β_N are smooth functions of their arguments, they are bounded in ε when μ, d and e satisfy (5.32), and they do not depend on the derivatives of μ, d and e.

Finally,

$$\begin{aligned}\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w}) Z_0 = \varepsilon^2 C_3 \Bigg[& \rho^2 a_0 \ddot{e} + \lambda_1 e - 2(\mathrm{Tr}_{\bar{g}}\, \bar{h} - \bar{h}_{00}) \left(\int \partial_{ii} \omega Z_0 \right) d_N + \alpha_0(\rho y_0) \\ & + \sum_i (\dot{d}_i^2 - \tfrac{1}{3} R_{ikil} d_k d_l \ + \ a_{Nk}^{ii} d_k d_{0N} + 4\bar{h}_{0j} d_j d_{0N}) \left(\int \partial_{ii} w Z_0 \right) \\ & + \varepsilon^2 \beta_0(\rho y_0; \mu, d, e) \Bigg] \\ & + \varepsilon^4 r.\end{aligned} \tag{5.40}$$

In (5.40), a_0 is the function defined in (2.10) and $\bar{h}$ is the second fundamental form of Σ as defined in (4.9). Again α_0 denotes an explicit smooth function, uniformly bounded in ε, and β_0 is a smooth function of its arguments, which is bounded in ε when μ, d and e satisfy (5.32), and it does not depend on the derivatives of μ, d and e.

In (5.38), (5.39) and (5.40), the term r denotes a sum of functions of the form (5.36).

We postpone the proof of (5.38), (5.39) and (5.40) to the Appendix, Section 9.

Thanks to the choice of the parameters performed in (5.37), from the expansion given in (5.33) we conclude that the error $S_\varepsilon(\mathtt{w})$, computed in (5.33), reduces to

$$S_\varepsilon(\mathtt{w}) = \varepsilon S_0 + \varepsilon[\rho^2 a_0 \ddot{e} + \lambda_1 e]\chi_\varepsilon Z_0 + \varepsilon^2 S_1, \tag{5.41}$$

where S_0 is a smooth function of ρy_0, uniformly bounded in ε. Observe that S_0 does not depend on μ, d and e. Furthermore, S_0 satisfies, for all $i = 0, 1, \ldots, N+1$,

$$\int_{\mathcal{D}_{y_0}} S_0 Z_i \, dy = 0 \quad \text{for all } y_0,$$

and

$$\|S_0\|_{**} \leq c$$

for some positive constant c independent of ε. In (5.41), a_0 is the function defined in (2.10), Z_0 is given by (2.12), and e is the parameter function which enters the definition (5.29) and whose $\|\cdot\|_e$ norm is bounded uniformly in ε (see (5.31)). On the other hand, S_1 depends on μ, d and e.

Now we introduce a further correction Π to $\mathtt{w}$, to get the final approximation $\mathtt{W} = \mathtt{w} + \Pi$ (5.26). The correction Π is chosen to reduce the size of the error (5.41), eliminating the term εS_0, as the unique solution of the following linear problem:

$$a_0 \partial_0^2 \Pi + \Delta_y \Pi + \tilde{\mathcal{A}} \Pi + p\omega^{p-1} \Pi = -\varepsilon S_0 + \sum \mathtt{c}_i Z_i \quad \text{in } \mathcal{D}, \tag{5.42}$$

$$\int_{\mathcal{D}_{y_0}} \Pi(y_0, y) Z_i \, dy = 0 \quad \forall y_0,\ \forall i = 0, \ldots, N+1 \tag{5.43}$$

and

$$\Pi(y_0, \bar{y}, y_N)_{|\partial \mathcal{D}_{y_0}} = 0 \quad \text{for all } y_0. \tag{5.44}$$

In (5.42), a_0 is defined as in (2.10), and $\tilde{\mathcal{A}}$ in (5.15). Taking into account the description of the linear operator (5.14) carried out at the beginning of this section, the assumptions of Proposition 3.2 are satisfied and the linear theory developed in Section 3 can be applied, given the estimate

$$\|\Pi\|_* \leq c\varepsilon \tag{5.45}$$

for some positive constant c. The linear operator in (5.42) depends on μ and d (but not on e). This implies that Π itself depends on μ and d. A direct analysis of (5.42), together with (5.14), shows that

$$\|\Pi_{\mu_1, d_1} - \Pi_{\mu_2, d_2}\|_* \leq c\varepsilon^2 \|(\mu_1 - \mu_2, d_1 - d_2)\|. \tag{5.46}$$

We next compute the size of $\mathtt{c}_i = \mathtt{c}_i(\rho y_0)$. Multiplying equation (5.42) with Z_i, and integrating on the section $\mathcal{D}_{y_0}$, we obtain, for all y_0,

$$\mathtt{c}_i \int_{\mathcal{D}_{y_0}} Z_i^2 = a_0 \int_{\mathcal{D}_{y_0}} \partial_0^2 \Pi Z_i + \int_{\mathcal{D}_{y_0}} (\Delta_y \Pi + p\omega^{p-1}\Pi) Z_i + \int_{\mathcal{D}_{y_0}} \tilde{\mathcal{A}}(\Pi) Z_i. \tag{5.47}$$

Taking into account (5.43) and (5.32), we have

$$\left|\int_{\mathcal{D}_{y_0}} \partial_0 \Pi Z_i\right| \leq o(1)\varepsilon^3, \quad \left|\int_{\mathcal{D}_{y_0}} \partial_0^2 \Pi Z_i\right| \leq o(1)\varepsilon^3,$$

where $o(1)$ denotes a small function of y_0. Furthermore, integrating by parts and using (5.43), we have

$$\left|\int_{\mathcal{D}_{y_0}} (\Delta_y \Pi + p\omega^{p-1}\Pi) Z_i\right| \leq o(1)\varepsilon^3.$$

Finally, from (5.14) we obtain

$$\left|\int_{\mathcal{D}_{y_0}} \tilde{\mathcal{A}}(\Pi) Z_i\right| \leq o(1)\varepsilon^3.$$

Thus we conclude that

$$\sup |\mathrm{c}_i| \leq o(1)\varepsilon^3. \tag{5.48}$$

Directly from (5.47) and (5.46) we see that $\mathrm{c}_i = \mathrm{c}_i[\mu, d]$ depends smoothly on μ, d and their derivatives. Indeed, we have

$$\|\mathrm{c}_i[\mu_1, d_1] - \mathrm{c}_i[\mu_2, d_2]\|_\infty \leq c\varepsilon^2 \|(\mu_1 - \mu_2, d_1 - d_2)\|. \tag{5.49}$$

Let $\psi := \partial_0 \Pi$. We have

$$a_0 \partial_0^2 \psi + \Delta_y \psi + \tilde{\mathcal{A}}\psi + p\omega^{p-1}\psi + \rho \dot{a}_0 \partial_0 \psi = h + \sum \partial_0 \mathrm{c}_i Z_i \quad \text{in } \mathcal{D} \tag{5.50}$$

with

$$h = -\varepsilon\rho\partial_0 S_0 - \partial_0 \tilde{\mathcal{A}}(\Pi),$$

$$\int_{\mathcal{D}_{y_0}} \psi(y_0, y) Z_i dy = o(1)\varepsilon \quad \forall y_0,\ \forall i = 0, \ldots, N+1 \tag{5.51}$$

and

$$\psi(y_0, \bar{y}, y_N)_{|\partial\mathcal{D}_{y_0}} - \partial_0(d_{\varepsilon N}/\mu_\varepsilon)\partial_N \Pi(y_0, \bar{y}, y_N)_{|\partial\mathcal{D}_{y_0}} = 0 \quad \text{for all } y_0. \tag{5.52}$$

Direct computations show that

$$\|h\|_{**} \leq C\varepsilon\rho$$

and condition (5.52) reduces to

$$\psi(y_0, \bar{y}, y_N)_{|\partial\mathcal{D}_{y_0}} = O(1)\varepsilon^{3-\frac{1}{N-2}},$$

where $O(1)$ denotes a smooth function of y_0, uniformly bounded in ε, for μ, d and e satisfying (5.32). We thus conclude that

$$\|\partial_0 \Pi\|_* \leq c\rho\varepsilon.$$

With this choice of Π we have

$$S_\varepsilon(\mathtt{W}) = \varepsilon^2 S_1 + \varepsilon[\rho^2 a_0 \ddot{e} + \lambda_1 e]\chi_\varepsilon Z_0 + N_1(\Pi) + \sum \mathtt{c}_i Z_i \tag{5.53}$$

(see (5.41)), where

$$N_1(\Pi) = \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}[(\mathtt{w} + \Pi)^{p-\varepsilon} - \mathtt{w}^{p-\varepsilon}] - p\omega^{p-1}\Pi. \tag{5.54}$$

Observe that S_1 depends smoothly on the parameters μ, d and e, and

$$\|S_1(\mu_1, d_1, e_1) - S_1(\mu_2, d_2, e_2)\|_{**} \le c\|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|. \tag{5.55}$$

We next estimate $\|N_1(\Pi)\|_{**}$. If $|y| \le \delta\varepsilon^{-1/2}$, we have

$$|N_1(\Pi)| \le c|\omega^{p-2}\Pi^2|.$$

Thus in this region, we have

$$\sup_{|y|<\delta\varepsilon^{-1/2}} |(1+|y|)^{N-2} N_1(\Pi)| \le c\varepsilon^2.$$

If now $|y| > \delta\varepsilon^{-1/2}$, then $|N_1(\Pi)| \le c|\Pi^p|$, so that

$$\sup_{|y|>\delta\varepsilon^{-1/2}} |(1+|y|)^{N-2} N_1(\Pi)| \le c\varepsilon^p \sup_{|y|>\delta\varepsilon^{-1/2}} |(1+|y|)^{-2+\frac{8}{N-2}}| \le c\varepsilon^{2+\frac{8}{N-2}}.$$

We conclude that

$$\|N_1(\Pi)\|_{**} \le c\|\omega^{p-2}\Pi^2\|_{**} \le c\varepsilon^2. \tag{5.56}$$

This concludes the construction of our approximation $\mathtt{W}$ (5.26) and the analysis of the error $S_\varepsilon(\mathtt{W})$ (5.53).

6. The gluing procedure

This section is devoted to a gluing procedure that reduces the full problem (2.1). A first observation is that on replacing u by $\rho^{(N-2)/2}u(\rho z)$ the problem becomes equivalent to

$$\begin{cases} \Delta u + \rho^{-\frac{N-2}{2}\varepsilon} u^{p-\varepsilon} = 0 & \text{in } \Omega_\varepsilon, \\ u > 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{6.1}$$

where $\Omega_\varepsilon = \rho^{-1}\Omega$.

The function $\mathtt{W}(y_0, y)$ built in the previous section in (5.26) defines an approximation W to a solution of (2.1) near the geodesic through the natural change of variables (5.4)–(5.3). More generally, let us denote by $z \in \mathbb{R}^{N+1}$ the original variable in Ω_ε. Then for a function $f(z)$ defined on a small neighborhood of Γ we use in this section the notation

$$f(z) = \tilde{\mu}_\varepsilon^{\frac{N-2}{2}}(\rho y_0)\tilde{f}(y_0, y) \quad \text{for } z = \rho^{-1}G(\rho y_0, \rho\tilde{\mu}_\varepsilon(\rho y_0)y + \varepsilon\tilde{d}_\varepsilon(\rho y_0))$$

or

$$\tilde{f}(y_0, y) = \tilde{\mu}_\varepsilon^{\frac{N-2}{2}}(\rho y_0) f(\rho^{-1} G(\rho y_0, \rho\tilde{\mu}_\varepsilon(\rho y_0) y + \varepsilon \tilde{d}_\varepsilon(\rho y_0)))$$

so that in particular W and $\mathtt{W}$ are linked as $\mathtt{W} = \tilde{W}$. In fact we recall that near Γ_ε, after setting in this language $v := \tilde{u}$, the equation in (6.1) becomes

$$S_\varepsilon(v) := \mathcal{A}v + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon} v^{p-\varepsilon} = 0, \tag{6.2}$$

where $\mathcal{A}$ is the operator defined in (5.14).

Let $\delta > 0$ be a fixed number with $4\delta < \hat{\delta}$, where $\hat{\delta}$ was chosen in (2.9). We consider a smooth cut-off function $\xi_\delta(s)$ such that $\xi_\delta(s) = 1$ if $0 < s < \delta$, and $= 0$ if $s > 2\delta$. Let us consider the cut-off function

$$\zeta_\delta^\varepsilon(y_0, y) = \zeta_\delta(|G(\rho y_0, \tilde{\mu}_\varepsilon(\rho y_0)\rho y + \varepsilon \tilde{d}_\varepsilon(\rho y_0))|),$$

and its pull-back to Ω_ε, supported near $\rho^{-1}\Gamma$, defined as

$$\eta_\delta^\varepsilon(z) = \zeta_\delta^\varepsilon(y_0, y) \quad \text{for } z = \rho^{-1} G(\rho y_0, \tilde{\mu}_\varepsilon(\rho y_0)\rho y + \varepsilon \tilde{d}_\varepsilon(\rho y_0)).$$

We observe that with this definition $\eta_\delta^\varepsilon(z)$ no longer depends on the parameter functions and it is well defined in the entire Ω_ε, by just extending it by zero outside the range of the variables (y_0, y). We define our global first approximation $\mathbf{w}(z)$ to a solution of (2.1) to be simply

$$\mathbf{w}(z) = \eta_\delta^\varepsilon(z)\tilde{\mathbf{w}}(z). \tag{6.3}$$

We look for a solution to problem (6.1) of the form $u = \mathbf{w} + \Phi$, namely

$$\begin{cases} \Delta\Phi + p\mathbf{w}^{p-1}\Phi + N(\Phi) + E = 0 & \text{in } \Omega_\varepsilon, \\ \Phi = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{6.4}$$

where

$$N(\Phi) = \rho^{-\frac{N-2}{2}\varepsilon}(\mathbf{w} + \Phi)^{p-\varepsilon} - \mathtt{w}^{p-\varepsilon} - p\mathbf{w}^{p-1}\Phi, \quad E = \Delta\mathbf{w} + \mathtt{w}^{p-\varepsilon}.$$

According to (6.2), near the geodesic $v = \tilde{u} + \tilde{\Phi}$ must then satisfy

$$\mathcal{A}\tilde{\Phi} + p\tilde{\mathbf{w}}^{p-1}\tilde{\Phi} + \mathtt{N}(\tilde{\Phi}) + S_\varepsilon(\tilde{\mathbf{w}}) = 0, \tag{6.5}$$

where now

$$\mathtt{N}(\tilde{\Phi}) = \tilde{\mu}_\varepsilon^{-\frac{N-2}{2}\varepsilon}(\tilde{\mathbf{w}} + \tilde{\Phi})^{p-\varepsilon} - \tilde{\mathtt{w}}^{p-\varepsilon} - p\tilde{\mathbf{w}}^{p-1}\tilde{\Phi}, \quad S_\varepsilon(\tilde{\mathbf{w}}) = \mathcal{A}\tilde{\mathbf{w}} + \tilde{\mathbf{w}}^{p-\varepsilon}.$$

We look for a solution Φ of (6.4) in the following form:

$$\Phi = \eta_{2\delta}\phi + \psi,$$

where the function ϕ is such that $\tilde{\phi}$ is in principle defined only in $\mathcal{D}$. It is immediate to check that Φ of this form will satisfy the above problem if the pair (ψ, ϕ) satisfies the following nonlinear coupled system:

$$\mathcal{A}\tilde{\phi} + p\tilde{\mathbf{w}}^{p-1}\tilde{\phi} = -\mathrm{N}(\zeta^{\varepsilon}_{2\delta}\tilde{\phi} + \tilde{\psi}) - \mathrm{E} - p\tilde{\mathbf{w}}^{p-1}\tilde{\psi} \quad \text{in } \mathcal{D}, \tag{6.6}$$

$$\tilde{\phi} = 0 \quad \text{on } \partial\mathcal{D}. \tag{6.7}$$

$$\begin{aligned} \Delta\psi + (1-\eta^{\varepsilon}_{2\delta})p\mathbf{w}^{p-1}\psi &= -2\nabla\phi\nabla\eta^{\varepsilon}_{2\delta} - \phi\Delta\eta^{\varepsilon}_{2\delta} \\ &\quad - (1-\eta^{\varepsilon}_{2\delta})N(\eta^{\varepsilon}_{2\delta}\phi + \psi) \quad \text{in } \Omega_\varepsilon, \\ \psi &= 0 \quad \text{on } \partial\Omega_\varepsilon. \end{aligned} \tag{6.8}$$

Given ϕ such that in $\mathcal{D}$, $\tilde{\phi}$ has a sufficiently small $\|\cdot\|_*$-norm, we first solve problem (6.8) for ψ.

Let us assume first that Ω is bounded. Since $\Omega_\varepsilon = \rho^{-1}\Omega$, the problem

$$-\Delta\psi = h \quad \text{in } \Omega_\varepsilon, \qquad \psi = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{6.9}$$

has a unique solution $\psi := (-\Delta)^{-1}(h)$ for each given $h \in L^\infty(\Omega_\varepsilon)$. Moreover

$$\|\psi\|_\infty \le C\left(\frac{N-1}{N-2}\right)^{-2}\|h\|_\infty.$$

Let us observe that, for instance,

$$\|\Delta\eta^{\varepsilon}_{2\delta}\phi\|_\infty \le C\rho^2\|\tilde{\phi}\|_{L^\infty(|y|>\delta\rho^{-1})} \le C\rho^{N-2}\|\tilde{\phi}\|_*.$$

We obtain similarly

$$\|\nabla\eta^{\varepsilon}_{2\delta}\nabla\phi\|_\infty \le C\rho^{N-2}\|\tilde{\phi}\|_*.$$

Let us now assume $\|\psi\|_\infty \le R\rho^{N-4}\|\tilde{\phi}\|_*$ and consider in this ball the operator

$$M(\psi) := (1-\eta^{\varepsilon}_{2\delta})N(\eta^{\varepsilon}_{2\delta}\phi + \psi) = (1-\eta^{\varepsilon}_{2\delta})(\eta^{\varepsilon}_{2\delta}\phi + \psi)^p.$$

We have

$$\begin{aligned} \|M(\psi_1) - M(\psi_2)\|_\infty &\le C(\|\tilde{\phi}\|_{L^\infty(|y|>\delta\rho^{-1})} + R\rho^{N-4}\|\tilde{\phi}\|_*)^{p-1}\|\psi_1 - \psi_2\|_\infty \\ &\le C(1+R)^{p-1}\rho^{\frac{4(N-4)}{N-2}}\|\phi\|_*^{p-1}\|\psi_1 - \psi_2\|_\infty. \end{aligned}$$

Observe that also

$$\|(1-\eta^{\varepsilon}_{2\delta})p\mathbf{w}^{p-1}\psi\|_\infty \le C\rho^4\|\psi\|_\infty.$$

By taking R suitably large but fixed, we see directly from an application of the contraction mapping principle that the fixed point problem, equivalent to (6.8),

$$\psi = (-\Delta)^{-1}(M(\psi) + (1-\eta^{\varepsilon}_{2\delta})p\mathbf{w}^{p-1}\psi + 2\nabla\phi\nabla\eta^{\varepsilon}_{2\delta} + \phi\Delta\eta^{\varepsilon}_{2\delta})$$

has a unique solution $\psi = \psi(\phi)$ with $\|\psi\|_\infty \leq R\rho^{N-4}\|\tilde{\phi}\|_*$, whenever $\|\tilde{\phi}\|_*$ is sufficiently small, independently of ε. Note that $\rho^{N-4} = \varepsilon^{N-3-2/(N-2)}$. In addition, the nonlinear operator ψ satisfies a Lipschitz condition of the form

$$\|\psi(\phi_1) - \psi(\phi_2)\|_\infty \leq C\varepsilon^{N-3-\frac{2}{N-2}}\|\phi_1 - \phi_2\|_*. \tag{6.10}$$

Let us now consider the case $\Omega = \mathbb{R}^N \setminus \Lambda$ with Λ bounded. In this case, exactly the same arguments go through. Indeed, let us pull back the equation for ψ to Ω in the following way: for $f(z)$ defined in Ω_ε let us write $\hat{f}(z) := f(z/\varepsilon)$. Equation (6.8) then becomes

$$\begin{aligned}\Delta\hat{\psi} &+ \rho^{-2}(1-\hat{\eta}^\varepsilon_{2\delta})p\hat{\mathbf{w}}^{p-1}\psi \\ &= -2\rho^{-2}\widehat{\nabla\phi}\widehat{\nabla\eta^\varepsilon_{2\delta}} - \hat{\phi}\rho^{-2}\widehat{\Delta\eta^\varepsilon_{2\delta}} - \rho^{-2}(1-\hat{\eta}^\varepsilon_{2\delta})(\hat{\eta}^\varepsilon_{2\delta}\hat{\phi}+\hat{\psi})^p \quad \text{in } \Omega, \\ &\hat{\psi} = 0 \quad \text{on } \partial\Omega,\end{aligned}$$

or

$$\Delta\hat{\psi} + O(\rho^2)\chi\psi = -2O(\rho^{N-6})\|\tilde{\phi}\|_*\chi - \rho^{-2}(O(\rho^{N-4})\|\tilde{\phi}\|_*\chi + \hat{\psi})^p \quad \text{in } \Omega,$$

where χ is just a function with bounded support. In the case of the exterior domain, after a Kelvin transform we see that the problem (in $\mathbb{R}^{N+1}$)

$$-\Delta\hat{\psi} = h \quad \text{in } \Omega, \qquad \hat{\psi} = 0 \quad \text{on } \partial\Omega, \tag{6.11}$$

has a solution $\hat{\psi} := (-\Delta)^{-1}(h)$ with

$$\|(1+|z|^{N-1})\hat{\psi}(z)\|_\infty \leq C\|(1+|z|^{N+3})h(z)\|_\infty < +\infty.$$

In this setting we can do a fixed point scheme similar to that before, the reason being that if

$$\|(1+|z|^{N-1})\hat{\psi}(z)\|_\infty \leq C\rho^{N-6}\|\tilde{\phi}\|_*,$$

then

$$|\hat{\psi}(z)|^p \leq \rho^{-2+(N-6)p}\|\tilde{\phi}\|_*^p(1+|z|)^{-p(N-1)}$$

and we also have $p(N-1) = (N+2)(N-1)/(N-2) > N-3$. Thus (6.8) can be solved in the same way as before, and the conclusion remains unchanged. It is worth observing that the energy of ψ in Ω_ε is small with ε indeed small in any case, provided that $\|\tilde{\phi}\|_*$ is bounded by some small fixed constant.

As a conclusion, substituting $\tilde{\psi} = \tilde{\psi}(\tilde{\phi})$ in equation (6.6), we have reduced the full problem (2.1) to solving the following (nonlocal) problem in $\mathcal{D}$:

$$\begin{aligned}\mathcal{A}\tilde{\phi} + p\tilde{\mathbf{w}}^{p-1}\tilde{\phi} &= -\mathrm{N}(\zeta^\varepsilon_{2\delta}\tilde{\phi} + \tilde{\psi}(\tilde{\phi})) - S_\varepsilon(\tilde{\mathbf{w}}) - p\tilde{\mathbf{w}}^{p-1}\tilde{\psi}(\phi) \quad \text{in } \mathcal{D}, \\ \tilde{\phi} &= 0 \quad \text{on } \partial\mathcal{D}.\end{aligned} \tag{6.12}$$

We will solve a projected version of this problem in the next section, and in Section 8 we will solve it in full.

7. The nonlinear projected problem

This section is devoted to solve a projected problem associated to (6.12). We shall relieve the notation in (6.12) dropping the tildes and write it as

$$L(\phi) = S_\varepsilon(\mathbf{w}) + N(\phi) \quad \text{in } \mathcal{D}, \tag{7.1}$$

$$\phi(y_0 + \rho^{-1}\ell, y) = \phi(y_0, y) \quad \text{for all } y_0, y, \tag{7.2}$$

$$\phi = 0 \quad \text{on } \partial\mathcal{D}, \tag{7.3}$$

where $L(\phi) = \mathcal{A}\phi + p\omega^{p-1}\phi$, with $\mathcal{A}$ defined in (5.14) and ω in (2.4), and $N(\phi)$ is given by

$$N(\phi) = p(\omega^{p-1} - \mathtt{w}^{p-1})\phi - \mathtt{N}(\zeta_{2\delta}^{\varepsilon}\phi + \psi(\phi)) + \zeta_{2\delta}^{\varepsilon} p\mathbf{w}^{p-1}\psi(\phi) \tag{7.4}$$

with

$$\mathtt{N}(\phi) = \tilde{\mu}_\varepsilon^{-\frac{N-2}{2}\varepsilon}(\mathbf{w} + \phi)^{p-\varepsilon} - \mathtt{w}^{p-\varepsilon} - p\mathbf{w}^{p-1}\tilde{\phi}.$$

Let us observe that $S_\varepsilon(\mathtt{W})$ can be decomposed in the following way:

$$S_\varepsilon(\mathtt{W}) = E + \{\varepsilon[\rho^2 a_0 \ddot{e}(\rho y_0) + \lambda_1 e(\rho y_0)]\}\chi_\varepsilon Z_0 \tag{7.5}$$

(see (5.53). The projected version of the problem is as follows: Given μ, d and e satisfying (5.32), the projected problem we want to solve is: find functions ϕ and $c_i(y_0)$ for $i = 0, \ldots, N+1$ so that

$$L(\phi) = E + N(\phi) + \sum_i c_i Z_i \quad \text{in } \mathcal{D}, \tag{7.6}$$

$$\phi(y_0 + \rho^{-1}\ell, y) = \phi(y_0, y) \quad \text{for all } y_0, y, \tag{7.7}$$

$$\phi = 0 \quad \text{on } \partial\mathcal{D}, \tag{7.8}$$

$$\int_{\mathcal{D}_{y_0}} \phi Z_i = 0 \quad \text{for all } i = 0, \ldots, N+1 \text{ and all } y_0. \tag{7.9}$$

Observe that the last term in (7.5) have been absorbed in $c_0 Z_0$.

For further reference, it is useful to point out the Lipschitz dependence of the error term S_1 on the parameters μ, d and e for the norms defined in (5.8), (5.9) and (5.31). We have the estimate

$$\|E(\mu_1, d_1, e_1) - E(\mu_1, d_1, e_1)\|_\infty \le c\varepsilon^2 \|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|. \tag{7.10}$$

This is a consequence of (5.53), (5.49), (5.46), (5.55). As already observed, we can apply the linear theory developed in Section 3. Given Proposition 3.2, solving (7.6)–(7.9) reduces to solving a fixed point problem, namely

$$\phi = T(E + N(\phi)) =: A(\phi), \tag{7.11}$$

where T is the operator defined in Proposition 3.2.

Consider the set

$$\mathcal{M} := \{\phi : \|\phi\|_* \le c\varepsilon^2\}$$

for a certain positive constant c.

We first show that A maps $\mathcal{M}$ in itself. Assume $\|\phi\|_* \leq c\varepsilon^2$. Then

$$\|A(\phi)\|_* \leq C\|E + N(\phi)\|_{**}.$$

We first estimate $\|E\|_{**}$. Given the definition (5.53) for S_1, we get

$$\|\chi_\varepsilon E\|_{**} \leq C\varepsilon^2. \tag{7.12}$$

Next we estimate $\|N(\phi)\|_{**}$. We have

$$\|N(\phi)\|_{**} \leq C[\|(\omega^{p-1} - \mathrm{w}^{p-1})\phi\|_{**} + \|\eta_{3\delta}^\varepsilon \mathrm{N}(\eta_{3\delta}^\varepsilon \phi + \psi(\phi))\|_{**} + \|\eta_{3\delta}^\varepsilon \mathbf{w}^{p-1}\psi(\phi)\|_{**}].$$

We get

$$\begin{aligned}\|(\omega^{p-1} - \mathrm{w}^{p-1})\phi\|_{**} &\leq C\|[(\omega + \varepsilon e Z_0 + \Pi)^{p-1} - \omega^{p-1}]\phi\|_{**}\\ &\leq C\|\omega^{p-2}(\varepsilon e Z_0 + \Pi)\phi\|_{**} \leq C\varepsilon\|\phi\|_*;\end{aligned}$$

furthermore

$$\begin{aligned}\|\zeta_{3\delta}^\varepsilon \mathrm{N}(\zeta_{3\delta}^\varepsilon \phi + \psi(\phi))\|_{**} &\leq C \sup_{|y|\leq c\varepsilon^{-1/2}} |(1+|y|)^{N-2}\omega^{p-2}(\phi+\psi)^2|\\ &\quad + \sup_{|y|\geq c\varepsilon^{-1/2}} (1+|y|)^{N-2}(|\phi|^p + |\psi|^p)\\ &\leq C\varepsilon^4\end{aligned}$$

and

$$\|\zeta_{3\delta}^\varepsilon \mathbf{w}^{p-1}\psi(\phi)\|_{**} \leq C\varepsilon^{N-3-\frac{2}{N-2}} \sup_{|y|\leq c\varepsilon^{-(N-1)/(N-2)}} (1+|y|)^{N-6}\|\phi\|_* \leq C\varepsilon^{2+\frac{2}{N-2}}\|\phi\|_*.$$

Thus we get

$$\|N(\phi)\|_{**} \leq C\varepsilon^3$$

for all $\|\phi\|_* \leq c\varepsilon^2$. Given (7.12), we conclude that $A(\phi) \in \mathcal{M}$ for any $\phi \in \mathcal{M}$, provided c in the definition of $\mathcal{M}$ is chosen large enough.

We next prove that A is a contraction mapping, so that the fixed point problem (7.11) can be uniquely solved in $\mathcal{M}$. This fact is a direct consequence of (6.10). Indeed, arguing as in the estimates above we obtain

$$\|A(\phi_1) - A(\phi_2)\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**} \leq C\varepsilon\|\phi_1 - \phi_2\|_*.$$

Emphasizing the dependence on μ, d, e, what we find for the linear operator T is the Lipschitz dependence

$$\|T_{\mu_1,d_1,e_1} - T_{\mu_2,d_2,e_2}\| \leq C\varepsilon\|(\mu_1-\mu_2, d_1-d_2, e_1-e_2)\|.$$

We recall that we have the Lipschitz dependence (7.10). Moreover, the operator N also has Lipschitz dependence on (μ, d, e). It is easily checked that for $\|\phi\|_* \leq C\varepsilon^2$ we have, with the obvious notation,

$$\|N_{(\mu_1,d_1,e_1)}(\phi) - N_{(\mu_2,d_2,e_2)}(\phi)\|_{**} \leq C\varepsilon^3\|(\mu_1-\mu_2, d_1-d_2, e_1-e_2)\|.$$

Hence from the fixed point characterization we see that

$$\|\phi_{(\mu_1,d_1,e_1)} - \phi_{(\mu_2,d_2,e_2)}\|_* \leq C\varepsilon^4 \|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|. \tag{7.13}$$

We have thus proved the following

Proposition 7.1. *There is a number $c > 0$ such that for all sufficiently small ε and all μ, d, e satisfying* (5.32)*, problem* (7.6)–(7.9) *has a unique solution $\phi = \phi(\mu, d, e)$ and $c_i = c_i(\mu, d, e)$ which satisfies*

$$\|\phi\|_* \leq c\varepsilon^2. \tag{7.14}$$

Moreover ϕ depends Lipschitz-continuously on μ, d and e in the sense of estimate (7.13).

8. The final adjustment of parameters: conclusion of the proof

In this section we will find equations relating μ, d and e to get all the coefficients c_i in (7.6) identically equal to zero. To do this, we multiply equation (7.6) by Z_i, for all $i = 0, \ldots, N+1$ (see (2.11) and (2.12)), and we integrate in y. Thus, the system

$$c_i(\rho y_0) = 0 \quad \text{for all } i = 0, \ldots, N+1$$

is equivalent to

$$\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{W}) Z_i \, dy + \int_{\mathcal{D}_{y_0}} (N(\phi) - \mathcal{A}\phi - \omega^{p-1}\phi) Z_i = 0 \quad \text{for all } i \text{ and } y_0,$$

where $S_\varepsilon(\mathtt{W})$ is defined in (5.53), $N(\phi)$ in (7.4), $\mathcal{A}$ in (5.14), and ω in (2.4).

Taking into account Section 7 and Proposition 7.1, we get

$$\int_{\mathcal{D}_{y_0}} (N(\phi) - \mathcal{A}\phi - \omega^{p-1}\phi) Z_i = \varepsilon^3 r,$$

where r is a sum of functions of the form

$$h_0(\rho y_0)[h_1(\mu, d, e, \dot{\mu}, \dot{d}, \dot{e}) + o(1) h_2(\mu, d, e, \dot{\mu}, \dot{d}, \dot{e}, \ddot{\mu}, \ddot{d}, \ddot{e})],$$

where h_0 is a smooth function uniformly bounded in ε, h_1 depends smoothly on μ, d, e and their first derivatives, it is bounded in the sense that

$$\|h_1\|_\infty \leq c\|(\mu, d, e)\|,$$

and it is compact, as a direct application of the Ascoli–Arzelà Theorem shows. The function h_2 depends on (μ, d, e), together with their first and second derivatives. An important remark is that h_2 depends linearly on $\ddot{\mu}$, $\ddot{d}$ and $\ddot{e}$. Furthermore it is Lipschitz, with

$$\|h_2(\mu_1, d_1, e_1) - h_2(\mu_2, d_2, e_2)\|_\infty \leq o(1)\|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|.$$

We next study $\int S_\varepsilon(\mathtt{W}) Z_i\, dy$, with $S_\varepsilon(\mathtt{W})$ given by (5.53). First we have

$$\int_{\mathcal{D}_{y_0}} \Big[N_1(\Pi) + \sum \mathtt{c}_i Z_i \Big] Z_j = \varepsilon^2 h_0(\rho y_0) + o(1)\varepsilon^3 r,$$

where $h_0(\rho y_0)$ is a smooth function of ρy_0, which does not depend on μ, d, e, and r is as before.

Taking into account the previous computation and the results of Section 5, (5.35), (5.38), (5.39), (5.40), we conclude that the equations

$$c_i = 0$$

are equivalent to the following limit system of $N+2$ nonlinear ordinary differential equations in the unknowns $\mu, d_1, \ldots, d_N, e$:

$$\begin{cases} L_{N+1}(\mu) := -C_2 \varepsilon^{1+\frac{2}{N-2}} \mu_0 \ddot{\mu} + A\mu + B d_N = \alpha_{N+1} + \varepsilon M_{N+1}, \\ L_N(d_N) := -C_1 \varpi \varepsilon \mu_0 \ddot{d}_N + B\mu + C d_N = \alpha_N + \varepsilon M_N, \\ L_k(d_k) := -\ddot{d}_k + \sum_{j=1}^{N-1} R_{0j0k} d_j = \alpha_k + \varepsilon M_k, \quad k = 1, \ldots, N-1, \\ L_0(e) := \rho^2 a_0 \ddot{e}(\rho y_0) + \lambda_1 e(\rho y_0) + \gamma_0 d_N = \alpha_0 + \varepsilon Q_0 + \varepsilon^2 M_0, \end{cases} \tag{8.1}$$

where μ, $d_1, \ldots, d_N$ and e satisfy periodic boundary conditions in $[-\ell, \ell]$. In (8.1), we have $A > 0$, $C > 0$ and $AC - B^2 > 0$. The functions α_i are explicit functions of x_0, smooth and uniformly bounded in ε. The function γ_0 is given by $\gamma_0 = 2(\mathrm{Tr}_{\bar{g}}\, \bar{h} - \bar{h}_{00})(\int \partial_{ii}\omega Z_0)$. The operators $M_i = M_i(\mu, d, e)$ can be decomposed in the following form:

$$M_i(f, e) = A_i(\mu, d, e) + K_i(\mu, d, e),$$

where K_i is uniformly bounded in $L^\infty(-\ell, \ell)$ for (μ, d, e) satisfying (5.32) and is also compact. The operator A_i depends on (μ, d, e) and their first and second derivatives and it is Lipschitz in this region, namely

$$\|A_i(\mu_1, d_1, e_1) - A_i(\mu_2, d_2, e_2)\|_\infty \le C o(1) \|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|.$$

We remark that the dependence on $\ddot{\mu}$, $\ddot{d}$ and $\ddot{e}$ is linear. Finally, the operator Q_0 is quadratic in d and it is uniformly bounded in $L^\infty(-\ell, \ell)$ for (μ, d, e) satisfying (5.32).

Our goal is now to solve (8.1) in μ, d and e. To do so, we first analyze the invertibility of the linear operators L_i.

We start with a linear theory, in L^∞ setting, for the problem of finding 2ℓ-periodic solutions of the problem

$$L_{N+1}(\mu) = h_1, \quad L_N(d) = h_2, \tag{8.2}$$

with h_1 and h_2 bounded. This is the content of the next lemma.

Lemma 8.1. *Assume that $A > 0$, $C > 0$ and $AC - B^2 > 0$ and that $\|h_1\|_\infty + \|h_2\|_\infty$ is bounded. Then there exist a 2ℓ-periodic solution (μ, d) to the above system and a constant c such that*

$$\|\mu\|_\infty + \|d\|_\infty + \varepsilon^{\frac{1}{2}+\frac{1}{N-2}} \|\dot{\mu}\|_\infty + \varepsilon^{\frac{1}{2}} \|\dot{d}\|_\infty \le c \left[\|h_1\|_\infty + \|h_2\|_\infty \right].$$

Proof. System (8.2) has a variational structure. The associated energy functional on the class of 2ℓ-periodic functions is positive, bounded from below away from zero and convex. Existence of solution thus follows.

In order to get the a priori estimate, we assume towards a contradiction that there exists a sequence (h_{1n}, h_{2n}) with

$$\|h_{1n}\|_\infty + \|h_{2n}\|_\infty \to 0$$

and a sequence of solutions (μ_n, d_n) with

$$\|\mu_n\|_\infty + \|d_n\|_\infty + \varepsilon^{\frac{1}{2}+\frac{1}{N-2}}\|\dot{\mu}_n\|_\infty + \varepsilon^{\frac{1}{2}}\|\dot{d}_n\|_\infty = 1.$$

Since $A > 0$ and $C > 0$, applying the maximum principle to each equation in the system, we see that $\|\mu_n\|_\infty \le c\|d_n\|_\infty$ and $\|d_n\|_\infty \le c\|\mu_n\|_\infty$. Hence we can assume $d_n(m_n) = \|d_n\|_\infty > \delta$ and $m_n \to m$. Scaling the system with $y = (x - m)/\varepsilon$, we find that the scaled functions, which we denote by $\hat{\mu}_n$ and $\hat{d}_n$, solve

$$\begin{aligned} -\varepsilon^{\frac{1}{N-2}} C_2 \hat{\mu}\ddot{\hat{\mu}}_n + A\hat{\mu}_n &= -B\hat{d}_n + o(1), \\ -C_1 \frac{A_2}{A_1}\hat{\mu}\ddot{\hat{d}}_n + C\hat{d}_n &= -B\hat{\mu}_n + o(1). \end{aligned} \tag{8.3}$$

From the second equation we deduce that $\|\hat{d}_n\|_\infty + \|\dot{\hat{d}}_n\|_\infty \le c$ and a direct application of the Ascoli–Arzelà Theorem implies that $\hat{d}_n \to \hat{d}$ uniformly on compact sets.

We state that

$$A\hat{\mu}_n \to -B\hat{d}. \tag{8.4}$$

Assume that this is not true. There exists a compact interval I and a sequence of points $x_n \in I$ such that

$$|A\hat{\mu}_n(x_n) + B\hat{d}(x_n)| > a \tag{8.5}$$

for a certain fixed positive constant a. Up to a subsequence, which we still denote x_n, we have $x_n \to x_0$. We now scale with $z = (y - x_0)/\varepsilon^{1/(N-2)}$, so that the scaled functions $\bar{\mu}_n$ and $\bar{d}_n$ satisfy

$$-C_2\hat{\mu}\ddot{\bar{\mu}}_n + A\bar{\mu}_n = -B\bar{d}_n + o(1).$$

In this scale, we get $\|\dot{\bar{d}}_n\|_\infty \le c\varepsilon^{1/2(N-2)} \to 0$. This implies that $\bar{d}_n$ converges uniformly over compact sets to a constant and this constant has to be $\hat{d}(x_0)$. Hence $A\bar{\mu}_n + B\bar{d}_n$ converges to 0 locally over compact sets. This is in contradiction with (8.5), and proves (8.4).

We now go back to (8.3), which reduces to saying that $\hat{d}$ solves

$$-C_1\hat{\mu}\ddot{\hat{d}} + (C - B^2/A)\hat{d} = 0.$$

Since $C - B^2/A > 0$, we conclude that $\hat{d} = 0$, a contradiction. $\square$

Concerning the invertibility of the operator L_0, we have the validity of the following lemma.

Lemma 8.2. *Assume that condition* (1.7) *holds. If $f \in C(-\ell, \ell) \cap L^\infty(-\ell, \ell)$ then there is a unique solution e of $L_0(e) = f$ which is 2ℓ-periodic and satisfies*

$$\rho^2 \|\ddot{e}\|_\infty + \rho \|\dot{e}\|_\infty + \|e\|_\infty \le C\rho^{-1} \|f\|_\infty.$$

Moreover, if f is in $C^2(-\ell, \ell)$, then

$$\rho^2 \|\ddot{e}\|_\infty + \rho \|\dot{e}\|_\infty + \|e\|_\infty \le C[\|\ddot{f}\|_\infty + \|\dot{f}\|_\infty + \|f\|_\infty].$$

Proof. Consider the following transformation:

$$l = \int_{-\ell}^{\ell} \frac{1}{\sqrt{a_0(s)}}\, ds, \quad t = \frac{\int_{-\ell}^{s} (\sqrt{a_0(\theta)})^{-1} d\theta}{l}, \quad \tilde{\lambda}_1 = \frac{l^2}{\pi^2} \lambda_1$$

and

$$y(t) = \tilde{e}(s).$$

Then the problem

$$L_0(\tilde{e}) = f, \quad \tilde{e}(-\ell) = \tilde{e}(\ell), \quad \dot{\tilde{e}}(-\ell) = \dot{\tilde{e}}(\ell)$$

reduces to

$$\rho^2 \ddot{y} + \tilde{\lambda}_1 \ddot{y} = \tilde{f}, \quad y(0) = y(\pi), \quad \dot{y}(0) = \dot{y}(\pi). \tag{8.6}$$

Thus (8.6) is solvable if and only if $\rho^2 \tilde{\lambda}_1 \ne \lambda_k$ for all $k \ge 0$, where λ_k is an infinite sequence of eigenvalues for (8.6), with $\tilde{f} = 0$, where $y_k(t)$ is an orthonormal basis of $L^2(0, \pi)$ formed by the eigenfunctions of

$$\ddot{y}_k + 4k^2 \ddot{y} = 0, \quad y_k(0) = y_k(\pi), \quad \dot{y}_k(0) = \dot{y}_k(\pi).$$

Furthermore,

$$\sqrt{\lambda_k} = 2k + O(1/k^3). \tag{8.7}$$

When (8.6) is solvable, its solution is given by

$$y(t) = \sum_{k=0}^{\infty} \frac{\tilde{f}_k}{\tilde{\lambda}_1 - 4k^2 \rho^2} y_k(t), \tag{8.8}$$

and $\|\tilde{f}\|_{L^2} = (\int_0^\pi \tilde{f}_k^2)^{1/2}$. Choose

$$|\rho^2 4k^2 - \tilde{\lambda}_1| \ge c\rho \tag{8.9}$$

for all k, where c is small. This corresponds precisely to the condition (1.7) in the statement of the theorem with

$$\kappa = \frac{\pi}{2} \sqrt{\lambda_1} \int_{-\ell}^{\ell} \frac{1}{\sqrt{a_0(s)}}\, ds. \tag{8.10}$$

From (8.9) we then find that $|\tilde{\lambda}_1 - \lambda_k \rho^2| \geq (c/2)\rho$ if ρ is also sufficiently small. It follows directly from expression (8.8) that $\|y\|_{L^\infty(0,\pi)} \leq C\rho^{-1}\|\tilde{f}\|_{L^\infty(0,\pi)}$. Observe also that

$$\|y'\|^2_{L^\infty(0,\pi)} \leq \sum_{k=0}^{\infty} |\tilde{f}_k|^2 \frac{1+|\lambda_k|^2}{(\tilde{\lambda}_1 - \lambda_k\rho^2)^2} \leq C\sum_{k=0}^{\infty}(1+k^4)|\tilde{f}_k|^2.$$

Hence

$$\rho\|y'\|_{L^\infty(0,\pi)} + \|y\|_{L^\infty(0,\pi)} \leq C\rho^{-1}\|\tilde{f}\|_{L^\infty(0,\pi)}.$$

Moreover, if $\tilde{f}$ is in $C^2(0,\pi)$ with $f(0) = f(\pi)$, $f'(0) = f'(\pi)$, then the sum $\sum_k k^4 \tilde{f}_k^2$ is finite and bounded by the C^2-norm of $\tilde{f}$. This automatically implies

$$\rho^2\|y''\|_{L^\infty(0,\pi)} + \|y'\|_{L^\infty(0,\pi)} + \|y\|_{L^\infty(0,\pi)} \leq C\|\tilde{f}\|_{C^2(0,\pi)},$$

and the proof is complete. □

We now conclude with

Proof of Theorem 1.1. Since the geodesic Γ is nondegenerate, the linear operator L_k is invertible in the set of 2ℓ-periodic functions. More precisely, there is a positive constant C such that for any $f \in L^\infty(-\ell,\ell)$, there exists a 2ℓ-periodic function d_k such that $L_k(d_k) = f$ and

$$\|\ddot{d}_k\|_\infty + \|\dot{d}_k\|_\infty + \|d_k\|_\infty \leq C\|f\|_\infty.$$

Define $\tilde{\mu}_0, \tilde{d}_{0N}, \tilde{d}_{0k}$ to be a solution of

$$L_{N+1}(\tilde{\mu}_0) = \alpha_{N+1}, \qquad L_N(\tilde{d}_{0N}) = \alpha_N,$$
$$L_k(\tilde{d}_{0k}) = \alpha_k \quad \text{for all } k = 1,\dots,N-1.$$

Thus we have

$$\varepsilon\|\ddot{\tilde{d}}_{0N}\|_\infty + \varepsilon^{1/2}\|\dot{\tilde{d}}_{0N}\|_\infty + \|\tilde{d}_{0N}\|_\infty \leq c, \qquad \|\ddot{\tilde{d}}_{0k}\|_\infty + \|\dot{\tilde{d}}_{0k}\|_\infty + \|\tilde{d}_{0k}\|_\infty \leq c$$

and

$$\varepsilon^{1+\frac{1}{N-2}}\|\ddot{\tilde{\mu}}_0\|_\infty + \varepsilon^{\frac{1}{2}+\frac{1}{N-2}}\|\dot{\tilde{\mu}}_0\|_\infty + \|\tilde{\mu}_0\|_\infty \leq c.$$

We now solve $L_0(\tilde{E}_0) = -2(\mathrm{Tr}_{\bar{g}}\,\bar{h} - \bar{h}_{00})(\int \partial_{ii}\omega Z_0)\tilde{d}_{0N} + \alpha_0 + \varepsilon Q_0(\tilde{d}_0)$, where $\tilde{d}_0 = (\tilde{d}_{01},\dots,\tilde{d}_{0N})$. Since the right hand side is regular, by Lemma 8.2 we have

$$\varepsilon^{2+\frac{2}{N-2}}\|\ddot{e}_0\|_\infty + \|E_0\|_\infty \leq c.$$

We have

$$\|(\tilde{\mu}_0, \tilde{d}_0, \tilde{E}_0)\| \leq c.$$

Define

$$\mu = \tilde{\mu}_0 + \tilde{\mu}_1, \quad d = \tilde{d}_0 + \tilde{d}_1, \quad e = \tilde{E}_0 + \tilde{e}_1.$$

The system (8.1) reduces to

$$\begin{cases} L_{N+1}(\tilde{\mu}_1) = \varepsilon M_{N+1}, \quad L_N(\tilde{d}_{1N}) = \varepsilon M_N, \\ L_k(\tilde{d}_{1k}) = \varepsilon M_k, \quad k = 1, \ldots, N-1, \\ L_0(\tilde{e}_1) = -2(\mathrm{Tr}_{\bar{g}}\, \bar{h} - \bar{h}_{00})\left(\displaystyle\int \partial_{ii}\omega Z_0\right)\tilde{d}_{1N} + \varepsilon^2 M_0. \end{cases} \tag{8.11}$$

Let us observe now that the linear operator

$$\mathcal{L}(\mu_1, d_1, e_1) = \big(L_{N+1}(\mu_1), L_N(d_{1N}), L_{N-1}(d_{1(N-1)}), \ldots, L_1(d_{11}), L_0(e_1)\big)$$

is invertible with bounds for $L(\mu_1, d_1, e_1) = (f, g, h)$ given by

$$\|(\mu_1, d_1, e_1)\| \le C[\|f\|_\infty + \|g\|_\infty + \varepsilon^{-\frac{N-1}{N-2}}\|h\|_\infty].$$

It then follows from the contraction mapping principle that, given $\sigma > 0$, the problem

$$[\mathcal{L} + (\varepsilon M_{N+1}, \varepsilon M_N, \varepsilon M_{N-1}, \ldots, \varepsilon M_1, \varepsilon^2 M_0)](\mu_1, d_1, e_1) = (f, g, h)$$

is uniquely solvable for $\|(\mu_1, d_1, e_1)\| \le c\varepsilon^\sigma$ if $\|f\|_\infty < \varepsilon^{\sigma+\rho}$, $\|g\|_\infty < \varepsilon^{\sigma+\rho}$, $\|h\|_2 < \varepsilon^{\sigma+\rho-(N-1)/(N-2)}$, for some $\rho > 0$. The desired result for the full problem (8.11) then follows directly from Schauder's fixed point theorem. In fact we get $\|(\tilde{\mu}_1, \tilde{d}_1, \tilde{e}_1)\| = O(\varepsilon^{(N-3)/(N-2)})$ for the solution. □

9. Appendix

Proof of (5.33)

We write

$$\begin{aligned} S_\varepsilon(\mathrm{w}) = {} & S_\varepsilon(\tilde{\omega}) + \{\rho^2 a_0 \ddot{e}_\varepsilon(\rho y_0) + \lambda_1 e_\varepsilon(\rho y_0)\}\chi_\varepsilon Z_0 + \tilde{A}(e_\varepsilon \chi_\varepsilon Z_0) \\ & + 2e_\varepsilon \nabla\chi_\varepsilon \nabla Z_0 + N_0(e_\varepsilon \chi_\varepsilon Z_0), \end{aligned} \tag{9.1}$$

where

$$N_0(e_\varepsilon \chi_\varepsilon Z_0) = \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}[(\tilde{\omega} + e_\varepsilon \chi_\varepsilon Z_0)^{p-\varepsilon} - \tilde{\omega}^{p-\varepsilon}] - p e_\varepsilon \omega^{p-1}\chi_\varepsilon Z_0. \tag{9.2}$$

We start by analyzing $S_\varepsilon(\tilde{\omega})$. Expanding $S_\varepsilon(\tilde{\omega})$ in ε and taking into account that

$$\Delta[(1+\alpha_\varepsilon)\omega] + \mu_\varepsilon^{-\frac{N-2}{2}\varepsilon}[(1+\alpha_\varepsilon)\omega]^p = 0 \quad \text{in } \mathbb{R}^N, \tag{9.3}$$

we have

$$\begin{aligned} S_\varepsilon(\tilde{\omega}) = {} & \sum_{k=0}^{5} \mathcal{A}_k \omega - p\omega^{p-1}\bar{\omega} - \varepsilon\omega^p \log\omega \\ & + B(\omega) - \mathcal{A}(\bar{\omega}) + \alpha_\varepsilon \mathcal{A}(\omega - \bar{\omega}) + a_0 \partial_0^2[\alpha_\varepsilon(\omega - \bar{\omega})] \\ & + b(\rho y_0, y; \mu, d)\varepsilon^2\omega^p, \end{aligned} \tag{9.4}$$

where the operators $\mathcal{A}_k$ and $\mathcal{A}$ are defined in Lemmas 5.1 and 5.14, the operator B is given by (5.16) and b is a sum of functions of the form

$$b_0(\rho y_0) b_1(\mu, d)$$

with b_0 a smooth function of ρy_0, uniformly bounded in ε together with its derivatives, and b_1 a smooth function of its arguments, uniformly bounded in ε. Note that b_1 does not depend on the derivatives of its arguments.

The main part in (9.4) is

$$e_0 := \sum_{k=0}^{5} \mathcal{A}_k \omega - p\omega^{p-1}\bar{\omega} - \varepsilon\omega^p \log\omega. \tag{9.5}$$

Indeed, $B(\omega)$ is of lower order with respect to $\sum_{k=0}^{5} \mathcal{A}_k\omega$ as shown by Lemma 5.1, and so is the term given by $\mathcal{A}(\bar{\omega})$ since $\bar{\omega} = O(\varepsilon)\omega$ and also the term $\alpha_\varepsilon \mathcal{A}(\omega - \bar{\omega})$ since $\alpha_\varepsilon = O(\varepsilon|\log\varepsilon|)$ as $\varepsilon \to 0$. Observe furthermore that $\partial_0^2 \alpha_\varepsilon = \rho^2 O(\alpha_\varepsilon)$, so $a_0\partial_0^2[\alpha_\varepsilon(\omega - \bar{\omega})] = o(1)\rho^2\omega$. Summarizing, we can write

$$S_\varepsilon(\tilde{\omega}) = e_0 + \varepsilon^2 b(\rho y_0; \mu, d)\omega^p + \varepsilon^3 r, \tag{9.6}$$

where r is a sum of functions of the form

$$h_0(\rho y_0) f_1(\mu, d, \dot{\mu}, \dot{d}) f_2(y)$$

with h_0 a smooth function uniformly bounded in ε, f_1 a smooth function of its arguments, homogeneous of degree 3, uniformly bounded in ε and

$$\sup\,(1 + |y|^{N-2})|f_2(y)| < +\infty.$$

By means of Lemma 5.1 and taking into account notation (5.3), we can expand the first term in (9.5) in powers of ε:

$$\begin{aligned}
\sum_{k=0}^{5} \mathcal{A}_k(\omega) = {} & \varepsilon[-2\bar{h}_{ij}\tilde{d}_N \partial_{ij}\omega] + \varepsilon^{1+\frac{1}{N-2}} \tilde{\mu}[-2\bar{h}_{ij} y_N \partial_{ij}\omega + \mathrm{Tr}_{\bar{g}}\, \bar{h}\partial_N\omega] \\
& + \varepsilon^2\Big[\sum_{ij} (\dot{\tilde{d}}_i \dot{\tilde{d}}_j - \tfrac{1}{3} R_{ijkl}\tilde{d}_k\tilde{d}_l + a_{Nk}^{ij}\tilde{d}_k\tilde{d}_N + 4\bar{h}_{0j}\tilde{d}_i\tilde{d}_N)\partial_{ij}\omega\Big] \\
& + \varepsilon^{2+\frac{1}{N-2}}\big[-\tilde{\mu} D_y\omega \cdot \ddot{\tilde{d}} - \tfrac{1}{3}\tilde{\mu} R_{ijkl} y_k \tilde{d}_l \partial_{ij}\omega + 2\tilde{\mu} a_{Nk}^{ij} y_k \tilde{d}_N \partial_{ij}\omega \\
& + \tilde{\mu}(\tfrac{2}{3} R_{ijik} + R_{0j0k})\tilde{d}_k \partial_j\omega + 4\bar{h}_{0j}\big(\tilde{\mu} y_N D_y(\partial_j\omega)\dot{\delta} + \dot{\tilde{\mu}}\tilde{d}_N(\gamma\partial_j\omega + D_y(\partial_j\omega)y)\big) \\
& + b_N^j \tilde{\mu}\tilde{d}_N \partial_j\omega - \mathrm{Tr}_{\bar{g}}\, \bar{k}\tilde{\mu}\tilde{d}_N\partial_N\omega - 2\dot{\tilde{\mu}} D_y Z_{N+1} \cdot \dot{\tilde{d}}\big] \\
& + \varepsilon^{2+\frac{2}{N-2}}\big[-\ddot{\tilde{\mu}}\tilde{\mu} Z_{N+1} \\
& + \tilde{\mu}^2\big(-\tfrac{1}{3} R_{ikjl} y_k y_l \partial_{ij}\omega + (\tfrac{2}{3} R_{ijik} + R_{0j0k}) y_k \partial_j\omega + b_N^j y_N \partial_j\omega - \mathrm{Tr}_{\bar{g}}\, \bar{k} y_N \partial_N\omega\big) \\
& + 4\bar{h}_{0j}\tilde{\tilde{\mu}}\dot{\tilde{\mu}} y_N(\gamma\partial_j\omega + D_y(\partial_j\omega)\cdot y) \\
& + (\dot{\tilde{\mu}})^2\big(D_{yy}\omega[y]^2 + 2(1+\gamma)D_y\omega\cdot y + \gamma(1+\gamma)\omega\big)\big] \\
& + \varepsilon^3 r,
\end{aligned} \tag{9.7}$$

where r denotes a sum of functions of the form

$$h_0(\rho y_0)[f_1(\nu, d, \dot{\mu}, \dot{d}) + o(1) f_2(\mu, d, \dot{\mu}, \dot{d}, \ddot{\mu}, \ddot{d})] f_3(y)$$

with h_0 a smooth function of ρy_0 uniformly bounded in ε, f_1, f_2 smooth functions of their arguments, f_1 homogeneous of degree 3, f_2 linear in the variables $(\ddot{\mu}, \ddot{d})$, and

$$\sup\,(1 + |y|^{N-2})|f_3(y)| < +\infty.$$

The previous expansion, together with (9.5), (9.6) and the notation (5.3), gives a precise description of the first term $S_\varepsilon(\tilde{\omega})$ in (9.1). Let us now consider the term $\mathcal{A}(e_\varepsilon \chi_\varepsilon Z_0)$. Arguing as before, we have

$$\mathcal{A}(e_\varepsilon \chi_\varepsilon Z_0) = \sum_{k=0}^{5} \mathcal{A}_k(e_\varepsilon Z_0) + \varepsilon^3 r,$$

where r is a sum of functions of the form

$$h_0(\rho y_0)[f_1(\nu, d, e, \dot{\mu}, \dot{d}, \dot{e}) + o(1) f_2(\mu, d, e, \dot{\mu}, \dot{d}, \dot{e}, \ddot{\mu}, \ddot{d}, \ddot{e})] f_3(y)$$

with h_0 a smooth function of ρy_0 uniformly bounded in ε, f_1, f_2 smooth functions of their arguments, f_1 homogeneous of degree 3, f_2 linear in the variables $(\ddot{\mu}, \ddot{d}, \ddot{e})$, and

$$\sup\,(1 + |y|^{N-2})|f_3(y)| < +\infty.$$

Let us then consider the term $\sum_{k=0}^{5} \mathcal{A}_k(e_\varepsilon Z_0)$. Directly from Lemma 5.1 and taking into account (5.30), we obtain

$$\sum_{k=0}^{5} \mathcal{A}_k(e_\varepsilon Z_0) = \varepsilon \tilde{e} A + \varepsilon^{2+\frac{1}{N-2}} \dot{\tilde{e}} B,$$

where

$$\begin{aligned}
A = {} & \varepsilon[-2\bar{h}_{ij} \tilde{d}_N \partial_{ij} Z_0] + \varepsilon^{1+\frac{1}{N-2}} \tilde{\mu}[-2\bar{h}_{ij} y_N \partial_{ij} Z_0 + \mathrm{Tr}_{\bar{g}}\, \bar{h} \partial_N Z_0] \\
& + \varepsilon^2 \Big[\sum_{ij} (\dot{\tilde{d}}_i \dot{\tilde{d}}_j - \tfrac{1}{3} R_{ijkl} \tilde{d}_k \tilde{d}_l + a_{Nk}^{ij} \tilde{d}_k \tilde{d}_N + 4\bar{h}_{0j} \tilde{d}_i \tilde{d}_N) \partial_{ij} Z_0 \Big] \\
& + \varepsilon^{2+\frac{1}{N-2}} \big[-\tilde{\mu} D_y Z_0 \cdot \ddot{\tilde{d}} - \tfrac{1}{3} \tilde{\mu} R_{ijkl} y_k \tilde{d}_l \partial_{ij} Z_0 + 2\tilde{\mu} a_{Nk}^{ij} y_k \tilde{d}_N \partial_{ij} Z_0 \\
& + \tilde{\mu} (\tfrac{2}{3} R_{ijik} + R_{0j0k}) \tilde{d}_k \partial_j Z_0 \\
& + 4\bar{h}_{0j} \big(\tilde{\mu} y_N D_y (\partial_j Z_0) \dot{\delta} + \dot{\tilde{\mu}} \tilde{d}_N (\gamma \partial_j Z_0 + D_y(\partial_j Z_0) y) \big) \\
& + b_N^j \tilde{\mu} \tilde{d}_N \partial_j Z_0 - \mathrm{Tr}_{\bar{g}}\, \bar{k} \tilde{\mu} \tilde{d}_N \partial_N Z_0 - 2\dot{\tilde{\mu}} (\gamma D_y Z_0 + D_{yy} Z_0[y]) \cdot \dot{\tilde{d}} \big] \\
& + \varepsilon^{2+\frac{2}{N-2}} \big[-\ddot{\tilde{\mu}} \tilde{\mu} Z_{N+1} \\
& + \tilde{\mu}^2 \big(-\tfrac{1}{3} R_{ikjl} y_k y_l \partial_{ij} Z_0 + (\tfrac{2}{3} R_{ijik} + R_{0j0k}) y_k \partial_j Z_0 + b_N^j y_N \partial_j Z_0 - \mathrm{Tr}_{\bar{g}}\, \bar{k} y_N \partial_N Z_0 \big) \\
& + \bar{h}_{0j} \tilde{\mu} \dot{\tilde{\mu}} y_N (\gamma \partial_j Z_0 + D_y(\partial_j Z_0) \cdot y) \\
& + (\dot{\tilde{\mu}})^2 \big(D_{yy} Z_0[y]^2 + 2(1+\gamma) D_y Z_0 \cdot y + \gamma(1+\gamma) Z_0 \big) \big] \\
& + \varepsilon^3 r
\end{aligned}$$

and r is as before. On the other hand,

$$B = \varepsilon[-2\tilde{\mu} D_y Z_0 \cdot \dot{\delta} - 4\bar{h}_{0j} \tilde{\mu} \tilde{d}_N \partial_j Z_0] \\ + \varepsilon^{1+\frac{1}{N-2}} [-2\tilde{\mu}\dot{\tilde{\mu}} D_y Z_0 \cdot y - 2\gamma \tilde{\mu}\dot{\tilde{\mu}} Z_0 - 4(\tilde{\mu})^2 \bar{h}_{0j} y_N \partial_j Z_0] + \varepsilon^2 r,$$

with r as before.

Expanding in ε the term $N_0(e_\varepsilon \chi_\varepsilon Z_0)$ defined in (9.2), we get

$$N_0(e_\varepsilon \chi_\varepsilon Z_0) = \varepsilon^2 [p(p-1) E_0^2 \omega^{p-2} Z_0^2 + p E_0 \omega^{p-1} \log \omega Z_0] + \varepsilon^3 |\log \varepsilon| r, \tag{9.8}$$

where r is a sum of functions of the form

$$h_0(\rho y_0) h_1(\mu, d, e) h_2(y)$$

with h_0 a smooth function, uniformly bounded in ε, h_1 a smooth function of its arguments and $\sup\, (1+|y|)^{N+2} |h_2|(y) \le C$. Summing up all the computation, we obtain the proof of (5.33). □

Proof of (5.35), (5.38), (5.39), (5.40)

The proof consists of two steps. In the first step we compute the expansion in ε of the projections assuming that

$$\mu_\varepsilon = \varepsilon^{\frac{N-1}{N-2}} \tilde{\mu}, \quad d_{\varepsilon N} = \varepsilon \tilde{d}_N, \quad d_{\varepsilon j} = \varepsilon d_j, \quad e_\varepsilon = \varepsilon \tilde{e}.$$

In the second part, we will choose μ_1, d_{N1} and e_1 to get the above expansion when μ, d and e are defined as in (5.4), (5.3), (5.6), (5.29) and (5.30).

Step 1. We start with the projection of the nonlinear part

$$h = -p\omega^{p-1} \bar{\omega} - \varepsilon \omega^p \log \omega.$$

We have the following facts: as $\varepsilon \to 0$,

$$\int_{\mathcal{D}_{y_0}} h Z_{N+1}\, dy = \varepsilon \left[A_2 \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-2} - A_3 + \varepsilon^{\frac{1}{N-2}} \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-1} g_{N+1} \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right) \right], \tag{9.9}$$

$$\int_{\mathcal{D}_{y_0}} h Z_N\, dy = \varepsilon^{1+\frac{1}{N-2}} \left[-A_1 \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-1} + \varepsilon^{\frac{1}{N-2}} \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right)^{N} g_N \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right) \right], \tag{9.10}$$

$$\int_{\mathcal{D}_{y_0}} h Z_k\, dy = \varepsilon^{2+\frac{3}{N-2}} g_k \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right) \quad \text{for } k = 1, \dots, N-1, \tag{9.11}$$

$$\int_{\mathcal{D}_{y_0}} h Z_0\, dy = \varepsilon \left[-A_4 \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-2} - A_5 + \varepsilon^{\frac{1}{N-2}} \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right)^{N-1} g_0 \left(\frac{\tilde{\mu}}{\tilde{d}_N} \right) \right]. \tag{9.12}$$

In these formulas, the functions g_i are smooth functions with $g_i(0) \neq 0$ and A_i are positive constants.

We first prove (9.10). By Taylor expansion we have

$$
\begin{aligned}
& - p \int_{\mathcal{D}_{y_0}} \bar{\omega}\omega^{p-1} Z_N \\
&= p c_N^{\frac{N+2}{2}} \int_{\mathcal{D}_{y_0}} \frac{N-2}{(1+|\bar{y}|^2 + |y_N + 2\varepsilon^{-1/(N-2)}\tilde{d}_N/\tilde{\mu}|^2)^{\frac{N-2}{2}}} \frac{y_N}{(1+|y|^2)^{\frac{N+4}{2}}}\, dy \\
&= \varepsilon^{1+\frac{1}{N-2}} \left[-A_1 \left(\frac{\tilde{\mu}}{\tilde{d}_N}\right)^{N-1} + \varepsilon^{\frac{1}{N-2}} \left(\frac{\tilde{\mu}}{\tilde{d}_N}\right)^{N} g_N\left(\frac{\tilde{\mu}}{\tilde{d}_N}\right) \right].
\end{aligned}
$$

The constant A_1 which appears in (9.9) is precisely given by

$$
A_1 = \frac{p c_N^{\frac{N+2}{2}} (N-2)^2}{2^{N-1}} \int \frac{y_N^2}{(1+|y|^2)^{\frac{N+4}{2}}}.
$$

Furthermore, we have

$$
-\varepsilon \int_{\mathcal{D}_{y_0}} \omega^p \log \omega \, Z_N = \varepsilon^{2+\frac{2}{N-2}} O\left(\left(\frac{\tilde{\mu}}{\tilde{d}_N}\right)^{N} \right).
$$

This proves (9.10). Concerning the projection along Z_{N+1}, arguing as before we get

$$
-p \int_{\mathcal{D}_{y_0}} \bar{\omega}\omega^{p-1} Z_{N+1} = \varepsilon \left[A_2 \left(\frac{\tilde{\mu}}{\tilde{d}_N}\right)^{N-2} + \varepsilon^{\frac{1}{N-2}} \left(\frac{\tilde{\mu}}{\tilde{d}_N}\right)^{N-1} g_N\left(\frac{\tilde{\mu}}{\tilde{d}_N}\right) \right]
$$

for a positive constant A_2 which can be computed explicitly.

Finally, we get

$$
-\varepsilon \int_{\mathcal{D}_{y_0}} \omega^p \log \omega \, Z_{N+1} = -\varepsilon A_3 + \varepsilon^{2+\frac{2}{N-2}} O\left(\left(\frac{\tilde{\mu}}{\tilde{d}_N}\right)^{N} \right)
$$

where A_3 is the positive constant given by

$$
\begin{aligned}
A_3 &= \int \omega^p \log \omega \, Z_{N+1} = \frac{N-2}{2} \int \omega^{p+1} \log \omega + \int \log \omega \, \nabla \left(\frac{\omega^{p+1}}{p+1} \right) \cdot y \\
&= -\frac{1}{p+1} \int \omega^p \nabla \omega \cdot y = \frac{N}{(p+1)^2} \int \omega^{p+1}.
\end{aligned}
$$

This proves (9.9). Estimate (9.12) follows in a similar way. Finally, (9.11) which follows from the observation that

$$
p \int \omega^{p-1} \bar{\omega} \, Z_k = \int \omega^p \log \omega \, Z_k = 0 \quad \text{for all } k = 1, \dots, N-1,
$$

due to symmetry.

We continue with the projections of $S := S_\varepsilon(\mathtt{w}) - h$. We have

$$
\begin{aligned}
Z_{N+1} = \varepsilon^2 &\bigg[\int \Upsilon_\varepsilon Z_{N+1}(1 + o(\varepsilon)) \bigg] \\
&+ \varepsilon^{2+\frac{2}{N-2}} \bigg[-C_2 \tilde{\mu} \ddot{\tilde{\mu}} + (\dot{\tilde{\mu}})^2 \int [D_{yy}\omega y^2 + 2(1+\gamma) D_y \omega y + \gamma(1+\gamma)\omega] Z_{N+1} \\
&- (\tilde{\mu})^2 \bigg[\operatorname{Tr}_{\bar{g}} \bar{k} \int y_N \partial_N \omega Z_{N+1} + \tfrac{1}{3} R_{ikjl} \int y_k y_l \partial_{ij} \omega Z_{N+1} \bigg]\bigg] \\
&+ \varepsilon^3 r
\end{aligned} \tag{9.13}
$$

where r is a sum of functions of the form (5.36).

Concerning the projection along Z_N, we get at main order

$$
\begin{aligned}
\int_{\mathcal{D}_{y_0}} S Z_N &= \varepsilon^{1+\frac{1}{N-2}} \tilde{\mu} \bigg[-2\bar{h}_{ii} \int y_N \partial_N \omega \partial_{ii} \omega + \operatorname{Tr}_{\bar{g}} \bar{h} \int (\partial_N \omega)^2 \bigg] \\
&\quad + \varepsilon^{2+\frac{1}{N-2}} \bigg[-C_1 \tilde{\mu} \ddot{\tilde{d}}_N - 2\dot{\tilde{\mu}} \int D_y Z_{N+1} [\dot{\tilde{d}}] Z_N \\
&\quad + 4\bar{h}_{0j} \bigg(\tilde{\mu} d_j \int y_N \partial_{jj} \omega Z_N + \dot{\tilde{\mu}} d_N \int \partial_N \partial_j \omega y_N \partial_N \omega \bigg) - C_1 \tilde{\mu} d_N \operatorname{Tr}_{\bar{g}} \bar{k} \\
&\quad - \tilde{A}_1 \bigg(\frac{\tilde{\mu}}{\tilde{d}_N} \bigg)^{N-1} \tilde{e} - 2\bar{h}_{00} \tilde{e} \tilde{d}_N \int y_N \omega^{p-1} Z_0 Z_N \bigg] \\
&\quad + \varepsilon^{3+\frac{2}{N-2}} r \\
&= \varepsilon^{1+\frac{1}{N-2}} C_1 \tilde{\mu} \bar{h}_{00} + \varepsilon^{2+\frac{1}{N-2}} C_1 \bigg[-\tilde{\mu} \ddot{\tilde{d}}_N - \operatorname{Tr}_{\bar{g}} \bar{k} \tilde{\mu} \tilde{d}_N + 2\bar{h}_{0j} \tilde{\mu} d_j \\
&\quad - A_1 \bigg(\frac{\tilde{\mu}}{\tilde{d}_N} \bigg)^{N-1} \tilde{e} - 2\bar{h}_{00} \tilde{e} \tilde{d}_N \int y_N \omega^{p-1} Z_0 Z_N \bigg] + \varepsilon^{3+\frac{2}{N-2}} r,
\end{aligned} \tag{9.14}
$$

where we use

$$
\int y_N \partial_{jj} \omega \partial_N \omega = \frac{1}{2} C_1, \qquad \int \partial_j Z_{N+1} \partial_N = 0, \qquad \text{for all } j.
$$

We now handle the projection along Z_k for $k = 1, \ldots, N-1$. First we write

$$
\begin{aligned}
\int_{\mathcal{D}_{y_0}} S Z_k &= \varepsilon^{2+\frac{1}{N-2}} \tilde{\mu} \bigg[-C_1 \ddot{d}_k + \bigg(-\tfrac{2}{3} R_{iljm} \int y_m \partial_{ij} \omega Z_k + C_1 (\tfrac{2}{3} R_{ijil} + R_{0j0l}) \bigg) d_l \\
&\quad + \tilde{d}_N \bigg(2a_{Nl}^{ij} \int y_l \partial_{ij} \omega Z_k + b_N^j C_1 \bigg) + \dot{\tilde{d}}_N \bigg(4\bar{h}_{0k} \int y_N \partial_{Nk} \omega Z_k \bigg) \bigg] \\
&\quad + \varepsilon^{3+\frac{2}{N-2}} r \\
&= \varepsilon^{2+\frac{1}{N-2}} \tilde{\mu} C_1 [-\ddot{d}_k + R_{0j0l} d_l + \gamma_{0k} \tilde{d}_N + \gamma_{1k} \dot{\tilde{d}}_N] + \varepsilon^{3+\frac{2}{N-2}} r
\end{aligned} \tag{9.15}
$$

since

$$-\tfrac{2}{3}R_{iljm}d_l\int_{D_N} y_m\partial_{ij}\omega Z_k = -\frac{2}{3}\Big[R_{ilik}\int_{D_N} y_k\partial_{ii}\omega Z_k + R_{ilki}\int_{D_N} y_i\partial_{ik}\omega Z_k$$
$$+ R_{kljj}\int_{D_N} y_j\partial_{kj}\omega Z_k\Big]d_l$$
$$= -\tfrac{1}{3}C_1[R_{ilik}-R_{ilki}]d_l = -\tfrac{2}{3}C_1R_{ilik}d_l.$$

In (9.15), γ_{0k} and γ_{1k} denote smooth explicit functions of ρy_0.

Finally, using the orthogonality in L^2 of Z_0 with respect to Z_i, for $i = 1,\ldots,N+1$, direct computations show

$$\int_{\mathcal{D}_{y_0}} SZ_0 = \varepsilon C_3[-2(\mathrm{Tr}_{\bar g}\,\bar h - \bar h_{00})\tilde d_N]$$
$$+\varepsilon^2 C_3\Big[\rho^2 a_0\ddot{\tilde e} + \lambda_1\tilde e + \dot{\tilde d}_i^2 - \tfrac{1}{3}R_{ikil}d_kd_l + a^{ii}_{Nk}d_k\tilde d_N + 4\bar h_{0j}d_j\tilde d_N + \int \Upsilon_\varepsilon Z_0\Big]$$
$$+\varepsilon^{2+\frac{2}{N-2}}[(\dot{\tilde\mu})^2 + f_1(\rho y_0)\tilde\mu^2 + f_2(\rho y_0)\tilde\mu\dot{\tilde\mu}] + \varepsilon^3 r, \tag{9.16}$$

where f_i are explicit smooth functions, uniformly bounded in ε, and r is as before.

Summing up the previous calculations, we conclude that at main order

$$\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w})Z_{N+1}\,dy = \varepsilon\Big[A_2\Big(\frac{\tilde\mu}{\tilde d_N}\Big)^{N-2} - A_3 + \varepsilon^{\frac{1}{N-2}}\Big(\frac{\tilde\mu}{\tilde d_N}\Big)^{N-1} g_{N+1}\Big(\frac{\tilde\mu}{\tilde d_N}\Big)\Big](1+o(1)),$$

$$\varpi\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w})Z_N\,dy$$
$$= \varepsilon^{1+\frac{1}{N-2}}\Big[C_1\frac{A_2}{A_1}\bar h_{00}\tilde\mu - A_1\Big(\frac{\tilde\mu}{\tilde d_N}\Big)^{N-1} + \varepsilon^{\frac{1}{N-2}}\Big(\frac{\tilde\mu}{\tilde d_N}\Big)^{N}\tilde g_N\Big(\frac{\tilde\mu}{\tilde d_N}\Big)\Big](1+o(1)),$$

$$\int_{\mathcal{D}_{y_0}} S_\varepsilon(\mathtt{w})Z_0\,dy = \varepsilon\Big[\lambda_1\tilde e - 2(\mathrm{Tr}_{\bar g}\,\bar h - \bar h_{00})\Big(\int\partial_{ii}\omega Z_0\Big)\tilde d_N - A_4\Big(\frac{\tilde\mu}{\tilde d_N}\Big)^{N-2} - A_5$$
$$+\varepsilon^{\frac{1}{N-2}}\Big(\frac{\tilde\mu}{\tilde d_N}\Big)^{N-1}\tilde g_0\Big(\frac{\tilde\mu}{\tilde d_N}\Big)\Big](1+o(1)).$$

Step 2. Let now $(\mu^0_\varepsilon, d^0_{\varepsilon N}, e^0_\varepsilon) \in (0,\infty)\times(0,\infty)\times\mathbb{R}$ be the solution to the following system of nonlinear equations:

$$\begin{cases} A_2\Big(\dfrac{\mu}{d_N}\Big)^{N-2} - A_3 + \varepsilon^{\frac{1}{N-2}}\Big(\dfrac{\mu}{d_N}\Big)^{N-1}\tilde g_{N+1}\Big(\dfrac{\mu}{d_N}\Big) = 0,\\ C_1\dfrac{A_2}{A_1}\bar h_{00}\mu - A_2\Big(\dfrac{\mu}{d_N}\Big)^{N-1} + \varepsilon^{\frac{1}{N-2}}\Big(\dfrac{\mu}{d_N}\Big)^{N}\tilde g_N\Big(\dfrac{\mu}{d_N}\Big) = 0,\\ \lambda_1 e - 2(\mathrm{Tr}_{\bar g}\,\bar h - \bar h_{00})\Big(\displaystyle\int\partial_{ii}\omega Z_0\Big)d_N - A_4\Big(\dfrac{\mu}{d_N}\Big)^{N-2} - A_5\\ \quad + \varepsilon^{\frac{1}{N-2}}\Big(\dfrac{\mu}{d_N}\Big)^{N-1}\tilde g_0\Big(\dfrac{\mu}{d_N}\Big) = 0. \end{cases} \tag{9.17}$$

It is easy to show that the solution $(\mu_\varepsilon^0, d_{\varepsilon N}^0, e_\varepsilon^0)$ has the form

$$\hat{\mu} = \mu_0 + \varepsilon^{\frac{1}{N-2}}\mu_1, \quad \hat{d}_N = d_0 + \varepsilon^{\frac{1}{N-2}}d_{1N}, \quad \hat{e} = e_0 + \varepsilon^{\frac{1}{N-2}}e_1,$$

where μ_0, d_0, E_0 is the solution to

$$F(\mu, d_N, e) := \begin{bmatrix} A_2(\mu/d_N)^{N-2} - A_3 \\ C_1 \frac{A_2}{A_1}\bar{h}_{00}\mu - A_2(\mu/d_N)^{N-1} \\ \lambda_1 e - 2(\mathrm{Tr}_{\bar{g}}\,\bar{h} - \bar{h}_{00})(\int \partial_{ii}\omega Z_0)d_N - \ A_4(\mu/d_N)^{N-2} - A_5 \end{bmatrix} = 0.$$

Observe that $\mu_0 > 0$ and $d_0 > 0$. Direct computations show that

$$\begin{aligned} F_0 &:= \nabla_{\mu,d_N,e}F(\mu_0, d_0, E_0) \\ &= \begin{bmatrix} (N-2)A_2\mu_0^{N-3}/d_0^{N-2} & -(N-2)A_2\mu_0^{N-2}/d_0^{N-1} & 0 \\ -(N-2)A_2\mu_0^{N-2}/d_0^{N-1} & (N-1)A_2\mu_0^{N-1}/d_0^{N} & 0 \\ 0 & -2(\mathrm{Tr}_{\bar{g}}\,\bar{h} - \bar{h}_{00})\int \partial_{ii}\omega Z_0 & \lambda_1 \end{bmatrix}. \end{aligned}$$

Since

$$\det(\nabla_{\mu,d_N,e}F(\mu_0, d_0, E_0)) = (N-2)A_2C_1\lambda_1\frac{\mu_0^{N-2}}{d_0^{N-1}}\bar{h}_{00} > 0,$$

system (9.17) is equivalent to a fixed point problem, which is uniquely solvable in the set

$$\{(\mu_1, d_{1N}, e_1) : \|\mu_1\|_\infty \le \delta,\ \|d_{1N}\|_\infty \le \delta,\ \|e_1\|_\infty \le \delta\}$$

for some proper small $\delta > 0$.

We deduce the expansions (5.38), (5.39) and (5.40), with

$$A = (N-2)A_2\frac{\mu_0^{N-3}}{d_0^{N-2}} > 0, \quad B = -(N-2)A_2\frac{\mu_0^{N-2}}{d_0^{N-1}}, \quad C = N-1)A_2\frac{\mu_0^{N-1}}{d_0^{N}} > 0.$$

An easy computation shows that $AC - B^2 > 0$. This concludes the proof. □

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