J. Eur. Math. Soc. 13, 119–174

DOI 10.4171/JEMS/246

Veronica Felli · Alberto Ferrero · Susanna Terracini

Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential

Received October 14, 2008 and in revised form March 13, 2009

Abstract. Asymptotics of solutions to Schrödinger equations with singular magnetic and electric potentials is investigated. By using an Almgren type monotonicity formula, separation of variables, and an iterative Brezis–Kato type procedure, we describe the exact behavior near the singularity of solutions to linear and semilinear (critical and subcritical) elliptic equations with an inverse square electric potential and a singular magnetic potential with homogeneity of order -1.

Keywords. Singular electromagnetic potentials, Hardy's inequality, Schrödinger operators

1. Introduction

In quantum mechanics, the hamiltonian of a nonrelativistic charged particle in an electromagnetic field has the form $(-i\nabla+\boldsymbol{\mathcal{A}})^2+V$, where $V:\mathbb{R}^N\to\mathbb{R}$ is the electric potential and $\boldsymbol{\mathcal{A}}:\mathbb{R}^N\to\mathbb{R}^N$ is a magnetic potential associated to the magnetic field $B=\operatorname{curl}\boldsymbol{\mathcal{A}}$. For $N=2,3$, "curl" denotes the usual curl operator, whereas for $N>3$ by $B=\operatorname{curl}\boldsymbol{\mathcal{A}}$ we mean the 2-form (B_{jk}) with $B_{jk}:=\partial_j\mathcal{A}_k-\partial_k\mathcal{A}_j$, where $\boldsymbol{\mathcal{A}}=(\mathcal{A}_j)_{j=1,\dots,N}$. Linear and nonlinear elliptic equations associated to electromagnetic hamiltonians have been the object of a wide recent mathematical research; we cite, among others, [2, 7, 8, 9, 10, 17].

In this paper we are concerned with singular homogeneous electromagnetic potentials $(\boldsymbol{\mathcal{A}},V)$ which make the operator invariant by scaling, namely of the form

$$\boldsymbol{\mathcal{A}}(x)=\frac{\mathbf{A}(x/|x|)}{|x|}\quad\text{and}\quad V(x)=-\frac{a(x/|x|)}{|x|^2}$$

in $\mathbb{R}^N$, where $N\geq 2$, $\mathbf{A}\in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$, and $a\in L^\infty(\mathbb{S}^{N-1},\mathbb{R})$. A prototype in dimension 2 is given by potentials associated to thin solenoids: if the radius of the solenoid tends to zero while the flux through it remains constant, then the particle is subject to a δ-type magnetic field, which is called an *Aharonov–Bohm* field. A vector potential associated to the Aharonov–Bohm magnetic field in $\mathbb{R}^2$ has the form

$$\boldsymbol{\mathcal{A}}(x_1,x_2)=\alpha\left(-\frac{x_2}{|x|^2},\frac{x_1}{|x|^2}\right),\qquad(x_1,x_2)\in\mathbb{R}^2,\tag{1}$$

V. Felli, A. Ferrero, S. Terracini: Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Via Cozzi 53, 20125 Milano, Italy;
e-mail: veronica.felli@unimib.it, alberto.ferrero@unimib.it, susanna.terracini@unimib.it

Mathematics Subject Classification (2010): 35J10, 35B40, 83C50, 35J60

with $\alpha \in \mathbb{R}$ representing the circulation of $\mathcal{A}$ around the solenoid. We notice that the potential in (1) is singular at 0, homogeneous of degree -1 and satisfies the following transversality condition:

$$\mathbf{A}(\theta) \cdot \theta = 0 \quad \text{for all } \theta \in \mathbb{S}^{N-1}.$$

We refer to [3, 15, 23] for properties of Aharonov–Bohm magnetic potentials and related Hardy inequalities. In the present paper, we consider, for $N \geq 2$, a larger class of singular vector potentials, characterized by the presence of a homogeneous isolated singularity of order -1 and by the transversality (or Poincaré) condition (we address the reader to [16] and [26, §8.4.2] for details about the transversal or Poincaré gauge). Such a class includes, for $N = 2$, the Aharonov–Bohm magnetic potential (1). The Aharonov–Bohm potential in dimension $N = 3$ is singular on a straight line and is not covered by the analysis performed here, which only allows treating isolated singularities. In a forthcoming paper, we will extend the present results to potentials with cylindrical singularity including the 3-dimensional Aharonov–Bohm case.

Singular homogeneous electric potentials which scale as the laplacian arise in nonrelativistic molecular physics, where the interaction between an electric charge and the dipole moment $\mathbf{D} \in \mathbb{R}^N$ of a molecule is described by an inverse square potential with an anisotropic coupling strength of the form

$$V(x) = -\frac{\lambda(x \cdot \mathbf{d})}{|x|^3} \quad \text{in } \mathbb{R}^N,$$

where $\lambda > 0$ is proportional to the magnitude of the dipole moment $\mathbf{D}$ and $\mathbf{d} = \mathbf{D}/|\mathbf{D}|$ denotes the orientation of $\mathbf{D}$ (see [12, 13, 21]). We notice that the above electric potential is singular at 0 and homogeneous of degree -2.

We aim to describe the asymptotic behavior near the singularity of solutions to equations associated to the following class of Schrödinger operators with singular homogeneous electromagnetic potentials:

$$\mathcal{L}_{\mathbf{A},a} := \left(-i\nabla + \frac{\mathbf{A}(x/|x|)}{|x|}\right)^2 - \frac{a(x/|x|)}{|x|^2}.$$

We study both linear and nonlinear equations obtained as perturbations of the operator $\mathcal{L}_{\mathbf{A},a}$ in a domain $\Omega \subset \mathbb{R}^N$ containing either the origin or a neighborhood of ∞. More precisely, we deal with linear equations of the type

$$\mathcal{L}_{\mathbf{A},a} u = h(x) u \quad \text{in } \Omega \tag{2}$$

where $h \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\})$ is negligible with respect to the inverse square potential $|x|^{-2}$ near the singularity, and semilinear equations

$$\mathcal{L}_{\mathbf{A},a} u(x) = f(x, u(x)) \tag{3}$$

with f having at most critical growth.

Regularity properties of solutions to Schrödinger equations with less singular magnetic and electric potentials have been studied by several authors. In particular, in [7], boundedness and decay at ∞ of solutions are proved in dimensions $N \geq 3$ for L^2_{loc} magnetic potentials and electric potentials with $L^{N/2}$ negative part. It is also worth quoting [18] and [17], where, in dimensions $N \geq 3$, local boundedness and, respectively, a unique continuation property are established under the assumption that the electric potential and the square of the magnetic one belong to the Kato class. In [18] the continuity of solutions is also obtained under restricted assumptions on the potentials.

Due to the presence of a stronger singularity which keeps potentials in $\mathcal{L}_{\mathbf{A},a}$ out of the Kato class, it is natural to expect that solutions to equations (2) and (3) behave singularly at the origin: our purpose is to describe the rate and the shape of the singularity of solutions, by relating them to the eigenvalues and the eigenfunctions of a Schrödinger operator on the sphere $\mathbb{S}^{N-1}$ corresponding to the angular part of $\mathcal{L}_{\mathbf{A},a}$.

As remarked in [11, 13] for the case $\mathbf{A} = 0$ (i.e. no magnetic vector potential), the estimate of the behavior of solutions to elliptic equations with singular potentials near the singularities has several important applications to the study of spectral properties of the associated Schrödinger operator, such as essential self-adjointness, positivity, etc. In [12], the exact asymptotic behavior near the singularity of solutions to Schrödinger equations with singular dipole-type electric potentials is established, using separation of variables combined with a comparison method. Comparison and maximum principles play a crucial role also in [24], where the existence of the limit at the singularity of any quotient of two positive solutions to Fuchsian type elliptic equations is proved. In the presence of a singular magnetic potential, comparison methods are no more available, preventing us from a direct extension of the results of [12, 24]. This difficulty is overcome by an Almgren type monotonicity formula (see [1, 14]) and blow-up methods which allow avoiding the use of comparison methods.

1.1. Assumptions and functional setting

As already mentioned, we shall deal with electromagnetic potentials $(\boldsymbol{\mathcal{A}}, V)$ in $\mathbb{R}^N$, $N \geq 2$, satisfying the following assumptions:

(A.1) $\quad \boldsymbol{\mathcal{A}}(x) = \dfrac{\mathbf{A}(x/|x|)}{|x|} \quad$ and $\quad V(x) = -\dfrac{a(x/|x|)}{|x|^2}$ **(homogeneity)**,

(A.2) $\quad \mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N) \quad$ and $\quad a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$ **(regularity of angular coefficients)**,

(A.3) $\quad \mathbf{A}(\theta) \cdot \theta = 0 \quad$ for all $\theta \in \mathbb{S}^{N-1}$ **(transversality)**.

Under assumption (**A.3**), the operator $\mathcal{L}_{\mathbf{A},a}$ acts on functions $u : \mathbb{R}^N \to \mathbb{C}$ as

$$\mathcal{L}_{\mathbf{A},a} u = -\Delta u - \frac{a(x/|x|) - |\mathbf{A}(x/|x|)|^2 + i \operatorname{div}_{\mathbb{S}^{N-1}} \mathbf{A}(x/|x|)}{|x|^2} u - 2i \frac{\mathbf{A}(x/|x|)}{|x|} \cdot \nabla u,$$

where $\operatorname{div}_{\mathbb{S}^{N-1}} \mathbf{A}$ denotes the Riemannian divergence of $\mathbf{A}$ on the unit sphere $\mathbb{S}^{N-1}$ endowed with the standard metric.

The positivity properties of the Schrödinger operator $\mathcal{L}_{\mathbf{A},a}$ are strongly related to the first eigenvalue of the angular component of the operator on the sphere $\mathbb{S}^{N-1}$. More precisely, the positivity of the quadratic form associated to $\mathcal{L}_{\mathbf{A},a}$ is ensured under the assumption

$$\textbf{(A.4)} \qquad \mu_1(\mathbf{A},a) > -\left(\frac{N-2}{2}\right)^2 \qquad \textbf{(positive definiteness)}$$

(see Lemma 2.2), where $\mu_1(\mathbf{A},a)$ is the first eigenvalue of the angular component of the operator on the sphere $\mathbb{S}^{N-1}$, i.e. of the operator

$$L_{\mathbf{A},a} := (-i\nabla_{\mathbb{S}^{N-1}} + \mathbf{A})^2 - a.$$

When dealing with the nonlinear problem (3) we introduce the stronger condition

$$\textbf{(A.5)} \qquad \mu_1(0,a) > -\left(\frac{N-2}{2}\right)^2.$$

From the diamagnetic inequality it follows that $\mu_1(0,a) \le \mu_1(\mathbf{A},a)$ with equality holding if and only if $\operatorname{curl}(\mathbf{A}/|x|) = 0$ in the sense of distributions (see Lemma A.2 in the Appendix). In particular the assumption (**A.5**) is in general stronger than (**A.4**).

The spectrum of the angular operator $L_{\mathbf{A},a}$ is discrete and consists of a nondecreasing sequence of eigenvalues $\mu_1(\mathbf{A},a) \le \mu_2(\mathbf{A},a) \le \cdots$ diverging to $+\infty$ (see Lemma A.5). Condition (**A.4**) is fundamental to introduce a proper functional setting in which to frame our analysis. Let us define $\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})$ as the completion of $C^\infty_{\mathrm{c}}(\mathbb{R}^N \setminus \{0\},\mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})} := \left(\int_{\mathbb{R}^N}\left(|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\right)dx\right)^{1/2}. \tag{4}$$

It is easy to verify that

$$\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C}) = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\},\mathbb{C}) : u/|x| \in L^2(\mathbb{R}^N,\mathbb{C}) \text{ and } \nabla u \in L^2(\mathbb{R}^N,\mathbb{C}^N)\right\}.$$

The following lemma ensures that, under assumption (**A.4**), the space $\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})$ coincides with the Hilbert space generated by the quadratic form $Q_{\mathbf{A},a}$ associated to the operator $\mathcal{L}_{\mathbf{A},a}$,

$$\begin{aligned} &Q_{\mathbf{A},a} : \mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C}) \to \mathbb{R},\\ &Q_{\mathbf{A},a}(u) := \int_{\mathbb{R}^N}\left[\left|\left(\nabla + i\,\frac{\mathbf{A}(x/|x|)}{|x|}\right)u(x)\right|^2 - \frac{a(x/|x|)}{|x|^2}|u(x)|^2\right]dx. \end{aligned} \tag{5}$$

Lemma 1.1. *Assume that $N \ge 2$ and* (**A.2**)–(**A.4**) *hold. Then*

(i) $\displaystyle\inf_{u\in\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{Q_{\mathbf{A},a}(u)}{\int_{\mathbb{R}^N}|x|^{-2}|u(x)|^2\,dx} > 0,$

(ii) $Q_{\mathbf{A},a}$ *is positive definite in* $\mathcal{D}_*^{1,2}(\mathbb{R}^N,\mathbb{C})$*, i.e.* $\displaystyle \inf_{u\in\mathcal{D}_*^{1,2}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{Q_{\mathbf{A},a}(u)}{\|u\|^2_{\mathcal{D}_*^{1,2}(\mathbb{R}^N,\mathbb{C})}} > 0,$

(iii) $\mathcal{D}_*^{1,2}(\mathbb{R}^N,\mathbb{C}) = \mathcal{D}_{\mathbf{A},a}^{1,2}(\mathbb{R}^N)$*, where* $\mathcal{D}_{\mathbf{A},a}^{1,2}(\mathbb{R}^N)$ *is the completion of* $C_c^\infty(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ *with respect to the norm*

$$\|u\|_{\mathcal{D}_{\mathbf{A},a}^{1,2}(\mathbb{R}^N)} := (Q_{\mathbf{A},a}(u))^{1/2}.$$

Moreover the norms $\|\cdot\|_{\mathcal{D}_*^{1,2}(\mathbb{R}^N,\mathbb{C})}$ *and* $\|\cdot\|_{\mathcal{D}_{\mathbf{A},a}^{1,2}(\mathbb{R}^N)}$ *are equivalent.*

In any open bounded domain $\Omega \subset \mathbb{R}^N$ containing 0, we introduce the function space $H^1_*(\Omega,\mathbb{C})$ as the completion of

$$\{u \in H^1(\Omega,\mathbb{C}) \cap C^\infty(\Omega,\mathbb{C}) : u \text{ vanishes in a neighborhood of } 0\}$$

with respect to the norm

$$\|u\|_{H^1_*(\Omega,\mathbb{C})} = \big(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^N)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} + \|u/|x|\|^2_{L^2(\Omega,\mathbb{C})}\big)^{1/2}.$$

It is easy to verify that

$$H^1_*(\Omega,\mathbb{C}) = \big\{u \in H^1(\Omega,\mathbb{C}) : u/|x| \in L^2(\Omega,\mathbb{C})\big\}.$$

If $N \geq 3$, then $H^1_*(\Omega,\mathbb{C}) = H^1(\Omega,\mathbb{C})$ and their norms are equivalent, as one can easily deduce from the Hardy type inequality with boundary terms from [27] (see (131)) and continuity of Sobolev trace imbeddings. On the other hand, if $N = 2$, then $H^1_*(\Omega,\mathbb{C})$ is strictly smaller than $H^1(\Omega,\mathbb{C})$.

For any h satisfying

$$h \in L^\infty_{\rm loc}(\Omega\setminus\{0\},\mathbb{C}), \qquad |h(x)| = O(|x|^{-2+\varepsilon}) \quad \text{as } |x| \to 0 \text{ for some } \varepsilon > 0, \tag{6}$$

we introduce the notion of weak solution to (2): we say that a function $u \in H^1_*(\Omega,\mathbb{C})$ is an $H^1_*(\Omega,\mathbb{C})$-*weak solution* to (2) if, for all $w \in H^1_0(\Omega,\mathbb{C})$ such that $w/|x| \in L^2(\Omega,\mathbb{C})$,

$$\mathcal{Q}^\Omega_{\mathbf{A},a}(u,w) = \int_\Omega h(x)u(x)\overline{w(x)}\,dx,$$

where $\mathcal{Q}^\Omega_{\mathbf{A},a} : H^1_*(\Omega,\mathbb{C}) \times H^1_*(\Omega,\mathbb{C}) \to \mathbb{C}$ is defined by

$$\begin{aligned}\mathcal{Q}^\Omega_{\mathbf{A},a}(u,w) := &\int_\Omega \left(\nabla u(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right)\cdot\overline{\left(\nabla w(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}w(x)\right)}\,dx\\ &- \int_\Omega \frac{a(x/|x|)}{|x|^2}u(x)\overline{w(x)}\,dx.\end{aligned}$$

In an analogous way, we define the notion of weak solution to (3) in a bounded domain for every Carathéodory function $f : \Omega\times\mathbb{C} \to \mathbb{C}$ satisfying the growth restriction

$$\left|\frac{f(x,z)}{z}\right| \leq \begin{cases} C_f(1+|z|^{2^*-2}) & \text{if } N \geq 3,\\ C_f(1+|z|^{p-2}) \text{ for some } p > 2 & \text{if } N = 2,\end{cases} \tag{7}$$

for a.e. $x \in \Omega$ and for all $z \in \mathbb{C} \setminus \{0\}$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent and the constant $C_f > 0$ is independent of $x \in \Omega$ and $z \in \mathbb{C} \setminus \{0\}$: we say that a function $u \in H^1_*(\Omega, \mathbb{C})$ is an $H^1_*(\Omega, \mathbb{C})$-*weak solution* to (3) if, for all $w \in H^1_0(\Omega, \mathbb{C})$ such that $w/|x| \in L^2(\Omega, \mathbb{C})$,

$$\mathcal{Q}^{\Omega}_{\mathbf{A},a}(u, w) = \int_{\Omega} f(x, u(x))\overline{w(x)}\, dx.$$

Regularity of solutions to (2) or (3) outside the singularity follows from classical elliptic regularity theory, as described in the following remark.

Remark 1.2. If $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$, $a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$, and $h \in L^\infty_{\text{loc}}(\Omega \setminus \{0\})$, then, from standard regularity theory and bootstrap arguments, it follows that any $H^1_*(\Omega, \mathbb{C})$-weak solution u of (2) satisfies $u \in W^{2,p}_{\text{loc}}(\Omega \setminus \{0\})$ for any $1 \le p < \infty$ and in particular $u \in C^{1,\tau}_{\text{loc}}(\Omega \setminus \{0\}, \mathbb{C})$ for any $\tau \in (0, 1)$. The Brezis–Kato technique introduced in [4], standard regularity theory, and bootstrap arguments, lead to the same conclusion also for $H^1_*(\Omega, \mathbb{C})$-weak solutions to (3) with f as in (7).

1.2. Statement of the main results

The following theorem provides a classification of the behavior of any solution u to (2) near the singularity based on the limit as $r \to 0^+$ of the *Almgren frequency* function (see [14])

$$\mathcal{N}_{u,h}(r) = \frac{r \int_{B_r} \left[\left| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right|^2 - \frac{a(x/|x|)}{|x|^2} |u(x)|^2 - \Re(h(x)) |u(x)|^2 \right] dx}{\int_{\partial B_r} |u(x)|^2\, dS}, \tag{8}$$

where, for any $r > 0$, B_r denotes the ball $\{x \in \mathbb{R}^N : |x| < r\}$.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded open set containing* 0*, let* (**A.1**)–(**A.4**) *hold, and let u be an $H^1_*(\Omega, \mathbb{C})$-weak solution to* (2)*, $u \not\equiv 0$, with h satisfying* (6)*. Then, for $\mathcal{N}_{u,h}(r)$ as in* (8)*, there exists $k_0 \in \mathbb{N}$, $k_0 \ge 1$, such that*

$$\lim_{r \to 0^+} \mathcal{N}_{u,h}(r) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)}. \tag{9}$$

Furthermore, if γ denotes the limit in (9)*, $m \ge 1$ is the multiplicity of the eigenvalue $\mu_{k_0}(\mathbf{A}, a)$, and $\{\psi_i : j_0 \le i \le j_0 + m - 1\}$ ($j_0 \le k_0 \le j_0 + m - 1$) is an $L^2(\mathbb{S}^{N-1}, \mathbb{C})$-orthonormal basis for the eigenspace of the operator $L_{\mathbf{A},a}$ associated to $\mu_{k_0}(\mathbf{A}, a)$, then*

$$\lambda^{-\gamma} u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1}, \mathbb{C}) \text{ as } \lambda \to 0^+, \tag{10}$$

and

$$\lambda^{1-\gamma}\nabla u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i(\gamma\psi_i(\theta)\theta + \nabla_{\mathbb{S}^{N-1}}\psi_i(\theta)) \quad \text{in } C^{0,\tau}(\mathbb{S}^{N-1}, \mathbb{C}^N) \text{ as } \lambda \to 0^+, \tag{11}$$

for any $\tau \in (0, 1)$*, where*

$$\beta_i = \int_{\mathbb{S}^{N-1}} \left[R^{-\gamma}u(R\theta) + \int_0^R \frac{h(s\theta)u(s\theta)}{2\gamma + N - 2}\left(s^{1-\gamma} - \frac{s^{\gamma+N-1}}{R^{2\gamma+N-2}}\right) ds\right]\overline{\psi_i(\theta)}\, dS(\theta) \tag{12}$$

for all $R > 0$ *such that* $\overline{B_R} = \{x \in \mathbb{R}^N : |x| \leq R\} \subset \Omega$ *and* $(\beta_{j_0}, \beta_{j_0+1}, \dots, \beta_{j_0+m-1}) \neq (0, 0, \dots, 0)$.

We notice that (12) is actually a *Cauchy integral type formula* for u which allows retracing the behavior of u at the singularity from the values of u along any circle centered at 0, up to some term depending on the perturbation h.

An application of Theorem 1.3 to the special case of Aharonov–Bohm magnetic fields in $\mathbb{R}^2$ of the form (1) is described in Section 7.

Theorem 1.3 implies a *strong unique continuation property* as the following corollary states. Moreover, if $\gamma > 0$ (as happens e.g. under assumption (**A.4**) in dimension $N = 2$) then the solutions to (2) are Hölder continuous for $0 < \gamma < 1$ and Lipschitz continuous for $\gamma \geq 1$.

Corollary 1.4. *Suppose that all the assumptions of Theorem* 1.3 *hold true. Let* γ *denote the limit in* (9) *and* u *be an* $H^1_*(\Omega, \mathbb{C})$*-weak solution to* (2).

(i) *If* $u(x) = O(|x|^k)$ *as* $|x| \to 0$ *for all* $k \in \mathbb{N}$*, then* $u \equiv 0$ *in* Ω.
(ii) *If* $0 < \gamma < 1$ *then* $u \in C^{0,\gamma}_{\text{loc}}(\Omega, \mathbb{C})$.
(iii) *If* $\gamma \geq 1$ *then* u *is locally Lipschitz continuous in* Ω.

We notice that the unique continuation property proved in [17] for electromagnetic potentials in the Kato class does not contain the result stated in part (i) of Corollary 1.4 for singular homogeneous magnetic potentials. We also remark that the monotonicity argument used to prove Theorem 1.3 (see Sections 5 and 6) actually applies when perturbing the magnetic homogeneous potential with a nonsingular term, namely with a magnetic potential of the form

$$\boldsymbol{\mathcal{A}}(x) = \frac{\mathbf{A}(x/|x|)}{|x|} + \mathbf{b}(x) \tag{13}$$

where $\mathbf{b} \in C^1(\Omega \setminus \{0\}, \mathbb{C}^N)$ satisfies $|\mathbf{b}(x)| = O(|x|^{-1+\varepsilon})$ and $|\nabla\mathbf{b}(x)| = O(|x|^{-2+\varepsilon})$ as $|x| \to 0$ for some $\varepsilon > 0$ as $|x| \to 0$. For the sake of simplicity, we omit the details of case (13), which can be treated following closely the strategy developed in Sections 5 and 6.

Due to the homogeneity of the potentials, the Schrödinger operators $\mathcal{L}_{\mathbf{A},a}$ are invariant under the Kelvin transform

$$\tilde{u}(x) = |x|^{-(N-2)}u(x/|x|^2),$$

which is an isomorphism of $\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})$. Indeed, if $u \in H^1_*(\Omega,\mathbb{C})$ weakly solves (2) in a bounded open set Ω containing 0, then its Kelvin transform $\tilde{u}$ weakly solves (2) with h replaced by $|x|^{-4}h(x/|x|^2)$ in the exterior domain $\widetilde{\Omega} = \{x \in \mathbb{R}^N : x/|x|^2 \in \Omega\}$. By a weak solution of problem (2) with h satisfying

$$h \in L^\infty_{\rm loc}(\Omega,\mathbb{C}), \qquad h(x) = O(|x|^{-2-\varepsilon}) \quad \text{as } |x| \to +\infty \text{ for some } \varepsilon > 0, \tag{14}$$

in an *exterior domain* Ω (i.e. a domain Ω such that $\mathbb{R}^N \setminus B_{R_0} \subset \Omega \subset \mathbb{R}^N \setminus B_{R_1}$ for some $R_0 > R_1 > 0$), we mean a function u such that $u/|x| \in L^2(\Omega,\mathbb{C})$, $\nabla u \in L^2(\Omega,\mathbb{C}^N)$, and

$$\mathcal{Q}^\Omega_{\mathbf{A},a}(u,w) = \int_\Omega h(x)u(x)\overline{w(x)}\,dx$$

for any $w \in \mathcal{D}^{1,2}_*(\Omega,\mathbb{C})$, where $\mathcal{D}^{1,2}_*(\Omega,\mathbb{C})$ is the completion of $C^\infty_{\rm c}(\Omega,\mathbb{C})$ with respect to the norm $\|u\|_{\mathcal{D}^{1,2}_*(\Omega)} := (\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^N)} + \|u/|x|\|^2_{L^2(\Omega,\mathbb{C})})^{1/2}$.

Theorem 1.3 and invariance under the Kelvin transform provide the following description of the behavior of solutions to (2) as $|x| \to \infty$. The Almgren frequency type function in exterior domains has the form

$$\widetilde{\mathcal{N}}_{u,h}(r) = \frac{r\int_{\mathbb{R}^N\setminus B_r}\Big[\big|\nabla u(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\big|^2 - \frac{a(x/|x|)}{|x|^2}|u(x)|^2 - \Re(h(x))|u(x)|^2\Big]\,dx}{\int_{\partial B_r}|u(x)|^2\,dS}. \tag{15}$$

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open set such that $\mathbb{R}^N \setminus B_{R_0} \subset \Omega \subset \mathbb{R}^N \setminus B_{R_1}$ for some $R_0 > R_1 > 0$, let* (**A.1**)*–*(**A.4**) *hold, and let u be a weak solution to* (2)*, $u \not\equiv 0$, with h satisfying* (14)*. Then, for $\widetilde{\mathcal{N}}_{u,h}$ as in* (15)*, there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that*

$$\lim_{r\to+\infty}\widetilde{\mathcal{N}}_{u,h}(r) = \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A},a)}. \tag{16}$$

Moreover, if $\tilde{\gamma}$ denotes the limit in (16)*, $m \geq 1$ is the multiplicity of the eigenvalue $\mu_{k_0}(\mathbf{A},a)$, and $\{\psi_i : j_0 \leq i \leq j_0+m-1\}$ ($j_0 \leq k_0 \leq j_0+m-1$) is an $L^2(\mathbb{S}^{N-1},\mathbb{C})$-orthonormal basis for the eigenspace of $L_{\mathbf{A},a}$ associated to $\mu_{k_0}(\mathbf{A},a)$, then*

$$\lambda^{\tilde{\gamma}}u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1}\widetilde{\beta}_i\psi_i(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1},\mathbb{C}) \text{ as } \lambda \to +\infty$$

and

$$\lambda^{\tilde{\gamma}+1}\nabla u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1}\widetilde{\beta}_i(-\tilde{\gamma}\psi_i(\theta)\theta + \nabla_{\mathbb{S}^{N-1}}\psi_i(\theta)) \;\; \text{in } C^{0,\tau}(\mathbb{S}^{N-1},\mathbb{C}^N) \text{ as } \lambda \to +\infty,$$

for every $\tau \in (0,1)$, where

$$\widetilde{\beta}_i = \int_{\mathbb{S}^{N-1}}\left[R^{\tilde{\gamma}}u(R\theta) + \int_R^{+\infty}\frac{h(s\theta)u(s\theta)}{2\tilde{\gamma}-N+2}(s^{\tilde{\gamma}+1} - R^{2\tilde{\gamma}-N+2}s^{-\tilde{\gamma}+N-1})\,ds\right]\overline{\psi_i(\theta)}\,dS(\theta)$$

for all $R > 0$ such that $\mathbb{R}^N \setminus B_R \subset \Omega$ and $(\widetilde{\beta}_{j_0},\ldots,\widetilde{\beta}_{j_0+m-1}) \neq (0,\ldots,0)$.

A Brezis–Kato type iteration (see [4]) allows us to obtain the asymptotics of solutions also for semilinear problems with at most critical growth. In order to start such an iterative procedure, we require assumption (**A.5**) which allows transforming equation (3) into a degenerate elliptic equation without singular potentials to which the Brezis–Kato method applies successfully (see Lemmas 9.1 and 10.3). The iteration scheme developed in Sections 9 and 10 provides an upper bound for solutions and then reduces the semilinear problem to a linear one with enough control on the perturbing potential at the singularity to apply Theorem 1.3 and to recover the exact asymptotic behavior, as stated in the following theorem.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set containing* 0*, let* (**A.1**)–(**A.3**) *and* (**A.5**) *hold, and let u be an $H^1_*(\Omega, \mathbb{C})$-weak solution to* (3)*, $u \not\equiv 0$, with f being a Carathéodory function satisfying* (7)*. Then there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that*

$$\lim_{r\to 0^+} \mathcal{N}_{u, f(\cdot,u)/u}(r) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)}. \tag{17}$$

Furthermore, if γ denotes the limit in (17)*, $m \geq 1$ is the multiplicity of the eigenvalue $\mu_{k_0}(\mathbf{A}, a)$, and $\{\psi_i : j_0 \leq i \leq j_0 + m - 1\}$ ($j_0 \leq k_0 \leq j_0 + m - 1$) is an $L^2(\mathbb{S}^{N-1}, \mathbb{C})$-orthonormal basis for the eigenspace of $L_{\mathbf{A},a}$ associated to $\mu_{k_0}(\mathbf{A}, a)$, then*

$$\lambda^{-\gamma} u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1}, \mathbb{C}) \text{ as } \lambda \to 0^+$$

and

$$\lambda^{1-\gamma} \nabla u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i (\gamma \psi_i(\theta)\theta + \nabla_{\mathbb{S}^{N-1}} \psi_i(\theta)) \quad \text{in } C^{0,\tau}(\mathbb{S}^{N-1}, \mathbb{C}^N) \text{ as } \lambda \to 0^+,$$

for any $\tau \in (0, 1)$, where

$$\beta_i = \int_{\mathbb{S}^{N-1}} \left[R^{-\gamma} u(R\theta) + \int_0^R \frac{f(s\theta, u(s\theta))}{2\gamma + N - 2} \left(s^{1-\gamma} - \frac{s^{\gamma+N-1}}{R^{2\gamma+N-2}} \right) ds \right] \overline{\psi_i(\theta)}\, dS(\theta)$$

for all $R > 0$ such that $\overline{B_R} \subset \Omega$ and $(\beta_{j_0}, \dots, \beta_{j_0+m-1}) \neq (0, \dots, 0)$.

Similar conclusions to those in Corollary 1.4 can be deduced from the above theorem for solutions to semilinear equations of type (3): under the same assumption as in Theorem 1.6, if $\gamma > 0$ then the solutions to (3) are γ-Hölder continuous for $0 < \gamma < 1$ and Lipschitz continuous for $\gamma \geq 1$.

The following result is the counterpart of Theorem 1.6 in exterior domains.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open set such that $\mathbb{R}^N \setminus B_{R_0} \subset \Omega \subset \mathbb{R}^N \setminus B_{R_1}$ for some $R_0 > R_1 > 0$, let* (**A.1**)–(**A.3**) *and* (**A.5**) *hold, and let u be a weak solution to* (3) *in Ω, $u \not\equiv 0$, with f satisfying, for some $\widetilde{C}_f > 0$,*

$$\left| \frac{f(x, z)}{z} \right| \leq \begin{cases} \widetilde{C}_f (|x|^{-4} + |z|^{2^*-2}) & \text{if } N \geq 3, \\ \widetilde{C}_f |x|^{-4}(1 + |z|^{p-2}) \text{ for some } p > 2 & \text{if } N = 2, \end{cases}$$

for a.e. $x \in \Omega$ and for all $z \in \mathbb{C} \setminus \{0\}$. Then there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that

$$\lim_{r\to+\infty} \widetilde{\mathcal{N}}_{u,f(\cdot,u)/u}(r) = \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A},a)}. \tag{18}$$

Moreover, if $\tilde{\gamma}$ denotes the limit in (18)*, $m \geq 1$ is the multiplicity of the eigenvalue $\mu_{k_0}(\mathbf{A},a)$, and $\{\psi_i : j_0 \leq i \leq j_0+m-1\}$ ($j_0 \leq k_0 \leq j_0+m-1$) is an $L^2(\mathbb{S}^{N-1},\mathbb{C})$-orthonormal basis for the eigenspace of the operator $L_{\mathbf{A},a}$ associated to $\mu_{k_0}(\mathbf{A},a)$, then*

$$\lambda^{\tilde{\gamma}} u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1} \widetilde{\beta}_i \psi_i(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1},\mathbb{C}) \text{ as } \lambda \to +\infty$$

and

$$\lambda^{\tilde{\gamma}+1} \nabla u(\lambda\theta) \to \sum_{i=j_0}^{j_0+m-1} \widetilde{\beta}_i \big(-\tilde{\gamma}\psi_i(\theta)\theta + \nabla_{\mathbb{S}^{N-1}}\psi_i(\theta)\big) \text{ in } C^{0,\tau}(\mathbb{S}^{N-1},\mathbb{C}^N) \text{ as } \lambda \to +\infty,$$

for every $\tau \in (0,1)$, where

$$\widetilde{\beta}_i = \int_{\mathbb{S}^{N-1}} \left[R^{\tilde{\gamma}} u(R\theta) + \int_R^{+\infty} \frac{f(s\theta,u(s\theta))}{2\tilde{\gamma}-N+2} (s^{\tilde{\gamma}+1} - R^{2\tilde{\gamma}-N+2} s^{-\tilde{\gamma}+N-1})\, ds \right] \overline{\psi_i(\theta)}\, dS(\theta)$$

for all $R > 0$ such that $\mathbb{R}^N \setminus B_R \subset \Omega$ and $(\widetilde{\beta}_{j_0}, \dots, \widetilde{\beta}_{j_0+m-1}) \neq (0,\dots,0)$.

The paper is organized as follows. In Section 2 we prove Lemma 1.1 and discuss the relation between the positivity of the quadratic form associated to $\mathcal{L}_{\mathbf{A},a}$ and the first eigenvalue of the angular operator on the sphere $\mathbb{S}^{N-1}$. In Section 3 we prove a Hardy type inequality with boundary terms and singular electromagnetic potential, while in Section 4 we derive a Pohozaev type identity for solutions to (2). Section 5 contains an Almgren type monotonicity formula, which is used in Section 6 together with a blow-up method to prove Theorems 1.3 and 1.5. Section 7 contains an application of Theorem 1.3 to Aharonov–Bohm magnetic potentials. In Section 8 we prove a Hardy–Sobolev inequality with magnetic potentials which is needed in Section 9 to start a Brezis–Kato iteration procedure in order to obtain a priori pointwise bounds for solutions to the nonlinear equation and to prove Theorems 1.6 and 1.7 in dimension $N \geq 3$. The proofs of Theorems 1.6 and 1.7 in dimension $N = 2$ can be found in Section 10. In a final appendix, we recall well known results such as the diamagnetic inequality, Hardy's inequality with boundary terms, and the description of the spectrum of the angular operator $L_{\mathbf{A},a}$.

Notation. We list some notation used throughout the paper.

- For all $r > 0$, B_r denotes the ball $\{x \in \mathbb{R}^N : |x| < r\}$ in $\mathbb{R}^N$ with center at 0 and radius r.
- For all $r > 0$, $\overline{B_r} = \{x \in \mathbb{R}^N : |x| \leq r\}$ denotes the closure of B_r.
- dS denotes the volume element on the spheres ∂B_r, $r > 0$.
- For every $z \in \mathbb{C}$, $\Re z$ denotes its real part and $\Im z$ its imaginary part.
- For every $z \in \mathbb{C}$, $\overline{z}$ denotes its complex conjugate.

2. Positivity of the quadratic form

In this section, we study the quadratic form associated to the Schrödinger operator $\mathcal{L}_{\mathbf{A},a}$ and defined in (5). To study the sign of $Q_{\mathbf{A},a}$, we define the first eigenvalue of $Q_{\mathbf{A},a}$ with respect to the Hardy singular weight as

$$\lambda_1(\mathbf{A},a) := \inf_{u\in\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{Q_{\mathbf{A},a}(u)}{\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,dx}$$

and discuss the relation between $\lambda_1(\mathbf{A},a)$ and the first eigenvalue of the angular component of the operator on the sphere $\mathbb{S}^{N-1}$, i.e. of the operator

$$L_{\mathbf{A},a} = (-i\nabla_{\mathbb{S}^{N-1}} + \mathbf{A})^2 - a = -\Delta_{\mathbb{S}^{N-1}} - (a(\theta) - |\mathbf{A}|^2 + i\,\mathrm{div}_{\mathbb{S}^{N-1}}\mathbf{A}) - 2i\mathbf{A}\cdot\nabla_{\mathbb{S}^{N-1}}.$$

We notice that, by (**A.2**), $\lambda_1(\mathbf{A},a)$ is well defined and finite. Let us introduce the Sobolev space

$$H^1_{\mathbf{A}}(\mathbb{S}^{N-1}) := \left\{\psi\in L^2(\mathbb{S}^{N-1},\mathbb{C}) : \nabla_{\mathbb{S}^{N-1}}\psi + i\mathbf{A}(\theta)\psi \in L^2(\mathbb{S}^{N-1},\mathbb{C}^N)\right\}, \tag{19}$$

endowed with the norm

$$\|\psi\|_{H^1_{\mathbf{A}}(\mathbb{S}^{N-1})} := \left(\int_{\mathbb{S}^{N-1}}\left[|(\nabla_{\mathbb{S}^{N-1}} + i\mathbf{A}(\theta))\psi(\theta)|^2 + |\psi(\theta)|^2\right]dS(\theta)\right)^{1/2}, \tag{20}$$

dS denoting the volume element on the sphere $\mathbb{S}^{N-1}$. If $\mathbf{A}\in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$, then $H^1_{\mathbf{A}}(\mathbb{S}^{N-1})$ is equal to the classical Sobolev space $H^1(\mathbb{S}^{N-1},\mathbb{C})$ and its norm is equivalent to the $H^1(\mathbb{S}^{N-1},\mathbb{C})$-norm (see Lemma A.4 in the appendix).

Under assumption (**A.2**), the operator $L_{\mathbf{A},a}$ on $\mathbb{S}^{N-1}$ admits a diverging sequence of real eigenvalues $\mu_1(\mathbf{A},a)\le\mu_2(\mathbf{A},a)\le\cdots$, the first of which can be characterized as

$$\mu_1(\mathbf{A},a) = \min_{\psi\in H^1_{\mathbf{A}}(\mathbb{S}^{N-1})\setminus\{0\}} \frac{\int_{\mathbb{S}^{N-1}}\left[|(\nabla_{\mathbb{S}^{N-1}} + i\mathbf{A}(\theta))\psi(\theta)|^2 - a(\theta)|\psi(\theta)|^2\right]dS(\theta)}{\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2\,dS(\theta)} \tag{21}$$

(see Lemma A.5). The relation between $\lambda_1(\mathbf{A},a)$ and $\mu_1(\mathbf{A},a)$ is clarified in the following lemma.

Lemma 2.1. *If $N\ge 2$, and* (**A.2**) *and* (**A.3**) *hold, then*

$$\lambda_1(\mathbf{A},a) = \mu_1(\mathbf{A},a) + \left(\frac{N-2}{2}\right)^2.$$

Proof. Let $\psi\in H^1_{\mathbf{A}}(\mathbb{S}^{N-1})$, $\psi\not\equiv 0$, attain $\mu_1(\mathbf{A},a)$ and let $\varphi\in C^\infty_c((0,+\infty),\mathbb{R})$ so that $\widetilde{\varphi}: x\mapsto\varphi(|x|)\in C^\infty_c(\mathbb{R}^N\setminus\{0\},\mathbb{R})$. If $u(x)=\varphi(|x|)\psi(x/|x|)$, then

$$\left(\nabla + i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u(x) = \varphi'(|x|)\psi\left(\frac{x}{|x|}\right)\frac{x}{|x|} + \frac{1}{|x|}\varphi(|x|)\nabla_{\mathbb{S}^{N-1}}\psi\left(\frac{x}{|x|}\right) + \frac{i}{|x|}\mathbf{A}\left(\frac{x}{|x|}\right)\varphi(|x|)\psi\left(\frac{x}{|x|}\right)$$

and, by assumption (**A.3**),

$$\left|\left(\nabla + i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u(x)\right|^2 = |\varphi'(|x|)|^2\left|\psi\left(\frac{x}{|x|}\right)\right|^2 + \frac{|\varphi(|x|)|^2}{|x|^2}\left|\nabla_{\mathbb{S}^{N-1}}\psi\left(\frac{x}{|x|}\right) + i\mathbf{A}\left(\frac{x}{|x|}\right)\psi\left(\frac{x}{|x|}\right)\right|^2.$$

Therefore, from the definition of $\lambda_1(\mathbf{A}, a)$ it follows that

$$\begin{aligned}
&\lambda_1(\mathbf{A}, a)\left(\int_0^{+\infty} r^{N-1}\frac{|\varphi(r)|^2}{r^2}\,dr\right)\left(\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2\,dS(\theta)\right)\\
&\leq \int_{\mathbb{R}^N}\left[\left|\left(\nabla + i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u(x)\right|^2 - a\left(\frac{x}{|x|}\right)\frac{|u(x)|^2}{|x|^2}\right]dx\\
&= \left(\int_0^{+\infty} r^{N-1}|\varphi'(r)|^2\,dr\right)\left(\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2\,dS(\theta)\right)\\
&+\left(\int_0^{+\infty} r^{N-1}\frac{|\varphi(r)|^2}{r^2}\,dr\right)\left(\int_{\mathbb{S}^{N-1}}[|\nabla_{\mathbb{S}^{N-1}}\psi(\theta) + i\mathbf{A}(\theta)\psi(\theta)|^2 - a(\theta)|\psi(\theta)|^2]\,dS(\theta)\right)\\
&= \left(\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2\,dS(\theta)\right)\left[\int_0^{+\infty} r^{N-1}|\varphi'(r)|^2\,dr + \mu_1(\mathbf{A}, a)\int_0^{+\infty} r^{N-1}\frac{|\varphi(r)|^2}{r^2}\,dr\right].
\end{aligned}$$

Hence

$$\lambda_1(\mathbf{A}, a) - \mu_1(\mathbf{A}, a) \leq \frac{\int_0^{+\infty} r^{N-1}|\varphi'(r)|^2\,dr}{\int_0^{+\infty} r^{N-3}|\varphi(r)|^2\,dr} = \frac{\int_{\mathbb{R}^N}|\nabla\widetilde{\varphi}(x)|^2\,dx}{\int_{\mathbb{R}^N}\frac{|\widetilde{\varphi}(x)|^2}{|x|^2}\,dx}$$

for every radial function $\widetilde{\varphi} \in C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$. Hence by Schwarz symmetrization

$$\begin{aligned}
\lambda_1(\mathbf{A}, a) - \mu_1(\mathbf{A}, a) &\leq \inf_{\substack{\widetilde{\varphi}\in C_c^\infty(\mathbb{R}^N\setminus\{0\},\mathbb{R})\setminus\{0\}\\ \widetilde{\varphi}\text{ radial}}} \frac{\int_{\mathbb{R}^N}|\nabla\widetilde{\varphi}(x)|^2\,dx}{\int_{\mathbb{R}^N}\frac{|\widetilde{\varphi}(x)|^2}{|x|^2}\,dx}\\
&= \inf_{v\in C_c^\infty(\mathbb{R}^N\setminus\{0\},\mathbb{R})\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla v(x)|^2\,dx}{\int_{\mathbb{R}^N}\frac{|v(x)|^2}{|x|^2}\,dx} = \left(\frac{N-2}{2}\right)^2,
\end{aligned}$$

where the last identity is due to the optimality of the classical best Hardy constant for $N \geq 3$ and to direct calculations for $N = 2$. In order to prove the reverse inequality, let $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$. The magnetic gradient of u can be written in polar coordinates as

$$\nabla u(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}u(x) = (\partial_r u(r,\theta))\theta + \frac{1}{r}\nabla_{\mathbb{S}^{N-1}}u(r,\theta) + i\frac{u(r,\theta)}{r}\mathbf{A}(\theta),$$
$$r = |x|, \quad \theta = \frac{x}{|x|}.$$

By assumption (**A.3**),

$$\left|\nabla u(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right|^2 = |\partial_r u(r,\theta)|^2 + \frac{1}{r^2}|\nabla_{\mathbb{S}^{N-1}} u(r,\theta) + i\mathbf{A}(\theta)u(r,\theta)|^2, \quad (22)$$

hence

$$Q_{\mathbf{A},a}(u) = \int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty} r^{N-1}|\partial_r u(r,\theta)|^2\,dr\right)dS(\theta)$$
$$+\int_0^{+\infty}\frac{r^{N-1}}{r^2}\left(\int_{\mathbb{S}^{N-1}}[|\nabla_{\mathbb{S}^{N-1}} u(r,\theta) + i\mathbf{A}(\theta)u(r,\theta)|^2 - a(\theta)|u(r,\theta)|^2]\,dS(\theta)\right)dr. \quad (23)$$

For all $\theta \in \mathbb{S}^{N-1}$, let $\varphi_\theta \in C^\infty_c((0,+\infty),\mathbb{C})$ be defined by $\varphi_\theta(r) = u(r,\theta)$, and let $\widetilde{\varphi}_\theta \in C^\infty_c(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ be the radially symmetric function given by $\widetilde{\varphi}_\theta(x) = \varphi_\theta(|x|)$. If $N \geq 3$, Hardy's inequality yields

$$\begin{aligned}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty} r^{N-1}|\partial_r u(r,\theta)|^2\,dr\right)dS(\theta) &= \int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty} r^{N-1}|\varphi_\theta'(r)|^2\,dr\right)dS(\theta)\\
&= \frac{1}{\omega_{N-1}}\int_{\mathbb{S}^{N-1}}\left(\int_{\mathbb{R}^N}|\nabla\widetilde{\varphi}_\theta(x)|^2\,dx\right)dS(\theta)\\
&\geq \frac{1}{\omega_{N-1}}\left(\frac{N-2}{2}\right)^2\int_{\mathbb{S}^{N-1}}\left(\int_{\mathbb{R}^N}\frac{|\widetilde{\varphi}_\theta(x)|^2}{|x|^2}\,dx\right)dS(\theta)\\
&= \left(\frac{N-2}{2}\right)^2\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty}\frac{r^{N-1}}{r^2}|u(r,\theta)|^2\,dr\right)dS(\theta)\\
&= \left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,dx, \qquad (24)\end{aligned}$$

where ω_{N-1} denotes the volume of the unit sphere $\mathbb{S}^{N-1}$, i.e. $\omega_{N-1} = \int_{\mathbb{S}^{N-1}} dS(\theta)$. For $N = 2$, (24) holds trivially. On the other hand, from the definition of $\mu_1(\mathbf{A},a)$ it follows that

$$\int_{\mathbb{S}^{N-1}}[|\nabla_{\mathbb{S}^{N-1}} u(r,\theta) + i\mathbf{A}(\theta)u(r,\theta)|^2 - a(\theta)|u(r,\theta)|^2]\,dS(\theta)$$
$$\geq \mu_1(\mathbf{A},a)\int_{\mathbb{S}^{N-1}}|u(r,\theta)|^2\,dS(\theta). \quad (25)$$

From (23)–(25), we deduce that

$$Q_{\mathbf{A},a}(u) \geq \left[\left(\frac{N-2}{2}\right)^2 + \mu_1(\mathbf{A},a)\right]\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,dx \quad \text{for all } u \in C^\infty_c(\mathbb{R}^N\setminus\{0\},\mathbb{C}),$$

which, by density of $C^\infty_c(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ in $\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})$, implies

$$\lambda_1(\mathbf{A},a) \geq \left(\frac{N-2}{2}\right)^2 + \mu_1(\mathbf{A},a),$$

thus completing the proof. □

The relation between the positivity of $Q_{\mathbf{A},a}$ and the values $\mu_1(\mathbf{A}, a)$, $\lambda_1(\mathbf{A}, a)$ is described in the following lemma.

Lemma 2.2. *If $N \geq 2$, and* (**A.2**) *and* (**A.3**) *hold, then the following conditions are equivalent:*

(i) $Q_{\mathbf{A},a}$ *is positive definite in* $\mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C})$*, i.e.* $\displaystyle\inf_{u\in\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{Q_{\mathbf{A},a}(u)}{\|u\|^2_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})}} > 0$*;*

(ii) $\lambda_1(\mathbf{A}, a) > 0$*;*

(iii) $\mu_1(\mathbf{A}, a) > -\left(\frac{N-2}{2}\right)^2$.

Proof. The equivalence between (ii) and (iii) is an immediate consequence of Lemma 2.1. The fact that (i) implies (ii) follows easily from (4). It remains to prove that (ii) implies (i). One can proceed as in the proof of [25, Proposition 1.3]. For completeness we give the details. Assume (ii) and suppose towards a contradiction that (i) is not true. Then for any $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C})$ such that

$$\begin{aligned}
Q_{\mathbf{A},a}(u_\varepsilon) &< \varepsilon\|u_\varepsilon\|^2_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})} \\
&\leq 2(\|\mathbf{A}\|^2_{L^\infty(\mathbb{S}^{N-1},\mathbb{R}^N)} + 1)\varepsilon \\
&\quad\times \left(\int_{\mathbb{R}^N} \left|\left(\nabla + i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u(x)\right|^2 dx + \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2}\, dx\right)
\end{aligned}$$

and hence, for ε small,

$$\lambda_1\left(\mathbf{A}, \frac{1}{1 - 2\varepsilon(\|\mathbf{A}\|^2_{L^\infty(\mathbb{S}^{N-1},\mathbb{R}^N)} + 1)}a\right) < \frac{2\varepsilon(\|\mathbf{A}\|^2_{L^\infty(\mathbb{S}^{N-1},\mathbb{R}^N)} + 1)}{1 - 2\varepsilon(\|\mathbf{A}\|^2_{L^\infty(\mathbb{S}^{N-1},\mathbb{R}^N)} + 1)}.$$

On the other hand, from the characterization of $\lambda_1(\mathbf{A}, a)$ given in Lemma 2.1, the map $a \mapsto \lambda_1(\mathbf{A}, a)$ is continuous with respect to the $L^\infty(\mathbb{S}^{N-1})$-norm and hence, letting $\varepsilon \to 0$, we obtain $\lambda_1(\mathbf{A}, a) \leq 0$, a contradiction. □

The previous lemma allows relating $\mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C})$ to the Hilbert space $\mathcal{D}^{1,2}_{\mathbf{A},a}(\mathbb{R}^N)$ generated by the quadratic form $Q_{\mathbf{A},a}$, thus proving Lemma 1.1.

Proof of Lemma 1.1. (i) follows from Lemma 2.1 and assumption (**A.4**). (ii) is a direct consequence of Lemma 2.2 and (**A.4**). From (ii) we deduce that $(Q_{\mathbf{A},a}(\cdot))^{1/2}$ defines a norm in $C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ which is equivalent to $\|\cdot\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})}$. Hence completing $C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to the norms $(Q_{\mathbf{A},a}(\cdot))^{1/2}$ and $\|\cdot\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})}$ yields two coinciding spaces with equivalent norms. □

By Hardy type inequalities, it is possible to compare the function space $\mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C})$ with the classical Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C})$ defined as the completion of $C^\infty_c(\mathbb{R}^N, \mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{C})} := \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx\right)^{1/2}$$

and with the space $\mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^N)$ given by the completion of $C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to the magnetic Dirichlet norm

$$\|u\|_{\mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \left|\nabla u(x) + i\,\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right|^2 dx\right)^{1/2}.$$

The presence of a vector potential satisfying a suitable nondegeneracy condition allows recovering a Hardy inequality even for $N = 2$. Indeed, if $N = 2$ and (**A.3**) holds, and

$$\Phi_{\mathbf{A}} := \frac{1}{2\pi}\int_0^{2\pi} \alpha(t)\,dt \notin \mathbb{Z}, \qquad \text{where } \alpha(t) := \mathbf{A}(\cos t, \sin t)\cdot(-\sin t, \cos t), \tag{26}$$

then functions in $\mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^2)$ satisfy the following Hardy inequality:

$$\Big(\min_{k\in\mathbb{Z}} |k - \Phi_{\mathbf{A}}|\Big)^2 \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\,dx \le \int_{\mathbb{R}^2} \left|\nabla u(x) + i\,\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right|^2 dx, \tag{27}$$

$(\min_{k\in\mathbb{Z}} |k - \Phi_{\mathbf{A}}|)^2$ being the best constant, as proved in [19]. It is easy to verify that, for $N = 2$,

$$\mu_1(\mathbf{A}, 0) = \min_{\substack{\psi\in H^1((0,2\pi),\mathbb{C})\\ \psi(0)=\psi(2\pi)}} \frac{\int_0^{2\pi} |\psi'(t) + i\alpha(t)\psi(t)|^2\,dt}{\int_0^{2\pi} |\psi(t)|^2\,dt},$$

where $\alpha(t) := \mathbf{A}(\cos t, \sin t)\cdot(-\sin t, \cos t)$. Furthermore, $\mu_1(\mathbf{A}, 0) > 0$ if and only if (26) holds. Combining Lemma 2.1 (in the case $N = 2$ and $a \equiv 0$) with [19], we conclude that, for $N = 2$,

$$\mu_1(\mathbf{A}, 0) = \Big(\min_{k\in\mathbb{Z}} |k - \Phi_{\mathbf{A}}|\Big)^2. \tag{28}$$

Lemma 2.3. (i) *If $N \ge 3$ then $\mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C}) = \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C})$ and the norms $\|\cdot\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})}$ and $\|\cdot\|_{\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{C})}$ are equivalent.*

(ii) *If $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ and either $N \ge 3$, or $N = 2$ and* (**A.3**) *and* (26) *hold, then $\mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C}) = \mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^N)$ with equivalent norms.*

Proof. By the classical Hardy inequality, for $N \ge 3$ the norms $\|\cdot\|_{\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{C})}$ and $\|\cdot\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})}$ are equivalent over the space $C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$. The proof of (i) then follows by completion after observing that, for $N \ge 3$, $C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C})$.

In order to prove (ii), let $u \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$. Then

$$\begin{aligned}
\|u\|_{\mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^N)} &= \left\|\nabla u + i\,\frac{\mathbf{A}(x/|x|)}{|x|}u\right\|_{L^2(\mathbb{R}^N,\mathbb{C}^N)} \\
&\le \|\nabla u\|_{L^2(\mathbb{R}^N,\mathbb{C}^N)} + \left\|\frac{\mathbf{A}(x/|x|)}{|x|}u\right\|_{L^2(\mathbb{R}^N,\mathbb{C}^N)} \\
&\le \|\nabla u\|_{L^2(\mathbb{R}^N,\mathbb{C}^N)} + \sup_{\mathbb{S}^{N-1}} |\mathbf{A}|\left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,dx\right)^{1/2} \le \text{const}\,\|u\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})}.
\end{aligned}$$

On the other hand, by the diamagnetic inequality in Lemma A.1, classical Hardy inequality for $N \geq 3$, and (27) for $N = 2$, we have

$$
\begin{aligned}
\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{C})} &= \|\nabla u\|_{L^2(\mathbb{R}^N,\mathbb{C}^N)} \\
&\leq \left\| \nabla u + i\,\frac{\mathbf{A}(x/|x|)}{|x|} u \right\|_{L^2(\mathbb{R}^N,\mathbb{C}^N)} + \left\| \frac{\mathbf{A}(x/|x|)}{|x|} u \right\|_{L^2(\mathbb{R}^N,\mathbb{C}^N)} \\
&\leq \|u\|_{\mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^N)} + \sup_{\mathbb{S}^{N-1}} |\mathbf{A}| \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\, dx \right)^{1/2} \\
&\leq \|u\|_{\mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^N)} + \text{const}\, \big\| \nabla |u| \big\|_{L^2(\mathbb{R}^N,\mathbb{C}^N)} \leq (1 + \text{const}) \|u\|_{\mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^N)}.
\end{aligned}
$$

The above inequalities show that $\|\cdot\|_{\mathcal{D}^{1,2}_*(\mathbb{R}^N,\mathbb{C})}$ and $\|\cdot\|_{\mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^N)}$ are equivalent norms over the space $C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$. The lemma then follows immediately from the definition of the spaces $\mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C})$ and $\mathcal{D}^{1,2}_{\mathbf{A}}(\mathbb{R}^N)$. □

3. A Hardy type inequality with boundary terms

We extend to singular electromagnetic potentials the Hardy type inequality with boundary terms proved by Wang and Zhu in [27] (see Lemma A.3 in the Appendix).

Lemma 3.1. *If $N \geq 2$, and* (**A.2**) *and* (**A.3**) *hold, then*

$$
\int_{B_r} \left[\left| \left(\nabla + i\,\frac{\mathbf{A}(x/|x|)}{|x|} \right) u \right|^2 - \frac{a(x/|x|)}{|x|^2} |u|^2 \right] dx + \frac{N-2}{2r} \int_{\partial B_r} |u(x)|^2\, dS
\geq \left(\mu_1(\mathbf{A}, a) + \left(\frac{N-2}{2} \right)^2 \right) \int_{B_r} \frac{|u(x)|^2}{|x|^2}\, dx \tag{29}
$$

for all $r > 0$ and $u \in H^1_*(B_r, \mathbb{C})$.

Proof. By scaling, it is enough to prove (29) for $r = 1$. Let $u \in C^\infty(B_1, \mathbb{C}) \cap H^1_*(B_1, \mathbb{C})$ with $0 \notin \operatorname{supp} u$. Passing to polar coordinates and using (22), we have

$$
\begin{aligned}
&\int_{B_1} \left[\left| \left(\nabla + i\,\frac{\mathbf{A}(x/|x|)}{|x|} \right) u \right|^2 - \frac{a(x/|x|)}{|x|^2} |u|^2 \right] dx + \frac{N-2}{2} \int_{\partial B_1} |u(x)|^2\, dS \\
&\quad = \int_{\mathbb{S}^{N-1}} \left(\int_0^1 r^{N-1} |\partial_r u(r,\theta)|^2\, dr \right) dS(\theta) + \frac{N-2}{2} \int_{\mathbb{S}^{N-1}} |u(1,\theta)|^2\, dS(\theta) \\
&\quad\quad + \int_0^1 \frac{r^{N-1}}{r^2} \left(\int_{\mathbb{S}^{N-1}} \big[|\nabla_{\mathbb{S}^{N-1}} u(r,\theta) + i\mathbf{A}(\theta) u(r,\theta)|^2 - a(\theta) |u(r,\theta)|^2 \big]\, dS(\theta) \right) dr.
\end{aligned} \tag{30}
$$

For all $\theta \in \mathbb{S}^{N-1}$, let $\varphi_\theta \in C^\infty_{\rm c}((0, +\infty), \mathbb{C})$ be defined by $\varphi_\theta(r) = u(r, \theta)$, and let $\widetilde{\varphi}_\theta \in C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ be the radially symmetric function given by $\widetilde{\varphi}_\theta(x) = \varphi_\theta(|x|)$.

The Hardy inequality with boundary term proved in [27] (see Lemma A.3 in the Appendix) yields, for $N \geq 3$,

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}}\biggl(\int_0^1 r^{N-1}|\partial_r u(r,\theta)|^2\,dr + \frac{N-2}{2}|u(1,\theta)|^2\biggr)\,dS(\theta)\\
&\quad= \int_{\mathbb{S}^{N-1}}\biggl(\int_0^1 r^{N-1}|\varphi_\theta'(r)|^2\,dr + \frac{N-2}{2}|\varphi_\theta(1)|^2\biggr)\,dS(\theta)\\
&\quad= \frac{1}{\omega_{N-1}}\int_{\mathbb{S}^{N-1}}\biggl(\int_{B_1}|\nabla\widetilde{\varphi}_\theta(x)|^2\,dx + \frac{N-2}{2}\int_{\partial B_1}|\widetilde{\varphi}_\theta(x)|^2\,dS\biggr)\,dS(\theta)\\
&\quad\geq \frac{1}{\omega_{N-1}}\Bigl(\frac{N-2}{2}\Bigr)^2\int_{\mathbb{S}^{N-1}}\biggl(\int_{B_1}\frac{|\widetilde{\varphi}_\theta(x)|^2}{|x|^2}\,dx\biggr)\,dS(\theta)\\
&\quad= \Bigl(\frac{N-2}{2}\Bigr)^2\int_{\mathbb{S}^{N-1}}\biggl(\int_0^1\frac{r^{N-1}}{r^2}|u(r,\theta)|^2\,dr\biggr)\,dS(\theta)\\
&\quad= \Bigl(\frac{N-2}{2}\Bigr)^2\int_{B_1}\frac{|u(x)|^2}{|x|^2}\,dx. \qquad (31)
\end{aligned}$$

On the other hand, (31) trivially holds also for $N = 2$. From (30), (31), and (25), we deduce that

$$\begin{aligned}
&\int_{B_1}\biggl[\Bigl|\Bigl(\nabla + i\frac{\mathbf{A}(x/|x|)}{|x|}\Bigr)u\Bigr|^2 - \frac{a(x/|x|)}{|x|^2}|u|^2\biggr]\,dx + \frac{N-2}{2}\int_{\partial B_1}|u(x)|^2\,dS\\
&\qquad\qquad \geq \biggl[\Bigl(\frac{N-2}{2}\Bigr)^2 + \mu_1(\mathbf{A},a)\biggr]\int_{B_1}\frac{|u(x)|^2}{|x|^2}\,dx
\end{aligned}$$

for all $u \in C^\infty(B_1,\mathbb{C}) \cap H^1_*(B_1,\mathbb{C})$ with $0 \notin \operatorname{supp} u$, which, by density, yields the stated inequality for all $H^1_*(B_r,\mathbb{C})$-functions for $r = 1$. □

Remark 3.2. In view of (28), Lemma 3.1 for $N = 2$ and $a \equiv 0$ yields

$$\int_{B_r}\biggl[\Bigl|\Bigl(\nabla + i\frac{\mathbf{A}(x/|x|)}{|x|}\Bigr)u\Bigr|^2\,dx \geq \Bigl(\min_{k\in\mathbb{Z}}|k-\Phi_{\mathbf{A}}|\Bigr)^2\int_{B_r}\frac{|u(x)|^2}{|x|^2}\,dx$$

for all $r > 0$ and $u \in H^1_*(B_r,\mathbb{C})$.

4. A Pohozaev type identity

Solutions to (2) satisfy the following Pohozaev type identity.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set such that $0 \in \Omega$. Let a, $\mathbf{A}$ satisfy (**A.2**), and let u be a weak $H^1_*(\Omega,\mathbb{C})$-solution to (2) in Ω, with h satisfying (6). Then*

$$-\frac{N-2}{2}\int_{B_r}\left[\left|\left(\nabla+i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u\right|^2-\frac{a(x/|x|)}{|x|^2}|u|^2\right]dx$$
$$+\frac{r}{2}\int_{\partial B_r}\left[\left|\left(\nabla+i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u\right|^2-\frac{a(x/|x|)}{|x|^2}|u|^2\right]dS$$
$$=r\int_{\partial B_r}\left|\frac{\partial u}{\partial\nu}\right|^2dS+\int_{B_r}\Re\big(h(x)u(x)(x\cdot\overline{\nabla u(x)})\big)\,dx \qquad (32)$$

for all $r>0$ such that $\overline{B_r}\subset\Omega$, where $\nu=\nu(x)$ is the outer normal vector $\nu(x)=x/|x|$.

Proof. Let $r>0$ be such that $\overline{B_r}\subset\Omega$. Since

$$\int_0^r\left[\int_{\partial B_s}\left[\left|\left(\nabla+i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u\right|^2+\frac{|u|^2}{|x|^2}+\left|\frac{\partial u}{\partial\nu}\right|^2\right]dS\right]ds$$
$$=\int_{B_r}\left[\left|\left(\nabla+i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u\right|^2+\frac{|u|^2}{|x|^2}+\left|\frac{\partial u}{\partial\nu}\right|^2\right]dx<+\infty$$

there exists a sequence $\{\delta_n\}_{n\in\mathbb{N}}\subset(0,r)$ such that $\lim_{n\to+\infty}\delta_n=0$ and

$$\delta_n\int_{\partial B_{\delta_n}}\left[\left|\left(\nabla+i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u\right|^2+\frac{|u|^2}{|x|^2}+\left|\frac{\partial u}{\partial\nu}\right|^2\right]dS\to0\quad\text{as } n\to+\infty. \qquad (33)$$

From classical elliptic regularity theory, $u\in W^{2,p}_{\mathrm{loc}}(\Omega\setminus\{0\})$ for all $p\in[1,\infty)$ and $u\in C^{1,\tau}_{\mathrm{loc}}(\Omega\setminus\{0\},\mathbb{C})$ for any $\tau\in(0,1)$ (see Remark 1.2), hence we can multiply equation (2) by $x\cdot\overline{\nabla u(x)}$, integrate over $B_r\setminus B_{\delta_n}$, and take the real part, thus obtaining

$$\int_{B_r\setminus B_{\delta_n}}\Re\big(\nabla u(x)\cdot\nabla(x\cdot\overline{\nabla u(x)})\big)\,dx$$
$$+\int_{B_r\setminus B_{\delta_n}}\frac{|\mathbf{A}(x/|x|)|^2-a(x/|x|)}{|x|^2}\Re\big(u(x)(x\cdot\overline{\nabla u(x)})\big)\,dx$$
$$+\int_{B_r\setminus B_{\delta_n}}\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\Im\big(\overline{u}(x)\nabla(\nabla u(x)\cdot x)\big)\,dx$$
$$+\int_{B_r\setminus B_{\delta_n}}\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\Im\big((\overline{\nabla u(x)}\cdot x)\nabla u(x)\big)\,dx$$
$$=r\int_{\partial B_r}\left|\frac{\partial u}{\partial\nu}\right|^2dS-\delta_n\int_{\partial B_{\delta_n}}\left|\frac{\partial u}{\partial\nu}\right|^2dS+\int_{B_r\setminus B_{\delta_n}}\Re\big(h(x)u(x)(x\cdot\overline{\nabla u(x)})\big)\,dx. \qquad (34)$$

Integration by parts yields

$$
\begin{aligned}
\int_{B_r\setminus B_{\delta_n}} &\nabla u(x)\cdot\nabla(x\cdot\overline{\nabla u(x)})\,dx \\
&= -(N-1)\int_{B_r\setminus B_{\delta_n}}|\nabla u(x)|^2\,dx + r\int_{\partial B_r}|\nabla u(x)|^2\,dS \\
&\quad - \delta_n\int_{\partial B_{\delta_n}}|\nabla u(x)|^2\,dS - \sum_{i,j=1}^{N}\int_{B_r\setminus B_{\delta_n}} x_j\,\overline{\frac{\partial u}{\partial x_i}}\,\frac{\partial^2 u}{\partial x_i\partial x_j}\,dx. \qquad (35)
\end{aligned}
$$

A further integration by parts leads to

$$
\begin{aligned}
\sum_{i,j=1}^{N}\int_{B_r\setminus B_{\delta_n}} x_j\,\overline{\frac{\partial u}{\partial x_i}}\,\frac{\partial^2 u}{\partial x_i\partial x_j}\,dx &= -N\int_{B_r\setminus B_{\delta_n}}|\nabla u(x)|^2\,dx + r\int_{\partial B_r}|\nabla u(x)|^2\,dS \\
&\quad - \delta_n\int_{\partial B_{\delta_n}}|\nabla u(x)|^2\,dS - \sum_{i,j=1}^{N}\int_{B_r\setminus B_{\delta_n}} x_j\,\frac{\partial u}{\partial x_i}\,\overline{\frac{\partial^2 u}{\partial x_i\partial x_j}}\,dx
\end{aligned}
$$

and hence

$$
\begin{aligned}
\sum_{i,j=1}^{N}\int_{B_r\setminus B_{\delta_n}} &\Re\left(x_j\,\overline{\frac{\partial u}{\partial x_i}}\,\frac{\partial^2 u}{\partial x_i\partial x_j}\right)dx \\
&= -\frac{N}{2}\int_{B_r\setminus B_{\delta_n}}|\nabla u(x)|^2\,dx + \frac{r}{2}\int_{\partial B_r}|\nabla u(x)|^2\,dS - \frac{\delta_n}{2}\int_{\partial B_{\delta_n}}|\nabla u(x)|^2\,dS. \qquad (36)
\end{aligned}
$$

Collecting (35) and (36) we obtain

$$
\begin{aligned}
\int_{B_r\setminus B_{\delta_n}}\Re\big(\nabla u(x)\cdot\nabla(x\cdot\overline{\nabla u(x)})\big)\,dx &= -\frac{N-2}{2}\int_{B_r\setminus B_{\delta_n}}|\nabla u(x)|^2\,dx \\
&\quad + \frac{r}{2}\int_{\partial B_r}|\nabla u(x)|^2\,dS - \frac{\delta_n}{2}\int_{\partial B_{\delta_n}}|\nabla u(x)|^2\,dS. \qquad (37)
\end{aligned}
$$

Letting $f(\theta) = |\mathbf{A}(\theta)|^2 - a(\theta)$, we find that $f \in L^\infty(\mathbb{S}^{N-1},\mathbb{R})$ and, passing to polar coordinates $r = |x|, \theta = x/|x|$, and observing that $\partial_r u(r,\theta) = \nabla u(r\theta)\cdot\theta$,

$$
\begin{aligned}
\int_{B_r\setminus B_{\delta_n}} &\frac{f(x/|x|)}{|x|^2}u(x)(x\cdot\overline{\nabla u(x)})\,dx = \int_{\mathbb{S}^{N-1}} f(\theta)\left[\int_{\delta_n}^{r} s^{N-2}u(s\theta)\overline{\partial_s u(s\theta)}\,ds\right]dS(\theta) \\
&= \int_{\mathbb{S}^{N-1}} f(\theta)\Big[r^{N-2}|u(r\theta)|^2 - \delta_n^{N-2}|u(\delta_n\theta)|^2 \\
&\quad - (N-2)\int_{\delta_n}^{r} s^{N-3}|u(s\theta)|^2\,ds - \int_{\delta_n}^{r} s^{N-2}\overline{u(s\theta)}\partial_s u(s\theta)\,ds\Big]\,dS(\theta) \\
&= r\int_{\partial B_r}\frac{f(x/|x|)}{|x|^2}|u(x)|^2\,dS - \delta_n\int_{\partial B_{\delta_n}}\frac{f(x/|x|)}{|x|^2}|u(x)|^2\,dS \\
&\quad - (N-2)\int_{B_r\setminus B_{\delta_n}}\frac{f(x/|x|)}{|x|^2}|u(x)|^2\,dx - \int_{B_r\setminus B_{\delta_n}}\frac{f(x/|x|)}{|x|^2}\overline{u(x)}(x\cdot\nabla u(x))\,dx,
\end{aligned}
$$

thus leading to

$$\begin{aligned}\int_{B_r\setminus B_{\delta_n}} &\frac{|\mathbf{A}(x/|x|)|^2-a(x/|x|)}{|x|^2}\Re\big(u(x)(x\cdot\overline{\nabla u(x)})\big)\,dx\\ &= -\frac{N-2}{2}\int_{B_r\setminus B_{\delta_n}} \frac{|\mathbf{A}(x/|x|)|^2-a(x/|x|)}{|x|^2}|u(x)|^2\,dx\\ &\quad+\frac{r}{2}\int_{\partial B_r}\frac{|\mathbf{A}(x/|x|)|^2-a(x/|x|)}{|x|^2}|u(x)|^2\,dS\\ &\quad-\frac{\delta_n}{2}\int_{\partial B_{\delta_n}}\frac{|\mathbf{A}(x/|x|)|^2-a(x/|x|)}{|x|^2}|u(x)|^2\,dS. \end{aligned}\tag{38}$$

From integration by parts it follows that

$$\begin{aligned}\int_{B_r\setminus B_{\delta_n}} &\overline{u(x)}\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla(\nabla u(x)\cdot x)\,dx\\ &= -(N-2)\int_{B_r\setminus B_{\delta_n}}\overline{u(x)}\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla u(x)\,dx\\ &\quad+r\int_{\partial B_r}\overline{u(x)}\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla u(x)\,dS-\delta_n\int_{\partial B_{\delta_n}}\overline{u(x)}\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla u(x)\,dS\\ &\quad-\int_{B_r\setminus B_{\delta_n}}\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla u(x)(x\cdot\overline{\nabla u(x)})\,dx \end{aligned}$$

and therefore

$$\begin{aligned}\int_{B_r\setminus B_{\delta_n}} &\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\Im(\overline{u}(x)\nabla(\nabla u(x)\cdot x))\,dx+\int_{B_r\setminus B_{\delta_n}}\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\Im\big((\overline{\nabla u(x)}\cdot x)\nabla u(x)\big)\,dx\\ &= -(N-2)\int_{B_r\setminus B_{\delta_n}}\Im\left(\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla u(x)\overline{u(x)}\right)dx\\ &\quad+r\int_{\partial B_r}\Im\left(\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla u(x)\overline{u(x)}\right)dS\\ &\quad-\delta_n\int_{\partial B_{\delta_n}}\Im\left(\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla u(x)\overline{u(x)}\right)dS. \end{aligned}\tag{39}$$

Putting together (34) and (37)–(39) and taking into account that

$$\left|\left(\nabla+i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u\right|^2=|\nabla u|^2+2\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\Im(\overline{u}\nabla u)+\frac{|\mathbf{A}(x/|x|)|^2}{|x|^2}|u|^2,$$

we obtain

$$
\begin{aligned}
-\frac{N-2}{2}\int_{B_r\setminus B_{\delta_n}}&\left[\left|\left(\nabla+i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u\right|^2-\frac{a(x/|x|)}{|x|^2}|u|^2\right]dx\\
&+\frac{r}{2}\int_{\partial B_r}\left[\left|\left(\nabla+i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u\right|^2-\frac{a(x/|x|)}{|x|^2}|u|^2\right]dS\\
&-\frac{\delta_n}{2}\int_{\partial B_{\delta_n}}\left[\left|\left(\nabla+i\frac{\mathbf{A}(x/|x|)}{|x|}\right)u\right|^2-\frac{a(x/|x|)}{|x|^2}|u|^2\right]dS\\
=r\int_{\partial B_r}&\left|\frac{\partial u}{\partial\nu}\right|^2 dS-\delta_n\int_{\partial B_{\delta_n}}\left|\frac{\partial u}{\partial\nu}\right|^2 dS+\int_{B_r\setminus B_{\delta_n}}\Re\big(h(x)u(x)(x\cdot\overline{\nabla u(x)})\big)\,dx.
\end{aligned}
$$

Letting $n\to+\infty$ in the above identity and using (33) we obtain (32). □

5. The Almgren type frequency function

Let u be an $H^1_*(\Omega,\mathbb{C})$-weak solution to equation (2) in a bounded domain $\Omega\subset\mathbb{R}^N$ containing the origin with h satisfying (6). Let $\overline{R}>0$ be such that $\overline{B_{\overline{R}}}\subseteq\Omega$. Thus, the following functions are well defined for every $r\in(0,\overline{R}]$:

$$
D(r)=\frac{1}{r^{N-2}}\int_{B_r}\left[\left|\nabla u(x)+i\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right|^2-\frac{a(x/|x|)}{|x|^2}|u(x)|^2-\Re(h(x))|u(x)|^2\right]dx \tag{40}
$$

and

$$
H(r)=\frac{1}{r^{N-1}}\int_{\partial B_r}|u|^2\,dS. \tag{41}
$$

We are going to study the regularity of D and H. We first differentiate H.

Lemma 5.1. *Let $\Omega\subset\mathbb{R}^N$, $N\geq 2$, be a bounded open set such that $0\in\Omega$. Let $a,\mathbf{A}$ satisfy* (**A.2**)*, and let u be a weak $H^1_*(\Omega,\mathbb{C})$-solution to* (2) *in Ω, with h satisfying* (6)*. If H is the function defined in* (41)*, then $H\in C^1(0,\overline{R})$ and*

$$
H'(r)=\frac{2}{r^{N-1}}\int_{\partial B_r}\Re\left(u\frac{\partial\overline{u}}{\partial\nu}\right)dS\quad\text{for every } r\in(0,\overline{R}). \tag{42}
$$

Proof. Fix $r_0\in(0,\overline{R})$ and consider the limit

$$
\lim_{r\to r_0}\frac{H(r)-H(r_0)}{r-r_0}=\lim_{r\to r_0}\int_{\partial B_1}\frac{|u(r\theta)|^2-|u(r_0\theta)|^2}{r-r_0}\,dS(\theta). \tag{43}
$$

Since $u\in C^1(B_{\overline{R}}\setminus\{0\},\mathbb{C})$ (see Remark 1.2), for every $\theta\in\partial B_1$ we have

$$
\lim_{r\to r_0}\frac{|u(r\theta)|^2-|u(r_0\theta)|^2}{r-r_0}=2\Re\left(\frac{\partial\overline{u}}{\partial\nu}(r_0\theta)u(r_0\theta)\right). \tag{44}
$$

On the other hand, for any $r \in (r_0/2, \overline{R})$ and $\theta \in \partial B_1$,

$$\left|\frac{|u(r\theta)|^2 - |u(r_0\theta)|^2}{r - r_0}\right| \leq 2 \sup_{B_{\overline{R}} \setminus B_{r_0/2}} |u| \cdot \sup_{B_{\overline{R}} \setminus B_{r_0/2}} |\nabla u|,$$

and hence, by (43), (44), and the Dominated Convergence Theorem, we obtain

$$H'(r_0) = \int_{\partial B_1} 2\Re\left(\frac{\partial \overline{u}}{\partial \nu}(r_0\theta)u(r_0\theta)\right) dS(\theta) = \frac{2}{r_0^{N-1}} \int_{\partial B_{r_0}} \Re\left(u \frac{\partial \overline{u}}{\partial \nu}\right) dS.$$

The continuity of H' on the interval $(0, \overline{R})$ follows from this representation, the fact that $u \in C^1(B_{\overline{R}} \setminus \{0\}, \mathbb{C})$, and the Dominated Convergence Theorem. □

In the lemma below, we study the regularity of the function D.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set such that $0 \in \Omega$. Let a, $\mathbf{A}$ satisfy* (**A.2**)*, and let u be a weak $H^1_*(\Omega, \mathbb{C})$-solution to* (2) *in Ω, with h satisfying* (6)*. If D is the function defined in* (40)*, then $D \in W^{1,1}_{\mathrm{loc}}(0, \overline{R})$. Moreover*

$$D'(r) = \frac{2}{r^{N-1}}\left[r \int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}\right|^2 dS + \int_{B_r} \Re\big(h(x)\overline{u(x)}(x \cdot \nabla u(x))\big)\, dx \right.$$
$$\left. + \frac{N-2}{2} \int_{B_r} \Re(h(x))|u(x)|^2\, dx - \frac{r}{2} \int_{\partial B_r} \Re(h(x))|u(x)|^2\, dS \right] \qquad (45)$$

in the distributional sense and for a.e. $r \in (0, \overline{R})$.

Proof. For any $r \in (0, \overline{R})$ let

$$I(r) = \int_{B_r} \left[\left|\nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x)\right|^2 - \frac{a(x/|x|)}{|x|^2}|u(x)|^2 - \Re(h(x))|u(x)|^2\right] dx$$
$$= \int_0^r \left(\int_{\partial B_\rho} \left[\left|\nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x)\right|^2 - \frac{a(x/|x|)}{|x|^2}|u(x)|^2 - \Re(h(x))|u(x)|^2\right] dS\right) d\rho. \qquad (46)$$

From the fact that $u \in H^1_*(B_{\overline{R}}, \mathbb{C})$, we deduce that $I \in W^{1,1}(0, \overline{R})$ and

$$I'(r) = \int_{\partial B_r} \left[\left|\nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x)\right|^2 - \frac{a(x/|x|)}{|x|^2}|u(x)|^2 - \Re(h(x))|u(x)|^2\right] dS \qquad (47)$$

for a.e. $r \in (0, \overline{R})$ and in the distributional sense. Therefore by (32), (46), and (47), we deduce that $D \in W^{1,1}_{\mathrm{loc}}(0, \overline{R})$ and

$$D'(r) = r^{1-N}[-(N-2)I(r) + rI'(r)]$$
$$= r^{1-N}\left[2r \int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}\right|^2 dS + 2 \int_{B_r} \Re\big(h(x)\overline{u(x)}(x \cdot \nabla u(x))\big)\, dx \right.$$
$$\left. + (N-2) \int_{B_r} \Re(h(x))|u(x)|^2\, dx - r \int_{\partial B_r} \Re(h(x))|u(x)|^2\, dS \right] \qquad (48)$$

for a.e. $r \in (0, \overline{R})$ and in the distributional sense. □

We now show that $H(r)$ does not vanish for every $r > 0$ sufficiently close to zero.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set such that $0 \in \Omega$, let $a, \mathbf{A}$ satisfy* (**A.2**)–(**A.4**)*, and let $u \not\equiv 0$ be an $H^1_*(\Omega, \mathbb{C})$-weak solution to* (2) *in Ω, with h satisfying* (6)*. Let $H = H(r)$ be the function defined in* (41)*. Then there exists $\overline{r} > 0$ such that $H(r) > 0$ for any $r \in (0, \overline{r})$.*

Proof. Suppose towards a contradiction that there exists a sequence $r_n \to 0^+$ such that $H(r_n) = 0$. Then for any n, $u \equiv 0$ on ∂B_{r_n}. Multiplying both sides of (2) by $\overline{u}$ and integrating by parts over B_{r_n} we obtain

$$\int_{B_{r_n}} \left| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right|^2 dx - \int_{B_{r_n}} \frac{a(x/|x|)}{|x|^2} |u(x)|^2 \, dx$$
$$= \int_{B_{r_n}} h(x)|u(x)|^2 \, dx + \int_{\partial B_{r_n}} \frac{\partial u}{\partial \nu} \overline{u} \, dS = \int_{B_{r_n}} h(x)|u(x)|^2 \, dx.$$

Taking the real part on both sides it follows that

$$\int_{B_{r_n}} \left| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right|^2 dx - \int_{B_{r_n}} \frac{a(x/|x|)}{|x|^2} |u(x)|^2 \, dx = \int_{B_{r_n}} \Re(h(x))|u(x)|^2 \, dx.$$

Since $u \equiv 0$ on ∂B_{r_n}, Lemma 3.1 and (6) yield, for some positive constant $c_h > 0$ depending only on h,

$$\begin{aligned} 0 \geq & \int_{B_{r_n}} \left| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right|^2 dx \\ & - \int_{B_{r_n}} \frac{a(x/|x|)}{|x|^2} |u(x)|^2 \, dx - c_h r_n^{\varepsilon} \int_{B_{r_n}} \frac{|u(x)|^2}{|x|^2} \, dx \\ \geq & \left(\mu_1(\mathbf{A}, a) + \left(\frac{N-2}{2} \right)^2 - c_h r_n^{\varepsilon} \right) \int_{B_{r_n}} \frac{|u(x)|^2}{|x|^2} \, dx. \end{aligned} \tag{49}$$

Since $\mu_1(\mathbf{A}, a) + \left(\frac{N-2}{2}\right)^2 > 0$ and $r_n \to 0^+$, we conclude that $u \equiv 0$ in B_{r_n} for n sufficiently large. Since $u \equiv 0$ in a neighborhood of the origin, we may apply, away from the origin, a unique continuation principle for second order elliptic equations with locally bounded coefficients (see e.g. [28]) to conclude that $u \equiv 0$ in Ω, a contradiction. □

By virtue of Lemma 5.3, the *Almgren type frequency function*

$$\mathcal{N}(r) = \mathcal{N}_{u,h}(r) = \frac{D(r)}{H(r)} \tag{50}$$

is well defined in a suitably small interval $(0, \bar{r})$. Combining Lemmas 5.1 and 5.2, we compute the derivative of $\mathcal{N}$.

Lemma 5.4. *Let* $\Omega \subset \mathbb{R}^N$*,* $N \geq 2$*, be a bounded open set such that* $0 \in \Omega$*, let* $a, \mathbf{A}$ *satisfy* (**A.2**)–(**A.4**)*, and let* $u \not\equiv 0$ *be an* $H^1_*(\Omega, \mathbb{C})$*-weak solution to* (2) *in* Ω*, with* h *satisfying* (6)*. Then, letting* $\mathcal{N}$ *be as in* (50)*, we have* $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}(0, \bar{r})$ *and*

$$\mathcal{N}'(r) = \frac{2r\Big[\Big(\int_{\partial B_r} \big|\frac{\partial u}{\partial \nu}\big|^2 dS\Big) \cdot \Big(\int_{\partial B_r} |u|^2 dS\Big) - \Big(\int_{\partial B_r} \Re\big(u \frac{\partial \overline{u}}{\partial \nu}\big) dS\Big)^2\Big]}{\Big(\int_{\partial B_r} |u|^2 dS\Big)^2}$$
$$+ \frac{2\Big[\int_{B_r} \Re\big(h(x)\overline{u(x)}(x \cdot \nabla u(x))\big) dx + \frac{N-2}{2} \int_{B_r} \Re(h(x))|u(x)|^2 dx - \frac{r}{2} \int_{\partial B_r} \Re(h(x))|u(x)|^2 dS\Big]}{\int_{\partial B_r} |u|^2 dS} \tag{51}$$

in the distributional sense and for a.e. $r \in (0, \bar{r})$*.*

Proof. From Lemmas 5.1–5.3, it follows that $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}(0, \bar{r})$. Multiplying both sides of (2) by $\overline{u}$, integrating by parts, and taking the real part we obtain the identity

$$\int_{B_r} \left[\left| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right|^2 - \frac{a(x/|x|)}{|x|^2} |u(x)|^2 - \Re(h(x))|u(x)|^2 \right] dx = \int_{\partial B_r} \Re\left(u \frac{\partial \overline{u}}{\partial \nu} \right) dS.$$

Therefore, by (40) and (42) we infer that

$$D(r) = \frac{1}{2} r H'(r) \tag{52}$$

for every $r \in (0, \bar{r})$. From (52) we have

$$\mathcal{N}'(r) = \frac{D'(r)H(r) - D(r)H'(r)}{(H(r))^2} = \frac{D'(r)H(r) - \frac{1}{2}r(H'(r))^2}{(H(r))^2}$$

and, using (42) and (45), the assertion of the lemma easily follows. □

We now prove that $\mathcal{N}(r)$ admits a finite limit as $r \to 0^+$.

Lemma 5.5. *Under the same assumptions as in Lemma* 5.4*, the limit*

$$\gamma := \lim_{r \to 0^+} \mathcal{N}(r)$$

exists and is finite.

Proof. We start by proving that $\mathcal{N}(r)$ is bounded from below as $r \to 0^+$. By Lemma 3.1, proceeding as in (49) we arrive, for some positive constant $c_h > 0$ depending only on h, at

$$\begin{aligned}\int_{B_r}\left|\nabla u(x)+i\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right|^2 dx-\int_{B_r}\frac{a(x/|x|)}{|x|^2}|u(x)|^2\,dx-\int_{B_r}\Re(h(x))|u(x)|^2\,dx\\ \geq -\frac{N-2}{2r}\int_{\partial B_r}|u(x)|^2\,dS+\left(\mu_1(\mathbf{A},a)+\left(\frac{N-2}{2}\right)^2-c_h r^{\varepsilon}\right)\int_{B_r}\frac{|u(x)|^2}{|x|^2}\,dx\\ > -\frac{N-2}{2r}\int_{\partial B_r}|u(x)|^2\,dS \qquad (53)\end{aligned}$$

for $r>0$ sufficiently small. This together with (40)–(41) yields

$$\mathcal{N}(r) > -\frac{N-2}{2} \qquad (54)$$

for any $r>0$ sufficiently close to zero. Thanks to (6), for some $C_1>0$, we estimate

$$\begin{aligned}\bigg|\int_{B_r}\Re\big(h(x)\overline{u(x)}(x\cdot\nabla u(x))\big)\,dx+\frac{N-2}{2}\int_{B_r}\Re(h(x))|u(x)|^2\,dx\\ -\frac{r}{2}\int_{\partial B_r}\Re(h(x))|u(x)|^2\,dS\bigg|\\ \leq C_1 r^{\varepsilon}\bigg(\int_{B_r}\left|\nabla u+i\frac{\mathbf{A}(x/|x|)}{|x|}u\right|^2 dx+\int_{B_r}\frac{|u(x)|^2}{|x|^2}\,dx+r^{N-2}H(r)\bigg).\end{aligned}$$

Together with (53), this implies that there exist $C_2>0$ and $\tilde r>0$ such that, for any $r\in(0,\tilde r)$,

$$\begin{aligned}\bigg|\int_{B_r}\Re\big(h(x)\overline{u(x)}(x\cdot\nabla u(x))\big)\,dx+\frac{N-2}{2}\int_{B_r}\Re(h(x))|u(x)|^2\,dx\\ -\frac{r}{2}\int_{\partial B_r}\Re(h(x))|u(x)|^2\,dS\bigg|\leq C_2 r^{\varepsilon+N-2}\bigg[D(r)+\frac{N-2}{2}H(r)\bigg].\end{aligned}$$

Therefore, for any $r\in(0,\tilde r)$, we have

$$\begin{aligned}\left|\frac{\int_{B_r}\Re\big(h(x)\overline{u(x)}(x\cdot\nabla u(x))\big)\,dx+\frac{N-2}{2}\int_{B_r}\Re(h(x))|u(x)|^2\,dx-\frac{r}{2}\int_{\partial B_r}\Re(h(x))|u(x)|^2\,dS}{\int_{\partial B_r}|u(x)|^2\,dS}\right|\\ \leq C_2 r^{-1+\varepsilon}\frac{D(r)+\frac{N-2}{2}H(r)}{H(r)}\leq C_2 r^{-1+\varepsilon}\mathcal{N}(r)+C_2\frac{N-2}{2}r^{-1+\varepsilon}. \qquad (55)\end{aligned}$$

By Lemma 5.4 and Schwarz's inequality, one sees that

$$\begin{aligned}&\mathcal{N}'(r)\\ &\geq 2\frac{\int_{B_r}\Re\big(h(x)\overline{u(x)}(x\cdot\nabla u(x))\big)\,dx+\frac{N-2}{2}\int_{B_r}\Re(h(x))|u(x)|^2\,dx-\frac{r}{2}\int_{\partial B_r}\Re(h(x))|u(x)|^2\,dS}{\int_{\partial B_r}|u(x)|^2\,dS}\end{aligned}$$

and hence by (55) we obtain

$$\mathcal{N}'(r) \geq -2C_2 r^{-1+\varepsilon}\mathcal{N}(r) - C_2(N-2)r^{-1+\varepsilon} \tag{56}$$

for any $r \in (0, \tilde{r})$. After integration it follows that, for some $C_3 > 0$,

$$\mathcal{N}(r) \leq \mathcal{N}(\tilde{r})e^{(2C_2/\varepsilon)(\tilde{r}^\varepsilon - r^\varepsilon)} + (N-2)C_2 e^{-(2C_2/\varepsilon)r^\varepsilon}\int_r^{\tilde{r}} s^{\varepsilon-1}e^{(2C_2/\varepsilon)s^\varepsilon}\,ds \leq C_3 \tag{57}$$

for any $r \in (0, \tilde{r})$. This shows that the left hand side of (55) belongs to $L^1(0, \tilde{r})$. In particular by Lemma 5.4 and Schwarz's inequality we see that $\mathcal{N}'$ is the sum of a nonnegative function and an L^1-function. Therefore

$$\mathcal{N}(r) = \mathcal{N}(\tilde{r}) - \int_r^{\tilde{r}} \mathcal{N}'(s)\,ds$$

admits a limit as $r \to 0^+$, which is necessarily finite in view of (54) and (57). □

A first consequence of the above analysis of the Almgren frequency function is the following estimate of $H(r)$.

Lemma 5.6. *Under the same assumptions as in Lemma* 5.4*, let* $\gamma := \lim_{r\to 0^+}\mathcal{N}(r)$ *be as in Lemma* 5.5*. Then there exists a constant* $K_1 > 0$ *such that*

$$H(r) \leq K_1 r^{2\gamma} \quad \textit{for all } r \in (0, \bar{r}). \tag{58}$$

On the other hand for any $\sigma > 0$ *there exists a constant* $K_2(\sigma) > 0$ *depending on* σ *such that*

$$H(r) \geq K_2(\sigma) r^{2\gamma+\sigma} \quad \textit{for all } r \in (0, \bar{r}). \tag{59}$$

Proof. We start by proving (58). Since, by Lemma 5.5, $\mathcal{N}' \in L^1(0, \bar{r})$ and $\mathcal{N}$ is bounded, by (56) we infer that

$$\mathcal{N}(r) - \gamma = \int_0^r \mathcal{N}'(s)\,ds \geq -C_4 r^\varepsilon \tag{60}$$

for some constant $C_4 > 0$ and $r \in (0, \tilde{r})$ with $0 < \tilde{r} < \bar{r}$. Therefore by (52) and (60) we deduce that for $r \in (0, \tilde{r})$,

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} \geq \frac{2\gamma}{r} - 2C_4 r^{-1+\varepsilon}.$$

Then (58) follows immediately after integration of the previous differential inequality over the interval $(r, \tilde{r})$ and by continuity of H outside 0.

Let us prove (59). Since $\gamma = \lim_{r\to 0^+}\mathcal{N}(r)$, for any $\sigma > 0$ there exists $r_\sigma > 0$ such that $\mathcal{N}(r) < \gamma + \sigma/2$ for any $r \in (0, r_\sigma)$ and hence

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} < \frac{2\gamma + \sigma}{r} \quad \text{for all } r \in (0, r_\sigma).$$

Integrating over the interval (r, r_σ) and by the continuity of H outside 0, we obtain (59) for some constant $K_2(\sigma)$ depending on σ. □

6. Proofs of Theorems 1.3 and 1.5

In this section we use the monotonicity properties established in Section 5 combined with a blow-up technique to deduce the asymptotics of solutions near the singularity and to prove Theorems 1.3 and 1.5.

Lemma 6.1. *Let* $\Omega \subset \mathbb{R}^N$*,* $N \geq 2$*, be a bounded open set containing* 0*, let* $a, \mathbf{A}$ *be such that* (**A.2**)–(**A.4**) *hold, and let h be as in* (6)*. For* $u \in H^1_*(\Omega, \mathbb{C})$ *weakly solving* (2)*,* $u \not\equiv 0$*, let* $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ *be as in Lemma* 5.5*. Then*

(i) *there exists* $k_0 \in \mathbb{N}$ *such that*

$$\gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)};$$

(ii) *for every sequence* $\lambda_n \to 0^+$*, there exist a subsequence* $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ *and an eigenfunction* ψ *of the operator* $L_{\mathbf{A},a}$ *associated to the eigenvalue* $\mu_{k_0}(\mathbf{A}, a)$ *such that* $\|\psi\|_{L^2(\mathbb{S}^{N-1},\mathbb{C})} = 1$ *and*

$$\frac{u(\lambda_{n_k} x)}{\sqrt{H(\lambda_{n_k})}} \to |x|^\gamma \psi\left(\frac{x}{|x|}\right)$$

weakly in $H^1(B_1, \mathbb{C})$*, strongly in* $H^1(B_r, \mathbb{C})$ *for every* $0 < r < 1$*, and in* $C^{1,\tau}_{\rm loc}(B_1 \setminus \{0\}, \mathbb{C})$ *for any* $\tau \in (0, 1)$*.*

Proof. Let us set

$$w^\lambda(x) = \frac{u(\lambda x)}{\sqrt{H(\lambda)}}.$$

We notice that $\int_{\partial B_1} |w^\lambda|^2 \, dS = 1$. Moreover, by scaling and (57),

$$\int_{B_1} \left| \nabla w^\lambda(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} w^\lambda(x) \right|^2 dx - \int_{B_1} \frac{a(x/|x|)}{|x|^2} |w^\lambda(x)|^2 \, dx - \int_{B_1} \lambda^2 \Re(h(\lambda x)) |w^\lambda(x)|^2 \, dx = \mathcal{N}(\lambda) \leq \text{const.} \tag{61}$$

Hence, by (29) and (6) there exists $c_h > 0$ such that

$$\left(\mu_1(\mathbf{A}, a) + \left(\frac{N-2}{2}\right)^2 - c_h \lambda^\varepsilon\right) \int_{B_1} \frac{|w^\lambda(x)|^2}{|x|^2} \, dx \leq \frac{N-2}{2} + \mathcal{N}(\lambda),$$

and consequently there exist $\bar\lambda > 0$ and const > 0 such that

$$\int_{B_1} \frac{|w^\lambda(x)|^2}{|x|^2} \, dx \leq \text{const} \quad \text{for every } 0 < \lambda < \bar\lambda,$$

which, in view of (61), implies that $\{w^\lambda\}_{\lambda\in(0,\bar\lambda)}$ is bounded in $H^1_*(B_1, \mathbb{C})$.

Therefore, for any given sequence $\lambda_n \to 0^+$, there exists a subsequence $\lambda_{n_k} \to 0^+$ such that $w^{\lambda_{n_k}} \rightharpoonup w$ weakly in $H^1_*(B_1, \mathbb{C})$ for some $w \in H^1_*(B_1, \mathbb{C})$. We notice that $H^1_*(B_1, \mathbb{C})$ is continuously imbedded into $H^1(B_1, \mathbb{C})$, hence $w^{\lambda_{n_k}} \rightharpoonup w$ weakly also in $H^1(B_1, \mathbb{C})$. Due to compactness of the trace imbedding $H^1(B_1, \mathbb{C}) \hookrightarrow L^2(\partial B_1, \mathbb{C})$, we obtain $\int_{\partial B_1} |w|^2\, dS = 1$. In particular $w \not\equiv 0$. Furthermore, weak convergence allows passing to the weak limit in the equation

$$\mathcal{L}_{\mathbf{A},a} w^{\lambda_{n_k}}(x) = \lambda_{n_k}^2 h(\lambda_{n_k} x) w^{\lambda_{n_k}}(x) \tag{62}$$

which holds in a weak sense in $B_{\overline{R}/\lambda_{n_k}} \supset B_1$ (see the beginning of Section 5 for the definition of $\overline{R}$), thus yielding

$$\mathcal{L}_{\mathbf{A},a} w(x) = 0 \quad \text{in } B_1. \tag{63}$$

A bootstrap argument and classical regularity theory lead to

$$w^{\lambda_{n_k}} \to w \quad \text{in } C^{1,\tau}_{\mathrm{loc}}(B_1 \setminus \{0\}, \mathbb{C})$$

for any $\tau \in (0, 1)$ and

$$w^{\lambda_{n_k}} \to w \quad \text{in } H^1(B_r, \mathbb{C}) \text{ and in } H^1_*(B_r, \mathbb{C}) \tag{64}$$

for any $r \in (0, 1)$. Since the functions $w^{\lambda_{n_k}}$ solve equation (62), for any $r \in (0, 1)$ we may define the functions

$$\begin{aligned} D_k(r) = {}& \frac{1}{r^{N-2}} \int_{B_r} \left[\left| \nabla w^{\lambda_{n_k}}(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} w^{\lambda_{n_k}}(x) \right|^2 \right] dx \\ & - \frac{1}{r^{N-2}} \int_{B_r} \left[\frac{a(x/|x|)}{|x|^2} |w^{\lambda_{n_k}}(x)|^2 + \lambda_{n_k}^2 \Re(h(\lambda_{n_k} x)) |w^{\lambda_{n_k}}(x)|^2 \right] dx \end{aligned}$$

and

$$H_k(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |w^{\lambda_{n_k}}|^2\, dS.$$

On the other hand, since w solves (63), we put

$$D_w(r) = \frac{1}{r^{N-2}} \int_{B_r} \left[\left| \nabla w(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} w(x) \right|^2 - \frac{a(x/|x|)}{|x|^2} |w(x)|^2 \right] dx \tag{65}$$

for all $r \in (0, 1)$ and

$$H_w(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} |w|^2\, dS \quad \text{for all } r \in (0, 1). \tag{66}$$

Using a change of variables, one sees that

$$\mathcal{N}_k(r) := \frac{D_k(r)}{H_k(r)} = \frac{D(\lambda_{n_k} r)}{H(\lambda_{n_k} r)} = \mathcal{N}(\lambda_{n_k} r) \quad \text{for all } r \in (0, 1). \tag{67}$$

By (6) and (64), for any fixed $r \in (0, 1)$ we have

$$D_k(r) \to D_w(r). \tag{68}$$

On the other hand, by compactness of the trace imbedding $H^1(B_r, \mathbb{C}) \hookrightarrow L^2(\partial B_r, \mathbb{C})$, we also have

$$H_k(r) \to H_w(r) \quad \text{for any fixed } r \in (0, 1). \tag{69}$$

From (29) it follows that $D_w(r) > -\frac{N-2}{2} H_w(r)$ for all $r \in (0, 1)$. Therefore, if, for some $r \in (0, 1)$, $H_w(r) = 0$ then $D_w(r) > 0$, and passing to the limit in (67) would give a contradiction with Lemma 5.5. Hence $H_w(r) > 0$ for all $r \in (0, 1)$. Thus the function

$$\mathcal{N}_w(r) := \frac{D_w(r)}{H_w(r)}$$

is well defined for $r \in (0, 1)$. This, together with (67)–(69) and Lemma 5.5, shows that

$$\mathcal{N}_w(r) = \lim_{k\to\infty} \mathcal{N}(\lambda_{n_k} r) = \gamma \tag{70}$$

for all $r \in (0, 1)$. Therefore $\mathcal{N}_w$ is constant in $(0, 1)$ and hence $\mathcal{N}_w'(r) = 0$ for any $r \in (0, 1)$. By (63) and Lemma 5.4 with $h \equiv 0$, we obtain

$$\left(\int_{\partial B_r} \left|\frac{\partial w}{\partial \nu}\right|^2 dS\right) \cdot \left(\int_{\partial B_r} |w|^2\, dS\right) - \left(\int_{\partial B_r} \Re\left(w \frac{\partial \overline{w}}{\partial \nu}\right) dS\right)^2 = 0 \quad \text{for all } r \in (0, 1),$$

i.e.

$$\left|\int_{\partial B_r} \Re\left(w \frac{\partial \overline{w}}{\partial \nu}\right) dS\right|^2 = \|w\|^2_{L^2(\partial B_r, \mathbb{C})} \cdot \left\|\frac{\partial w}{\partial \nu}\right\|^2_{L^2(\partial B_r, \mathbb{C})}.$$

This shows that w and $\partial w/\partial \nu$ have the same direction as vectors in $L^2(\partial B_r, \mathbb{C})$ and hence there exists a real valued function $\eta = \eta(r)$ such that $\frac{\partial w}{\partial \nu}(r, \theta) = \eta(r) w(r, \theta)$ for $r \in (0, 1)$. After integration we obtain

$$w(r, \theta) = e^{\int_1^r \eta(s)\, ds} w(1, \theta) = \varphi(r)\psi(\theta), \quad r \in (0, 1),\ \theta \in \mathbb{S}^{N-1}, \tag{71}$$

where we put $\varphi(r) = e^{\int_1^r \eta(s)\, ds}$ and $\psi(\theta) = w(1, \theta)$. Since

$$\mathcal{L}_{\mathbf{A},a} w = -\frac{\partial^2 w}{\partial r^2} - \frac{N-1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2} L_{\mathbf{A},a} w,$$

(71) yields

$$\left(-\varphi''(r) - \frac{N-1}{r}\varphi'(r)\right)\psi(\theta) + \frac{\varphi(r)}{r^2} L_{\mathbf{A},a}\psi(\theta) = 0.$$

Taking r fixed we deduce that ψ is an eigenfunction of the operator $L_{\mathbf{A},a}$. If $\mu_{k_0}(\mathbf{A}, a)$ is the corresponding eigenvalue then $\varphi(r)$ solves the equation

$$-\varphi''(r) - \frac{N-1}{r}\varphi(r) + \frac{\mu_{k_0}(\mathbf{A}, a)}{r^2}\varphi(r) = 0$$

and hence $\varphi(r)$ is of the form

$$\varphi(r) = c_1 r^{\sigma_{k_0}^+} + c_2 r^{\sigma_{k_0}^-}$$

for some $c_1, c_2 \in \mathbb{R}$, where

$$\sigma_{k_0}^+ = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)},$$
$$\sigma_{k_0}^- = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)}.$$

Since $|x|^{-1}(|x|^{\sigma_{k_0}^-}\psi(x/|x|)) \notin L^2(B_1, \mathbb{C})$ and hence $|x|^{\sigma_{k_0}^-}\psi(x/|x|) \notin H^1_*(B_1, \mathbb{C})$, it follows that $c_2 = 0$ and $\varphi(r) = c_1 r^{\sigma_{k_0}^+}$. Since $\varphi(1) = 1$, we obtain $c_1 = 1$ and then

$$w(r, \theta) = r^{\sigma_{k_0}^+}\psi(\theta) \quad \text{for all } r \in (0, 1) \text{ and } \theta \in \mathbb{S}^{N-1}. \tag{72}$$

It remains to prove part (i). Since w solves (63), after integration by parts

$$\int_{B_r}\left[\left|\nabla w(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}w(x)\right|^2 - \frac{a(x/|x|)}{|x|^2}|w(x)|^2\right]dx = \int_{\partial B_r}\frac{\partial w}{\partial \nu}\overline{w}\,dS.$$

Therefore, by (65), (66), (70), and (72), it follows that

$$\gamma = \mathcal{N}_w(r) = \frac{D_w(r)}{H_w(r)} = \frac{r\int_{\partial B_r}\frac{\partial w}{\partial \nu}\overline{w}\,dS}{\int_{\partial B_r}|w|^2\,dS} = \sigma_{k_0}^+.$$

This completes the proof of the lemma. □

A further step towards a priori bounds for solutions to (2) lies in uniformly estimating the supremum of $|u|$ on ∂B_r with $H(r)$.

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set containing* 0*, let a, $\mathbf{A}$ be such that* (**A.2**)–(**A.4**) *hold, and let h be as in* (6)*. Then, for any $H^1_*(\Omega, \mathbb{C})$-weak solution u to* (2) *there exist $\bar{s} > 0$ and $C > 0$ such that*

$$\sup_{\partial B_s}|u|^2 \leq \frac{C}{s^{N-1}}\int_{\partial B_s}|u|^2\,dS \quad \textit{for every } 0 < s < \bar{s}.$$

Proof. Let $\gamma = \lim_{r\to 0^+}\mathcal{N}(r)$ be as in Lemma 5.5 and $k_0 \in \mathbb{N}$ be such that

$$\gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)}$$

(see Lemma 6.1). Denote by $\mathcal{A}_0$ the eigenspace of the operator $L_{\mathbf{A},a}$ associated to the eigenvalue $\mu_{k_0}(\mathbf{A}, a)$. Since $\dim \mathcal{A}_0$ is finite, it is easy to verify that

$$\Lambda = \sup_{v \in \mathcal{A}_0\setminus\{0\}}\frac{\sup_{\mathbb{S}^{N-1}}|v|^2}{\int_{\mathbb{S}^{N-1}}|v|^2\,dS} < +\infty.$$

Let $\widetilde{C} > 2^{N-1}\Lambda$. We claim that there exists $\bar{\lambda}$ such that

$$\sup_{\partial B_{1/2}} |w^\lambda|^2 \leq \widetilde{C} \int_{\partial B_{1/2}} |w^\lambda|^2 \, dS \quad \text{for every } \lambda \in (0, \bar{\lambda}). \tag{73}$$

To prove (73), assume towards a contradiction that there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \to 0^+$ and

$$\sup_{\partial B_{1/2}} |w^{\lambda_n}|^2 > \widetilde{C} \int_{\partial B_{1/2}} |w^{\lambda_n}|^2 \, dS. \tag{74}$$

Lemma 6.1 implies that there exist a subsequence $\{\lambda_{n_j}\}_{j\in\mathbb{N}}$ and an eigenfunction $\psi \in \mathcal{A}_0$ such that $\|\psi\|^2_{L^2(\mathbb{S}^{N-1},\mathbb{C})} = 1$ and $w^{\lambda_{n_j}} \to |x|^\gamma \psi(x/|x|)$ weakly in $H^1(B_1, \mathbb{C})$ and in $C^{1,\tau}_{\rm loc}(B_1 \setminus \{0\}, \mathbb{C})$ for any $\tau \in (0, 1)$. Passing to the limit in (74), this would imply that

$$\sup_{\mathbb{S}^{N-1}} |\psi|^2 \geq \frac{\widetilde{C}}{2^{N-1}} \int_{\mathbb{S}^{N-1}} |\psi|^2 \, dS > \Lambda \int_{\mathbb{S}^{N-1}} |\psi|^2 \, dS,$$

giving rise to a contradiction with the definition of Λ. Claim (73) is thereby proved.

Estimate (73) can be written as

$$\sup_{\partial B_{\lambda/2}} |u|^2 \leq \frac{\widetilde{C}}{\lambda^{N-1}} \int_{\partial B_{\lambda/2}} |u|^2 \, dS \quad \text{for every } \lambda \in (0, \bar{\lambda}).$$

Choosing $\bar{s} = \frac{1}{2}\bar{\lambda}$ and $C = 2^{1-N}\widetilde{C}$, the conclusion follows. □

From Lemmas 5.6 and 6.2 we deduce the following pointwise estimate for solutions to (2).

Corollary 6.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set containing* 0*, let a, $\mathbf{A}$ be such that* (**A.2**)–(**A.4**) *hold, and let h be as in* (6)*. Then, for any $H^1_*(\Omega, \mathbb{C})$-weak solution u to* (2) *there exist $\bar{s} > 0$ and $\bar{C} > 0$ such that*

$$|u(x)| \leq \bar{C}|x|^\gamma \quad \textit{for every } x \in B_{\bar{s}},$$

where $\gamma = \lim_{r\to 0^+} \mathcal{N}(r)$ is as in Lemma 5.5.

Proof. This follows from (58) and Lemma 6.2. □

Let us now describe the behavior of $H(r)$ as $r \to 0^+$.

Lemma 6.4. *Under the same assumptions as in Lemma* 5.4 *and letting $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ be as in Lemma* 5.5*, the limit*

$$\lim_{r\to 0^+} r^{-2\gamma} H(r)$$

exists and is finite.

Proof. In view of (58) it is sufficient to prove that the limit exists. By (41), (52), and Lemma 5.5 we have

$$\frac{d}{dr}\frac{H(r)}{r^{2\gamma}} = -2\gamma r^{-2\gamma-1}H(r) + r^{-2\gamma}H'(r) = 2r^{-2\gamma-1}(D(r) - \gamma H(r))$$
$$= 2r^{-2\gamma-1}H(r)\int_0^r \mathcal{N}'(s)\,ds.$$

Denote by $M_1(r)$ and $M_2(r)$ respectively the first and the second term on the right hand side of (51). After integration over $(r, \tilde{r})$, we obtain

$$\frac{H(\tilde{r})}{\tilde{r}^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{\tilde{r}} 2s^{-2\gamma-1}H(s)\left(\int_0^s M_1(t)\,dt\right)ds$$
$$+ \int_r^{\tilde{r}} 2s^{-2\gamma-1}H(s)\left(\int_0^s M_2(t)\,dt\right)ds. \tag{75}$$

By Schwarz's inequality we have $M_1(t) \geq 0$ and hence the limit

$$\lim_{r\to 0^+}\int_r^{\tilde{r}} 2s^{-2\gamma-1}H(s)\left(\int_0^s M_1(t)\,dt\right)ds$$

exists. On the other hand, by (55) and (58) we deduce that $|M_2(r)| = O(r^{-1+\varepsilon})$ and $H(r) = O(r^{2\gamma})$ as $r \to 0^+$. Therefore, if $\tilde{r}$ is sufficiently small, for some const > 0 we have

$$\left|s^{-2\gamma-1}H(s)\left(\int_0^s M_2(t)\,dt\right)\right| \leq \frac{\text{const}}{\varepsilon}s^{-1+\varepsilon}$$

for all $r \in (0, \tilde{r})$, which proves that $s^{-2\gamma-1}H(s)(\int_0^s M_2(t)\,dt) \in L^1(0, \tilde{r})$. We may conclude that both terms on the right hand side of (75) admit a limit as $r \to 0^+$, thus completing the proof of the lemma. □

The limit $\lim_{r\to 0^+} r^{-2\gamma}H(r)$ is indeed strictly positive, as we prove in the following lemma.

Lemma 6.5. *Under the same assumptions as in Lemma* 5.4 *and letting* $\gamma := \lim_{r\to 0^+}\mathcal{N}(r)$ *be as in Lemma* 5.5*, we have*

$$\lim_{r\to 0^+} r^{-2\gamma}H(r) > 0.$$

Proof. Let us fix $R > 0$ such that $\overline{B_R} \subset \Omega$. For any $k \in \mathbb{N} \setminus \{0\}$, let ψ_k be an L^2-normalized eigenfunction of the operator $L_{\mathbf{A},a}$ on the sphere associated to the k-th eigenvalue $\mu_k(\mathbf{A}, a)$, i.e. satisfying

$$\begin{cases} L_{\mathbf{A},a}\psi_k(\theta) = \mu_k(\mathbf{A}, a)\psi_k(\theta) & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} |\psi_k(\theta)|^2\,dS(\theta) = 1. \end{cases} \tag{76}$$

We can choose the functions ψ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{N-1}, \mathbb{C})$, hence u and hu can be expanded as

$$u(x) = u(\lambda\theta) = \sum_{k=1}^{\infty} \varphi_k(\lambda)\psi_k(\theta), \qquad h(x)u(x) = h(\lambda\theta)u(\lambda\theta) = \sum_{k=1}^{\infty} \zeta_k(\lambda)\psi_k(\theta), \tag{77}$$

where $\lambda = |x| \in (0, R]$, $\theta = x/|x| \in \mathbb{S}^{N-1}$, and

$$\varphi_k(\lambda) = \int_{\mathbb{S}^{N-1}} u(\lambda\theta)\overline{\psi_k(\theta)}\, dS(\theta), \qquad \zeta_k(\lambda) = \int_{\mathbb{S}^{N-1}} h(\lambda\theta)u(\lambda\theta)\overline{\psi_k(\theta)}\, dS(\theta). \tag{78}$$

Equations (2) and (76) imply that, for every k,

$$-\varphi_k''(\lambda) - \frac{N-1}{\lambda}\varphi_k'(\lambda) + \frac{\mu_k(\mathbf{A}, a)}{\lambda^2}\varphi_k(\lambda) = \zeta_k(\lambda) \quad \text{in } (0, R).$$

A direct calculation shows that, for some $c_1^k, c_2^k \in \mathbb{R}$,

$$\varphi_k(\lambda) = \lambda^{\sigma_k^+}\left(c_1^k + \int_\lambda^R \frac{s^{-\sigma_k^+ + 1}}{\sigma_k^+ - \sigma_k^-}\zeta_k(s)\, ds\right) + \lambda^{\sigma_k^-}\left(c_2^k + \int_\lambda^R \frac{s^{-\sigma_k^- + 1}}{\sigma_k^- - \sigma_k^+}\zeta_k(s)\, ds\right), \tag{79}$$

where

$$\sigma_k^+ = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)},$$
$$\sigma_k^- = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)}.$$

In view of Lemma 6.1, there exist $j_0, m \in \mathbb{N}$, $j_0, m \geq 1$, such that m is the multiplicity of the eigenvalue $\mu_{j_0}(\mathbf{A}, a) = \mu_{j_0+1}(\mathbf{A}, a) = \cdots = \mu_{j_0+m-1}(\mathbf{A}, a)$ and

$$\gamma = \lim_{r\to 0^+} \mathcal{N}(r) = \sigma_i^+, \qquad i = j_0, \ldots, j_0 + m - 1. \tag{80}$$

The Parseval identity yields

$$H(\lambda) = \int_{\mathbb{S}^{N-1}} |u(\lambda\theta)|^2\, dS(\theta) = \sum_{k=1}^{\infty} |\varphi_k(\lambda)|^2 \quad \text{for all } 0 < \lambda \leq R. \tag{81}$$

Assume for contradiction that $\lim_{\lambda\to 0^+} \lambda^{-2\gamma} H(\lambda) = 0$ and fix $i \in \{j_0, \ldots, j_0 + m - 1\}$. Then (80) and (81) imply that

$$\lim_{\lambda\to 0^+} \lambda^{-\sigma_i^+}\varphi_i(\lambda) = 0. \tag{82}$$

From (6) and Corollary 6.3, we obtain

$$\zeta_i(\lambda) = O(\lambda^{-2+\varepsilon+\sigma_i^+}) \quad \text{as } \lambda \to 0^+, \tag{83}$$

and consequently the functions

$$s \mapsto \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s) \quad \text{and} \quad s \mapsto \frac{s^{-\sigma_i^- +1}}{\sigma_i^- - \sigma_i^+}\zeta_i(s)$$

belong to $L^1((0, R), \mathbb{C})$. Hence

$$\lambda^{\sigma_i^+}\left(c_1^i + \int_\lambda^R \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds\right) = o(\lambda^{\sigma_i^-}) \quad \text{as } \lambda \to 0^+,$$

and thus, since $u/|x| \in L^2(B_R, \mathbb{C})$ and $|x|^{\sigma_i^-}/|x| \notin L^2(B_R, \mathbb{C})$, we conclude that

$$c_2^i = -\int_0^R \frac{s^{-\sigma_i^- +1}}{\sigma_i^- - \sigma_i^+}\zeta_i(s)\,ds.$$

Using (83), we then deduce that

$$\lambda^{\sigma_i^-}\left(c_2^i + \int_\lambda^R \frac{s^{-\sigma_i^- +1}}{\sigma_i^- - \sigma_i^+}\zeta_i(s)\,ds\right) = \lambda^{\sigma_i^-}\left(\int_0^\lambda \frac{s^{-\sigma_i^- +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds\right) = O(\lambda^{\sigma_i^+ + \varepsilon}) \tag{84}$$

as $\lambda \to 0^+$. From (79), (82), and (84), we obtain

$$c_1^i + \int_0^R \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds = 0,$$

thus implying, together with (83),

$$\lambda^{\sigma_i^+}\left(c_1^i + \int_\lambda^R \frac{s^{-\sigma_i^+ +1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds\right) = \lambda^{\sigma_i^+}\int_0^\lambda \frac{s^{-\sigma_i^+ +1}}{\sigma_i^- - \sigma_i^+}\zeta_i(s)\,ds = O(\lambda^{\sigma_i^+ + \varepsilon}) \tag{85}$$

as $\lambda \to 0^+$. Collecting (79), (84), and (85), we conclude that

$$\varphi_i(\lambda) = O(\lambda^{\sigma_i^+ + \varepsilon}) \quad \text{as } \lambda \to 0^+ \text{ for every } i \in \{j_0, \dots, j_0 + m - 1\},$$

namely, setting $u^\lambda(\theta) = u(\lambda\theta)$,

$$(u^\lambda, \psi)_{L^2(\mathbb{S}^{N-1},\mathbb{C})} = O(\lambda^{\gamma+\varepsilon}) \quad \text{as } \lambda \to 0^+$$

for every $\psi \in \mathcal{A}_0$, where $\mathcal{A}_0$ is the eigenspace of $L_{\mathbf{A},a}$ associated to the eigenvalue $\mu_{j_0}(\mathbf{A}, a) = \mu_{j_0+1}(\mathbf{A}, a) = \cdots = \mu_{j_0+m-1}(\mathbf{A}, a)$. Let $w^\lambda(\theta) = (H(\lambda))^{-1/2}u(\lambda\theta)$. From (59), there exists $C(\varepsilon) > 0$ such that $\sqrt{H(\lambda)} \geq C(\varepsilon)\lambda^{\gamma+\varepsilon/2}$ for λ small, and therefore

$$(w^\lambda, \psi)_{L^2(\mathbb{S}^{N-1},\mathbb{C})} = O(\lambda^{\varepsilon/2}) = o(1) \quad \text{as } \lambda \to 0^+ \tag{86}$$

for every $\psi \in \mathcal{A}_0$. From Lemma 6.1, for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_j}\}_{j\in\mathbb{N}}$ and an eigenfunction $\widetilde{\psi} \in \mathcal{A}_0$ such that

$$\int_{\mathbb{S}^{N-1}} |\widetilde{\psi}(\theta)|^2\,dS = 1 \quad \text{and} \quad w^{\lambda_{n_j}} \to \widetilde{\psi} \quad \text{in } L^2(\mathbb{S}^{N-1}, \mathbb{C}). \tag{87}$$

From (86) and (87), we infer that

$$0 = \lim_{j \to +\infty} (w^{\lambda_{n_j}}, \widetilde{\psi})_{L^2(\mathbb{S}^{N-1},\mathbb{C})} = \|\widetilde{\psi}\|^2_{L^2(\mathbb{S}^{N-1},\mathbb{C})} = 1,$$

thus reaching a contradiction. □

The analysis carried out in this section leads to a complete description of the behavior of solutions to (2) near the singularity and hence to the proof of Theorem 1.3.

Proof of Theorem 1.3. Identity (9) follows from part (i) of Lemma 6.1, thus there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that

$$\lim_{r \to 0^+} \mathcal{N}_{u,h}(r) = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)} =: \gamma.$$

Let us denote by m the multiplicity of $\mu_{k_0}(\mathbf{A}, a)$, so that, for some $j_0 \in \mathbb{N}$, $j_0 \geq 1$, $j_0 \leq k_0 \leq j_0 + m - 1$, $\mu_{j_0}(\mathbf{A}, a) = \mu_{j_0+1}(\mathbf{A}, a) = \cdots = \mu_{j_0+m-1}(\mathbf{A}, a)$ and let $\{\psi_i : j_0 \leq i \leq j_0 + m - 1\}$ be an $L^2(\mathbb{S}^{N-1}, \mathbb{C})$-orthonormal basis for the eigenspace of $L_{\mathbf{A},a}$ associated to $\mu_{k_0}(\mathbf{A}, a)$. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ be such that $\lim_{n\to+\infty} \lambda_n = 0$. Then, from Lemmas 6.1(ii), 6.4, and 6.5, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and m real numbers $\beta_{j_0}, \dots, \beta_{j_0+m-1} \in \mathbb{R}$ such that $(\beta_{j_0}, \dots, \beta_{j_0+m-1}) \neq (0, \dots, 0)$ and

$$\lambda_{n_k}^{-\gamma} u(\lambda_{n_k}\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i \psi_i(\theta) \quad \text{in } C^{1,\tau}(\mathbb{S}^{N-1}, \mathbb{C}) \text{ as } k \to +\infty \tag{88}$$

and

$$\lambda_{n_k}^{1-\gamma} \nabla u(\lambda_{n_k}\theta) \to \sum_{i=j_0}^{j_0+m-1} \beta_i (\gamma \psi_i(\theta)\theta + \nabla_{\mathbb{S}^{N-1}} \psi_i(\theta)) \quad \text{in } C^{0,\tau}(\mathbb{S}^{N-1}, \mathbb{C}^N) \text{ as } k \to +\infty, \tag{89}$$

for any $\tau \in (0, 1)$. We now prove that the β_i's depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$.

Let us fix $R > 0$ such that $\overline{B_R} \subset \Omega$. Defining φ_i and ζ_i as in (78) and expanding u as in (77), from (88) it follows that, for any $i = j_0, \dots, j_0 + m - 1$,

$$\lambda_{n_k}^{-\gamma} \varphi_i(\lambda_{n_k}) = \int_{\mathbb{S}^{N-1}} \frac{u(\lambda_{n_k}\theta)}{\lambda_{n_k}^{\gamma}} \overline{\psi_i(\theta)}\, dS(\theta) \to \sum_{j=j_0}^{j_0+m-1} \beta_j \int_{\mathbb{S}^{N-1}} \psi_j(\theta)\overline{\psi_i(\theta)}\, dS(\theta) = \beta_i \tag{90}$$

as $k \to +\infty$. As deduced in the proof of Lemma 6.5, for any $i = j_0, \dots, j_0 + m - 1$ and $\lambda \in (0, R]$ we have

$$\varphi_i(\lambda) = \lambda^{\sigma_i^+}\left(c_1^i + \int_\lambda^R \frac{s^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds\right) + \lambda^{\sigma_i^-}\left(\int_0^\lambda \frac{s^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds\right)$$
$$= \lambda^{\sigma_i^+}\left(c_1^i + \int_\lambda^R \frac{s^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds\right) + O(\lambda^{\sigma_i^+ + \varepsilon}) \quad \text{as } \lambda \to 0^+, \tag{91}$$

for some $c_1^i \in \mathbb{R}$, where

$$\sigma_i^+ = \gamma = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)},$$
$$\sigma_i^- = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_{k_0}(\mathbf{A}, a)}.$$

Choosing $\lambda = R$ in the first line of (91), we obtain

$$c_1^i = R^{-\sigma_i^+}\varphi_i(R) - R^{\sigma_i^- - \sigma_i^+}\int_0^R \frac{s^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds.$$

Hence (91) yields

$$\lambda^{-\gamma}\varphi_i(\lambda) \to R^{-\sigma_i^+}\varphi_i(R) - R^{\sigma_i^- - \sigma_i^+}\int_0^R \frac{s^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds + \int_0^R \frac{s^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds$$

as $\lambda \to 0^+$, and therefore, from (90) we deduce that

$$\beta_i = R^{-\gamma}\int_{\mathbb{S}^{N-1}} u(R\theta)\overline{\psi_i(\theta)}\,dS(\theta)$$
$$- R^{-2\gamma - N + 2}\int_0^R \frac{s^{\gamma + N - 1}}{2\gamma + N - 2}\left(\int_{\mathbb{S}^{N-1}} h(s\eta)u(s\eta)\overline{\psi_i(\eta)}\,dS(\eta)\right)ds$$
$$+ \int_0^R \frac{s^{1-\gamma}}{2\gamma + N - 2}\left(\int_{\mathbb{S}^{N-1}} h(s\eta)u(s\eta)\overline{\psi_i(\eta)}\,dS(\eta)\right)ds.$$

In particular the β_i's depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$, thus implying that the convergences in (88) and (89) actually hold as $\lambda \to 0^+$ and proving the theorem. □

Proof of Corollary 1.4. Statement (i) follows directly from (10). Statement (iii) is an immediate consequence of (10) and (11). To prove (ii), we notice that classical elliptic regularity theory yields Hölder continuity away from 0, so it remains to prove that u is Hölder continuous in every $\overline{B_r} \subset \Omega$. Assume towards a contradiction that there exist sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset \overline{B_r}$ such that

$$\lim_{n\to+\infty} \frac{|u(x_n) - u(y_n)|}{|x_n - y_n|^\gamma} = +\infty. \tag{92}$$

Hölder continuity away from 0 implies that either $|x_n| \to 0$ or $|y_n| \to 0$ along a subsequence. We can assume, without loss of generality, that $|y_n| \to 0$ and $|x_n| \geq |y_n|$. Two cases can occur.

Case 1: there exists a positive constant $c > 1$ such that $|x_n|/|y_n| \leq c$. Then $|x_n| \to 0$ and, letting $\lambda_n = 2c|x_n|$ and observing that $x_n/\lambda_n,\, y_n/\lambda_n \in \overline{B}_{1/(2c)} \setminus B_{1/(2c^2)} \Subset B_1 \setminus \{0\}$, from Lemmas 6.1(ii), 6.4 and 6.5 it follows that

$$\lim_{n\to+\infty} \frac{\left|\lambda_n^{-\gamma} u\left(\lambda_n \frac{x_n}{\lambda_n}\right) - (2c)^{-\gamma}\psi\left(\frac{x_n}{|x_n|}\right) - \lambda_n^{-\gamma} u\left(\lambda_n \frac{y_n}{\lambda_n}\right) + \frac{|y_n|^\gamma}{\lambda_n^\gamma}\psi\left(\frac{y_n}{|y_n|}\right)\right|}{\left|\frac{x_n}{\lambda_n} - \frac{y_n}{\lambda_n}\right|^\gamma} = 0$$

for some eigenfunction ψ of $L_{\mathbf{A},a}$. Since the function $|x|^\gamma \psi(x/|x|)$ is Hölder continuous away from 0, from above we conclude that

$$\frac{|u(x_n) - u(y_n)|}{|x_n - y_n|^\gamma} = \frac{\left|\lambda_n^{-\gamma} u\left(\lambda_n \frac{x_n}{\lambda_n}\right) - \lambda_n^{-\gamma} u\left(\lambda_n \frac{y_n}{\lambda_n}\right)\right|}{\left|\frac{x_n}{\lambda_n} - \frac{y_n}{\lambda_n}\right|^\gamma}$$

is bounded uniformly in n, thus giving rise to a contradiction.

Case 2: There exist subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{y_{n_k}\}_{k\in\mathbb{N}}$ such that $|x_{n_k}|/|y_{n_k}| \to +\infty$. In particular $|y_{n_k}| = o(|x_{n_k}|)$ as $k \to +\infty$. From (92) we deduce that $|x_{n_k}| \to 0$ as $k \to +\infty$ and by Corollary 6.3,

$$\frac{|u(x_{n_k}) - u(y_{n_k})|}{|x_{n_k} - y_{n_k}|^\gamma} = |x_{n_k}|^{-\gamma} \frac{|u(x_{n_k}) - u(y_{n_k})|}{\left|\frac{x_{n_k}}{|x_{n_k}|} - \frac{y_{n_k}}{|x_{n_k}|}\right|^\gamma} \leq \text{const}|x_{n_k}|^{-\gamma} \frac{|x_{n_k}|^\gamma + |y_{n_k}|^\gamma}{\left|\frac{x_{n_k}}{|x_{n_k}|} - \frac{y_{n_k}}{|x_{n_k}|}\right|^\gamma} \leq \text{const},$$

in contradiction with (92). □

Invariance under Kelvin's transform allows rewriting equations in exterior domains as equations in bounded neighborhoods of 0, thus reducing the problem of asymptotics at infinity to the problem of asymptotics at 0. Hence we can deduce Theorem 1.5 from Theorem 1.3.

Proof of Theorem 1.5. Let u be a weak solution of (2) where Ω is an exterior domain as in the statement of the theorem. Let v be the Kelvin transform of u, i.e.

$$v(x) = |x|^{2-N} u(x/|x|^2), \qquad x \in \widetilde{\Omega} = \left\{x \in \mathbb{R}^N : x/|x|^2 \in \Omega\right\}. \tag{93}$$

If we put $y = x/|x|^2$, then

$$\Delta u(x) = |y|^{N+2} \Delta v(y) \quad \text{for all } y \in \widetilde{\Omega}, \tag{94}$$

and

$$\begin{aligned} &\frac{a(x/|x|) - |\mathbf{A}(x/|x|)|^2 + i\,\mathrm{div}_{\mathbb{S}^{N-1}} \mathbf{A}(x/|x|)}{|x|^2} u(x) \\ &\quad = |y|^{N+2} \frac{a(y/|y|) - |\mathbf{A}(y/|y|)|^2 + i\,\mathrm{div}_{\mathbb{S}^{N-1}} \mathbf{A}(y/|y|)}{|y|^2} v(y) \quad \text{for all } y \in \widetilde{\Omega}. \end{aligned} \tag{95}$$

Moreover, by the transversality assumption (**A.3**) we also have

$$\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla u(x)=|y|^{N+2}\frac{\mathbf{A}(y/|y|)}{|y|}\cdot\nabla v(y)\quad\text{for all } y\in\widetilde{\Omega}. \tag{96}$$

Therefore, by (93)–(96) we obtain

$$\mathcal{L}_{\mathbf{A},a}v(y)=|y|^{-4}h(y/|y|^2)v(y)\quad\text{in }\widetilde{\Omega}\setminus\{0\}. \tag{97}$$

From a direct computation we infer that $\nabla v\in L^2(\widetilde{\Omega},\mathbb{C}^N)$, $v/|x|\in L^2(\widetilde{\Omega},\mathbb{C})$, and hence $v\in H^1_*(\widetilde{\Omega},\mathbb{C})$. This is sufficient for proving that v is an H^1_*-weak solution of equation (97) in $\widetilde{\Omega}$.

On the other hand, by (14),

$$\big||y|^{-4}h(y/|y|^2)\big|=O(|y|^{-2+\varepsilon})\quad\text{as } |y|\to0^+,$$

and hence v satisfies all the assumptions of Theorem 1.3. Then (16) and the asymptotic estimate for u follow from Theorem 1.3, (93), and the fact that

$$\mathcal{N}_{v,|y|^{-4}h(y/|y|^2)}(r)=\widetilde{\mathcal{N}}_{u,h}(1/r)-N+2 \tag{98}$$

with $\widetilde{\mathcal{N}}_{u,h}$ as in (15). To prove the estimate on the gradient one may proceed as follows. Let $\tilde{\gamma}$ be as in the statement of the theorem and let $\gamma=\lim_{r\to0^+}\mathcal{N}_{v,|y|^{-4}h(y/|y|^2)}(r)$. From (98) it follows that $\gamma=\tilde{\gamma}-N+2$, hence by (93) we have

$$\lambda^{1-\gamma}\nabla v(\lambda\theta)=(2-N)\lambda^{-\tilde{\gamma}}u(\theta/\lambda)\theta+\lambda^{-\tilde{\gamma}-1}\nabla u(\theta/\lambda)-2\lambda^{-\tilde{\gamma}-1}(\nabla u(\theta/\lambda)\cdot\theta)\theta \tag{99}$$

for any λ such that $B_\lambda\subset\widetilde{\Omega}$ and for any $\theta\in\mathbb{S}^{N-1}$. Applying Theorem 1.3 to the function v, from the previous identity we infer

$$(2-N)\lambda^{-\tilde{\gamma}}u(\theta/\lambda)-\lambda^{-\tilde{\gamma}-1}(\nabla u(\theta/\lambda)\cdot\theta)\to\gamma\sum_{i=j_0}^{j_0+m-1}\widetilde{\beta}_i\psi_i(\theta)$$

in $C^{0,\tau}(\mathbb{S}^{N-1},\mathbb{C})$ for any $\tau\in(0,1)$ as $\lambda\to0^+$. From the first part of the theorem we also have

$$\lambda^{-\tilde{\gamma}}u(\theta/\lambda)\to\sum_{i=j_0}^{j_0+m-1}\widetilde{\beta}_i\psi_i(\theta), \tag{100}$$

from which we obtain

$$\lambda^{-\tilde{\gamma}-1}(\nabla u(\theta/\lambda)\cdot\theta)\to-\tilde{\gamma}\sum_{i=j_0}^{j_0+m-1}\widetilde{\beta}_i\psi_i(\theta) \tag{101}$$

in $C^{0,\tau}(\mathbb{S}^{N-1},\mathbb{C})$ for any $\tau\in(0,1)$ as $\lambda\to0^+$. Letting $\lambda\to0^+$ in (99), applying again Theorem 1.3 to v and using (100)–(101) we deduce that

$$\lambda^{-\tilde{\gamma}-1}\nabla u(\theta/\lambda)\to\sum_{i=j_0}^{j_0+m-1}\widetilde{\beta}_i\big(-\tilde{\gamma}\psi_i(\theta)\theta+\nabla_{\mathbb{S}^{N-1}}\psi_i(\theta)\big)$$

in $C^{0,\tau}(\mathbb{S}^{N-1},\mathbb{C}^N)$ for any $\tau\in(0,1)$ as $\lambda\to0^+$. By replacing λ with $1/\lambda$ we obtain the desired estimate. □

7. An example: Aharonov–Bohm magnetic potentials in dimension 2

In this section we discuss an application of Theorem 1.3 to Schrödinger equations with Aharonov–Bohm vector potentials (1), i.e. we let $N = 2$, $\mathbf{A}(\cos t, \sin t) = \alpha(-\sin t, \cos t)$, $a(\cos t, \sin t) = a_0$ for some $a_0 \in \mathbb{R}$, and consider the corresponding equation

$$\left(-i\nabla + \alpha\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)\right)^2 u - \frac{a_0}{|x|^2}u = hu,$$

with $x = (x_1, x_2)$ in a bounded domain of $\mathbb{R}^2$ containing 0, and h satisfying (6). In this case, an explicit calculation yields

$$\{\mu_k(\mathbf{A}, a) : k \in \mathbb{N} \setminus \{0\}\} = \{(\alpha - j)^2 - a_0 : j \in \mathbb{Z}\},$$

hence, in particular,

$$\mu_1(\mathbf{A}, a) = (\operatorname{dist}(\alpha, \mathbb{Z}))^2 - a_0.$$

If $\operatorname{dist}(\alpha, \mathbb{Z}) \neq 1/2$, then all eigenvalues are simple and the eigenspace associated to the eigenvalue $(\alpha - j)^2 - a_0$ is generated by $\psi(\cos t, \sin t) = e^{-ijt}$. If $\operatorname{dist}(\alpha, \mathbb{Z}) = 1/2$, then all eigenvalues have multiplicity 2. Theorem 1.3 hence yields:

(i) if $a_0 < (\operatorname{dist}(\alpha, \mathbb{Z}))^2$ and $\operatorname{dist}(\alpha, \mathbb{Z}) \neq 1/2$, then there exist $j_0 \in \mathbb{Z}$ and $\beta \in \mathbb{C}$ such that

$$\lambda^{-\sqrt{(\alpha - j_0)^2 - a_0}} u(\lambda \cos t, \lambda \sin t) \to \beta e^{-ij_0 t} \quad \text{as } \lambda \to 0^+,$$

in $C^{1,\tau}(0, 2\pi, \mathbb{C})$ for all $\tau \in (0, 1)$;

(ii) if $a_0 < (\operatorname{dist}(\alpha, \mathbb{Z}))^2$ and $\operatorname{dist}(\alpha, \mathbb{Z}) = 1/2$, then there exist $j_0 \in \mathbb{Z}$ and $\beta_1, \beta_2 \in \mathbb{C}$ such that $2\alpha - j_0 \in \mathbb{Z}$ and

$$\lambda^{-\sqrt{(\alpha - j_0)^2 - a_0}} u(\lambda \cos t, \lambda \sin t) \to \beta_1 e^{-ij_0 t} + \beta_2 e^{-i(2\alpha - j_0)t} \quad \text{as } \lambda \to 0^+,$$

in $C^{1,\tau}(0, 2\pi, \mathbb{C})$ for all $\tau \in (0, 1)$.

The constants β, β_1, β_2 can be computed as in (12). Furthermore, in view of Corollary 1.4, if $(\operatorname{dist}(\alpha, \mathbb{Z}))^2 < 1 + a_0$ then $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega, \mathbb{C})$ with $\gamma = \sqrt{(\operatorname{dist}(\alpha, \mathbb{Z}))^2 - a_0}$, whereas u is locally Lipschitz continuous in Ω if $(\operatorname{dist}(\alpha, \mathbb{Z}))^2 \geq 1 + a_0$.

8. Magnetic Hardy–Sobolev type inequalities

This section is devoted to the proof of a weighted electromagnetic Hardy–Sobolev inequality in dimension $N \geq 3$. We start by observing that, from Lemma 2.2 and the classical Sobolev inequality, the following electromagnetic Hardy–Sobolev inequality holds.

Proposition 8.1. *Let $N \geq 3$ and let a, $\mathbf{A}$ satisfy* (**A.2**)–(**A.4**). *Then*

$$S(\mathbf{A}, a) := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \frac{Q_{\mathbf{A},a}(u)}{(\int_{\mathbb{R}^N} |u(x)|^{2^*} dx)^{2/2^*}} > 0.$$

Proof. This follows from Lemma 2.2, Lemma 2.3(i), and Sobolev's inequality. □

We assume $N \geq 3$ and (**A.5**) so that the number

$$\sigma = \sigma(a, N) := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1(0, a)} \tag{102}$$

is well defined. Let $\phi \in H^1(\mathbb{S}^{N-1}, \mathbb{R})$ with $\|\phi\|_{L^2(\mathbb{S}^{N-1},\mathbb{R})} = 1$ be the first positive eigenfunction of the eigenvalue problem

$$-\Delta_{\mathbb{S}^{N-1}}\phi(\theta) - a(\theta)\phi(\theta) = \mu_1(0, a)\phi(\theta) \quad \text{in } \mathbb{S}^{N-1}.$$

We recall from [12, Lemma 2.1] that $\mu_1(0, a)$ is simple and $\min_{\mathbb{S}^{N-1}} \phi > 0$. Let

$$w(x) = |x|^\sigma \phi(x/|x|) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\} \tag{103}$$

and introduce the weighted space $\mathcal{D}^{1,2}_w(\mathbb{R}^N, \mathbb{C})$ as the closure of $C^\infty_{\rm c}(\mathbb{R}^N, \mathbb{C})$ with respect to the norm

$$\|v\|_{\mathcal{D}^{1,2}_w(\mathbb{R}^N,\mathbb{C})} := \left(\int_{\mathbb{R}^N} w^2(x)\big|\nabla v(x)\big|^2\,dx\right)^{1/2}.$$

By the Caffarelli–Kohn–Nirenberg inequality (see [5] and [6]), $v \in \mathcal{D}^{1,2}_w(\mathbb{R}^N, \mathbb{C})$ if and only if $wv \in \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C})$ and there exists $\mathcal{C}_w > 0$ such that

$$\mathcal{C}_w \int_{\mathbb{R}^N} w^2(x)\frac{|v(x)|^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^N} w^2(x)|\nabla v(x)|^2\,dx$$

for every $v \in \mathcal{D}^{1,2}_w(\mathbb{R}^N, \mathbb{C})$.

Proposition 8.2. *Let $N \geq 3$, let a, $\mathbf{A}$ satisfy* (**A.2**)*,* (**A.3**)*, and* (**A.5**)*, and let w be the function defined in* (103)*. Then*

$$\int_{\mathbb{R}^N} w^2(x)\left|\nabla v(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}v(x)\right|^2 dx \geq S(\mathbf{A}, a)\left(\int_{\mathbb{R}^N} w^{2^*}(x)|v(x)|^{2^*}\,dx\right)^{2/2^*} \tag{104}$$

for all $v \in \mathcal{D}^{1,2}_w(\mathbb{R}^N, \mathbb{C})$.

Proof. First of all, one can check by explicit computation that w solves the equation

$$-\Delta w(x) - \frac{a(x/|x|)}{|x|^2}w(x) = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{105}$$

Let $v \in C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \subset \mathcal{D}^{1,2}_w(\mathbb{R}^N, \mathbb{C})$ so that the function $u(x) := w(x)v(x)$ is in $C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\}, \mathbb{C}) \subset \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C})$. By (105) and integration by parts we have

$$\int_{\mathbb{R}^N} \nabla w(x)\nabla(w(x)|v(x)|^2)\,dx - \int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^2}w^2(x)|v(x)|^2\,dx = 0. \tag{106}$$

By a direct computation we infer

$$\nabla w\nabla(w|v|^2) = |\nabla w|^2|v|^2 + w\nabla w(\overline{v}\nabla v + v\nabla\overline{v}) \tag{107}$$

and

$$\left|\nabla u + i\frac{\mathbf{A}(x/|x|)}{|x|}u\right|^2 = |\nabla w|^2|v|^2 + w\nabla w(\overline{v}\nabla v + v\nabla\overline{v}) + w^2|\nabla v|^2 - 2\Im\left(\frac{\mathbf{A}(x/|x|)}{|x|}w^2 v\nabla\overline{v}\right) + \frac{|\mathbf{A}(x/|x|)|^2}{|x|^2}w^2|v|^2. \tag{108}$$

From (106)–(108), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\left|\nabla u(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right|^2 dx - \int_{\mathbb{R}^N}\frac{a(x/|x|)}{|x|^2}|u(x)|^2\,dx\\ = \int_{\mathbb{R}^N} w^2(x)|\nabla v(x)|^2\,dx - \int_{\mathbb{R}^N} 2\Im\left(\frac{\mathbf{A}(x/|x|)}{|x|}w^2(x)v(x)\nabla\overline{v}(x)\right)dx\\ + \int_{\mathbb{R}^N}\frac{|\mathbf{A}(x/|x|)|^2}{|x|^2}w^2(x)|v(x)|^2\,dx\\ = \int_{\mathbb{R}^N} w^2(x)\left|\nabla v(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}v(x)\right|^2 dx.\end{aligned}$$

By the above identity and Proposition 8.1, we obtain (104) for any $v \in C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$. By a density argument (see [6, Lemma 2.1]), we deduce that inequality (104) holds for any $v \in \mathcal{D}_w^{1,2}(\mathbb{R}^N, \mathbb{C})$. □

9. A Brezis–Kato type lemma for $N \geq 3$

This section is devoted to the proof of a Brezis–Kato type result in dimension $N \geq 3$. Let w be the function defined in (103). We define the weighted space $H_w^1(\Omega, \mathbb{C})$ as the closure of $H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C})$ with respect to the norm

$$\|v\|_{H_w^1(\Omega,\mathbb{C})} := \left(\int_\Omega w^2(x)\big[|\nabla v(x)|^2 + |v(x)|^2\big]\,dx\right)^{1/2}, \tag{109}$$

and the space $\mathcal{D}_w^{1,2}(\Omega, \mathbb{C})$ as the closure of $C_c^\infty(\Omega, \mathbb{C})$ with respect to

$$\|v\|_{\mathcal{D}_w^{1,2}(\Omega,\mathbb{C})} := \left(\int_\Omega w^2(x)|\nabla v(x)|^2\,dx\right)^{1/2}.$$

It is easy to verify that $v \in H_w^1(\Omega, \mathbb{C})$ if and only if $wv \in H^1(\Omega, \mathbb{C})$. For $N \geq 3$ and any $q \geq 1$, we also denote as $L^q(w^{2^*}, \Omega, \mathbb{C})$ the weighted L^q-space endowed with the norm

$$\|v\|_{L^q(w^{2^*},\Omega,\mathbb{C})} := \left(\int_\Omega w^{2^*}(x)|v(x)|^q\,dx\right)^{1/q},$$

where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. We say that a function $V \in L^1_{\mathrm{loc}}(\Omega \setminus \{0\}, \mathbb{C})$ is *form-bounded with respect to the weight w* if

$$\sup_{u \in H_w^1(\Omega,\mathbb{C})\setminus\{0\}} \frac{\int_\Omega w^{2^*}(x)|V(x)|\,|u(x)|^2\,dx}{\|u\|^2_{H_w^1(\Omega,\mathbb{C})}} < +\infty.$$

Lemma 9.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set containing* 0*, let* (**A.2**)*,* (**A.3**)*, and* (**A.5**) *hold, and let $v \in H^1_w(\Omega, \mathbb{C}) \cap L^q(w^{2^*}, \Omega, \mathbb{C})$, $q > 2$, be a weak solution to*

$$-\operatorname{div}(w^2(x)\nabla v(x)) - \frac{2i\frac{\mathbf{A}(x/|x|)}{\phi(x/|x|)}\nabla_{\mathbb{S}^{N-1}}\phi\big(\frac{x}{|x|}\big) - |\mathbf{A}\big(\frac{x}{|x|}\big)|^2 + i\operatorname{div}_{\mathbb{S}^{N-1}}\mathbf{A}\big(\frac{x}{|x|}\big)}{|x|^2}w^2(x)v(x)$$
$$- 2iw^2(x)\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla v(x) = w^{2^*}(x)V(x)v(x) \quad \text{in } \Omega, \tag{110}$$

where V is form-bounded with respect to the weight w and $(\Re(V))_+ \in L^s(w^{2^*}, \Omega, \mathbb{C})$ for some $s > N/2$. Then, for any $\Omega' \Subset \Omega$ such that $0 \in \Omega'$, we have $v \in L^{2^*q/2}(w^{2^*}, \Omega', \mathbb{C})$ and

$$\|v\|_{L^{2^*q/2}(w^{2^*},\Omega',\mathbb{C})}$$
$$\leq S(\mathbf{A},a)^{-1/q}\|v\|_{L^q(w^{2^*},\Omega,\mathbb{C})}\left(\frac{32}{C(q)}\frac{M^{2-2^*}(\widetilde{C}(\Omega,\Omega'))^{\sigma(2-2^*)}}{(\operatorname{dist}(\Omega',\partial\Omega))^2} + \frac{2\ell_q}{C(q)}\right)^{1/q}, \tag{111}$$

where $C(q) := \min\left\{\frac{1}{4}, \frac{4}{q+4}\right\}$, $M = \min_{\mathbb{S}^{N-1}}\phi > 0$,

$$\widetilde{C}(\Omega,\Omega') = \begin{cases} \operatorname{diam}\Omega & \text{if } \mu_1(0,a) \leq 0, \\ \operatorname{dist}(0, \mathbb{R}^N \setminus \Omega') & \text{if } \mu_1(0,a) > 0, \end{cases}$$

$$\ell_q = \left[\max\left\{\frac{8}{S(\mathbf{A},a)}\|(\Re(V))_+\|^{2s/N}_{L^s(w^{2^*},\Omega,\mathbb{C})}, \frac{q+4}{2S(\mathbf{A},a)}\|(\Re(V))_+\|^{2s/N}_{L^s(w^{2^*},\Omega,\mathbb{C})}\right\}\right]^{\frac{N}{2s-N}}.$$

Proof. Hölder's inequality and (104) yield, for any $u \in \mathcal{D}^{1,2}_w(\Omega, \mathbb{C})$,

$$\int_\Omega w^{2^*}(x)(\Re(V(x)))_+|u(x)|^2\,dx$$
$$\leq \ell_q\int_{(\Re(V(x)))_+\leq\ell_q} w^{2^*}(x)|u(x)|^2\,dx + \int_{(\Re(V(x)))_+\geq\ell_q} w^{2^*-2}(x)(\Re(V(x)))_+w^2(x)|u(x)|^2\,dx$$
$$\leq \ell_q\int_\Omega w^{2^*}(x)|u(x)|^2\,dx + \left(\int_\Omega w^{2^*}(x)|u(x)|^{2^*}\,dx\right)^{2/2^*} \times\left(\int_{(\Re(V(x)))_+\geq\ell_q} w^{2^*}(x)(\Re(V(x)))_+^{N/2}\,dx\right)^{2/N}$$
$$\leq \frac{1}{S(\mathbf{A},a)}\left(\int_\Omega w^2(x)\left|\nabla u(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right|^2dx\right) \times\left(\int_{(\Re(V(x)))_+\geq\ell_q} w^{2^*}(x)(\Re(V(x)))_+^{N/2}\,dx\right)^{2/N} + \ell_q\int_\Omega w^{2^*}(x)|u(x)|^2\,dx. \tag{112}$$

By Hölder's inequality and by the choice of ℓ_q it follows that

$$\begin{aligned}
&\int_{(\Re(V(x)))_+\geq \ell_q} w^{2^*}(x)(\Re(V(x)))_+^{N/2}\,dx\\
&\quad\leq \left(\int_\Omega w^{2^*}(x)(\Re(V(x)))_+^s\,dx\right)^{\frac{N}{2s}}\left(\int_{(\Re(V(x)))_+\geq \ell_q} w^{2^*}(x)\,dx\right)^{\frac{2s-N}{2s}}\\
&\quad\leq \left(\int_\Omega w^{2^*}(x)(\Re(V(x)))_+^s\,dx\right)^{\frac{N}{2s}}\left(\int_{(\Re(V(x)))_+\geq \ell_q}\left(\frac{(\Re(V(x)))_+}{\ell_q}\right)^s w^{2^*}(x)\,dx\right)^{\frac{2s-N}{2s}}\\
&\quad\leq \|(\Re(V))_+\|^s_{L^s(w^{2^*},\Omega,\mathbb{C})}\ell_q^{-s+N/2}\leq \min\left\{\frac{S(\mathbf{A},a)}{8},\frac{2S(\mathbf{A},a)}{q+4}\right\}^{N/2},
\end{aligned}\tag{113}$$

and hence from (112) we find that for any $u\in\mathcal{D}^{1,2}_w(\Omega,\mathbb{C})$,

$$\begin{aligned}
\int_\Omega w^{2^*}(x)(\Re(V(x)))_+|u(x)|^2\,dx &\leq \ell_q\int_\Omega w^{2^*}(x)|u(x)|^2\,dx\\
&\quad+\min\left\{\frac{1}{8},\frac{2}{q+4}\right\}\left(\int_\Omega w^2(x)\left|\nabla u(x)+i\,\frac{\mathbf{A}(x/|x|)}{|x|}u(x)\right|^2dx\right).
\end{aligned}\tag{114}$$

Let $\eta\in C^\infty_{\mathrm{c}}(\Omega,\mathbb{R})$ be a nonnegative cut-off function such that

$$\operatorname{supp}(\eta)\Subset\Omega,\qquad \eta\equiv 1\quad\text{on }\Omega',\qquad |\nabla\eta(x)|\leq\frac{2}{\operatorname{dist}(\Omega',\partial\Omega)}.$$

Set $v^n:=\min(n,|v|)\in H^1_w(\Omega,\mathbb{C})$. Let us test (110) with $\eta^2(v^n)^{q-2}\bar v\in\mathcal{D}^{1,2}_w(\Omega,\mathbb{C})$ and take the real part. Observing that $\Re(\bar v\nabla v)=|v|\nabla|v|$ and using the elementary inequality $2ab\leq\frac{1}{2}a^2+2b^2$ and the diamagnetic inequality (see Lemma A.1), we thus obtain

$$\begin{aligned}
(q-2)\int_\Omega & w^2(x)\eta^2(x)(v^n(x))^{q-2}\chi_{\{y\in\Omega\,:\,|v(y)|<n\}}(x)|\nabla|v|(x)|^2\,dx\\
&+\int_\Omega w^2(x)\eta^2(x)(v^n(x))^{q-2}|\nabla v(x)|^2\,dx\\
&+\int_\Omega\frac{|\mathbf{A}(x/|x|)|^2}{|x|^2}w^2(x)\eta^2(x)(v^n(x))^{q-2}|v(x)|^2\,dx\\
&+2\int_\Omega w^2(x)\eta^2(x)(v^n(x))^{q-2}\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\Im(\bar v(x)\nabla v(x))\,dx\\
=&\int_\Omega w^{2^*}(x)\Re(V(x))\eta^2(x)|v(x)|^2(v^n(x))^{q-2}\,dx\\
&-2\int_\Omega w^2(x)\eta(x)(v^n(x))^{q-2}|v(x)|\nabla|v|(x)\cdot\nabla\eta(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_\Omega w^{2^*}(x)\Re(V(x))\eta^2(x)|v(x)|^2(v^n(x))^{q-2}\,dx \\
&\quad + 2\int_\Omega w^2(x)|\nabla\eta(x)|^2(v^n(x))^{q-2}|v(x)|^2\,dx \\
&\quad + \frac{1}{2}\int_\Omega w^2(x)\eta^2(x)(v^n(x))^{q-2}\left|\nabla v(x) + i\,\frac{\mathbf{A}(x/|x|)}{|x|}v(x)\right|^2 dx
\end{aligned}
$$

and hence

$$
\begin{aligned}
(q-2)\int_\Omega & w^2(x)\eta^2(x)(v^n(x))^{q-2}\chi_{\{y\in\Omega\,:\,|v(y)|<n\}}(x)|\nabla|v|(x)|^2\,dx \\
&+ \frac{1}{2}\int_\Omega w^2(x)\eta^2(x)(v^n(x))^{q-2}\left|\nabla v(x) + i\,\frac{\mathbf{A}(x/|x|)}{|x|}v(x)\right|^2 dx \\
&\qquad \le \int_\Omega w^{2^*}(x)\Re(V(x))\eta^2(x)|v(x)|^2(v^n(x))^{q-2}\,dx \\
&\qquad\quad + 2\int_\Omega w^2(x)|\nabla\eta(x)|^2(v^n(x))^{q-2}|v(x)|^2\,dx. \qquad (115)
\end{aligned}
$$

Furthermore, by the diamagnetic inequality (see Lemma A.1) we have

$$
\begin{aligned}
&\left|\nabla\big((v^n)^{q/2-1}v\eta\big) + i\,\frac{\mathbf{A}(x/|x|)}{|x|}(v^n)^{q/2-1}v\eta\right|^2 \\
&\quad = |\nabla((v^n)^{q/2-1}v\eta)|^2 + 2\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\eta^2(v^n)^{q-2}\Im(\bar v\nabla v) + \frac{|\mathbf{A}(x/|x|)|^2}{|x|^2}(v^n)^{q-2}\eta^2|v|^2 \\
&\quad \le \frac{(q+4)(q-2)}{4}(v^n)^{q-2}\eta^2|\nabla v^n|^2 + 2(v^n)^{q-2}\eta^2\left|\nabla v + i\,\frac{\mathbf{A}(x/|x|)}{|x|}v\right|^2 \\
&\qquad + \frac{q+2}{2}(v^n)^{q-2}|v|^2|\nabla\eta|^2. \qquad (116)
\end{aligned}
$$

Letting $C(q) := \min\left\{\frac{1}{4}, \frac{4}{q+4}\right\}$, from (115) and (116) we obtain

$$
\begin{aligned}
C(q)\int_\Omega & w^2(x)\left|\nabla((v^n)^{q/2-1}v\eta)(x) + i\,\frac{\mathbf{A}(x/|x|)}{|x|}(v^n(x))^{q/2-1}v(x)\eta(x)\right|^2 dx \\
&\le \int_\Omega w^{2^*}(x)\Re(V(x))\eta^2(x)|v(x)|^2(v^n(x))^{q-2}\,dx \\
&\quad + 2\int_\Omega w^2(x)(v^n(x))^{q-2}|v(x)|^2|\nabla\eta(x)|^2\,dx \\
&\quad + C(q)\frac{q+2}{2}\int_\Omega w^2(x)(v^n(x))^{q-2}|v(x)|^2|\nabla\eta(x)|^2\,dx. \qquad (117)
\end{aligned}
$$

Estimate (114) applied to $\eta(v^n)^{q/2-1}v$ gives

$$\int_\Omega w^{2^*}(x)(\Re(V(x)))_+|\eta(x)(v^n(x))^{q/2-1}v(x)|^2\,dx$$
$$\le \ell_q\int_\Omega w^{2^*}(x)|\eta(x)(v^n(x))^{q/2-1}v(x)|^2\,dx+\min\left\{\frac{1}{8},\frac{2}{q+4}\right\}$$
$$\times\left(\int_\Omega w^2(x)\left|\nabla(\eta(v^n)^{q/2-1}v)(x)+i\frac{\mathbf{A}(x/|x|)}{|x|}\eta(x)(v^n(x))^{q/2-1}v(x)\right|^2dx\right). \tag{118}$$

Using (118) to estimate the term with V in (117), (104) yields

$$\left(\int_\Omega w^{2^*}(x)|v^n(x)|^{(q/2-1)2^*}|v(x)|^{2^*}\eta^{2^*}(x)\,dx\right)^{2/2^*}$$
$$\le\frac{2\ell_q}{C(q)S(\mathbf{A},a)}\int_\Omega w^{2^*}(x)\eta^2(x)|v^n(x)|^{q-2}|v(x)|^2\,dx$$
$$+\frac{4+C(q)(q+2)}{C(q)S(\mathbf{A},a)}\int_\Omega w^2(x)|v^n(x)|^{q-2}|v(x)|^2|\nabla\eta(x)|^2\,dx$$
$$\le\frac{2\ell_q}{C(q)S(\mathbf{A},a)}\int_\Omega w^{2^*}(x)\eta^2(x)|v^n(x)|^{q-2}|v(x)|^2\,dx$$
$$+\frac{8}{C(q)S(\mathbf{A},a)}\int_\Omega w^2(x)|v^n(x)|^{q-2}|v(x)|^2|\nabla\eta(x)|^2\,dx.$$

Letting $n\to\infty$ in the above inequality, (111) follows. □

Remark 9.2. It is possible to extend the result of Lemma 9.1 also to the case

$$(\Re(V))_+\in L^{N/2}(w^{2^*},\Omega,\mathbb{C})$$

and obtain estimate (111). Indeed, by the previous summability assumption on $(\Re(V))_+$, it is possible to find ℓ_q such that

$$\int_{(\Re(V(x)))_+\ge\ell_q} w^{2^*}(x)(\Re(V(x)))_+^{N/2}\,dx\le\min\left\{\frac{S(\mathbf{A},a)}{8},\frac{2S(\mathbf{A},a)}{q+4}\right\}^{N/2}.$$

But we have no control on the constant ℓ_q in terms of q as in Lemma 9.1 since it is not possible to apply Hölder's inequality in (113) when $s=N/2$. The rest of the proof in the case $s=N/2$ coincides with the proof of Lemma 9.1.

The previous lemma allows starting a Brezis–Kato type iteration.

Theorem 9.3. *Let $\Omega\subset\mathbb{R}^N$, $N\ge3$, be a bounded open set containing* 0*, and let* (**A.2**)*,* (**A.3**)*, and* (**A.5**) *hold.*

(i) *If V is form-bounded with respect to the weight w and $(\Re(V))_+ \in L^s(w^{2^*}, \Omega, \mathbb{C})$ for some $s > N/2$, then, for any $\Omega' \Subset \Omega$, there exists a positive constant*

$$C_\infty = C_\infty\big(N, \mathbf{A}, a, \|(\Re(V))_+\|_{L^s(w^{2^*},\Omega,\mathbb{C})}, \mathrm{dist}(\Omega', \partial\Omega), \widetilde{C}(\Omega, \Omega')\big),$$

depending only on the indicated quantities, such that for any $H^1(\Omega, \mathbb{C})$-weak solution u to

$$\mathcal{L}_{\mathbf{A},a} u(x) = w^{2^*-2}(x) V(x) u(x) \quad \text{in } \Omega, \tag{119}$$

we have $|x|^{-\sigma} u \in L^\infty(\Omega', \mathbb{C})$ and

$$\big\| |x|^{-\sigma} u \big\|_{L^\infty(\Omega',\mathbb{C})} \le C_\infty \|u\|_{L^{2^*}(\Omega,\mathbb{C})}.$$

(ii) *If V is form-bounded with respect to the weight w and $(\Re(V))_+ \in L^{N/2}(w^{2^*}, \Omega, \mathbb{C})$, then, for any $\Omega' \Subset \Omega$ and for any $s \ge 1$, there exists a positive constant*

$$C_s = C_s\big(N, \mathbf{A}, a, \|(\Re(V))_+\|_{L^{N/2}(w^{2^*},\Omega,\mathbb{C})}, s, \mathrm{dist}(\Omega', \partial\Omega), \widetilde{C}(\Omega, \Omega')\big),$$

depending only on the indicated quantities, such that for any $H^1(\Omega, \mathbb{C})$-weak solution u to (119) *in Ω we have $|x|^{-\sigma} u \in L^s(w^{2^*}, \Omega', \mathbb{C})$ and*

$$\big\| |x|^{-\sigma} u \big\|_{L^s(w^{2^*},\Omega',\mathbb{C})} \le C_s \|u\|_{L^{2^*}(\Omega,\mathbb{C})}.$$

Proof. (i) Let u be an $H^1(\Omega, \mathbb{C})$-weak solution to (119). It is easy to verify that the function $v := w^{-1} u$ belongs to $H^1_w(\Omega, \mathbb{C})$ and is a weak solution to (110). Let $R > 0$ be such that

$$\Omega' \Subset \Omega' + B(0, 2R) \Subset \Omega.$$

Using Lemma 9.1 in $\Omega_1 := \Omega' + B(0, R(2 - r_1)) \Subset \Omega' + B(0, 2R)$, $r_1 = 1$, with $q = q_1 = 2^*$, we infer that $v \in L^{(2^*)^2/2}(w^{2^*}, \Omega_1, \mathbb{C})$ and

$$\begin{aligned}
&\|v\|_{L^{(2^*)^2/2}(w^{2^*},\Omega_1,\mathbb{C})} \\
&\quad \le S(\mathbf{A}, a)^{-1/q_1} \|v\|_{L^{2^*}(w^{2^*},\Omega,\mathbb{C})} \left(\frac{32}{C(q_1)} \frac{M^{2-2^*} \big(\widetilde{C}(\Omega, \Omega')\big)^{\sigma(2-2^*)}}{(Rr_1)^2} + \frac{2\ell_{q_1}}{C(q_1)} \right)^{1/q_1}.
\end{aligned}$$

Using again Lemma 9.1 in $\Omega_2 := \Omega' + B(0, R(2 - r_1 - r_2)) \Subset \Omega_1$, $r_2 = 1/4$, with $q = q_2 = (2^*)^2/2$, we infer that $v \in L^{(2^*)^3/4}(w^{2^*}, \Omega_2, \mathbb{C})$ and

$$\begin{aligned}
&\|v\|_{L^{(2^*)^3/4}(w^{2^*},\Omega_2,\mathbb{C})} \\
&\quad \le S(\mathbf{A}, a)^{-1/q_2} \left(\frac{32}{C(q_2)} \frac{M^{2-2^*} (\widetilde{C}(\Omega, \Omega'))^{\sigma(2-2^*)}}{(Rr_2)^2} + \frac{2\ell_{q_2}}{C(q_2)} \right)^{1/q_2} \|v\|_{L^{q_2}(w^{2^*},\Omega_1,\mathbb{C})} \\
&\quad \le S(\mathbf{A}, a)^{-(1/q_1+1/q_2)} \left(\frac{32}{C(q_1)} \frac{M^{2-2^*} (\widetilde{C}(\Omega, \Omega'))^{\sigma(2-2^*)}}{(Rr_1)^2} + \frac{2\ell_{q_1}}{C(q_1)} \right)^{1/q_1} \\
&\qquad \times \left(\frac{32}{C(q_2)} \frac{M^{2-2^*} (\widetilde{C}(\Omega, \Omega'))^{\sigma(2-2^*)}}{(Rr_2)^2} + \frac{2\ell_{q_2}}{C(q_2)} \right)^{1/q_2} \|v\|_{L^{2^*}(w^{2^*},\Omega,\mathbb{C})}.
\end{aligned}$$

Setting, for any $n \in \mathbb{N}$, $n \geq 1$,

$$q_n = 2\Big(\frac{2^*}{2}\Big)^n, \quad \Omega_n := \Omega' + B\Big(0, R\Big(2 - \sum_{k=1}^{n} r_k\Big)\Big), \quad \text{and} \quad r_n = \frac{1}{n^2},$$

and using iteratively Lemma 9.1, we deduce that, for any $n \geq 1$,

$$\|v\|_{L^{q_{n+1}}(w^{2^*},\Omega',\mathbb{C})} \leq \|v\|_{L^{q_{n+1}}(w^{2^*},\Omega_n,\mathbb{C})} \leq \|v\|_{L^{2^*}(w^{2^*},\Omega,\mathbb{C})}(S(\mathbf{A},a))^{-\sum_{k=1}^{n} 1/q_k} \times \prod_{k=1}^{n}\Big(\frac{32}{C(q_k)}\frac{M^{2-2^*}(\widetilde{C}(\Omega,\Omega'))^{\sigma(2-2^*)}}{(Rr_k)^2} + \frac{2\ell_{q_k}}{C(q_k)}\Big)^{1/q_k}. \tag{120}$$

We notice that

$$\prod_{k=1}^{n}\Big(\frac{32}{C(q_k)}\frac{M^{2-2^*}(\widetilde{C}(\Omega,\Omega'))^{\sigma(2-2^*)}}{(Rr_k)^2} + \frac{2\ell_{q_k}}{C(q_k)}\Big)^{1/q_k} = \exp\Big[\sum_{k=1}^{n} b_k\Big]$$

where

$$b_k = \frac{1}{q_k}\log\Big(\frac{32}{C(q_k)}\frac{M^{2-2^*}(\widetilde{C}(\Omega,\Omega'))^{\sigma(2-2^*)}}{(Rr_k)^2} + \frac{2\ell_{q_k}}{C(q_k)}\Big),$$

and, for some constant $C = C(N, \mathbf{A}, a, \|(\Re(V))_+\|_{L^s(w^{2^*},\Omega,\mathbb{C})}, \operatorname{dist}(\Omega',\partial\Omega), \widetilde{C}(\Omega,\Omega'))$,

$$b_k \sim \frac{1}{2}\Big(\frac{2}{2^*}\Big)^k \log\Big[C\Big(2\Big(\frac{2^*}{2}\Big)^k\Big)^{\frac{2s}{2s-N}}\Big] \quad \text{as } k \to +\infty.$$

Hence $\sum_{n=1}^{\infty} b_n$ converges to some positive sum depending only on N, $\mathbf{A}$, a, $\|(\Re(V))_+\|_{L^s(w^{2^*},\Omega,\mathbb{C})}$, $\operatorname{dist}(\Omega',\partial\Omega)$, $\widetilde{C}(\Omega,\Omega')$, hence

$$\lim_{n\to+\infty}(S(\mathbf{A},a))^{-\sum_{k=1}^{n} 1/q_k}\prod_{k=1}^{n}\Big(\frac{32}{C(q_k)}\frac{M^{2-2^*}\big(\widetilde{C}(\Omega,\Omega')\big)^{\sigma(2-2^*)}}{(Rr_k)^2} + \frac{2\ell_{q_k}}{C(q_k)}\Big)^{1/q_k}$$

is finite and depends only on the same quantities. Hence, from (120), we deduce that there exists a positive constant C (depending only on the same quantities) such that

$$\|v\|_{L^{q_{n+1}}(w^{2^*},\Omega',\mathbb{C})} \leq C\|v\|_{L^{2^*}(w^{2^*},\Omega,\mathbb{C})} \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to +\infty$ we deduce that $|v|$ is essentially bounded in Ω' with respect to the measure $w^{2^*}\,dx$ and

$$\|v\|_{L^\infty(w^{2^*},\Omega',\mathbb{C})} \leq C\|v\|_{L^{2^*}(w^{2^*},\Omega,\mathbb{C})} = C\|u\|_{L^{2^*}(\Omega,\mathbb{C})},$$

where $\|v\|_{L^\infty(w^{2^*},\Omega',\mathbb{C})}$ denotes the essential supremum of v with respect to the measure $w^{2^*}\,dx$. Since $w^{2^*}\,dx$ is absolutely continuous with respect to the Lebesgue measure and vice versa, we have $\|v\|_{L^\infty(w^{2^*},\Omega',\mathbb{C})} = \|v\|_{L^\infty(\Omega',\mathbb{C})}$, hence $v \in L^\infty(\Omega',\mathbb{C})$ and

$$\|v\|_{L^\infty(\Omega',\mathbb{C})} \leq C\|u\|_{L^{2^*}(\Omega,\mathbb{C})},$$

thus completing the proof of part (i). We recall that for any $x \in \Omega \setminus \{0\}$ we have

$$|x|^{-\sigma} u(x) = w^{-1}(x)\phi(x/|x|)u(x) = \phi(x/|x|)v(x) \le (\max_{\mathbb{S}^{N-1}} \phi)v(x).$$

(ii) Since $u \in H^1(\Omega, \mathbb{C})$ is a weak solution to (119), the function $v := w^{-1}u$ is an $H^1_w(\Omega, \mathbb{C})$-weak solution of (110). Using Remark 9.2 and the iterative scheme used to prove part (i), for any $1 \le s < \infty$, after a finite number of iterations we arrive at $v \in L^s(w^{2^*}, \Omega', \mathbb{C})$ and

$$\|v\|_{L^s(w^{2^*},\Omega',\mathbb{C})} \le C_s \|v\|_{L^{2^*}(w^{2^*},\Omega,\mathbb{C})}.$$

This completes the proof. □

Applying Theorem 9.3 to the nonlinear equation (3), we can obtain a pointwise estimate for solutions to (3).

Theorem 9.4. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 3$, be a bounded open set containing 0, and let (**A.2**), (**A.3**), and (**A.5**) hold. Let u be an $H^1(\Omega, \mathbb{C})$-weak solution of (3) with $f(x,u)$ satisfying (7). Then for any $\Omega' \Subset \Omega$ there exists a positive constant*

$$\widetilde{C}_\infty = \widetilde{C}_\infty\big(N, \mathbf{A}, a, C_f, \operatorname{dist}(\Omega', \partial\Omega), \widetilde{C}(\Omega, \Omega')\big),$$

depending only on the indicated quantities, such that $|x|^{-\sigma}u \in L^\infty(\Omega', \mathbb{C})$ and

$$\big\||x|^{-\sigma}u\big\|_{L^\infty(\Omega',\mathbb{C})} \le \widetilde{C}_\infty \|u\|_{L^{2^*}(\Omega,\mathbb{C})}. \tag{121}$$

Proof. If we put

$$V(x) := \begin{cases} w^{2-2^*} \dfrac{f(x,u(x))}{u(x)} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0, \end{cases}$$

then, by (7) and the Sobolev imbedding $H^1(\Omega, \mathbb{C}) \subset L^{2^*}(\Omega, \mathbb{C})$, we conclude that $V \in L^{N/2}(w^{2^*}, \Omega, \mathbb{C})$ and u weakly solves

$$\mathcal{L}_{\mathbf{A},a} u(x) = w^{2^*-2} V(x) u(x) \quad \text{in } \Omega.$$

From Theorem 9.3(ii), it follows that $|x|^{-\sigma}u \in L^s(w^{2^*}, \Omega', \mathbb{C})$ for any $\Omega' \Subset \Omega$ and any $s \ge 1$. Fix now $s_0 = N/2 + \varepsilon_0$ with $0 < \varepsilon_0 < N(N-2)/(4|\sigma|)$. By (7) we easily deduce that $V \in L^{s_0}(w^{2^*}, \Omega', \mathbb{C})$. The assertion now follows from Theorem 9.3(i). □

The a priori estimate of solutions to the nonlinear problem obtained above allows deducing Theorem 1.6 from Theorem 1.3.

Proof of Theorem 1.6 for $N \ge 3$. Note that all the assumptions of Theorem 9.4 are satisfied and hence

$$|u(x)| = O(|x|^\sigma) \quad \text{as } |x| \to 0, \tag{122}$$

where $\sigma > -(N-2)/2$ is defined by (102). Therefore, by (7) and (122),

$$\left|\frac{f(x,u)}{u}\right| \le \text{const}\Big(1 + |x|^{-2+\frac{4}{N-2}\sqrt{(\frac{N-2}{2})^2+\mu_1(0,a)}}\Big)$$

for some constant const > 0. Hence, the function

$$h(x) := \begin{cases} f(x, u(x))/u(x) & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0, \end{cases}$$

satisfies $h(x) = O(|x|^{-2+\varepsilon})$ as $|x| \to 0^+$ for some $\varepsilon > 0$. On the other hand, by Remark 1.2 we also have $u \in L^\infty_{\text{loc}}(\Omega \setminus \{0\})$ and in turn by (7), $h \in L^\infty_{\text{loc}}(\Omega \setminus \{0\})$. This shows that all the assumptions of Theorem 1.3 are satisfied and the assertion of Theorem 1.6 follows in the case $N \geq 3$. The proof for $N = 2$ is postponed to Section 10. □

Proof of Theorem 1.7 for $N \geq 3$. This follows from Theorems 1.5 and 1.6 by the use of the Kelvin transform. □

Since the proof of the pointwise a priori estimate (121) (and then of Theorems 1.6 and 1.7) in dimension $N = 2$ starts from a different inequality than (104) and requires a somewhat different notation, we devote the next section to a sketched description of the modifications to be made in the above argument to treat the case $N = 2$.

10. A Brezis–Kato type lemma in dimension $N = 2$

Similarly to Section 9, for $N = 2$ we define the spaces $\mathcal{D}^{1,2}_*(\Omega, \mathbb{C})$ and $\mathcal{D}^{1,2}_{*,w}(\Omega, \mathbb{C})$ as the completion of $C^\infty_c(\Omega \setminus \{0\}, \mathbb{C})$ respectively with the norms

$$\|u\|_{\mathcal{D}^{1,2}_*(\Omega,\mathbb{C})} := \left(\int_\Omega \left(|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\right) dx\right)^{1/2}$$

and

$$\|v\|_{\mathcal{D}^{1,2}_{*,w}(\Omega,\mathbb{C})} := \left(\int_\Omega w^2 \left(|\nabla u(x)|^2 + \frac{|u(x)|^2}{|x|^2}\right) dx\right)^{1/2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain containing the origin and w is defined by (103). We observe that the space $\mathcal{D}^{1,2}_*(\Omega, \mathbb{C})$ is smaller than $H^1_0(\Omega, \mathbb{C})$. Moreover, it is easy to verify that $v \in \mathcal{D}^{1,2}_{*,w}(\Omega, \mathbb{C})$ if and only if $wv \in \mathcal{D}^{1,2}_*(\Omega, \mathbb{C})$. Similarly, we define the space $H^1_{*,w}(\Omega, \mathbb{C})$ as the completion of

$$\{v \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : v \text{ vanishes in a neighborhood of } 0\}$$

with respect to the norm

$$\|v\|_{H^1_{*,w}(\Omega,\mathbb{C})} := \left(\int_\Omega w^2 \left[|\nabla v(x)|^2 + \frac{|v(x)|^2}{|x|^2} + |v(x)|^2\right] dx\right)^{1/2}.$$

The following weighted Poincaré–Sobolev inequality holds.

Proposition 10.1. *Let $N = 2$ and let a, $\mathbf{A}$ satisfy* (**A.2**)*,* (**A.3**)*, and* (**A.5**)*. Then, for any $1 \le p < \infty$,*

$$S(\mathbf{A}, a, p, \Omega) = \inf_{u \in \mathcal{D}^{1,2}_*(\Omega,\mathbb{C})\setminus\{0\}} \frac{\int_\Omega \big[\big|\big(\nabla + i\frac{\mathbf{A}(x/|x|)}{|x|}\big)u(x)\big|^2 - \frac{a(x/|x|)}{|x|^2}|u(x)|^2\big]\,dx}{\big(\int_\Omega |u(x)|^p\,dx\big)^{2/p}} > 0. \tag{123}$$

Moreover

$$\int_\Omega w^2 \left|\nabla v(x) + i\frac{\mathbf{A}(x/|x|)}{|x|}v(x)\right|^2 dx \ge S(\mathbf{A}, a, p, \Omega)\left(\int_\Omega w^p |v(x)|^p\,dx\right)^{2/p} \tag{124}$$

for all $v \in \mathcal{D}^{1,2}_{*,\sigma}(\Omega, \mathbb{C})$.

Proof. Inequality (123) follows from Lemma 2.2 and the classical Poincaré–Sobolev inequality. To obtain the second part of the statement, by density it is sufficient to prove inequality (124) for functions $v \in C^\infty_c(\Omega \setminus \{0\}, \mathbb{C})$, which one can easily do by following the same procedure developed in the proof of Proposition 8.2. □

Remark 10.2. We notice that the constant in (124) depends on the domain Ω, unlike the constant appearing in (104) in the case $N = 3$ and $p = 2^*$. Moreover $S(\mathbf{A}, a, p, \Omega)$ is decreasing with respect to Ω, i.e. if $\Omega_1 \subset \Omega_2$ then $S(\mathbf{A}, a, p, \Omega_1) \ge S(\mathbf{A}, a, p, \Omega_2)$.

We are now ready to prove the following 2-dimensional version of Lemma 9.1.

Lemma 10.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set containing* 0*, let* (**A.2**)*,* (**A.3**)*, and* (**A.5**) *hold, and, for some $p, q > 2$, let $v \in H^1_{*,w}(\Omega, \mathbb{C}) \cap L^q(w^p, \Omega, \mathbb{C})$ be a weak solution to*

$$\begin{aligned} &-\operatorname{div}(w^2(x)\nabla v(x)) \\ &\quad - \frac{2i\frac{\mathbf{A}(x/|x|)}{\phi(x/|x|)}\nabla_{\mathbb{S}^{N-1}}\phi(x/|x|) - |\mathbf{A}(x/|x|)|^2 + i\operatorname{div}_{\mathbb{S}^{N-1}}\mathbf{A}(x/|x|)}{|x|^2} w^2(x)v(x) \\ &\quad - 2i w^2(x)\frac{\mathbf{A}(x/|x|)}{|x|}\cdot\nabla v(x) = w^p(x)V(x)v(x) \quad \text{in } \Omega, \end{aligned}$$

where V is form-bounded with respect to the weight w and $(\Re(V))_+ \in L^s(w^p, \Omega, \mathbb{C})$ for some $s > p/(p-2)$. Then, for any $\Omega' \Subset \Omega$ such that $0 \in \Omega'$, it follows that $v \in L^{pq/2}(w^p, \Omega', \mathbb{C})$ and

$$\begin{aligned} &\|v\|_{L^{pq/2}(w^p,\Omega',\mathbb{C})} \\ &\quad \le S(\mathbf{A}, a, p, \Omega)^{-1/q}\|v\|_{L^q(w^p,\Omega,\mathbb{C})}\left(\frac{32}{C(q)}\frac{M^{2-p}(\widetilde{C}(\Omega,\Omega'))^{\sigma(2-p)}}{(\operatorname{dist}(\Omega',\partial\Omega))^2} + \frac{2\ell_q}{C(q)}\right)^{1/q}, \end{aligned}$$

where $C(q) := \min\{\frac{1}{4}, \frac{4}{q+4}\}$, $\widetilde{C}(\Omega, \Omega') = \operatorname{dist}(0, \mathbb{R}^N \setminus \Omega')$, $M = \min_{\mathbb{S}^{N-1}} \phi > 0$, and

$$\ell_q = \left[\max\left\{\frac{8\|(\Re(V))_+\|^{s(p-2)/p}_{L^s(w^p,\Omega,\mathbb{C})}}{S(\mathbf{A}, a, p, \Omega)}, \frac{q+4}{2S(\mathbf{A}, a, p, \Omega)}\|(\Re(V))_+\|^{s(p-2)/p}_{L^s(w^p,\Omega,\mathbb{C})}\right\}\right]^{\frac{p}{s(p-2)-p}}.$$

Proof. Proceed as in the proof of Lemma 9.1, using (124) in place of (104). □

The counterpart in dimension $N = 2$ of Theorem 9.3 is the following Brezis–Kato type result.

Theorem 10.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set containing 0, let* (**A.2**)*,* (**A.3**)*, and* (**A.5**) *hold, and let $p > 2$.*

(i) *If V is form-bounded with respect to the weight w and $(\Re(V))_+ \in L^s(w^p, \Omega, \mathbb{C})$ for some $s > p/(p-2)$, then for any $\Omega' \Subset \Omega$ there exists a positive constant*

$$C_{\infty,2} = C_{\infty,2}\big(\Omega, p, \mathbf{A}, a, \|(\Re(V))_+\|_{L^s(w^p,\Omega,\mathbb{C})}, \operatorname{dist}(\Omega', \partial\Omega), \widetilde{C}(\Omega, \Omega')\big),$$

depending only on the indicated quantities, such that for any $H^1_*(\Omega, \mathbb{C})$-weak solution u to

$$\mathcal{L}_{\mathbf{A},a} u(x) = w^{p-2} V(x) u(x) \quad \text{in } \Omega, \tag{125}$$

we have $|x|^{-\sigma} u \in L^\infty(\Omega', \mathbb{C})$ and

$$\big\| |x|^{-\sigma} u \big\|_{L^\infty(\Omega',\mathbb{C})} \leq C_{\infty,2} \|u\|_{L^p(\Omega,\mathbb{C})}.$$

(ii) *If V is form-bounded with respect to the weight w and $(\Re(V))_+ \in L^{p/(p-2)}(w^p, \Omega, \mathbb{C})$, then for any $\Omega' \Subset \Omega$ and for any $1 \leq s < \infty$ there exists a positive constant*

$$C_{s,2} = C_{s,2}(\Omega, p, \mathbf{A}, a, \|(\Re(V))_+\|_{L^s(w^p,\Omega,\mathbb{C})}, \operatorname{dist}(\Omega', \partial\Omega), \widetilde{C}(\Omega, \Omega')),$$

depending only on the indicated quantities, such that for any $H^1_*(\Omega, \mathbb{C})$-weak solution u to (125) *in Ω we have $|x|^{-\sigma} u \in L^s(w^p, \Omega', \mathbb{C})$ and*

$$\big\| |x|^{-\sigma} u \big\|_{L^s(w^p,\Omega',\mathbb{C})} \leq C_{s,2} \|u\|_{L^p(\Omega,\mathbb{C})}.$$

Proof. This theorem can be proved by iterating the estimate proved in Lemma 10.3 and following the same scheme as in the proof of Theorem 9.3. We notice that the constants $S(\mathbf{A}, a, p, \Omega_i)$ appearing at each step (at a negative power) can be uniformly controlled with $S(\mathbf{A}, a, p, \Omega)$ in view of Remark 10.2. □

From the above analysis, Theorems 1.6 and 1.7 in dimension $N = 2$ follow.

Proof of Theorem 1.6 for $N = 2$. Arguing as in the proof of Theorem 9.4, from Theorem 10.4 we deduce that $|u(x)| = O(|x|^\sigma)$ as $|x| \to 0$. In particular, from (7), the function $\frac{f(x,u(x))}{u(x)} \chi_{\{x\,:\,u(x) \neq 0\}}$ is bounded. The conclusion then follows from Theorem 1.3. □

Proof of Theorem 1.7 for $N = 2$. As in dimension $N \geq 3$, the conclusion follows from Theorems 1.5 and 1.6 by the use of the Kelvin transform. □

Appendix

We recall the following well known result proved in [22].

Lemma A.1 (Diamagnetic inequality). *Let* $N \geq 2$. *If* $u \in \mathcal{D}^{1,2}_*(\mathbb{R}^N, \mathbb{C})$ *then*

$$|\nabla |u|(x)| \leq \left| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right| \quad \textit{for a.e. } x \in \mathbb{R}^N.$$

Proof. We only give the idea of the proof. We have

$$\begin{aligned}|\nabla |u|(x)| &= \left| \Re\left(\frac{\overline{u}(x)}{|u(x)|} \nabla u(x) \right) \right| \\ &\leq \left| \Re\left(\left(\nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right) \frac{\overline{u}(x)}{|u(x)|} \right) \right| \leq \left| \nabla u(x) + i \frac{\mathbf{A}(x/|x|)}{|x|} u(x) \right| \quad (126)\end{aligned}$$

for a.e. $x \in \mathbb{R}^N$. □

An analogous result can be easily shown also for $H^1_*(\Omega, \mathbb{C})$-functions. The following lemma allows comparing assumptions (**A.4**) and (**A.5**).

Lemma A.2. *Let* $N \geq 2$ *and assume* (**A.2**) *and* (**A.3**) *hold. Then* $\mu_1(\mathbf{A}, a) \geq \mu_1(0, a)$ *with equality holding if and only if* $\operatorname{curl}(\mathbf{A}/|x|) = 0$ *in the distributional sense.*

Proof. The fact that $\mu_1(\mathbf{A}, a) \geq \mu_1(0, a)$ follows by (21) and the diamagnetic inequality on the sphere,

$$\left| \nabla_{\mathbb{S}^{N-1}} |\psi|(\theta) \right| \leq |\nabla_{\mathbb{S}^{N-1}} \psi(\theta) + i A(\theta) \psi(\theta)| \quad \text{for a.e. } \theta \in \mathbb{S}^{N-1}, \quad (127)$$

which holds for any function $\psi \in H^1(\mathbb{S}^{N-1})$. Indeed, if $\psi_1 \in H^1(\mathbb{S}^{N-1})$ is a nontrivial eigenfunction of $\mu_1(\mathbf{A}, a)$ then

$$\begin{aligned}\mu_1(\mathbf{A}, a) &= \frac{\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} \psi_1(\theta) + i \mathbf{A}(\theta) \psi_1(\theta)|^2 \, dS - \int_{\mathbb{S}^{N-1}} a(\theta) |\psi_1(\theta)|^2 \, dS}{\int_{\mathbb{S}^{N-1}} |\psi_1(\theta)|^2 \, dS} \\ &\geq \frac{\int_{\mathbb{S}^{N-1}} \left| \nabla_{\mathbb{S}^{N-1}} |\psi_1|(\theta) \right|^2 dS - \int_{\mathbb{S}^{N-1}} a(\theta) |\psi_1(\theta)|^2 \, dS}{\int_{\mathbb{S}^{N-1}} |\psi_1(\theta)|^2 \, dS} \geq \mu_1(0, a). \quad (128)\end{aligned}$$

We start by assuming that $\mu_1(\mathbf{A}, a) = \mu_1(0, a)$. Let ψ_1 be as in (128) so that by (127) we infer

$$|\nabla_{\mathbb{S}^{N-1}} \psi_1(\theta) + i \mathbf{A}(\theta) \psi_1(\theta)| = \left| \nabla_{\mathbb{S}^{N-1}} |\psi_1|(\theta) \right| \quad \text{for a.e. } \theta \in \mathbb{S}^{N-1}. \quad (129)$$

Similarly to (126) we have

$$\begin{aligned}\left| \nabla_{\mathbb{S}^{N-1}} |\psi_1|(\theta) \right| &\leq \left| \Re\left(\frac{\overline{\psi}_1(\theta)}{|\psi_1(\theta)|} (\nabla_{\mathbb{S}^{N-1}} \psi_1(\theta) + i \mathbf{A}(\theta) \psi_1(\theta)) \right) \right| \\ &\leq |\nabla_{\mathbb{S}^{N-1}} \psi_1(\theta) + i \mathbf{A}(\theta) \psi_1(\theta)|, \quad (130)\end{aligned}$$

which together with (129) gives

$$\Im\big(\overline{\psi}_1(\theta)(\nabla_{\mathbb{S}^{N-1}}\psi_1(\theta)+i\mathbf{A}(\theta)\psi_1(\theta))\big)=0 \quad \text{for a.e. } \theta\in\mathbb{S}^{N-1}$$

and in turn

$$\mathbf{A}(\theta)=-\Im\bigg(\frac{\nabla_{\mathbb{S}^{N-1}}\psi_1(\theta)}{\psi_1(\theta)}\bigg) \quad \text{for a.e. } \theta\in\mathbb{S}^{N-1}.$$

This implies

$$\frac{\mathbf{A}(x/|x|)}{|x|}=-\Im\bigg(\frac{\nabla(\psi_1(x/|x|))}{\psi_1(x/|x|)}\bigg) \quad \text{for a.e. } x\in\mathbb{R}^N.$$

By direct computation this gives $\operatorname{curl}(\mathbf{A}/|x|)=0$ in the distributional sense.

Suppose now that $\operatorname{curl}(\mathbf{A}/|x|)=0$ in the distributional sense and let us prove that $\mu_1(\mathbf{A},a)=\mu_1(0,a)$. By [20] there exists $\phi\in L^1_{\rm loc}(\mathbb{R}^N)$ such that $\nabla\phi=\mathbf{A}/|x|$ in the distributional sense. From (**A.3**) it follows that $\phi(x)=\phi(x/|x|)$ and $\nabla_{\mathbb{S}^{N-1}}\phi=\mathbf{A}$. Let Ψ be a nontrivial eigenfunction of $\mu_1(0,a)$ and define the angular function $\psi(\theta)$ by

$$\psi(\theta)=e^{-i\phi(\theta)}\Psi(\theta).$$

Then

$$\begin{aligned}\mu_1(\mathbf{A},a)&\le\frac{\int_{\mathbb{S}^{N-1}}|\nabla_{\mathbb{S}^{N-1}}\psi(\theta)+i\mathbf{A}(\theta)\psi(\theta)|^2\,dS-\int_{\mathbb{S}^{N-1}}a(\theta)|\psi(\theta)|^2\,dS}{\int_{\mathbb{S}^{N-1}}|\psi(\theta)|^2\,dS}\\&=\frac{\int_{\mathbb{S}^{N-1}}|\nabla_{\mathbb{S}^{N-1}}\Psi(\theta)|^2\,dS-\int_{\mathbb{S}^{N-1}}a(\theta)|\Psi(\theta)|^2\,dS}{\int_{\mathbb{S}^{N-1}}|\Psi(\theta)|^2\,dS}=\mu_1(0,a).\end{aligned}$$

Since the reverse inequality is always satisfied, the proof is complete. □

The following Hardy type inequality with boundary terms is due to Wang and Zhu [27].

Lemma A.3 (Wang and Zhu). *For every $r>0$ and $u\in H^1(B_r,\mathbb{C})$,*

$$\int_{B_r}|\nabla u(x)|^2\,dx+\frac{N-2}{2r}\int_{\partial B_r}|u(x)|^2\,dS\ge\bigg(\frac{N-2}{2}\bigg)^2\int_{B_r}\frac{|u(x)|^2}{|x|^2}\,dx. \tag{131}$$

Proof. See [27, Theorem 1.1]. □

The following lemma establishes the relation between the classical H^1-space on the sphere and its magnetic counterpart.

Lemma A.4. *If $N\ge 2$ and $\mathbf{A}\in L^\infty(\mathbb{S}^{N-1},\mathbb{R}^N)$, then the space $H^1_{\mathbf{A}}(\mathbb{S}^{N-1})$ defined in* (19)–(20) *coincides with the Sobolev space*

$$H^1(\mathbb{S}^{N-1},\mathbb{C}):=\{\psi\in L^2(\mathbb{S}^{N-1},\mathbb{C}):\nabla_{\mathbb{S}^{N-1}}\psi\in L^2(\mathbb{S}^{N-1},\mathbb{C}^N)\}.$$

Moreover the norms $\|\cdot\|_{H^1_{\mathbf{A}}(\mathbb{S}^{N-1})}$ and

$$\|\cdot\|_{H^1(\mathbb{S}^{N-1},\mathbb{C})}:=(\|\nabla_{\mathbb{S}^{N-1}}\cdot\|^2_{L^2(\mathbb{S}^{N-1},\mathbb{C}^N)}+\|\cdot\|^2_{L^2(\mathbb{S}^{N-1},\mathbb{C})})^{1/2}$$

are equivalent.

Proof. This follows easily from the boundedness of the function $\theta\mapsto|\mathbf{A}(\theta)|$. □

We finally describe the spectrum of the angular operator $L_{\mathbf{A},a}$.

Lemma A.5. *Let* $a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$ *and* $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$*. Then the spectrum of the operator* $L_{\mathbf{A},a}$ *on* $\mathbb{S}^{N-1}$ *consists of a diverging sequence of real eigenvalues with finite multiplicity* $\mu_1(\mathbf{A}, a) \le \mu_2(\mathbf{A}, a) \le \cdots$*, the first of which admits the variational characterization* (21).

Proof. For $\lambda = 1 + \|a\|_{L^\infty(\mathbb{S}^{N-1},\mathbb{R})}$, the operator $T : L^2(\mathbb{S}^{N-1}, \mathbb{C}) \to L^2(\mathbb{S}^{N-1}, \mathbb{C})$ defined as

$$Tf = u \quad \text{if and only if} \quad (-i\nabla_{\mathbb{S}^{N-1}} + \mathbf{A})^2 u - au + \lambda u = f$$

is well defined, symmetric, and compact. The lemma then follows from classical spectral theory. □

Acknowledgments. The authors wish to express their gratitude to the unknown referee for his/her helpful remarks, which stimulated them to revise and improve the paper, both in the results and in the exposition.

The first and third authors were supported by Italy MIUR, national project "Variational Methods and Nonlinear Differential Equations".

Note added in proof. After the present paper was accepted for publication, Prof. F. Pacard brought to our attention that related asymptotic expansions were obtained in previous works such as [R. Mazzeo, *Elliptic theory of differential edge operators. I*, Comm. Partial Differential Equations **16**, 1615–1664 (1991)] and [R. Mazzeo, *Regularity for the singular Yamabe problem*, Indiana Univ. Math. J. **40**, 1277–1299 (1991)] for elliptic equations on manifolds with conical singularities by Mellin transform methods. The common aspects and differences between our results and the results of the aforementioned papers are discussed in the addendum [V. Felli, A. Ferrero, S. Terracini, *Addendum to "Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential"*, http://arxiv.org/abs/1007.4434], where some variants and improvements are obtained under weaker assumptions on the perturbing potential h. In the addendum, it is also pointed out that a great advantage of the monotonicity approach lies in its applicability to semilinear problems, for which it allows one to directly prove (without passing through Brezis–Kato iteration) sharper a priori pointwise bounds and asymptotics.

References

[1] Almgren, F. J. Jr.: *Q* valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two. Bull. Amer. Math. Soc. **8**, 327–328 (1983) Zbl 0529.49021 MR 0684900

[2] Arioli, G., Szulkin, A.: A semilinear Schrödinger equation in the presence of a magnetic field. Arch. Ration. Mech. Anal. **170**, 277–295 (2003) Zbl 1051.35082 MR 2022133

[3] Balinsky, A. A.: Hardy type inequalities for Aharonov–Bohm magnetic potentials with multiple singularities. Math. Res. Lett. **10**, 169–176 (2003) Zbl 1048.35141 MR 1981894

[4] Brézis, H., Kato, T.: Remarks on the Schrödinger operator with singular complex potentials. J. Math. Pures Appl. (9) **58**, 137–151 (1979) Zbl 0408.35025 MR 0539217

[5] Caffarelli, L., Kohn, R., Nirenberg, L.: First order interpolation inequalities with weights. Compos. Math. **53**, 259–275 (1984) Zbl 0563.46024 MR 0768824

[6] Catrina, F., Wang, Z.-Q.: On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. Comm. Pure Appl. Math. **54**, 229–258 (2001) Zbl 1072.35506 MR 1794994

[7] Chabrowski, J., Szulkin, A.: On the Schrödinger equation involving a critical Sobolev exponent and magnetic field. Topol. Methods Nonlinear Anal. **25**, 3–21 (2005) Zbl 1176.35022 MR 2133390

[8] Cingolani, S.: Semiclassical stationary states of nonlinear Schrödinger equations with an external magnetic field. J. Differential Equations **188**, 52–79 (2003) Zbl 1062.81056 MR 1954508

[9] Cingolani, S., Secchi, S.: Semiclassical limit for nonlinear Schrödinger equations with electromagnetic fields. J. Math. Anal. Appl. **275**, 108–130 (2002) Zbl 1014.35087 MR 1941775

[10] Esteban, M. J., Lions, P.-L.: Stationary solutions of nonlinear Schrödinger equations with an external magnetic field. In: Partial Differential Equations and the Calculus of Variations, Vol. I, Progr. Nonlinear Differential Equations Appl. 1, Birkhäuser Boston, Boston, MA, 401–449 (1989) Zbl 0702.35067 MR 1034014

[11] Felli, V., Marchini, E. M., Terracini, S.: On Schrödinger operators with multipolar inverse-square potentials. J. Funct. Anal. **250**, 265–316 (2007) Zbl pre05206079 MR 2352482

[12] Felli, V., Marchini, E. M., Terracini, S.: On the behavior of solutions to Schrödinger equations with dipole-type potentials near the singularity. Discrete Contin. Dynam. Systems **21**, 91–119 (2008) Zbl 1141.35362 MR 2379458

[13] Felli, V., Marchini, E. M., Terracini, S.: On Schrödinger operators with multisingular inverse-square anisotropic potentials. Indiana Univ. Math. J **58**, 617–676 (2009) Zbl 1169.35013 MR 2514383

[14] Garofalo, N., Lin, F.-H.: Monotonicity properties of variational integrals, A_p weights and unique continuation. Indiana Univ. Math. J. **35**, 245–268 (1986) Zbl 0678.35015

[15] Hoffmann-Ostenhof, M., Hoffmann-Ostenhof, T., Laptev, A., Tidblom, J.: Many-particle Hardy inequalities. J. London Math. Soc. (2) **77**, 99–114 (2008) Zbl 1135.35012 MR 2389919

[16] Jackson, J. D.: From Lorenz to Coulomb and other explicit gauge transformations. Amer. J. Phys. **70**, 917–928 (2002)

[17] Kurata, K.: A unique continuation theorem for the Schrödinger equation with singular magnetic field. Proc. Amer. Math. Soc. **125**, 853–860 Zbl 0887(1997)35026 MR 1363173

[18] Kurata, K.: Local boundedness and continuity for weak solutions of $-(\nabla - ib)^2 u + Vu = 0$. Math. Z. **224**, 641–653 (1997) Zbl 0876.35028 MR 1452052

[19] Laptev, A., Weidl, T.: Hardy inequalities for magnetic Dirichlet forms. In: Mathematical Results in Quantum Mechanics (Prague, 1998), Oper. Theory Adv. Appl. 108, Birkhäuser, Basel, 299–305 (1999) MR 1708811

[20] Leinfelder, H.: Gauge invariance of Schrödinger operators and related spectral properties. J. Operator Theory **9**, 163–179 Zbl 0528(1983)35024 0528.35024

[21] Lévy-Leblond, J. M.: Electron capture by polar molecules. Phys. Rev. **153**, 1–4 (1967)

[22] Lieb, E. H., Loss, M.: Analysis. Grad. Stud. Math. 14, Amer. Math. Soc. (1997) Zbl 0873.26002 MR 1415616

[23] Melgaard, M., Ouhabaz, E.-M., Rozenblum, G.: Negative discrete spectrum of perturbed multivortex Aharonov–Bohm Hamiltonians. Ann. Henri Poincaré **5**, 979–1012 (2004) Zbl 1059.81049 MR 2136197

[24] Pinchover, Y.: On positive Liouville theorems and asymptotic behavior of solutions of Fuchsian type elliptic operators. Ann. Inst. H. Poincaré Anal. Non Linéaire **11**, 313–341 (1994) Zbl 0837.35010 MR 1277898

[25] Terracini, S.: On positive entire solutions to a class of equations with a singular coefficient and a critical exponent. Adv. Differential Equations **1**, 241–264 (1996) Zbl 0847.35045 MR 1364003

[26] Thaller, B.: The Dirac Equation. Texts Monogr. Phys., Springer, Berlin (1992) Zbl 0765.47023 MR 1219537

[27] Wang, Z.-Q., Zhu, M.: Hardy inequalities with boundary terms. Electron. J. Differential Equations **2003**, no. 43, 8 pp. Zbl 1109.35316 MR 1971109

[28] Wolff, T. H.: A property of measures in $\mathbb{R}^N$ and an application to unique continuation. Geom. Funct. Anal. **2**, 225–284 (1992) Zbl 0780.35015 MR 1159832