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Simon Brendle

A note on Ricci flow and optimal transportation

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Abstract. We describe a new link between Perelman's monotonicity formula for the reduced volume and ideas from optimal transport theory.

1. Introduction

In this note, we describe an interpolation inequality in the setting of Ricci flow and $\mathcal{L}$-distance. This inequality is motivated by the following classical inequality due to Prékopa and Leindler:

Theorem 1 (Prékopa [9]; Leindler [5]). *Fix a real number* $0 < \lambda < 1$. *Moreover, let* $u_1, u_2, v : \mathbb{R}^n \to \mathbb{R}$ *be nonnegative measurable functions satisfying*

$$v((1-\lambda)x + \lambda y) \geq u_1(x)^{1-\lambda} u_2(y)^\lambda$$

for all points $x, y \in \mathbb{R}^n$. *Then*

$$\int_{\mathbb{R}^n} v \geq \left(\int_{\mathbb{R}^n} u_1\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} u_2\right)^\lambda.$$

Cordero-Erausquin, McCann, and Schmuckenschläger [2], [3] have generalized this inequality to Riemannian manifolds. The proof employs techniques from optimal transport theory.

Our goal in this paper is to replace the Riemannian distance by Perelman's $\mathcal{L}$-distance (cf. [8]). The theory of $\mathcal{L}$-optimal transport was developed in recent work of Topping [11] (see also [6], [7]). Among other things, Topping proved an important monotonicity formula for the $\mathcal{L}$-Wasserstein distance on the space of probability measures. Lott [6] established a convexity property for the entropy along $\mathcal{L}$-Wasserstein geodesics.

To fix notation, let M be a compact manifold of dimension n, and let $g(t)$, $t \in [0, T]$, be a one-parameter family of metrics on M. We assume that the metrics $g(t)$ evolve by backward Ricci flow, i.e.

$$\frac{\partial}{\partial t} g(t) = 2\,\mathrm{Ric}_{g(t)}\,.$$

S. Brendle: Department of Mathematics, Stanford University, Stanford, CA 94305, USA;
e-mail: brendle@math.stanford.edu

This evolution equation was introduced in a seminal paper by R. Hamilton [4]. For an introduction to Ricci flow, see e.g. [1] or [10]. Following Perelman [8], we define the $\mathcal{L}$-*length* of a path $\gamma : [\tau_1, \tau_2] \to M$ by

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{t}\,(R_{g(t)}(\gamma(t)) + |\gamma'(t)|^2_{g(t)})\, dt,$$

where $R_{g(t)}$ denotes the scalar curvature of the metric $g(t)$. Moreover, the $\mathcal{L}$-distance is defined by

$$Q(x, \tau_1; y, \tau_2) = \inf\{\mathcal{L}(\gamma) : \gamma : [\tau_1, \tau_2] \to M,\ \gamma(\tau_1) = x,\ \gamma(\tau_2) = y\}.$$

A path $\gamma : [\tau_1, \tau_2] \to M$ is called an $\mathcal{L}$-*geodesic* if the first variation of $\mathcal{L}$ is zero. For each tangent vector $Z \in T_x M$, we define

$$\mathcal{L}_{\tau_1,\tau_2} \exp_x(Z) = \gamma(\tau_2),$$

where $\gamma : [\tau_1, \tau_2] \to M$ is the unique $\mathcal{L}$-geodesic satisfying $\gamma(\tau_1) = x$ and $\sqrt{\tau_1}\,\gamma'(\tau_1) = Z$. The map $\mathcal{L}_{\tau_1,\tau_2} \exp_x : T_x M \to M$ is called the $\mathcal{L}$-*exponential map*.

The following is the main result of this note:

Theorem 2. *Fix real numbers* τ_1, τ_2, τ *such that* $0 < \tau_1 < \tau < \tau_2 < T$. *For abbreviation, we write*

$$\frac{1}{\sqrt{\tau}} = \frac{1-\lambda}{\sqrt{\tau_1}} + \frac{\lambda}{\sqrt{\tau_2}},$$

where $0 < \lambda < 1$. *Let* $u_1, u_2, v : M \to \mathbb{R}$ *be nonnegative measurable functions such that*

$$\begin{aligned}\left(\frac{\tau}{\tau_1^{1-\lambda}\tau_2^{\lambda}}\right)^{n/2} v(\gamma(\tau)) \geq\ & \exp\left(-\frac{1-\lambda}{2\sqrt{\tau_1}} Q(\gamma(\tau_1), \tau_1; \gamma(\tau), \tau)\right) u_1(\gamma(\tau_1))^{1-\lambda} \\ & \cdot \exp\left(\frac{\lambda}{2\sqrt{\tau_2}} Q(\gamma(\tau), \tau; \gamma(\tau_2), \tau_2)\right) u_2(\gamma(\tau_2))^{\lambda}\end{aligned}$$

for every minimizing $\mathcal{L}$-*geodesic* $\gamma : [\tau_1, \tau_2] \to M$. *Then*

$$\int_M v\, d\mathrm{vol}_{g(\tau)} \geq \left(\int_M u_1\, d\mathrm{vol}_{g(\tau_1)}\right)^{1-\lambda} \left(\int_M u_2\, d\mathrm{vol}_{g(\tau_2)}\right)^{\lambda}.$$

By sending $\tau_1 \to 0$, we recover the monotonicity of Perelman's reduced volume. This is discussed in Section 3.

2. Proof of Theorem 2

In order to prove Theorem 2, we make extensive use of Topping's notion of $\mathcal{L}$-optimal transportation (cf. [11]; see also [6]). Without loss of generality, we may assume that

$$\int_M u_1(x)\, d\mathrm{vol}_{g(\tau_1)} = \int_M u_2(x)\, d\mathrm{vol}_{g(\tau_2)} = 1.$$

We define Borel probability measures ν_1 and ν_2 by

$$d\nu_1 = u_1\, d\mathrm{vol}_{g(\tau_1)} \quad \text{and} \quad d\nu_2 = u_2\, d\mathrm{vol}_{g(\tau_2)}.$$

Topping introduced a notion of reflexive function (see [11, Definition 2.1] for a precise definition). This notion is analogous to the concept of c-concavity in the classical theory of optimal transportation. Topping then proved the following existence theorem (cf. [11, Section 2]; see also [12, Theorem 10.28]):

Proposition 3 (P. Topping [11]). *We can find a reflexive function $\varphi : M \to \mathbb{R}$, a Borel set $K \subset M$, and a Borel map $F : M \to M$ with the following properties:*

(i) $\nu_2 = F_\# \nu_1$.
(ii) *The set $M \setminus K$ has measure zero.*
(iii) *The function φ is differentiable at each point $x \in K$.*
(iv) *If $x \in K$ and $y = F(x)$, then the function $Q(\cdot, \tau_1; y, \tau_2) - \varphi$ attains its global minimum at the point x. In particular, the function $Q(\cdot, \tau_1; y, \tau_2) - \varphi$ is differentiable at x, and its gradient is equal to zero.*

For each $t \in [\tau_1, \tau_2]$, we define a Borel map $F_t : M \to M$ by

$$F_t(x) = \mathcal{L}_{\tau_1,t} \exp_x\Big(-\frac{1}{2}\nabla\varphi(x)\Big)$$

for $x \in K$. The following result is a consequence of property (iv) in Proposition 3 (cf. [11, Lemma 2.4]).

Proposition 4. *We have $F_{\tau_1}(x) = x$ and $F_{\tau_2}(x) = F(x)$ for all $x \in K$. Moreover, for each point $x \in K$, the path $t \mapsto F_t(x)$ has minimal $\mathcal{L}$-length among all paths joining (x, τ_1) and $(F(x), \tau_2)$.*

Since φ is reflexive, the function φ is semiconcave (cf. [11, Lemma 2.10]). By Theorem 14.1 in [12], we can find a Borel set $\tilde{K} \subset K$ with the following properties:

- The set $M \setminus \tilde{K}$ has measure zero.
- For each point $x \in \tilde{K}$, the function φ admits a Taylor expansion of order two around x.

For each point $x \in \tilde{K}$, we denote by $\nabla\varphi(x)$ and $(\mathrm{Hess}\,\varphi)_x$ the gradient and Hessian of the function φ with respect to the metric $g(\tau_1)$. Theorem 14.1 in [12] guarantees that $(\mathrm{Hess}\,\varphi)_x$ is symmetric.

We next describe the volume distortion coefficients associated with the map F_t. To that end, we fix a point $x \in \tilde{K}$ and a time $t \in (\tau_1, \tau_2]$. The linearization of the $\mathcal{L}$-exponential map $\mathcal{L}_{\tau_1,t} \exp_x$ gives a linear transformation

$$D(\mathcal{L}_{\tau_1,t} \exp_x)_{-\frac{1}{2}\nabla\varphi(x)} : (T_x M, g(\tau_1)) \to (T_{F_t(x)} M, g(t)).$$

Moreover, the Hessian of the function $Q(\cdot, \tau_1; F_t(x), t) - \varphi$ at the point x defines a symmetric linear transformation from the tangent space $(T_x M, g(\tau_1))$ into itself.

Let us denote by $\Psi_{x,t} : (T_x M, g(\tau_1)) \to (T_{F_t(x)} M, g(t))$ the composition of these two linear transformations; that is,

$$\Psi_{x,t} = \frac{1}{2} D(\mathcal{L}_{\tau_1,t} \exp_x)_{-\frac{1}{2}\nabla\varphi(x)} \circ [\mathrm{Hess}(Q(\cdot, \tau_1; F_t(x), t) - \varphi)]_x$$

(cf. [11, Lemma 2.13]). We note that $\Psi_{x,t}$ can be characterized in terms of $\mathcal{L}$-Jacobi fields; see [11, Lemma 2.18] for details. Finally, we define the *volume distortion coefficients* by

$$\mathcal{J}(x, t) = \det \Psi_{x,t}$$

for each point $x \in \tilde{K}$ and each $t \in (\tau_1, \tau_2]$.

Proposition 5. *For each point $x \in \tilde{K}$, we have*

$$\begin{aligned} &\tau^{-n/2} \exp\Big(-\frac{1-\lambda}{2\sqrt{\tau_1}} Q(x, \tau_1; F_\tau(x), \tau)\Big) \mathcal{J}(x, \tau) \\ &\qquad \geq \tau_1^{-n(1-\lambda)/2} \tau_2^{-n\lambda/2} \exp\Big(-\frac{\lambda}{2\sqrt{\tau_2}} Q(F_\tau(x), \tau; F(x), \tau_2)\Big) \mathcal{J}(x, \tau_2)^\lambda. \end{aligned}$$

Proof. Fix a point $x \in \tilde{K}$, and let $\{e_1, \dots, e_n\}$ be a basis of $T_x M$ which is orthonormal with respect to the metric $g(\tau_1)$. We next consider the path

$$\gamma : [\tau_1, \tau_2] \to M, \quad t \mapsto F_t(x).$$

By Lemma 4, the path γ has minimal $\mathcal{L}$-length among all paths joining (x, τ_1) to $(F(x), \tau_2)$.

Let $\{E_1(t), \dots, E_n(t)\}$ be vector fields along γ such that $E_i(\tau_1) = e_i$ and

$$\langle D_t E_i(t), E_j(t)\rangle_{g(t)} + \mathrm{Ric}_{g(t)}(E_i(t), E_j(t)) = 0$$

for all $t \in [\tau_1, \tau_2]$. For each $t \in [\tau_1, \tau_2]$, the vectors $\{E_1(t), \dots, E_n(t)\}$ are orthonormal with respect to the metric $g(t)$.

Let $\{Y_1(t), \dots, Y_n(t)\}$ be $\mathcal{L}$-Jacobi fields along γ satisfying the initial conditions

$$Y_j(\tau_1) = e_j \quad \text{and} \quad \langle e_i, D_t Y_j(\tau_1)\rangle_{g(\tau_1)} = -\frac{1}{2\sqrt{\tau_1}} (\mathrm{Hess}\,\varphi)_x(e_i, e_j).$$

For each $t \in [\tau_1, \tau_2]$, we define an $n \times n$ matrix $A(t)$ by

$$a_{ij}(t) = \langle E_i(t), Y_j(t) \rangle_{g(t)}.$$

It follows from the initial conditions for Y_j that $a_{ij}(\tau_1) = \delta_{ij}$ and

$$a'_{ij}(\tau_1) = \mathrm{Ric}_{g(\tau_1)}(e_i, e_j) - \frac{1}{2\sqrt{\tau_1}} (\mathrm{Hess}\,\varphi)_x(e_i, e_j).$$

In particular, the matrix $A'(\tau_1)A(\tau_1)^{-1}$ is symmetric. Moreover, it was shown by Topping [11] that

$$A''(t) + \frac{1}{2t} A'(t) = M(t)A(t)$$

for all $t \in [\tau_1, \tau_2]$. Here, $M(t)$ is a symmetric $n \times n$ matrix, whose trace is given by

$$\begin{aligned} 2\mathrm{tr}(M(t)) = &\frac{\partial}{\partial t} R_{g(t)}(\gamma(t)) + 2\langle \nabla R_{g(t)}(\gamma(t)), \gamma'(t) \rangle_{g(t)} \\ &- 2\mathrm{Ric}_{g(t)}(\gamma'(t), \gamma'(t)) + \frac{1}{t} R_{g(t)}(\gamma(t)). \end{aligned} \tag{1}$$

Arguing as in the proof of Lemma 3.1 in [11], we obtain

$$\begin{aligned} &t^{-3/2} \frac{d}{dt}\left[t^{3/2} \frac{d}{dt} \log\det A(t) \right] \\ &\quad= \frac{d^2}{dt^2} \log\det A(t) + \frac{3}{2t} \frac{d}{dt} \log\det A(t) \\ &\quad= \mathrm{tr}(A''(t)A(t)^{-1}) - \mathrm{tr}(A'(t)A(t)^{-1}A'(t)A(t)^{-1}) + \frac{3}{2t} \mathrm{tr}(A'(t)A(t)^{-1}) \\ &\quad= \mathrm{tr}(M(t)) - \mathrm{tr}(A'(t)A(t)^{-1}A'(t)A(t)^{-1}) + \frac{1}{t} \mathrm{tr}(A'(t)A(t)^{-1}) \\ &\quad= \mathrm{tr}(M(t)) - \mathrm{tr}\left[\left(A'(t)A(t)^{-1} - \frac{1}{2t} I \right)^2 \right] + \frac{n}{4t^2}. \end{aligned}$$

Note that the matrix $A'(\tau_1)A(\tau_1)^{-1}$ is symmetric. Moreover, the matrix $M(t)$ is symmetric for each $t \in [\tau_1, \tau_2]$. Consequently, the matrix $A'(t)A(t)^{-1}$ is symmetric for all $t \in [\tau_1, \tau_2]$. Hence, we obtain

$$t^{-3/2} \frac{d}{dt}\left[t^{3/2} \frac{d}{dt} \log\det A(t) \right] \leq \mathrm{tr}(M(t)) + \frac{n}{4t^2} \tag{2}$$

for all $t \in [\tau_1, \tau_2]$. On the other hand, we have

$$\frac{d}{dt} Q(x, \tau_1; F_t(x), t) = \sqrt{t}(R_{g(t)}(\gamma(t)) + |\gamma'(t)|^2_{g(t)})$$

by definition of the $\mathcal{L}$-distance. This implies

$$\begin{aligned}
&t^{-3/2}\frac{d}{dt}\bigg[t^{3/2}\frac{d}{dt}(t^{-1/2}Q(x,\tau_1;F_t(x),t))\bigg]\\
&\quad= t^{-1}\frac{d}{dt}\bigg[t^{1/2}\frac{d}{dt}Q(x,\tau_1;F_t(x),t)\bigg] = t^{-1}\frac{d}{dt}[t(R_{g(t)}(\gamma(t))+|\gamma'(t)|^2_{g(t)})]\\
&\quad= \frac{\partial}{\partial t}R_{g(t)}(\gamma(t)) + 2\langle\nabla R_{g(t)}(\gamma(t)),\gamma'(t)\rangle_{g(t)} - 2\mathrm{Ric}_{g(t)}(\gamma'(t),\gamma'(t)) + \frac{1}{t}R_{g(t)}(\gamma(t))
\end{aligned}$$

(cf. [8, equation (7.3)]). Using (1), we obtain

$$t^{-3/2}\frac{d}{dt}\bigg[t^{3/2}\frac{d}{dt}(t^{-1/2}Q(x,\tau_1;F_t(x),t))\bigg] = 2\mathrm{tr}(M(t)). \tag{3}$$

Putting these facts together, we conclude that

$$t^{-3/2}\frac{d}{dt}\bigg[t^{3/2}\frac{d}{dt}\bigg(\frac{n}{2}\log t + \frac{1}{2}t^{-1/2}Q(x,\tau_1;F_t(x),t) - \log\det A(t)\bigg)\bigg] \geq 0.$$

Hence, if we write

$$\frac{n}{2}\log t + \frac{1}{2}t^{-1/2}Q(x,\tau_1;F_t(x),t) - \log\det A(t) = h(t^{-1/2}),$$

then the function h is convex. Using the relation $\tau^{-1/2} = (1-\lambda)\tau_1^{-1/2} + \lambda\tau_2^{-1/2}$, we obtain

$$h(\tau^{-1/2}) \leq (1-\lambda)h(\tau_1^{-1/2}) + \lambda h(\tau_2^{-1/2}),$$

hence

$$\begin{aligned}
&\tau^{-n/2}\exp\bigg(-\frac{1}{2\sqrt{\tau}}Q(x,\tau_1;F_\tau(x),\tau)\bigg)\det A(\tau)\\
&\qquad\geq \tau_1^{-n(1-\lambda)/2}\tau_2^{-n\lambda/2}\exp\bigg(-\frac{\lambda}{2\sqrt{\tau_2}}Q(x,\tau_1;F(x),\tau_2)\bigg)(\det A(\tau_2))^\lambda.
\end{aligned}$$

Moreover, we have

$$Q(x,\tau_1;F(x),\tau_2) = Q(x,\tau_1;F_\tau(x),\tau) + Q(F_\tau(x),\tau;F(x),\tau_2)$$

since γ is a minimizing $\mathcal{L}$-geodesic. Hence, we obtain

$$\begin{aligned}
&\tau^{-n/2}\exp\bigg(-\frac{1-\lambda}{2\sqrt{\tau_1}}Q(x,\tau_1;F_\tau(x),\tau)\bigg)\det A(\tau)\\
&\qquad\geq \tau_1^{-n(1-\lambda)/2}\tau_2^{-n\lambda/2}\exp\bigg(-\frac{\lambda}{2\sqrt{\tau_2}}Q(F_\tau(x),\tau;F(x),\tau_2)\bigg)(\det A(\tau_2))^\lambda.
\end{aligned}$$

On the other hand, it follows from Lemma 2.18 in [11] that $\Psi_{x,t}(e_j) = Y_j(t)$ for all $t \in (\tau_1,\tau_2]$. From this, we deduce that $\langle E_i(t),\Psi_{x,t}(e_j)\rangle_{g(t)} = a_{ij}(t)$, hence $\mathcal{J}(x,t) =$

det $A(t)$ for all $t \in (\tau_1, \tau_2]$. Putting these facts together, the assertion follows. This completes the proof of Proposition 5.

We next consider the interpolant measure $\nu = (F_\tau)_\# \nu_1$. It follows from work of Topping that ν is absolutely continuous with respect to the volume measure (cf. [11, Lemma 2.17]). Hence, we may write $d\nu = u \, d\mathrm{vol}_{g(\tau)}$ for some Borel measurable function u. Using Proposition 5, we obtain a lower bound for the density u.

Proposition 6. *There exists a Borel set $\hat{K} \subset \tilde{K}$ such that $M \setminus \hat{K}$ has measure zero and*

$$\begin{aligned}\Big(\frac{\tau}{\tau_1^{1-\lambda}\tau_2^{\lambda}}\Big)^{n/2} u(F_\tau(x)) \leq \exp\Big(-\frac{1-\lambda}{2\sqrt{\tau_1}} Q(x,\tau_1; F_\tau(x),\tau)\Big) u_1(x)^{1-\lambda} \\ \cdot \exp\Big(\frac{\lambda}{2\sqrt{\tau_2}} Q(F_\tau(x),\tau; F(x),\tau_2)\Big) u_2(F(x))^{\lambda}\end{aligned}$$

for all $x \in \hat{K}$.

Proof. It follows from Theorem 2.14 in [11] that

$$u_1(x) = u_2(F(x))\mathcal{J}(x,\tau_2) > 0$$

for almost all $x \in \tilde{K}$. Applying the analogous reasoning to the interpolant measure ν yields

$$u_1(x) = u(F_\tau(x))\mathcal{J}(x,\tau) > 0$$

for almost all $x \in \tilde{K}$. Using Proposition 5, we obtain

$$\begin{aligned}&\tau^{-n/2} \exp\Big(-\frac{1-\lambda}{2\sqrt{\tau_1}} Q(x,\tau_1; F_\tau(x),\tau)\Big) \frac{u_1(x)}{u(F_\tau(x))} \\ &\quad \geq \tau_1^{-n(1-\lambda)/2} \tau_2^{-n\lambda/2} \exp\Big(-\frac{\lambda}{2\sqrt{\tau_2}} Q(F_\tau(x),\tau; F(x),\tau_2)\Big) \Big(\frac{u_1(x)}{u_2(F(x))}\Big)^{\lambda}\end{aligned}$$

for almost all $x \in \tilde{K}$. Rearranging terms, the assertion follows.

Corollary 7. *We have*

$$\int_M v \, d\mathrm{vol}_{g(\tau)} \geq 1.$$

Proof. Fix a point $x \in \hat{K}$. By Lemma 4, the path $t \mapsto F_t(x)$ is a minimizing $\mathcal{L}$-geodesic. Therefore, we have

$$\begin{aligned}\Big(\frac{\tau}{\tau_1^{1-\lambda}\tau_2^{\lambda}}\Big)^{n/2} v(F_\tau(x)) \geq \exp\Big(-\frac{1-\lambda}{2\sqrt{\tau_1}} Q(x,\tau_1; F_\tau(x),\tau)\Big) u_1(x)^{1-\lambda} \\ \cdot \exp\Big(\frac{\lambda}{2\sqrt{\tau_2}} Q(F_\tau(x),\tau; F(x),\tau_2)\Big) u_2(F(x))^{\lambda}.\end{aligned}$$

Using Proposition 6, we conclude that

$$v(F_\tau(x)) \geq u(F_\tau(x))$$

for all $x \in \hat{K}$. This implies

$$\int_M v \, d\mathrm{vol}_{g(\tau)} \geq \int_{F_\tau(\hat{K})} v \, d\mathrm{vol}_{g(\tau)} \geq \int_{F_\tau(\hat{K})} u \, d\mathrm{vol}_{g(\tau)} = \nu(F_\tau(\hat{K})).$$

Moreover, we have

$$\nu(F_\tau(\hat{K})) = \nu_1[F_\tau^{-1}(F_\tau(\hat{K}))] \geq \nu_1(\hat{K}) = 1$$

by definition of ν. Putting these facts together, the assertion follows.

3. Relation to Perelman's reduced volume

In this final section, we discuss how Theorem 2 is related to the monotonicity of Perelman's reduced volume. The strategy is to fix τ and τ_2, and pass to the limit as $\tau_1 \to 0$.

Let us fix a point $p \in M$ and real numbers $0 < \tau < \tau_2 < T$. We define a function v by

$$v = \tau^{-n/2} \exp\Big(-\frac{1}{2\sqrt{\tau}} Q(p, 0; \cdot, \tau)\Big).$$

For $\tau_1 > 0$ sufficiently small, we denote by $B(p, \sqrt{\tau_1})$ the geodesic ball of radius $\sqrt{\tau_1}$ in the metric $g(0)$. We can find a positive constant N such that $Q(p, 0; x, \tau_1) \leq N\sqrt{\tau_1}$ and $Q(x, \tau_1; p, 2\tau_1) \leq N\sqrt{\tau_1}$ for all points $x \in B(p, \sqrt{\tau_1})$. Note that the constant N is independent of τ_1.

As above, we write

$$\frac{1}{\sqrt{\tau}} = \frac{1-\lambda}{\sqrt{\tau_1}} + \frac{\lambda}{\sqrt{\tau_2}},$$

where $0 < \lambda < 1$. We now specify the functions u_1 and u_2. We define

$$u_1 = \tau_1^{-n/2} \exp\Big(-\frac{N\sqrt{\tau_1}}{2(1-\lambda)}\Big(\frac{1}{\sqrt{\tau}} + \frac{\lambda}{\sqrt{\tau_2}}\Big)\Big) \mathbb{1}_{B(p,\sqrt{\tau_1})},$$

$$u_2 = \tau_2^{-n/2} \exp\Big(-\frac{1}{2\sqrt{\tau_2}} Q(p, 2\tau_1; \cdot, \tau_2)\Big).$$

In the next step, we verify that u_1, u_2, v satisfy the assumptions of Theorem 2.

Proposition 8. *We have*

$$\Big(\frac{\tau}{\tau_1^{1-\lambda}\tau_2^{\lambda}}\Big)^{n/2} v(\gamma(\tau)) \geq \exp\Big(-\frac{1-\lambda}{2\sqrt{\tau_1}} Q(\gamma(\tau_1), \tau_1; \gamma(\tau), \tau)\Big) u_1(\gamma(\tau_1))^{1-\lambda}$$
$$\cdot \exp\Big(\frac{\lambda}{2\sqrt{\tau_2}} Q(\gamma(\tau), \tau; \gamma(\tau_2), \tau_2)\Big) u_2(\gamma(\tau_2))^{\lambda}$$

for every minimizing $\mathcal{L}$-geodesic $\gamma : [\tau_1, \tau_2] \to M$.

Proof. If $\gamma(0) \notin B(p, \sqrt{\tau_1})$, then $u_1(\gamma(0)) = 0$ and the assertion is trivial. Hence, it suffices to consider the case $\gamma(0) \in B(p, \sqrt{\tau_1})$. In this case, we have

$$\begin{aligned} Q(p, 0; \gamma(\tau), \tau) &\leq Q(\gamma(\tau_1), \tau_1; \gamma(\tau), \tau) + Q(p, 0; \gamma(\tau_1), \tau_1) \\ &\leq Q(\gamma(\tau_1), \tau_1; \gamma(\tau), \tau) + N\sqrt{\tau_1} \end{aligned}$$

and

$$\begin{aligned} Q(p, 2\tau_1; \gamma(\tau_2), \tau_2) &\geq Q(\gamma(\tau_1), \tau_1; \gamma(\tau_2), \tau_2) - Q(\gamma(\tau_1), \tau_1; p, 2\tau_1) \\ &\geq Q(\gamma(\tau_1), \tau_1; \gamma(\tau_2), \tau_2) - N\sqrt{\tau_1}. \end{aligned}$$

This implies

$$v(\gamma(\tau), \tau) \geq \tau^{-n/2} \exp\Big(-\frac{N\sqrt{\tau_1}}{2\sqrt{\tau}}\Big) \exp\Big(-\frac{1}{2\sqrt{\tau}} Q(\gamma(\tau_1), \tau_1; \gamma(\tau), \tau)\Big)$$

and

$$u_2(\gamma(\tau_2)) \leq \tau_2^{-n/2} \exp\Big(\frac{N\sqrt{\tau_1}}{2\sqrt{\tau_2}}\Big) \exp\Big(-\frac{1}{2\sqrt{\tau_2}} Q(\gamma(\tau_1), \tau_1; \gamma(\tau_2), \tau_2)\Big).$$

Moreover, we have

$$Q(\gamma(\tau_1), \tau_1; \gamma(\tau_2), \tau_2) = Q(\gamma(\tau_1), \tau_1; \gamma(\tau), \tau) + Q(\gamma(\tau), \tau; \gamma(\tau_2), \tau_2)$$

since γ has minimal $\mathcal{L}$-length. Putting these facts together, we obtain

$$\begin{aligned} v(\gamma(\tau)) \geq \Big(\frac{\tau_2^{\lambda}}{\tau}\Big)^{n/2} &\exp\Big(-\frac{N\sqrt{\tau_1}}{2}\Big(\frac{1}{\sqrt{\tau}} + \frac{\lambda}{\sqrt{\tau_2}}\Big)\Big) \\ &\cdot \exp\Big(-\frac{1-\lambda}{2\sqrt{\tau_1}} Q(\gamma(\tau_1), \tau_1; \gamma(\tau), \tau)\Big) \\ &\cdot \exp\Big(\frac{\lambda}{2\sqrt{\tau_2}} Q(\gamma(\tau), \tau; \gamma(\tau_2), \tau_2)\Big) u_2(\gamma(\tau_2))^{\lambda}. \end{aligned}$$

From this, the assertion follows.

Let $\tilde{V}(\tau)$ denote the reduced volume at time τ. Using Theorem 2, we obtain

$$\tilde{V}(\tau) = \int_M v \, d\mathrm{vol}_{g(\tau)} \geq \Big(\int_M u_1 \, d\mathrm{vol}_{g(\tau_1)}\Big)^{1-\lambda} \Big(\int_M u_2 \, d\mathrm{vol}_{g(\tau_2)}\Big)^{\lambda}.$$

We now fix τ and τ_2, and pass to the limit as $\tau_1 \to 0$. Clearly,

$$1 - \lambda = \sqrt{\tau_1}\Big(\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{\tau_2}}\Big) + O(\tau_1).$$

This implies

$$-\frac{N\sqrt{\tau_1}}{2(1-\lambda)}\Big(\frac{1}{\sqrt{\tau}} + \frac{\lambda}{\sqrt{\tau_2}}\Big) \to -\frac{N}{2}\Big(\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{\tau_2}}\Big)^{-1}\Big(\frac{1}{\sqrt{\tau}} + \frac{1}{\sqrt{\tau_2}}\Big)$$

as $\tau_1 \to 0$. Hence, the integral $\int_M u_1 \, d\text{vol}_{g(\tau_1)}$ converges to a positive real number as $\tau_1 \to 0$. Since $1 - \lambda \to 0$, we conclude that

$$\left(\int_M u_1 \, d\text{vol}_{g(\tau_1)} \right)^{1-\lambda} \to 1 \quad \text{as } \tau_1 \to 0.$$

Moreover, we have

$$\left(\int_M u_2 \, d\text{vol}_{g(\tau_2)} \right)^{\lambda} \to \tilde{V}(\tau_2) \quad \text{as } \tau_1 \to 0.$$

Putting these facts together, we obtain

$$\tilde{V}(\tau) = \int_M v \, d\text{vol}_{g(\tau)} \geq \tilde{V}(\tau_2).$$

Thus, Theorem 2 implies the monotonicity of the reduced volume.

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