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Kazhdan–Lusztig basis and a geometric filtration of an affine Hecke algebra, II

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Abstract. An affine Hecke algebra can be realized as an equivariant K-group of the corresponding Steinberg variety. This gives rise naturally to some two-sided ideals of the affine Hecke algebra by means of the closures of nilpotent orbits of the corresponding Lie algebra. In this paper we will show that the two-sided ideals are in fact the two-sided ideals of the affine Hecke algebra defined through the two-sided cells of the corresponding affine Weyl group after the two kinds of ideals are tensored by $\mathbb{Q}$. This proves a weak form of a conjecture of Ginzburg proposed in 1987.

0. Introduction

Let H be an affine Hecke algebra over the ring $\mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in an indeterminate v with integer coefficients. The affine Hecke algebra has a Kazhdan–Lusztig basis. The basis has many remarkable properties and plays an important role in representation theory. Also, Kazhdan and Lusztig and Ginzburg gave a geometric realization of H, which is the key to the proof by Kazhdan and Lusztig of the Deligne–Langlands conjecture on classification of irreducible modules of affine Hecke algebras over $\mathbb{C}$ at non-roots of 1. This geometric construction of H has some two-sided ideals defined naturally by means of the nilpotent variety of the corresponding Lie algebra. The two-sided ideals form a nice filtration of the affine Hecke algebra. In [G2] Ginzburg conjectured that the two-sided ideals are in fact the two-sided ideals of the affine Hecke algebra defined through two-sided cells of the corresponding affine Weyl group (see also [L6, T2]). The conjecture is known to be true for the trivial nilpotent orbit {0} (see Corollary 8.13 in [L6] and Theorem 7.4 in [X1]) and for type A [TX]. Other evidence is showed in [L6, Corollary 9.13]. We will prove the two kinds of two-sided ideals coincide after they are tensored by $\mathbb{Q}$ (see Theorem 1.5 in Section 1). This proves a weak form of Ginzburg's conjecture.

1. Affine Hecke algebra

1.1. Let G be a simply connected simple algebraic group over the complex number field $\mathbb{C}$. The Weyl group W_0 acts naturally on the character group X of a maximal tours

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of G. The semidirect product $W = W_0 \ltimes X$ with respect to this action is called an (extended) *affine Weyl group*. Let H be the associated Hecke algebra over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ (v an indeterminate) with parameter v^2. Thus H has an $\mathcal{A}$-basis $\{T_w \mid w \in W\}$ and its multiplication is defined by the relations $(T_s - v^2)(T_s + 1) = 0$ if s is a simple reflection and $T_w T_u = T_{wu}$ if $l(wu) = l(w) + l(u)$, where l is the length function of W.

1.2. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$, and $\mathcal{B}$ the variety of all Borel subalgebras of $\mathfrak{g}$. The *Steinberg variety* Z is the subvariety of $\mathcal{N} \times \mathcal{B} \times \mathcal{B}$ consisting of all triples $(n, \mathfrak{b}, \mathfrak{b}')$, $n \in \mathfrak{b} \cap \mathfrak{b}' \cap \mathcal{N}$, $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$. Let $\Lambda = \{(n, \mathfrak{b}) \mid n \in \mathcal{N} \cap \mathfrak{b}, \mathfrak{b} \in \mathcal{B}\}$ be the cotangent bundle of $\mathcal{B}$. Clearly Z can be regarded as a subvariety of $\Lambda \times \Lambda$ via the imbedding $Z \to \Lambda \times \Lambda$, $(n, \mathfrak{b}, \mathfrak{b}') \mapsto (n, \mathfrak{b}, n, \mathfrak{b}')$. Define a $G \times \mathbb{C}^*$-action on Λ by $(g, z) : (n, \mathfrak{b}) \mapsto (z^{-2}\mathrm{ad}(g)n, \mathrm{ad}(g)\mathfrak{b})$. Let $G \times \mathbb{C}^*$ act on $\Lambda \times \Lambda$ diagonally; then Z is a $G \times \mathbb{C}^*$-stable subvariety of $\Lambda \times \Lambda$. For $1 \le i < j \le 3$, let p_{ij} be the projection from $\Lambda \times \Lambda \times \Lambda$ to its (i, j)-factor. Note that the restriction of p_{13} gives rise to a proper morphism $p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z) \to Z$. Let $K^{G\times\mathbb{C}^*}(Z) = K^{G\times\mathbb{C}^*}(\Lambda \times \Lambda; Z)$ be the Grothendieck group of the category of $G \times \mathbb{C}^*$-equivariant coherent sheaves on $\Lambda \times \Lambda$ with support in Z. We define the convolution product

$$* : K^{G\times\mathbb{C}^*}(Z) \times K^{G\times\mathbb{C}^*}(Z) \to K^{G\times\mathbb{C}^*}(Z), \qquad \mathscr{F} * \mathscr{G} = (p_{13})_*(p_{12}^*\mathscr{F} \otimes_{\mathcal{O}_{\Lambda\times\Lambda\times\Lambda}} p_{23}^*\mathscr{G}),$$

where $\mathcal{O}_{\Lambda\times\Lambda\times\Lambda}$ is the structure sheaf of $\Lambda \times \Lambda \times \Lambda$. This endows $K^{G\times\mathbb{C}^*}(Z)$ with an associative algebra structure over the representation ring $R_{G\times\mathbb{C}^*}$ of $G \times \mathbb{C}^*$. We shall regard the indeterminate v as the representation $G \times \mathbb{C}^* \to \mathbb{C}^*$, $(g, z) \mapsto z$. Then $R_{G\times\mathbb{C}^*}$ is identified with $\mathcal{A} \otimes_{\mathbb{Z}} R_G$. In particular, $K^{G\times\mathbb{C}^*}(Z)$ is an $\mathcal{A}$-algebra. Moreover, as an $\mathcal{A}$-algebra, $K^{G\times\mathbb{C}^*}(Z)$ is isomorphic to the Hecke algebra H (see [G1, KL2] or [CG, L6]). We shall identify $K^{G\times\mathbb{C}^*}(Z)$ with H.

1.3. Let $\mathcal{C}$ and $\mathcal{C}'$ be two G-orbits in $\mathcal{N}$. We say that $\mathcal{C} \le \mathcal{C}'$ if $\mathcal{C}$ is in the closure of $\mathcal{C}'$. This defines a partial order on the set of G-orbits in $\mathcal{N}$. Given a locally closed G-stable subvariety of $\mathcal{N}$, we set $Z_Y = \{(n, \mathfrak{b}, \mathfrak{b}') \in Z \mid n \in Y\}$.

If Y is closed, then the inclusion $i_Y : Z_Y \to Z$ induces a map $(i_Y)_* : K^{G\times\mathbb{C}^*}(Z_Y) \to K^{G\times\mathbb{C}^*}(Z)$ (see [G1, KL2]). The image H_Y of $(i_Y)_*$ is in fact a two-sided ideal of $K^{G\times\mathbb{C}^*}(Z)$ (see [L6, Corollary 9.13]), which is generated by $G \times \mathbb{C}^*$-equivariant sheaves supported on Z_Y. It is conjectured that this ideal is spanned by elements in a Kazhdan–Lusztig basis (see [G2, L6, T2]).

1.4. Let $C_w = v^{-l(w)} \sum_{y \le w} P_{y,w}(v^2) T_y$, where $P_{y,w}$ are the Kazhdan–Lusztig polynomials. Then the elements C_w ($w \in W$) form an $\mathcal{A}$-basis of H, called a *Kazhdan–Lusztig basis* of H. Define $w \le_{LR} u$ if $a_w \ne 0$ in the expression $hC_uh' = \sum_{z \in W} a_z C_z$ ($a_z \in \mathcal{A}$) for some h, h' in H. This defines a preorder on W. The corresponding equivalence classes are called *two-sided cells* and the preorder gives rise to a partial order $\le_{LR}$ on the set of two-sided cells of W. (See [KL1].) For an element w in W and a two-sided cell c of W we shall write $w \le_{LR} c$ if $w \le_{LR} u$ for some (equivalent any) u in c.

Lusztig established a bijection between the set of G-orbits in $\mathcal{N}$ and the set of two-sided cells of W (see [L4, Theorem 4.8]). Lusztig's bijection preserves the partial orders we have defined: this was conjectured by Lusztig and verified by Bezrukavnikov (see [B, Theorem 4(b)]). Perhaps this bijection is at the heart of the theory of cells in affine Weyl groups; many deep results are related to it. Now we can state the main result of this paper.

Theorem 1.5. *Let $\mathcal{C}$ be a G-orbit in $\mathcal{N}$ and c the two-sided cell of W corresponding to $\mathcal{C}$ under Lusztig's bijection. Then the elements C_w ($w \leq_{LR} c$) form a $\mathbb{Q}[v, v^{-1}]$-basis of $H_{\bar{\mathcal{C}}} \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\bar{\mathcal{C}}$ denotes the closure of $\mathcal{C}$ and $H_{\bar{\mathcal{C}}}$ is the image of the map $(i_{\bar{\mathcal{C}}})_* : K^{G\times\mathbb{C}^*}(Z_{\bar{\mathcal{C}}}) \to K^{G\times\mathbb{C}^*}(Z) = H$.*

Remark. In [B] Bezrukavnikov established a closely related result, which involves affine flag manifolds, derived categories and the Springer resolution (see Theorem 4(a) there). Bezrukavnikov's result deals with canonical left cells and suggests a very nice possible approach to Theorem 1.5. We will discuss this approach in Section 3. I am very grateful to the referee for pointing out this approach.

2. Proof of the theorem

2.1. Before proving the theorem we need to recall some results about representations of an affine Hecke algebra. Let $\mathbf{H} = \mathbb{C}[v, v^{-1}] \otimes_{\mathcal{A}} H$ and for any nonzero complex number q set $\mathbf{H}_q = \mathbf{H} \otimes_{\mathbb{C}[v,v^{-1}]} \mathbb{C}$, where $\mathbb{C}$ is regarded as a $\mathbb{C}[v, v^{-1}]$-algebra by specializing v to a square root of q.

For any G-stable locally closed subvariety Y of $\mathcal{N}$ we set $\mathbf{K}^{G\times\mathbb{C}^*}(Z_Y) = K^{G\times\mathbb{C}^*}(Z_Y) \otimes \mathbb{C}$. If Y is closed, then the inclusion $i_Y : Z_Y \to Z$ induces an injective map $(i_Y)_* : \mathbf{K}^{G\times\mathbb{C}^*}(Z_Y) \hookrightarrow \mathbf{K}^{G\times\mathbb{C}^*}(Z) = \mathbf{H}$. If Y is a closed subset of $\mathcal{N}$, we shall identify $\mathbf{K}^{G\times\mathbb{C}^*}(Z_Y)$ with the image of $(i_Y)_*$, which is a two-sided ideal of $\mathbf{H}$. See [KL2, 5.3] or [L6, Corollary 9.13].

Let s be a semisimple element of G, and n a nilpotent element in $\mathcal{N}$ such that $\mathrm{ad}(s)n = qn$, where q is in $\mathbb{C}^*$. Let $\mathcal{B}_n^s$ be the subvariety of $\mathcal{B}$ consisting of the Borel subalgebras containing n and fixed by s. Then the component group $A(s, n) = C_G(s, n)/C_G(s, n)^o$ of the simultaneous centralizer in G of s and n acts on the total complex homology group $H_*(\mathcal{B}_n^s)$. Let ρ be a representation of $A(s, n)$ appearing in the space $H_*(\mathcal{B}_n^s)$. It is known that if $\sum_{w\in W_0} q^{l(w)} \neq 0$ then the isomorphism classes of irreducible representations of $\mathbf{H}_q$ are in one-to-one correspondence to the G-conjugacy classes of all the triples (s, n, ρ), where $s \in G$ is semisimple, $n \in \mathcal{N}$ satisfies $\mathrm{ad}(s)n = qn$, and ρ is an irreducible representation of $A(s, n)$ appearing in $H_*(\mathcal{B}_n^s)$. See [KL2, X3].

Remark. In the proof of this section we shall often use arguments from [KL2] although the setting there is different from ours. In [KL2] equivariant topological K-homology $K_{\mathrm{top}}(\ )$ is considered, while we consider equivariant algebraic K-theory $K(\ )$. We explain why the arguments of [KL2] work in the present paper. Besides the fact that algebraic K-theory and topological K-theory share many properties (one may compare [KL2] with [Th1, Th2, CG]), the key reason is that $K(\mathcal{B}_n^s) \simeq K_{\mathrm{top}}(\mathcal{B}_n^s)$ and $K(\mathcal{B}_n^s) \otimes \mathbb{C} \simeq K_{\mathrm{top}}(\mathcal{B}_n^s) \otimes \mathbb{C}$, as explained in [L5, p. 80]. (The isomorphisms rely on the results in [DLP].) One may see that the properties of $K_{\mathrm{top}}(\mathcal{B}_n^s) \otimes \mathbb{C}$ play a key role in the arguments of [KL2].

2.2. From now on we assume that q is not a root of 1. Let $L_q(s, n, \rho)$ be an irreducible representation of $\mathbf{H}_q$ corresponding to the triple (s, n, ρ). Kazhdan and Lusztig constructed a standard module $M(s, n, q, \rho)$ over $\mathbf{H}_q$ such that $L_q(s, n, \rho)$ is the unique simple quotient of $M(s, n, q, \rho)$ (see [KL2, 5.12(b) and Theorem 7.12]). We shall write $M_q(s, n, \rho)$ for $M(s, n, q, \rho)$. The following simple fact will be needed.

(a) Let $\mathcal{C}$ be a G-orbit in $\mathcal{N}$. Then the image $H_{\bar{\mathcal{C}}}$ of $(i_{\bar{\mathcal{C}}})_*$ acts on $M_q(s, n, \rho)$ and $L_q(s, n, \rho)$ by zero if n is not in $\bar{\mathcal{C}}$.

Proof. Clearly $Y = \bar{\mathcal{C}} \cup (\overline{G.n} - G.n)$ is closed. If n is not in $\bar{\mathcal{C}}$, then the complement in $X = \bar{\mathcal{C}} \cup \overline{G.n}$ of Y is $G.n$. Recall that $\mathbf{K}^{G\times\mathbb{C}^*}(Z_{Y'})$ is regarded as a two-sided ideal of $\mathbf{H}$ for any closed subset Y' of $\mathcal{N}$ (see 2.1). According to [KL2, 5.3(c), (d) and (e)], the inclusions $i : Y \hookrightarrow X$ and $j : G.n \hookrightarrow X$ induce an exact sequence of $\mathbf{H}$-bimodules

$$0 \to \mathbf{K}^{G\times\mathbb{C}^*}(Z_Y) \to \mathbf{K}^{G\times\mathbb{C}^*}(Z_X) \to \mathbf{K}^{G\times\mathbb{C}^*}(Z_{G.n}) \to 0.$$

Using [KL2, 5.3(e)] we know the inclusion $k : \bar{\mathcal{C}} \to Y$ induces an injective $\mathbf{H}$-bimodule homomorphism $k_* : \mathbf{K}^{G\times\mathbb{C}^*}(Z_{\bar{\mathcal{C}}}) \to \mathbf{K}^{G\times\mathbb{C}^*}(Z_Y)$. Since $M_q(s, n, \rho)$ is a quotient module of $\mathbf{K}^{G\times\mathbb{C}^*}(Z_{G.n})$ (cf. proof of 5.13 in [KL2]), the statement (a) then follows from the exact sequence above.

2.3. Let J_c be the based ring of a two-sided cell c of W, which has a $\mathbb{Z}$-basis $\{t_w \mid w \in c\}$. Let D_c be the set of distinguished involutions in c. For $x, y \in W$, we write $C_xC_y = \sum_{z\in W} h_{x,y,z}C_z,\ h_{x,y,z} \in \mathcal{A}$. The map

$$\varphi_c(C_w) = \sum_{\substack{d\in D_c\\ u\in W\\ a(d)=a(u)}} h_{w,d,u}t_u, \qquad w \in W,$$

defines an $\mathcal{A}$-algebra homomorphism $H \to J_c \otimes_{\mathbb{Z}} \mathcal{A}$, where $a : W \to \mathbb{N}$ is the a-function defined in [L1, 2.1]. The homomorphism φ_c induces a $\mathbb{C}$-algebra homomorphism $\varphi_{c,q} : \mathbf{H}_q \to \mathbf{J}_c = J_c\otimes_{\mathbb{Z}}\mathbb{C}$. If E is a $\mathbf{J}_c$-module, then through $\varphi_{c,q}$, E gets an $\mathbf{H}_q$-module structure, which will be denoted by E_q. See [L2, L3].

Let $\mathcal{C}$ be the nilpotent orbit corresponding to c. According to [L4, Theorems 4.2 and 4.8], the map $E \to E_q$ defines a bijection between the isomorphism classes of simple $\mathbf{J}_c$-modules and the isomorphism classes of standard modules $M_q(s, n, \rho)$ with n in $\mathcal{C}$. The following fact will be needed.

(a) Let c be a two-sided cell of W and $\mathcal{C}$ the corresponding nilpotent class. Let $M_q(s, n, \rho)$ be a standard module with n in a nilpotent class $\mathcal{C}'$. If $C_wM_q(s, n, \rho) \neq 0$ for some $w \in c$, then $\bar{\mathcal{C}}' \subseteq \bar{\mathcal{C}}$.

Proof. Let c' be the two-sided cell corresponding to $\mathcal{C}'$. Then $M_q(s, n, \rho)$ is isomorphic to E_q for some simple $\mathbf{J}_{c'}$-module E. Thus $C_wM_q(s, n, \rho) \neq 0$ implies that $\varphi_{c',q}(C_w)E \neq 0$. So $h_{w,d,u} \neq 0$ for some distinguished involution $d \in c'$ and some $u \in c'$. We then have $c' \leq_{LR} c$. By [B, Theorem 4(b)] we know that $\bar{\mathcal{C}}' \subseteq \bar{\mathcal{C}}$. The statement is proved.

Now we start to prove Theorem 1.5.

2.4. We first show that $H_{\bar{\mathcal{C}}}$ is contained in the two-sided ideal $H^{\leq c}$ of H spanned by all C_w ($w \leq_{LR} c$).

Let $\mathcal{C} = G.n$ and recall that $H_{\bar{\mathcal{C}}}$ stands for the image of $(i_{\bar{\mathcal{C}}})_* : K^{G\times\mathbb{C}^*}(Z_{\bar{\mathcal{C}}}) \to K^{G\times\mathbb{C}^*}(Z) = H$. If $H_{\bar{\mathcal{C}}}$ were not contained in the $\mathcal{A}$-submodule $H^{\leq c}$ of H, we could find $x \in W$ such that $x \not\leq_{LR} c$ and C_x appears in $H_{\bar{\mathcal{C}}}$. (We say that C_x *appears* in $H_{\bar{\mathcal{C}}}$ if there exists an element $\sum_{w\in W} a_w C_w$ ($a_w \in \mathcal{A}$) in $H_{\bar{\mathcal{C}}}$ such that $a_x \neq 0$.) Choose $x \in W$ such that C_x appears in $H_{\bar{\mathcal{C}}}$, $x \not\leq_{LR} c$ and x is highest with respect to the preorder $\leq_{LR}$ and to $H_{\bar{\mathcal{C}}}$ in the following sense: whenever C_w appears in $H_{\bar{\mathcal{C}}}$, then either w and x are in the same two-sided cell or $x \not\leq_{LR} w$. Let c' be the two-sided cell containing x. We then have $c' \not\leq_{LR} c$.

Choose an element $h = \sum_{w\in W} a_w C_w$ ($a_w \in \mathcal{A}$) in $H_{\bar{\mathcal{C}}}$ such that $h_{c'} = \sum_{w\in c'} a_w C_w$ is nonzero. We have $\varphi_{c'}(h) = \varphi_{c'}(h_{c'})$.

We claim that $\varphi_{c'}(h_{c'})$ is nonzero. Let $u \in c'$ be such that a_u has the highest degree (as a Laurent polynomial in v) among all $a_w,\ w \in c'$. Let d be the distinguished involution such that d and u are in the same left cell. It is known that for any distinguished involution d', the degree $h_{w,d',u}$ is less than the degree of $h_{u,d,u}$ if either $w \neq u$ or $d' \neq d$ (see [L2, Theorems 1.8 and 1.10]). Thus the degree of $a_w h_{w,d',u}$ is less than the degree of $a_u h_{u,d,u}$ if either $w \neq u$ or $d' \neq d$. Hence $\varphi_{c'}(h'_c)$ is nonzero.

Clearly, there are only finitely many q such that $\varphi_{c',q}(h_{c'})$ is zero after specializing v to a square root of q. According to [BO, Theorem 4], the ring $\mathbf{J}_{c'}$ is semisimple, that is, its Jacobson radical is zero. So we can find a nonzero q in $\mathbb{C}$ of infinite order and a simple $\mathbf{J}_{c'}$-module E' such that $\varphi_{c',q}(h) = \varphi_{c',q}(h_{c'})$ is nonzero and its action on E' is nonzero.

According to [L4, Theorems 4.2 and 4.8], E'_q is isomorphic to a standard module $M_q(s', n', \rho)$ with n' in the nilpotent orbit $\mathcal{C}'$ corresponding to c'. Since $c' \not\leq_{LR} c$, $\mathcal{C}'$ is not in the closure of $\mathcal{C}$ (see [B, Theorem 4(b)]), so by 2.2(a), the image $H_{\bar{\mathcal{C}}}$ of $(i_{\bar{\mathcal{C}}})_*$ acts on E'_q by zero. This contradicts that the action of $\varphi_{c',q}(h)$ on E' is nonzero. Therefore $H_{\bar{\mathcal{C}}}$ is contained in the two-sided ideal $H^{\leq c}$.

2.5. In this subsection all tensor products are over $\mathbb{Z}$ except when other specifications are given.

Now we show that $H^{\leq c} \otimes \mathbb{Q}$ is equal to $H_{\bar{\mathcal{C}}} \otimes \mathbb{Q}$. If $\mathcal{C}$ is regular, then $\bar{\mathcal{C}}$ is the whole nilpotent cone and the corresponding two-sided cell c contains the neutral element e; in this case, both $H_{\bar{\mathcal{C}}}$ and $H^{\leq c}$ are the whole Hecke algebra.

We use induction on the partial order $\leq_{LR}$ in the set of all two-sided cells of W. Assume that for all c' with $c \leq_{LR} c'$ and $c' \neq c$, we have $H_{\bar{\mathcal{C}'}} \otimes \mathbb{Q} = H^{\leq c'} \otimes \mathbb{Q}$, where $\mathcal{C}'$ is the nilpotent orbit corresponding to c'.

We need to show $H_{\bar{\mathcal{C}}} \otimes \mathbb{Q} = H^{\leq c} \otimes \mathbb{Q}$. Let c' be a two-sided cell different from c such that $c \leq_{LR} c'$ but there is no two-sided cell c'' between c and c', i.e. no c'' such that $c \leq_{LR} c'' \leq_{LR} c'$ and $c \neq c'' \neq c'$.

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{C}(v)$. We first show that $\mathbb{F} \otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}} = \mathbb{F} \otimes_{\mathcal{A}} H^{\leq c}$. Assume this were not true. Note that $\mathbb{F}$ is isomorphic to $\mathbb{C}$ (noncanonically), so we can apply the results in [KL2]. By 2.4 and induction hypothesis, there would exist $w \in c$ such that C_w is contained in $\mathbb{F} \otimes_{\mathcal{A}} H_{\bar{\mathcal{C}'}}$ but not in $\mathbb{F} \otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}}$.

We claim that C_w is not contained in $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}'-\mathcal{C}'}$. Let $\mathcal{C}_i$ $(i=1,\dots,k)$ be nilpotent classes such that $\bar{\mathcal{C}}'-\mathcal{C}'$ is the union of $\bar{\mathcal{C}}_1,\dots,\ \bar{\mathcal{C}}_k$ and $\bar{\mathcal{C}}_i\not\subseteq\bar{\mathcal{C}}_j$ whenever $1\le i\ne j\le k$. By the choice of $\mathcal{C}'$, we have $\mathcal{C}=\mathcal{C}_i$ for some i. It is no harm to assume that $\mathcal{C}=\mathcal{C}_1$. It is known that $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}'-\mathcal{C}'}$ is the sum of all $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_i}$, $1\le i\le k$ (see [KL2, 5.3(e)]). Since $\bar{\mathcal{C}}\not\subseteq\bar{\mathcal{C}}_i$ for $i\ne 1$, by [B, Theorem 4(b)] we know that C_w is not in $H^{\le c_i}$, where c_i is the two-sided cell corresponding to $\mathcal{C}_i$. By 2.4 we see that $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_i}$ $(i\ge 1)$ does not contain C_w. Assume that C_w were contained in $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}'-\mathcal{C}'}$. Then there would exist a subset J of $\{1,\dots,k\}$ and $h_i\in\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_i}$ $(i\in J)$ such that $C_w=\sum_{i\in J}h_i$ and $h_i\notin H_{\bar{\mathcal{C}}_{i'}}$ for different i,i' in J. We may choose such a J so that $\sum_{i\in J}i$ is minimal possible. Let j be the largest number in J. Then $j>1$ since C_w is not contained in $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_1}$ (recall that $\mathcal{C}_1=\mathcal{C}$).

Let $\mathcal{C}_j'$ be a nilpotent class in $\bar{\mathcal{C}}_j$ such that h_j is in $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_j'}$ but not in $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_j'-\mathcal{C}_j'}$. Thus the image in $M_{\mathcal{C}_j'}=\mathbb{F}\otimes_{\mathcal{A}} K^{G\times\mathbb{C}^*}(Z_{\mathcal{C}_j'})=\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}'}/\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_j'-\mathcal{C}_j'}$ of h_j is nonzero. According to [KL2, Corollary 5.9], the action of each nonzero element in $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_j'}\setminus\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_j'-\mathcal{C}_j'}$ on $M_{\mathcal{C}'}$ is nonzero. The argument for [KL2, Proposition 5.13] implies that each nonzero element in $M_{\mathcal{C}_j'}$ would have nonzero image in some standard quotient module of $M_{\mathcal{C}_j'}$. Thus the action of h_j on some standard quotient module $M_{v^2}(s,n_j',\rho')$ of $M_{\mathcal{C}_j'}$ is nonzero, where $n_j'\in\mathcal{C}_j'$. Note that $\bar{\mathcal{C}}_j'\not\subseteq\bar{\mathcal{C}}_i$ for any $i\in J$ with $i\ne j$ since h_j is not in $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}_i}$ if $i\ne j$. By 2.2(a), h_i acts on $M_{v^2}(s,n_j',\rho')$ by zero if $i\ne j$. So $C_wM_{v^2}(s,n_j',\rho')=h_jM_{v^2}(s,n_j',\rho')\ne 0$. By 2.3(a), we get $\bar{\mathcal{C}}_j'\subseteq\bar{\mathcal{C}}=\mathcal{C}_1$. This contradicts that $\sum_{i\in J}i$ is minimal and $j>1$. Therefore C_w is not contained in $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}'-\mathcal{C}'}$.

Thus the image in $M_{\mathcal{C}'}=\mathbb{F}\otimes_{\mathcal{A}} K^{G\times\mathbb{C}^*}(Z_{\mathcal{C}'})=\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}'}/\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}'-\mathcal{C}'}$ of C_w is nonzero. According to [KL2, Corollary 5.9], the action of each nonzero element in $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}'}\setminus\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}'-\mathcal{C}'}$ on $M_{\mathcal{C}'}$ is nonzero. The argument for [KL2, Proposition 5.13] implies that each nonzero element in $M_{\mathcal{C}'}$ would have nonzero image in some standard quotient module of $M_{\mathcal{C}'}$. Thus the action of C_w on some standard quotient module $M_{v^2}(s,n',\rho)$ of $M_{\mathcal{C}'}$ is nonzero, where $n'\in\mathcal{C}'$. According to 2.3(a), we have $\bar{\mathcal{C}}'\subseteq\bar{\mathcal{C}}$. By Theorem 4(b) in [B], we get $c'\le_{LR}c$. This contradicts our assumption $c'\ne c\le_{LR}c'$. So we have $\mathbb{F}\otimes_{\mathcal{A}} H_{\bar{\mathcal{C}}}=\mathbb{F}\otimes_{\mathcal{A}} H^{\le c}$.

Thus for each $w\in c$, we can find a nonzero $a\in\mathbb{F}$ such that aC_w is in $H_{\bar{\mathcal{C}}}$. Clearly, we must have $a\in\mathcal{A}$. Now we show that $\mathbf{K}^{G\times\mathbb{C}^*}(Z_Y)$ is a free $\mathbb{C}[v,v^{-1}]$-module for any G-stable locally closed subvariety Y of $\mathcal{N}$. According to [KL2, 5.3] we may assume that Y is a nilpotent orbit $\mathcal{C}$. It is enough to show that the completion of $\mathbf{K}^{G\times\mathbb{C}^*}(Z_{\mathcal{C}})$ at any semisimple class in $G\times\mathbb{C}^*$ is free over $\mathbb{C}[v,v^{-1}]$. Using [KL2, 5.6] it is enough to show that the right hand side of 5.6(a) in [KL2] is free. This follows from [KL2, (l3)]; the assumption there is satisfied by [KL2, 4.1]. Using [KL2, 5.3] we know that as a free $\mathbb{C}[v,v^{-1}]$-module, $H_{\bar{\mathcal{C}}'}\otimes\mathbb{C}$ is a direct sum of $H_{\bar{\mathcal{C}}}\otimes\mathbb{C}$ and $\mathbf{K}^{G\times\mathbb{C}^*}(Z_{\bar{\mathcal{C}}'-\bar{\mathcal{C}}})$. By assumption, $H_{\bar{\mathcal{C}}'}\otimes\mathbb{Q}=H^{\le c'}\otimes\mathbb{Q}$, thus $H_{\bar{\mathcal{C}}'}\otimes\mathbb{Q}$ is a free $\mathbb{Q}[v,v^{-1}]$-module and contains C_w. These imply that if $aC_w\in H_{\bar{\mathcal{C}}}$ for some nonzero $a\in\mathcal{A}$ then $C_w\in H_{\bar{\mathcal{C}}}\otimes\mathbb{C}$. Therefore we can find a nonzero complex number a such that aC_w is in $H_{\bar{\mathcal{C}}}$. Obviously $a \in \mathbb{Z}$. Thus $H^{\leq c} \otimes \mathbb{Q}$ is contained in $H_{\bar{\mathcal{C}}} \otimes \mathbb{Q}$. By 2.4 we then have $H^{\leq c} \otimes \mathbb{Q} = H_{\bar{\mathcal{C}}} \otimes \mathbb{Q}$. Theorem 1.5 is proved.

3. An approach based on Theorem 4(a) in [B]

In this section we discuss a nice possible approach to the main result of the present paper based on Theorem 4(a) in [B]; this was suggested by the referee. Let Γ be the union of all canonical left cells of W, and I the left ideal of H generated by all C_w, $w \notin \Gamma$. Then $M = H/I$ is the anti-spherical module. Moreover, the images in M of all C_w, $w \in \Gamma$, form a basis of M. For each two-sided cell c of W, let $M_{\leq c}$ be the submodule of M spanned by the images of all C_w, $w \in \Gamma$ and $w \leq_{LR} c$.

According to Arkhipov and Bezrukavnikov (see Subsection 1.1.2 in [AB]), as an H-module, M is isomorphic to $K^{G\times\mathbb{C}^*}(\Lambda)$ (see Subsection 10.1 in [L6] for the definition of the H-module structure on $K^{G\times\mathbb{C}^*}(\Lambda)$). Let $\mathcal{C}$ be the nilpotent class in $\mathcal{N}$ corresponding to the two-sided cell c under Lusztig's bijection. Let $\Lambda_{\bar{\mathcal{C}}} = \{(N, \mathfrak{b}) \in \Lambda \mid N \in \bar{\mathcal{C}}\}$. Then the inclusion $j_{\bar{\mathcal{C}}} : \Lambda_{\bar{\mathcal{C}}} \to \Lambda$ induces an H-module homomorphism $(j_{\bar{\mathcal{C}}})_* : K^{G\times\mathbb{C}^*}(\Lambda_{\bar{\mathcal{C}}}) \to K^{G\times\mathbb{C}^*}(\Lambda)$. A variation of Theorem 4(a) in [B] implies that the image $\mathrm{Im}(j_{\bar{\mathcal{C}}})_*$ of $(j_{\bar{\mathcal{C}}})_*$ is $M_{\leq c}$ if we identify M with $K^{G\times\mathbb{C}^*}(\Lambda)$.

Since each left cell in a two-sided cell has a nonempty intersection with any right cell in the same two-sided cell, we see that for a two-sided cell c, the two-sided ideal $H^{\leq c}$ of H spanned by all C_w ($w \leq_{LR} c$) is the annihilator of $M/M_{\leq c}$.

Let $\mathcal{C}$ be the nilpotent class corresponding to the two-sided cell c. Then naturally one hopes to prove that the image $\mathrm{Im}(i_{\bar{\mathcal{C}}})_*$ of the map $(i_{\bar{\mathcal{C}}})_* : K^{G\times\mathbb{C}^*}(Z_{\bar{\mathcal{C}}}) \to K^{G\times\mathbb{C}^*}(Z) = H$ coincides with the two-sided ideal $H^{\leq c}$ by using the above characterizations for $M_{\leq c}$ and $H^{\leq c}$. A natural way to reach this coincidence is to prove the following two statements:

(a) $K^{G\times\mathbb{C}^*}(\Lambda \setminus \Lambda_{\bar{\mathcal{C}}})$ is isomorphic to $K^{G\times\mathbb{C}^*}(\Lambda)/\mathrm{Im}(j_{\bar{\mathcal{C}}})_*$.
(b) If $x \in K^{G\times\mathbb{C}^*}(Z)$ annihilates $K^{G\times\mathbb{C}^*}(\Lambda)/\mathrm{Im}(j_{\bar{\mathcal{C}}})_*$, then $x \in \mathrm{Im}(i_{\bar{\mathcal{C}}})_*$.

(a) implies that the image $\mathrm{Im}(i_{\bar{\mathcal{C}}})_*$ is in $H^{\leq c}$, and (b) implies that this image contains $H^{\leq c}$.

Unfortunately, the author has not been able to prove these two statements. See comments in Subsection 4.2 for some ideas.

4. Some comments

4.1. If one can show that $K^{G\times\mathbb{C}^*}(Z_{\mathcal{C}})$ is a free $\mathbb{Z}$-module for any nilpotent orbit $\mathcal{C}$, then the argument in 2.5 shows that the image of $(i_{\bar{\mathcal{C}}})_*$ in $H = K^{G\times\mathbb{C}^*}(Z)$ contains $H^{\leq c}$, where c is the two-sided cell corresponding to $\mathcal{C}$. Then Ginzburg's conjecture would be proved. In fact, it seems that one can expect more. More precisely, it is likely the following result is true.

(a) $K^{G\times\mathbb{C}^*}(Z_{\mathcal{C}})$ is a free $\mathcal{A}$-module and $K_1^{G\times\mathbb{C}^*}(Z_{\mathcal{C}})=0$ for all nilpotent orbits $\mathcal{C}$. (We refer to [CG, Section 5.2] and [Q] for the definition of the functor K_i^G.)

If (a) is true, then we also have

(b) The map $(i_{\bar{\mathcal{C}}})_*: K^{G\times\mathbb{C}^*}(Z_{\bar{\mathcal{C}}})\to K^{G\times\mathbb{C}^*}(Z)$ is injective.

We explain some evidence for (a) and prove it for $G = GL_n(\mathbb{C})$, $Sp_4(\mathbb{C})$ and type G_2. Let N be a nilpotent element in $\mathcal{C}$, and $\mathcal{B}_N$ be the variety of Borel subalgebras of $\mathfrak{g}$ containing N. By the Jacobson–Morozov theorem, there exists a homomorphism $\varphi: SL_2(\mathbb{C})\to G$ such that $d\varphi\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)=N$. For z in $\mathbb{C}^*$, let $d_z=\left(\begin{smallmatrix}z&0\\0&z^{-1}\end{smallmatrix}\right)$. Following Kazhdan and Lusztig [KL2, 2.4], we define $Q_N=\{(g,z)\in G\times\mathbb{C}^*\mid \mathrm{ad}(g)N=z^2N\}$. Then Q_N is a closed subgroup of $G\times\mathbb{C}^*$. Let $x=(g,z)\in Q_N$ act on $(G\times\mathbb{C}^*)\times\mathcal{B}_N\times\mathcal{B}_N$ by $x(y,\mathfrak{b},\mathfrak{b}')=(yx^{-1},\mathrm{ad}(g)\mathfrak{b},\mathrm{ad}(g)\mathfrak{b}')$. Then $Z_{\mathcal{C}}$ is isomorphic to the quotient space $Q_N\backslash((G\times\mathbb{C}^*)\times\mathcal{B}_N\times\mathcal{B}_N)$. Thus we have $K_i^{G\times\mathbb{C}^*}(Z_{\mathcal{C}})=K_i^{Q_N}(\mathcal{B}_N\times\mathcal{B}_N)$ (see [KL2, 5.5] and [Th1, Prop. 6.2]). It is known that $Q_\varphi=\{(g,z)\in G\times\mathbb{C}^*\mid g\varphi(x)g^{-1}=\varphi(d_zxd_z^{-1})\text{ for all } x\in SL_2(\mathbb{C})\}$ is a maximal reductive subgroup of Q_N (see [KL2, 2.4(d)]). So we have $K_i^{Q_N}(\mathcal{B}_N\times\mathcal{B}_N)=K_i^{Q_\varphi}(\mathcal{B}_N\times\mathcal{B}_N)$ (see [CG, 5.2.18]).

Let P be the parabolic subgroup of G associated to N (see [DLP, 1.12]). Then we know that the intersection $\mathcal{B}_{N,\mathscr{O}}$ of $\mathcal{B}_N$ with any P-orbit $\mathscr{O}$ on $\mathcal{B}$ is smooth. The torus $\mathcal{D}=\{\varphi(d_z)\mid z\in\mathbb{C}^*\}$ is a subgroup of P and acts on $\mathcal{B}_{N,\mathscr{O}}$, and $\mathcal{B}_{N,\mathscr{O}}$ is a vector bundle over the $\mathcal{D}$-fixed point set $\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}$ (see [DLP, 3.4(d)]). Since the action of Q_φ on $\mathcal{B}_{N,\mathscr{O}}$ commutes with the action of $\mathcal{D}$, according to [BB], this vector bundle is isomorphic to a Q_φ-stable subbundle of $T(\mathcal{B}_{N,\mathscr{O}})|_{\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}}$, where $T(\mathcal{B}_{N,\mathscr{O}})$ is the tangent bundle of $\mathcal{B}_{N,\mathscr{O}}$. Thus the vector bundle is Q_φ-equivariant, so that the computation of $K_i^{Q_\varphi}(\mathcal{B}_N\times\mathcal{B}_N)$ is reduced to the computation of $K_i^{Q_\varphi}(\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}\times\mathcal{B}_{N,\mathscr{O}'}^{\mathcal{D}})$ for various P-orbits $\mathscr{O}$, $\mathscr{O}'$ on $\mathcal{B}$ (see Theorems 2.7 and 4.1 in [Th1], or Theorems 5.4.17 and 5.2.14 in [CG]). Note that $C_\varphi=\{g\varphi(d_z^{-1})\mid(g,z)\in Q_\varphi\}$ is a maximal reductive subgroup of the centralizer $C_G(N)$ of N (see [BV, 2.4]) and the map $(g,z)\mapsto(g\varphi(d_z^{-1}),z)$ defines an isomorphism from Q_φ to $C_\varphi\times\mathbb{C}^*$. Thus we have $K_i^{Q_\varphi}(\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}\times\mathcal{B}_{N,\mathscr{O}'}^{\mathcal{D}})=K_i^{C_\varphi\times\mathbb{C}^*}(\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}\times\mathcal{B}_{N,\mathscr{O}'}^{\mathcal{D}})$. Now the factor $\mathbb{C}^*$ and the group $\mathcal{D}$ act on $\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}\times\mathcal{B}_{N,\mathscr{O}'}^{\mathcal{D}}$ trivially, we therefore have $K_i^{Q_\varphi}(\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}\times\mathcal{B}_{N,\mathscr{O}'}^{\mathcal{D}})=K_i^{C_\varphi}(\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}\times\mathcal{B}_{N,\mathscr{O}'}^{\mathcal{D}})\otimes R_{\mathbb{C}*}$ (see [CG, (5.2.4)], the argument there works for higher K-groups). Note that we have identified $R_{\mathbb{C}*}$ with $\mathcal{A}=\mathbb{Z}[v,v^{-1}]$. Thus the statement (a) is equivalent to the following one.

(c) $K_i^{C_\varphi}(\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}\times\mathcal{B}_{N,\mathscr{O}'}^{\mathcal{D}})$ is a free $\mathbb{Z}$-module for $i=0$ and is 0 for $i=1$.

The statement (c) seems much easier to access. The variety $\mathcal{B}_{N,\mathscr{O}}^{\mathcal{D}}$ and its fixed point set $\mathcal{B}_{N,\mathscr{O}}^{s,\mathcal{D}}$ for any semisimple element s in C_φ are smooth and have good homology properties. See [DLP].

4.2. Replacing Z by Λ, we can state the counterparts of 4.1(a), 4.1(b) and 4.1(c) as follows.

(a) $K^{G\times\mathbb{C}^*}(\Lambda_{\mathcal{C}})$ is a free $\mathcal{A}$-module and $K_1^{G\times\mathbb{C}^*}(\Lambda_{\mathcal{C}})=0$ for all nilpotent orbits $\mathcal{C}$.

If (a) is true, then we have

(b) The map $(i_{\bar{\mathcal{C}}})_*: K^{G\times\mathbb{C}^*}(\Lambda_{\bar{\mathcal{C}}})\to K^{G\times\mathbb{C}^*}(\Lambda)$ is injective.

As in 4.1, the statement (a) is equivalent to the following one:

(c) $K_i^{C_\varphi}(\mathcal{B}_{N,\mathcal{O}}^{\mathcal{D}})$ is a free $\mathbb{Z}$-module for $i=0$ and is 0 for $i=1$.

It is easy to check that the statement (a) implies 3(a). Also (a) is helpful to understand the statement 3(b).

Proposition 4.3. *The statements* 4.1(a) *and* 4.2(a) *are true for $GL_n(\mathbb{C})$, $Sp_4(\mathbb{C})$ and type G_2. In particular, Ginzburg's conjecture is true in these cases.*

Proof. We only need to prove statements 4.1(c) and 4.2(c). For $G=GL_n(\mathbb{C})$, we know that $\mathcal{B}_{N,\mathcal{O}}^{\mathcal{D}}$ has an α-partition into subsets which are affine space bundles over the flag variety $\mathcal{B}'$ of C_φ (see Theorems 2.2 and 2.4(a) in [X2]). In this case, 4.1(a) and 4.2(a) are true since we are reduced to computing $K_i^{C_\varphi}(\mathcal{B}'\times\mathcal{B}')$ and $K_i^{C_\varphi}(\mathcal{B}')$ (cf. [CG, Lemma 5.5.1] and the argument for [L7, Lemma 1.6]). For $G=Sp_4(\mathbb{C})$ or type G_2, we know that $\mathcal{B}_{N,\mathcal{O}}^{\mathcal{D}}$ is either empty or the flag variety of C_φ if N is not subregular (see Prop. 4.2(i) and Section 4.4 in [X2]). In this case, we are also reduced to computing $K_i^{C_\varphi}(\mathcal{B}'\times\mathcal{B}')$ and $K_i^{C_\varphi}(\mathcal{B}')$ (loc.cit.), so 4.1(a) and 4.2(a) are true. If N is subregular, then $\mathcal{B}_N$ is a Dynkin curve and it is easy to see that $\mathcal{B}_{N,\mathcal{O}}^{\mathcal{D}}$ is either a projective line or a finite set (see Prop. 4.2(ii) and Section 4.4 in [X2] for a computable description of $\mathcal{B}_N$). The computation for $K_i^{C_\varphi}(\mathcal{B}_{N,\mathcal{O}}^{\mathcal{D}}\times\mathcal{B}_{N,\mathcal{O}'}^{\mathcal{D}})$ and $K_i^{C_\varphi}(\mathcal{B}_{N,\mathcal{O}}^{\mathcal{D}})$ is easy, they are free $\mathbb{Z}$-modules for $i=0$ (see 4.3(b) and 4.4 in [X2]), and are 0 for $i=1$ (since this is true for a projective line and a finite set). The proposition is proved.

Remark. For $GL_n(\mathbb{C})$, this proposition also provides another proof for the main result of [TX], where results of [T1] are used.

Proposition 4.4. *Assume that C_φ is connected. Then*

(a) *$K^{C_\varphi}(\mathcal{B}_N\times\mathcal{B}_N)$ is a free $\mathbb{Z}$-module.*
(b) *$K^{Q_\varphi}(\mathcal{B}_N\times\mathcal{B}_N)$ is a free $\mathcal{A}$-module. That is, $K^{G\times\mathbb{C}^*}(Z_{G.N})$ is a free $\mathcal{A}$-module.*

Proof. Let T be a maximal torus of C_φ. According to [Th2, (1.11)], we have a split monomorphism $K^{C_\varphi}(\mathcal{B}_N\times\mathcal{B}_N)\to K^T(\mathcal{B}_N\times\mathcal{B}_N)$. Similar to the argument for [L7, Lemma 1.13(d)], we see that $K^T(\mathcal{B}_N\times\mathcal{B}_N)$ is a free R_T-module. (a) follows.

The reasoning for (b) is similar since Q_φ is isomorphic to $C_\varphi\times\mathbb{C}^*$ and the monomorphism $K^{Q_\varphi}(\mathcal{B}_N\times\mathcal{B}_N)\to K^{T\times\mathbb{C}^*}(\mathcal{B}_N\times\mathcal{B}_N)$ is split. The proposition is proved.

Remark. If $G=GL_n(\mathbb{C})$, then all C_φ are connected and have simply connected derived group. In this case $K^{Q_\varphi}(\mathcal{B}_N\times\mathcal{B}_N)$ is a free R_{Q_φ}-module since $R_{Q_\varphi}=R_{C_\varphi}\otimes\mathcal{A}$ and $R_{T\times\mathbb{C}^*}$ is a free $R_{C_\varphi}\otimes\mathcal{A}$-module. Combining this, Subsection 2.4 and the argument in Subsection 2.5 we obtain a different proof of the main result in [TX].

4.5. The K-groups $K^F(\mathcal{B}_N)$ and $K^F(\mathcal{B}_N \times \mathcal{B}_N)$ are important in representation theory of affine Hecke algebras for F being $Q_\varphi,\ C_\varphi$ or a torus of Q_φ (see [KL2, L7]). For the nilpotent element N, in [L4, 10.5] Lusztig conjectured that there exists a finite C_φ-set Y which plays a key role in understanding the based ring of the two-sided cell corresponding to $G.N$. It seems that as R_{C_φ}-modules, $K^{C_\varphi}(Y)$ and $K^{C_\varphi}(Y \times Y)$ are isomorphic to $K^{C_\varphi}(\mathcal{B}_N)$ and $K^{C_\varphi}(\mathcal{B}_N \times \mathcal{B}_N)$ respectively. Let $X = \mathcal{B}_N$ or $\mathcal{B}_N \times \mathcal{B}_N$. In view of [L4, 10.5] one may hope to find a canonical $\mathbb{Z}$-basis of $K^{C_\varphi}(X)$ and a canonical $\mathcal{A}$-basis of $K^{Q_\varphi}(X)$ in the spirit of [L6, L7]. Moreover, there should exist a natural bijection between the elements of the canonical basis of $K^F(\mathcal{B}_N \times \mathcal{B}_N)$ ($F = C_\varphi$ or Q_φ) and the elements of the two-sided cell corresponding to $G.N$.

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