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Differences of random Cantor sets and lower spectral radii

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Abstract. We investigate the question under which conditions the algebraic difference between two independent random Cantor sets C_1 and C_2 almost surely contains an interval, and when not. The natural condition is whether the sum $d_1 + d_2$ of the Hausdorff dimensions of the sets is smaller (no interval) or larger (an interval) than 1. Palis conjectured that *generically* it should be true that $d_1 + d_2 > 1$ should imply that $C_1 - C_2$ contains an interval. We prove that for 2-adic random Cantor sets generated by a vector of probabilities (p_0, p_1) the interior of the region where the Palis conjecture does not hold is given by those p_0, p_1 which satisfy $p_0 + p_1 > \sqrt{2}$ and $p_0 p_1 (1 + p_0^2 + p_1^2) < 1$. We furthermore prove a general result which characterizes the interval/no interval property in terms of the lower spectral radius of a set of 2×2 matrices.

1. Introduction

The *algebraic difference* of two sets A, B of real numbers is defined as

$$A - B := \{x - y : x \in A, y \in B\}.$$

An interesting situation arises when A and B are relatively small, and $A - B$ large. For example, A and B are Cantor sets, but $A - B$ contains an interval. Whether this will happen or not depends on the size of A and B . For instance, if the sum of the Hausdorff dimensions of A and B is smaller than 1, then $A - B$ will have a Hausdorff dimension smaller than 1, and cannot contain an interval. A well known conjecture by Palis ([Pal87]) states that—conversely—if

$$\dim_{\text{H}} A + \dim_{\text{H}} B > 1, \tag{1}$$

then *generically* it should be true that $A - B$ contains an interval. In this paper we will follow [Lar90] and [DS08] and interpret ‘generically’ as ‘almost surely’ with respect to a probability measure. The central question in this paper is:

Under which conditions does the algebraic difference between two independent random Cantor sets almost surely contain an interval, and when not?

Here we will consider a canonical class of random Cantor sets, which randomize the classical triadic Cantor set C in a natural way. In this introduction we will give a loose

description. Instead of discarding the middle interval and keeping the left and the right interval at every step in the construction of C by decreasing intersections of unions of triadic intervals, we do the following: fix three numbers p_0 , p_1 and p_2 between 0 and 1. Then at every step, retain the left interval with probability p_0 (discard it with probability $1 - p_0$), the middle with probability p_1 and the right interval with probability p_2 , independently of each other, and of the actions in other intervals at all levels. (See also Figure 1, where we use trees to describe this recursive construction).

More generally we consider the M -adic case for integers $M = 2, 3, \dots$, where intervals are recursively divided into M subintervals of equal length, which are retained with survival probabilities p_0, \dots, p_{M-1} .

It turns out that the cyclic correlation coefficients γ_k , defined by

$$\gamma_k := \sum_{i=0}^{M-1} p_i p_{i+k}, \quad (2)$$

for $k = 0, \dots, M - 1$ play an important role (here the indices $i + k$ should be taken modulo M). Indeed, the main result in [DS08] is the following.

Theorem 1.1 ([DS08]). *Consider two independent random Cantor sets F_1 and F_2 with survival probabilities p_0, \dots, p_{M-1} .*

- (a) *If $\gamma_k > 1$ for all $k = 0, \dots, M - 1$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.*
- (b) *If $\gamma_k, \gamma_{k+1} < 1$ for some k , then $F_1 - F_2$ contains no interval a.s.*

This implies that if F_1 and F_2 are two independent copies of the random triadic Cantor set described above, then their difference $F_1 - F_2$ contains an interval a.s. if $p_0 p_1 + p_1 p_2 + p_2 p_0 > 1$, and does not contain an interval a.s. if $p_0 p_1 + p_1 p_2 + p_2 p_0 < 1$. However, the triadic case is special, and Theorem 1.1 gives only a partial solution to, e.g., the dyadic case, where intervals are split in two all the time, and retained with probability p_0 and p_1 . Here Theorem 1.1 merely implies that $F_1 - F_2$ contains an interval a.s. if $2p_0 p_1 > 1$, and does not contain an interval if $p_0^2 + p_1^2 < 1$. In Section 8 we will fill the gap, and completely classify dyadic random Cantor sets with respect to this property (except on the separating curve).

The tool that is used in the proof of this result is that of higher order Cantor sets, which has been introduced in [DS08]. The same tool will enable us in Section 7 to obtain a general classification result in terms of the lower spectral radius of a certain set of matrices.

A major problem that arises is that the ‘independent interval’ property gets lost if one passes to higher order Cantor sets. It is therefore important (and not just for the sake of generalization) to consider a more complex mechanism to generate random Cantor sets. The obvious way to allow for dependence is to define a joint survival distribution μ on the set of all subsets of $\{0, \dots, M - 1\}$. In Section 4 we give a version of Theorem 1.1 for this case.

Finally we remark that the recent paper [MSS09] considers the weaker property of the difference $F_1 - F_2$ having positive Lebesgue measure. This time it is not the minimal γ_k

which is determining, but the *product* of the γ_k 's. Thus one obtains interesting examples of differences of Cantor sets which have positive Lebesgue measure, but do not contain an interval.

2. Construction

The construction of M -adic Cantor sets is intimately related to M -ary trees and M -ary expansions of numbers.

Let $M \geq 2$ be an integer. An M -ary tree is a tree in which every node has precisely M children. The nodes are conveniently identified with strings over an alphabet of size M ; we use the alphabet $\mathbb{A} := \{0, \dots, M - 1\}$.

Strings over \mathbb{A} of length n are denoted as $i_n = i_1 \dots i_n$, where $i_1, \dots, i_n \in \mathbb{A}$. The empty string is denoted by \emptyset and has length 0. The concatenation of strings $i_1 \dots i_n$ and $j_1 \dots j_n$ is simply denoted by $i_1 \dots i_n j_1 \dots j_n$.

The M -ary tree \mathcal{T} is defined as the set of all strings over the alphabet \mathbb{A} . The root node is the empty string \emptyset . The *children* of each node $i_1 \dots i_n \in \mathcal{T}$ are the nodes $i_1 \dots i_n i_{n+1}$ for all $i_{n+1} \in \mathbb{A}$. The *level* of a node corresponds to its length as a string. For each $n \geq 0$, the set of all nodes at level n is denoted by \mathcal{T}_n . We thus have

$$\mathcal{T} = \bigcup_{n \geq 0} \mathcal{T}_n = \bigcup_{n \geq 0} \bigcup_{i_1 \in \mathbb{A}} \dots \bigcup_{i_n \in \mathbb{A}} \{i_1 \dots i_n\}.$$

Strings over the alphabet \mathbb{A} can also be interpreted as M -ary expansions of numbers. For all $i_1 \dots i_n \in \mathcal{T}$ we let $[i_1 \dots i_n]_M$ denote the value of $i_1 \dots i_n$ as an M -ary number:

$$[i_1 \dots i_n]_M := \sum_{k=1}^n M^{n-k} i_k. \tag{3}$$

Consequently, $[i_1 \dots i_n]_M$ takes its value in the range $0, \dots, M^n - 1$.

2.1. Random Cantor sets

We consider the construction of *random M -adic Cantor sets* on the interval $[0, 1]$. The construction is iterative: we start with the entire interval $[0, 1]$, and at each level of the construction, the intervals surviving so far are ‘split’ into M equally sized closed subintervals, of which a certain random subset is allowed to survive at the next level. The random Cantor set is a stochastic object which consists of those points in $[0, 1]$ that persist at all levels. Here, when we speak of ‘splitting’ a closed set into some smaller closed sets, it should be understood that the smaller sets need not be disjoint, but their interiors are.

We consider the probability measure \mathbb{P}_μ on the space of $\{0, 1\}$ -labelled trees $\{0, 1\}^{\mathcal{T}}$, where we label each node $i_1 \dots i_n \in \mathcal{T}$ with $X_{i_1 \dots i_n} \in \{0, 1\}$ and μ is a probability measure on $2^{2^{\mathbb{A}}}$ called the *joint survival measure*. It is of course determined by its restriction

to $2^{\mathbb{A}}$, which we also denote μ , and call the *joint survival distribution*. The measure \mathbb{P}_μ is defined by requiring that $\mathbb{P}_\mu(X_\emptyset = 1) = 1$ and that for all $i_1 \dots i_n \in \mathcal{T}$ the random sets

$$\{i_{n+1} \in \mathbb{A} : X_{i_1 \dots i_n i_{n+1}} = 1\} \tag{4}$$

are independent and identically distributed according to μ .

The n^{th} level M -adic subintervals of $[0, 1]$ are defined by

$$I_{i_1 \dots i_n} := \frac{1}{M^n} [[i_1 \dots i_n]_M, [i_1 \dots i_n]_M + 1] \tag{5}$$

for all $i_1 \dots i_n \in \mathcal{T}$. The n^{th} level intervals that survive in the n^{th} level approximation of the random Cantor set are the ones that are indexed by the nodes in the *level n survival set*

$$S_n := \{i_1 \dots i_n : X_{i_1} = X_{i_1 i_2} = \dots = X_{i_1 \dots i_n} = 1\} \tag{6}$$

for all $n \geq 0$. The random Cantor set F is given as the intersection of all its n^{th} level approximations, which we denote by F^n :

$$F := \bigcap_{n=0}^{\infty} F^n = \bigcap_{n=0}^{\infty} \bigcup_{i_1 \dots i_n \in S_n} I_{i_1 \dots i_n}.$$

An important property of random Cantor sets is their self-similarity: conditional on the survival of any n^{th} level M -adic interval, the process starting at that interval (scaled by M^n) has the same distribution as the whole process, which starts at $I_\emptyset = [0, 1]$.

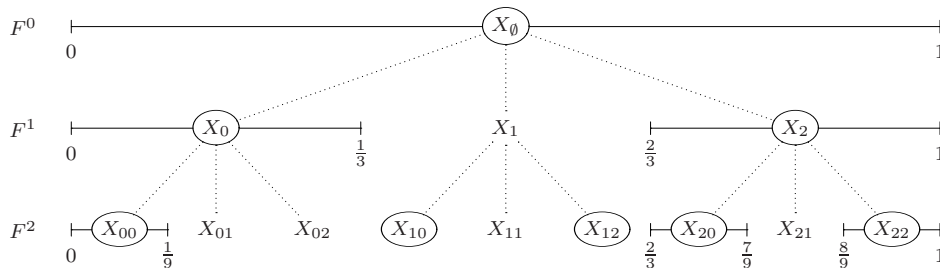


Fig. 1. The first three levels of a realization of a tree labelled by $(X_{i_1 \dots i_n})$ with $\mathbf{p} = (1, 0, 1/2)$, with the surviving intervals in the approximations F^n . The encircled $X_{i_1 \dots i_n}$ correspond to nodes obtaining the value 1.

The vector of *marginal probabilities* $\mathbf{p} := (p_0, \dots, p_{M-1})$ is defined by

$$p_i := \mathbb{P}_\mu(X_i = 1) \tag{7}$$

for all $i \in \mathbb{A}$. Note that these marginal probabilities do not need to sum up to 1. The joint survival distribution μ can be chosen such that the $X_i, i \in \mathbb{A}$, are M independent

Bernoulli variables; the respective probabilities of success then equal the marginal probabilities p_0, \dots, p_{M-1} . In this case we call F an ‘independent interval’ Cantor set.

The traditional deterministic triadic (so $M = 3$) Cantor set is obtained with the measure μ defined by $\mu(\{0, 2\}) = 1$. Its vector of marginal probabilities is $\mathbf{p} = (1, 0, 1)$.

The number $\#S_n$ of level n intervals selected in F^n is a branching process with, as offspring distribution, the distribution of $\#S_1$. Since the F^n are non-increasing, $F = \emptyset$ if and only if the branching process $(\#S_n)$ dies out. Since $\mathbb{E}_\mu[\#S_1] = p_0 + \dots + p_{M-1} = \|\mathbf{p}\|_1$, it follows that $F \neq \emptyset$ with positive probability if and only if

$$\|\mathbf{p}\|_1 > 1 \quad \text{or} \quad \mathbb{P}_\mu(\#S_1 = 1) = 1.$$

Discarding the uninteresting case on the right, we will assume henceforth that

$$\|\mathbf{p}\|_1 > 1. \tag{8}$$

2.2. Algebraic difference

We consider the algebraic difference $F_1 - F_2$ between two independent random M -adic Cantor sets F_1 and F_2 . In general, we denote the joint survival distribution of F_1 by μ and that of F_2 by λ . The corresponding marginal distributions will be denoted by \mathbf{p} and \mathbf{q} respectively. In Section 6 and further we will restrict ourselves to the *symmetric case*, where $\mathbf{p} = \mathbf{q}$. The algebraic difference $F_1 - F_2$ can be seen as a 45° projection of the Cartesian product $F_1 \times F_2$. Thus $F_1 - F_2$ is defined on the product space of the probability spaces of F_1 and F_2 . We will use $\mathbb{P} := \mathbb{P}_\mu \times \mathbb{P}_\lambda$ to denote the corresponding product measure and \mathbb{E} to denote expectations with respect to this probability.

3. Triangles and expectations

Let F_1 and F_2 be two independent M -adic random Cantor sets with joint survival distributions μ and λ , respectively. Denote by F_1^n and F_2^n their n^{th} level approximations ($n \geq 0$) and define the following subsets of the unit square $[0, 1]^2$:

$$\Lambda^n := F_1^n \times F_2^n, \quad n \geq 0, \quad \Lambda := F_1 \times F_2 = \bigcap_{n=0}^\infty \Lambda^n.$$

Note that as $F_1^n \downarrow F_1$ and $F_2^n \downarrow F_2$, also $\Lambda^n \downarrow \Lambda$. Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the 45° projection given by $\phi(x, y) = x - y$; then $F_1 - F_2 = \phi(\Lambda)$. As ϕ is a continuous function and $\{\Lambda^n\}_{n=0}^\infty$ is a non-increasing sequence of compact sets, it follows that the algebraic difference $F_1 - F_2$ can be written as

$$F_1 - F_2 = \phi(\Lambda) = \phi\left(\bigcap_{n=0}^\infty \Lambda^n\right) = \bigcap_{n=0}^\infty \phi(\Lambda^n) = \bigcap_{n=0}^\infty \phi(F_1^n \times F_2^n) = \bigcap_{n=0}^\infty (F_1^n - F_2^n).$$

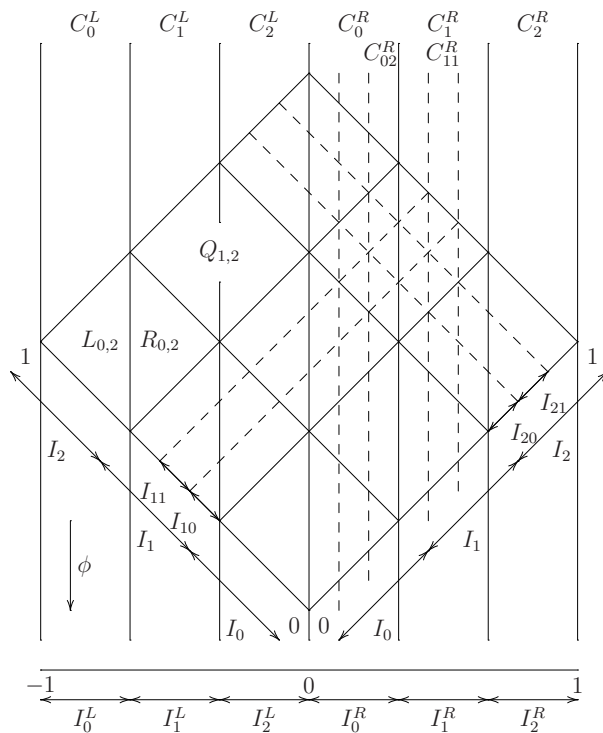


Fig. 2. An illustration for $M = 3$ of the unit square $[0, 1]^2$ rotated by 45° , being projected by ϕ to a $\sqrt{2}$ -scaled-down version of $[-1, 1]$. The columns $C_{k_n}^U$ split the n^{th} level squares $Q_{i_n, j_n} = I_{i_n} \times I_{j_n}$ into the ‘left’ and ‘right’ triangles L_{i_n, j_n} and R_{i_n, j_n} .

3.1. Squares, columns and triangles

The Λ^n are unions of M -adic squares

$$Q_{i_1 \dots i_n, j_1 \dots j_n} := I_{i_1 \dots i_n} \times I_{j_1 \dots j_n}$$

with $i_1 \dots i_n, j_1 \dots j_n \in \mathcal{T}_n$ and $n \geq 0$. See Figure 2 for a graphical representation of these M -adic squares and their ϕ -projections. Observe that the projections $\phi(Q_{i_1 \dots i_n, j_1 \dots j_n})$ are equal to unions of two subsequent level n M -adic intervals in $[-1, 1]$.

In order to be able to represent the M -adic intervals that are in $[-1, 0]$, we generalize our notation of M -adic intervals on $[0, 1]$ to the entire real line. For any $n \geq 0$ and $k \in \mathbb{Z}$ we define

$$I_n(k) := \frac{1}{M^n} [k, k + 1] = \left[\frac{k}{M^n}, \frac{k + 1}{M^n} \right]. \tag{9}$$

Note that $I_n(k + M^n i) = I_{k_1 \dots k_n} + i$ for all $k = [k_1 \dots k_n]_M$ and $i \in \mathbb{Z}$. See also Figure 3. The inverse images of these intervals under ϕ form diagonal ‘columns’ in the plane \mathbb{R}^2 ,

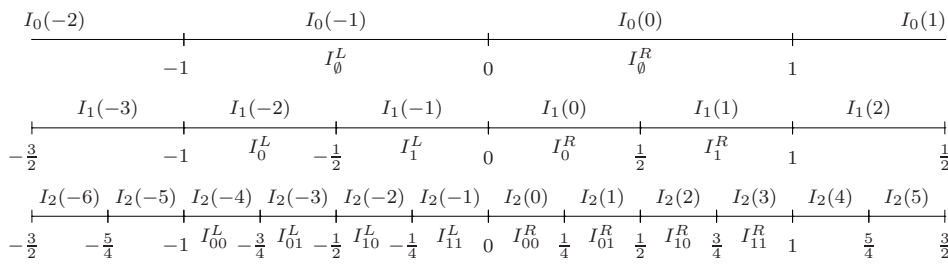


Fig. 3. Two ways of enumerating M -adic intervals (with $M = 2$) — $I_n(i)$, $i \in \mathbb{Z}$, and $I_{\underline{k}_n}^U$, with $U \in \{L, R\}$ and $\underline{k}_n \in \mathcal{T}$. Here $n = 0, 1, 2$.

denoted by

$$C_n(k) := \phi^{-1}(I_n(k))$$

for all $k \in \mathbb{Z}$.

When rotating the unit square $[0, 1]^2$ by 45° , as in Figure 2, the columns with ϕ -image in $[-1, 0]$ intersect with the ‘left’ half of the unit square and those with ϕ -image in $[0, 1]$ intersect with the ‘right’ half. For this reason we distinguish between ‘left’ and ‘right’ M -adic intervals and columns by defining, for any $k_1 \dots k_n \in \mathcal{T}$,

$$I_{k_1 \dots k_n}^L := I_{k_1 \dots k_n} - 1 = I_n([k_1 \dots k_n]_M - M^n), \quad I_{k_1 \dots k_n}^R := I_{k_1 \dots k_n} = I_n([k_1 \dots k_n]_M),$$

$$C_{k_1 \dots k_n}^L := C_n([k_1 \dots k_n]_M - M^n), \quad C_{k_1 \dots k_n}^R := C_n([k_1 \dots k_n]_M).$$

In fact, any n^{th} level M -adic square $Q_{i_1 \dots i_n, j_1 \dots j_n}$ is split into a ‘left’ and a ‘right’ triangle by the M -adic columns. These triangles are called L -triangles and R -triangles, and are denoted by

$$L_{i_1 \dots i_n, j_1 \dots j_n} := Q_{i_1 \dots i_n, j_1 \dots j_n} \cap C_n([i_1 \dots i_n]_M - [j_1 \dots j_n]_M - 1),$$

$$R_{i_1 \dots i_n, j_1 \dots j_n} := Q_{i_1 \dots i_n, j_1 \dots j_n} \cap C_n([i_1 \dots i_n]_M - [j_1 \dots j_n]_M),$$
(10)

for any $i_1 \dots i_n, j_1 \dots j_n \in \mathcal{T}$.

3.2. Triangle counts

For all $U, V \in \{L, R\}$ and $\underline{k}_n \in \mathcal{T}$ we let

$$Z^{UV}(\underline{k}_n) := \#\{(i_n, \underline{j}_n) : Q_{i_n, \underline{j}_n} \subseteq \Lambda^n, V_{i_n, \underline{j}_n} \subseteq C_{\underline{k}_n}^U\}$$

denote the number of level n V -triangles in $\Lambda^n \cap C_{\underline{k}_n}^U$. Note that these V -triangles have been generated by the level 0 U -triangle. We also denote the total number of V -triangles in columns $C_{\underline{k}_n}^L$ and $C_{\underline{k}_n}^R$ together by

$$Z^V(\underline{k}_n) := Z^{LV}(\underline{k}_n) + Z^{RV}(\underline{k}_n),$$

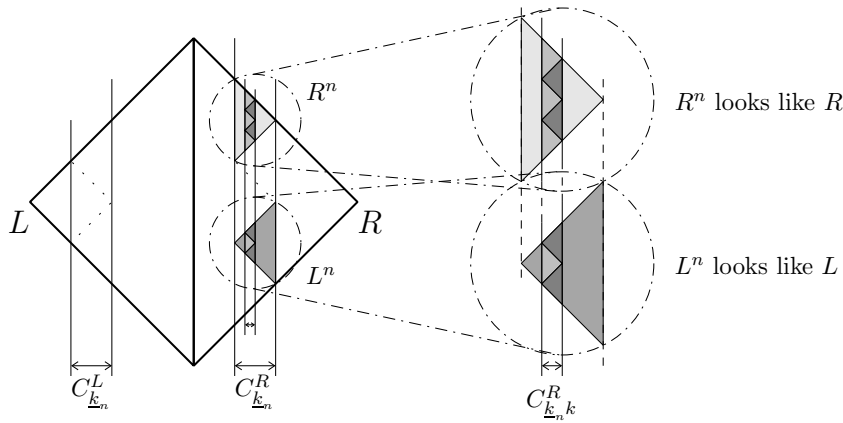


Fig. 4. A comparison of a level n left triangle L^n and right triangle R^n with the level 0 left triangle L and right triangle R . In particular, the expected number of (level $n + 1$) V -triangles in the intersection of the subcolumn $C_{k_n}^R$ with the triangles L^n and R^n together equals $\mathbb{E}Z^V(k)$, the expected number of (level 1) V -triangles in C_k^L and C_k^R together.

for all $k_n \in \mathcal{T}$. These triangle counts (and their self-similarity property) are illustrated in Figure 4.

An important observation is that an M -adic interval $I_{k_n}^U$ is absent in $\phi(\Lambda^n)$ exactly when there are no triangles in the corresponding column $C_{k_n}^U$ in Λ^n :

$$I_{k_n}^U \not\subseteq \phi(\Lambda^n) \Leftrightarrow Z^{UL}(k_n) = Z^{UR}(k_n) = 0. \tag{11}$$

The triangle counts $Z^{UV}(k_n)$, with k_1, k_2, \dots a fixed path, constitute a two-type branching process in a varying environment with interaction: the interaction comes from the dependence between triangles that are *aligned*, i.e., triangles contained in respective squares $Q_{i_1 \dots i_n, j_1 \dots j_n}$ and $Q_{i'_1 \dots i'_n, j'_1 \dots j'_n}$ with $i_1 \dots i_n = i'_1 \dots i'_n$ or $j_1 \dots j_n = j'_1 \dots j'_n$. The *expectation matrices* of the two type branching process are given by

$$\mathcal{M}(k_n) := \begin{bmatrix} \mathbb{E}Z^{LL}(k_n) & \mathbb{E}Z^{LR}(k_n) \\ \mathbb{E}Z^{RL}(k_n) & \mathbb{E}Z^{RR}(k_n) \end{bmatrix}, \tag{12}$$

where $k_n \in \mathcal{T}$. These matrices satisfy the basic relation

$$\mathcal{M}(k_1 \dots k_n) = \mathcal{M}(k_1) \cdots \mathcal{M}(k_n) \tag{13}$$

for all $k_1 \dots k_n \in \mathcal{T}$.

3.3. Correlation coefficients

Define the *cyclic cross-correlation coefficients*

$$\gamma_k := \sum_{i=0}^{M-1} q_i p_{i+k}, \tag{14}$$

where the indices of p should be taken modulo M , and $k \in \mathbb{A}$.

This definition is extended to \mathbb{Z} by setting $\gamma_{k+iM} := \gamma_k$ for all $i \in \mathbb{Z}$. For the symmetric case $\mathbf{p} = \mathbf{q}$, these coefficients are called the *cyclic auto-correlation coefficients*. For brevity, however, we will use the shorter term *correlation coefficients*. The smallest correlation coefficient value is denoted by

$$\gamma := \min_{k \in \mathbb{A}} \gamma_k. \tag{15}$$

Lemma 3.1 below motivates the definition of the correlation coefficients, as they are in fact the triangle count expectations $\mathbb{E}Z^V(k)$, $V \in \{L, R\}$.

For convenience, we define the vector $\mathbf{e} := [1 \ 1]$.

Lemma 3.1 ([DS08]). *For all $k \in \mathbb{A}$ we have*

$$\mathbf{e}\mathcal{M}(k) = [1 \ 1]\mathcal{M}(k) = [\mathbb{E}Z^L(k) \ \mathbb{E}Z^R(k)] = [\gamma_{k+1} \ \gamma_k].$$

Proof. As in [DS08] this follows from (10) with $n = 1$,

$$\mathbb{P}(Q_{i,j} \subseteq \Lambda^1) = \mathbb{P}(I_i \subseteq F_1^1, I_j \subseteq F_2^1) = \mathbb{P}_\mu(I_i \subseteq F_1^1)\mathbb{P}_\lambda(I_j \subseteq F_2^1) = p_i q_j$$

and some careful bookkeeping. □

A property that in general holds *only* for the symmetric case (i.e., $\mathbf{p} = \mathbf{q}$) is that γ_0 is the largest of the auto-correlation coefficients:

$$0 \leq \gamma_k \leq \gamma_0. \tag{16}$$

This follows easily from the Cauchy–Schwarz inequality.

4. The basic result with joint survival distributions

In this section we generalize Theorem 1.1 of [DS08] to joint survival distributions, *and* to the asymmetric case.

In the setting of general joint survival distributions μ and λ , the condition stated below is sufficient for the theorem to hold.

For a joint survival distribution $\mu : 2^{\mathbb{A}} \rightarrow [0, 1]$ we define its *marginal support* by

$$\text{Supp}_m(\mu) := \bigcup \{S \subseteq \mathbb{A} : \mu(S) > 0\}. \tag{17}$$

In other words, the marginal support is the set of $i \in \mathbb{A}$ for which $p_i = \mathbb{P}_\mu(X_i = 1) > 0$. For example, take $M = 4$ and μ defined by $\mu(\{0, 3\}) = \mu(\{1, 3\}) = \mu(\{3\}) = 1/3$; then $\text{Supp}_m(\mu) = \{0, 1, 3\}$.

Condition 4.1. A joint survival distribution $\mu : 2^{\mathbb{A}} \rightarrow [0, 1]$ satisfies the *joint survival condition (JSC)* if it assigns a positive probability to its marginal support: $\mu(\text{Supp}_m(\mu)) > 0$.

In the ‘independent interval’ case, μ satisfies the joint survival condition since in that case $\mu(\text{Supp}_m(\mu)) = \prod_{i \in \text{Supp}_m(\mu)} p_i > 0$. In the example above, where $\text{Supp}_m(\mu) = \{0, 1, 3\}$, the JSC is not satisfied.

Theorem 4.1. Consider two independent random Cantor sets F_1 and F_2 whose joint survival distributions satisfy Condition 4.1, the joint survival condition.

- (a) If $\gamma_k > 1$ for all $k \in \mathbb{A}$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.
- (b) If $\gamma_k, \gamma_{k+1} < 1$ for some $k \in \mathbb{A}$, then $F_1 - F_2$ contains no interval a.s.

We will give a proof of this result in the next section.

The joint survival condition is not a necessary condition. Here is an example where the JSC does not hold, but where $F_1 - F_2$ contains an interval: take $M = 5$, and $\mu(\{0, 1, 2, 3\}) = p = 1 - \mu(\{1, 2, 3, 4\})$ where p is arbitrary between 0 and 1. Then $\gamma = 3$ for all p , but the JSC does not hold. However, any realisation Λ^n contains the *deterministic* set $\tilde{\Lambda}^n$ generated by $\tilde{\mu}$ defined by $\tilde{\mu}(\{1, 2, 3\}) = 1$. A simple geometric analysis shows that $\tilde{F} = [-1/2, 1/2]$, and hence F must contain this interval. (An algebraic alternative is to use Theorem 2 of [DS08]: the collection of reduced matrices of $\tilde{\mu}$ is $\{T_3, T_9, T_{12}\}$, and since this set is closed under (mutual) multiplications, this theorem tells us that \tilde{F} will contain an interval.) See [DD10] for a weaker condition than the joint survival condition under which Theorem 4.1 still holds.

5. Proof of Theorem 4.1

A proof for the symmetric case and ‘independent interval’ Cantor sets is given in [DS08]. An extension of the proof to the asymmetric case and general survival distributions satisfying the joint survival condition is easy for part (b) (where, moreover, the JSC is not needed), but for part (a) several complications arise. The proof for this part is based on the following observations. The process of n^{th} level M -adic squares that are surviving in the level n approximations Λ^n inherits the self-similarity property of the individual random Cantor sets F_1 and F_2 : conditional on the survival of an n^{th} level M -adic square $Q_{i_1 \dots i_n, j_1 \dots j_n}$, the (scaled) process starting at this surviving square has the same distribution as the whole process, which starts at $[0, 1]^2$. Moreover, conditional on the survival of a set of n^{th} level M -adic squares that are pairwise *unaligned*, the processes in each of these squares are independent.

The columns behave very inhomogeneously: for every n there are columns which contain at most *one* triangle. This observation led to the idea in [DS08] to pair unaligned left and right triangles that survive in the *same* column into what are called Δ -pairs. The main idea of the proof for part (a) is to show that with positive probability a Δ -pair will occur in some column C of some Λ^m (Lemma 5.2), and that conditional on this, the pairs in all subcolumns will grow exponentially, so the projection of the Δ -pairs within $C \cap \Lambda$ will be an M -adic interval (Lemma 5.4). We will follow the structure of the proof in [DS08]; each lemma there corresponds to a subsection with lemma here.

5.1. Joint growth of Δ -pairs

For the ' k_n th subcolumn of a level m Δ -pair' (L^m, R^m) that is contained in a column $C = C_{L_m}^U$ we will write

$$C_{L_m k_n}^U \cap (L^m \cup R^m)$$

for the intersection of the Δ -pair with $C_{L_m k_n}^U$, the k_n th subcolumn of $C_{L_m}^U$.

For such a Δ -pair (L^m, R^m) , the distribution of the number of level $m+n$ V -triangles surviving in Λ^{m+n} in the k_n th subcolumn of (L^m, R^m) , conditional on the survival of (L^m, R^m) in Λ^m , is independent of m , the particular choice of the column C and the Δ -pair in it. Therefore, we can unambiguously denote a random variable having this distribution by

$$\tilde{Z}^V(k_n) \tag{18}$$

for all $V \in \{L, R\}$ and $k_n \in \mathcal{T}$.

In general $\tilde{Z}^V(k_n)$ does not have the distribution of $Z^V(k_n)$ because there is possible dependence between the offspring generation of the two level 0 triangles, whereas there is no dependence between the offspring generation of the L -triangle and the R -triangle of a level m Δ -pair by unalignedness. Essentially, $\tilde{Z}^V(k_n)$ has the distribution of the sum of independent copies of the random variables $Z^{RV}(k_n)$ and $Z^{LV}(k_n)$, but $Z^V(k_n)$ is just the sum of these random variables, which may be dependent. (Compare the level 0 triangles with the Δ -pair in Figure 5.) Clearly, the $Z^{UV}(k_n)$ are dependent if they count triangles that are aligned, e.g., $Z^{RR}(k_n)$ and $Z^{LL}(k_n)$ will in general not be independent. However,

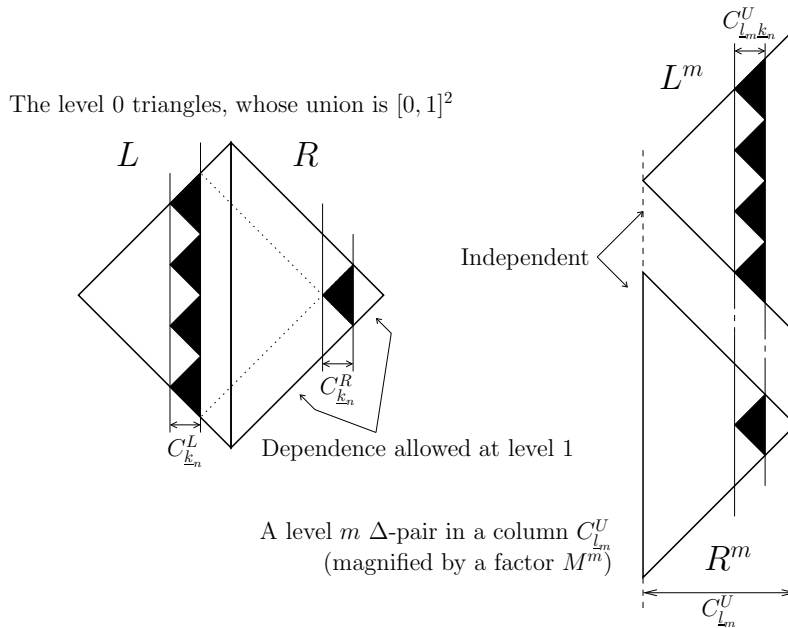


Fig. 5. The distinction between level 0 triangles and a Δ -pair.

$Z^{RV}(k_n)$ and $Z^{LV}(k_n)$ will never be aligned. But there can also be dependence between the offspring of level 0 squares—if that is dictated by the joint survival distributions—which induces dependence between the random variables $Z^{UV}(k_n)$, at any level n . See Figure 5. Despite these differences, both do have the same expected value (by linearity of expectations), and this is all that is needed in the proof.

Let

$$\tilde{N}(k_n) = \min\{\tilde{Z}^L(k_n), \tilde{Z}^R(k_n)\} \tag{19}$$

for all $k_n \in \mathcal{T}$. This is the distribution of the minimum number of triangles of each triangle type that survive in the k_n th subcolumn of a Δ -pair. The next lemma states that with positive probability the growth of $\tilde{N}(k_n)$ for all k_n up to a certain level n is exponential.

Lemma 5.1 (Extension of Lemma 1 in [DS08]). *If $\gamma > 1$, and the joint survival distribution(s) satisfy the joint survival condition, then for all $n \geq 0$,*

$$\mathbb{P}(\tilde{N}(k_m) \geq \gamma^m \text{ for all } k_m \in \mathcal{T}_m \text{ for all } 0 \leq m \leq n) > 0.$$

Proof. The proof is similar to the proof of Lemma 1 in [DS08], but it shows where and how the JSC emerges for general survival distributions.

For a joint survival distribution μ we uniquely define the joint survival distribution μ^* by requiring that $\mu^*(\text{Supp}_m(\mu)) = 1$. (See (17) for the definition of the marginal support $\text{Supp}_m(\mu)$.) For example, if $\mathbf{p} = (1, 1/\pi, 0, 1/2)$, then $\text{Supp}_m(\mu) = \{0, 1, 3\}$ and $\mathbf{p}^* = (1, 1, 0, 1)$. Note that $\text{Supp}_m(\mu) = \text{Supp}_m(\mu^*)$. We obtain μ^*, λ^* from μ, λ in this way. We mark all entities that refer to the substitution of μ, λ by μ^*, λ^* with a ^{*} superscript. Note that $\mathcal{M}^*(k_n) \geq \mathcal{M}(k_n)$ componentwise for all $k_n \in \mathcal{T}$.

Consider for each $n \geq 0$ the event

$$J_n := \bigcap_{m=0}^n \bigcup \{Q_{i_m, j_m} \subseteq \Lambda^m : i_m, j_m \in \mathcal{T}_m \text{ with } p_{i_m}, p_{j_m} > 0\}.$$

This is the event that in the first n steps of the construction, all squares that have positive marginal survival probability do survive all jointly. Let a be the number of level 1 squares having positive marginal survival probability. For each $n \geq 0$ the number of such squares at level n is given by a^n and the event J_n has probability $\mathbb{P}(J_n) = \mathbb{P}(J_1)^{1+a+\dots+a^{n-1}}$ where $\mathbb{P}(J_1) = \mu(\text{Supp}_m(\mu))\lambda(\text{Supp}_m(\lambda))$. At this point we use the joint survival condition, since it implies that $\mathbb{P}(J_1) > 0$ and thus $\mathbb{P}(J_n) > 0$. Note that by construction we have $\mathbb{P}^*(J_n) = 1$.

Let $\tilde{N}(k_m) = \min\{\tilde{Z}^L(k_m), \tilde{Z}^R(k_m)\}$. By the self-similarity of the process and the requirement that the process runs independently in the triangles of a Δ -pair, the event that in the first $n \geq 0$ sublevels of the surviving Δ -pair all triangles (in the Δ -pair) that have positive probability to survive, do survive simultaneously, occurs with probability at least $(\mathbb{P}(J_n))^2 > 0$. Conditional on this latter event—which has positive probability—the following chain of componentwise (in)equalities holds:

$$\begin{aligned} [\tilde{Z}^L(k_m) \tilde{Z}^R(k_m)] &= [\tilde{Z}^{*,L}(k_m) \tilde{Z}^{*,R}(k_m)] \\ &= [\mathbb{E}\tilde{Z}^{*,L}(k_m) \mathbb{E}\tilde{Z}^{*,R}(k_m)] = [\mathbb{E}Z^{*,L}(k_m) \mathbb{E}Z^{*,R}(k_m)] \\ &= \mathbf{e}\mathcal{M}^*(k_m) \geq \mathbf{e}\mathcal{M}(k_m) \geq \gamma \mathbf{e}\mathcal{M}(k_2) \cdots \mathcal{M}(k_m) \geq \cdots \geq \gamma^m \mathbf{e} \end{aligned}$$

for all $0 \leq m \leq n$ and $\underline{k}_m \in \mathcal{T}_m$, and where \mathbf{e} is the 2-dimensional all-one row vector. This directly implies the statement of the lemma. \square

5.2. Existence of a Δ -pair

Lemma 5.2 (Extension of Lemma 2 in [DS08]). *If $\gamma > 1$, then*

$$p_\Delta := \mathbb{P}(\exists m \geq 0 \text{ such that there exists a level } m \text{ } \Delta\text{-pair in } \Lambda^m) > 0.$$

Proof. We will show that one can take $m = 2$: if $\gamma > 1$, then

$$\mathbb{P}(\text{there exists a level } 2 \text{ } \Delta\text{-pair in } C_{00}^R \cap \Lambda^2) > 0.$$

The corresponding geometric structure is depicted in Figure 2.

From $\gamma_0 = \sum_{i=0}^{M-1} \mathbb{P}(Q_{i,i} \subseteq \Lambda_1) > 1$ we may conclude that there exist distinct $a, b \in \{0, \dots, M - 1\}$ such that

$$p^{ab} := \mathbb{P}(Q_{a,a}, Q_{b,b} \subseteq \Lambda^1) > 0.$$

From $\gamma_1 = \mathbb{P}(Q_{0,M-1} \subseteq \Lambda^1) + \sum_{i=1}^{M-1} \mathbb{P}(Q_{i,i-1} \subseteq \Lambda^1) > 1$ we may conclude that there exists at least one $c \in \{1, \dots, M - 1\}$ such that $p^c := \mathbb{P}(Q_{c,c-1} \subseteq \Lambda^1) > 0$.

Using the independence of the process in the unaligned squares $Q_{a,a}$ and $Q_{b,b}$ and the self-similarity of the process, we now conclude that

$$\begin{aligned} &\mathbb{P}(Q_{aa,aa}, Q_{bc,b(c-1)} \subseteq \Lambda^2) \\ &= \mathbb{P}(Q_{aa,aa}, Q_{bc,b(c-1)} \subseteq \Lambda^2 \mid Q_{a,a}, Q_{b,b} \subseteq \Lambda^1) \mathbb{P}(Q_{a,a}, Q_{b,b} \subseteq \Lambda^1) \\ &= \mathbb{P}(Q_{aa,aa} \subseteq \Lambda^2 \mid Q_{a,a}, Q_{b,b} \subseteq \Lambda^1) \mathbb{P}(Q_{bc,b(c-1)} \subseteq \Lambda^2 \mid Q_{a,a}, Q_{b,b} \subseteq \Lambda^1) p^{ab} \\ &= \mathbb{P}(Q_{a,a} \subseteq \Lambda^1) \mathbb{P}(Q_{c,c-1} \subseteq \Lambda^1) p^{ab} \geq p^{ab} p^c p^{ab} > 0, \end{aligned}$$

which finishes the proof, as $(L_{bc,b(c-1)}, R_{aa,aa})$ is a level 2 Δ -pair in C_{00}^R . \square

5.3. Unaligned triangles

The following lemma is purely combinatorial, and serves to obtain independence between the triangles in a Δ -pair.

Lemma 5.3 (Lemma 3 in [DS08]). *We are given N distinct odd numbers o_1, \dots, o_N and N distinct even numbers e_1, \dots, e_N . Then we can couple the odd numbers with the even numbers and we can colour the N couples with three colours (say \mathfrak{r} , \mathfrak{g} and \mathfrak{b}) such that no two numbers in pairs of the same colour are adjacent and all colours are used for at least $\lfloor N/3 \rfloor$ pairs. That is, there exists a permutation π of $\{1, \dots, N\}$ such that we can colour the pairs*

$$(e_1, o_{\pi(1)}), \dots, (e_N, o_{\pi(N)})$$

with the three colours such that with each colour we paint at least $\lfloor N/3 \rfloor$ pairs and for any (also if $\ell = k$) $(e_k, o_{\pi(k)})$ and $(e_\ell, o_{\pi(\ell)})$ having the same colour it is true that

$$|e_\ell - o_{\pi(k)}| > 1.$$

5.4. Exponential growth

Lemma 5.4 (Adaptation of Lemma 4 of [DS08]). *If $\gamma > 1$, then there exists an $1 < \eta < \gamma$ such that*

$$p_I := \mathbb{P}(\tilde{N}(\underline{k}_n) \geq \eta^n \text{ for all } \underline{k}_n \in \mathcal{T}_n \text{ for all } n \geq 0) > 0.$$

Proof. We can follow literally the proof of Lemma 4 in [DS08], except that we define here the sets

$$A_n := \{\tilde{N}(\underline{k}_m) \geq \eta^m \text{ for all } \underline{k}_m \in \mathcal{T}_m \text{ for all } 0 \leq m \leq n\}, \quad (20)$$

instead of $A_n = \{\tilde{N}(\underline{k}_m) \geq \eta^m \text{ for all } \underline{k}_m \in \mathcal{T}_m\}$. The (rather embarrassing) reason is that the equality $\mathbb{P}(A_{n+1}^c \mid A_r \cap \dots \cap A_n) = \mathbb{P}(A_{n+1}^c \mid A_n)$ on the bottom of page 10 in [DS08] is wrong in general. All one needs is that the equality sign can be replaced by a \leq sign, but since proving this seems to be rather involved (although intuitively obvious) we chose to redefine A_n as in (20), since this conveniently leads to a sequence of decreasing sets with intersection the set in the statement of Lemma 5.4. The proof then continues as in [DS08], deducing from Lemmas 5.1 and 5.3 that

$$p_I = \mathbb{P}\left(\bigcap_{n \geq 0} A_n\right) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n) > 0,$$

which finishes the proof. \square

Lemma 5.4 ensures that with positive probability p_I the offspring in all subcolumns of a surviving Δ -pair never dies out. Lemma 5.2 ensures that with positive probability p_Δ a surviving Δ -pair exists. Using the self-similarity of the process, we thus have the following corollary:

Corollary 5.5. *If $\gamma > 1$, then for all $i_n, j_n \in \mathcal{T}$,*

$$\mathbb{P}(\phi(\Lambda \cap Q_{i_n, j_n}) \text{ contains an interval} \mid Q_{i_n, j_n} \subseteq \Lambda^n) \geq p_\Delta p_I > 0.$$

As the next step, it is shown in [DS08] that when $\Lambda \neq \emptyset$, the maximum number of unaligned surviving squares at level n grows to infinity as $n \rightarrow \infty$. In each unaligned surviving square the process runs independently and identically distributed to the process starting at $[0, 1]^2$. Because the number of these squares grows arbitrarily large, and in each square there is a positive probability that its projection contains a non-empty interval, this implies that almost surely the projection of Λ contains an interval.

6. Higher order Cantor sets

From now on we restrict ourselves to the symmetric case, that is, the vectors of marginal probabilities satisfy $\mathbf{p} = \mathbf{q}$. Whether similar results can be obtained for $\mathbf{p} \neq \mathbf{q}$ is yet unknown. The major difficulty is that relation (16) will no longer hold (in general) in the asymmetric case.

Essentially, the n^{th} order random Cantor set is constructed by ‘collapsing’ n steps of its construction into one step. We denote all entities of an n^{th} order random Cantor set with a superscript (n) .

The alphabet $\mathbb{A}^{(n)}$ of the n^{th} order random Cantor set is $\{0, \dots, M^n - 1\}$, with elements $i^{(n)} = [i_1 \dots i_n]_M$. To reduce the overloaded notation we will omit the superscript here, and simply write $i = i^{(n)}$.

The joint survival distribution $\mu^{(n)} : 2^{\mathbb{A}^{(n)}} \rightarrow [0, 1]$, which by definition is the distribution of the sets

$$\{i_{m+1} \in \mathbb{A}^{(n)} : X_{i_1 \dots i_m i_{m+1}}^{(n)} = 1\}$$

for all $i_1 \dots i_m \in \mathcal{T}^{(n)}$, is determined uniquely by requiring that

$$X_i^{(n)} \sim \prod_{d=1}^n X_{i_1 \dots i_d} = X_{i_1} X_{i_1 i_2} \cdots X_{i_1 \dots i_n}$$

for all $i = [i_1 \dots i_n]_M \in \mathbb{A}^{(n)}$, where the $X_{i_1 \dots i_d}$ are defined in (4).

It is clear that the higher order marginal probabilities are given by

$$p_i^{(n)} := \mathbb{P}_{\mu^{(n)}}(X_i^{(n)} = 1) = \prod_{d=1}^n \mathbb{P}_{\mu}(X_{i_1 \dots i_d} = 1) = \prod_{d=1}^n p_{i_d} \tag{21}$$

for all $i = [i_1 \dots i_n]_M \in \mathbb{A}^{(n)}$.

6.1. Joint survival

Note that when μ describes an ‘independent interval’ Cantor set, it is *not* true in general that $\mu^{(n)}$ corresponds to an ‘independent interval’ Cantor set with marginals $p_i^{(n)}$. This is because e.g. the products $X_0 X_{00}$ and $X_0 X_{01}$ are not independent: they share the X_0 term.

However, the joint survival condition (Condition 4.1) nicely propagates to higher order Cantor sets. Suppose μ satisfies the JSC. We have

$$\text{Supp}_m(\mu^{(n)}) = \left\{ i = [i_1 \dots i_n]_M \in \mathbb{A}^{(n)} : p_i^{(n)} = \prod_{d=1}^n p_{i_d} > 0 \right\},$$

which implies that $\mu^{(n)}$ satisfies the joint survival condition as well:

$$\mu^{(n)}(\text{Supp}_m(\mu^{(n)})) = (\mu(\text{Supp}_m(\mu)))^{1+a+a^2+\dots+a^{n-1}} > 0,$$

where $a := \#\text{Supp}_m(\mu)$ is the cardinality of the marginal support of μ .

The key observation regarding higher order Cantor sets is that for all $n \geq 1$,

$$F^{(n)} \sim \bigcap_{m=1}^{\infty} F^{n \cdot m} = F, \tag{22}$$

hence statements such as Theorem 4.1 can be applied to higher order correlation coefficients $\gamma_k^{(n)}$ in order to get results not only for $F_1^{(n)} - F_2^{(n)}$, but for $F_1 - F_2$ as well.

6.2. Expectation matrices

The expectation matrices of the higher order Cantor sets satisfy the following factorization property: for all $k_1 \dots k_n \in \mathcal{T}$,

$$\mathcal{M}^{(n)}([k_1 \dots k_n]_M) = \mathcal{M}(k_1 \dots k_n) = \mathcal{M}(k_1) \cdots \mathcal{M}(k_n). \tag{23}$$

For the $\gamma_k^{(n)}$ we can use Lemma 3.1, which relates the $\gamma_k^{(n)}$ to the expectation matrices. Recall that $\mathbf{e} := [1 \ 1]$. For all $k_1 \dots k_n \in \mathcal{T}$,

$$[\gamma_{k+1}^{(n)} \ \gamma_k^{(n)}] = \mathbf{e} \mathcal{M}^{(n)}(k) = \mathbf{e} \mathcal{M}(k_1) \cdots \mathcal{M}(k_n), \tag{24}$$

where $k = [k_1 \dots k_n]_M$. We define $\gamma_k^{(0)} = 1$ for all $k \in \mathbb{Z}$. A direct consequence of (24) is the following recursive relation between $\gamma_k^{(n)}$ of different orders, written as a matrix multiplication:

$$[\gamma_{M^{n-m}k+l+1}^{(n)} \ \gamma_{M^{n-m}k+l}^{(n)}] = [\gamma_{k+1}^{(m)} \ \gamma_k^{(m)}] \mathcal{M}^{(n-m)}(l) \tag{25}$$

for all $0 \leq m \leq n$, $k \in \mathbb{A}^{(m)}$ and $l \in \mathbb{A}^{(n-m)}$. In fact we can take $k \in \mathbb{Z}$ here because, by definition, $\gamma_{k+M^m i}^{(m)} = \gamma_k^{(m)}$ for all $i \in \mathbb{Z}$, $m \geq 0$ and $k \in \mathbb{Z}$. Note that in particular

$$[\gamma_{Mk+l+1}^{(n+1)} \ \gamma_{Mk+l}^{(n+1)}] = [\gamma_{k+1}^{(n)} \ \gamma_k^{(n)}] \mathcal{M}(l) \tag{26}$$

for all $n \geq 0$, $k \in \mathbb{Z}$ and $l \in \mathbb{A}$.

6.3. An alternative notation

In (12) the expectations $\mathbb{E}Z^{UV}(k_n)$ are put into the matrices $\mathcal{M}(k_n)$. In this section we discuss another way to alias these expectations in such a way that equivalent entries get the same alias. This aliasing scheme also provides an alternative representation of recursion relation (26).

The expectation matrices $\mathcal{M}(k)$ together contain $4M$ entries, but not all entries are distinct. The number of left triangles in column k for example equals the number of right triangles in column $k + 1$. The reason is simple: the left triangles in column k and the right triangles in column $k + 1$ are the respective halves of the same M -adic squares—those that have the same projection under ϕ . The following equations point out which triangle count expectations are always equal to each other, and define the aliasing scheme m_e , $e \in \{-M, \dots, M\}$, for those e satisfying $|e| < M$:

$$\begin{aligned} m_{k-M} &:= \mathbb{E}Z^{LL}(k-1) = \mathbb{E}Z^{LR}(k), & 0 < k < M, \\ m_0 &:= \mathbb{E}Z^{LL}(M-1) = \mathbb{E}Z^{RR}(0), \\ m_k &:= \mathbb{E}Z^{RL}(k-1) = \mathbb{E}Z^{RR}(k), & 0 < k < M. \end{aligned}$$

The remaining two aliases, m_e with $|e| = M$, are given by

$$m_{-M} := \mathbb{E}Z^{LR}(0) = 0, \quad m_M := \mathbb{E}Z^{RL}(M-1) = 0.$$

These expectations are always zero because there are no right triangles in the leftmost column and no left triangles in the rightmost column.

The matrices $\mathcal{M}(k)$ take their entries from these m_e as follows:

$$\mathcal{M}(k) = \begin{bmatrix} m_{k+1-M} & m_{k-M} \\ m_{k+1} & m_k \end{bmatrix} \tag{27}$$

for $k \in \{0, \dots, M - 1\}$. Consequently, recursion relation (26) can be written using the m_e as

$$\gamma_{Mk+l}^{(n+1)} = m_{l-M} \gamma_{k+1}^{(n)} + m_l \gamma_k^{(n)} \tag{28}$$

for any $n \geq 0, k \in \mathbb{Z}$ and $l \in \{0, \dots, M\}$. Note in particular that the case $l = M$ is allowed and valid here; it corresponds to the *left* column of (26) with $l = M - 1$ there.

Relation (25) can also be written in terms of the m_e , though the higher order version $m_e^{(n-m)}$ is required as we are dealing with entries from $\mathcal{M}^{(n-m)}(l)$ instead of $\mathcal{M}(l)$. The equation that relates the n^{th} order to the m^{th} order is

$$\gamma_{M^{n-m}k+l}^{(n)} = m_{l-M^{n-m}}^{(n-m)} \gamma_{k+1}^{(m)} + m_l^{(n-m)} \gamma_k^{(m)} \tag{29}$$

for all $0 \leq m \leq n, k \in \mathbb{Z}$ and $l \in \{0, \dots, M^{n-m}\}$.

An explicit formula for the m_e can also be given; some careful bookkeeping yields the following set of formulas:

$$m_e = \sum_{i,j \in \mathbb{A}: i-j=e} p_i p_j = \sum_{i=\max(0,e)}^{\min(M,M+e)-1} p_i p_{i-e} = \sum_{j=\max(0,-e)}^{\min(M,M-e)-1} p_{j+e} p_j \tag{30}$$

for $e \in \{-M, \dots, M\}$. For $|e| = M$, these are sums over empty sets, which by convention evaluate to 0.

From equation (30) it is immediately clear that $m_e = m_{-e}$ for all $e \in \{-M, \dots, M\}$, so m_0, \dots, m_M is all we need to encode the matrices.

6.4. Bounded skewness

The scope of Theorem 4.1 can be extended if eventually for some n all $\gamma_k^{(n)}$ are strictly greater than 1, or when two subsequent ones, say $\gamma_k^{(n)}$ and $\gamma_{k+1}^{(n)}$, are strictly less than 1. If it were possible that the ratio between two successive $\gamma_k^{(n)}$ could become arbitrarily large as $n \rightarrow \infty$, it could be hard to show that we would eventually end up in one of these cases; perhaps one $\gamma_k^{(n)}$ would always remain below 1, while all the others would already be above 1.

The next lemma shows that the scenario of unbounded neighbour ratio is not the case if all m_e are positive for $|e| < M$. In the next section this lemma will play a key role in proving that—except for a (topologically) negligible set of marginal probabilities \mathbf{p} —Theorem 4.1 can always be fruitfully applied to higher order Cantor sets for orders n that are large enough.

Suppose that $m_e > 0$ for $|e| < M$. Then

$$R := \min_{\substack{0 < k < M \\ k' \in \{k-1, k+1\}}} \frac{\min(m_{k-M}, m_k)}{\max(m_{k'-M}, m_{k'})} \tag{31}$$

is well defined, and $R \in (0, 1]$.

Lemma 6.1. *If $m_e > 0$ for $e = 1, \dots, M - 1$, then*

$$\frac{\min(\gamma_k^{(n)}, \gamma_{k+1}^{(n)})}{\max(\gamma_k^{(n)}, \gamma_{k+1}^{(n)})} \geq R \tag{32}$$

for all $n \geq 0$ and $k \in \mathbb{Z}$, where R is defined as in (31).

Proof. Assume that $m_e > 0$ for all $|e| < M$. Note that by applying a simple inductive argument over n to recursion relation (28) with initial conditions $\gamma_k^{(0)} = 1$ for all $k \in \mathbb{Z}$ —and using the fact that for $l \in \{0, \dots, M\}$, m_{l-M} and m_l are never both equal to zero—it follows that $\gamma_k^{(n)} > 0$ for all $n \geq 0$ and $k \in \mathbb{Z}$. Hence the fraction in (32) is well defined.

An equivalent formulation of (32) is that for all $n \geq 0$ and all neighbour indices $k, k' \in \mathbb{Z}$ —i.e., $|k - k'| = 1$ —we have $\gamma_k^{(n)}/\gamma_{k'}^{(n)} \geq R$. We will prove this version, as it turns out to be slightly more convenient to work with.

We will use induction on n . For $n = 0$ we have $\gamma_k^{(0)}/\gamma_{k'}^{(0)} = 1/1 = 1 \geq R$. Now assume that the lemma holds for an $n \geq 0$. Let $k, k' \in \mathbb{Z}$ with $|k - k'| = 1$ be arbitrary.

First assume that M does *not* divide k ; then we can write $k = Mk_1 + k_2$ for some $k_1 \in \mathbb{Z}$ and some $0 < k_2 < M$. If we define k'_2 by setting $k'_2 - k_2 = k' - k$, then $k' = Mk_1 + k'_2$ with $0 \leq k'_2 \leq M$. Using recursion relation (28), we obtain

$$\begin{aligned} \frac{\gamma_k^{(n+1)}}{\gamma_{k'}^{(n+1)}} &= \frac{\gamma_{Mk_1+k_2}^{(n+1)}}{\gamma_{Mk_1+k'_2}^{(n+1)}} = \frac{m_{k_2-M}\gamma_{k_1+1}^{(n)} + m_{k_2}\gamma_{k_1}^{(n)}}{m_{k'_2-M}\gamma_{k_1+1}^{(n)} + m_{k'_2}\gamma_{k_1}^{(n)}} \\ &\geq \frac{\min(m_{k_2-M}, m_{k_2})(\gamma_{k_1+1}^{(n)} + \gamma_{k_1}^{(n)})}{\max(m_{k'_2-M}, m_{k'_2})(\gamma_{k_1+1}^{(n)} + \gamma_{k_1}^{(n)})} \geq R. \end{aligned}$$

Note how the fact that $0 < k_2 < M$ is used in the last inequality. Also, we did not need to use the induction hypothesis for this case.

Now assume that M *does* divide k , and write $k = Mk_1$ for some $k_1 \in \mathbb{Z}$. Because $m_e + m_{e-M} = \gamma_e \leq \gamma_0 = m_0$ for all $e \in \{0, \dots, M\}$ —here we use equation (16)—we have the bounds

$$\begin{aligned} \gamma_{k+1}^{(n+1)} &= \gamma_{Mk_1+1}^{(n+1)} = m_{1-M}\gamma_{k_1+1}^{(n)} + m_1\gamma_{k_1}^{(n)} \leq m_0 \max(\gamma_{k_1}^{(n)}, \gamma_{k_1+1}^{(n)}), \\ \gamma_{k-1}^{(n+1)} &= \gamma_{M(k_1-1)+(M-1)}^{(n+1)} = m_{-1}\gamma_{k_1}^{(n)} + m_{M-1}\gamma_{k_1-1}^{(n)} \leq m_0 \max(\gamma_{k_1}^{(n)}, \gamma_{k_1-1}^{(n)}). \end{aligned}$$

If we define k'_1 by setting $k'_1 - k_1 = k' - k$, k'_1 and k_1 are neighbours, we can use these bounds as follows:

$$\frac{\gamma_k^{(n+1)}}{\gamma_{k'}^{(n+1)}} = \frac{\gamma_{Mk_1}^{(n+1)}}{\gamma_{k'}^{(n+1)}} \geq \frac{m_0 \gamma_{k_1}^{(n)}}{m_0 \max(\gamma_{k_1}^{(n)}, \gamma_{k'_1}^{(n)})} = \min\left(1, \frac{\gamma_{k_1}^{(n)}}{\gamma_{k'_1}^{(n)}}\right) \geq R,$$

where in the last step we used the induction hypothesis. □

7. The lower spectral radius connection

In the previous section higher order Cantor sets were introduced. The corresponding higher order expectation matrices $\mathcal{M}^{(n)}(k)$ are products of the first order matrices $\mathcal{M}(k)$, as (24) shows. It will turn out that, for most \mathbf{p} the lower spectral radius of the set of matrices $\mathcal{M}(k)$, $k \in \mathbb{A}$, captures exactly the information needed to determine whether $F_1 - F_2$ contains an interval or not. The following definition generalizes the concept of ‘spectral radius’ to a *set* of matrices.

Definition 7.1 ([Gur95]). Let $\|\cdot\|$ be a submultiplicative norm on $\mathbb{R}^{d \times d}$ and $\Sigma \subseteq \mathbb{R}^{d \times d}$ a finite non-empty set of square matrices. The *lower spectral radius* of Σ is defined by

$$\underline{\rho}(\Sigma) := \liminf_{n \rightarrow \infty} \underline{\rho}_n(\Sigma, \|\cdot\|), \quad \underline{\rho}_n(\Sigma, \|\cdot\|) := \min_{A_1, \dots, A_n \in \Sigma} \|A_1 \cdots A_n\|^{1/n}. \quad (33)$$

It is easily seen that the definition of the lower spectral radius is independent of the particular choice of matrix norm.

7.1. A spectral radius characterization

For the algebraic difference between two M -adic random Cantor sets, we are interested in the lower spectral radius of the set of matrices

$$\Sigma_{\mathcal{M}} := \{\mathcal{M}(0), \dots, \mathcal{M}(M - 1)\}. \quad (34)$$

The following theorem provides a generalization of Theorem 4.1 for the symmetric case using the concept of the lower spectral radius.

First we need yet another notion. We call the collection $\{\mathcal{M}(0), \dots, \mathcal{M}(M - 1)\}$ of matrices *irreducible* if

$$\begin{aligned} \mathbb{E}Z^{LR}(k) &> 0, & k = 1, \dots, M - 1, \\ \mathbb{E}Z^{RL}(k) &> 0, & k = 0, \dots, M - 2. \end{aligned}$$

Note that the constraints on k are natural, because we *always* have $\mathbb{E}Z^{LR}(0) = 0$ and $\mathbb{E}Z^{RL}(M - 1) = 0$.

Theorem 7.1. Consider the algebraic difference $F_1 - F_2$ between two M -adic independent random Cantor sets F_1 and F_2 whose joint survival distributions satisfy the joint survival condition, have equal marginal probabilities and lead to an irreducible collection $\Sigma_{\mathcal{M}}$ as in (34).

- (a) If $\underline{\rho}(\Sigma_{\mathcal{M}}) > 1$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.
- (b) If $\underline{\rho}(\Sigma_{\mathcal{M}}) < 1$, then $F_1 - F_2$ contains no intervals a.s.

Proof. Let $\|\cdot\|_1$ denote the maximum absolute column sum norm.

First assume that $\underline{\rho}(\Sigma_{\mathcal{M}}) < 1$. Then there exists a number n such that $\underline{\rho}_n(\Sigma_{\mathcal{M}}, \|\cdot\|_1) < 1$. In particular, there exist $k_1, \dots, k_n \in \mathbb{A}$ such that

$$\|\mathcal{M}(k_1) \cdots \mathcal{M}(k_n)\|_1 < 1.$$

It follows that for $k = [k_1 \dots k_n]_M \in \mathbb{A}^{(n)}$,

$$\max(\gamma_{k+1}^{(n)}, \gamma_k^{(n)}) = \|\mathcal{M}_k^{(n)}\|_1 = \|\mathcal{M}(k_1) \cdots \mathcal{M}(k_n)\|_1 < 1. \tag{35}$$

From Theorem 4.1 statement (b) follows.

Now assume that $\underline{\rho}(\Sigma_{\mathcal{M}}) \geq 1 + \delta$ with $\delta > 0$. Then there exist infinitely many n such that for all $k_1, \dots, k_n \in \mathbb{A}$,

$$\|\mathcal{M}(k_1) \cdots \mathcal{M}(k_n)\|_1^{1/n} \geq \underline{\rho}_n(\Sigma_{\mathcal{M}}, \|\cdot\|_1) > 1 + \delta/2.$$

This implies that there exists an n such that for all $k = [k_1 \dots k_n]_M \in \mathbb{A}^{(n)}$,

$$\max(\gamma_{k+1}^{(n)}, \gamma_k^{(n)}) = \|\mathcal{M}_k^{(n)}\|_1 = \|\mathcal{M}(k_1) \cdots \mathcal{M}(k_n)\|_1 > \frac{1}{R},$$

where by irreducibility we can take the constant R as in (31). Using Lemma 6.1, this implies that for this n and all $k \in \mathbb{A}^{(n)}$,

$$\gamma_k^{(n)} \geq \frac{\min(\gamma_k^{(n)}, \gamma_{k+1}^{(n)})}{\max(\gamma_k^{(n)}, \gamma_{k+1}^{(n)})} \cdot \max(\gamma_k^{(n)}, \gamma_{k+1}^{(n)}) > R \cdot \frac{1}{R} = 1,$$

so statement (a) also follows by applying Theorem 4.1. □

We remark that the assumptions $\mathbf{p} = \mathbf{q}$ and irreducibility are used only in the second part of the proof, hence these conditions are not necessary for statement (b) in the theorem.

7.2. Scope of the theorem

We consider the following two questions:

- When do we get $\underline{\rho}(\Sigma_{\mathcal{M}}) = 1$ or a reducible $\Sigma_{\mathcal{M}}$, the cases the theorem says nothing about?
- How can we calculate $\underline{\rho}(\Sigma_{\mathcal{M}})$? Is there an explicit expression or algorithm for calculating it?

There is good and there is bad news here: the good news is that the cases $\underline{\rho}(\Sigma_{\mathcal{M}}) = 1$ and $m_e = 0$ for some $|e| < M$ happen only for a very limited set of vectors of marginal probabilities \mathbf{p} in $[0, 1]^M$, but the bad news is that the lower spectral radius is in general hard to calculate (see [TB97]).

First note that $m_e = 0$ for some $|e| < M$ happens only when at least one of the p_i , $i \in \mathbb{A}$, is zero. Thus

$$E_0 := \{\mathbf{p} \in [0, 1]^M : \Sigma_{\mathcal{M}} \text{ reducible}\}$$

has dimension at most $M - 1$, and hence has empty interior and Lebesgue measure zero. The other exceptional set in the theorem is

$$E_1 := \{\mathbf{p} \in [0, 1]^M : \underline{\rho}(\Sigma_{\mathcal{M}}) = 1\}. \tag{36}$$

The lower spectral radius has the following *scaling property* with respect to the vector of marginal probabilities: if $\tilde{\mathbf{p}} = c\mathbf{p}$ for some $c \geq 0$, then $\tilde{\mathcal{M}}(k) = c^2\mathcal{M}(k)$ for all $k \in \mathbb{A}$ and hence $\underline{\rho}(\Sigma_{\tilde{\mathcal{M}}}) = c^2\underline{\rho}(\Sigma_{\mathcal{M}})$. Thus for each vector of marginal probabilities \mathbf{p} at most one scalar multiple is in E_1 . Similarly, if $\tilde{\mathbf{p}} \geq \mathbf{p}$ componentwise, then $\tilde{\mathcal{M}}(k) \geq \mathcal{M}(k)$ componentwise and hence $\underline{\rho}(\Sigma_{\tilde{\mathcal{M}}}) \geq \underline{\rho}(\Sigma_{\mathcal{M}})$. Combining this with the scaling property, it follows that if $\tilde{\mathbf{p}} > \mathbf{p}$ componentwise, then also $\underline{\rho}(\Sigma_{\tilde{\mathcal{M}}}) > \underline{\rho}(\Sigma_{\mathcal{M}})$. From the scaling property it also follows that E_1 has empty interior.

With respect to the irreducibility condition for $\Sigma_{\mathcal{M}}$ it is interesting to consider the second example given in [DS08, Section 7]. Here $\mathbf{p} = (1, 0, p, 0, 1)$ with $p \in [0, 1]$ (see Figure 6). The corresponding (reducible) set of expectation matrices is given by

$$(\mathcal{M}(k))_{k \in \mathbb{A}} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 2+p^2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2p & 0 \end{bmatrix}, \begin{bmatrix} 2p & 0 \\ 0 & 2p \end{bmatrix}, \begin{bmatrix} 0 & 2p \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2+p^2 & 0 \\ 0 & 1 \end{bmatrix} \right),$$

hence $(\gamma_k)_{k \in \mathbb{A}} = (2+p^2, 1, 2p, 2p, 1)$.

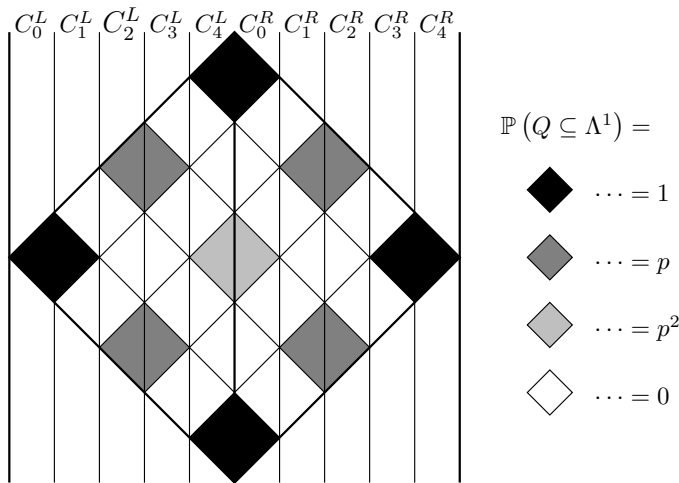


Fig. 6. The parameterized family $\mathbf{p} = (1, 0, p, 0, 1)$, $p \in [0, 1]$.

The level 1 expectation matrices have the property that each row and column contains at most one non-zero element. We will call matrices having this property *permutative matrices*. Clearly any product of permutative matrices is permutative again. Moreover, letting $\pi(A)$ denote the product of the non-zero entries of a permutative matrix A , it follows that

$$\pi(\mathcal{M}(k_1 \dots k_n)) = \prod_{i=1}^n \pi(\mathcal{M}(k_i))$$

for all $k_1 \dots k_n \in \mathcal{T}$. It is easy to see that for any permutative $D \times D$ matrix A ,

$$\|A\|_\infty = \|A\|_1 \geq \sqrt[D]{|\pi(A)|}.$$

If $p \geq 1/2$, then $\pi(\mathcal{M}(k)) \geq 2p$ for all $k \in \mathbb{A}$. Altogether, by plugging the two equations above into the definition of the lower spectral radius, we find that

$$\underline{\rho}(\Sigma_{\mathcal{M}}) \geq \sqrt{2p} > 1 \quad \text{if } p > 1/2.$$

However, in [DS08] it is shown that for all $p < 1$ the algebraic difference $F_1 - F_2$ contains no interval a.s. This example thus shows that at least some irreducibility condition is necessary in Theorem 7.1.

8. Classifying 2-adic random Cantor sets

In this section we consider the symmetric case for $M = 2$. For short we write $(\alpha, \beta) := (p_0, p_1) = (q_0, q_1)$ for the marginals. Note that

$$p_0 + p_1 = \mu(\{0\}) + \mu(\{1\}) + 2\mu(\{0, 1\}).$$

Since we require $p_0 + p_1 > 1$ (recall (8)), it follows that $\mu(\{0, 1\}) > 0$, and so the joint survival condition is always satisfied.

8.1. Expectations

The expectation matrices are given by

$$\mathcal{M}(0) = \begin{bmatrix} \alpha\beta & 0 \\ \alpha\beta & \alpha^2 + \beta^2 \end{bmatrix}, \quad \mathcal{M}(1) = \begin{bmatrix} \alpha^2 + \beta^2 & \alpha\beta \\ 0 & \alpha\beta \end{bmatrix}, \quad (37)$$

and the γ_k and m_e are given by

$$\gamma_0 = \alpha^2 + \beta^2, \quad \gamma_1 = 2\alpha\beta, \quad m_0 = \alpha^2 + \beta^2, \quad m_1 = \alpha\beta, \quad m_2 = 0.$$

Recursion relation (28) hence becomes

$$\begin{aligned} \gamma_{2k}^{(n+1)} &= m_0 \gamma_k^{(n)} = (\alpha^2 + \beta^2) \gamma_k^{(n)}, \\ \gamma_{2k+1}^{(n+1)} &= m_1 (\gamma_k^{(n)} + \gamma_{k+1}^{(n)}) = \alpha\beta (\gamma_k^{(n)} + \gamma_{k+1}^{(n)}), \end{aligned} \quad (38)$$

for all $n \geq 0$ and $k \in \mathbb{Z}$.

Note that $p_0 + p_1 > 1$ implies that $\alpha, \beta > 0$. Therefore both $m_0 > 0$ and $m_1 > 0$.

8.2. Neighbour bounds

The skewness lower bound from Lemma 6.1 is reduced to the simple expression

$$R = \frac{m_1}{m_0} = \frac{\alpha\beta}{\alpha^2 + \beta^2}. \tag{39}$$

The following lemma provides a bound that is even sharper than that of Lemma 6.1 and shows a specific ordering of neighbouring $\gamma_k^{(n)}$ that appears in the symmetric 2-adic algebraic difference. See also Figure 7.

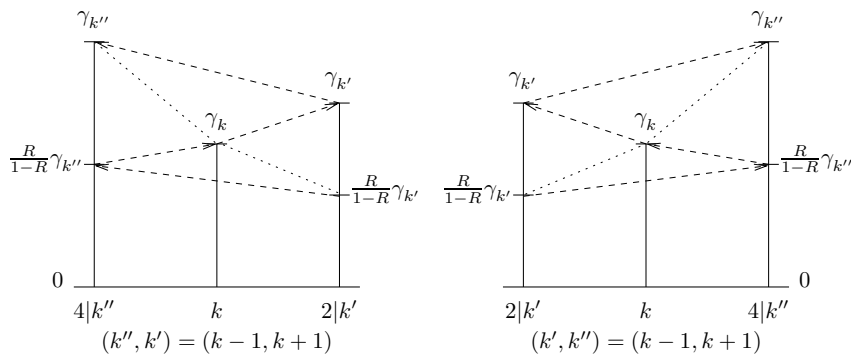


Fig. 7. A graphical representation of Lemma 8.1, with the γ_k displayed as a bar graph. The situation is shown for the two solutions to the equation $\{k', k''\} = \{k - 1, k + 1\}$. The arrows indicate the order of the inequalities in the lemma.

Lemma 8.1. Consider the algebraic difference between two independent 2-adic random Cantor sets that have equal vectors of marginal probabilities. Then

$$\frac{R}{1 - R} \gamma_{k'}^{(n)} \leq \frac{R}{1 - R} \gamma_{k''}^{(n)} \leq \gamma_k^{(n)} \leq \gamma_{k'}^{(n)} \leq \gamma_{k''}^{(n)} \tag{40}$$

for all $n \geq 0$, odd $k \in \mathbb{Z}$ and $\{k', k''\} = \{k - 1, k + 1\}$ such that $4 | k''$ and $2 | k'$.

Proof. First note that since $m_0 = \gamma_0 \geq \gamma_1 = m_{-1} + m_1 = 2m_1$, we have $R \leq 1/2$, so the fraction $R/(1 - R)$ is always well defined.

The proof is by induction on n . Since $\gamma_k^{(0)} = 1$ for all $k \in \mathbb{Z}$, the case $n = 0$ is trivial. Now assume that (40) holds for some $n \geq 0$, and let k, k', k'' be as stated. Note that $k'/2$ is odd and $k''/2$ even, and $|k'/2 - k''/2| = 1$. The first and the last inequality of (40) follow from

$$\gamma_{k'}^{(n+1)} \stackrel{(38)}{=} m_0 \gamma_{k'/2}^{(n)} \stackrel{(40)}{\leq} m_0 \gamma_{k''/2}^{(n)} \stackrel{(38)}{=} \gamma_{k''}^{(n+1)}. \tag{41}$$

The middle two inequalities of (40) follow from

$$\begin{aligned} \frac{R}{1-R} \gamma_{k''}^{(n+1)} &\stackrel{(39)}{=} \frac{m_1}{m_0} \left(\frac{R}{1-R} + 1 \right) \gamma_{k''}^{(n+1)} \stackrel{(38)}{=} m_1 \left(\frac{R}{1-R} + 1 \right) \gamma_{k''/2}^{(n)} \\ &\stackrel{(40)}{\leq} m_1 (\gamma_{k'/2}^{(n)} + \gamma_{k''/2}^{(n)}) \stackrel{(38)}{=} \gamma_k^{(n+1)} \stackrel{(40)}{\leq} m_1 \left(1 + \frac{1-R}{R} \right) \gamma_{k'/2}^{(n)} \\ &\stackrel{(38)}{=} \frac{m_1}{m_0} \left(1 + \frac{1-R}{R} \right) \gamma_{k'}^{(n+1)} \stackrel{(39)}{=} \gamma_{k'}^{(n+1)}, \end{aligned}$$

where in the last inequality we used (40), multiplied by a factor $(1 - R)/R$. □

8.3. The smallest correlation coefficient

Lemma 8.1 allows us to pinpoint for each $n \geq 0$ a k_n such that $\gamma_{k_n}^{(n)}$ equals the minimal value

$$\gamma^{(n)} = \min_{k \in \mathbb{Z}} \gamma_k^{(n)} = \min_{k \in \mathbb{A}^{(n)}} \gamma_k^{(n)}.$$

Obviously k_n should be an odd number, but we can say more: one of the two neighbours of $2k_n$ will turn out to serve well as k_{n+1} . Even though this selection procedure is very ‘local’, it will still pinpoint a global minimum.

In order to decide which neighbour of $2k_n$ has to be chosen, an auxiliary sequence (k'_n) is defined. This is done in such a way that k'_n is the neighbour of k_n that has the smallest value of $\gamma_{k'_n}^{(n)}$. Define the sequences (k_n) and (k'_n) by

$$\begin{aligned} k_0 &:= 0, & k_{n+1} &:= 2k_n + (k'_n - k_n) = k'_n + k_n, \\ k'_0 &:= 1, & k'_{n+1} &:= 2k_n, \end{aligned} \tag{42}$$

for all $n \geq 0$. Note that this indeed makes k'_n a neighbour of k_n for all $n \geq 0$, but their relative order alternates for subsequent n : $k'_{2n} = k_{2n} + 1$, but $k'_{2n+1} = k_{2n+1} - 1$ for all $n \geq 0$.

We mention that the sequence (k_n) is also known as the Jacobsthal sequence, since $k_{n+2} = k_{n+1} + 2k_n$ for $n \geq 0$ and $k_0 = 0, k_1 = 1$.

Lemma 8.2. *Let $(k_n)_{n \geq 0}$ and $(k'_n)_{n \geq 0}$ be as defined in (42). Then for all $n \geq 0$,*

$$\gamma_{k_n}^{(n)} = \min_{k \in \mathbb{Z}} \gamma_k^{(n)} = \min_{k \in \mathbb{Z}} \gamma_{2k+1}^{(n)}, \quad \gamma_{k'_n}^{(n)} = \min_{k \in \mathbb{Z}} \gamma_{2k}^{(n)}. \tag{43}$$

Proof. The proof is by induction on n . For $n = 0$ the statements are trivial because $\gamma_k^{(0)} = 1$ for all $k \in \mathbb{Z}$. Now suppose that the statement of the lemma holds for a certain $n \geq 0$. Then, by using the induction hypothesis, the last equality of (43) follows from

$$\min_{k \in \mathbb{Z}} \gamma_{2k}^{(n+1)} = m_0 \min_{k \in \mathbb{Z}} \gamma_k^{(n)} = m_0 \gamma_{k_n}^{(n)} = \gamma_{2k_n}^{(n+1)} = \gamma_{k'_{n+1}}^{(n+1)}. \tag{44}$$

Note that by the induction hypothesis

$$\gamma_{k_n}^{(n)} = \min_{k \in \mathbb{Z}} \gamma_{2k+1}^{(n)} \quad \text{and} \quad \gamma_{k'_n}^{(n)} = \min_{k \in \mathbb{Z}} \gamma_{2k}^{(n)},$$

thus by observing that of any two consecutive integers one is always even and one is always odd, it follows that

$$\min_{k \in \mathbb{Z}} (\gamma_k^{(n)} + \gamma_{k+1}^{(n)}) = \gamma_{k_n}^{(n)} + \gamma_{k'_n}^{(n)}.$$

By applying Lemma 8.1, the first, double equality from (43) follows:

$$\begin{aligned} \min_{k \in \mathbb{Z}} \gamma_k^{(n+1)} &= \min_{k \in \mathbb{Z}} \gamma_{2k+1}^{(n+1)} = m_1 \min_{k \in \mathbb{Z}} (\gamma_k^{(n)} + \gamma_{k+1}^{(n)}) \\ &= m_1 (\gamma_{k_n}^{(n)} + \gamma_{k'_n}^{(n)}) = \gamma_{2k_n + (k'_n - k_n)}^{(n+1)} = \gamma_{k_{n+1}}^{(n+1)}. \end{aligned} \quad (45)$$

□

8.4. Limit behaviour of $\gamma^{(n)}$

Define the sequence $(a_n)_{n \geq 0}$ of minimum values by setting

$$a_n := \gamma_{k_n}^{(n)} \quad (46)$$

for all $n \geq 0$. Using (45) and (44) the following recurrence relation is obtained:

$$\begin{aligned} a_{n+2} &= \gamma_{k_{n+2}}^{(n+2)} = m_1 (\gamma_{k_{n+1}}^{(n+1)} + \gamma_{k'_{n+1}}^{(n+1)}) \\ &= m_1 (\gamma_{k_{n+1}}^{(n+1)} + m_0 \gamma_{k_n}^{(n)}) = m_1 a_{n+1} + m_1 m_0 a_n \end{aligned} \quad (47)$$

for all $n \geq 0$. The initial conditions are given by

$$a_0 = \gamma_{k_0}^{(0)} = 1, \quad a_1 = \gamma_{k_1}^{(1)} = \gamma_1 = 2m_1. \quad (48)$$

The characteristic polynomial of this linear recurrence relation is $h(x) = x^2 - m_1 x - m_0 m_1$ with roots

$$x_{\pm} = \frac{1}{2} m_1 \pm \sqrt{m_1 m_0 + \frac{1}{4} m_1^2}.$$

Since $m_1 > 0$, x_+ and x_- are two distinct roots of the characteristic equation, so the general form of the solution to the recurrence relation is

$$a_n = c_+ x_+^n + c_- x_-^n$$

for all $n \geq 0$, where the constants c_+ and c_- are determined by the initial conditions. Since $x_+ > 0$ is the largest zero of the parabola h we have

$$|x_+| < 1 \Leftrightarrow h(1) > 0 \Leftrightarrow m_1(m_0 + 1) < 1,$$

thus we can conclude that

$$\lim_{n \rightarrow \infty} \gamma_{k_n}^{(n)} = \lim_{n \rightarrow \infty} a_n = \begin{cases} 0, & |x_+| < 1 \\ \infty, & |x_+| > 1 \end{cases} = \begin{cases} 0, & m_1(m_0 + 1) < 1, \\ \infty, & m_1(m_0 + 1) > 1. \end{cases} \quad (49)$$

Using the rightmost equations in (44), we find that the neighbour sequence $\gamma_{k'_n}^{(n)}$ behaves in exactly the same way.

Altogether this leads to the following result:

Theorem 8.3. *Consider the algebraic difference $F_1 - F_2$ between two independent 2-adic random Cantor sets F_1 and F_2 whose joint survival distributions have the same marginal probability vectors.*

- If $C > 1$, then $F_1 - F_2$ contains an interval a.s. on $\{F_1 - F_2 \neq \emptyset\}$.
- If $C < 1$, then $F_1 - F_2$ contains no interval a.s.

Here the number C is defined by

$$C := m_1(1 + m_0) = \alpha\beta(1 + \alpha^2 + \beta^2) = p_0 p_1(1 + p_0^2 + p_1^2). \quad (50)$$

Proof. We will show that the conditions of Theorem 4.1 do hold for Cantor sets of appropriate higher order. We already remarked that the JSC holds for 2-adic Cantor sets, and so it holds for all higher order Cantor sets.

If $m_1(1 + m_0) < 1$, then $\gamma_{k_n}^{(n)} \rightarrow 0$ and $\gamma_{k'_n}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$, so for some n large enough, $\gamma_{k_n}^{(n)}$ and $\gamma_{k'_n}^{(n)}$ are both strictly smaller than 1.

If $m_1(1 + m_0) > 1$, then $\gamma_{k_n}^{(n)} \rightarrow \infty$ as $n \rightarrow \infty$, so for some n large enough, $\gamma_{k_n}^{(n)}$ is strictly larger than 1, and by Lemma 8.2 this holds for all $\gamma_k^{(n)}$. \square

Remark 8.4. The number C in Theorem 8.3 is not the spectral radius of the collection

$$\Sigma_{\mathcal{M}} = \{\mathcal{M}(0), \mathcal{M}(1)\}.$$

In fact, let $\lambda_{PF}(A)$ denote the Perron–Frobenius eigenvalue of a matrix A . It is well known that $\lambda_{PF}(A)$ is sandwiched between the smallest and the largest column sum of A . Combining this with Lemma 6.1 we obtain

$$\gamma_{k_n}^{(n)} = \min_{k \in \mathbb{Z}} \min\{\gamma_k^{(n)}, \gamma_{k+1}^{(n)}\} \leq \min_{k \in \mathbb{Z}} \lambda_{PF}(\mathcal{M}^{(n)}k) \leq \frac{1}{R} \min_{k \in \mathbb{Z}} \min\{\gamma_k^{(n)}, \gamma_{k+1}^{(n)}\} = \frac{1}{R} \gamma_{k_n}^{(n)}.$$

According to Theorem B.1 of [Gur95], $(\min_{k \in \mathbb{Z}} \lambda_{PF}(\mathcal{M}(k)))^{1/n}$, which is nothing other than the n^{th} root of the smallest spectral radius of all length n products of matrices from Σ , converges to $\underline{\rho}(\Sigma_{\mathcal{M}})$. Together with our results above, it follows that the lower spectral radius of $\Sigma_{\mathcal{M}} = \{\mathcal{M}(0), \mathcal{M}(1)\}$ is equal to

$$\lim_{n \rightarrow \infty} a_n^{1/n} = \lim_{n \rightarrow \infty} (c_+ x_+^n + c_- x_-^n)^{1/n} = x_+ = \frac{1}{2} m_1 + \sqrt{m_1 m_0 + \frac{1}{4} m_1^2}.$$

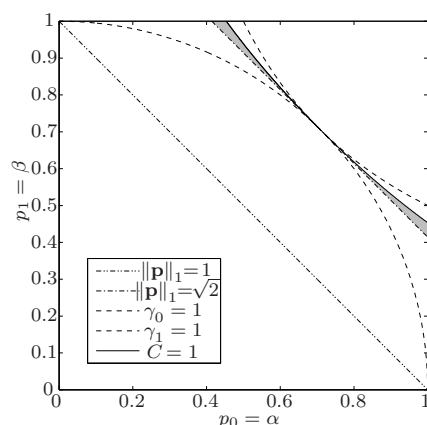


Fig. 8. Classification of the 2-adic symmetric algebraic difference in the (p_0, p_1) plane: (1) Below $\|\mathbf{p}\|_1 = 1$ a.s. $F_1 = F_2 = \emptyset$. (2) Below $\|\mathbf{p}\|_1 = \sqrt{2}$ there are no intervals because $\dim_{\mathbb{H}}(F_1 - F_2) < 1$. (3) Below $\gamma_0 = 1$ the ‘no intervals’ part of Theorem 4.1 holds. (4) Above $\gamma_1 = 1$ the ‘intervals’ part holds. (5) The line $C = 1$ is the separating boundary of Theorem 8.3. The area where the Palis conjecture fails is shaded grey.

This might be of independent interest: we have shown that for $\max\{2b, 1 - 2b\} \leq a \leq 1 + b^2$ the lower spectral radius of the collection consisting of

$$\mathcal{M}(0) = \begin{bmatrix} a & b \\ 0 & b \end{bmatrix}, \quad \mathcal{M}(1) = \begin{bmatrix} b & 0 \\ b & a \end{bmatrix} \quad (51)$$

is equal to $\frac{1}{2}b + \frac{1}{2}\sqrt{4ab + b^2}$.

Figure 8 gives an overview of boundaries in the space of vectors of marginal probabilities \mathbf{p} that separate areas where different sets of conditions imply the absence or presence of intervals. The figure also indicates the area where the Palis conjecture fails, i.e., the area where (1) does not imply that $F_1 - F_2$ contains an interval (on $\{F_1 - F_2 \neq \emptyset\}$).

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