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G_δ ideals of compact sets

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Abstract. We investigate the structure of G_δ ideals of compact sets. We define a class of G_δ ideals of compact sets that, on the one hand, avoids certain phenomena present among general G_δ ideals of compact sets and, on the other hand, includes all naturally occurring G_δ ideals of compact sets. We prove structural theorems for ideals in this class, and we describe how this class is placed among all G_δ ideals. In particular, we establish a result representing ideals in this class via the meager ideal. This result is analogous to Choquet's theorem representing alternating capacities of order ∞ via Borel probability measures. Methods coming from the structure theory of Banach spaces are used in constructing important examples of G_δ ideals outside of our class.

Keywords. Ideals of compact sets

1. Introduction

In the present paper, E stands for a compact metric space and $\mathcal{K}(E)$ denotes the compact space of all compact subsets of E equipped with the Vietoris topology. A subfamily of $\mathcal{K}(E)$ is called *downward closed* if it is closed under taking compact subsets. A downward closed subfamily of $\mathcal{K}(E)$ that is also closed under taking finite unions is called an *ideal of compact sets*. A downward closed subfamily of $\mathcal{K}(E)$ that is closed under taking countable unions provided the union is compact is called a *σ-ideal of compact sets*.

The study of definable ideals of compact sets is by now a classical subject in descriptive set theory. For a comprehensive recent survey of this field the reader can consult [12]. Of particular interest among definable ideals of compact sets are coanalytic σ-ideals mostly because of a wide range of examples belonging to this class and because of the theory that can be developed for it. By a dichotomy proved in [8], coanalytic σ-ideals fall into two major subclasses: they are either coanalytic complete or else they are G_δ. This paper investigates the structure of the latter class of G_δ σ-ideals of compact sets.

The following definition will be crucial in our considerations. A set $\mathcal{I} \subseteq \mathcal{K}(E)$ is said to have *property* $(*)$ if for any sequence $K_n \in \mathcal{I}$, $n \in \omega$, there exists a G_δ set $G \subseteq E$ such that $\bigcup_n K_n \subseteq G$ and each compact subset of G is in $\mathcal{I}$.

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As will be proved in Proposition 2.1, families of compact sets with $(*)$ are σ-ideals and, if they are additionally assumed to be coanalytic, they are G_δ. Thus, $(*)$ can be viewed as a strong version of σ-completeness for G_δ ideals. Some vague analogies with the results of [18] can be taken to indicate that coanalytic σ-ideals of compact sets with $(*)$ are to general coanalytic σ-ideals of compact sets what analytic P-ideals of subsets of ω are to general analytic ideals of subsets of ω.

A note on terminology. As said above, coanalytic families with $(*)$ are automatically G_δ σ-ideals. Thus, there is a range of names such families can be called. We will refer to them as *G_δ ideals with $(*)$*.

In Section 2, we present examples of G_δ ideals with property $(*)$ and general facts about this property. Recently, a new phenomenon among G_δ ideals of compact sets was discovered by Mátrai in [14]. He constructed a G_δ ideal of compact subsets of 2^ω containing all singletons and such that each dense G_δ subset of 2^ω contains a compact set not in the ideal. At this point, this property seems somewhat pathological for G_δ ideals, and condition $(*)$ delineates a natural class of G_δ ideals avoiding it. (Sections 2, 3, and 4 of the present paper were, however, mostly completed before the appearance of [14].) The aim of presenting the examples in Section 2 is to show that natural G_δ ideals do have property $(*)$. Furthermore, we show in Proposition 2.2 that all calibrated, thin families of compact sets have $(*)$. On the other hand, we give an example in Proposition 2.4 of a calibrated G_δ ideal that does not have $(*)$.

In Sections 3 and 4, we study the structure of G_δ ideals with $(*)$. The following operation will be fundamental in a representation theorem for such ideals. For $A \subseteq E$, let

$$A^* = \{K \in \mathcal{K}(E) : K \cap A \neq \emptyset\}. \tag{1.1}$$

In Theorem 3.2 we represent each G_δ ideal with property $(*)$ via the nowhere dense ideal by showing that a compact set K is in the ideal if and only if K^* is nowhere dense in some fixed, but depending on the ideal, compact subset of $\mathcal{K}(E)$. In fact, G_δ ideals with $(*)$ are the only ideals that can be represented in this fashion. Such a representation is new even for classical ideals like, for example, the ideal of compact measure zero sets or of compact zero-dimensional sets. This theorem, though not its proof, is analogous to the classical theorem of Choquet [2] (see also [13, pp. 30–35]) that gives a characterization of ideals of compact sets that can be represented as follows: a compact set K is in the ideal if and only if K^* has measure zero with respect to some Borel probability measure on $\mathcal{K}(E)$; those are precisely the ideals of zero sets with respect to an alternating capacity of order ∞. Thus, our representation result gives a meager ideal analogue of this measure ideal theorem; in our result, condition $(*)$ provides the appropriate characterization. As a consequence to this theorem, we find in Theorem 3.2 a representation for G_δ ideals $\mathcal{I}$ with calibration and $(*)$, which uses the following operation $A \mapsto A^+$ in addition to the operation $A \mapsto A^*$ defined above:

$$A^+ = \{K \in \mathcal{K}(E) : K \cap A \text{ is not covered by countably many elements of } \mathcal{I}\}. \tag{1.2}$$

Next we prove in Theorem 4.1 that all G_δ ideals with $(*)$ are Tukey reducible to the nowhere dense ideal. This gives, for ideals with $(*)$, an affirmative answer to a question of Louveau and Veličković [10]. (After the present work was completed, Justin Moore and the author proved [16], using one of the ideals constructed in Section 6 of the present paper, that the question has a negative answer in general.)

In Sections 5 and 6, we investigate the placement of the class of ideals with $(*)$ within the class of all G_δ ideals. This is done by introducing a natural transfinite rank, with respect to a given ideal, on all open, downward closed families of compact sets. The rank quantifies the degree of closedness under taking unions such a family enjoys. We prove in Theorem 5.4 that property $(*)$ for an ideal is equivalent to having open, downward closed approximations with the highest possible rank ω_1. For each $\alpha < \omega_1$, we exhibit in Theorem 6.1 a G_δ ideal, necessarily without $(*)$, for whose open, downward closed approximations the highest value of the rank is precisely α. The combinatorial objects used in this construction are analogous to objects that come up in the structure theory of Banach spaces.

Notation. By ω we denote the set of all natural numbers including 0. If $n \in \omega$, we identify n with the set $\{0, \dots, n-1\}$. In particular, $0 = \emptyset$. Similarly, if $\bar{s}$ is a finite sequence whose domain is $n \in \omega$ and if $m \leq n$, then $\bar{s}\restriction m$ is the sequence obtained from $\bar{s}$ by keeping its first m elements. As usual, if $\bar{s}$ is a finite sequence and $\bar{t}$ is another finite sequence, by $\bar{s}^\frown\bar{t}$ we denote the sequence obtained from $\bar{s}$ by extending it by $\bar{t}$. If $\bar{s}$ is a sequence of elements of a set A and $a \in A$, then $\bar{s}^\frown a$ stands for the sequence $\bar{s}^\frown\bar{t}$, where $\bar{t} = (a)$.

2. Basic facts and examples

Part (i) of the following proposition shows that $(*)$ can be viewed as a strong version of σ-completeness, and the proof of the coanalytic part of (ii) shows that it can be thought of as a separation principle. The proposition below should be compared with a result due to Dougherty, Kechris, Louveau, and Woodin, [6], [8], that for $\mathcal{I} \subseteq \mathcal{K}(E)$, $\mathcal{I}$ is an analytic σ-ideal if and only if $\mathcal{I}$ is a G_δ ideal.

Proposition 2.1. (i) *Let $\mathcal{I} \subseteq \mathcal{K}(E)$ have property $(*)$. Then $\mathcal{I}$ is a σ-ideal.*
(ii) *If, additionally, $\mathcal{I}$ is assumed to be analytic or coanalytic, then it is a G_δ.*

Proof. Note that $(*)$ immediately implies that $\mathcal{I}$ is downward closed and that if a compact set K can be covered by countably many members of $\mathcal{I}$, then $K \in \mathcal{I}$. So (i) follows. If $\mathcal{I}$ is analytic, then it is a G_δ by (i) and [8]. Assume now that it is coanalytic. Then, if it is not G_δ, by Hurewicz's theorem there is a continuous map $f : 2^\omega \to \mathcal{K}(E)$ such that $f(x) \in \mathcal{I}$ iff $x \in \mathbf{Q}$ where $\mathbf{Q}$ consists of all sequences in 2^ω which are eventually 0. By $(*)$ we can find a G_δ $G \subseteq E$ such that $\bigcup_{x\in\mathbf{Q}} f(x) \subseteq G$ and $\mathcal{K}(G) \subseteq \mathcal{I}$. But $\mathcal{K}(G)$ is a G_δ, so $f^{-1}(\mathcal{K}(G))$ is a G_δ subset of 2^ω. On the other hand, it equals $\mathbf{Q}$, contradiction. □

We now list some classes of G_δ ideals of compact subsets of a compact space E that have property $(*)$. All these ideals occur naturally in various parts of mathematics. The new point proved here is that they fulfill $(*)$. (There exist overlaps among the classes of ideals listed below, which I have not investigated.) On the other hand, as follows from [14], there do exist G_δ ideals of compact subsets of 2^ω that do not have $(*)$.

Examples of G_δ ideals with $(*)$. For all the ideals listed below it is straightforward to check directly from their definitions that they are analytic or coanalytic. Thus, in view of Proposition 2.1, we only need to check property $(*)$ for them. These arguments will follow the list below.

1. Compact nowhere dense subsets of E; more generally, for a non-empty σ-compact set $\mathcal{F} \subseteq \mathcal{K}(E)$, the ideal
$$\{K \in \mathcal{K}(E) : \forall L \in \mathcal{F}\ K \cap L \text{ is meager in } L\}. \tag{2.1}$$
Consideration of the general formula (2.1) was proposed by Alain Louveau. For examples of ideals given by formula (2.1) in the special case of countable $\mathcal{F}$ see [9, Theorem 2].
2. Compact measure zero sets with respect to a finite Borel measure on E; more generally, for a non-empty σ-compact (in the weak* topology) set $\mathcal{M}$ of finite Borel measures on E, the ideal
$$\{K \in \mathcal{K}(E) : \forall \mu \in \mathcal{M}\ \mu(K) = 0\}. \tag{2.2}$$
For ideals given by the general formula (2.2) see [3], [4, p. 31], [8].
3. Compact zero sets with respect to a Hausdorff measure; see [4, p. 31].
4. Given $n \in \omega$, compact subsets of E of topological dimension $\leq n$; for definiteness we consider here, say, the covering dimension. See [15, p. 152].
5. $\mathcal{Z}$-sets for $E = [0,1]^\omega$. Recall that a compact set $K \subseteq [0,1]^\omega$ is a *$\mathcal{Z}$-set* if for any continuous function $f : [0,1]^\omega \to [0,1]^\omega$ and for any $\epsilon > 0$ there exists a continuous function $g : [0,1]^\omega \to [0,1]^\omega$ that is uniformly ϵ-close to f and whose range is disjoint from K. See [15, p. 307].
6. The ideals associated with analytic P-ideals of subsets of ω as follows: let I be an analytic P-ideal of subsets of ω, for $K \in \mathcal{K}(2^\omega)$, and put $K \in \mathcal{I}$ iff for some $x \in I$, $\{y \cap x : y \in K\}$ is meager in $\mathcal{P}(x)$ (see [18, p. 55]). Recall that an ideal of subsets of ω is a *P-ideal* if for any $x_n \in I$, $n \in \omega$, there exists $x \in I$ with $x_n \setminus x$ finite for each n.

We will now give the arguments that the ideals above fulfill $(*)$.

1. Obviously, if $\mathcal{F} = \{E\}$, formula (2.1) gives the ideal of all compact nowhere dense subsets of E. So it suffices to check that ideals given by (2.1) have $(*)$. In fact, it will be enough to do it only for $\mathcal{F}$ that is assumed to be compact. Indeed, if $\mathcal{F} = \bigcup_n \mathcal{F}_n$ with $\mathcal{F}_n$ compact, and for each n the ideal defined via (2.1) using $\mathcal{F}_n$ has $(*)$, then it immediately follows that so does the ideal defined using $\mathcal{F}$.

Fix therefore a compact non-empty family $\mathcal{F}$ and let $\mathcal{I}$ be the ideal defined by (2.1) using $\mathcal{F}$. Fix a countable topological basis $\mathcal{B}$ of E. For $U, V \in \mathcal{B}$ with $\overline{U} \subseteq V$ set
$$\mathcal{F}_{U,V} = \overline{\{\overline{L \cap V} : L \in \mathcal{F} \text{ and } L \cap \overline{U} \neq \emptyset\}}.$$

By compactness of $\mathcal{F}$ it follows that

$$\forall L' \in \mathcal{F}_{U,V}\ \exists L \in \mathcal{F}\ (L \cap \overline{U} \neq \emptyset \text{ and } L \cap V \subseteq L'). \tag{2.3}$$

Moreover, using (2.3), the following equality is rather easy to show and we leave proving it to the reader:

$$\mathcal{I} = \{K \in \mathcal{K}(E) : \forall U, V \in \mathcal{B}, \overline{U} \subseteq V\ \forall L \in \mathcal{F}_{U,V}\ L \not\subseteq K\}. \tag{2.4}$$

Thus, by (2.4), it suffices to show that given $K_n \in \mathcal{I}$, $n \in \omega$, and $U, V \in \mathcal{B}$ with $\overline{U} \subseteq V$, there exists an open $O \subseteq E$ with $\bigcup_n K_n \subseteq O$ and such that O contains no elements of $\mathcal{F}_{U,V}$.

To produce such a set O, we will need the following fact. If $O' \subseteq E$ is open and such that $\overline{O'}$ does not contain an element of $\mathcal{F}_{U,V}$, then for any $K \in \mathcal{I}$, $\overline{O'} \cup K$ does not contain an element of $\mathcal{F}_{U,V}$. To see this, assume that $L' \in \mathcal{F}_{U,V}$ is such that $L' \subseteq \overline{O'} \cup K$. Using (2.3) pick $L \in \mathcal{F}$ with $L \cap \overline{U} \neq \emptyset$ and with $L \cap V \subseteq L'$. If $L \cap V \subseteq \overline{O'}$, then $\overline{L \cap V} \subseteq \overline{O'}$, contradicting our assumption on O'. Thus, $\emptyset \neq (L \cap V) \setminus \overline{O'}$. It follows that $(L \cap V) \setminus \overline{O'}$ is a non-empty relatively open subset of $L \in \mathcal{F}$ and it is clearly contained in K, so $K \notin \mathcal{I}$, contradiction.

Now since $K_0 \in \mathcal{I}$, it does not contain elements from $\mathcal{F}_{U,V}$. Using compactness of K_0 and of $\mathcal{F}_{U,V}$, we find an open set O_0 with $K_0 \subseteq O_0$ and with $\overline{O}_0$ not containing an element of $\mathcal{F}_{U,V}$. Assume we have an open set O_n with $K_0 \cup \cdots \cup K_n \subseteq O_n$ and with $\overline{O}_n$ not containing sets from $\mathcal{F}_{U,V}$. Using the fact proved above, we see that $\overline{O}_n \cup K_{n+1}$ does not contain sets from $\mathcal{F}_{U,V}$. Now using compactness of $\overline{O}_n \cup K_{n+1}$ and of $\mathcal{F}_{U,V}$, we find an open set O_{n+1} containing $\overline{O}_n \cup K_{n+1}$ and such that no compact subset of $\overline{O}_{n+1}$ is in $\mathcal{F}_{U,V}$. Once this recursive construction is carried out, let $O = \bigcup_n O_n$. This open set is clearly as required.

2. As in point 1, we can limit our considerations to the situation when $\mathcal{F}$ is compact. If K_n, $n \in \omega$, are compact and $\mu(K_n) = 0$ for each n, then, given n and $\epsilon > 0$, by compactness of $\mathcal{F}$ we can find a single open set O_n such that $K_n \subseteq O_n$ and $\mu(\overline{O}_n) < \epsilon/2^{n+1}$ (see [7, Theorem 17.20(iii)]). Therefore, $O = \bigcup_n O_n$ is an open set containing $\bigcup_n K_n$ and with measure $< \epsilon$ with respect to each $\mu \in \mathcal{M}$. Repeating this construction for a sequence of positive reals ϵ converging to 0 and taking the intersection of the resulting open sets gives a G_δ with the required properties.

3. Essentially the same proof as for 2 works here.

4. We only need to point out here that the notion of dimension applies to all subsets of the given ambient compact metric space, that the union of countably many compact sets of dimension $\leq n$ has dimension $\leq n$ (see [15, Theorem 3.2.8]), that each set of dimension $\leq n$ is contained in a G_δ set of dimension $\leq n$, as follows from [15, Theorem 3.2.5], and that subsets of sets of dimension $\leq n$ have dimension $\leq n$ (see [15, Theorem 3.2.9]).

5. This follows from Corollary 5.3.6, Lemma 5.1.3(2), and Lemma 5.1.7(6) of [15] and the observation that the subset $s = (0,1)^\omega$ of $[0,1]^\omega$ is a G_δ.

6. Let $K_n \in \mathcal{I}$, $n \in \omega$, and let $x_n \in I$ be such that $\{y \cap x_n : y \in K_n\}$ is meager in $\mathcal{P}(x_n)$. Since I is a P-ideal, there is $x \in I$ with $x_n \setminus x$ finite for each n. Then $\{y \cap x : y \in K_n\}$

is meager in $\mathcal{P}(x)$ for each n. Let $G \subseteq \mathcal{P}(x)$ be a G_δ set such that $\mathcal{P}(x) \setminus G$ is dense in $\mathcal{P}(x)$ and

$$\Big\{y \cap x : y \in \bigcup_n K_n\Big\} \subseteq G.$$

Then $\{y \in \mathcal{P}(\omega) : y \cap x \in G\}$ is a G_δ, $\bigcup_n K_n$ is included in it, and all compact sets included in it are in $\mathcal{I}$ as witnessed by x.

We now point out one general condition implying property $(*)$. Recall that a family $\mathcal{I}$ of compact subsets of E is *thin* if there is no uncountable family of pairwise disjoint compact sets not in $\mathcal{I}$; and $\mathcal{I}$ is called *calibrated* if for any compact set $K \subseteq E$, if $\mathcal{K}(K \setminus \bigcup_n K_n) \subseteq \mathcal{I}$ for some $K_n \in \mathcal{I}$, $n \in \omega$, then $K \in \mathcal{I}$. So, for example the ideal of compact subsets of measure zero with respect to a Borel probability measure is calibrated, while the ideal of nowhere dense subsets of an uncountable compact space is not.

Proposition 2.2. *If $\mathcal{I} \subseteq \mathcal{K}(E)$ is calibrated and thin, then it has $(*)$.*

Proof. Let $K_n \in \mathcal{I}$, $n \in \omega$. Let $\{L_n : n \in \omega\}$ be a maximal family of compact subsets of $E \setminus \bigcup_n K_n$ which are pairwise disjoint and not in $\mathcal{I}$. This family is countable by thinness. Let $G = E \setminus \bigcup_n L_n$. Then G is a G_δ containing $\bigcup_n K_n$. Let $K \subseteq G$ be compact. By maximality of $\{L_n : n \in \omega\}$, $\mathcal{K}(K \setminus \bigcup_n K_n) \subseteq \mathcal{I}$. Thus, by calibration, $K \in \mathcal{I}$. So, G witnesses that $(*)$ holds for the sequence (K_n). □

As an application of property $(*)$, we deduce a result of Zelený [20]. The original argument in [20] was rather different and used the fact that coanalytic sets admit coanalytic ranks.

Corollary 2.3 (Zelený [20]). *Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be calibrated, thin, and coanalytic. Then $\mathcal{I}$ is G_δ.*

Proof. By Proposition 2.2, $\mathcal{I}$ has $(*)$ and hence by Proposition 2.1(ii), it is a G_δ. □

As shown in the proposition below, one cannot remove the assumption that $\mathcal{I}$ be thin in Proposition 2.2 even if $\mathcal{I}$ is assumed to be G_δ.

Proposition 2.4. *There exists a calibrated G_δ ideal of compact subsets of 2^ω that does not have $(*)$.*

The proof of Proposition 2.4 is postponed till the end of Section 6, where a general construction of G_δ ideals without $(*)$ is described.

3. Representations of ideals with $(*)$

We prove a representation theorem for G_δ ideals with $(*)$. This representation is somewhat analogous to Choquet's representation of alternating capacities of order ∞ (see [2], [13, pp. 30–35], [7, 30.4]). Its proof is, however, different.

It follows from Proposition 2.1(ii) that coanalytic families with $(*)$ are G_δ. However, we will carry out the proof of the representation assuming only coanalyticity of the family. This does not make the argument more complicated and, since the form of this representation implies easily that the family is a G_δ ideal, we will have yet another proof that coanalytic families with $(*)$ are G_δ ideals.

The operation A^* is defined in (1.1).

Theorem 3.1. *Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be coanalytic and non-empty. Then $\mathcal{I}$ has $(*)$ if and only if there exists a compact set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that*

$$K \in \mathcal{I} \;\Leftrightarrow\; K^* \cap \mathcal{F} \text{ is meager in } \mathcal{F}.$$

Proof. Assume that $\mathcal{I}$ is coanalytic, non-empty, and has $(*)$. We construct a family $\mathcal{F} \subseteq \mathcal{K}(E)$ as in the conclusion of the theorem. Let

$$E' = E \setminus \bigcup\{U : U \text{ open and } \overline{U} \in \mathcal{I}\}.$$

Clearly E' is compact. Since $\mathcal{I}$ is a σ-ideal, for $K \in \mathcal{K}(E)$ we see that $K \in \mathcal{I}$ if and only if $K \cap E' \in \mathcal{I} \cap \mathcal{K}(E')$. It follows that it suffices to find $\mathcal{F} \subseteq \mathcal{K}(E')$ as in the conclusion of the theorem for the coanalytic, non-empty family $\mathcal{I} \cap \mathcal{K}(E')$ having property $(*)$. Thus, we can, and will from now on, assume that $E' = E$. Note that since $\mathcal{I}$ is a σ-ideal, all sets in $\mathcal{I} \cap \mathcal{K}(E')$ are nowhere dense in E'. Therefore, we will assume that all sets in $\mathcal{I}$ are nowhere dense in E.

If $\mathcal{I} = \{\emptyset\}$, let $\mathcal{F} = \{E\}$. Thus, from this point on we assume that $\mathcal{I} \subseteq \mathcal{K}(E)$ is coanalytic, has property $(*)$, consists of only nowhere dense sets, and contains $\{x\}$ for some $x \in E$. Recall first the operation A^+ defined in (1.2). In the course of the argument, we will prove the following condition, which is therefore also equivalent to $(*)$ and is of some interest:

- there exist compact $\mathcal{F}_n \subseteq \mathcal{K}(E)$, $n \in \omega$, that are *upward closed* (that is, closed under taking compact supersets) with $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$ and such that for $K \in \mathcal{K}(E)$,

$$K \in \mathcal{I} \;\Rightarrow\; \forall n \; K^* \cap \mathcal{F}_n \text{ is meager in } \mathcal{F}_n,$$
$$K \notin \mathcal{I} \;\Rightarrow\; \exists n \; K^+ \cap \mathcal{F}_n \text{ is dense in } \mathcal{F}_n.$$

Note that, since K^* is compact and contains K^+, the second part of the above condition implies that $K^* \supseteq \mathcal{F}_n$ for all but finitely many n.

Fix a continuous surjection $f : \omega^\omega \to \mathcal{K}(E) \setminus \mathcal{I}$. Such a function exists since $\mathcal{I}$ is coanalytic. First we note that for any non-empty, closed set $F \subseteq \omega^\omega$, there exists a non-empty, relatively open set $U \subseteq F$ such that

$$\forall (K_l)_{l\in\omega} \subseteq \mathcal{I} \; \exists L \in \mathcal{K}(E) \; \Big(L \cap \bigcup_l K_l = \emptyset \text{ and } \forall x \in U \; f(x) \cap L \neq \emptyset\Big). \tag{3.1}$$

Assume towards a contradiction that this fails. Let U_i, $i \in \omega$, be an open basis for F consisting of non-empty sets. The failure of the above condition allows us to pick for each i a sequence $(K^i_l) \subseteq \mathcal{I}$ so that for each compact $L \subseteq E \setminus \bigcup_l K^i_l$ there exists $x \in U_i$ with $f(x) \cap L = \emptyset$. Let now (K_n) enumerate $\{K^i_l : l, i \in \omega\}$. By $(*)$, we can now pick a G_δ set G so that $\bigcup_n K_n \subseteq G$ and $\mathcal{K}(G) \subseteq \mathcal{I}$. Let L_j, $j \in \omega$, be compact and chosen so that $E \setminus G = \bigcup_j L_j$. Note that for each $x \in F$, $f(x) \notin \mathcal{I}$, and therefore, $f(x)$ intersects some L_j. Additionally, for each j, the set $\{x \in F : f(x) \cap L_j \neq \emptyset\}$ is closed in F. Thus, by the Baire category theorem, there exist i and j with $U_i \subseteq \{x \in F : f(x) \cap L_j \neq \emptyset\}$, which gives an immediate contradiction with the fact that $L_j \subseteq E \setminus \bigcup_n K_n^i$.

For $F \subseteq \omega^\omega$ closed define

$$\widetilde{F} = F \setminus \bigcup \{U : U \subseteq F \text{ relatively open and fulfilling (3.1)}\}.$$

Define further $F^0 = \omega^\omega$, $F^{\alpha+1} = \widetilde{F^\alpha}$, and $F^\lambda = \bigcap_{\alpha<\lambda} F^\alpha$ if λ is a limit ordinal. There exists a countable ordinal β such that $F^{\beta+1} = F^\beta$ and by the argument from the previous paragraph $F^\beta = \emptyset$. It follows that ω^ω can be represented as a countable union of (not necessarily open) sets fulfilling (3.1). Putting them in one sequence, we obtain F_n, $n \in \omega$, whose union is ω^ω and with (3.1) holding for each F_n. Let $\mathcal{L}'_n = f[F_n]$.

Now for each n let

$$\mathcal{K}'_n = \bigcap \{K^* : K \in \mathcal{L}'_n\}.$$

Note that each $\mathcal{K}'_n$ is compact. We claim that

$$\forall (K_l) \left((K_l) \subseteq \mathcal{I} \Rightarrow \mathcal{K}'_n \not\subseteq \bigcup_l K_l^* \right). \tag{3.2}$$

Indeed, if $\mathcal{K}'_n \subseteq \bigcup_l K_l^*$ for some $(K_l) \subseteq \mathcal{I}$, then for each $L \in \mathcal{K}'_n$, $L \cap K_l \neq \emptyset$ for some l. This means, by the definition of $\mathcal{K}'_n$, that for each compact L,

$$(\forall K \in \mathcal{L}'_n \; L \cap K \neq \emptyset) \;\Rightarrow\; L \cap \bigcup_l K_l \neq \emptyset.$$

This however directly contradicts (3.1) for F_n since $\mathcal{L}'_n = f[F_n]$.

Let

$$\mathcal{K}_n = \mathcal{K}'_n \setminus \bigcup \left\{ \mathcal{U} : \mathcal{U} \subseteq \mathcal{K}'_n \text{ relatively open and } \exists (K_l) \subseteq \mathcal{I} \; \mathcal{U} \subseteq \bigcup_l K_l^* \right\}.$$

Then $\mathcal{K}_n$ is compact and by (3.2) non-empty. Note also that

$$K \in \mathcal{I} \;\Rightarrow\; K^* \cap \mathcal{K}_n \text{ is meager in } \mathcal{K}_n. \tag{3.3}$$

Otherwise, since K^* is compact, there would be a $\mathcal{V} \subseteq \mathcal{K}_n$ non-empty, relatively open in $\mathcal{K}_n$ such that $\mathcal{V} \subseteq K^*$. Then it is easy to see that

$$\mathcal{V} \subseteq \bigcup \left\{ \mathcal{U} : \mathcal{U} \subseteq \mathcal{K}'_n \text{ relatively open and such that } \exists (K_l) \subseteq \mathcal{I} \; \mathcal{U} \subseteq \bigcup_l K_l^* \right\},$$

so $\mathcal{V} \cap \mathcal{K}_n = \emptyset$, contradiction. Note also that for compact K,

$$K \notin \mathcal{I} \;\Rightarrow\; \exists n \; \mathcal{K}_n \subseteq K^*. \tag{3.4}$$

Indeed, if $K \notin \mathcal{I}$, then for some n_0, $K \in \mathcal{L}'_{n_0}$, so $\mathcal{K}_{n_0} \subseteq \mathcal{K}'_{n_0} \subseteq K^*$.

Now we use the sets $\mathcal{K}_n$ to produce $\mathcal{F}_n$ as in the condition at the beginning of the proof. Let V_i^n, $i \in \omega$, be an open basis of $\mathcal{K}_n$ consisting of non-empty sets. Let $\mathcal{F}_n$ be the closure of the set

$$\{L \in \mathcal{K}(E) : \forall i, j \leq n \; V_i^j \nsubseteq (\overline{E \setminus L})^*\}. \tag{3.5}$$

It is easy to check that each $\mathcal{F}_n$ is upward closed and that $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$.

Let $K \in \mathcal{I}$, and let $n \in \omega$. Let L be in the set given by (3.5). Since, by (3.3), for all $j \leq n$, $K^* \cap \mathcal{K}_j$ is nowhere dense in $\mathcal{K}_j$, we see that, for all $i, j \leq n$,

$$((\overline{E \setminus L}) \cup K)^*$$

does not contain V_i^j. Therefore, we can find an open set U containing K and such that for all $i, j \leq n$ we still have

$$V_i^j \nsubseteq (\overline{E \setminus L} \cup \overline{U})^* = (\overline{E \setminus (L \setminus U)})^*. \tag{3.6}$$

Let $L' = L \setminus U$. By (3.6) and the definitions of $\mathcal{F}_n$ and U, we see that L' is in $\mathcal{F}_n \setminus K^*$. Since $\mathcal{F}_n$ is upward closed, for any finite set $F \subseteq E \setminus U$, we find that $L'' = L' \cup F$ is an element of $\mathcal{F}_n \setminus K^*$. By manipulating the sets U and F (making $\overline{U}$ close to K and F close to $\overline{L \cap U}$ in the Vietoris topology), we can make L'' as close to L as we wish. (We use here the assumption that all elements of $\mathcal{I}$, in particular K, are nowhere dense.) Thus, we have found an element of $\mathcal{F}_n \setminus K^*$ arbitrarily close to an arbitrary element of the set given by (3.5). Since the set defined by (3.5) is dense in $\mathcal{F}_n$, we have just shown that $\mathcal{F}_n \setminus K^*$ is dense in $\mathcal{F}_n$. Since it is also open, $K^* \cap \mathcal{F}_n$ is meager in $\mathcal{F}_n$.

Let now $K \in \mathcal{K}(E) \setminus \mathcal{I}$. Then, by (3.4), for some n and some (in fact, all) $i \leq n$, we have $V_i^n \subseteq K^*$. Let L be in the set given by (3.5). We will show that $K \cap L \notin \mathcal{I}$, hence K^+ is dense in $\mathcal{F}_n$. Note that

$$K^* \subseteq (\overline{E \setminus L})^* \cup (K \cap L)^*.$$

The first of the sets in the union on the right hand side does not contain V_i^n by (3.5) and is compact. Therefore, the second set, $(K \cap L)^*$, intersected with $\mathcal{K}_n$ is not meager in $\mathcal{K}_n$ as $V_i^n \subseteq K^*$. It follows by (3.3) that $K \cap L \notin \mathcal{I}$. Thus, the required properties of the sequence $\mathcal{F}_n$ are proved.

Fix now x_0 with $\{x_0\} \in \mathcal{I}$. Note that since $\{x_0\}$ is nowhere dense, x_0 is not isolated. Thus, we can find sets $W_n \subseteq E$, $n \in \omega$, such that each W_n is the interior of its closure, each neighborhood of x_0 contains all but finitely many sets W_n, and $\overline{W}_{n+1}$ is properly included in W_n. Define

$$\mathcal{F} = \overline{\bigcup_n \{F \setminus W_n : F \in \mathcal{F}_n\}}.$$

We claim that $\mathcal{F}$ is such that $K \in \mathcal{I}$ if and only if $K^* \cap \mathcal{F}$ is meager.

Assume that $K \in \mathcal{I}$, and suppose towards a contradiction that $K^* \cap \mathcal{F}$ is not meager in $\mathcal{F}$, that is, it has non-empty interior in $\mathcal{F}$. Then, since $\bigcup_n \{F \setminus W_n : F \in \mathcal{F}_n\}$ is dense in $\mathcal{F}$, for some n the relative interior of

$$K^* \cap \{F \setminus W_n : F \in \mathcal{F}_n\}$$

in the set $\{F \setminus W_n : F \in \mathcal{F}_n\}$ is non-empty. This implies that there are open sets $U, U_0, \dots, U_p \subseteq E$ with $U_0, \dots, U_p \subseteq U$ and such that the set

$$\{F \setminus W_n : F \in \mathcal{F}_n,\ F \setminus W_n \subseteq U,\ \text{and}\ \forall i \leq p\ (F \setminus W_n) \cap U_i \neq \emptyset\} \tag{3.7}$$

is non-empty and included in K^*. Non-emptiness of (3.7) implies that $U_i \setminus W_n \neq \emptyset$ for each $i \leq p$. Since W_n is the interior of its closure, we can find non-empty open sets U_i', $i \leq p$, with $U_i' \subseteq U_i$ and $U_i' \cap W_n = \emptyset$. Using closedness of $\mathcal{F}_n$ under taking compact supersets and non-emptiness of each U_i' and of the set (3.7), we see that for some $F \in \mathcal{F}_n$,

$$F \setminus W_n \subseteq U \quad \text{and} \quad \forall i \leq p\ (F \setminus W_n) \cap U_i' \neq \emptyset.$$

Thus, the set

$$\{F \in \mathcal{F}_n : F \subseteq U \cup W_n \text{ and } \forall i \leq p\ F \cap U_i' \neq \emptyset\} \tag{3.8}$$

is non-empty. Also it is included in K^* since each of its elements contains an element of the set (3.7) and since K^* is upward closed. Thus, $K^* \cap \mathcal{F}_n$ is not meager as it contains the non-empty set (3.8) relatively open in $\mathcal{F}_n$, contradicting $K \in \mathcal{I}$.

Now assume that $K \notin \mathcal{I}$. Since $\mathcal{I}$ is a σ-ideal and $\{x_0\} \in \mathcal{I}$, there exists $m \geq 1$ such that $K \setminus W_{m-1} \notin \mathcal{I}$. Thus, there exists n_0 with $(K \setminus W_{m-1})^* \supseteq \mathcal{F}_{n_0}$. Since the sequence $(\mathcal{F}_n)$ is decreasing, we can assume that $n_0 \geq m$. It follows that

$$\{F \setminus W_{n_0} : F \in \mathcal{F}_{n_0}\} \subseteq K^*.$$

Using compactness of K^* and the fact that the sequence $(\mathcal{F}_n)$ is decreasing as is the sequence (W_n), we deduce from the formula above that

$$\overline{\bigcup_{n \geq n_0} \{F \setminus W_n : F \in \mathcal{F}_n\}} \subseteq K^*. \tag{3.9}$$

Pick now a non-empty open set U with $U \subseteq W_{n_0-1} \setminus W_{n_0}$. Since all sets in $\mathcal{I}$ are nowhere dense, U contains a compact set $M \notin \mathcal{I}$. Consequently, there exists $q \in \omega$ with $M^* \supseteq \mathcal{F}_q$. Since $\mathcal{F}_{n_0} \subseteq \mathcal{F}_q$ or $\mathcal{F}_q \subseteq \mathcal{F}_{n_0}$, we see that some element of $\mathcal{F}_{n_0}$ intersects M, and therefore it intersects U. Thus, since $W_{n_0} \cap U = \emptyset$, the set $\{F \in \mathcal{F} : F \cap U \neq \emptyset\}$ is non-empty. It is also clearly open in $\mathcal{F}$. Now, from (3.9) and the fact that $U \subseteq W_k$ for all $k < n_0$, we get

$$\{F \in \mathcal{F} : F \cap U \neq \emptyset\} \subseteq \overline{\bigcup_{n \geq n_0} \{F \setminus W_n : F \in \mathcal{F}_n\}} \subseteq K^*,$$

which proves that $K^* \cap \mathcal{F}$ is not meager.

Now for the opposite implication in the theorem. Let $\mathcal{F} \subseteq \mathcal{K}(E)$ be as on the right hand side of the equivalence of the theorem. Let K_m, $m \in \omega$, be such that $K_m^* \cap \mathcal{F}$ is meager in $\mathcal{F}$ for each m. We can pick $L_k \in \mathcal{F}$, $k \in \omega$, so that for each k, m, $L_k \notin K_m^*$ and $\{L_k : k \in \omega\}$ is dense in $\mathcal{F}$. Define $G = E \setminus \bigcup_k L_k$. Clearly G is a G_δ and contains $\bigcup_m K_m$. Moreover, if $K \subseteq G$ is compact, then K^* is compact and does not contain any of the sets L_k; thus, $K^* \cap \mathcal{F}$ is nowhere dense in $\mathcal{F}$. □

After the present work was completed Maya Saran proved in [17] that given a compact family $\mathcal{F}$ as in Theorem 3.1 above and assuming that $\mathcal{I}$ consists only of nowhere dense sets, one can modify $\mathcal{F}$ to obtain an *upward closed* compact family that still represents $\mathcal{I}$ by the formula from this theorem. Note that the assumption that sets in $\mathcal{I}$ are nowhere dense is necessary here.

We will now give a representation of calibrated G_δ ideals with $(*)$. (The definition of a calibrated family is given just before Proposition 2.2.) This representation provides a characterization of such ideals. Note that by Proposition 2.4 property $(*)$ cannot be removed from the left hand side of the equivalence in the theorem below.

Recall the operations A^* and A^+ defined in (1.1) and (1.2). Note that $A^+ \subseteq A^*$. Furthermore, if $K \in \mathcal{K}(E)$ and $\mathcal{I}$ is a σ-ideal, then $K^+ = \{L \in \mathcal{K}(E) : L \cap K \notin \mathcal{I}\}$.

Theorem 3.2. *Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be coanalytic and non-empty. Then $\mathcal{I}$ has $(*)$ and is calibrated if and only if there exists a compact set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that*

$$\begin{aligned} K \in \mathcal{I} \ &\Leftrightarrow\ K^* \cap \mathcal{F} \textit{ is meager in } \mathcal{F} \\ &\Leftrightarrow\ K^+ \cap \mathcal{F} \textit{ is meager in } \mathcal{F}. \end{aligned}$$

Proof. We first show the implication from left to right. Since $\mathcal{I}$ is coanalytic and has $(*)$, there exists a compact set $\mathcal{F} \subseteq \mathcal{K}(E)$ as in Theorem 3.1. We prove that the displayed condition holds for this compact set. Of course, it suffices to see the second equivalence. The direction $\Rightarrow$ is obvious since $K \in \mathcal{I}$ implies that $K^* \cap \mathcal{F}$ is meager in $\mathcal{F}$ and $K^+ \subseteq K^*$. To prove $\Leftarrow$, assume that $K \notin \mathcal{I}$. Let $\mathcal{U} \subseteq \mathcal{F}$ be the relative interior in $\mathcal{F}$ of $\overline{(K^* \setminus K^+) \cap \mathcal{F}}$. Pick $K_l \in (K^* \setminus K^+) \cap \mathcal{F}$ so that $\{K_l : l \in \omega\}$ is dense in $\mathcal{U}$. Of course, if $\mathcal{U} = \emptyset$, then we do not pick the sequence (K_l). Note that for all l, $K_l \cap K \in \mathcal{I}$. Since $K \notin \mathcal{I}$, using calibration of $\mathcal{I}$, we get a compact set $L \notin \mathcal{I}$ with $L \subseteq K \setminus \bigcup_l K_l$. Then $L^* \subseteq K^*$ and $K_l \notin L^*$ for each l. Now, $L^* \cap \mathcal{F}$ is not meager in $\mathcal{F}$, so it has a non-empty interior in $\mathcal{F}$. Let us denote this interior by $\mathcal{V}$. Since for all l, $K_l \notin \mathcal{V}$, we have $\mathcal{V} \cap \mathcal{U} = \emptyset$. Thus, $K^* \setminus K^+$ is nowhere dense in $\mathcal{V}$. Since $\mathcal{V} \subseteq K^*$, this implies that $K^+ \cap \mathcal{V}$ contains a dense open subset of $\mathcal{V}$, hence K^+ has non-empty interior in $\mathcal{F}$.

Now, we prove the other implication. By Theorem 3.1, $\mathcal{I}$ has property $(*)$. To see calibration, let K and K_p, $p \in \omega$, be elements of $\mathcal{K}(E)$ such that $K \notin \mathcal{I}$, $K_p \in \mathcal{I}$, $K_p \subseteq K$. We show that there is $M \in \mathcal{K}(E)$ such that $M \notin \mathcal{I}$ and $M \subseteq K \setminus \bigcup_p K_p$. Since $K \notin \mathcal{I}$, by our assumption $K^+ \cap \mathcal{F}$ is not meager in $\mathcal{F}$. Since $\mathcal{I}$ has property $(*)$, we can find a G_δ set G such that $\bigcup_p K_p \subseteq G$ and $\mathcal{K}(G) \subseteq \mathcal{I}$. Fix now compact sets M_k, $k \in \omega$, so that $K \setminus G = \bigcup_k M_k$. For any $L \in \mathcal{K}(E)$, if $L \in K^+$, then $L \cap K \notin \mathcal{I}$, which implies $L \cap K \not\subseteq G$. Thus, $K^+ \subseteq \bigcup_k M_k^*$. Since $K^+ \cap \mathcal{F}$ is not meager in $\mathcal{F}$, we see that for some k_0, $M_{k_0}^* \cap \mathcal{F}$ is not meager in $\mathcal{F}$, so $M_{k_0} \notin \mathcal{I}$. As $M_{k_0} \subseteq K \setminus \bigcup_p K_p$, we are done. □

We point out that both conditions in Theorem 3.2 are equivalent to the existence of a closed set $\mathcal{F} \subseteq \mathcal{K}(E)$ such that

$$\begin{aligned} K \in \mathcal{I} \ &\Leftrightarrow\ K^* \cap \mathcal{F} \text{ is meager in } \mathcal{F} \\ &\Leftrightarrow\ \{(L_1, L_2) : K \cap L_1 \cap L_2 \neq \emptyset\} \text{ is meager in } \mathcal{F} \times \mathcal{F}. \end{aligned}$$

Indeed, assuming that a compact family $\mathcal{F} \subseteq \mathcal{K}(E)$ is such that, for $K \in \mathcal{K}(E)$, $K \in \mathcal{I}$ if and only if $K^* \cap \mathcal{F}$ is meager in $\mathcal{F}$, we have

$$\begin{aligned} K^+ \cap \mathcal{F} &= \{L_1 \in \mathcal{F} : L_1 \cap K \notin \mathcal{I}\} \\ &= \{L_1 \in \mathcal{F} : \{L_2 \in \mathcal{F} : (L_1 \cap K) \cap L_2 \neq \emptyset\} \text{ is not meager in } \mathcal{F}\}. \end{aligned}$$

From this, by the Kuratowski–Ulam theorem, we infer that $K^+ \cap \mathcal{F}$ is meager in $\mathcal{F}$ precisely when $\{(L_1, L_2) : K \cap L_1 \cap L_2 \neq \emptyset\}$ is meager in $\mathcal{F} \times \mathcal{F}$. It follows that the condition above is equivalent to the second condition in the conclusion of Theorem 3.2.

4. Property $(*)$ and Tukey reduction

As is customary, I will denote by NWD the ideal of compact nowhere dense subsets of 2^ω. It is a G_δ ideal with $(*)$.

Given two partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$ we say that P is *Tukey reducible to* Q, in symbols $P \leq_T Q$, if there exists a function $f : P \to Q$ such that for each $q \in Q$, $\{p \in P : f(p) \leq_Q q\}$ is bounded in P. Below we consider ideals of compact sets as partial orders where the order relation is inclusion between sets in the ideal. Tukey reduction among ideals of this sort has been studied in a number of papers (see for example [5], [10], and [19]). It is a question of some interest whether each G_δ ideal of compact sets is Tukey reducible to NWD (see [10, p. 194, Question 3]). The following theorem shows that it is so for G_δ ideals with $(*)$. However, after completion of the present work, Justin Moore and the author [16] showed that the answer to this question is negative for general G_δ ideals of compact sets.

Theorem 4.1. *If $\mathcal{I} \subseteq \mathcal{K}(E)$ is a G_δ ideal with $(*)$, then $\mathcal{I} \leq_T$* NWD.

Proof. Let X be a metric compact space with a metric d. Let $L \subseteq K \subseteq X$ be both compact, and let $f \in \omega^\omega$ be increasing. Define, taking $1/0 = \infty$,

$$U^{L,K,f} = \{x \in X : x \notin L \text{ and } \forall n\ (d(x, L) \leq 1/n \Rightarrow d(x, K) < 1/f(n))\}$$

for $L \neq \emptyset$, and if $L = \emptyset$, let

$$U^{\emptyset,K,f} = \{x \in X : d(x, K) < 1/f(0)\}.$$

Claim. (i) *$U^{L,K,f}$ is open.*
(ii) $K \cap U^{L,K,f} = K \setminus L$.
(iii) *If M is a compact subset of X with $M \cap K = L$, then $M \cap U^{L,K,f} = \emptyset$ for some increasing $f \in \omega^\omega$.*

Proof of Claim. All this is completely clear for $L = \emptyset$. So assume $L \neq \emptyset$. We leave the proof of (i) and (ii) to the reader. To see (iii), let M be compact with $M \cap K = L$. Define $f(0) = 0$ and for $n > 0$ let

$$f(n) = 1 + \max\left\{\left[\left(\inf\left\{d(x, K) : x \in M,\ d(x, L) \geq \frac{1}{n+1}\right\}\right)^{-1}\right],\ f(n-1)\right\},$$

where $[\cdot]$ stands for the integer part function, the infimum is taken to be ∞ if the set to which it is applied is empty, and, of course, $1/\infty = 0$.

Assume that $x_0 \notin L$ and

$$\forall n \; (d(x_0, L) \leq 1/n \Rightarrow d(x_0, K) < 1/f(n)). \tag{4.1}$$

We need to see that $x_0 \notin M$. Assume towards a contradiction that $x_0 \in M$. Since $x_0 \notin L$, we can find an n_0 such that $d(x_0, L) \leq 1/n_0$ and $d(x_0, L) > 1/(n_0+1)$. Then from (4.1) we have $d(x_0, K) < 1/f(n_0)$, so

$$f(n_0) < 1/d(x_0, K) \leq \left(\inf\left\{d(x, K) : x \in M,\; d(x, L) \geq \frac{1}{n_0+1}\right\}\right)^{-1} \leq f(n_0),$$

contradiction. The claim is proved.

Fix now a compact set $\mathcal{F} \subseteq \mathcal{K}(E)$ with the properties as in Theorem 3.1. We will now find a Tukey reduction from $\mathcal{I}$ to

$$\{F \in \mathcal{K}(\mathcal{F}) : F \text{ is nowhere dense in } \mathcal{F}\} \times \omega^\omega,$$

where ω^ω is taken with the order of pointwise inequality. This will be enough since by [5] the ideal of compact nowhere dense subsets of any compact metric space is Tukey reducible to NWD and

$$\text{NWD} \times \omega^\omega \leq_T \text{NWD} \times \text{NWD} \leq_T \text{NWD}.$$

Pick $K \in \mathcal{I}$. Let

$$\mathcal{L}^K = K^* \cap \mathcal{F}.$$

We will apply the Claim to $X = \mathcal{K}(E)$ with some fixed metric d. Claim (iii) allows us to pick $f^K \in \omega^\omega$ increasing so that

$$K^* \cap U^{\mathcal{L}^K, \mathcal{F}, f^K} = \emptyset. \tag{4.2}$$

Define $\phi : \mathcal{I} \to \text{NWD} \times \omega^\omega$ by

$$\phi(K) = (\mathcal{L}^K, f^K).$$

We check now that ϕ is a Tukey reduction. It suffices to show that given any $F \subseteq \mathcal{F}$ compact nowhere dense in $\mathcal{F}$ and $f \in \omega^\omega$, which can be assumed to be increasing, the set $\{K : \mathcal{L}^K \subseteq F,\; f^K \leq f\}$ is bounded in $\mathcal{I}$, that is, all of its members are included in a fixed element of $\mathcal{I}$.

If $\mathcal{L}^K \subseteq F$ and $f^K \leq f$, then

$$\begin{aligned}\mathcal{U} = U^{F, \mathcal{F}, f} &= \{K' : K' \notin F \text{ and } \forall n \; (d(K', F) \leq 1/n \Rightarrow d(K', \mathcal{F}) < 1/f(n))\} \\ &\subseteq \{K' : K' \notin \mathcal{L}^K \text{ and } \forall n \; (d(K', \mathcal{L}^K) \leq 1/n \Rightarrow d(K', \mathcal{F}) < 1/f^K(n))\} \\ &= U^{\mathcal{L}^K, \mathcal{F}, f^K}.\end{aligned}$$

Note now that

(a) $\mathcal{U}$ is open by Claim (i).
(b) For K with $\mathcal{L}^K \subseteq F$ and $f^K \leq f$, we have $K^* \cap \mathcal{U} = \emptyset$ by the above calculation and by (4.2).
(c) $\mathcal{F} \setminus F = \mathcal{F} \cap \mathcal{U}$ by Claim (ii).

By (a), $\{K \in \mathcal{K}(E) : K^* \cap \mathcal{U} = \emptyset\}$ is compact, and therefore so is

$$M = \bigcup\{K \in \mathcal{K}(E) : K^* \cap \mathcal{U} = \emptyset\}.$$

Moreover, $M^* = \bigcup\{K^* : K^* \cap \mathcal{U} = \emptyset\}$, hence $M^* \cap \mathcal{U} = \emptyset$. Now, it follows from (c) that $M^* \cap \mathcal{F} \subseteq F$, which is nowhere dense in $\mathcal{F}$. Thus, $M \in \mathcal{I}$. Furthermore, for each $K \in \mathcal{K}(E)$ with $\mathcal{L}^K \subseteq F$ and $f^K \leq f$, by (b), we have $K \subseteq M$. This finishes the proof. □

One can give an alternative proof of Theorem 4.1 above. Instead of applying Theorem 3.1, one can apply its consequence, the condition of Theorem 5.4(iii) together with a criterion of Fremlin from [5, Proposition 3C] used much like in the proof of [5, Corollary 3E].

5. Property $(*)$ and transfinite ranks

In this section, we will introduce a rank measuring to what extent closedness of a G_δ ideal $\mathcal{I}$ under taking finite unions is reflected by approximations to $\mathcal{I}$, where by an *approximation* to $\mathcal{I}$ we understand a downward closed, open set $\mathcal{U} \subseteq \mathcal{K}(E)$ with $\mathcal{I} \subseteq \mathcal{U}$.

Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be downward closed and non-empty, and let $\mathcal{U} \subseteq \mathcal{K}(E)$ be open and downward closed. Define

$$\begin{aligned} \mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq 0 \;&\Leftrightarrow\; \mathcal{U} \neq \emptyset; \\ \mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \alpha + 1 \;&\Leftrightarrow\; \forall K \in \mathcal{I}\, \exists U \subseteq E \text{ open } (K \subseteq U \text{ and} \\ &\qquad \mathrm{add}_{\mathcal{I}}(\{L \in \mathcal{K}(E) : \overline{U} \cup L \in \mathcal{U}\}) \geq \alpha); \\ \mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \lambda \;&\Leftrightarrow\; \mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \alpha \text{ for each } \alpha < \lambda \text{ if } \lambda \text{ is limit.} \end{aligned}$$

Note that the set $\{L \in \mathcal{K}(E) : \overline{U} \cup L \in \mathcal{U}\}$ in the definition above is open and downward closed if $\mathcal{U}$ is, so the application of $\mathrm{add}_{\mathcal{I}}$ to it is justified. Note also that the last two clauses of the above definition can be replaced by one,

$$\begin{aligned} \mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \alpha \;&\Leftrightarrow\; \forall \beta < \alpha\, \forall K \in \mathcal{I}\, \exists U \subseteq E \text{ open } (K \subseteq U \text{ and} \\ &\qquad \mathrm{add}_{\mathcal{I}}(\{L \in \mathcal{K}(E) : \overline{U} \cup L \in \mathcal{U}\}) \geq \beta). \end{aligned}$$

It follows immediately from the definition that, for $\mathcal{U} \subseteq \mathcal{K}(E)$ open and downward closed and for an ordinal β, $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \beta$ implies $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \alpha$ for all $\alpha \leq \beta$. (We are using here $\mathcal{I} \neq \emptyset$.)

We point out that $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq 1$ is equivalent to saying that $\mathcal{I} \subseteq \mathcal{U}$, and that $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq 2$ is equivalent to the condition that for any $K \in \mathcal{I}$ there exists an open set $U \subseteq E$ such that $K \subseteq U$ and $\overline{U} \cup L \in \mathcal{U}$ for all $L \in \mathcal{I}$.

Some conditions ensuring some degree of closedness under taking unions of open, downward closed families containing a given ideal have been considered before. For example, they are implicit in the proof of [8, Lemma 7] and explicit in [10, p. 194]. Note however that these conditions are very strong, in particular, each of them easily implies $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \omega_1$.

The following lemma is easily proved by induction on the rank.

Lemma 5.1. *Let $\mathcal{I}$ and $\mathcal{J}$ be downward closed, non-empty families of compact subsets of E and let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{K}(E)$ be open and downward closed. If $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{U} \subseteq \mathcal{V}$, then*

$$\mathrm{add}_{\mathcal{J}}(\mathcal{U}) \leq \mathrm{add}_{\mathcal{I}}(\mathcal{V}).$$

We have the following lemma saying that $\mathrm{add}_{\mathcal{I}}(\mathcal{U})$ after reaching ω_1 is arbitrarily large.

Lemma 5.2. *Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be downward closed and non-empty, and let $\mathcal{U} \subseteq \mathcal{K}(E)$ be downward closed and open. Then $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \omega_1$ implies $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \alpha$ for each ordinal α.*

Proof. A moment's thought tells us that it suffices to show that $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \omega_1$ implies $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \omega_1 + 1$. For each $\alpha < \omega_1$, let

$$\mathcal{U}^\alpha = \{K \in \mathcal{U} : \exists U \supseteq K \text{ open } (\mathrm{add}_{\mathcal{I}}(\{L \in \mathcal{K}(E) : \overline{U} \cup L \in \mathcal{U}\}) \geq \alpha)\}.$$

By our assumption that $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \omega_1$, each $\mathcal{U}^\alpha$ contains $\mathcal{I}$. Moreover $\mathcal{U}^\beta \subseteq \mathcal{U}^\alpha$ if $\beta > \alpha$, and each $\mathcal{U}^\alpha$ is open in $\mathcal{K}(E)$. Since $\mathcal{K}(E)$ has a countable basis, it follows that there exists α_0 such that $\mathcal{U}^\alpha = \mathcal{U}^{\alpha_0}$ for $\alpha \geq \alpha_0$. This implies that

$$\{K \in \mathcal{U} : \exists U \supseteq K \text{ open } (\mathrm{add}_{\mathcal{I}}(\{L \in \mathcal{K}(E) : \overline{U} \cup L \in \mathcal{U}\}) \geq \omega_1)\}$$

contains $\mathcal{U}^{\alpha_0}$ and so contains $\mathcal{I}$, hence $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \omega_1 + 1$. □

We will view $\mathrm{add}_{\mathcal{I}}(\mathcal{U})$ as the ordinal $\sup\{\alpha < \omega_1 : \mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \alpha\}$. Note that the supremum is attained. In view of Lemma 5.2, this ordinal captures all the information contained in the rank.

Proposition 5.3. *Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be a non-empty family of sets.*

(i) *$\mathcal{I}$ is a downward closed G_δ if and only if $\mathcal{I} = \bigcap_n \mathcal{U}_n$ for a sequence of open, downward closed families $\mathcal{U}_n$.*
(ii) *If $\mathcal{I} = \bigcap_n \mathcal{U}_n$, with $\mathcal{U}_n \subseteq \mathcal{K}(E)$ open and downward closed, and $\mathrm{add}_{\mathcal{I}}(\mathcal{U}_n) \geq 2$ for each n, then $\mathcal{I}$ is a σ-ideal.*

Proof. We leave proving (i), which is well known, to the reader. To see (ii) assume that $\mathcal{I}$ is not a σ-ideal. By an observation due to Kechris [6] (see also [11, Lemma 2.1]) we can find compact sets K_0 and L_0 such that $L_0 \subseteq K_0$, $L_0 \in \mathcal{I}$, $K_0 \notin \mathcal{I}$, and for each open set V with $L_0 \subseteq V$, $K_0 \setminus V \in \mathcal{I}$. Since $K_0 \notin \mathcal{I}$, we can fix an n with $K_0 \notin \mathcal{U}_n$. Since $\mathrm{add}_{\mathcal{I}}(\mathcal{U}_n) \geq 2$, there exists an open $V \supseteq L_0$ such that $\overline{V} \cup K \in \mathcal{U}_n$ for each $K \in \mathcal{I}$. Since $K_0 \setminus V \in \mathcal{I}$, $\overline{V} \cup (K_0 \setminus V) \in \mathcal{U}_n$. But then $K_0 \subseteq \overline{V} \cup (K_0 \setminus V) \in \mathcal{U}_n$, so $K \in \mathcal{U}_n$, contradiction. □

The ideal $\mathcal{I}_1$ constructed in Theorem 6.1 below shows that the implication in point (ii) of the proposition above cannot be reversed even if $\mathcal{I}$ is assumed to be G_δ.

In light of Proposition 5.3, it is natural to consider the following ordinal as a rank measuring additivity of a downward closed G_δ family $\mathcal{I}$ of compact sets:

$$\sup\Big\{\alpha < \omega_1 : \exists (\mathcal{U}_n)_{n\in\omega} \Big(\mathcal{U}_n \subseteq \mathcal{K}(E) \text{ open and downward closed}, \\ \mathcal{I} = \bigcap_n \mathcal{U}_n, \text{ and } \mathrm{add}_{\mathcal{I}}(\mathcal{U}_n) \geq \alpha \text{ for each } n\Big)\Big\}. \quad (5.1)$$

The theorem below shows that property $(*)$ is equivalent to having the largest possible value ω_1 of this rank.

Theorem 5.4. *For non-empty $\mathcal{I} \subseteq \mathcal{K}(E)$ the following conditions are equivalent:*

(i) *$\mathcal{I}$ is G_δ and has $(*)$;*

(ii) *for each $\alpha < \omega_1$ there exist $\mathcal{U}_n \subseteq \mathcal{K}(E)$, $n \in \omega$, open and downward closed and such that $\mathcal{I} = \bigcap_n \mathcal{U}_n$ and $\mathrm{add}_{\mathcal{I}}(\mathcal{U}_n) \geq \alpha$;*

(iii) *there exist $\mathcal{U}_n \subseteq \mathcal{K}(E)$, $n \in \omega$, open and downward closed and such that $\mathcal{I} = \bigcap_n \mathcal{U}_n$ and for each $K \in \mathcal{U}_n$ there exists m with $K \cup L \in \mathcal{U}_n$ for each $L \in \mathcal{U}_m$.*

Proof. (ii)⇒(i). We will produce a countable family $\mathcal{A}$ of open, downward closed families $\mathcal{U}$ with

$$\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \omega_1 \quad \text{and} \quad \mathcal{I} = \bigcap \{\mathcal{U} : \mathcal{U} \in \mathcal{A}\}.$$

Let $\mathcal{K} \subseteq \mathcal{K}(E)$ be a non-empty compact set disjoint from $\mathcal{I}$. We start by proving that there exists $V \subseteq \mathcal{K}$ relatively open and non-empty such that

$$\forall \alpha < \omega_1 \; \exists \mathcal{U} \; (\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \alpha \text{ and } \mathcal{U} \cap V = \emptyset). \tag{5.2}$$

Fix a countable basis $\mathcal{B}$ of the topology on $\mathcal{K}$ consisting of non-empty sets. By (ii), for each $\alpha < \omega_1$ there exist $\mathcal{U}_n^\alpha$ with $\mathrm{add}_{\mathcal{I}}(\mathcal{U}_n^\alpha) \geq \alpha$ and $\mathcal{I} = \bigcap_n \mathcal{U}_n^\alpha$. Thus, we can find $V_\alpha \in \mathcal{B}$ such that $V_\alpha \cap \mathcal{U}_n^\alpha = \emptyset$ for some n. Since $\mathcal{B}$ is countable, there exists a fixed $V \in \mathcal{B}$ with $V = V_\alpha$ for uncountably many $\alpha < \omega_1$. This V clearly works.

Now fix V as above and let

$$\mathcal{V} = \{K \in \mathcal{K}(E) : \forall L \in \overline{V} \; L \not\subseteq K\}.$$

This set is open and downward closed and, by (5.2), it contains for each $\alpha < \omega_1$ an open and downward closed family $\mathcal{U}$ with $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \alpha$. Since, by Lemma 5.1, the rank $\mathrm{add}_{\mathcal{I}}$ is larger on bigger families, we have $\mathrm{add}_{\mathcal{I}}(\mathcal{V}) \geq \omega_1$. Note also that $\mathcal{V} \cap V = \emptyset$. Now iterating this argument one produces a transfinite sequence of open and downward closed sets $\mathcal{V}_\xi$ and sets $V_\xi \subseteq \mathcal{K}$ (each of which is the intersection of a closed subset and an open subset of $\mathcal{K}$) with $\xi < \xi_0$ for some countable ordinal ξ_0 such that $\mathrm{add}_{\mathcal{I}}(\mathcal{V}_\xi) \geq \omega_1$, $\mathcal{V}_\xi \cap V_\xi = \emptyset$, and $\mathcal{K} = \bigcup_{\xi < \xi_0} V_\xi$. In particular, $\mathcal{K} \cap \bigcap_{\xi < \xi_0} \mathcal{V}_\xi = \emptyset$. Since by (ii) the complement of $\mathcal{I}$ is a countable union of compact sets, we obtain the desired family $\mathcal{A}$.

Having a countable family $\mathcal{A}$ of open, downward closed $\mathcal{U}$ with $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \omega_1$ and with $\mathcal{I} = \bigcap \{\mathcal{U} : \mathcal{U} \in \mathcal{A}\}$, to show $(*)$ for $\mathcal{I}$ it suffices to prove the following: if $\mathcal{U}$ is an open, downward closed family with $\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \omega_1$ and $K_n \in \mathcal{I}$, $n \in \omega$, then there exists an open $W \subseteq E$ such that $\bigcup_n K_n \subseteq W$ and $\mathcal{K}(W) \subseteq \mathcal{U}$. We now prove this statement. We construct open, downward closed $\mathcal{W}_n$ with $\mathrm{add}_{\mathcal{I}}(\mathcal{W}_n) \geq \omega_1$ and open sets $W_n \subseteq E$ as follows. Put $\mathcal{W}_0 = \mathcal{U}$. Assume $\mathcal{W}_n$ has been defined. Since $\mathrm{add}_{\mathcal{I}}(\mathcal{W}_n) \geq \omega_1$, by Lemma 5.2 we have $\mathrm{add}_{\mathcal{I}}(\mathcal{W}_n) \geq \omega_1 + 1$, and therefore the family

$$\{K \in \mathcal{W}_n : \exists U \supseteq K \text{ open } (\mathrm{add}_{\mathcal{I}}(\{L \in \mathcal{K}(E) : \overline{U} \cup L \in \mathcal{W}_n\}) \geq \omega_1)\}$$

contains $\mathcal{I}$. Pick $W_n \subseteq E$ open so that $K_n \subseteq W_n$ and

$$\mathrm{add}_{\mathcal{I}}(\{L \in \mathcal{K}(E) : \overline{W}_n \cup L \in \mathcal{W}_n\}) \geq \omega_1.$$

Let

$$\mathcal{W}_{n+1} = \{L \in \mathcal{K}(E) : \overline{W}_n \cup L \in \mathcal{W}_n\}.$$

After completing the above construction, define $W = \bigcup_n W_n$. Clearly $\bigcup_n K_n \subseteq W$. It remains to show that $\mathcal{K}(W) \subseteq \mathcal{U}$. Let $K \subseteq W$ be compact. Fix N with $K \subseteq W_0 \cup \cdots \cup W_N$. By the very definitions of W_N and $\mathcal{W}_N$ we obtain $\overline{W}_N \in \mathcal{W}_N$ and then inductively, assuming $\overline{W_{n+1} \cup \cdots \cup W_N} \in \mathcal{W}_{n+1}$, we get

$$\overline{W}_n \cup (\overline{W_{n+1} \cup \cdots \cup W_N}) \in \mathcal{W}_n.$$

Thus, ultimately we have

$$\overline{W_0 \cup \cdots \cup W_N} \in \mathcal{W}_0 = \mathcal{U}.$$

Therefore $K \in \mathcal{U}$.

(iii)⇒(ii). From the definition of the rank $\mathrm{add}_{\mathcal{I}}$, one checks that, for each ordinal $\alpha < \omega_1$, for a sequence $\mathcal{U}_n$ as in (iii) we have $\mathrm{add}_{\mathcal{I}}(\mathcal{U}_n) \geq \alpha$ for each n. This is done by induction on α for all n simultaneously.

(i)⇒(iii). Since $\mathcal{I}$ has property $(*)$ and is G_δ, we can find a compact subset $\mathcal{F}$ of $\mathcal{K}(E)$ as in Theorem 3.1. Let V_i, $i \in \omega$, be an open basis of $\mathcal{F}$ consisting of non-empty sets. Now for $K \in \mathcal{K}(E)$, let $K \in \mathcal{U}_n$ precisely when

$$\forall i \leq n \; V_i \setminus K^* \neq \emptyset.$$

It is easy to check that each $\mathcal{U}_n$ is downward closed, open, and that $\mathcal{I} = \bigcap_n \mathcal{U}_n$. To see that the sequence $(\mathcal{U}_n)_n$ fulfills the condition in point (iii), let $K \in \mathcal{U}_n$. Find $m \geq n$ large enough so that for each $i \leq n$ there exists $i' \leq m$ with $V_{i'} \subseteq V_i \setminus K^*$. Then clearly $K \cup L \in \mathcal{U}_n$ for any $L \in \mathcal{U}_m$. □

The implication (iii)⇒(i) in the theorem above can be established directly with a compactness argument. In fact, this argument is implicit in the proof of [8, Lemma 7] and can be used to show the following. If $\mathcal{I} = \bigcap_n \mathcal{U}_n$ and for each n and each $K \in \mathcal{U}_n$, the set $\{L \in \mathcal{K}(E) : K \cup L \in \mathcal{U}_n\}$ contains $\mathcal{I}$, then $\mathcal{I}$ has $(*)$. Note that the property of $\mathcal{U}_n$ in the assumption of this implication easily gives that $\mathrm{add}_{\mathcal{I}}(\mathcal{U}_n) \geq \omega_1$.

Games

The results of Section 6 will need a closer analysis of the rank $\mathrm{add}_{\mathcal{I}}$. Originally, this goal was achieved by introducing a new rank $\mathrm{der}_{\mathcal{I}}$, whose definition and basic properties are stated in the following subsection. Subsequently, Alain Louveau indicated that the same goal can be reached with a game-theoretic approach. Since this point of view gives a more intuitive presentation, we opted for it in this final version of the paper. The definitions of the games below and modifications of the proofs of Lemmas 6.3 and 6.4 to the game-theoretic context are due to Louveau.

Given a downward closed, non-empty family $\mathcal{I} \subseteq \mathcal{K}(E)$, an open, downward closed family $\mathcal{U} \subseteq \mathcal{K}(E)$, and an ordinal α, we define the following game $G_\alpha(\mathcal{I}, \mathcal{U})$. It is played by two players. Player I plays $K_i \in \mathcal{I}$ and ordinals $\xi_i < \alpha$ so that $\xi_{i+1} < \xi_i$ for all $i \geq 0$, while Player II plays open sets $U_i \subseteq E$ so that $K_i \subseteq U_i$ and $\overline{U}_i \subseteq U_{i+1}$ for all $i \geq 0$.

The players take turns with Player I making the first move. Note that the empty sequence is the only run of the game $G_0(\mathcal{I},\mathcal{U})$. A run of the game $G_\alpha(\mathcal{I},\mathcal{U})$ terminates with a move of Player II after finitely many steps since there is no strictly decreasing sequence of ordinals. If $\alpha = 0$, we declare Player II the winner of the run of the game $G_0(\mathcal{I},\mathcal{U})$ if and only if $\mathcal{U} \neq \emptyset$. (So saying that Player II has a winning strategy in $G_0(\mathcal{I},\mathcal{U})$ is a complicated way of expressing the condition $\mathcal{U} \neq \emptyset$.) If $\alpha > 0$ and U_p is this last move of Player II, Player II is declared the winner of the run of $G_\alpha(\mathcal{I},\mathcal{U})$ if $\overline{U_p} \in \mathcal{U}$.

We have the following lemma connecting the games $G_\alpha(\mathcal{I},\mathcal{U})$ with the rank $\mathrm{add}_{\mathcal{I}}(\mathcal{U})$. This is the only result about the games that will be used in what follows.

Lemma 5.5. *Let $\mathcal{I}$ be a downward closed, non-empty family of compact subsets of E and let $\mathcal{U}$ be a downward closed, open family of compact subsets of E. Then for each ordinal α,*

$$\mathrm{add}_{\mathcal{I}}(\mathcal{U}) \geq \alpha \;\Leftrightarrow\; \textit{II has a winning strategy in } G_\alpha(\mathcal{I},\mathcal{U}), \tag{5.3}$$

Proof. The lemma is proved by a straightforward induction on α. Indeed, for $\alpha = 0$, (5.3) holds since both sides of it are equivalent to $\mathcal{U} \neq \emptyset$. To see that (5.3) holds for limit α assuming it holds for all ordinals $< \alpha$ it suffices to notice that if Player II has a winning strategy in $G_\xi(\mathcal{I},\mathcal{U})$ for each $\xi < \alpha$, then he also has a winning strategy in $G_\alpha(\mathcal{I},\mathcal{U})$. Showing (5.3) for $\alpha + 1$ assuming that it holds for α requires only noticing that after Players I and II make their first moves (K_0, ξ_0), U_0 in $G_{\alpha+1}(\mathcal{I},\mathcal{U})$, the rest of the run of the game is a run of the game

$$G_{\xi_0}(\mathcal{I}, \{L \in \mathcal{K}(E) : \overline{U}_0 \cup L \in \mathcal{U}\}). \qquad \square$$

There is another game connected directly to property $(*)$. The game $G(\mathcal{I},\mathcal{U})$ is played by two players taking turns with Player I playing $K_i \in \mathcal{I}$ and Player II playing $U_i \subseteq E$ open so that $K_i \subseteq U_i$ and $\overline{U}_i \subseteq U_{i+1}$ for all $i \geq 0$. Player II wins a run of $G(\mathcal{I},\mathcal{U})$ if $\overline{U}_i \in \mathcal{U}$ for each i. The following result can be proved.

(i) *Player II has a winning strategy in $G(\mathcal{I},\mathcal{U})$ if and only if he has a winning strategy in $G_\alpha(\mathcal{I},\mathcal{U})$ for each $\alpha < \omega_1$.*
(ii) *$\mathcal{I}$ has property $(*)$ if and only if $\mathcal{I} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$ with II having a winning strategy in $G(\mathcal{I},\mathcal{U}_n)$ for each n.*

We omit a detailed proof of the above fact leaving it to the reader. We only notice that the proof of (i) consists of a standard argument used in analyzing open games. Point (ii) follows from (i), Lemma 5.5, and Theorem 5.4.

Another rank

Let us make the following observation. Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be downward closed and non-empty. An application of the Baire category theorem proves that the following two conditions are equivalent.

- If $K \in \mathcal{K}(E)$, $K_n \in \mathcal{I}$ for $n \in \omega$, and $K = \bigcup_n K_n$, then $K \in \mathcal{I}$.
- For any $K \in \mathcal{K}(E) \setminus \mathcal{I}$, there exists $L \subseteq K$ with $\emptyset \neq L \in \mathcal{K}(E)$ such that for any open set U if $U \cap L \neq \emptyset$, then $\overline{U \cap L} \notin \mathcal{I}$.

Of course, the first of these conditions says that $\mathcal{I}$ is a σ-ideal, while the second one asserts the existence of a perfect, with respect to $\mathcal{I}$, part in each compact set not in $\mathcal{I}$. The two conditions give rise to two ranks on approximations of a downward closed, non-empty family $\mathcal{I}$. Approximations are understood as either downward closed, open sets containing $\mathcal{I}$ or upward closed, closed sets disjoint from $\mathcal{I}$, and the two ranks measure invariance of the first type of approximations under taking unions, and invariance of the second type of approximations under taking perfect parts. The first of these ranks, called $\mathrm{add}_\mathcal{I}$, was introduced above. The second one, called $\mathrm{der}_\mathcal{I}$, is introduced below. This second rank will not be used in the remainder of this paper but may be of some independent interest.

Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be downward closed and non-empty. For $\mathcal{F} \subseteq \mathcal{K}(E)$ upward closed, that is, closed under taking compact supersets, define

$$\mathrm{d}_\mathcal{I}(\mathcal{F}) = \{L \in \mathcal{K}(E) : \exists K \in \mathcal{F}\ L \subseteq K \text{ and } \overline{K \setminus L} \in \mathcal{I}\}.$$

Note that $\mathrm{d}_\mathcal{I}(\mathcal{F})$ is upward closed, $\mathrm{d}_\mathcal{I}(\mathcal{F}) \supseteq \mathcal{F}$, and if $\mathcal{I}$ is an ideal, then $\mathrm{d}_\mathcal{I}(\mathrm{d}_\mathcal{I}(\mathcal{F})) = \mathrm{d}_\mathcal{I}(\mathcal{F})$.

Let $\mathcal{F} \subseteq \mathcal{K}(E)$ be closed and upward closed. For $\alpha < \omega_1$ define $\mathrm{d}^\alpha_\mathcal{I}(\mathcal{F})$ as follows:

$$\begin{aligned} \mathrm{d}^0_\mathcal{I}(\mathcal{F}) &= \mathcal{F}; \\ \mathrm{d}^{\alpha+1}_\mathcal{I}(\mathcal{F}) &= \overline{\mathrm{d}_\mathcal{I}(\mathrm{d}^\alpha_\mathcal{I}(\mathcal{F}))}; \\ \mathrm{d}^\lambda_\mathcal{I}(\mathcal{F}) &= \bigcup_{\alpha<\lambda} \mathrm{d}^\alpha_\mathcal{I}(\mathcal{F}) \quad \text{if } \lambda \text{ is limit.} \end{aligned}$$

The last two equalities in the definition above can be replaced by one,

$$\mathrm{d}^\alpha_\mathcal{I}(\mathcal{F}) = \bigcup_{\beta<\alpha} \overline{\mathrm{d}_\mathcal{I}(\mathrm{d}^\beta_\mathcal{I}(\mathcal{F}))}.$$

Now define a rank $\mathrm{der}_\mathcal{I}$ on closed, upward closed subsets $\mathcal{F}$ of $\mathcal{K}(E)$ by letting

$$\mathrm{der}_\mathcal{I}(\mathcal{F}) \geq \alpha \iff \mathrm{d}^\alpha_\mathcal{I}(\mathcal{F}) \cap \mathcal{I} = \emptyset.$$

The following fact, whose proof we omit, shows that the two ranks introduced in this section are closely related.

Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be downward closed and non-empty, and let $\mathcal{U} \subseteq \mathcal{K}(E)$ be open and downward closed. Then

$$\mathrm{add}_\mathcal{I}(\mathcal{U}) = 1 + \mathrm{der}_\mathcal{I}(\mathcal{K}(E) \setminus \mathcal{U}).$$

6. Unboundedness of the rank

The aim of this section is to prove that there exist G_δ ideals of compact sets with arbitrary countable positive value of the rank defined by (5.1). For the sake of the short discussion below, set

$$\mathrm{add}(\mathcal{I}) = \sup\Big\{\alpha < \omega_1 : \exists (\mathcal{U}_n)_{n\in\omega} \Big(\mathcal{U}_n \subseteq \mathcal{K}(E) \text{ open and downward closed,} \\ \mathcal{I} = \bigcap_n \mathcal{U}_n, \text{ and } \mathrm{add}_\mathcal{I}(\mathcal{U}_n) \geq \alpha \text{ for each } n\Big)\Big\}$$

for a non-empty, downward closed G_δ family $\mathcal{I} \subseteq \mathcal{K}(E)$. By Proposition 5.3 for any such $\mathcal{I}$ we have $0 < \mathrm{add}(\mathcal{I}) \leq \omega_1$. By Theorem 5.4, $\mathrm{add}(\mathcal{I}) = \omega_1$ is equivalent to $\mathcal{I}$ having property $(*)$, and Section 2 gives a large number of examples of G_δ ideals with $(*)$. In the theorem below we complete the picture, that is, for each ordinal α with $1 < \alpha < \omega_1$ we construct a G_δ ideal $\mathcal{I}_\alpha$ with $\mathrm{add}(\mathcal{I}_\alpha) = \alpha$. The main combinatorial tool in the construction will be certain objects coming from the structure theory of Banach spaces.

Theorem 6.1. *For each $0 < \alpha < \omega_1$, there exists an ideal $\mathcal{I}_\alpha$ of compact subsets of 2^ω with the following three properties:*

(i) *there exist open, downward closed $\mathcal{U}_n \subseteq \mathcal{K}(E)$, $n \in \omega$, with*

$$\mathcal{I}_\alpha = \bigcap_n \mathcal{U}_n \quad \textit{and} \quad \mathrm{add}_{\mathcal{I}_\alpha}(\mathcal{U}_n) = \alpha \textit{ for each } n;$$

(ii) *if $\mathcal{I}_\alpha = \bigcap_n \mathcal{V}_n$ where each $\mathcal{V}_n$ is open and downward closed, then there exists n with $\mathrm{add}_{\mathcal{I}_\alpha}(\mathcal{V}_n) \leq \alpha$;*

(iii) *$\mathcal{I}_\alpha$ contains all singletons of 2^ω, and each non-meager subset of 2^ω that has the Baire property contains a compact subset not in $\mathcal{I}_\alpha$.*

It can be shown that for Mátrai's G_δ ideal $\mathcal{J}$ from [14] the following is true: if $\mathcal{J} = \bigcap_n \mathcal{V}_n$ with $\mathcal{V}_n \subseteq \mathcal{K}(E)$ open and downward closed, then $\mathrm{add}_{\mathcal{I}}(\mathcal{V}_n) = 1$ for some n. In this respect $\mathcal{J}$ is similar to the ideal $\mathcal{I}_1$ from the theorem above. These two ideals are however distinct. For example, one can check that $\mathcal{I}_1$ is translation invariant, while $\mathcal{J}$ is not (when 2^ω is treated as the infinite countable product of $\mathbb{Z}/2$). In fact, each ideal $\mathcal{I}_\alpha$ from Theorem 6.1 is translation invariant.

We now introduce combinatorial objects, called block sequences, families of which will be important in defining ideals of compact sets whose existence is asserted in Theorem 6.1. These objects resemble block sequences used in the study of Banach spaces (see [1]).

Let FF be the family of all non-empty partial functions from ω to $\{0, 1\}$ whose domains are finite intervals. For two such functions s and t we write

$$s < t \Leftrightarrow (i < j \text{ for all } i \in \mathrm{dom}(s) \text{ and } j \in \mathrm{dom}(t)).$$

By a *block sequence* we mean a sequence $\bar{s} = (s_0, \ldots, s_n)$ of elements of FF with $s_0 < s_1 < \cdots < s_n$. A block sequence can be empty. For two block sequences $\bar{s} = (s_0, \ldots, s_m)$ and $\bar{t} = (t_0, \ldots, t_n)$, let

$$\bar{s} < \bar{t} \Leftrightarrow (s_i < t_j \text{ for all } i \leq m \text{ and } j \leq n).$$

In particular, $\bar{s} < \emptyset < \bar{s}$ for any block sequence $\bar{s}$. For a block sequence $\bar{s} = (s_0, \ldots, s_n)$, define the domain of $\bar{s}$ by

$$\mathrm{dom}(\bar{s}) = \mathrm{dom}(s_0) \cup \cdots \cup \mathrm{dom}(s_n).$$

We now introduce a certain operation on families of block sequences. Let B be a non-empty family of non-empty block sequences. Define

$$i(B) = B \cup \{\bar{s} : \bar{s} \text{ is a block sequence and } \exists n\ (\{s \in \mathrm{FF} : \bar{s}^\frown s \in B,\ \min\mathrm{dom}(s) \leq n\} \text{ is infinite})\}.$$

Define $i^0(B) = B$ and, for a countable ordinal α,

$$i^{\alpha+1}(B) = i(i^\alpha(B)),$$

and, for a limit countable ordinal λ,

$$i^\lambda(B) = \bigcup_{\alpha<\lambda} i^\alpha(B).$$

Let

$$r(B) = \sup\{\alpha < \omega_1 : \emptyset \notin i^\alpha(B)\}.$$

Note that since all sequences in B are assumed to be non-empty, the set $\{\alpha < \omega_1 : \emptyset \notin i^\alpha(B)\}$ is non-empty and the sup in the above definition is an ordinal $\leq \omega_1$.

We now show how to associate an ideal of compact sets with each family of block sequences. We treat block sequences as codes for sets that are both closed and open, for short clopen sets, and certain sequences of block sequences as codes for closed sets. Here is how the decoding is done. For a block sequence $\bar{s} = (s_0, \ldots, s_n)$, let

$$]\bar{s}[= \{x \in 2^\omega : \exists i \leq n\ s_i \subseteq x\}.$$

Let us explicitly point out that $]\emptyset[= \emptyset$. Note that the set $]\bar{s}[$ is clopen. An infinite sequence $(\bar{s}_k)_k$ of block sequences is called *increasing* if for each k we have $\bar{s}_k < \bar{s}_{k+1}$. Similarly a finite, possibly empty, sequence $(\bar{s}_k)_{k<N}$ of block sequences is called increasing if for each $k < N-1$ we have $\bar{s}_k < \bar{s}_{k+1}$. For an increasing sequence of block sequences $(\bar{s}_k)_k$, or $(\bar{s}_k)_{k<N}$ if the sequence is finite, let

$$[(\bar{s}_k)_k] = \bigcap_k]\bar{s}_k[\quad \text{and} \quad [(\bar{s}_k)_{k<N}] = \bigcap_{k<N}]\bar{s}_k[. \tag{6.1}$$

These sets are clearly closed. We explicitly point out that for the empty increasing sequence $\emptyset$ we have $[\emptyset] = 2^\omega$. Note however that $(\emptyset)$, where $\emptyset$ is the empty block sequence, is an increasing sequence of block sequences and $[(\emptyset)] = \emptyset$.

Let B be a non-empty family of non-empty block sequences. Let $\mathrm{IN}(B)$ stand for the set of all increasing, finite or infinite, sequences with entries in B. Define

$$\mathcal{R}(B) = \{[S] : S \in \mathrm{IN}(B)\},$$
$$\mathcal{I}(B) = \{K \in \mathcal{K}(2^\omega) : \forall L \in \mathcal{R}(B)\ K \cap L \text{ is meager in } L\}.$$

Note that since $\emptyset \in \mathrm{IN}(B)$, we have $2^\omega = [\emptyset] \in \mathcal{R}(B)$. Thus, all sets in $\mathcal{I}(B)$ are nowhere dense and the following lemma is obvious.

Lemma 6.2. *For any non-empty family B of non-empty block sequences, $\mathcal{I}(B)$ is a σ-ideal of compact sets such that all sets in $\mathcal{I}(B)$ are nowhere dense.*

Lemma 6.3. *Let B be a non-empty family of non-empty block sequences and let α be a countable ordinal. If $r(B) \geq \alpha$, then there exist open, downward closed families $\mathcal{U}_n$, $n \in \omega$, such that $\mathcal{I}(B) = \bigcap_n \mathcal{U}_n$ and $\mathrm{add}_{\mathcal{I}(B)}(\mathcal{U}_n) \geq \alpha$ for each n.*

Proof. We start with a general estimate on the rank of certain downward closed, open families of closed sets. Let a non-empty family B' of non-empty block sequences be given. Consider the family

$$\mathcal{U} = \{K \in \mathcal{K}(2^\omega) : \exists O \text{ clopen } (K \subseteq O \text{ and } \forall L \in \mathcal{R}(B')\ L \nsubseteq O)\}. \tag{6.2}$$

Claim. $\mathrm{add}_{\mathcal{I}(B')}(\mathcal{U}) \geq r(B')$.

Proof of Claim. For a countable ordinal β define

$$\mathrm{IN}_\beta = \mathrm{IN}(B') \cup \{(\bar{s}_0, \dots, \bar{s}_J) : \bar{s}_0 < \cdots < \bar{s}_J, \forall k < J\ \bar{s}_k \in B', \text{ and } \bar{s}_J \in i^\beta(B')\}.$$

Note that $\mathrm{IN}_0 = \mathrm{IN}(B')$.

We now establish the following observation.

Observation 1. *Let $S_n \in \mathrm{IN}_\beta$, $n \in \omega$, be such that $[S_n] \to L$ as $n \to \infty$ for some compact set $L \subseteq 2^\omega$. Then L contains a set of the form $[T]$ for some $T \in \mathrm{IN}_{\beta+1}$.*

We split the proof into several cases. We say that an increasing sequence of block sequences S has *length m* if $S = (\bar{s}_0, \dots, \bar{s}_{m-1})$, and $\bar{s}_i$ is called the i-th element of S. If S is infinite, we say that it has infinite length.

Case 1: For each m there exists M such that for each S_n of length $> m$ the maximum of the domain of the m-th element of S_n is $\leq M$.

In this case, it is easy to check that there exists $T \in \mathrm{IN}_\beta \subseteq \mathrm{IN}_{\beta+1}$ such that $[S_n] \to [T] = L$ as $n \to \infty$. To see this, we consider two cases. If there exists N such that for infinitely many n the length of S_n is $\leq N$, then by going to a subsequence we can assume that all S_n have the same finite length. Then T can be taken to be a sequence of the same length that is the constant value of a subsequence of $(S_n)_n$. If the lengths of S_n go to infinity as $n \to \infty$, then a sequence of infinite length T can be found that is the limit of a subsequence of $(S_n)_n$, in which case $T \in \mathrm{IN}(B') \subseteq \mathrm{IN}_\beta$.

Case 2: There exists m for which there exists a subsequence S_{n_k}, $k \in \omega$, of the sequence S_n, $n \in \omega$, such that each S_{n_k} has length $> m$ and the maxima of the domains of the m-th elements of S_{n_k} go to ∞ as $k \to \infty$.

Fix $m = m_0$ smallest with the property as in Case 2. Assume also, by going to a subsequence, that each S_n has length $> m_0$ and the maxima of the domains of the m_0-th elements of S_n go to ∞ as $n \to \infty$. Let $(s_0^n, \dots, s_{k_n}^n)$ be the block sequence that is the m_0-th element of S_n.

Subcase 2a: $\min s_{k_n}^n \to \infty$ as $n \to \infty$.

In this case, it is easy to see that there exists $T \in \mathrm{IN}(B') \subseteq \mathrm{IN}_{\beta+1}$ with length m_0 such that $[T] = L$. The sequence T is obtained by taking the limit of a subsequence of $(S_n \restriction m_0)_n$. (Note that m_0 may be 0, in which case $T = \emptyset$ and $[T] = L = 2^\omega$.)

Subcase 2b: There exists N such that for infinitely many n we have $\min s^n_{k_n} \leq N$.

By going to a subsequence, we can assume that for all n we have $\min s^n_{k_n} \leq N$. In this case, it is easy to see, using the fact that $i^{\beta+1}(B') = i(i^\beta(B'))$, that there exists $T \in \mathrm{IN}_{\beta+1}$ of length m_0 such that $[T] \subseteq L$. The sequence T is obtained by taking the limit of a subsequence of

$$((S_n \restriction m_0)^\frown((s_0, \dots, s_{k_n-1})))_n.$$

(Note that k_n may be 0. In this case, $(s_0, \dots, s_{k_n-1})$ is the empty sequence, so $[(S_n \restriction m_0)^\frown((s_0, \dots, s_{k_n-1}))] = \emptyset$ and $\emptyset \in i^{\beta+1}(B')$.) Thus, Observation 1 is proved.

Let

$$\mathcal{U}_\beta = \{O : O \subseteq 2^\omega \text{ clopen and } [S] \nsubseteq O \text{ for } S \in \mathrm{IN}_\beta)\}.$$

Note that $\mathcal{U}_0$ is the family of all clopen sets in the family $\mathcal{U}$ given by (6.2). We make the following observation.

Observation 2. *Assume $K \in \mathcal{I}(B')$ and $O \in \mathcal{U}_\beta$. Let $\xi < \beta$. There exists $O' \in \mathcal{U}_\xi$ such that $K \cup O \subseteq O'$.*

To prove this observation, assume towards a contradiction that the conclusion fails. This assumption allows us to find a sequence $S_n \in \mathrm{IN}_\xi$, $n \in \omega$, and a compact set $L \subseteq K \cup O$ such that $[S_n] \to L$ as $n \to \infty$. Now, by Observation 1, L contains a set of the form $[T]$ with $T \in \mathrm{IN}_{\xi+1} \subseteq \mathrm{IN}_\beta$. Note that by the definition of IN_β, for such T either $T \in \mathrm{IN}(B')$ or $[T]$ is clopen. Thus, in either case, no relatively open, non-empty subset of $[T]$ is contained in K. (We are using here the assumption that $K \in \mathcal{I}(B')$ and also its consequence: K is nowhere dense.) It follows that $[T] \setminus O$ is empty, implying that $[T] \subseteq O$ and contradicting $O \in \mathcal{U}_\beta$.

Now we construct a winning strategy for Player II in $G_\alpha(\mathcal{I}(B'), \mathcal{U})$, which will establish the Claim by Lemma 5.5. The existence of a winning strategy for Player II in $G_0(\mathcal{I}(B'), \mathcal{U})$ is clear since $\mathcal{U} \neq \emptyset$. Let $\alpha > 0$. Assume that Player I in her first move plays $K_0 \in \mathcal{I}(B')$ and $\xi_0 < \alpha$. By Observation 2 with $O = \emptyset \in \mathcal{U}_\alpha$, there is $U_0 \in \mathcal{U}_{\xi_0}$ with $K_0 \subseteq U_0$. This is Player II's move. Assume $n > 0$ and suppose that Player I played K_{n-1} and ξ_{n-1} in her $n-1$-st move and that Player II responded with $U_{n-1} \in \mathcal{U}_{\xi_{n-1}}$. In the n-th move, if it occurs, let Player I play $K_n \in \mathcal{I}(B')$ and $\xi_n < \xi_{n-1}$. Using Observation 2, Player II responds by playing $U_n \in \mathcal{U}_{\xi_n}$ with $K_n \cup U_{n-1} \subseteq U_n$. This is a winning strategy since the last move Player II makes, after Player I played K_p and $\xi_p = 0$, is $U_p \in \mathcal{U}_0 \subseteq \mathcal{U}$. The Claim is proved.

Let now B be a non-empty family of block sequences with $r(B) \geq \alpha$. Let $s \in \mathrm{FF}$. Put

$$B_s = \{(s)\} \cup \{\bar{s} \in B : (s) < \bar{s}\}.$$

It is easy to check that $r(B) \geq \alpha$ implies $r(B_s) \geq \alpha$. Let

$$\mathcal{U}_s = \{K \in \mathcal{K}(2^\omega) : \exists O \text{ clopen } (K \subseteq O \text{ and } \forall L \in \mathcal{R}(B_s)\ L \nsubseteq O)\}.$$

Note further that since each element of $\mathcal{R}(B_s)$ is a non-empty clopen set or is a non-empty relatively clopen subset of an element of $\mathcal{R}(B)$, we have $\mathcal{I}(B) \subseteq \mathcal{I}(B_s)$. Using this inclusion along with Lemma 5.1 and with the Claim, we get

$$\alpha \leq \mathrm{add}_{\mathcal{I}(B_s)}(\mathcal{U}_s) \leq \mathrm{add}_{\mathcal{I}(B)}(\mathcal{U}_s). \tag{6.3}$$

We now show that

$$\mathcal{I}(B) = \bigcap_s \mathcal{U}_s.$$

The inclusion $\subseteq$ is clear from (6.3) since $\alpha \geq 1$. To see the other inclusion, let $K \in \mathcal{K}(E) \setminus \mathcal{I}(B)$. Then there is $(\bar{t}_k)_k \in \mathrm{IN}(B)$, finite or infinite, with $K \cap [(\bar{t}_k)_k]$ having non-empty interior in $[(\bar{t}_k)_k]$. This easily translates into the existence of $s : \{0, \dots, n-1\} \to \{0, 1\}$ for some $n \in \omega$ for which

$$](s)[\subseteq K \tag{6.4}$$

or for which there is k_0 such that $(s) < \bar{t}_{k_0}$ and

$$[(\bar{t}'_k)_k] \subseteq K, \tag{6.5}$$

where $\bar{t}'_0 = (s)$ and $\bar{t}'_k = \bar{t}_{k_0+k-1}$ for $k > 0$. Note further that $](s)[, [(\bar{t}'_k)_k] \in \mathcal{R}(B_s)$ and so $K \notin \mathcal{U}_s$ by (6.4) or (6.5), as required. □

Lemma 6.4. *Let B be a non-empty family of non-empty block sequences and let α be a countable ordinal. Assume that all finite subsets of 2^ω are in $\mathcal{I}(B)$. Assume further that for each $N \in \omega$,*

$$r(\{\bar{s} : \bar{s} \in B \text{ and } \min(\mathrm{dom}(\bar{s})) \geq N\}) \leq \alpha.$$

If $\mathcal{I}(B) = \bigcap_n \mathcal{V}_n$ for some open, downward closed families $\mathcal{V}_n$, $n \in \omega$, then for some n, $\mathrm{add}_{\mathcal{I}(B)}(\mathcal{V}_n) \leq \alpha$.

Proof. Let $\mathcal{V}_n$, $n \in \omega$, be open, downward closed and such that $\mathcal{I} = \bigcap_n \mathcal{V}_n$. Consider

$$f : \{(\bar{s}_n) \in B^\omega : \bar{s}_0 < \bar{s}_1 < \cdots\} \to \mathcal{K}(E) \setminus \mathcal{I}$$

given by $f((\bar{s}_n)) = [(\bar{s}_n)]$. This is a continuous function if B^ω is given the product topology with B being discrete, and the domain of f is G_δ. By the Baire category theorem, there exists n_0 such that $f^{-1}(\mathcal{K}(E) \setminus \mathcal{V}_{n_0})$ has non-empty interior in the domain of f. It is easy to check that in this situation there are $\bar{s}_0, \dots, \bar{s}_m \in B$ with $\bar{s}_0 < \cdots < \bar{s}_m$ and

$$\{[(\bar{s}_0, \dots, \bar{s}_m, \bar{s})] : \bar{s} \in B \text{ and } \bar{s}_m < \bar{s}\} \subseteq \mathcal{K}(E) \setminus \mathcal{V}_{n_0}. \tag{6.6}$$

Set

$$B_0 = \{\bar{s} \in B : \bar{s} > \bar{s}_m\} \quad \text{and} \quad S_0 = (\bar{s}_0, \dots, \bar{s}_m).$$

By our assumption we have $r(B_0) \leq \alpha$. So α, n_0, B_0, and S_0 are fixed.

We claim that $\mathrm{add}_{\mathcal{I}(B)}(\mathcal{V}_{n_0}) \leq \alpha$. In order to prove this inequality, in light of Lemma 5.5, it will suffice to find a winning strategy for Player I in the game $G_{\alpha+1}(\mathcal{I}(B), \mathcal{V}_{n_0})$.

We will need some auxiliary objects. For $\bar{s} \in i^{\alpha+1}(B_0) \setminus B_0$, let $r(\bar{s})$ be the unique ξ such that $\bar{s} \in i^{\xi+1}(B_0) \setminus i^{\xi}(B_0)$. To each such $\bar{s}$ we associate

$$(t_m(\bar{s}))_{m\in\omega},\ F(\bar{s}),\ x(\bar{s}),$$

where $(t_m(\bar{s}))_m$ is a sequence of elements of FF, $F(\bar{s})$ is a finite subset of 2^ω, and $x(\bar{s})$ is a function from $\omega \setminus \{0, \dots, k-1\}$, for some $k \in \omega$, to 2. This assignment is produced as follows. Setting $\xi = r(\bar{s})$, we require, for each m, $\bar{s}^\frown t_m(\bar{s}) \in i^{\xi}(B_0)$ and, if $\xi > 0$, $\bar{s}^\frown t_m(\bar{s}) \notin B_0$. Furthermore, setting $k = \min \mathrm{dom}(x(\bar{s}))$, we require $k = \min \mathrm{dom}(t_m(\bar{s}))$ for each m, $\max \mathrm{dom}(t_m(\bar{s})) \to \infty$ as $m \to \infty$, and $x(\bar{s})(j) = t_m(\bar{s})(j)$ for $j \geq k$ and large enough m. Note that the sequence $(t_m(\bar{s}))_m$ and the function $x(\bar{s})$ exist by the definition of the operation i and by compactness of 2^ω. Finally, we let

$$F(\bar{s}) = \{x \in [S_0] : x \restriction \mathrm{dom}(x(\bar{s})) = x(\bar{s})\}.$$

Note that each $F(\bar{s})$ is finite, so it is an element of $\mathcal{I}(B)$.

At stage n of the game, Player I will produce a block sequence $\bar{s}_n$. As long as $\bar{s}_n \in i^{\alpha+1}(B_0) \setminus B_0$, this block sequence will determine Player I's n-th move in $G_{\alpha+1}(\mathcal{I}(B), \mathcal{V}_{n_0})$ by the formulas

$$K_n = F(\bar{s}_n) \quad \text{and} \quad \xi_n = r(\bar{s}_n). \tag{6.7}$$

At stage 0, Player I chooses the empty block sequence as $\bar{s}_0$. By assumption, $\bar{s}_0 = \emptyset \in i^{\alpha+1}(B_0) \setminus B_0$, so $r(\emptyset)$ is defined and is $\leq \alpha$. Thus, formulas (6.7) give a legal move of Player I in the game. After stage n is completed, Player I has produced $\bar{s}_0, \dots, \bar{s}_n$ and her moves are given by formulas (6.7). They were answered by Player II playing open sets $U_0, \dots, U_n$ with $\overline{U_i} \subseteq U_{i+1}$ for all $i < n$. We assume recursively that for each $i < n$

$$[S_0^\frown \bar{s}_{i+1}] \subseteq U_i. \tag{6.8}$$

To determine $\bar{s}_{n+1}$ for Player I, find $q \in \omega$ large enough so that

$$\{x \in 2^\omega : \exists y \in K_n\ x\restriction q = y \restriction q\} \subseteq U_n. \tag{6.9}$$

Now find m large enough so that

$$t_m(\bar{s}_n)(j) = x(\bar{s}_n)(j)$$

for all $j \in \mathrm{dom}(x(\bar{s}_n))$ with $j < q$. We let

$$\bar{s}_{n+1} = \bar{s}_n^\frown t_m(\bar{s}_n).$$

Note now that since $K_n = F(\bar{s}_n)$, it follows from (6.9) that

$$[S_0^\frown (t_m(\bar{s}_n))] = \{x \in [S_0] : x \restriction \mathrm{dom}(t_m(\bar{s}_n)) = t_m(\bar{s}_n)\} \subseteq U_n.$$

It follows from this inclusion, from the inductive assumption (6.8), and from $U_i \subseteq U_n$ for $i < n$ that

$$[S_0^\frown \bar{s}_{n+1}] = [S_0^\frown \bar{s}_n] \cup [S_0^\frown (t_m(\bar{s}_n))] \subseteq U_n.$$

Thus, (6.8) holds for $i = n$. Note that the above procedure of finding $\bar{s}_{n+1}$ can be performed for an arbitrary value of $\xi_n \geq 0$. However, if additionally $\xi_n > 0$, then, by the definition of $t_m(\bar{s}_n)$, we see that $r(\bar{s}_{n+1})$ is defined and is $< \xi_n$. Thus, Player I can make her $n+1$-st move according to formulas (6.7).

We claim that the above description produces a winning strategy for Player I. Indeed, assume a run of the game is played according to this strategy. Assume we reached a stage p at which Player I played $\xi_p = 0$. Note that after Player II made his last, p-th, move U_p, Player I can still produce a block sequence $\bar{s}_{p+1}$, which is then an element of B_0. Since from (6.8) for $i = p$ we have $[S_0^\frown \bar{s}_{p+1}] \subseteq U_p$, by (6.6) we obtain $\overline{U}_p \notin \mathcal{V}_{n_0}$, witnessing the defeat of Player II. □

A family B of block sequences is called a *hedge* if for any $t_0 < t_1 < t_2 < \cdots$ with $t_i \in \mathrm{FF}$ there exists $k \in \omega$ with $(t_0, \ldots, t_k) \in B$. The reader may compare this definition with the definitions of a front and a barrier from [1].

Lemma 6.5. *Let B be a hedge. Each non-meager subset of 2^ω that has the Baire property contains a compact subset not in $\mathcal{I}(B)$.*

Proof. It will suffice to prove the conclusion for a G_δ set that is dense in some open non-empty subset of 2^ω. Let now U_n, $n \in \omega$, be open subsets of 2^ω and let $s \in \mathrm{FF}$ be such that each U_n is dense in $](s)[$ and the G_δ set in question is $\bigcap_n U_n$. For each n pick $t_k^n \in \mathrm{FF}$, $k \in \omega$, such that $s < t_0^n < t_1^n < \cdots$ and, for each k,

$$](s)[\, \cap \,](t_0^n, \ldots, t_k^n)[\, \subseteq U_n.$$

This is easily done using density of U_n in $](s)[$. Note that our assumption that B is a hedge implies that for any sequence $t_0 < t_1 < \cdots$ of elements of FF and any k_0 there exists $k_1 \geq k_0$ such that $(t_{k_0}, \ldots, t_{k_1}) \in B$. Using this reformulation, we can recursively find $\bar{s}_0, \bar{s}_1, \ldots \in B$ such that $\bar{s}_0 < \bar{s}_1 < \cdots$ and for each n,

$$\bar{s}_n = (t_{k_0}^n, \ldots, t_{k_1}^n)$$

for some $k_0 \leq k_1$ depending on n. It follows that $](s)[\, \cap \,]\bar{s}_n[\, \subseteq U_n$, and therefore

$$\emptyset \neq \,](s)[\, \cap [(\bar{s}_n)_n] = \,](s)[\, \cap \bigcap_n \,]\bar{s}_n[\, \subseteq \bigcap_n U_n.$$

Since $(\bar{s}_n)_n \in \mathrm{IN}(B)$, the above inclusion shows that $\bigcap_n U_n$ contains a non-empty relatively clopen subset of a set from $\mathcal{R}(B)$, hence a compact set not in $\mathcal{I}(B)$. □

Finally a general lemma.

Lemma 6.6. *Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be downward closed and non-empty, and let $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{K}(E)$ be open and downward closed. Let α be an ordinal. If $\mathrm{add}_{\mathcal{I}}(\mathcal{U}_1) \geq \alpha$ and $\mathrm{add}_{\mathcal{I}}(\mathcal{U}_2) \geq \alpha$, then $\mathrm{add}_{\mathcal{I}}(\mathcal{U}_1 \cap \mathcal{U}_2) \geq \alpha$.*

Proof. This is proved by induction. For $\alpha = 0$ it amounts to the observation that if $\mathcal{U}_1$ and $\mathcal{U}_2$ are non-empty, then, since $\mathcal{U}_1$ and $\mathcal{U}_2$ are downward closed, $\emptyset \in \mathcal{U}_1$ and $\emptyset \in \mathcal{U}_2$, so $\mathcal{U}_1 \cap \mathcal{U}_2$ is non-empty. The rest of the induction argument is equally easy and is left to the reader. □

Proof of Theorem 6.1. For a family B of block sequences and $n \in \omega$, let

$$(B)_n = \{\bar{s} \in B : n < \min \mathrm{dom}(\bar{s})\}.$$

For each $0 < \alpha < \omega_1$, we will produce a family of block sequences B_α such that

(a) each block sequence in B_α has length ≥ 2,
(b) B_α is a hedge,
(c) $r(B_\alpha) \geq \alpha$,
(d) $r((B_\alpha)_n) \leq \alpha$ for each $n \in \omega$.

It follows from the first of these conditions that no set in $\mathcal{R}(B_\alpha)$ has isolated points; thus, $\mathcal{I}(B_\alpha)$ contains all singletons of elements of 2^ω. This, together with (b) which, by Lemma 6.5, implies that each non-meager subset of 2^ω with Baire property contains an element not in $\mathcal{I}(B_\alpha)$, gives (iii) of the theorem. Condition (d) combined with Lemma 6.4 and (iii) of the theorem yields (ii) of the theorem. Condition (c) and Lemma 6.3 produce a sequence $\mathcal{U}_n$ of open, downward closed families with $\bigcap_n \mathcal{U}_n = \mathcal{I}(B_\alpha)$ and $\mathrm{add}_{\mathcal{I}(B_\alpha)}(\mathcal{U}_n) \geq \alpha$. By Lemma 6.6, we can assume that for each n we have $\mathcal{U}_n \supseteq \mathcal{U}_{n+1}$. By (ii), we know that for some n, $\mathrm{add}_{\mathcal{I}(B_\alpha)}(\mathcal{U}_n) \leq \alpha$, hence, by Lemma 5.1, this inequality holds for all large enough n. We get (i) of the theorem by deleting a finite initial segment of the sequence $(\mathcal{U}_n)_n$. Thus, the theorem will be proved once the sets B_α have been produced.

Let

$$B_1 = \{(s_0, s_1) : s_0, s_1 \in \mathrm{FF} \text{ and } s_0 < s_1\}. \tag{6.10}$$

One easily checks conditions (a)–(d).

Assume the construction has been carried out for all $\alpha < \alpha_0$. Let s_k, $k \in \omega$, be an injective enumeration of FF. Let B_k, $k \in \omega$, be hedges fulfilling (a) and such that $r(B_k) \leq r(B_{k+1})$, for any $\alpha < \alpha_0$ there is k with $r(B_k) \geq \alpha$, and for each k and n we have $\alpha_0 > r((B_k)_n)$. Such families exist by our recursive assumption. Define

$$B_{\alpha_0} = \bigcup_k \{(s_k)^\frown \bar{t} : \bar{t} \in (B_k)_{\max \mathrm{dom}(s_k)}\}.$$

Conditions (a) and (b) for B_{α_0} follow readily from the definition of this family. To show (c), note that, by our choice of B_k and since $r(B_k) \leq r((B_k)_{\max \mathrm{dom}(s_k)})$, for each $\alpha < \alpha_0$ there are only finitely many k with $(s_k) \in i^\alpha(B_{\alpha_0})$. Thus, the smallest ordinal α with $(s_k) \in i^\alpha(B_{\alpha_0})$ for infinitely many k is $\geq \alpha_0$. It follows that $\emptyset \notin i^{\alpha_0}(B_{\alpha_0})$, hence $r(B_{\alpha_0}) \geq \alpha_0$. On the other hand, again by the choice of B_k, for each $n \in \omega$, $(s_k) \in i^{\alpha_0}((B_{\alpha_0})_n)$ for each k such that $\min \mathrm{dom}(s_k) > n$. Since there exist infinitely many k with $\min \mathrm{dom}(s_k) = n + 1$, we get $\emptyset \in i^{\alpha_0+1}((B_{\alpha_0})_n)$. Thus, $r((B_{\alpha_0})_n) \leq \alpha_0$. □

We still need to prove Proposition 2.4. This proposition shows that in Proposition 2.2 one cannot remove the assumption of thinness, that is, for G_δ ideals calibration alone does not imply property $(*)$. One can modify the ideals $\mathcal{I}_\alpha$ from Theorem 6.1 so that they retain the properties from that theorem and additionally become calibrated. However, we will only present an argument showing calibration of the ideal $\mathcal{I}_1$.

Proof of Proposition 2.4. Recall that the ideal $\mathcal{I}_1$ from Theorem 6.1 is defined to be $\mathcal{I}_1 = \mathcal{I}(B_1)$, where B_1 is given by (6.10). We prove that this ideal is calibrated. This will suffice since it follows from Theorems 6.1 and 5.4 that $\mathcal{I}_1$ does not have $(*)$. Thus, we need to prove that if $K_n \in \mathcal{I}_1$, $n \in \omega$, and $K \in \mathcal{K}(2^\omega) \setminus \mathcal{I}_1$, then $K \setminus \bigcup_n K_n$ contains a compact set not in $\mathcal{I}_1$. Since each compact set not in $\mathcal{I}_1$ contains a set of the form

$$L = [((s), (s_0^0, s_1^0), (s_0^1, s_1^1), (s_0^2, s_1^2), \dots)] \tag{6.11}$$

where, for each n, $s, s_0^n, s_1^n \in \mathrm{FF}$ and $s < s_0^0 < s_1^0 < s_0^1 < s_1^1 < \cdots$, and each compact set in $\mathcal{I}_1$ has nowhere dense intersection with such a set L, it suffices to show that for L as above and for any compact sets $L_n \subseteq L$, $n \in \omega$, that are nowhere dense in L there is a set of the form (6.11) contained in $L \setminus \bigcup_n L_n$. This is done as follows. Since L_0 is nowhere dense in L, it is easy to see that there exist $m_0 \in \omega$ and $i_0, \dots, i_{n_0} \in \{0, 1\}$ such that

$$\{x \in 2^\omega : s \subseteq x \text{ and } s_{i_0}^0{}^\frown \cdots {}^\frown s_{i_{m_0}}^{m_0} \subseteq x\} \cap L_0 = \emptyset. \tag{6.12}$$

By the same argument, there exist $n_0 > m_0$ and $i_{m_0+1}, \dots, i_{n_0} \in \{0, 1\}$ such that

$$\{x \in 2^\omega : s \subseteq x \text{ and } s_{i_{m_0+1}}^{m_0+1}{}^\frown \cdots {}^\frown s_{i_{n_0}}^{n_0} \subseteq x\} \cap L_0 = \emptyset. \tag{6.13}$$

Note that $s_{i_0}^0{}^\frown \cdots {}^\frown s_{i_{m_0}}^{m_0}$ and $s_{i_{m_0+1}}^{m_0+1}{}^\frown \cdots {}^\frown s_{i_{n_0}}^{n_0}$ may not be elements of FF since their domains may not be intervals. Let t_0^0 be an element of FF that extends $s_{i_0}^0{}^\frown \cdots {}^\frown s_{i_{m_0}}^{m_0}$ and with the interval $[\min \mathrm{dom}(s_{i_0}^0), \max \mathrm{dom}(s_{i_{m_0}}^{m_0})]$ as its domain. Similarly let t_1^0 be an element of FF extending $s_{i_{m_0+1}}^{m_0+1}{}^\frown \cdots {}^\frown s_{i_{n_0}}^{n_0}$ and whose domain is the interval $[\min \mathrm{dom}(s_{i_{m_0+1}}^{m_0+1}), \max \mathrm{dom}(s_{i_{n_0}}^{n_0})]$. Then, by (6.12) and (6.13), we have

$$[(s), (t_0^0, t_1^0)] \cap L_0 = \emptyset.$$

In a similar fashion, we find $t_0^1, t_1^1 \in \mathrm{FF}$ and $m_1, n_1 \in \omega$ so that $n_0 < m_1 < n_1$, t_0^1 extends $s_{i_{n_0+1}}^{n_0+1}{}^\frown \cdots {}^\frown s_{i_{m_1}}^{m_1}$ for some choice of $i_{n_0+1}, \dots, i_{m_1} \in \{0, 1\}$ and its domain is the interval $[\min \mathrm{dom}(s_{i_{n_0+1}}^{n_0+1}), \max \mathrm{dom}(s_{i_{m_1}}^{m_1})]$, t_1^1 extends $s_{i_{m_1+1}}^{m_1+1}{}^\frown \cdots {}^\frown s_{i_{n_1}}^{n_1}$ again for some choice of $i_{m_1+1}, \dots, i_{n_1} \in \{0, 1\}$ and its domain is the interval $[\min \mathrm{dom}(s_{i_{m_1+1}}^{m_1+1}), \max \mathrm{dom}(s_{i_{n_1}}^{n_1})]$, and finally $[(s), (t_0^1, t_1^1)] \cap L_1 = \emptyset$. Continuing in this way, we obtain $t_0^n, t_1^n \in \mathrm{FF}$ so that in the end we have

$$[(s), (t_0^0, t_1^0), (t_0^1, t_1^1), \dots] \cap \bigcup_n L_n = \emptyset.$$

Note that our choice of $t_0^0, t_1^0, t_0^1, t_1^1, \dots$ was arranged so that

$$[(s), (t_0^0, t_1^0), (t_0^1, t_1^1), \dots] \subseteq L.$$

Obviously, the set $[(s), (t_0^0, t_1^0), (t_0^1, t_1^1), \dots]$ is of the form (6.11), so we are done. □

7. Questions

The following question is motivated by Propositions 2.2 and 2.4.

1. Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be a thin G_δ ideal. Does $\mathcal{I}$ have $(*)$?

An affirmative answer to the following question would generalize Theorem 3.1. Note that this theorem gives an affirmative answer to the question when the set A is compact. It also shows that there exists $\mathcal{F}$ for which the implication $\Rightarrow$ holds.

2. Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be a G_δ ideal with $(*)$. Is it true that there exists a compact family $\mathcal{F} \subseteq \mathcal{K}(E)$ such that for any G_δ set $A \subseteq E$,

   A can be covered by countably many elements of $\mathcal{I}$ if and only if $A^* \cap \mathcal{F}$ is meager in $\mathcal{F}$?

An affirmative answer to the next question would generalize Theorem 3.2. Again, this theorem provides an affirmative answer in the case when A is compact. Also it shows that there exists $\mathcal{F}$ for which the implication $\Rightarrow$ holds in the first equivalence mentioned in the question and the implication $\Leftarrow$ holds in the second one.

3. Let $\mathcal{I} \subseteq \mathcal{K}(E)$ be a calibrated G_δ ideal with $(*)$. Is it true that there exists a compact family $\mathcal{F} \subseteq \mathcal{K}(E)$ such that for any G_δ set $A \subseteq E$,

   A can be covered by countably many elements of $\mathcal{I}$ if and only if $A^* \cap \mathcal{F}$ is meager in $\mathcal{F}$, and each compact subset of A is in $\mathcal{I}$ if and only if $A^+ \cap \mathcal{F}$ is meager in $\mathcal{F}$?

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