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Scalar curvature and connected sums of self-dual 4-manifolds

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Abstract. Under a reasonable vanishing hypothesis, Donaldson and Friedman proved that the connected sum of two self-dual Riemannian 4-manifolds is again self-dual. Here we prove that the same result can be extended to the positive scalar curvature case. This is an analogue of the classical theorem of Gromov–Lawson and Schoen–Yau in the self-dual category. The proof is based on twistor theory.

1. Introduction

Let (M, g) be an oriented Riemannian n-manifold. Then the Riemann curvature tensor viewed as an operator decomposes as $\mathcal{R} = U \oplus Z \oplus W$ where

$$U = \frac{s}{2n(n-1)} g \bullet g \quad \text{and} \quad Z = \frac{1}{n-2} \mathring{\mathrm{Ric}}\, g,$$

s is the scalar curvature, $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{s}{n} g$ is the trace-free Ricci tensor, "$\bullet$" is the Kulkarni–Nomizu product, and W is the *Weyl tensor* which is defined to be what is left over from the first two pieces.

When we restrict ourselves to dimension $n = 4$, the Hodge star operator $* : \Lambda^2 \to \Lambda^2$ is an involution and has ± 1-eigenspaces decomposing the space of two-forms as $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, yielding a decomposition of any operator acting on this space. In particular $W_\pm : \Lambda^2_\pm \to \Lambda^2_\pm$ are called the self-dual and anti-self-dual pieces of the Weyl curvature operator. We say that g is a *self-dual* (resp. *anti-self-dual*) *metric* if W_- (resp. W_+) vanishes. In this case [AHS] constructs a complex 3-manifold Z called the *twistor space* of (M^4, g), which comes with a fibration by holomorphically embedded rational curves,

$$\begin{array}{ccll} \mathbb{CP}_1 & \to & Z & \text{Complex 3-manifold} \\ & & \downarrow & \\ & & M^4 & \text{Riemannian 4-manifold} \end{array}$$

This construction drew the attention of geometers, and many examples of self-dual metrics and related twistor spaces were given afterwards. One result proved to be a quite effective way to produce infinitely many examples and became a cornerstone in the field:

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Theorem 2.1 (Donaldson–Friedman, 1989, [DF]). *If (M_1, g_1) and (M_2, g_2) are compact self-dual Riemannian 4-manifolds with $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$, then the connected sum $M_1 \# M_2$ also admits a self-dual metric.*

The idea of the proof is to work upstairs in the complex category rather than downstairs. One glues the blown up twistor spaces from their exceptional divisors to obtain a singular complex space $Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2$. Then using the Kodaira–Spencer deformation theory extended by R. Friedman to singular spaces, one obtains a smooth complex manifold, which turns out to be the twistor space of the connected sum.

When working in differential geometry, one often deals with the moduli space of metrics of a certain kind. The situation is also the same for the self-dual theory. Many people, in particular LeBrun, Joyce, Pedersen, Poon, Honda, have obtained results on the space of positive scalar curvature self-dual (PSC-SD) metrics on various kinds of manifolds. Since the positivity of the scalar curvature imposes some topological restrictions on the moduli space, people often find it convenient to work under this assumption.

However one realizes that there is no connected sum theorem for self-dual positive scalar curvature metrics. Donaldson–Friedman's Theorem 2.1 does not make any statement about the scalar curvature of the metrics produced. Therefore we have attacked the problem of determining the sign of the scalar curvature for the metrics produced over the connected sum, beginning by proving the following, using techniques similar to that of [Le]:

Theorem 4.3 (Vanishing theorem). *Let $\omega : \mathcal{Z} \to \mathcal{U}$ be a 1-parameter standard deformation of $Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2$, and $\mathcal{U} \subset \mathbb{C}$ be a neighborhood of the origin. Let $L \to \mathcal{Z}$ be the holomorphic line bundle defined by*

$$\mathcal{O}(L^*) = \mathscr{I}_{\widetilde{Z}_1}(K_{\mathcal{Z}}^{1/2}).$$

If $(M_i, [g_i])$ has positive scalar curvature, then by possibly replacing $\mathcal{U}$ with a smaller neighborhood of $0 \in \mathbb{C}$ and simultaneously replacing $\mathcal{Z}$ with its inverse image we can arrange for our complex 4-fold $\mathcal{Z}$ to satisfy

$$H^1(\mathcal{Z}, \mathcal{O}(L^*)) = H^2(\mathcal{Z}, \mathcal{O}(L^*)) = 0.$$

The proof makes use of the Leray spectral sequence, homological algebra and Kodaira–Spencer deformation theory, involving many steps. Using this technical theorem we next prove that the Donaldson–Friedman theorem can be generalized to the positive scalar curvature (PSC) case:

Theorem 6.1. *Let (M_1, g_1) and (M_2, g_2) be compact self-dual Riemannian 4-manifolds with $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ for their twistor spaces. Moreover suppose that they have positive scalar curvature. Then, for all sufficiently small $\mathfrak{t} > 0$, the self-dual conformal class $[g_{\mathfrak{t}}]$ obtained on $M_1 \# M_2$ by the Donaldson–Friedman Theorem 2.1 contains a metric of positive scalar curvature.*

We work with the self-dual conformal classes constructed in the Donaldson–Friedman Theorem 2.1. Conformal Green's functions are used to detect the sign of the scalar curvature of these metrics. Positivity for the scalar curvature is characterized by nontriviality of the Green's functions. Then the vanishing theorem 4.3 will provide the Serre–Horrocks vector bundle construction, which gives the Serre class, a substitute for the Green's function introduced by Atiyah. Nontriviality of the Serre class will provide the nontriviality of the extension described by it.

In §2–§3 we review the background material. In §4 the vanishing theorem is proven, and finally in §5–§6 the sign of the scalar curvature is detected.

2. The Donaldson–Friedman construction

One of the main improvements in the field of self-dual Riemannian 4-manifolds is the connected sum theorem of Donaldson and Friedman [DF] published in 1989. If M_1 and M_2 admit self-dual metrics, then under certain circumstances their connected sum admits one too. This helped create many examples of self-dual manifolds. We state this more precisely:

Theorem 2.1 (Donaldson–Friedman [DF]). *Let (M_1, g_1) and (M_2, g_2) be compact self-dual Riemannian 4-manifolds and Z_i denote the corresponding twistor spaces. Suppose that $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ for $i = 1, 2$. Then there are self-dual conformal classes on $M_1 \# M_2$ whose twistor spaces arise as fibers in a 1-parameter standard deformation of $Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2$.*

We devote the rest of this section to understand the statement and the ideas in the proof of this theorem since our main result (6.1) is a generalization of this celebrated theorem. See [Kal] for details.

The idea is to work upstairs in the complex category rather than downstairs. So let $p_i \in M_i$ be arbitrary points in the manifolds. Consider their inverse images $C_i \approx \mathbb{CP}_1$ under the twistor fibration, which are twistor lines, i.e. rational curves invariant under the involution. Blow up the twistor spaces Z_i along these rational curves. Denote the exceptional divisors by $Q_i \approx \mathbb{CP}_1 \times \mathbb{CP}_1$ and the blown up twistor spaces by $\widetilde{Z}_i = \mathrm{Bl}(Z_i, C_i)$. The normal bundle for the exceptional divisor Q_2 in $\widetilde{Z}_2$ is computed to be

$$NQ_2 = N_{Q_2/\widetilde{Z}_2} \approx \mathcal{O}(1, -1)$$

where the second component is the fiber direction in the blow up process. We then construct the complex analytic space Z_0 by identifying Q_1 and Q_2 so that it has a normal crossing singularity,

$$Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2.$$

Carrying out this identification needs some care. We interchange the components of $\mathbb{CP}_1 \times \mathbb{CP}_1$ in the gluing process so that the normal bundles $N_{Q_1/\widetilde{Z}_1}$ and $N_{Q_2/\widetilde{Z}_2}$ are dual to each other. Moreover we should respect the real structures. The real structures σ_1 and σ_2 must agree on Q obtained by identifying Q_1 with Q_2, so that the real structures extend over Z_0 and form the anti-holomorphic involution $\sigma_0 : Z_0 \to Z_0$.

Now we will be trying to deform the singular space Z_0, for which Kodaira–Spencer's standard deformation theory does not work since it only applies to manifolds tells nothing about deformations of singular spaces. We must use the theory of deformations of compact reduced complex analytic spaces, due to Friedman [F]. This generalized theory is quite parallel to the theory of manifolds. The basic modification is that the roles of $H^i(\Theta)$ are now taken up by the groups $T^i = \mathrm{Ext}^i(\Omega^1, \mathcal{O})$.

We have assumed $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ so that the deformations of Z_i are unobstructed. Donaldson and Friedman are able to show that $T^2_{Z_0} = \mathrm{Ext}^2_{Z_0}(\Omega^1, \mathcal{O}) = 0$ so the deformations of the singular space are unobstructed. We have a versal family of deformations of Z_0. This family is parameterized by a neighborhood of the origin in $\mathrm{Ext}^1_{Z_0}(\Omega^1, \mathcal{O})$. The generic fiber is nonsingular and the real structure σ_0 extends to the total space of this family.

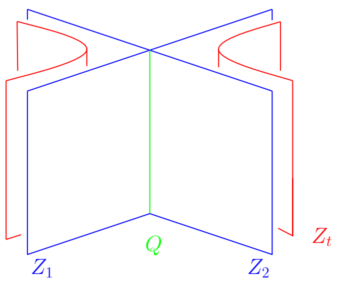
Instead of working with the entire versal family, it is convenient to work with certain subfamilies, called *standard deformations*.

Definition 2.2 ([Le]). A 1-*parameter standard deformation* of Z_0 is a flat proper holomorphic map $\omega : \mathcal{Z} \to \mathcal{U} \subset \mathbb{C}$ of a complex 4-manifold to a neighborhood of 0, with an anti-holomorphic involution σ of $\mathcal{Z}$ such that

- $\omega^{-1}(0) = Z_0$,
- $\sigma|_{Z_0} = \sigma_0$,
- σ descends to the complex conjugation in $\mathcal{U}$,
- ω is a submersion away from $Q \subset Z_0$,
- ω is modeled by $(x, y, z, w) \mapsto xy$ near any point of Q.

Then for sufficiently small, nonzero, real $t \in \mathcal{U}$ the complex space $Z_t = \omega^{-1}(t)$ is smooth and one can show that it is the twistor space of a self-dual metric on $M_1 \# M_2$.

$$\omega : \quad \begin{array}{l} \mathcal{Z} \to \mathcal{U} \\ Z_0 \mapsto 0 \end{array} \quad \text{for } Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2$$



3. The Leray spectral sequence

In this section we will go over the tools that we will be using from the theory of spectral sequences. Consult [H, GH] for details and proofs.

Given a continuous map $f : X \to Y$ between topological spaces, and a sheaf $\mathcal{F}$ over X, the *q-th direct image sheaf* is the sheaf $R^q(f_*\mathcal{F})$ on Y associated to the presheaf

$V \to H^q(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. This is actually the right derived functor of the functor f_*. The Leray spectral sequence is a spectral sequence $\{E_r\}$ with

$$E_2^{p,q} = H^p(Y, R^q(f_*\mathcal{F})) \quad \text{and} \quad E_\infty^{p,q} = H^{p+q}(X, \mathcal{F}).$$

The first page of this spectral sequence reads

$$\begin{array}{c|cccc} & \vdots & \vdots & \vdots & \\ & H^0(Y, R^2(f_*\mathcal{F})) & H^1(Y, R^2(f_*\mathcal{F})) & H^2(Y, R^2(f_*\mathcal{F})) & \cdots \\ & H^0(Y, R^1(f_*\mathcal{F})) & H^1(Y, R^1(f_*\mathcal{F})) & H^2(Y, R^1(f_*\mathcal{F})) & \cdots \\ E_2 & H^0(Y, R^0(f_*\mathcal{F})) & H^1(Y, R^0(f_*\mathcal{F})) & H^2(Y, R^0(f_*\mathcal{F})) & \cdots \\ \hline \end{array}$$

A degenerate case is when $R^i(f_*\mathcal{F}) = 0$ for all $i > 0$.

When the spectral sequence degenerates this way, the second and succeeding rows of the first page vanish. Since $V \to H^0(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$ is the presheaf of the direct image sheaf, we have $R^0 f_* = f_*$. So the first row consists of $H^i(Y, f_*\mathcal{F})$'s. Vanishing of the differentials causes immediate convergence to $E_\infty^{i,0} = H^{i+0}(X, \mathcal{F})$. So we get

Proposition 3.1. *If $R^i(f_*\mathcal{F}) = 0$ for all $i > 0$, then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ naturally for all $i \geq 0$.*

The following proposition gives another sufficient condition for this degeneration. See [V] for a sketch of proof.

Proposition 3.2 (Small fiber theorem). *Let $f : X \to Y$ be a holomorphic, proper and submersive map between complex manifolds, and $\mathcal{F}$ a coherent analytic sheaf or a holomorphic vector bundle on X. Then $H^i(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) = 0$ for all $y \in Y$ implies that $R^i(f_*\mathcal{F}) = 0$.*

We next state a very useful lemma which we will be using often.

Lemma 3.3 (Projection formula [H]). *If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}$-module, and if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) = f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E},$$

in particular for $\mathcal{F} = \mathcal{O}_X$, we have $f_* f^*\mathcal{E} = f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{E}$.

As an application of 3.1–3.3 and the weak version of Zariski's main theorem [H] one obtains the following classical result.

Proposition 3.4. *Let Z be a complex n-manifold with a complex k-dimensional submanifold V. Let $\widetilde{Z}$ denote the blow up of Z along V, with blow up map $\pi : \widetilde{Z} \to Z$. Let $\mathcal{G}$ denote a coherent analytic sheaf (or a vector bundle) over Z. Then we can compute the cohomology of $\mathcal{G}$ on either side, i.e.*

$$H^i(\widetilde{Z}, \pi^*\mathcal{G}) = H^i(Z, \mathcal{G}).$$

4. Vanishing theorem

In this section, we are going to prove that a certain cohomology group of a line bundle vanishes. For that we need some lemmas. Let $\omega : \mathcal{Z} \to \mathcal{U}$ be a 1-paramater standard deformation of Z_0, where $\mathcal{U} \subset \mathbb{C}$ is an open disk about the origin. Then the invertible sheaf $K_{\mathcal{Z}}$ has a square root as a holomorphic line bundle as follows.

We are going to show that the Stiefel–Whitney class $w_2(K_{\mathcal{Z}})$ vanishes. We write $\mathcal{Z} = \mathcal{U}_1 \cup \mathcal{U}_2$ where $\mathcal{U}_i$ is a tubular neighborhood of $\widetilde{Z}_i$, and $\mathcal{U}_1 \cap \mathcal{U}_2$ is a tubular neighborhood of $Q = \widetilde{Z}_1 \cap \widetilde{Z}_2$. So $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_1 \cap \mathcal{U}_2$ are deformation retracts of $\widetilde{Z}_1$, $\widetilde{Z}_2$ and Q respectively. Since $Q \approx \mathbb{P}_1 \times \mathbb{P}_1$ is simply connected, $H^1(\mathcal{U}_1 \cap \mathcal{U}_2, \mathbb{Z}_2) = 0$ and consequently the map r_{12} in the Mayer–Vietoris exact sequence

$$\begin{array}{ccccccc} \cdots \to & H^1(\mathcal{U}_1 \cap \mathcal{U}_2, \mathbb{Z}_2) & \to & H^2(\mathcal{U}_1 \cup \mathcal{U}_2, \mathbb{Z}_2) & \xrightarrow{r_{12}} & H^2(\mathcal{U}_1, \mathbb{Z}_2) \oplus H^2(\mathcal{U}_2, \mathbb{Z}_2) & \to \cdots \\ & \parallel & & \Psi & & & \\ & 0 & & w_2(K_{\mathcal{Z}}) & & & \end{array}$$

is injective. Therefore it is enough to see that the restrictions $r_i(w_2(K_{\mathcal{Z}})) \in H^2(\mathcal{U}_i, \mathbb{Z}_2)$ are zero. For that, we need to see that $K_{\mathcal{Z}}|_{\widetilde{Z}_i}$ has a square root. We have

$$K_{\mathcal{Z}}|_{\widetilde{Z}_1} = (K_{\widetilde{Z}_1} - \widetilde{Z}_1)|_{\widetilde{Z}_1} = (K_{\widetilde{Z}_1} + Q)|_{\widetilde{Z}_1} = ((\pi^* K_{Z_1} + Q) + Q)|_{\widetilde{Z}_1} = 2(\pi^* K_{Z_1}^{1/2} + Q)|_{\widetilde{Z}_1}$$

where the first equality is the application of the adjunction formula on $\widetilde{Z}_1$, the second comes from the linear equivalence of 0 with Z_t on $\widetilde{Z}_1$, and Z_t with Z_0,

$$0 = \mathcal{O}(Z_t)|_{\widetilde{Z}_1} = \mathcal{O}(Z_0)|_{\widetilde{Z}_1} = \mathcal{O}(\widetilde{Z}_1 + \widetilde{Z}_2)|_{\widetilde{Z}_1} = \mathcal{O}(\widetilde{Z}_1 + Q)|_{\widetilde{Z}_1},$$

and the third is the change of the canonical bundle under the blow up along a submanifold [GH]. K_{Z_1} has a natural square root,[1] so $\pi^* K_{Z_1}^{1/2} \otimes [Q]$ is a square root of $K_{\mathcal{Z}}$ on $\widetilde{Z}_1$. Similarly on $\widetilde{Z}_2$, so $K_{\mathcal{Z}}$ has a square root $K_{\mathcal{Z}}^{1/2}$.

Next, we are going to state the semicontinuity principle and Hitchin's vanishing theorem, which are involved in the proof of the vanishing theorem.

Lemma 4.1 (Semicontinuity principle [V]). *Let $\phi : \mathcal{X} \to \mathcal{B}$ be a map of a family of compact, complex manifolds with fiber X_b for $b \in \mathcal{B}$. Let $\mathcal{F}$ be a holomorphic vector bundle over $\mathcal{X}$. Then the function $b \mapsto h^q(X_b, \mathcal{F}|_{X_b})$ is upper semicontinuous, i.e. $h^q(X_b, \mathcal{F}|_{X_b}) \leq h^q(X_0, \mathcal{F}|_{X_0})$ for b in a neighborhood of $0 \in \mathcal{B}$.*

Lemma 4.2 (Hitchin's vanishing theorem [Hi2]). *Let Z be the twistor space of an oriented self-dual Riemannian manifold of positive scalar curvature with canonical bundle K, then*

$$h^0(Z, \mathcal{O}(K^{n/2})) = h^1(Z, \mathcal{O}(K^{n/2})) = 0 \quad \textit{for all } n \geq 1.$$

[1] The dual of the tangent bundle over the fibers is a canonical square root; equivalently, it is a spin manifold. Check [AHS] or for a recent exposition [Kal].

Theorem 4.3 (Vanishing theorem). *Let $\omega : \mathcal{Z} \to \mathcal{U}$ be a* 1*-parameter standard deformation of* $Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2$*, and* $\mathcal{U} \subset \mathbb{C}$ *be a neighborhood of the origin. Let* $L \to \mathcal{Z}$ *be the holomorphic line bundle defined by*

$$\mathcal{O}(L^*) = \mathscr{I}_{\widetilde{Z}_1}(K_{\mathcal{Z}}^{1/2}).$$

If $(M_i, [g_i])$ *has positive scalar curvature, then by possibly replacing* $\mathcal{U}$ *with a smaller neighborhood of* $0 \in \mathbb{C}$ *and simultaneously replacing* $\mathcal{Z}$ *with its inverse image we can arrange for our complex* 4*-fold* $\mathcal{Z}$ *to satisfy*

$$H^1(\mathcal{Z}, \mathcal{O}(L^*)) = H^2(\mathcal{Z}, \mathcal{O}(L^*)) = 0.$$

Proof. The proof proceeds by analogy to the techniques in [Le], and consists of several steps.

1. It is enough to *show that* $H^j(Z_0, \mathcal{O}(L^*)) = 0$ *for* $j = 1, 2$, since that would imply $h^j(Z_t, \mathcal{O}(L^*)) \le 0$ for $j = 1, 2$ in a neighborhood by the semicontinuity principle. Intuitively, this means that the fibers are too small, so we can apply Proposition 3.2 to see $R^j\omega_*\mathcal{O}(L^*) = 0$ for $j = 1, 2$. The first page of the Leray spectral sequence reads

$$\begin{array}{c|cccc}
 & \vdots & \vdots & \vdots & \\
 & H^0(\mathcal{U}, R^3\omega_*\mathcal{O}(L^*)) & H^1(\mathcal{U}, R^3\omega_*\mathcal{O}(L^*)) & H^2(\mathcal{U}, R^3\omega_*\mathcal{O}(L^*)) & \cdots \\
 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & \cdots \\
E_2 & H^0(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) & H^1(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) & H^2(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) & \cdots \\
\hline
\end{array}$$

Remember that

$$E_2^{p,q} = H^p(\mathcal{U}, R^q\omega_*\mathcal{O}(L^*)), \qquad E_\infty^{p,q} = H^{p+q}(\mathcal{Z}, \mathcal{O}(L^*))$$

and that the differential d_2 maps $E_2^{p,q}$ into $E_2^{p+2,q-1}$. Vanishing of the second row implies the immediate convergence of the first row till the third column because of the differentials, so

$$E_\infty^{p,0} = E_2^{p,0}, \quad \text{i.e.} \quad H^{p+0}(\mathcal{Z}, \mathcal{O}(L^*)) = H^p(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) \quad \text{for } p \le 3,$$

hence $H^p(\mathcal{Z}, \mathcal{O}(L^*)) = H^p(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*))$ for $p \le 3$.

Since $\mathcal{U}$ is one-dimensional, $\omega : \mathcal{Z} \to \mathcal{U}$ has to be a flat morphism, so the sheaf $\omega_*\mathcal{O}(L^*)$ is coherent [Gun, BaSt]. Since $\mathcal{U}$ is an open subset of $\mathbb{C}$, it is Stein, and the so called Theorem B of Stein manifold theory characterizes them as possessing vanishing higher dimensional ($p > 0$) coherent sheaf cohomology [Lew, H, Gun, BaSt]. So $H^p(\mathcal{U}, \omega_*\mathcal{O}(L^*)) = 0$ for $p > 0$. This tells us that $H^p(\mathcal{Z}, \mathcal{O}(L^*)) = 0$ for $0 < p \le 3$.

2. Over Z_0, we have the *Mayer–Vietoris like* sheaf exact sequence

$$0 \to \mathcal{O}_{Z_0}(L^*) \to \nu_*\mathcal{O}_{\widetilde{Z}_1}(L^*) \oplus \nu_*\mathcal{O}_{\widetilde{Z}_2}(L^*) \to \mathcal{O}_Q(L^*) \to 0$$

where $\nu : \widetilde{Z}_1 \sqcup \widetilde{Z}_2 \to Z_0$ is the inclusion map on each of the two components of the disjoint union $\widetilde{Z}_1 \sqcup \widetilde{Z}_2$. This gives the long exact cohomology sequence piece

$$\begin{aligned} 0 \to H^1(\mathcal{O}_{Z_0}(L^*)) &\to H^1(Z_0, \nu_*\mathcal{O}_{\widetilde{Z}_1}(L^*) \oplus \nu_*\mathcal{O}_{\widetilde{Z}_2}(L^*)) \to H^1(\mathcal{O}_Q(L^*)) \\ &\to H^2(\mathcal{O}_{Z_0}(L^*)) \to H^2(Z_0, \nu_*\mathcal{O}_{\widetilde{Z}_1}(L^*) \oplus \nu_*\mathcal{O}_{\widetilde{Z}_2}(L^*)) \to 0 \qquad (*) \end{aligned}$$

due to the fact that:

3. $H^0(\mathcal{O}_Q(L^*)) = H^2(\mathcal{O}_Q(L^*)) = 0$: To see this, we have to understand the restriction of $\mathcal{O}(L^*)$ to Q:

$$\begin{aligned} L^*|_Q &= (\tfrac{1}{2}K_{\mathcal{Z}} - \widetilde{Z}_1)|_{\widetilde{Z}_2}|_Q = (\tfrac{1}{2}(K_{\widetilde{Z}_2} - \widetilde{Z}_2) - \widetilde{Z}_1)|_{\widetilde{Z}_2}|_Q = (\tfrac{1}{2}(K_{\widetilde{Z}_2} + Q) - Q)|_{\widetilde{Z}_2}|_Q \\ &= \tfrac{1}{2}(K_{\widetilde{Z}_2} - Q)|_{\widetilde{Z}_2}|_Q = \tfrac{1}{2}(K_Q - Q - Q)|_{\widetilde{Z}_2}|_Q = (\tfrac{1}{2}K_Q - Q)|_{\widetilde{Z}_2}|_Q \\ &= \tfrac{1}{2}K_Q|_Q \otimes NQ^{-1}_{\widetilde{Z}_2} = \mathcal{O}(-2,-2)^{1/2} \otimes \mathcal{O}(1,-1)^{-1} = \mathcal{O}(-2,0). \end{aligned}$$

Here, we know that the normal bundle of Q in $\widetilde{Z}_2$ is $\mathcal{O}(1,-1)$, where the second component is the fiber direction in the blowing up process. So the line bundle L^* is trivial on the fibers. Since $Q = \mathbb{P}_1 \times \mathbb{P}_1$, we have

$$\begin{aligned} H^0(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2,0)) &= H^0(\mathbb{P}_1 \times \mathbb{P}_1, \pi_1^*\mathcal{O}(-2)) = H^0(\mathbb{P}_1, \pi_{1*}\pi_1^*\mathcal{O}(-2)) \\ &= H^0(\mathbb{P}_1, \mathcal{O}(-2)) = 0 \end{aligned}$$

by the Leray spectral sequence and the projection formula since $H^k(\mathbb{P}_1, \mathcal{O}) = 0$ for $k > 0$. Similarly, for dimensional reasons,

$$H^2(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2,0)) = H^2(\mathbb{P}_1, \mathcal{O}(-2)) = 0.$$

4. $H^1(\widetilde{Z}_2, \mathcal{O}_{\widetilde{Z}_2}(L^*)) = H^2(\widetilde{Z}_2, \mathcal{O}_{\widetilde{Z}_2}(L^*)) = 0$: These are applications of Hitchin's second ($k = 1$) vanishing theorem and are going to help us simplify our exact sequence $(*)$. We have

$$\begin{aligned} H^1(\widetilde{Z}_2, \mathcal{O}_{\widetilde{Z}_2}(L^*)) &= H^1(\widetilde{Z}_2, \mathcal{O}(K_{\mathcal{Z}}^{1/2} - \widetilde{Z}_1)|_{\widetilde{Z}_2}) = H^1(\widetilde{Z}_2, \mathcal{O}(K_{\mathcal{Z}}^{1/2} - Q)|_{\widetilde{Z}_2}) \\ &= H^1(\widetilde{Z}_2, \pi^*K_{Z_2}^{1/2}) = H^1(Z_2, \pi_*\pi^*K_{Z_2}^{1/2}) = H^1(Z_2, K_{Z_2}^{1/2}) = 0 \end{aligned}$$

by the Leray spectral sequence, projection formula and Hitchin's vanishing theorem for Z_2, since it is the twistor space of a positive scalar curvature space. This implies $H^2(Z_2, K_{Z_2}^{1/2}) \approx H^1(Z_2, K_{Z_2}^{1/2})^* = 0$ by the Kodaira–Serre duality. Hence our exact cohomology sequence piece simplifies to

$$\begin{aligned} 0 \to H^1(\mathcal{O}_{Z_0}(L^*)) \to H^1(\widetilde{Z}_1, \mathcal{O}_{\widetilde{Z}_1}(L^*)) \to H^1(\mathcal{O}_Q(L^*)) \to H^2(\mathcal{O}_{Z_0}(L^*)) \\ \to H^2(\widetilde{Z}_1, \mathcal{O}_{\widetilde{Z}_1}(L^*)) \to 0. \end{aligned}$$

5. $H^k(\mathcal{O}_{\widetilde{Z}_1}(L^* \otimes [Q]^{-1}_{\widetilde{Z}_1})) = 0$ *for* $k = 1, 2, 3$: This technical result is needed to understand the exact sequence in the next step. First we simplify the sheaf as

$$(L^* - Q)|_{\widetilde{Z}_1} = (\tfrac{1}{2}K_{\mathcal{Z}} - \widetilde{Z}_1 - Q)|_{\widetilde{Z}_1} = \tfrac{1}{2}K_{\mathcal{Z}}|_{\widetilde{Z}_1} = \tfrac{1}{2}(K_{\widetilde{Z}_1} - \widetilde{Z}_1)|_{\widetilde{Z}_1} = \tfrac{1}{2}(K_{\widetilde{Z}_1} + Q)|_{\widetilde{Z}_1}$$

by using the definitions and the adjunction formula. So we have

$$\begin{aligned} H^k(\widetilde{Z}_1, L^* - Q) &= H^k(\widetilde{Z}_1, (K_{\widetilde{Z}_1} + Q)/2) \approx H^{3-k}(\widetilde{Z}_1, (K_{\widetilde{Z}_1} - Q)/2)^* \\ &= H^{3-k}(\widetilde{Z}_1, \tfrac{1}{2}\pi^* K_{Z_1})^* = H^{3-k}(Z_1, \tfrac{1}{2}\pi_*\pi^* K_{Z_1})^* \\ &= H^{3-k}(Z_1, K_{Z_1}^{1/2})^* \approx H^k(Z_1, K_{Z_1}^{1/2}) \end{aligned}$$

by the Serre duality, Leray spectral sequence, projection formula, and one of the last two terms vanishes in any case for $k = 1, 2, 3$. So we apply the Hitchin vanishing theorem for dimensions 0 and 1.

6. *Restriction maps to Q*: Consider the short exact sequence of sheaves on $\widetilde{Z}_1$,

$$0 \to \mathcal{O}_{\widetilde{Z}_1}(L^* \otimes [Q]^{-1}_{\widetilde{Z}_1}) \to \mathcal{O}_{\widetilde{Z}_1}(L^*) \to \mathcal{O}_Q(L^*) \to 0.$$

The previous step implies that the restriction maps

$$H^1(\mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{\text{restr}_1} H^1(\mathcal{O}_Q(L^*)), \qquad H^2(\mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{\text{restr}_2} H^2(\mathcal{O}_Q(L^*))$$

are isomorphisms. In particular $H^2(\mathcal{O}_{\widetilde{Z}_1}(L^*)) = 0$ by step 4. Incidentally, this exact sheaf sequence is a substitute of the Hitchin vanishing theorem for the $\widetilde{Z}_2$ components in the cohomology sequence. We also assume Hitchin's theorems for the $\widetilde{Z}_1$ component.

7. *Conclusion*: Our exact sequence piece $(*)$ reduces to

$$0 \to H^1(\mathcal{O}_{Z_0}(L^*)) \to H^1(\widetilde{Z}_1, \mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{\text{restr}_1} H^1(\mathcal{O}_Q(L^*)) \to H^2(\mathcal{O}_{Z_0}(L^*)) \to 0.$$

The isomorphism in the middle forces the rest of the maps to be zero and hence we get $H^1(\mathcal{O}_{Z_0}(L^*)) = H^2(\mathcal{O}_{Z_0}(L^*)) = 0$. □

The sections after this point are devoted to detecting the sign of the scalar curvature of the metric we consider on the connected sum. We use Green's functions for that purpose. Positivity for the scalar curvature is going to be characterized by nontriviality of the Green's functions. Then our vanishing theorem will provide the Serre–Horrocks vector bundle construction, which gives the Serre class, a substitute for the Green's function due to Atiyah [At]. And the nontriviality of the Serre class will provide the nontriviality of the extension described by it.

5. Characterizations of positivity

In this section, we define Green's functions. To get a unique Green's function, we need an operator which has a trivial kernel. So we begin with a compact Riemannian 4-manifold (M, g), and assume that its *Yamabe Laplacian* $\Delta + s/6$ has trivial kernel. This is automatic if g is conformally equivalent to a metric of positive scalar curvature, impossible if it is conformally equivalent to a metric of zero scalar curvature because of the Hodge Laplacian, and may or may not happen for a metric of negative scalar curvature. Since the Hodge Laplacian Δ is self-adjoint, $\Delta + s/6$ is also self-adjoint, implying that this equation has a trivial cokernel whenever it has a trivial kernel. Therefore it is a bijection and we have a unique smooth solution u for the equation $(\Delta + s/6)u = f$ for any smooth function f. It also follows that it has a unique distributional solution u for any distribution f. Let $y \in M$ be any point. Consider the Dirac delta distribution δ_y at y defined by

$$\delta_y : C^\infty(M) \to \mathbb{R}, \quad \delta_y(f) = f(y);$$

intuitively, it behaves like a "function" which is identically zero on $M - \{y\}$ and infinity at y with integral 1. Then there is a unique distributional solution G_y to the equation

$$(\Delta + s/6)G_y = \delta_y$$

called the *conformal Green's function* at y. Since δ_y is identically zero on $M - \{y\}$, elliptic regularity implies that G_y is smooth on $M - \{y\}$. The name comes from the fact that the Yamabe Laplacian is a *conformally invariant* differential operator as a map between sections of some real line bundles. For any nonvanishing smooth function u, we have a conformally equivalent metric $\tilde{g} = u^2 g$, and $u^{-1}G_y$ is the conformal Green's function for $(M, \tilde{g}, y)$ if G_y is the one for (M, g, y).

Any metric on a compact manifold is conformally equivalent to a metric of constant scalar curvature sign. Actually, thanks to the proof of the Yamabe Conjecture [Y, T, A, S] we can choose a metric of constant scalar curvature (CSC). Also if two metrics with scalar curvatures of fixed signs are conformally equivalent, then their scalar curvatures have the same sign.

An application of Hopf's strong maximum principle [PW] provides us with a criterion for determining the sign of the Yamabe constant using Green's functions.

Lemma 5.1 (Green's function characterization for the sign [Le]). *Let (M, g) be a compact Riemannian 4-manifold with* $\mathrm{Ker}(\Delta + s/6) = 0$*, i.e. the Yamabe Laplacian has trivial kernel, where $\Delta = d^*d$ as in* [At]. *Fix a point $y \in M$. Then for the conformal class* $[g]$ *we have the following assertions.*

1. *It does not contain a metric of zero scalar curvature.*
2. *It contains a metric of positive scalar curvature iff $G_y(x) \neq 0$ for all $x \in M - \{y\}$.*
3. *It contains a metric of negative scalar curvature iff $G_y(x) < 0$ for some $x \in M - \{y\}$.*

Now let (M^4, g) be a compact self-dual Riemannian manifold with twistor space Z. One of the basic facts of twistor theory [Hi2] is that for any open set $U \subset M$ with corresponding inverse image $\widetilde{U} \subset Z$ in the twistor space, there is a natural isomorphism

$$\mathrm{pen} : H^1(\widetilde{U}, \mathcal{O}(K^{1/2})) \xrightarrow{\sim} \{\text{smooth complex-valued solutions of } (\Delta + s/6)u = 0 \text{ in } U\},$$

which is called *the Penrose transform* [BaSi, Hi1, At], where $K = K_Z$. Since locally $\mathcal{O}(K^{1/2}) \approx \mathcal{O}(-2)$, for a cohomology class $\psi \in H^1(\widetilde{U}, \mathcal{O}(K^{1/2}))$, the value of the corresponding function pen_ψ at $x \in U$ is obtained by restricting ψ to the line $P_x \subset Z$ to obtain an element

$$\mathrm{pen}_\psi(x) = \psi|_{P_x} \in H^1(P_x, \mathcal{O}(K^{1/2})) \approx H^1(\mathbb{CP}_1, \mathcal{O}(-2)) \approx \mathbb{C}.$$

Note that pen_ψ is a section of a line bundle over M, but the choice of a metric g in the conformal class determines a canonical trivialization of this line bundle [Hi1] and pen_ψ then becomes an ordinary function. Taking $U = M - \{y\}$, by uniqueness of the conformal Green's function we have $(\Delta + s/6)G_y = 0$ on U and $G_y(x)$ as a function of x corresponds to a canonical element

$$\text{pen}^{-1}_{G_y} \in H^1(Z - P_y, \mathcal{O}(K^{1/2}))$$

where P_y is the twistor line over the point y.

The nature of this cohomology class was described by Atiyah. It involves the *Serre class* of a complex submanifold, a construction due to Serre [Ser] and Horrocks [Hor]. We now give the definition of the Serre class via the following lemma.

Lemma 5.2 (Serre–Horrocks vector bundle, Serre class). *Let W be a (possibly noncompact) complex manifold, let $V \subset W$ be a closed complex submanifold of complex codimension 2, and $N = N_{V/W}$ be the normal bundle of V. For any holomorphic line bundle $L \to W$ satisfying*

$$L|_V \approx \bigwedge^2 N \quad \textit{and} \quad H^1(W, \mathcal{O}(L^*)) = H^2(W, \mathcal{O}(L^*)) = 0$$

there is a rank-2 holomorphic vector bundle $E \to W$ called the Serre–Horrocks bundle *of (W, V, L), with a holomorphic section ζ satisfying*

$$\bigwedge^2 E \approx L, \quad d\zeta|_V : N \xrightarrow{\sim} E \textit{ and } \zeta = 0 \textit{ exactly on } V.$$

The pair (E, ζ) is unique up to isomorphism if we also impose that the isomorphism $\det d\zeta : \bigwedge^2 N \to \bigwedge^2 E|_V$ should agree with a given isomorphism $\bigwedge^2 N \to L|_V$. They also give rise to an extension

$$0 \to \mathcal{O}(L^*) \to \mathcal{O}(E^*) \xrightarrow{\cdot\zeta} \mathscr{I}_V \to 0,$$

the class of which is defined to be the Serre class $\lambda(V) \in \text{Ext}^1_W(\mathscr{I}_V, \mathcal{O}(L^*))$*, where $\mathscr{I}_V$ is the ideal sheaf of V, and this extension determines an element of $H^1(W - V, \mathcal{O}(L^*))$ by restricting to $W - V$.*

We are now ready to state the answer of Atiyah.

Theorem 5.3 ([At]). *Let (M^4, g) be a compact self-dual Riemannian manifold with twistor space Z, and assume that the conformally invariant Laplace operator $\Box_g = d^*d + s/6$ on M has no global nontrivial solution so that the Green's functions are well defined. Let $y \in M$ be any point, and $P_y \subset Z$ be the corresponding twistor line. Then the image of the Serre class $\lambda(P_y) \in \text{Ext}^1_Z(\mathscr{I}_{P_y}, \mathcal{O}(K^{1/2}))$ in $H^1(Z - P_y, \mathcal{O}(K^{1/2}))$ is the Penrose transform of the Green's function G_y times a non-zero constant. More precisely*

$$\text{pen}^{-1}_{G_y} = \frac{1}{4\pi^2}\lambda(P_y).$$

Now thanks to this remarkable result of Atiyah, one can substitute the Serre class for the Green's functions in the previous characterization (Lemma 5.1) and obtain a better criterion for positivity.

Proposition 5.4 (Cohomological characterization, [Le]). *Let (M^4, g) be a compact self-dual Riemannian manifold with twistor space Z. Let P_y be a twistor line in Z. Then the conformal class $[g]$ contains a metric of positive scalar curvature if and only if $H^1(Z, \mathcal{O}(K^{1/2})) = 0$, and the Serre–Horrocks vector bundle in Lemma* 5.2 *on Z taking $L = K^{-1/2}$ associated to P_y satisfies $E|_{P_x} \approx \mathcal{O}(1) \oplus \mathcal{O}(1)$ for every twistor line P_x.*

6. The sign of the scalar curvature

We are now ready to approach the problem of determining the sign of the Yamabe constant for the self-dual conformal classes constructed in Theorem 2.1. The techniques used here are analogous to the ones used by LeBrun [Le].

Theorem 6.1. *Let (M_1, g_1) and (M_2, g_2) be compact self-dual Riemannian* 4*-manifolds with $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ for their twistor spaces. Moreover suppose that they have positive scalar curvature. Then, for all sufficiently small $\mathfrak{t} > 0$, the self-dual conformal class $[g_\mathfrak{t}]$ obtained on $M_1 \# M_2$ by the Donaldson–Friedman Theorem* 2.1 *contains a metric of positive scalar curvature.*

Proof. Pick a point $y \in (M_1 \# M_2) \backslash M_1$. Consider the real twistor line $P_y \subset \widetilde{Z}_2$, and extend it as a 1-parameter family of twistor lines $P_{y_\mathfrak{t}} \subset Z_\mathfrak{t}$ for $\mathfrak{t}$ near $0 \in \mathbb{C}$ and such that $P_{y_\mathfrak{t}}$ is a real twistor line for $\mathfrak{t}$ real. By shrinking $\mathcal{U}$ if needed, we may arrange that $\mathcal{P} = \bigcup_\mathfrak{t} P_{y_\mathfrak{t}}$ is a closed codimension-2 submanifold of $\mathcal{Z}$ and $H^1(\mathcal{Z}, \mathcal{O}(L^*)) = H^2(\mathcal{Z}, \mathcal{O}(L^*)) = 0$ by the vanishing theorem 4.3. Next we check that $L|_\mathcal{P} \approx \bigwedge^2 N_\mathcal{P}$. Over a twistor line $P_{y_\mathfrak{t}}$ we have

$$\bigwedge^2 N_\mathcal{P}|_{P_{y_\mathfrak{t}}} = \bigwedge^2(\mathcal{O}(1) \oplus \mathcal{O}(1)) = \mathcal{O}_{P_{y_\mathfrak{t}}}(2)$$

by considering the first Chern classes. On the other hand, notice that the restriction of L^* to any smooth fiber $Z_\mathfrak{t}$, $\mathfrak{t} \neq 0$, is simply $K^{1/2}$:

$$L^*|_{Z_\mathfrak{t}} = (\tfrac{1}{2}K_\mathcal{Z} - \widetilde{Z}_1)|_{Z_\mathfrak{t}} = \tfrac{1}{2}K_\mathcal{Z}|_{Z_\mathfrak{t}} = \tfrac{1}{2}(K_{Z_\mathfrak{t}} - Z_\mathfrak{t})|_{Z_\mathfrak{t}} = \tfrac{1}{2}K_{Z_\mathfrak{t}}|_{Z_\mathfrak{t}}.$$

Here, $\widetilde{Z}_1|_{Z_\mathfrak{t}} = 0$ because $\widetilde{Z}_1$ and $Z_\mathfrak{t}$ do not intersect for $\mathfrak{t} \neq 0$. The normal bundle of $Z_\mathfrak{t}$ is trivial, because we have a standard deformation. Then

$$L|_{P_{y_\mathfrak{t}}} = K_{Z_\mathfrak{t}}^{-1/2}|_{P_{y_\mathfrak{t}}} = TF|_{P_{y_\mathfrak{t}}} = \mathcal{O}_{P_{y_\mathfrak{t}}}(2) \quad \text{for } \mathfrak{t} \neq 0$$

since TF is the tangent bundle of the fibers, the square root of the anti-canonical bundle. For the case $\mathfrak{t} = 0$, we need the fact that $L^*|_{\widetilde{Z}_2} = \pi^* K_{Z_2}^{1/2}$ which we have computed in step 4 of the proof of the vanishing theorem 4.3. This yields

$$L|_{P_{y_0}} = \pi^* K_{Z_2}^{-1/2}|_{\widetilde{Z}_2}|_{P_{y_0}} = \mathcal{O}_{P_{y_0}}(2).$$

Then the Serre–Horrocks construction (5.2) is available to obtain the holomorphic vector bundle $E \to \mathcal{Z}$ and a holomorphic section ζ vanishing exactly along $\mathcal{P}$; also, the corresponding extension

$$0 \to \mathcal{O}(L^*) \to \mathcal{O}(E^*) \xrightarrow{\cdot\zeta} \mathscr{I}_{\mathcal{P}} \to 0$$

gives us the Serre class $\lambda(\mathcal{P}) \in H^1(\mathcal{Z} - \mathcal{P}, \mathcal{O}(L^*))$.

Since $L^*|_{Z_t} = K_{Z_t}^{1/2}$ for $t \neq 0$ by the above computation, Proposition 5.3 of Atiyah tells us that the restriction of $\lambda(\mathcal{P})$ to Z_t, $t > 0$, has Penrose transform equal to a positive constant times the conformal Green's function of $(M_1 \# M_2, g_t, y_t)$ for any $t > 0$.

Now, we will restrict (E, ζ) to the two components of the divisor Z_0. First restrict to $\widetilde{Z}_2$. We have $L|_{P_{y_0}} = \mathcal{O}_{P_0}(2) = \bigwedge^2 N_{P_{y_0}/\widetilde{Z}_2}$ and

$$H^k(\widetilde{Z}_2, L^*) = H^k(\widetilde{Z}_2, \pi^* K_{Z_2}^{1/2}) = H^k(Z_2, \pi_* \pi^* K_{Z_2}^{1/2}) = H^k(Z_2, K_{Z_2}^{1/2}) = 0$$

for $k = 1, 2$ by the projection lemma, Leray spectral sequence and Hitchin's vanishing theorem for positive scalar curvature on M_2. So that we have the Serre–Horrocks bundle for the triple $(\widetilde{Z}_2, P_{y_0}, L|_{\widetilde{Z}_2} = \pi^* K_{Z_2}^{-1/2})$. On the other hand it is possible to construct the Serre–Horrocks bundle E_2 for the triple $(Z_2, P_{y_0}, K_{Z_2}^{-1/2})$ for which all conditions are already checked. In the construction of these Serre–Horrocks bundles, if we stick to a chosen isomorphism $\bigwedge^2 N \to L|_{P_{y_0}}$, these bundles are isomorphic by (5.2). The splitting type of E on the twistor lines corresponding to the points in $M_2 - \{y_0, p_2\}$ is supposed to be the same as the splitting type of E_2, which is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ since Z_2 already admits a self-dual metric of positive scalar curvature.

Secondly, we restrict (E, ζ) to $\widetilde{Z}_1$. Alternatively we restrict the Serre class $\lambda(\mathcal{P})$ to $H^1(\widetilde{Z}_1, \mathcal{O}(L^*))$ where

$$\begin{aligned} L^*|_{\widetilde{Z}_1} &= \tfrac{1}{2} K_{\mathcal{Z}} - \widetilde{Z}_1|_{\widetilde{Z}_1} = \tfrac{1}{2} K_{\mathcal{Z}} + Q|_{\widetilde{Z}_1} = \tfrac{1}{2}(K_{\widetilde{Z}_1} - \widetilde{Z}_1) + Q|_{\widetilde{Z}_1} \\ &= \tfrac{1}{2}(K_{\widetilde{Z}_1} + Q) + Q|_{\widetilde{Z}_1} = \tfrac{1}{2}(\pi^* K_{Z_1} + 2Q) + Q|_{\widetilde{Z}_1} = \pi^* \tfrac{1}{2} K_{Z_1} + 2Q|_{\widetilde{Z}_1}, \end{aligned}$$

and show that it is nonzero on every real twistor line away from Q. Remember that we have the restriction isomorphism obtained in step 4 of the proof of the vanishing theorem 4.3,

$$H^1(\mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{\sim} H^1(\mathcal{O}_Q(L^*)) \approx \mathbb{C},$$

as a consequence of Hitchin's vanishing theorems for positive scalar curvature on M_1, as mentioned in step 4, and $H^1(\mathcal{O}_Q(L^*)) = H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2, 0)) = \mathbb{C}$, as computed in step 4. This shows that if there is a rational curve of Q on which the Serre class is nonzero, then this class is nonzero and a generator of $H^1(\mathcal{O}_{\widetilde{Z}_1}(L^*))$. The Serre–Horrocks bundle construction on Z_2 shows that $E|_{C_2} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ where C_2 is the twistor line on which the blow up is done. We know that $Q = \mathbb{P}_1 \times \mathbb{P}_1 \approx \mathbb{P}(NC_2)$. So the exceptional divisor has one set of rational curves which are the fibers, and another set of rational curves, coming from the sections of the projective bundle $\mathbb{P}(NC_2)$. Take the zero section of $\mathbb{P}(NC_2)$, on which E has splitting type $\mathcal{O}(1) \oplus \mathcal{O}(1)$. So over the zero section in Q,

E is the same, hence has nontrivial splitting type. This shows that over this rational curve on Q, the Serre class is nonzero. Hence by the isomorphism above, the Serre class is (up to a constant) the nontrivial class in $H^1(\widetilde{Z}_1, \mathcal{O}(L^*)) \approx \mathbb{C}$.

Next we have to show that this nontrivial class is nonzero on every real twistor line in $\widetilde{Z}_1 - Q$ or $Z_1 - C_1$.[2] For this purpose consider the Serre–Horrocks vector bundle E_1 and its section ζ_1 for the triple $(Z_1, C_1, K_{Z_1}^{-1/2})$, so that $\pi^*\zeta_1$ is a section of π^*E_1 vanishing exactly along Q. Remember the construction of the line bundle associated to the divisor Q in $\widetilde{Z}_1$ [GH]. Consider the local defining functions $s_\alpha \in \mathfrak{M}^*(U_\alpha)$ of Q over some open cover $\{U_\alpha\}$ of $\widetilde{Z}_1$.[3] These functions are holomorphic and vanish to first order along Q. Then the corresponding line bundle is constructed via the transition functions $g_{\alpha\beta} = s_\alpha / s_\beta$. Since s_α's transform according to the transition functions, they constitute a holomorphic section s of this line bundle $[Q]$, which vanishes up to first order along Q. The set of local holomorphic sections of this bundle is denoted by $\mathcal{O}([Q])$; they are local functions with simple poles along Q. If we multiply $\pi^*\zeta_1$ with these functions, we will get a holomorphic section of π^*E_1 on the corresponding local open set, since ζ_1 has a nondegenerate zero on Q, so that it vanishes up to degree 1 there. This guarantees that the map is one-to-one, and the multiplication embeds $\mathcal{O}([Q])$ into π^*E_1. The quotient has rank 1, and the transition functions of π^*E_1 relative to a suitable trivialization will then look like

$$\begin{pmatrix} g_{\alpha\beta} & k_{\alpha\beta} \\ 0 & d_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} \end{pmatrix}$$

where $d_{\alpha\beta}$ stands for the determinant of the transition matrix of the bundle π^*E_1 in this coordinate chart. The bundle $\det \pi^*E_1 \otimes [Q]^{-1}$ has the right transition functions to be isomorphic to the quotient bundle. Hence we have the exact sequence

$$0 \to [Q] \to \pi^*E_1 \to \pi^*K^{-1/2} \otimes [Q]^{-1} \to 0,$$

since $\det E_1 = K_{Z_1}^{-1/2}$ as an essential feature of the Serre–Horrocks construction. This line bundle extension is classified by an element in

$$\mathrm{Ext}^1_{\widetilde{Z}_1}(\pi^*K^{-1/2} \otimes [Q]^{-1}, [Q]) \approx H^1(\widetilde{Z}_1, \pi^*K^{1/2} \otimes [Q]^2)$$

by [At]. If we restrict our exact sequence to $\widetilde{Z}_1 - Q = Z_1 - C_1$, since the bundle $[Q]$ is trivial on the complement of Q, this extension class will be the Serre class of the triple $(Z_1, C_1, K_{Z_1}^{-1/2})$. Finally, since M_1 has positive scalar curvature, this class is nonzero on every real twistor line in $Z_1 - C_1$. So nontriviality of the class forces nontriviality over the real twistor lines. In other words, E has a nontrivial splitting type over the real twistor lines of $\widetilde{Z}_1$.

Thus we have shown that the Serre–Horrocks vector bundle E determined by $\lambda(\mathcal{P})$ splits as $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on all the σ_0-invariant rational curves in Z_0 which are limits of real

[2] Thanks to C. LeBrun for this idea.

[3] Here, $\mathfrak{M}^*$ stands for the multiplicative sheaf of meromorphic functions which are not identically zero in the convention of [GH]. Actually the local defining functions here are holomorphic because Q is effective.

twistor lines in $\mathcal{Z}_{\mathrm{t}}$ as $\mathrm{t} \to 0$. It therefore has the same splitting type on all the real twistor lines of $\mathcal{Z}_{\mathrm{t}}$ for t small. Moreover,

$$h^j(Z_t, \mathcal{O}(L^*)) \leq h^j(Z_0, \mathcal{O}(L^*)) = 0 \quad \text{for } j = 1, 2$$

by the semicontinuity principle and the proof of the vanishing theorem 4.3. So via $L^*|_{Z_{\mathrm{t}}} \approx K^{1/2}$,

$$H^1(Z_{\mathrm{t}}, \mathcal{O}(K^{1/2})) \approx \mathrm{Ker}(\Delta + s/6) = 0.$$

Since we met the two conditions, the cohomological characterization 5.4 guarantees the positivity of the conformal class. □

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