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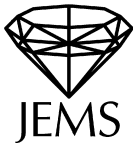

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Torelli theorem for stable curves

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Abstract. We study the Torelli morphism from the moduli space of stable curves to the moduli space of principally polarized stable semi-abelic pairs. We give two characterizations of its fibers, describe its injectivity locus, and give a sharp upper bound on the cardinality of finite fibers. We also bound the dimension of infinite fibers.

Keywords. Torelli map, Jacobian variety, theta divisor, stable curve, stable semi-abelic pair, compactified Picard scheme, semiabelian variety, moduli space, dual graph

Contents

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1. Introduction

1.1. Problems and results

In modern terms, the classical Torelli theorem ([T13], [ACGH]) asserts the injectivity of the Torelli map $\mathrm{t}_g : M_g \to A_g$ from the moduli scheme M_g, of smooth projective curves of genus g, to the moduli scheme A_g, of principally polarized abelian varieties of dimension g.

Context. It is well known that, if $g \geq 1$, the schemes M_g and A_g are not complete; the problem of finding good compactifications for them has been thoroughly investigated and solved in various ways. For M_g, the most widely studied compactification is the moduli space of Deligne–Mumford stable curves, $\overline{M}_g$.

Now, the Torelli map t_g does not extend to a regular map from $\overline{M}_g$ to A_g. More precisely, the largest subset of $\overline{M}_g$ admitting a regular map to A_g extending t_g is the locus of curves of compact type (i.e. every node is a separating node). Therefore the following question naturally arises: does there exist a good compactification of A_g which contains the image of an extended Torelli morphism from the whole of $\overline{M}_g$? If so, what are the properties of such an extended map?

It was known to D. Mumford that t_g extends to a morphism

$$\bar{\mathrm{t}}_g^{\mathrm{Vor}} : \overline{M}_g \to \overline{A}_g^{\mathrm{Vor}},$$

where $\overline{A}_g^{\mathrm{Vor}}$ is the second Voronoi toroidal compactification of A_g; see [AMRT], [Nam76b], [Nam80], [FC90]. On the other hand, the map $\bar{\mathrm{t}}_g^{\mathrm{Vor}}$ fails to be injective: if $g \geq 3$ it has positive-dimensional fibers over the locus of curves having a separating node (see [Nam80, Thm 9.30(vi)]). Furthermore, although $\bar{\mathrm{t}}_g^{\mathrm{Vor}}$ has finite fibers away from this locus, it still fails to be injective (see [V03]). The precise generalization of the Torelli theorem with respect to the above map $\bar{\mathrm{t}}_g^{\mathrm{Vor}}$ remains an open problem, since the pioneering work of Y. Namikawa.

In recent years, the space $\overline{A}_g^{\mathrm{Vor}}$ and the map $\bar{\mathrm{t}}_g^{\mathrm{Vor}}$ have been placed in a new modular framework by V. Alexeev ([Ale02], [Ale04]). As a consequence, there exists a different compactification of the Torelli morphism, whose geometric interpretation ties in well with the modular descriptions of $\overline{M}_g$ and of the compactified Jacobian. More precisely, in [Ale02] a new moduli space is constructed, the coarse moduli space $\overline{A}_g^{\mathrm{mod}}$, parametrizing principally polarized "semi-abelic stable pairs". The Voronoi compactification $\overline{A}_g^{\mathrm{Vor}}$ is shown to be the normalization of the irreducible component of $\overline{A}_g^{\mathrm{mod}}$ containing A_g; see Theorem 1.2.5 below. Next, in [Ale04], a new compactified Torelli morphism, $\bar{\mathrm{t}}_g$, factoring through $\bar{\mathrm{t}}_g^{\mathrm{Vor}}$, is defined:

$$\bar{\mathrm{t}}_g : \overline{M}_g \xrightarrow{\bar{\mathrm{t}}_g^{\mathrm{Vor}}} \overline{A}_g^{\mathrm{Vor}} \to \overline{A}_g^{\mathrm{mod}}.$$

$\bar{\mathrm{t}}_g$ is the map sending a stable curve X to the principally polarized semi-abelic stable pair

$(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X))$. Here $J(X)$ is the generalized Jacobian of X, $\overline{P_X^{g-1}}$ is a stable semi-abelic variety, called the compactified Picard scheme (in degree $g-1$), acted upon by $J(X)$; finally $\Theta(X) \subset \overline{P_X^{g-1}}$ is a Cartier, ample divisor, called the theta divisor. As proved in [Ale04], $\overline{P_X^{g-1}}$ coincides with the previously constructed compactified Picard schemes of [OS79], [Sim94], and [Cap94]; moreover the definition of the theta divisor extends the classical one very closely.

The main result. The goal of the present paper is to establish the precise analogue of the Torelli theorem for stable curves, using the compactified Torelli morphism $\bar{\mathrm{t}}_g$. This is done in Theorem 2.1.7, our main result, which characterizes curves having the same image via $\bar{\mathrm{t}}_g$. In particular we find that $\bar{\mathrm{t}}_g$ is injective at curves having 3-edge-connected dual graph (for example irreducible curves, or curves with two components meeting in at least three points). On the other hand $\bar{\mathrm{t}}_g$ fails to be injective at curves with two components meeting at two points, as soon as $g \geq 5$; see Theorem 5.1.5.

We actually obtain two different characterizations of curves having the same Torelli image: one is based on the classifying morphism of the generalized Jacobian (see Section 3), and the other, less sophisticated and more explicit, is of combinatorial type and we shall now illustrate it.

Let X and X' be two stable curves free from separating nodes (this is the key case); our main theorem states that $\bar{\mathrm{t}}_g(X) = \bar{\mathrm{t}}_g(X')$ if and only if X and X' are "C1-equivalent", i.e. if the following holds. First, X and X' have the same normalization, Y; let $\nu : Y \to X$ and $\nu' : Y \to X'$ be the normalization maps. Second, ν and ν' have the same "gluing set" $G \subset Y$, i.e. $\nu^{-1}(X_{\mathrm{sing}}) = \nu'^{-1}(X'_{\mathrm{sing}}) = G$. The third and last requirement is the interesting one, and can only be described after a preliminary step: we prove that the set X_{sing} of nodes of X has a remarkable partition into disjoint subsets, called "C1-sets", defined as follows. Two nodes of X belong to the same C1-set if the partial normalization of X at both of them is disconnected. Now, the gluing set G maps two-to-one onto X_{sing} and onto X'_{sing}, so the partitions of X_{sing} and of X'_{sing} into C1-sets induce each a partition on G, which we call the "C1-partition". We are ready to complete our main definition: two curves are C1-equivalent if their C1-partitions on G coincide; see Definition 2.1.1 and Section 2.2 for details.

Let us explain the close, yet not evident, connection between the C1-sets of X and the compactified Picard scheme $\overline{P_X^{g-1}}$. The scheme $\overline{P_X^{g-1}}$ is endowed with a canonical stratification with respect to the action of the Jacobian of X. Now, every codimension-one stratum ("C1" stands for "codimension one") is isomorphic to the Jacobian of the normalization of X at a uniquely determined C1-set; moreover, every C1-set can be recovered in this way (although different codimension-one strata may give the same C1-set).

Let us consider two simple cases. Let X be irreducible; then no partial normalization of X is disconnected, hence every C1-set has cardinality one. On the other hand $\overline{P_X^{g-1}}$ has a codimension-one stratum for every node of X. In this case the C1-partition completely determines X, as it identifies all pairs of branches over the nodes; we conclude that the Torelli map is injective on the locus of irreducible curves, a fact that, for $\bar{\mathrm{t}}_g^{\mathrm{Vor}}$, was well known to Namikawa.

The next case is more interesting; let X be a cycle of $h \geq 2$ smooth components, $C_1, \ldots, C_h$, with h nodes; then $G = \{p_1, q_1, \ldots, p_h, q_h\}$ with $p_i, q_i \in C_i$. Now every pair of nodes disconnects X, therefore there is only one C1-set, namely X_{sing}. On the other hand the scheme $\overline{P}_X^{g-1}$ is irreducible, and has a unique codimension-one stratum. We infer that all the curves of genus g whose normalization is $\bigsqcup_{i=1}^h C_i$ and whose gluing points are $\{p_1, q_1, \ldots, p_h, q_h\}$ are C1-equivalent, and hence they all have the same image via the Torelli map $\bar{\mathrm{t}}_g$. This case yields the simplest examples of non-isomorphic curves whose polarized compactified Jacobians are isomorphic.

Overview of the paper. In Section 2 we state our first version of the Torelli theorem, and prove a series of useful results of combinatorial type.

The proof of the main theorem, which occupies Section 4, is shaped as follows. The difficult part is the necessary condition: assume that two curves, stable and free from separating nodes, have the same image, denoted $(J \curvearrowright \overline{P}, \Theta)$, under the Torelli map; we must prove that they are C1-equivalent. First, the structure of J-scheme of $\overline{P}$ yields a stratification whose (unique) smallest stratum determines the normalization of the curves, apart from rational components. Second, the combinatorics of this stratification (the J-strata form a partially ordered set, by inclusion of closures) carries enough information about the combinatorics of the curves to determine the "cyclic equivalence class" (see 1.2.2) of their dual graphs. This second part requires a combinatorial analysis, carried out in Section 2. From these two steps one easily deduces that the two curves have the same normalization. It remains to prove that the gluing sets of the normalization maps are the same, together with their C1-partition. Here is where we use the theta divisor, Θ, its geometry and its connection with the Abel maps of the curves. See Subsection 4.2 for details on this part.

The proof of the converse (i.e. the fact that C1-equivalent curves have the same Torelli image) is based on the other, above mentioned, characterization of C1-equivalence, which we temporarily call "T-equivalence" (the "T" stands for Torelli). The crux of the matter is to prove that C1-equivalence and T-equivalence coincide; we do that in Section 3. Having done that, the proof of the sufficiency follows directly from the general theory of compactifications of principally polarized semiabelian varieties, on which our definition of T-equivalence is based.

The paper ends with a fifth section where we compute the upper bounds on the cardinality (Theorem 5.1.5), and on the dimension (Proposition 5.2.1), of the fibers of $\bar{\mathrm{t}}_g$. We prove that the finite fibers have cardinality at most $\lceil (g-2)!/2 \rceil$; in particular, since our bound is sharp, we find that, away from curves with a separating node, $\bar{\mathrm{t}}_g$ is injective if and only if $g \leq 4$. In Theorem 5.1.5 we give a geometric description of the injectivity locus of $\bar{\mathrm{t}}_g$.

1.2. Preliminaries

We work over an algebraically closed field k. A *variety* over k is a reduced scheme of finite type over k. A *curve* is a projective variety of pure dimension 1.

Throughout the paper X is a connected nodal curve of arithmetic genus g, and Y is a nodal curve, not necessarily connected. We denote by g_Y the arithmetic genus of Y.

A node n of Y is called *separating* if the number of connected components of $Y \setminus n$ is greater than the number of connected components of Y. We denote by Y_{sep} the set of separating nodes of Y.

For any subset $S \subset X_{\mathrm{sing}} := \{\text{nodes of } X\}$, we denote by $\nu_S : Y_S \to X$ the partial normalization of X at S. We denote by γ_S the number of connected components of Y_S. The (total) normalization of X will be denoted by

$$\nu : X^\nu \to X = \bigsqcup_{i=1}^{\gamma} C_i$$

where the C_i are the connected components of X^ν. The points $\nu^{-1}(X_{\mathrm{sing}}) \subset X^\nu$ will often be called *gluing points* of ν.

The dual graph of Y will be denoted by Γ_Y. The irreducible components of Y correspond to the vertices of Γ_Y, and we shall systematically identify these two sets. Likewise we shall identify the set of nodes of Y with the set $E(\Gamma_Y)$ of edges of Γ_Y.

A graph Γ is a *cycle* if it is connected and has h edges and h vertices (each of valency 2) for some $h \geq 1$. A curve whose dual graph is a cycle will be called a *cycle curve*.

1.2.1. *The graph $\Gamma_X(S)$ and the graph $\Gamma_X \setminus S$.* Let $S \subset X_{\mathrm{sing}}$ be a set of nodes of X; we associate to S a graph, $\Gamma_X(S)$, defined as follows. $\Gamma_X(S)$ is obtained from Γ_X by contracting to a point every edge not in S. In particular, the set of edges of $\Gamma_X(S)$ is naturally identified with S. Consider $\nu_S : Y_S \to X$ (the normalization of X at S). Then the vertices of $\Gamma_X(S)$ correspond to the connected components of Y_S. For example, $\Gamma_X(X_{\mathrm{sing}}) = \Gamma_X$, and $\Gamma_X(\emptyset)$ is a point.

The graph $\Gamma_X \setminus S$ is defined as the graph obtained from Γ_X by removing the edges in S and leaving everything else unchanged. Of course $\Gamma_X \setminus S$ is equal to the dual graph of Y_S.

The above notation was also used in [CV09].

1.2.2. In graph theory two graphs Γ and Γ' are called *cyclically equivalent* (or two-isomorphic), in symbols $\Gamma \equiv_{\mathrm{cyc}} \Gamma'$, if there exists a bijection $\epsilon : E(\Gamma) \to E(\Gamma')$ inducing a bijection between the cycles of Γ and the cycles of Γ'; such an ϵ will be called a *cyclic bijection*. In other words, if for any orientation on Γ there exists an orientation on Γ' such that the following diagram is commutative:

$$\begin{array}{ccc} C_1(\Gamma, \mathbb{Z}) & \xrightarrow[\cong]{\epsilon_C} & C_1(\Gamma', \mathbb{Z}) \\ \uparrow & & \uparrow \\ H_1(\Gamma, \mathbb{Z}) & \xrightarrow[\cong]{\epsilon_H} & H_1(\Gamma', \mathbb{Z}) \end{array}$$

where the vertical arrows are the inclusions, ϵ_C is the (linear) isomorphism induced by ϵ and ϵ_H the restriction of ϵ_C to $H_1(\Gamma, \mathbb{Z})$.

1.2.3. *The moduli space* $\overline{A}_g^{\mathrm{mod}}$

Definition 1.2.4 ([Ale02]). A *principally polarized stable semi-abelic pair* (ppSSAP for short) over k is a pair $(G \curvearrowright P, \Theta)$ where

(i) G is a semiabelian variety over k, that is, an algebraic group which is an extension of an abelian variety A by a torus T:

$$1 \to T \to G \to A \to 0.$$

(ii) P is a seminormal, connected, projective variety of pure dimension equal to $\dim G$.
(iii) G acts on P with finitely many orbits, and with connected and reduced stabilizers contained in the toric part T of G.
(iv) Θ is an effective ample Cartier divisor on P which does not contain any G-orbit, and such that $h^0(P, \mathcal{O}_P(\Theta)) = 1$.

A G-variety $(G \curvearrowright P)$ satisfying the first three properties above is called a *stable semi-abelic variety* (SSAV for short).

When G is an abelian variety, the word "semi-abelic" is replaced by "abelic".

A homomorphism $\Phi = (\phi_0, \phi_1) : (G \curvearrowright P, \Theta) \to (G' \curvearrowright P', \Theta')$ between two ppSSAP is given by a homomorphism of algebraic groups $\phi_0 : G \to G'$, and a morphism $\phi_1 : P \to P'$, satisfying the following two conditions:

(1) ϕ_0 and ϕ_1 are compatible with the actions of G on P and of G' on P'.
(2) $\phi_1^{-1}(\Theta') = \Theta$.

$\Phi = (\phi_0, \phi_1)$ is an isomorphism if ϕ_0 and ϕ_1 are isomorphisms.

One of the main results of [Ale02] is the following

Theorem 1.2.5. *There exists a projective scheme* $\overline{A}_g^{\mathrm{mod}}$ *which is a coarse moduli space for principally polarized stable semi-abelic pairs. Moreover the open subset parametrizing principally polarized stable abelic pairs is naturally isomorphic to* A_g. *The normalization of the irreducible component of* $\overline{A}_g^{\mathrm{mod}}$ *containing* A_g (*the "main component"*) *is isomorphic to the second toroidal Voronoi compactification* $\overline{A}_g^{\mathrm{Vor}}$.

To the best of our knowledge, it is not known whether the main component of $\overline{A}_g^{\mathrm{mod}}$ is normal; see [Bri07] for an expository account.

1.2.6. *The compactified Torelli map* $\bar{\mathrm{t}}_g : \overline{M}_g \to \overline{A}_g^{\mathrm{mod}}$. We shall now recall the modular description of the compactified Torelli map $\bar{\mathrm{t}}_g : \overline{M}_g \to \overline{A}_g^{\mathrm{mod}}$.

Definition 1.2.7. Let Y be a nodal curve of arithmetic genus g_Y. Let M be a line bundle on Y of multidegree $\underline{d}$ and degree $g_Y - 1$. We say that M, or its multidegree $\underline{d}$, is *semistable* if for every subcurve $Z \subset Y$ of arithmetic genus g_Z, we have

$$g_Z - 1 \leq d_Z, \tag{1.1}$$

where $d_Z := \deg_Z M$. We say that M, or its multidegree $\underline{d}$, is *stable* if equality holds in (1.1) exactly for every subcurve Z which is a union of connected components of Y. We denote by $\Sigma(Y)$ the set of stable multidegrees on Y.

We denote by $\mathrm{Pic}^{\underline{d}} Y$ the variety of line bundles of multidegree $\underline{d}$ on Y. The variety of line bundles having degree 0 on every component of Y, $\mathrm{Pic}^{\underline{0}} Y = J(Y)$, is identified with the generalized Jacobian. Using the notation of 1.2 and 1.2.1, we now recall some properties of the compactified Jacobian $\overline{P_X^{g-1}}$ (see [Ale04], [Cap07]).

Fact 1.2.8. *Let X be a connected nodal curve of genus g, and $J(X)$ its generalized Jacobian.*

(i) *$\overline{P_X^{g-1}}$ is a SSAV with respect to the natural action $J(X)$.*

(ii) *The orbits of the action of $J(X)$ give a stratification of $\overline{P_X^{g-1}}$:*

$$\overline{P_X^{g-1}} = \bigsqcup_{\substack{\emptyset \subseteq S \subseteq X_{\mathrm{sing}} \\ \underline{d} \in \Sigma(Y_S)}} P_S^{\underline{d}}, \tag{1.2}$$

where each stratum $P_S^{\underline{d}}$ is canonically isomorphic to $\mathrm{Pic}^{\underline{d}} Y_S$.

(iii) *$\Sigma(Y_S)$ is not empty if and only if Y_S has no separating node. In particular, if $\Sigma(Y_S)$ is not empty then $X_{\mathrm{sep}} \subseteq S$.*

(iv) *Each stratum $P_S^{\underline{d}}$ is a torsor under the generalized Jacobian $J(Y_S)$ of Y_S, and the action of $J(X)$ on $P_S^{\underline{d}}$ factorizes through the pull-back map $J(X) \twoheadrightarrow J(Y_S)$. Hence every non-empty stratum $P_S^{\underline{d}}$ has dimension*

$$\dim P_S^{\underline{d}} = \dim J(Y_S) = g - \#S + \gamma_S - 1 = g - b_1(\Gamma_X(S)). \tag{1.3}$$

(v) *If $P_{S'}^{\underline{d}'} \subset \overline{P_S^{\underline{d}}}$ then $S \subset S'$ and $\underline{d} \geq \underline{d}'$ (i.e. $d_i \geq d_i'$ for all $i = 1, \ldots, \gamma$).*

(vi) *The smooth locus P_X^{g-1} of $\overline{P_X^{g-1}}$ consists of the strata of maximal dimension:*

$$P_X^{g-1} = \bigsqcup_{\underline{d} \in \Sigma(Y_{X_{\mathrm{sep}}})} P_S^{\underline{d}}.$$

The irreducible components of $\overline{P_X^{g-1}}$ are the closures of the maximal dimension strata.

To give the definition of the theta divisor we introduce some notation. For any multidegree $\underline{d}$ on a curve Y and for any $r \geq 0$ we set

$$W_{\underline{d}}^r(Y) := \{L \in \mathrm{Pic}^{\underline{d}} Y : h^0(Y, L) > r\}; \tag{1.4}$$

when $r = 0$ the superscript is usually omitted: $W_{\underline{d}}^0(Y) = W_{\underline{d}}(Y)$.

The normalization of X at its set of separating nodes, X_{sep}, will be denoted by

$$\widetilde{X} = \bigsqcup_{i=1}^{\widetilde{\gamma}} \widetilde{X}_i \tag{1.5}$$

where the $\widetilde{X}_i$ are connected (and all free from separating nodes). Note that $\widetilde{\gamma} = \#X_{\mathrm{sep}} + 1$. We denote by $\widetilde{g}_i$ the arithmetic genus of $\widetilde{X}_i$.

The subsequent facts summarize results of [E97], [Ale04] and [Cap07].

Definition 1.2.9. The *theta divisor* $\Theta(X)$ of $\overline{P_X^{g-1}}$ is

$$\Theta(X) := \overline{\bigcup_{\underline{d}\in\Sigma(\widetilde{X})} W_{\underline{d}}(\widetilde{X})} \subset \overline{P_X^{g-1}}.$$

Fact 1.2.10. (i) *The pair* $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X))$ *is a ppSSAP. In particular* $\Theta(X)$ *is Cartier, ample and* $h^0(\overline{P_X^{g-1}}, \Theta(X)) = 1$.

(ii) *The stratification of* $\overline{P_X^{g-1}}$ *given by* 1.2.8(ii) *induces the stratification*

$$\Theta(X) = \bigsqcup_{\substack{\emptyset\subseteq S\subseteq X_{\mathrm{sing}} \\ \underline{d}\in\Sigma(Y_S)}} \Theta_S^{\underline{d}}, \tag{1.6}$$

where $\Theta_S^{\underline{d}} := \{M \in P_S^{\underline{d}} \,:\, h^0(Y_S, M) > 0\} \cong W^0_{\underline{d}}(Y_S)$ *is a divisor in* $P_S^{\underline{d}}$.

(iii) *Let* $Y_S = \bigsqcup_{i=1}^{\gamma_S} Y_i$ *be the decomposition of* Y_S *into connected components, and let* $\underline{d} \in \Sigma(Y_S)$. *The irreducible components of* $\Theta_S^{\underline{d}}$ *are given by*

$$(\Theta_S^{\underline{d}})_i = \{L \in P_S^{\underline{d}} \,:\, h^0(Y_i, L_{|Y_i}) > 0\}$$

for every $1 \le i \le \gamma_S$ *such that the arithmetic genus of* Y_i *is positive.*

Remark 1.2.11. From the description 1.2.8, we derive that there exists a unique $J(X)$-stratum in $\overline{P_X^{g-1}}$ contained in the closure of every other stratum, namely

$$P_{X_{\mathrm{sing}}}^{(g_1-1,\dots,g_\gamma-1)} = \prod_{i=1}^{\gamma} \mathrm{Pic}^{g_i-1} C_i.$$

We refer to this stratum as the *smallest stratum* of $\overline{P_X^{g-1}}$. Moreover, according to stratification (1.6), the restriction of $\Theta(X)$ to the smallest stratum is given by

$$\Theta(X)_{|P_{X_{\mathrm{sing}}}^{(g_1-1,\dots,g_\gamma-1)}} = \bigcup_{i=1}^{\gamma} \mathrm{Pic}^{g_1-1} C_1 \times \cdots \times \Theta(C_i) \times \cdots \times \mathrm{Pic}^{g_\gamma-1} C_\gamma. \tag{1.7}$$

We can now state the following result of Alexeev ([Ale04]):

Theorem 1.2.12. *The classical Torelli morphism is compactified by the morphism* $\bar{\mathrm{t}}_g : \overline{M}_g \to \overline{A}_g^{\mathrm{mod}}$ *which maps a stable curve* X *to* $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X))$.

1.3. First reductions

We shall now show that the ppSSAP $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X))$ depends only on the stabilization of every connected component of the partial normalization of X at its separating nodes. Most of what is in this subsection is well known to the experts.

We first recall the notion of stabilization. A connected nodal curve X of arithmetic genus $g \geq 0$ is called *stable* if each smooth rational component $E \subsetneq X$ meets the complementary subcurve $E^c = \overline{X \setminus E}$ in at least three points. So, when $g = 0$ the only stable curve is $\mathbb{P}^1$. If $g = 1$ a stable curve is either smooth or irreducible with one node. If $g \geq 2$ stable curves are Deligne–Mumford stable curves.

Given any nodal connected curve X, the *stabilization* of X is defined as the curve $\overline{X}$ obtained as follows. If X is stable then $X = \overline{X}$; otherwise let $E \subset X$ be an exceptional component (i.e. $E \subsetneq X$ such that $\#E \cap E^c \leq 2$ and $E \cong \mathbb{P}^1$), then we contract E to a point, thereby obtaining a new curve X_1. If X_1 is stable we let $X_1 = \overline{X}$, otherwise we choose an exceptional component of X_1 and contract it to a point. By iterating this process we certainly arrive at a stable curve $\overline{X}$. It is easy to check that $\overline{X}$ is unique up to isomorphism.

The stabilization of a disconnected curve will be defined as the union of the stabilizations of its connected components.

From the moduli properties of $\overline{A}_g^{\mathrm{mod}}$, and the fact that it is a projective scheme, one derives the following useful

Remark 1.3.1 (Invariance under stabilization). Let X be a connected nodal curve of arithmetic genus $g \geq 0$, and let $\overline{X}$ be its stabilization. Then

$$(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X)) \cong (J(\overline{X}) \curvearrowright \overline{P_{\overline{X}}^{g-1}}, \Theta(\overline{X})).$$

Now, we show how to deal with separating nodes. To do that we must deal with disconnected curves. Let $Y = \bigsqcup_{i=1}^h Y_i$ be such a curve and g_Y its arithmetic genus, so that $g_Y = \sum g_{Y_i} - h$. We have

$$\overline{P_Y^{g_Y-1}} = \prod_{i=1}^h \overline{P_{Y_i}^{g_{Y_i}-1}} \quad \text{and} \quad \Theta(Y) = \bigcup_{i=1}^h \pi_i^*(\Theta(Y_i)) \tag{1.8}$$

where $\pi_i : \overline{P_Y^{g_Y-1}} \to \overline{P_{Y_i}^{g_{Y_i}-1}}$ is the i-th projection.

The next lemma illustrates the recursive structure of $(\overline{P_X^{g-1}}, \Theta(X))$. For $S \subset X_{\mathrm{sing}}$ such that $\Sigma(Y_S)$ is non-empty (i.e. Y_S has no separating nodes), denote

$$\overline{P_S} := \overline{\bigcup_{\underline{d} \in \Sigma(Y_S)} P_S^{\underline{d}}} \subset \overline{P_X^{g-1}} \quad \text{and} \quad \overline{\Theta_S} := \Theta(X) \cap \overline{P_S}. \tag{1.9}$$

Lemma 1.3.2. *Under the assumptions as above, there is a natural isomorphism* $\overline{P_S} \cong \overline{P_{Y_S}^{g_{Y_S}-1}}$, *inducing an isomorphism between* $\overline{\Theta_S}$ *and* $\Theta(Y_S)$.

Proof. Recall that $\overline{P_X^{g-1}}$ is a GIT-quotient, $V_X \xrightarrow{q} \overline{P_X^{g-1}} = V_X/G$ where V_X is contained in a certain Hilbert scheme of curves in projective space (there are other descriptions of $\overline{P_X^{g-1}}$ as a GIT-quotient, to which the subsequent proof can be easily adjusted). Denote $V_Y := q^{-1}(\overline{P_S})$ so that V_Y is a G-invariant, reduced, closed subscheme of V_X and $\overline{P_S}$ is the GIT-quotient

$$V_Y \to V_Y/G = \overline{P_S}. \tag{1.10}$$

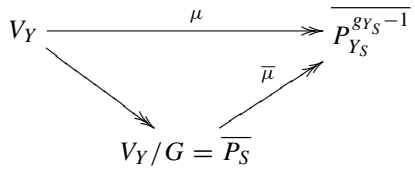
The restriction to V_Y of the universal family over the Hilbert scheme is a family of nodal curves $\mathcal{Z} \to V_Y$ endowed with a semistable line bundle $\mathcal{L} \to \mathcal{Z}$. Let Z be any fiber of $\mathcal{Z} \to V_Y$; then Z has X as stabilization, and the stabilization map $Z \to X$ blows up some set S' of nodes of X; note that S' certainly contains S. Therefore the exceptional divisors corresponding to $s \in S$ form a family over V_Y,

$$\mathcal{Z} \supset \mathcal{E}_S \to V_Y,$$

such that $\mathcal{E}_S = \bigsqcup_{s\in S} \mathcal{E}_s$ and every $\mathcal{E}_s$ is a $\mathbb{P}^1$-bundle over V_Y. Consider the family of curves obtained by removing $\mathcal{E}_S$:

$$\mathcal{Y} := \overline{\mathcal{Z} \smallsetminus \mathcal{E}_S} \to V_Y.$$

By construction the above is a family of nodal curves, all admitting a surjective map to Y_S which blows down some exceptional component (over a dense open subset of V_Y the fiber of $\mathcal{Y} \to V_Y$ is isomorphic to Y_S). The restriction $\mathcal{L}_\mathcal{Y}$ of $\mathcal{L}$ to $\mathcal{Y}$ is a relatively semistable line bundle. Therefore $\mathcal{L}_\mathcal{Y}$ determines a unique moduli map μ from V_Y to the compactified Picard variety of Y_S, i.e. $\mu : V_Y \to \overline{P_{Y_S}^{g_{Y_S}-1}}$. The map μ is of course G-invariant, and therefore it descends to a unique map $\overline{\mu} : V_Y/G \to \overline{P_{Y_S}^{g_{Y_S}-1}}$. Summarizing, we have a commutative diagram



By Fact 1.2.8 the morphism $\overline{\mu}$ is a bijection. Since $\overline{P_{Y_S}^{g_{Y_S}-1}}$ is seminormal, $\overline{\mu}$ is an isomorphism. Finally, by Fact 1.2.10 we conclude that $\overline{\mu}$ maps $\overline{\Theta_S}$ isomorphically to $\Theta(Y_S)$. □

We say that a ppSSAP $(G \curvearrowright P, \Theta)$ is *irreducible* if every irreducible component of P contains a unique irreducible component of Θ. In the next result we use the notation (1.5).

Corollary 1.3.3. (i) *If $X_{\mathrm{sep}} = \emptyset$ then $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X))$ is irreducible.*
(ii) *In general, we have the decomposition into irreducible non-trivial ppSSAP:*

$$(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X)) = \prod_{\widetilde{g}_i>0} (J(\widetilde{X}_i) \curvearrowright \overline{P_{\widetilde{X}_i}^{\widetilde{g}_i-1}}, \Theta(\widetilde{X}_i)).$$

Proof. The first assertion follows from [Cap07, Thm. 3.1.2]. For the second assertion, by 1.2.8 we have $J(X) = \prod_{i=1}^{\widetilde{\gamma}} J(\widetilde{X}_i)$. Now we apply Lemma 1.3.2 to $S = X_{\mathrm{sep}}$. Note that in this case $\overline{P_S} = \overline{P_X^{g-1}}$, and hence $\overline{\Theta_S} = \Theta(X)$. Therefore we get

$$(\overline{P_X^{g-1}}, \Theta(X)) \cong (\overline{P_{\widetilde{X}}^{g_{\widetilde{X}}-1}}, \Theta(\widetilde{X})) \cong \prod_{\widetilde{g}_i>0} (\overline{P_{\widetilde{X}_i}^{\widetilde{g}_i-1}}, \Theta(\widetilde{X}_i)). \qquad \square$$

2. Statement of the main theorem

2.1. C1-equivalence

Assume that $X_{\mathrm{sep}} = \emptyset$. We introduce two partially ordered sets (posets for short) associated to the stratification of $\overline{P_X^{g-1}}$ into $J(X)$-orbits, described in (1.2).

- The *poset of strata*, denoted $\mathcal{ST}_X$, is the set $\{P_S^d\}$ of all strata of $\overline{P_X^{g-1}}$, endowed with the following partial order:

$$P_S^d \geq P_T^e \;\Leftrightarrow\; \overline{P_S^d} \supset P_T^e. \tag{2.1}$$

- The *poset of (strata) supports*, denoted $\mathcal{SP}_X$, is the set of all subsets $S \subset X_{\mathrm{sing}}$ such that the partial normalization of X at S, Y_S, is free from separating nodes, or equivalently (recall 1.2.1):

$$\mathcal{SP}_X := \{S \subset E(\Gamma_X) : \Gamma_X \smallsetminus S \text{ has no separating edge}\}. \tag{2.2}$$

  Its partial order is defined as follows:

$$S \geq T \;\Leftrightarrow\; S \subseteq T.$$

  There is a natural map

$$\mathrm{Supp}_X : \mathcal{ST}_X \to \mathcal{SP}_X, \qquad P_S^d \mapsto S.$$

  Supp_X is order preserving (by Fact 1.2.8(v)), and surjective (by Fact 1.2.8(iii)).

We have the integer valued function, codim, on $\mathcal{SP}_X$ (cf. 1.2.1 and (1.3)):

$$\mathrm{codim}(S) := \dim J(X) - \dim J(Y_S) = b_1(\Gamma_X(S)). \tag{2.3}$$

Notice that $\mathrm{codim}(S)$ is the codimension in $\overline{P_X^{g-1}}$ of every stratum $P_S^d \in \mathrm{Supp}_X^{-1}(S)$. Moreover codim is strictly order reversing.

Lemma–Definition 2.1.1. *Assume $X_{\mathrm{sep}} = \emptyset$; let $S \in \mathcal{SP}_X$. We say that S is a* C1-set *if the two equivalent conditions below hold.*

(1) $\mathrm{codim}(S) = 1$.
(2) *The graph $\Gamma_X(S)$ (defined in 1.2.1) is a cycle.*

We denote by Set^1 *X the set of all C1-sets of X.*

Proof. The equivalence between (1) and (2) follows from (2.3), together with the fact that for any $S \subset X_{\mathrm{sing}}$ the graph $\Gamma_X(S)$ is connected and free from separating edges (because the same holds for Γ_X). □

2.1.2. Under the identification between the nodes of X and the edges of $\Gamma(X)$, our definition of C1-sets of X coincides with that of C1-sets of $\Gamma(X)$ given in [CV09, Def. 2.3.1]. The set of C1-sets of any graph Γ, which is a useful tool in graph theory, is denoted by $\mathrm{Set}^1\,\Gamma$; we shall, as usual, identify $\mathrm{Set}^1\,\Gamma_X = \mathrm{Set}^1\,X$. The following fact is a rephrasing of [CV09, Lemma 2.3.2].

Fact 2.1.3. *Let X be a connected curve free from separating nodes.*

(1) *Every node of X is contained in a unique C1-set.*
(2) *Two nodes of X belong to the same C1-set if and only if the corresponding edges of the dual graph Γ_X belong to the same cycles of Γ_X.*
(3) *Two nodes n_1 and n_2 of X belong to the same C1-set if and only if the normalization of X at n_1 and n_2 is disconnected.*

Remark 2.1.4. Therefore, if $X_{\mathrm{sep}} = \emptyset$ the C1-sets form a partition of X_{sing}. The preimage under the normalization map ν of this partition is a partition of the set of gluing points, $\nu^{-1}(X_{\mathrm{sing}}) \subset X^\nu$. We shall refer to this partition of $\nu^{-1}(X_{\mathrm{sing}})$ as the *C1-partition*.

The main result of this paper, Theorem 2.1.7 below, is based on the following

Definition 2.1.5. Let X and X' be connected nodal curves free from separating nodes; denote by $\nu : X^\nu \to X$ and $\nu' : X'^\nu \to X'$ their normalizations. X and X' are *C1-equivalent* if the following conditions hold:

(A) There exists an isomorphism $\phi : X^\nu \xrightarrow{\cong} X'^\nu$.
(B) There exists a bijection between their C1-sets, denoted by

$$\mathrm{Set}^1\,X \to \mathrm{Set}^1\,X', \qquad S \mapsto S',$$

such that $\phi(\nu^{-1}(S)) = \nu'^{-1}(S')$.

In general, two nodal curves Y and Y' are C1-equivalent if there exists a bijection between their connected components such that any two corresponding components are C1-equivalent.

With the terminology introduced in Remark 2.1.4, we can informally state that two curves free from separating nodes are C1-equivalent if and only if they have the same normalization Y, the same set of gluing points $G \subset Y$, and the same C1-partition of G.

Example 2.1.6. (1) Let X be irreducible. Then for every node $n \in X_{\mathrm{sing}}$ the set $\{n\}$ is a C1-set, and every C1-set of X is obtained in this way. It is clear that the only curve C1-equivalent to X is X itself.
(2) Let $X = C_1 \cup C_2$ be the union of two smooth components meeting at $\delta \geq 3$ nodes (the case $\delta = 2$ has to be treated apart, see below). Then again for every $n \in X_{\mathrm{sing}}$ we have $\{n\} \in \mathrm{Set}^1\,X$ so that $\mathrm{Set}^1\,X \cong X_{\mathrm{sing}}$. Also in this case X is the only curve in its C1-equivalent class. The same holds if the C_i have some node.

(3) Let X be such that its dual graph is a cycle of length at least 2. Now the only C1-set is the whole X_{sing} and, apart from some special cases, X will not be the unique curve in its C1-equivalence class; see Example 5.1.2 and Section 5 for details.

Theorem 2.1.7. *Let X and X' be two stable curves of genus g. Assume that X and X' are free from separating nodes. Then $\bar{\mathrm{t}}_g(X) = \bar{\mathrm{t}}_g(X')$ if and only if X and X' are C1-equivalent.*

In general, let $\widetilde{X}$ and $\widetilde{X}'$ be the normalizations of X and X' at their separating nodes. Then $\bar{\mathrm{t}}_g(X) = \bar{\mathrm{t}}_g(X')$ if and only if the stabilization of $\widetilde{X}$ is C1-equivalent to the stabilization of $\widetilde{X}'$.

By Example 2.1.6 we know that if X is irreducible, or if X is the union of two components meeting in at least three points, then the Torelli map is injective (i.e. $\mathrm{t}_g^{-1}(\mathrm{t}_g(X)) = \{X\}$). The locus of curves $X \in \overline{M}_g$ such that $\mathrm{t}_g^{-1}(\mathrm{t}_g(X)) = \{X\}$ will be characterized in Theorem 5.1.5. Theorem 2.1.7 will be proved in Section 4.

2.2. Some properties of C1-sets

Here are a few facts to be applied later.

Remark 2.2.1. Let $S \in \mathrm{Set}^1 X$ and consider Y_S, the normalization of X at S. By definition Y_S has $\#S$ connected components, and $\Gamma_X(S)$ can be viewed as the graph whose vertices are the connected components of Y_S, and whose edges correspond to S. Since $\Gamma_X(S)$ is a cycle, if X is stable every connected component of Y_S has positive arithmetic genus.

Lemma 2.2.2. *Let S and T be two distinct C1-sets of X. Then T is entirely contained in a unique connected component of Y_S.*

Proof. Recall that Y_S has $\#S$ connected components, all free from separating nodes. By Fact 2.1.3 the set T is contained in the singular locus of Y_S. Let $n_1, n_2 \in T$, and let X^* and Y_S^* be the normalizations at n_1 of, respectively, X and Y_S. By Fact 2.1.3(3), n_2 is a separating node of X^* and hence of Y_S^*. Since Y_S has no separating node we see that n_1 belongs to the same connected component as n_2. □

In the next lemma we use the notation of 2.1.2 and 2.2.1.

Lemma 2.2.3. *Let Γ be an oriented connected graph free from separating edges. Then the inclusion $H_1(\Gamma, \mathbb{Z}) \subset C_1(\Gamma, \mathbb{Z})$ factors naturally as*

$$H_1(\Gamma, \mathbb{Z}) \hookrightarrow \bigoplus_{S \in \mathrm{Set}^1 \Gamma} H_1(\Gamma(S), \mathbb{Z}) \hookrightarrow C_1(\Gamma, \mathbb{Z})$$

where the graphs $\Gamma(S)$ have the orientation induced by that of Γ.

Proof. Let $S \in \mathrm{Set}^1\,\Gamma$ and consider the natural map $\sigma_S : \Gamma \to \Gamma(S)$ contracting all edges not in S. Recall that $\Gamma(S)$ is a cycle whose set of edges is S. By Fact 2.1.3 we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1(\Gamma \smallsetminus S, \mathbb{Z}) & \longrightarrow & C_1(\Gamma, \mathbb{Z}) & \longrightarrow & C_1(\Gamma(S), \mathbb{Z}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H_1(\Gamma \smallsetminus S, \mathbb{Z}) & \longrightarrow & H_1(\Gamma, \mathbb{Z}) & \xrightarrow{\sigma_{S*}} & H_1(\Gamma(S), \mathbb{Z}) & \longrightarrow & 0 \end{array} \tag{2.4}$$

where $\Gamma \smallsetminus S \subset \Gamma$ is the subgraph obtained by removing S from $E(\Gamma)$. We claim that we have the commutative diagram

$$\begin{array}{ccc} H_1(\Gamma, \mathbb{Z}) & \hookrightarrow & \bigoplus_{S \in \mathrm{Set}^1\,\Gamma} H_1(\Gamma(S), \mathbb{Z}) \\ \downarrow & & \downarrow \\ C_1(\Gamma, \mathbb{Z}) & \hookleftarrow & \bigoplus_{S \in \mathrm{Set}^1\,\Gamma} C_1(\Gamma(S), \mathbb{Z}) \end{array} \tag{2.5}$$

where the vertical arrows are the usual inclusions. The bottom horizontal arrow is the obvious map mapping an edge $e \in E(\Gamma(S)) = S \subset E(\Gamma)$ to itself. It is injective because two different C1-sets of Γ are disjoint (by 2.1.3) (and surjective as Γ has no separating edges). Finally, the top horizontal arrow is the sum of the maps σ_{S*} defined in the previous diagram; it is injective because the diagram is clearly commutative and the other maps are injective. □

2.3. Gluing points and gluing data

Let X be such that $X_{\mathrm{sep}} = \emptyset$, and let $S \in \mathrm{Set}^1\,X$ be a C1-set of cardinality h. The partial normalization Y_S of X at S has a decomposition $Y_S = \bigsqcup_{i=1}^h Y_{S,i}$, with $Y_{S,i}$ connected and free from separating nodes. We denote by $Y_{S,i}^\nu$ the normalization of $Y_{S,i}$. We set

$$G_S := \nu^{-1}(S) \subset X^\nu. \tag{2.6}$$

Each of the connected components $Y_{S,i}$ of Y_S contains exactly two of the points in G_S, let us call them p_i and q_i. This enables us to define a unique fixed-point free involution on G_S, denoted ι_S, such that ι_S exchanges p_i and q_i for every $1 \leq i \leq h$.

The involutions ι_S and the curves $Y_{S,i}^\nu$ are the same for C1-equivalent curves, by the next result.

Lemma 2.3.1. *Let X be free from separating nodes. The data of X^ν and of the sets $G_S \subset X^\nu$ for every $S \in \mathrm{Set}^1\,X$ uniquely determine the curves $Y_{S,i}^\nu \subset X^\nu$ and the involution ι_S.*

Proof. Pick a C1-set S and let $h = \#S$. Denote $G_S := \{r_1, \ldots, r_{2h}\}$ and $Y_S = \bigsqcup_{i=1}^h Y_i$. We have

$$\#G_S \cap Y_i^\nu = 2 \tag{2.7}$$

for every i. Consider the point r_1 and let Y_1^ν be the component containing it. Let us show how to reconstruct Y_1^ν. Let $C_1 \subset X^\nu$ be the irreducible component containing r_1; of course $C_1 \subset Y_1^\nu$.

Now, by Lemma 2.2.2, for every $T \in \mathrm{Set}^1 X$ such that $S \neq T$, if $G_T \cap C_1 \neq \emptyset$ then T is entirely contained in the singular locus of Y_1. In particular every irreducible component of X^ν intersecting G_T is contained in Y_1^ν. Define the following subcurve Z_1 of $X^\nu = \bigsqcup C_i$:

$$Z_1 := C_1 \sqcup \bigsqcup_{\substack{\exists T \in \mathrm{Set}^1 X \smallsetminus \{S\}: \\ C_i \cap G_T \neq \emptyset, C_1 \cap G_T \neq \emptyset}} C_i.$$

We now argue as before, by replacing C_1 with Z_1. We find that if X has a C1-set $T \neq S$ such that G_T intersects Z_1, then again $T \subset (Y_1)_{\mathrm{sing}}$; therefore, by Lemma 2.2.2, every component of X^ν intersecting G_T is contained in Y_1^ν. We can hence inductively define the following subcurve of Y_1^ν. We rename $Z_0 := C_1$; next for $n \geq 1$ we set

$$Z_n := Z_{n-1} \sqcup \bigsqcup_{\substack{\exists T \in \mathrm{Set}^1 X \smallsetminus \{S\}: \\ C_i \cap G_T \neq \emptyset, Z_{n-1} \cap G_T \neq \emptyset}} C_i.$$

Since all of the nodes of Y_1 belong to some C1-set of X, for n large enough we have $Z_n = Z_{n+1} = \cdots = Y_1^\nu$. Hence Y_1^ν is uniquely determined. Now, by (2.7) we find that $Y_1^\nu \cap G_S = \{r_1, r_j\}$ for a unique $j \neq 1$; therefore we must have $\iota_S(r_1) = r_j$. This shows that the curves $Y_{S,i}^\nu$ are all determined, and so are the involutions ι_S. □

2.3.2. *Gluing data of X.* By Lemma 2.3.1, if X and X' are C1-equivalent for every pair of corresponding C1-sets S and S' the isomorphism between their normalizations preserves the decompositions $Y_S = \bigsqcup_{i=1}^h Y_{S,i}$ and $Y'_{S'} = \bigsqcup_{i=1}^h Y'_{S',i}$, as well as the involutions ι_S and $\iota_{S'}$. What extra data should one specify to reconstruct X from its C1-equivalence class? We now give an answer to this question. Fix $S \in \mathrm{Set}^1 X$, let $h = \#S$ and $Y_S = \bigsqcup_{i=1}^h Y_i$. By Lemma 2.3.1 the C1-equivalence class of X determines the involution ι_S of G_S. This enables us to write $G_S = \{p_1, q_1, \ldots, p_h, q_h\}$ with $p_i, q_i \in Y_i^\nu$. Of course this is not enough to determine how G_S is glued on X. To describe what is further needed, we introduce an abstract set of cardinality $2h$, denoted $G_h = \{s_1, t_1, \ldots, s_h, t_h\}$, endowed with the involution ι_h defined by $\iota_h(s_i) = t_i$ for every $1 \leq i \leq h$.

Pick either of the two cyclic orientations of $\Gamma_X(S)$. We claim that the gluing data of G_S determine, and are uniquely determined by, the following two items.

(1) A *marking* $\psi_S : (G_h, \iota_h) \xrightarrow{\cong} (G_S, \iota_S)$, where ψ_S is a bijection mapping the (unordered) pair (s_i, t_i) to the pair (p_i, q_i).
(2) A cyclic permutation of $\{1, \ldots, h\}$, denoted by σ_S, free from fixed points.

Indeed the points $\psi_S(s_i)$ and $\psi_S(t_i)$ correspond, respectively, to the sources and targets of the orientation of $\Gamma_X(S)$; the permutation σ_S is uniquely determined by the fact that the point $\psi_S(s_i)$ is glued to the point $\psi_S(t_{\sigma_S(i)})$. The opposite cyclic orientation of $\Gamma_X(S)$ corresponds to changing

$$(\sigma_S, \psi_S) \mapsto (\sigma_S^{-1}, \psi_S \circ \iota_h); \tag{2.8}$$

the above transformation defines an involution on the set of pairs (σ_S, ψ_S) as above. We call the equivalence class $[(\sigma_S, \psi_S)]$, with respect to the above involution, the *gluing data* of S on X.

Conversely, it is clear that a nodal curve X is uniquely determined, within its C1-equivalence class, by an equivalence class $[(\sigma_S, \psi_S)]$ for each C1-set $S \in \mathrm{Set}^1 X$. In fact, X is given as follows:

$$X = \frac{X^\nu}{\bigsqcup_{S\in \mathrm{Set}^1 X}\{\psi_S(s_i) = \psi_S(t_{\sigma_S(i)}) : 1 \leq i \leq \#S\}}.$$

The previous analysis would enable us to explicitly, and easily, bound the cardinality of any C1-equivalence class. We postpone this to the final section of the paper; see Lemma 5.1.6.

2.4. Dual graphs of C1-equivalent curves

In this subsection, we shall prove that two C1-equivalent curves have cyclically equivalent dual graphs. As a matter of fact, we will prove a slightly stronger result. We first need the following

Definition 2.4.1. Let Γ and Γ' be two graphs free from separating edges. We say that Γ and Γ' are *strongly cyclically equivalent* if they can be obtained from one another via iterated applications of the following move, called *twisting at a separating pair of edges*:

Fig. 1. A twisting at the separating pair of edges $\{e_1, e_2\}$.

The above picture means the following. Since (e_1, e_2) is a separating pair of edges, $\Gamma \smallsetminus \{e_1, e_2\}$ has two connected components; call them Γ_a and Γ_b. For $i = 1, 2$ let v_i^a (resp. v_i^b) be the vertex of Γ_a (resp. of Γ_b) adjacent to e_i. Then Γ' is obtained by joining the two graphs Γ_a and Γ_b by an edge e'_1 from v_1^a to v_2^b and by another edge e'_2 from v_2^a to v_1^b. Notice that if $v_1^a = v_2^a$ and $v_1^b = v_2^b$, our twisting operation does not change the isomorphism class of the graph.

Remark 2.4.2. If Γ and Γ' are strongly cyclically equivalent then they are cyclically equivalent.

This is intuitively clear. A cyclic bijection $E(\Gamma) \to E(\Gamma')$ can be obtained by mapping every separating pair of edges at which a twisting is performed to its image. To check

that this bijection preserves the cycles it suffices to observe that if two edges form a separating pair then they belong to the same cycles. Alternatively, the twisting at a separating pair of edges is a particular instance of the so-called second move of Whitney, which does not change the cyclic equivalence class of a graph (see [Whi33]).

Proposition 2.4.3. *Let X and X' be free from separating nodes and C1-equivalent. Then Γ_X and $\Gamma_{X'}$ are strongly cyclically equivalent (and hence cyclically equivalent).*

Proof. By the discussion in 2.3.2, it will be enough to show that for every C1-set $S \in \mathrm{Set}^1 X$, any two gluing data associated to S can be transformed into one another by a sequence of edge twistings of the type described in 2.4.1. Moreover, it is enough to consider one C1-set at a time, in fact by 2.2.2, the twisting at a separating pair of edges $\{e_1, e_2\}$ belonging to $S \in \mathrm{Set}^1 X$ does not affect the gluing data of the other C1-sets.

So let us fix $S \in \mathrm{Set}^1 X$ of cardinality h and let $[(\sigma_S, \psi_S)]$ be the gluing data of S on X. We consider two types of edge-twisting, as in 2.4.1:

(a) Fix a component Y_j of Y_S, exchange the two gluing points lying on Y_j, $\psi_S(s_j)$ and $\psi_S(t_j)$, and leave everything else unchanged. On Γ_X this operation corresponds to a twisting at the separating pair of edges of S that join Γ_{Y_j} to $\Gamma_{Y_S \smallsetminus Y_j}$ (both viewed as subgraphs of Γ_X). The gluing data are changed according to the rule

$$[(\sigma_S, \psi_S)] \mapsto [(\sigma_S, \psi_S \circ \mathrm{inv}_j)],$$

where inv_j is the involution of $\{s_1, t_1, \ldots, s_h, t_h\}$ exchanging s_j with t_j and fixing everything else.

(b) Fix a connected component Y_j of Y_S and an integer $1 \leq a \leq h-1$. Consider the curve

$$Z = Y_j \sqcup Y_{\sigma_S(j)} \sqcup \cdots \sqcup Y_{\sigma_S^a(j)} \subset Y_S.$$

Now change the gluing data between Z and $Y_S \smallsetminus Z$ by exchanging the two points of Z that are glued to $Y_S \smallsetminus Z$, and leaving everything else unchanged. On Γ_X this operation corresponds to a twisting at the separating pair of edges of S that join Γ_Z to $\Gamma_{Y_S \smallsetminus Z}$. The gluing data are changed according to the rule

$$[(\sigma_S, \psi_S)] \mapsto [(\tau_{j,a} \circ \sigma_S \circ \tau_{j,a}^{-1}, \psi_S \circ \mathrm{inv}_{j,a})],$$

where $\tau_{j,a}$ is the element of the symmetric group $\mathcal{S}_h$ defined by

$$\tau_{j,a} := \prod_{0 \leq b \leq \lfloor a-1/2 \rfloor} (\sigma_S^b(j) \sigma_S^{a-b}(j))$$

and $\mathrm{inv}_{j,a}$ is the involution of $\{s_1, t_1, \ldots, s_h, t_h\}$ that exchanges s_k with t_k for all $k = j, \sigma_S(j), \ldots, \sigma_S^a(j)$, and fixes all the other elements.

The proof consists in showing that all the possible gluing data of S can be obtained starting from $[(\sigma_S, \psi_S)]$ and performing operations of type (a) and (b).

First of all observe that, by iterating operations of type (a), it is possible to arbitrarily modify the marking ψ_S, while keeping the cyclic permutation σ_S fixed.

On the other hand, using the fact that any two cyclic permutations of the symmetric group $\mathcal{S}_h$ are conjugate, and that $\mathcal{S}_h$ is generated by transpositions, it will be enough to show that for any transposition $(jk) \in \mathcal{S}_h$, by iterating operations of type (b), we can pass from the gluing data $[(\sigma_S, \psi_S)]$ to gluing data of the form $[((jk) \circ \sigma_S \circ (jk)^{-1}, \psi'_S)]$ for some marking ψ'_S. If the transposition (jk) is such that $k = \sigma_S(j)$ (resp. $k = \sigma_S^2(j)$), then it is enough to apply the operation (b) with respect to the component Y_j and the integer $a = 1$ (resp. $a = 2$). In the other cases, we can write $k = \sigma_S^a(j)$ with $3 \leq a \leq h-1$ and then we apply the operation (b) two times: first with respect to the component $Y_{\sigma_S(j)}$ and the integer $a-2$; secondly with respect to the component Y_j and the integer a. After these two operations the cyclic permutation σ_S gets changed to $(jk) \circ \sigma_S \circ (jk)^{-1}$ since

$$(jk) = (j\sigma_S^a(j)) = \tau_{j,a} \circ \tau_{\sigma_S(j),a-2}. \qquad \square$$

3. T-equivalence: a second version of the Torelli theorem

3.0. The statement of Theorem 2.1.7 characterizes curves having isomorphic ppSSAV in terms of their normalization, and of the C1-partition of their gluing points, determined by the codimension-one strata of the compactified Picard scheme.

In this section we shall give a different characterization, based on the classifying morphism of the generalized Jacobian. From the general theory of semiabelian varieties, recall that the *generalized Jacobian* $J(X)$ of a nodal curve X is an extension

$$1 \to H^1(\Gamma_X, k^*) = \mathbb{G}_{\mathrm{m}}^{b_1(\Gamma_X)} \to J(X) \to J(X^\nu) = \prod_{i=1}^{\gamma} J(C_i) \to 0$$

(recall that $\bigsqcup_{i=1}^{\gamma} C_i = X^\nu$ is the normalization of X). The above extension is determined by the so-called classifying morphism, from the character group of the torus $H^1(\Gamma_X, k^*)$, i.e. from $H_1(\Gamma_X, \mathbb{Z})$, to the dual abelian variety of $J(X^\nu)$. Since $J(X^\nu)$ is polarized by the theta divisor, its dual variety can be canonically identified with $J(X^\nu)$ itself. So the classifying morphism in our case takes the form

$$c_X : H_1(\Gamma_X, \mathbb{Z}) \to J(X^\nu).$$

This morphism c_X will be explicitly described below. We shall use the groups of divisors and line bundles having degree 0 on every component:

$$\prod_{i=1}^{\gamma} \mathrm{Div}^0 C_i = \mathrm{Div}^{\underline{0}} X^\nu \to \mathrm{Pic}^{\underline{0}} X^\nu = \prod_{i=1}^{\gamma} \mathrm{Pic}^0 C_i = J(X^\nu).$$

3.1. Definition of T-equivalence

3.1.1. Fix an orientation of Γ_X and consider the source and target maps

$$s, t : E(\Gamma_X) \to V(\Gamma_X).$$

Now, $s(e)$ and $t(e)$ correspond naturally to the two points of X^ν lying over the node corresponding to e. We call such points $s_e, t_e \in X^\nu$. The usual boundary map is defined as follows:

$$\partial : C_1(\Gamma_X, \mathbb{Z}) \to C_0(\Gamma_X, \mathbb{Z}), \qquad e \mapsto t(e) - s(e),$$

and $H_1(\Gamma_X, \mathbb{Z}) = \ker \partial$. We now introduce the map

$$\widetilde{\eta_X} : C_1(\Gamma_X, \mathbb{Z}) \to \operatorname{Div}^0 X^\nu, \qquad e \mapsto t_e - s_e.$$

We will denote by η_X the restriction of $\widetilde{\eta_X}$ to $H_1(\Gamma_X, \mathbb{Z})$, which is easily seen to take values in the subgroup $\operatorname{Div}^{\underline{0}} X^\nu$ of divisors having degree 0 on every component.

Summarizing, we have a commutative diagram

$$\begin{array}{ccc} H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \operatorname{Div}^{\underline{0}} X^\nu \\ \downarrow & & \downarrow \\ C_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\widetilde{\eta_X}} & \operatorname{Div}^0 X^\nu \end{array}$$

The classifying morphism $c_X : H_1(\Gamma_X, \mathbb{Z}) \to J(X^\nu)$ of $J(X)$ is obtained by composing the homomorphism $\eta_X : H_1(\Gamma_X, \mathbb{Z}) \to \operatorname{Div}^{\underline{0}} X^\nu$ with the quotient map $\operatorname{Div}^{\underline{0}} X^\nu \to \operatorname{Pic}^{\underline{0}} X^\nu = J(X^\nu)$ sending a divisor to its linear equivalence class. See [Ale04, Sec. 2.4] or [Bri07, Sec. 1.3].

3.1.2. Recall the set-up and the notation described in 3.0. There are automorphisms of $\operatorname{Pic}^{\underline{0}} X^\nu$ and $\operatorname{Div}^{\underline{0}} X^\nu$ that do not change the isomorphism class of $J(X)$. We need to take those into account. In order to do that, consider the group $\mathrm{K}_\gamma := (\mathbb{Z}/2\mathbb{Z})^\gamma$; note that it acts diagonally as a subgroup of automorphisms, $\mathrm{K}_\gamma \hookrightarrow \operatorname{Aut}(\operatorname{Div}^{\underline{0}} X^\nu)$, $\mathrm{K}_\gamma \hookrightarrow \operatorname{Aut}(\operatorname{Div} X^\nu)$, and $\mathrm{K}_\gamma \hookrightarrow \operatorname{Aut}(\operatorname{Pic}^{\underline{0}} X^\nu)$, via multiplication by $+1$ or -1 on each factor. We shall usually identify K_γ with the image of the above monomorphisms.

For example, if $X^\nu = C_1 \cup C_2$ then $\mathrm{K}_2 \subset \operatorname{Aut}(\operatorname{Div}^{\underline{0}} X^\nu)$ is generated by the involutions $(D_1, D_2) \mapsto (-D_1, D_2)$ and $(D_1, D_2) \mapsto (D_1, -D_2)$.

Definition 3.1.3. We say that two nodal connected curves X and X' are *T-equivalent* if the following conditions hold:

(a) There exists an isomorphism $\phi : X^\nu \xrightarrow{\cong} X'^\nu$ between their normalizations.
(b) $\Gamma_X \equiv_{\mathrm{cyc}} \Gamma_{X'}$.
(c) For every orientation on Γ_X there exists an orientation on $\Gamma_{X'}$ and an automorphism $\alpha \in \mathrm{K}_\gamma \subset \operatorname{Aut}(\operatorname{Div}^{\underline{0}} X^\nu)$ such that the following diagram commutes:

$$\begin{array}{ccc} H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \operatorname{Div}^{\underline{0}} X^\nu \\ \cong \downarrow \epsilon_H & & \cong \downarrow \phi_D \circ \alpha \\ H_1(\Gamma_{X'}, \mathbb{Z}) & \xrightarrow{\eta_{X'}} & \operatorname{Div}^{\underline{0}} X'^\nu \end{array}$$

where ϵ_H is defined in 1.2.2 and $\phi_D : \operatorname{Div}^{\underline{0}} X^\nu \to \operatorname{Div}^{\underline{0}} X'^\nu$ is the isomorphism induced by ϕ.

We say that two disconnected nodal curves Y and Y' are T-equivalent if there exists a bijection between their connected components such that any two corresponding components are T-equivalent.

We shall prove in 3.2.1 that two curves free from separating nodes are T-equivalent if and only if they are C1-equivalent, thereby getting a new statement of Theorem 2.1.7. We first need some observations.

Remark 3.1.4. Let X and X' be T-equivalent and free from separating nodes. Then part (c) of the definition implies that

$$\phi(\nu^{-1}(X_{\mathrm{sing}})) = \nu'^{-1}(X'_{\mathrm{sing}}),$$

where $X^{\nu} \xrightarrow{\nu} X$ and $X'^{\nu} \xrightarrow{\nu} X'$ are the normalization maps.

Remark 3.1.5. Suppose that Γ_X and $\Gamma_{X'}$ are cyclically equivalent and fix a cyclic bijection $\epsilon : E(\Gamma_X) \to E(\Gamma_{X'})$. By [CV09, Cor. 2.3.5], ϵ induces a bijection from the C1-sets of X to those of X', mapping S to $\epsilon(S)$. For this bijection we shall always use the notation

$$\mathrm{Set}^1 X \to \mathrm{Set}^1 X', \qquad S \mapsto S'.$$

Lemma 3.1.6. *Let X and X' be T-equivalent connected curves, free from separating nodes; pick a pair of corresponding C1-sets, $S \in \mathrm{Set}^1 X$ and $S' \in \mathrm{Set}^1 X'$. Then the normalization of X at S is T-equivalent to the normalization of X' at S'.*

Proof. Let Y be the normalization of X at S, and Y' the normalization of X' at S'. It is obvious that Y and Y' have isomorphic normalizations. Observe that $\Gamma_Y = \Gamma_X \smallsetminus S$ and $\Gamma_{Y'} = \Gamma_{X'} \smallsetminus S'$. The bijection $\epsilon : E(\Gamma_X) \to E(\Gamma_{X'})$ maps the edges of S to the edges of S'; hence it induces a bijection $\epsilon_Y : E(\Gamma_Y) \to E(\Gamma_{Y'})$. To see that ϵ_Y induces a bijection on the cycles it suffices to observe that the cycles of $\Gamma_Y = \Gamma_X \smallsetminus S$ are precisely the cycles of Γ_X which do not contain S (by Fact 2.1.3), and the same holds for Y'. Therefore Γ_Y and $\Gamma_{Y'}$ are cyclically equivalent.

Finally, let us pick an orientation on Γ_Y and extend it to an orientation on Γ_X. The map η_Y naturally factors as

$$\eta_Y : H_1(\Gamma_X \smallsetminus S, \mathbb{Z}) \hookrightarrow H_1(\Gamma_X, \mathbb{Z}) \xrightarrow{\eta_X} \mathrm{Div}^{\underline{0}} X^{\nu}.$$

Choose an orientation on $\Gamma_{X'}$ so that condition (c) holds. Then we have a commutative diagram

$$\begin{array}{ccccc}
\eta_Y : H_1(\Gamma_Y, \mathbb{Z}) & \hookrightarrow & H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \mathrm{Div}^{\underline{0}} X^{\nu} \\
\cong \downarrow & & \cong \downarrow \epsilon_H & & \cong \downarrow \alpha \\
\eta_{Y'} : H_1(\Gamma_{Y'}, \mathbb{Z}) & \hookrightarrow & H_1(\Gamma_{X'}, \mathbb{Z}) & \xrightarrow{\eta_{X'}} & \mathrm{Div}^{\underline{0}} X'^{\nu}
\end{array}$$

This proves that condition (c) holds for Y and Y', so we are done. □

3.2. C1-equivalence equals T-equivalence

Proposition 3.2.1. *Let X and X' be connected curves free from separating nodes. Then X and X' are T-equivalent if and only if they are C1-equivalent.*

Proof. Suppose that X and X' are T-equivalent. Then property (A) of Definition 2.1.5 obviously holds. Let us simplify the notation by identifying $X^\nu = X'^\nu$. Since the dual graphs of X and X' are cyclically equivalent, we have a cardinality preserving bijection between the C1-sets of X and X', by Remark 3.1.5. To prove part (B) of Definition 2.1.5 let S, S' be any pair as in 3.1.5, and denote, as usual,

$$G_S := \nu^{-1}(S) \subset X^\nu \quad \text{and} \quad G_{S'} := \nu'^{-1}(S) \subset X^\nu.$$

We must prove that $G_S = G_{S'}$. Since X and X' are T-equivalent, by Remark 3.1.4 the gluing sets are the same:

$$G_{X_{\mathrm{sing}}} = G_{X'_{\mathrm{sing}}}. \tag{3.1}$$

Let Y be the normalization of X at S, and Y' the normalization of X' at S'. By Lemma 3.1.6, Y and Y' are T-equivalent. Now, the normalization of Y and Y' is X^ν, and by Remark 3.1.4 applied to Y and Y' we obtain

$$G_{Y_{\mathrm{sing}}} = G_{Y'_{\mathrm{sing}}} \subset X^\nu. \tag{3.2}$$

Now, it is clear that $G_S = G_{X_{\mathrm{sing}}} \smallsetminus G_{Y_{\mathrm{sing}}}$ and $G_{S'} = G_{X'_{\mathrm{sing}}} \smallsetminus G_{Y'_{\mathrm{sing}}}$. Therefore by (3.1) and (3.2) we get $G_S = G_{S'}$ as desired.

Conversely, assume that X and X' are C1-equivalent. By 2.4.3 their graphs are cyclically equivalent. Let us identify $X^\nu = X'^\nu$, so that by hypothesis $G_S = G_{S'}$ for every pair of corresponding C1-sets. It remains to prove that property (c) of Definition 3.1.3 holds.

We begin with a preliminary definition. From 3.1.2, recall that the group $\mathrm{K}_\gamma = (\mathbb{Z}/2\mathbb{Z})^\gamma$ acts as subgroup of automorphisms of $\operatorname{Div} X^\nu = \prod_{i=1}^\gamma \operatorname{Div} C_i$, by the natural diagonal action defined in 3.1.2 (so that any $\alpha \in \mathrm{K}_\gamma$ acts on each $\operatorname{Div} C_i$ either as the identity or as multiplication by -1). For every $S \in \mathrm{Set}^1 X$ denote as usual by $Y_1, \ldots, Y_h$ the connected components of Y_S and let Y_i^ν be the normalization of Y_i. We have $\operatorname{Aut}(\operatorname{Div} X^\nu) = \prod_{i=1}^h \operatorname{Aut}(\operatorname{Div} Y_i^\nu)$; we define a subgroup of K_γ,

$$\mathrm{K}_\gamma(S) := \{\alpha \in \mathrm{K}_\gamma \subset \operatorname{Aut}(\operatorname{Div} X^\nu) : \alpha_{|\operatorname{Div} Y_i^\nu} = \pm 1\}.$$

Let S and S' be corresponding C1-sets, as above. Let $\Gamma = \Gamma_X$ and $\Gamma' = \Gamma_{X'}$. The graphs $\Gamma(S)$ and $\Gamma'(S')$ are cycles of length $h = \#S = \#S'$, whose sets of edges are naturally identified with S and S' respectively. Hence there is a natural inclusion $C_1(\Gamma(S), \mathbb{Z}) \subset C_1(\Gamma, \mathbb{Z})$; ditto for S'. Set (notation in 3.1.1)

$$\widetilde{\eta(S)} := \widetilde{\eta_X}_{|C_1(\Gamma(S))} : C_1(\Gamma(S)) \to \operatorname{Div}^0 X^\nu \subset \operatorname{Div} X^\nu, \qquad e \mapsto t_e - s_e$$

(here and throughout the rest of the proof we omit $\mathbb{Z}$). For any orientation on $\Gamma(S)$ we let $\eta(S)$ be the restriction of $\widetilde{\eta(S)}$ to $H_1(\Gamma(S))$,

$$\widetilde{\eta_X}_{|H_1(\Gamma(S))} = \eta(S) : H_1(\Gamma(S)) \to \operatorname{Div} X^\nu. \tag{3.3}$$

We define $\eta(S') : H_1(\Gamma'(S')) \to \operatorname{Div} X^\nu$ analogously. Let us describe $\eta(S)$ and $\eta(S')$. As $\Gamma(S)$ is a cycle, for any choice of orientation we have a choice of two generators of $H_1(\Gamma(S)) \cong \mathbb{Z}$. We pick one of them and call it c_S. Write $G_S = \{p_1, q_1, \dots, p_h, q_h\}$ as in 2.3.2. Up to reordering the components $Y_1, \dots Y_h$ and switching p_i with q_i we may assume that

$$\eta(S)(c_S) = \sum_{i=1}^{h} (q_i - p_i). \tag{3.4}$$

Notice that the choice of orientation is essentially irrelevant: for any orientation and any generator $\widetilde{c_S}$ of $H_1(\Gamma(S))$ we have $\eta(S)(\widetilde{c_S}) = \pm \sum_{i=1}^{h} (q_i - p_i)$.

Similarly, choose an orientation for $\Gamma'(S')$ and pick a generator $c_{S'}$ of $H_1(\Gamma'(S'))$. Then one easily checks that there exists a partition $\{1, \dots, h\} = F \cup G$ into two disjoint sets, F and G, such that

$$\eta(S')(c_{S'}) = \sum_{i \in F} (q_i - p_i) + \sum_{i \in G} (p_i - q_i). \tag{3.5}$$

Let $\alpha(S) \in \mathrm{K}_\gamma(S) \subset \operatorname{Aut}(\operatorname{Div} X^\nu)$ be the automorphism whose restriction to $\operatorname{Div} Y_i^\nu$ is the identity for $i \in F$, and multiplication by -1 for $i \in G$. Now let

$$\epsilon(S) : H_1(\Gamma(S)) \xrightarrow{\cong} H_1(\Gamma'(S'))$$

be the isomorphism mapping c_S to $c_{S'}$. By construction $\eta(S) = \alpha(S) \circ \eta(S') \circ \epsilon(S)$, i.e. the map $\eta(S)$ factors as

$$\eta(S) : H_1(\Gamma(S)) \xrightarrow{\epsilon(S)} H_1(\Gamma(S')) \xrightarrow{\eta(S')} \operatorname{Div} X^\nu \xrightarrow{\alpha(S)} \operatorname{Div} X^\nu. \tag{3.6}$$

We repeat the above construction for every pair of corresponding C1-sets (S, S').

Using Lemma 2.2.3 and (3.3) we have

$$\eta_X = \Big(\bigoplus_{S \in \mathrm{Set}^1 X} \eta(S) \Big)_{|H_1(\Gamma)} \quad \text{and} \quad \eta_{X'} = \Big(\bigoplus_{S' \in \mathrm{Set}^1 X'} \eta(S') \Big)_{|H_1(\Gamma')}.$$

Now let

$$\alpha := \prod_{S \in \mathrm{Set}^1 X} \alpha(S) \in \mathrm{K}_\gamma$$

where the product above means composition of the $\alpha(S)$ in any chosen order. We claim that for every fixed $S \in \mathrm{Set}^1 X$ we have

$$\alpha \circ \eta(S') = \pm \alpha(S) \circ \eta(S').$$

Indeed, by 2.2.2, for any $T \in \mathrm{Set}^1 X$ with $T \neq S$, S is entirely contained in the singular locus of a unique connected component of Y_T, call it $Y_{T,1}$. Therefore the gluing set $G_{S'} = G_S$ is entirely contained in $Y_{T,1}^\nu$. By definition, $\alpha(T)$ acts either as the identity or as multiplication by -1 on every divisor of X^ν supported on $Y_{T,1}^\nu$; in particular $\alpha(T)$ acts as multiplication by ± 1 on $\eta(S')(c_{S'})$. The claim is proved.

As a consequence of this claim and of 3.6 we have

$$\alpha \circ \eta(S') \circ \epsilon(S) = \pm\eta(S).$$

Now, if for a certain S the above identity holds with a minus sign on the right, we change $\epsilon(S)$ into $-\epsilon(S)$, but we continue to denote it $\epsilon(S)$ for simplicity.

Using again Lemma 2.2.3 we let $\epsilon_X : H_1(\Gamma) \xrightarrow{\cong} H_1(\Gamma')$ be the restriction to $H_1(\Gamma)$ of the isomorphism

$$\bigoplus_{S\in \mathrm{Set}^1 X} \epsilon(S) : \bigoplus_{S\in \mathrm{Set}^1 \Gamma} H_1(\Gamma(S)) \xrightarrow{\cong} \bigoplus_{S'\in \mathrm{Set}^1 \Gamma'} H_1(\Gamma'(S')).$$

It is trivial to check that ϵ_X is an isomorphism. In fact by the proof of Proposition 2.4.3 it is clear that ϵ_X induces the given bijection between the C1-sets of X and X'. Altogether, we have a commutative diagram

$$\begin{array}{ccccc} \eta_X : H_1(\Gamma) & \hookrightarrow & \bigoplus_{S\in \mathrm{Set}^1 \Gamma} H_1(\Gamma(S)) & \xrightarrow{\oplus\eta(S)} & \mathrm{Div}\, X^\nu \\ \epsilon_X \downarrow \cong & & & & \alpha \downarrow \cong \\ \eta_{X'} : H_1(\Gamma') & \hookrightarrow & \bigoplus_{S'\in \mathrm{Set}^1 \Gamma'} H_1(\Gamma'(S')) & \xrightarrow{\oplus\eta(S')} & \mathrm{Div}\, X^\nu \end{array} \tag{3.7}$$

so we are done. □

4. Proof of the Main Theorem

The hard part of the proof of Theorem 2.1.7 is the necessary condition: if two stable curves with no separating nodes have the same image under the Torelli map, then they are C1-equivalent. The proof is given in Subsection 4.3 using the preliminary material of Subsections 4.1 and 4.2. The proof of the converse occupies Subsection 4.4.

4.1. Combinatorial preliminaries

In this subsection we fix a connected curve X free from separating nodes, and study the precise relation between the posets $\mathcal{ST}_X$ and $\mathcal{SP}_X$, defined in Subsection 2.1.

We will prove, in Lemma 4.1.6, that the support map $\mathrm{Supp}_X : \mathcal{ST}_X \to \mathcal{SP}_X$ is a quotient of posets, that is, given $S, T \in \mathcal{SP}_X$ we have $S \geq T$ if and only if there exist P_S^d and P_T^e in $\mathcal{ST}_X$ such that $P_S^d \geq P_T^e$. In particular, the poset $\mathcal{SP}_X$ is completely determined by $\mathcal{ST}_X$. This fact will play a crucial role later on, to recover the combinatorics of X from that of $\overline{P_X^{g-1}}$.

We shall here apply some combinatorial results obtained in [CV09], to which we refer for further details. First of all, observe that the poset $\mathcal{SP}_X$ can be defined purely in terms of the dual graph of X. Namely $\mathcal{SP}_X$ is equal to the poset $\mathcal{SP}_{\Gamma_X}$, defined in [CV09,

Def. 5.1.1] as the poset of all $S \subset E(\Gamma_X)$ such that $\Gamma_X \smallsetminus S$ is free from separating edges, ordered by reverse inclusion.

Next, we need to unravel the combinatorial nature of $\mathcal{ST}_X$; recall that its elements correspond to pairs $(S, \underline{d})$ where $S \in \mathcal{SP}_X$ and $\underline{d}$ is a stable multidegree on the curve Y_S. Now, it turns out that stable multidegrees can be defined in terms of so-called totally cyclic orientations on the graph Γ_X. To make this precise we introduce a new poset, $\mathcal{OP}_\Gamma$ (cf. [CV09, Subsec. 5.2]).

Definition 4.1.1. If Γ is a connected graph, an orientation of Γ is *totally cyclic* if there exists no proper non-empty subset $W \subset V(\Gamma)$ such that the edges between W and its complement $V(\Gamma) \smallsetminus W$ go all in the same direction.

If Γ is not connected, an orientation is totally cyclic if its restriction to each connected component of Γ is totally cyclic.

The poset $\mathcal{OP}_\Gamma$ is defined as the set

$$\mathcal{OP}_\Gamma = \{\phi_S : \phi_S \text{ is a totally cyclic orientation on } \Gamma \smallsetminus S,\ \forall S \in \mathcal{SP}_\Gamma\}$$

together with the following partial order:

$$\phi_S \geq \phi_T \Leftrightarrow S \subset T \text{ and } \phi_T = (\phi_S)_{|\Gamma \smallsetminus T}.$$

Remark 4.1.2. It is easy to check that if Γ admits some separating edge, then Γ admits no totally cyclic orientation. The converse also holds (see loc. cit.).

4.1.3. *Relation between $\mathcal{OP}_{\Gamma_X}$ and $\mathcal{ST}_X$.* How is the poset of totally cyclic orientations related to the poset $\mathcal{ST}_X$? This amounts to asking about the connection between totally cyclic orientations and stable multidegrees, which is well known to be the following.

Pick Y_S and any totally cyclic orientation ϕ_S on $\Gamma_{Y_S} = \Gamma \smallsetminus S$; for every vertex v_i denote by $d^+(\phi_S)_{v_i}$ the number of edges of $\Gamma \smallsetminus S$ that start from v_i according to ϕ_S. Now we define a multidegree $\underline{d}(\phi_S)$ on Y_S as follows:

$$\underline{d}(\phi_S)_{v_i} := g_i - 1 + d^+(\phi_S)_{v_i}, \qquad i = 1, \ldots, \gamma, \tag{4.1}$$

where g_i is the geometric genus of the component corresponding to v_i. Now:

A multidegree $\underline{d}$ is stable on Y_S if and only if there exists a totally cyclic orientation ϕ_S such that $\underline{d} = \underline{d}(\phi_S)$ (see [B77, Lemma 2.1] and [Cap07, Sec.1.3.2]).

Obviously, two totally cyclic orientations define the same multidegree if and only if the number of edges issuing from every vertex is the same. We shall regard two such orientations as equivalent:

Definition 4.1.4. Two orientations ϕ_S and ϕ_T of $\mathcal{OP}_\Gamma$ are equivalent if $S = T$ and if $d^+(\phi_S)_v = d^+(\phi_T)_v$ for every vertex v of Γ. The set of equivalence classes of orientations will be denoted by $\overline{\mathcal{OP}_\Gamma}$. The quotient map $\mathcal{OP}_\Gamma \to \overline{\mathcal{OP}_\Gamma}$ induces a unique poset structure on $\overline{\mathcal{OP}_\Gamma}$ such that two classes $[\phi_S], [\phi_T] \in \overline{\mathcal{OP}_\Gamma}$ satisfy $[\phi_S] \geq [\phi_T]$ if there exist respective representatives ϕ_S and ϕ_T such that $\phi_S \geq \phi_T$ in $\mathcal{OP}_\Gamma$.

The above definition coincides with [CV09, Def. 5.2.3].

We shall soon prove that there is a natural isomorphism of posets between $\overline{\mathcal{OP}_{\Gamma_X}}$ and $\mathcal{ST}_{\Gamma_X}$. Before doing that, we recall the key result about the relation between $\overline{\mathcal{OP}_{\Gamma_X}}$ and $\mathcal{SP}_{\Gamma_X}$.

Fact 4.1.5. *Let Γ be a connected graph free from separating edges; consider the natural maps*

$$\mathrm{Supp}_\Gamma : \mathcal{OP}_\Gamma \to \overline{\mathcal{OP}_\Gamma} \xrightarrow{\overline{\mathrm{Supp}_\Gamma}} \mathcal{SP}_\Gamma, \qquad \phi_S \mapsto [\phi_S] \mapsto S.$$

(1) *The maps Supp_Γ and $\overline{\mathrm{Supp}_\Gamma}$ are quotients of posets.*
(2) *The poset $\mathcal{SP}_\Gamma$ is completely determined, up to isomorphism, by the poset $\overline{\mathcal{OP}_\Gamma}$ (and conversely).*

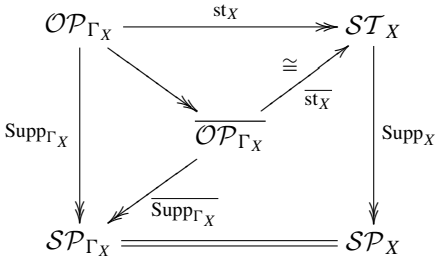
By [CV09, Lemma 5.3.1] the map Supp_Γ is a quotient of posets, hence so is $\overline{\mathrm{Supp}_\Gamma}$ (as $\mathcal{OP}_\Gamma \to \overline{\mathcal{OP}_\Gamma}$ is a quotient of posets by definition). Part (2) is the equivalence between (iii) and (v) in [CV09, Thm. 5.3.2].

Now, as explained in 4.1.3, to every $\phi_S \in \mathcal{OP}_{\Gamma_X}$ we can associate a stable multidegree $\underline{d}(\phi_S)$ of Y_S (see (4.1)); moreover two equivalent orientations define the same multidegree. This enables us to define two maps, st_X and $\overline{\mathrm{st}_X}$, as follows:

$$\mathrm{st}_X : \mathcal{OP}_{\Gamma_X} \to \overline{\mathcal{OP}_{\Gamma_X}} \xrightarrow{\overline{\mathrm{st}_X}} \mathcal{ST}_X, \qquad \phi_S \mapsto [\phi_S] \mapsto P_S^{\underline{d}(\phi_S)}. \tag{4.2}$$

Lemma 4.1.6. *Let X be connected and free from separating nodes. Then*

(1) *the map $\overline{\mathrm{st}_X} : \overline{\mathcal{OP}_{\Gamma_X}} \to \mathcal{ST}_X$ is an isomorphism of posets;*
(2) *there is a commutative diagram*



(4.3)

where every map is a quotient of posets. In particular the poset $\mathcal{SP}_X$ is completely determined (up to isomorphism) by the poset $\mathcal{ST}_X$.

Proof. The maps st_X and $\overline{\mathrm{st}_X}$ are surjective by what we said in 4.1.3. Moreover, by [Cap94, Prop. 5.1], they are morphisms of posets. From the definitions (4.1) and 4.1.4 it is clear that $\overline{\mathrm{st}_X}$ is bijective, and hence an isomorphism of posets.

The commutativity of the diagram is clear by what we said above. Finally, by Fact 4.1.5(2) we know that $\overline{\mathcal{OP}_{\Gamma_X}}$ completely determines $\mathcal{SP}_X$ as a poset, hence (2) follows from (1). □

4.2. Recovering gluing points from the theta divisor

Lemma 4.2.1. *Let $P^{\underline{d}}_S$ be a codimension-one stratum of $\overline{P^{g-1}_X}$ and let h be the number of irreducible components of $\Theta(X) \cap P^{\underline{d}}_S$. Then S is a C1-set of cardinality h.*

Proof. We have already proved most of the statement in 2.1. The only part that needs to be justified is the one concerning $\Theta(X)$. By 2.2.1 every connected component of Y_S has positive genus. Now, according to Fact 1.2.10(iii), the number h of irreducible components of $\Theta(X) \cap P^{\underline{d}}_S$ is equal to $\gamma_S = \#S$. □

4.2.2. The following set-up will be fixed throughout the rest of this subsection. X is a stable curve of genus g, $X_{\mathrm{sep}} = \emptyset$, and $S \in \mathrm{Set}^1 X$ is a C1-set of cardinality h. As usual $\nu_S : Y_S \to X$ denotes the normalization at S. We have $Y_S = \bigsqcup_{i=1}^h Y_i$, with Y_i connected, of arithmetic genus $g_i > 0$, free from separating nodes. The gluing set of ν_S is denoted $\{p_1, q_1, \ldots, p_h, q_h\}$ with $\nu_S(p_j) = \nu_S(q_{j+1})$ and $p_j, q_j \in Y_j$.

The pull-back via the partial normalization ν_S induces an exact sequence

$$0 \to k^* \to \mathrm{Pic}\, X \xrightarrow{\nu_S^*} \mathrm{Pic}\, Y_S = \prod_{i=1}^h \mathrm{Pic}\, Y_i \to 0.$$

In the following statement we use notation (1.4).

Lemma 4.2.3. *Fix $\underline{d} \in \Sigma(X)$. Let M be a general line bundle in $\mathrm{Pic}^{\underline{d}} Y_S$, $M_i := M_{|Y_i}$ and $\underline{d_i} := \underline{\deg}\, M_i$. Let y_i be a fixed smooth point of Y_i. Then for $i = 1, \ldots, h$ the following properties hold:*

(i) $h^0(Y_i, M_i) = 1$ *(hence $h^0(Y_S, M) = h$).*
(ii) *Set $\underline{d_i}(-y_i) := \underline{\deg}\, M_i(-y_i)$. Then $\underline{d_i}(-y_i)$ is semistable.*
(iii) *M_i does not have a base point in y_i (i.e. $h^0(Y_i, M_i(-y_i)) = 0$).*
(iv) *The restriction of the pull-back map, $\nu_S^* : W_{\underline{d}}(X) \to \mathrm{Pic}^{\underline{d}} Y_S$, is birational.*
(v) $\dim W^1_{\underline{d_i}}(Y_i) \leq g_i - 2$ *for every $1 \leq i \leq h$.*
(vi) *For any point p_k, define*

$$T_{p_k} = \left\{ M \in \mathrm{Pic}^{\underline{d}} Y_S \smallsetminus W^h_{\underline{d}}(Y_S) : \begin{array}{l} h^0(Y_S, M(-p_k)) = h \text{ and} \\ h^0(Y_S, M(-q_j)) < h, \ \forall 1 \leq j \leq h \end{array} \right\}. \quad (4.4)$$

Define T_{q_k} by replacing p_k with q_k and q_j with p_j in (4.4). Then

$$\mathrm{Pic}^{\underline{d}} Y_S \setminus \nu_S^*(W_{\underline{d}}(X)) = \bigcup_{k=1}^h (T_{p_k} \cup T_{q_k}).$$

Proof. Since $\underline{d}$ is stable, Theorem 3.1.2 of [Cap07] shows that $W_{\underline{d}}(X)$ is irreducible of dimension $g - 1$.

For any $M \in \mathrm{Pic}\, Y_S$ we set $F_M(X) := \{L \in \mathrm{Pic}\, X : \nu^* L = M\} \cong k^*$.

To prove (i), observe that the stability of $\underline{d}$ yields $\deg(M_i) = g_i$. Therefore the theorem of Riemann–Roch gives $h^0(Y_i, M_i) \geq 1$. Suppose, for a contradiction, that

$h^0(Y_i, M_i) > 1$ for some i. Then $h^0(Y_S, M) = \sum_{i=1}^h h^0(Y_i, M_i) \geq h+1$ for every $M \in \mathrm{Pic}^{\underline{d}} Y_S$.

This implies that $F_M(X) \subset W_{\underline{d}}(X)$ (indeed, there are at most h conditions on the global sections of M to descend to a global section of a fixed $L \in F_M(X)$). Therefore

$$\dim W_{\underline{d}}(X) = \dim \mathrm{Pic}^{\underline{d}} Y_S + 1 = \sum_{i=1}^h g_i + 1 = g,$$

a contradiction. This proves (i).

For (ii) and (iii), set $\underline{d}'_i = \underline{d}_i(-y_i)$; observe that $|\underline{d}'_i| = g_i - 1$. Let $Z \subset Y_i$ be a subcurve of Y_i and let $\widetilde{Z} := \nu(Z) \subset X$. Then, of course, $g_Z \leq g_{\widetilde{Z}}$. Denoting by $d'_{i,Z} = |(\underline{d}'_i)_{|Z}|$ the total degree of $\underline{d}'_i$ restricted to Z, and setting $d_{\widetilde{Z}} = |\underline{d}_Z|$, we have

$$d'_{i,Z} = \begin{cases} d_{\widetilde{Z}} \geq g_{\widetilde{Z}} \geq g_Z & \text{if } y_i \notin Z, \\ d_{\widetilde{Z}} - 1 \geq g_{\widetilde{Z}} - 1 \geq g_Z - 1 & \text{if } y_i \in Z, \end{cases}$$

where we used that $d_{\widetilde{Z}} \geq g_{\widetilde{Z}}$ ($\underline{d}$ is stable). So (ii) is proved. We can therefore use a result due to A. Beauville ([B77], see also Proposition 1.3.7 in [Cap07]), stating that every irreducible component of $W_{\underline{d}_i'}(Y_i)$ has dimension $g_i - 1$, and in particular $W_{\underline{d}_i'}(Y_i) \neq \mathrm{Pic}^{\underline{d}_i'}(Y_i)$. Therefore for general $M \in \mathrm{Pic}^{\underline{d}}(Y_S)$, we have $h^0(Y_i, M_i(-y_i)) = 0$, and this proves part (iii).

In order to prove (iv), we need to make the isomorphism $F_M(X) \cong k^*$ explicit. Any $c \in k^*$ determines a unique $L^c \in F_M(X)$, defined as follows. For every $j = 1, \ldots, h$ consider the two fibers of M over p_j and q_{j+1} (with $q_{h+1} = q_1$ as usual, recall that ν_S glues p_j with q_{j+1}) and fix an isomorphism between them. Then $L^c \in F_M(X)$ is obtained by gluing M_{p_j} to $M_{q_{j+1}}$ via the isomorphism

$$M_{p_j} \xrightarrow{\cong} M_{q_{j+1}} \quad \text{for } j = 1, \ldots, h-1, \qquad M_{p_h} \xrightarrow{\cdot c} M_{q_1},$$

where the last isomorphism is given by multiplication by c. Conversely, every $L \in F_M(X)$ is of type L^c, for a unique $c \in k^*$.

Now, by (i) we know that a general $M \in \mathrm{Pic}^{\underline{d}}(Y_S)$ does not belong to $W^h_{\underline{d}}(Y_S)$, i.e. we have $h^0(Y_i, M_i) = 1$ for all $i = 1, \ldots, h$. Take a generator, call it α_i, of $H^0(Y_i, M_i)$ and set $a_i^p := \alpha_i(p_i)$ and $a_i^q := \alpha_i(q_i)$. A section $\alpha = \sum_{i=1}^h x_i \alpha_i \in H^0(Y_S, M)$ descends to a section of $L^c \in F_M(X)$ on X if and only if it satisfies the following system of equations:

$$\begin{cases} x_i a_i^p = \alpha(p_i) = \alpha(q_{i+1}) = x_{i+1} a_{i+1}^q & \text{for } 1 \leq i \leq h-1, \\ c x_h a_h^p = c\alpha(p_h) = r(q_1) = x_1 a_1^q. \end{cases} \tag{4.5}$$

The above system of h equations in the h unknowns $x_1, \ldots, x_h$ admits a non-zero solution if and only if the determinant of the associated matrix is zero, that is, if and only if

$$c \cdot \prod_{i=1}^h a_i^p = \prod_{i=1}^h a_i^q. \tag{4.6}$$

Since a general $M \in \mathrm{Pic}^{\underline{d}}(Y_S)$ satisfies $a_i^p \neq 0$ and $a_i^q \neq 0$ for every i (by part (iii)), the above equation has a unique solution c and therefore $F_M(X)$ has a unique point in $W_{\underline{d}}(X)$. This proves (iv), since $\dim W_{\underline{d}}(X) = \dim \mathrm{Pic}^{\underline{d}}(Y_S) = g - 1$.

Now we prove (v). The fiber of the birational map $\nu^* : W_{\underline{d}}(X) \to \mathrm{Pic}^{\underline{d}} Y_S$ over $W^h_{\underline{d}}(Y_S)$ has dimension 1; hence, as $W_{\underline{d}}(X)$ is irreducible of dimension $\sum_{i=1}^h g_i$, we have $\dim W^h_{\underline{d}}(Y_S) \leq \sum_{i=1}^h g_i - 2$. Since $W^h_{\underline{d}}(Y) = \bigcup_{i=1}^h (\pi_i)^{-1}(W^1_{\underline{d_i}}(Y_i))$, where $\pi_i : \mathrm{Pic}^{\underline{d}}(Y_S) \to \mathrm{Pic}^{\underline{d_i}}(Y_i)$ is the projection, we deduce that $\dim W^1_{\underline{d_i}}(Y_i) \leq g_i - 2$.

Finally we prove (vi). As observed before, we have

$$\mathrm{Pic}^{\underline{d}} Y_S \smallsetminus \nu_S^*(W_{\underline{d}}(X)) \subset \mathrm{Pic}^{\underline{d}} Y_S \smallsetminus W^h_{\underline{d}}(Y_S).$$

With the above notation, a line bundle $M \in \mathrm{Pic}^{\underline{d}} Y_S \smallsetminus W^h_{\underline{d}}(Y_S)$ does not belong to $\nu^*(W_{\underline{d}}(X))$ if and only if the equation (4.6) does not admit a solution $c \in k^*$. This happens precisely when either $a_k^p = 0$ for at least one k and $a_i^q \neq 0$ for any i, or $a_k^q = 0$ for at least one k and $a_i^p \neq 0$ for any i. These conditions are easily seen to be equivalent to the fact that $M \in \bigcup_k (T_{p_k} \cup T_{q_k})$. □

Proposition 4.2.4. *Let X be such that $X_{\mathrm{sep}} = \emptyset$; pick $S \in \mathrm{Set}^1 X$ and $\underline{d} \in \Sigma(X)$. The image of the pull-back map $\nu_S^* : W_{\underline{d}}(X) \to \mathrm{Pic}^{\underline{d}} Y_S$ uniquely determines $\nu_S^{-1}(S)$, the gluing set of ν_S.*

Proof. Denote $\mathrm{Pic}^{\underline{d}} Y_S \smallsetminus \nu_S^*(W_{\underline{d}}(X)) = T_1 \sqcup T_2$ where using Lemma 4.2.3(vi) we have

$$T_1 := \bigcup_{k=1}^h T_{p_k}, \quad T_2 := \bigcup_{k=1}^h T_{q_k}, \tag{4.7}$$

for some set $\{p_1, \ldots, p_h, q_1, \ldots, q_h\}$ which we must prove is uniquely determined, up to reordering the p_i (or the q_i) among themselves. Notice that, for any such set, two different points p_k, p_j lie in two different connected components of Y_S, and the same holds for any two q_k, q_j. Therefore T_1 and T_2 are connected; on the other hand they obviously do not intersect, therefore they are uniquely determined. It thus suffices to prove that T_1 (and similarly T_2) determines a unique set of h smooth points of Y_S such that T_1 is expressed as in (4.7).

We begin with a preliminary analysis. Pick any smooth point of Y_S, let Y_k be the connected component on which it lies, and name the point y_k, for notational purposes. By Lemma 4.2.3(ii) the multidegree $\underline{d}'_k := \underline{d}_k(-y_k)$ is semistable on Y_k. Therefore we can apply Proposition 3.2.1 of [Cap07]. This implies that $W_{\underline{d}'_k}(Y_k)$ contains an irreducible component (of dimension $g_k - 1$) equal to the image of the $\underline{d}'_k$-th Abel map; we call this component A_k. We also know (loc. cit.) that A_k does not have a fixed base point,[1] and that $h^0(Y_k, L) = 1$ for the general $L \in A_k$.

We can thus define an irreducible effective divisor as follows:

$$D_{y_k} := \{M \in \mathrm{Pic}^{\underline{d}} Y_S : M_k(-y_k) \in A_k\}.$$

[1] $V \subset \mathrm{Pic}\, Y$ has a *fixed base point* if there exists a $y \in Y$ which is a base point for every $L \in V$.

Observe that D_{y_k} has no fixed base point other than y_k. Indeed, let $M \in D_{y_k}$ be a general point. If $j \neq k$ then M_j is general in $\mathrm{Pic}^{\underline{d}_j} Y_j$, hence by 4.2.3, M_j has no fixed base point and $h^0(Y_j, M_j) = 1$. On the other hand M_k varies in a set of dimension $g_k - 1$, therefore $h^0(Y_k, M_k) = 1$ by 4.2.3(v). Therefore, if every M_k had a base point in $r \neq y_k$, we would obtain

$$1 = h^0(M_k) = h^0(M_k(-y_k)) = h^0(M_k(-r)) = h^0(M_k(-y_k - r)). \tag{4.8}$$

But $M_k(-y_k) \in A_k$, so every element of A_k would have a base point in r, which is not possible (see above).

Summarizing, the general $M \in D_{y_k}$ has the following properties:

$$\begin{cases} h^0(Y_j, M_j) = 1 \quad \text{for any } j = 1, \dots, h, \\ h^0(Y_S, M) = h^0(Y_S, M(-y_k)) = h, \\ h^0(Y_S, M(-r)) < h^0(Y_S, M) \quad \forall r \neq y_k \text{ smooth point of } Y_S. \end{cases} \tag{4.9}$$

Now, back to the proof of the proposition; it suffices to concentrate on T_1. For a contradiction, suppose there are two different descriptions for T_1 as follows:

$$T_1 = \bigcup_{j=1}^{h} T_{p_j} = \bigcup_{j=1}^{h} T_{\tilde{p}_j};$$

we may assume $\tilde{p}_1 \notin \{p_1, \dots, p_h\}$. By (4.9) applied to $y_k = \tilde{p}_1$, together with 4.2.3(vi), we have

$$D_{\tilde{p}_1} \subset \overline{T_1}.$$

But then, since $T_1 = \bigcup_j T_{p_j}$, we conclude that $D_{\tilde{p}_1}$ has a fixed base point in some p_j, which is impossible by the last property in (4.9). □

4.3. Torelli theorem: proof of the necessary condition

By Corollary 1.3.3 and Remark 1.3.1, to prove the necessary condition of Theorem 2.1.7 it suffices to prove the following.

Let X and X' be stable curves of genus g free from separating nodes, and such that $\bar{\mathrm{t}}_g(X) = \bar{\mathrm{t}}_g(X')$. Then X and X' are C1-equivalent.

So, suppose we have an isomorphism

$$\Phi = (\phi_0, \phi_1) : (J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta(X)) \xrightarrow{\cong} (J(X') \curvearrowright \overline{P_{X'}^{g-1}}, \Theta(X')).$$

We divide the proof into several steps. In the first step we collect the combinatorial parts.

Step 1. (1) *The above isomorphism Φ induces a bijection*

$$\mathrm{Set}^1 X \to \mathrm{Set}^1 X', \qquad S \mapsto S',$$

such that $\#S = \#S'$ for every $S \in \mathrm{Set}^1 X$.

(2) *Γ and Γ' are cyclically equivalent.*

The isomorphism $\phi_1 : \overline{P^{g-1}_X} \xrightarrow{\cong} \overline{P^{g-1}_{X'}}$ induces an isomorphism between the posets of strata $\mathcal{ST}_X \cong \mathcal{ST}_{X'}$; hence, by Lemma 4.1.6, it induces an isomorphism

$$\mathcal{SP}_X \cong \mathcal{SP}_{X'}$$

of the posets of supports, compatible with the support maps. In particular, we have an induced bijection

$$\mathrm{Set}^1 X \to \mathrm{Set}^1 X', \qquad S \mapsto S'.$$

Let us show that this bijection preserves cardinalities. By what we just said, every stratum of type P^d_S is mapped isomorphically to a stratum of type $P^{d'}_{S'}$. Moreover, as the theta divisor of X is mapped isomorphically to the theta divisor of X', the intersection $\Theta(X) \cap P^d_S$ is mapped isomorphically to $\Theta(X') \cap P^{d'}_{S'}$; in particular the number of irreducible components of these two intersections is the same. Hence, by Lemma 4.2.1, S and S' have the same cardinality.

This proves the first item. At this point the fact Γ_X and $\Gamma_{X'}$ are cyclically equivalent follows immediately by what we just proved, thanks to the following immediate consequence of [CV09, Prop. 2.3.9(ii)] combined with [CV09, Thm. 5.3.2(i)–(iii)].

Fact 4.3.1. *Let Γ and Γ' be two connected graphs free from separating edges. Suppose that there exists an isomorphism of posets, $\mathcal{SP}_\Gamma \cong \mathcal{SP}_{\Gamma'}$, whose restriction to C1-sets, $\mathrm{Set}^1 \Gamma \cong \mathrm{Set}^1 \Gamma'$, preserves the cardinality. Then Γ and Γ' are cyclically equivalent.*

Step 2. $X^\nu \cong X'^\nu$.

By the previous step, the number of irreducible components of X^ν and X'^ν is the same; indeed, the number of edges and the first Betti number of Γ_X and $\Gamma_{X'}$ are the same, hence the number of vertices is the same. Denote by $X^\nu_+ \subset X^\nu$ and $X'^\nu_+ \subset X'^\nu$ the union of all components of positive genus. It is enough to show that

$$X^\nu_+ \cong X'^\nu_+. \tag{4.10}$$

In Remark 1.2.11 we saw that $\overline{P^{g-1}_X}$ has a unique stratum of smallest dimension, namely the unique stratum supported on X_{sing}. This smallest stratum is isomorphic to the product of the Jacobians of the components of X^ν, and hence to the product of the Jacobians of the components of X^ν having positive genus. It is clear that the smallest stratum of $\overline{P^{g-1}_X}$ is mapped by ϕ_1 to the smallest stratum of $\overline{P^{g-1}_{X'}}$. Recall now (1.7), expressing the restriction of the theta divisor to this smallest stratum in terms of the theta divisors of the components of X^ν. As a consequence the projection of the smallest stratum onto each of its factors determines the polarized Jacobian of all the positive genus components of the normalization. Hence, by the Torelli theorem for smooth curves, we conclude that the positive genus components of the normalizations of X and X' are isomorphic, so (4.10) is proved.

Step 3. *Condition* (B) *of Definition* 2.1.5 *holds.*

We use induction on the number of nodes. The base is the non-singular case, i.e. the classical Torelli theorem. From now on we assume X and X' are singular.

As usual, we denote the normalizations of X and X' both by X^ν.

Let $S \in \mathrm{Set}^1\, X$ and $S' \in \mathrm{Set}^1\, X'$ be a pair of corresponding C1-sets, under the bijection described in the first step; set $h := \#S = \#S'$. Let $\nu_S : Y_S \to X$ and $\nu_{S'} : Y'_{S'} \to X'$ be the partial normalizations at S and S', and call their arithmetic genus $g_S = g - h$. Recall that Y_S and $Y'_{S'}$ have h connected components, each of which is free from separating nodes and has positive arithmetic genus. We claim that Y_S and $Y'_{S'}$ are C1-equivalent.

Recall (see (1.9)) that we denote by $\overline{P_S} \subset \overline{P_X^{g-1}}$ and $\overline{P_{S'}} \subset \overline{P_{X'}^{g-1}}$ the closures of all strata supported, respectively, on S and S'. By what we said, the isomorphism ϕ_1 induces an isomorphism

$$\overline{P_S} \cong \overline{P_{S'}}. \tag{4.11}$$

By Lemma 1.3.2 we know that $\overline{P_S}$ together with the restriction of the theta divisor and the action of $J(Y_S)$ is naturally isomorphic to $(J(Y_S) \curvearrowright \overline{P_{Y_S}^{g_S-1}}, \Theta(Y_S))$; similarly for $\overline{P_{S'}}$. Therefore by (4.11) we have

$$(J(Y_S) \curvearrowright \overline{P_{Y_S}^{g_S-1}}, \Theta(Y_S)) \cong (J(Y'_{S'}) \curvearrowright \overline{P_{Y'_{S'}}^{g_S-1}}, \Theta(Y'_{S'})).$$

By Proposition 1.3.1, the same holds if Y_S and $Y'_{S'}$ are replaced by their stabilizations, $\overline{Y_S}$ and $\overline{Y'_{S'}}$. Therefore we can apply the induction hypothesis to $\overline{Y_S}$ and $\overline{Y'_{S'}}$ (which are stable, free from separating nodes, and have fewer nodes than X and X'). We thus conclude that $\overline{Y_S}$ is C1-equivalent , or T-equivalent, to $\overline{Y'_{S'}}$.

On the other hand the normalizations of Y_S and $Y'_{S'}$ are isomorphic, as they are equal to the normalizations of X and X'. Furthermore, as $\Gamma_X \equiv_{\mathrm{cyc}} \Gamma_{X'}$ (by Step 2) the dual graphs of Y_S and $Y'_{S'}$ are cyclically equivalent (by the same argument used for Lemma 3.1.6). Therefore, by Lemma 4.3.2, Y_S is T-equivalent, hence C1-equivalent, to $Y'_{S'}$. The claim is proved.

Next, consider the normalization maps

$$\nu : X^\nu \xrightarrow{\mu} Y_S \xrightarrow{\nu_S} X, \qquad \nu' : X^\nu \xrightarrow{\mu'} Y'_{S'} \xrightarrow{\nu_{S'}} X'$$

where μ and μ' are the normalizations of Y_S and $Y'_{S'}$. As Y_S and $Y'_{S'}$ are C1-equivalent, the gluing sets $\mu^{-1}((Y_S)_{\mathrm{sing}})$ and $\mu'^{-1}((Y'_{S'})_{\mathrm{sing}})$ are the same (cf. 3.1.4). The gluing sets of ν and ν' are obtained by adding to the above set the gluing sets of S and S'.

By Proposition 4.2.4, $\nu_S^{-1}(S)$ and $\nu_{S'}^{-1}(S')$ are uniquely determined by the ppSSAV of Y_S or of $Y_{S'}$, which are isomorphic. Therefore $(\overline{P_{Y_S}^{g_S-1}}, \Theta(Y_S))$ uniquely determines $\mu^{-1}(\nu_S^{-1}(S)) = \nu^{-1}(S)$ and $\mu'^{-1}(\nu_{S'}^{-1}(S')) = \nu'^{-1}(S')$ on X^ν. This says that, up to automorphisms of X^ν, the sets $\nu^{-1}(S)$ and $\nu'^{-1}(S')$ coincide. Denote $G_S := \nu^{-1}(S) = \nu'^{-1}(S')$. We also see that the gluing set of ν is equal to the gluing set of ν'; we call it $G_{X_{\mathrm{sing}}} = \nu^{-1}(X_{\mathrm{sing}}) = \nu'^{-1}(X'_{\mathrm{sing}})$. Of course $G_{X_{\mathrm{sing}}}$ is the disjoint union of all the gluing sets associated to all the C1-sets of X.

Now we apply the previous argument to every remaining pair of corresponding C1-sets, as follows. Pick a pair of corresponding C1-sets, U and U', with $U \neq S$. Then, as

before, Y_U and $Y'_{U'}$ are C1-equivalent, and their (same) ppSSAV uniquely determines

$$G_U := \nu^{-1}(U) = \nu'^{-1}(U') \subset G_{X_{\mathrm{sing}}} \smallsetminus G_S \subset X^\nu.$$

Therefore condition (B) of Definition 2.1.5 holds, i.e. X and X' are C1-equivalent. The proof is complete. □

We used the following basic fact.

Lemma 4.3.2. *Let X and X' be free from separating nodes; suppose that their stabilizations are T-equivalent and that $\Gamma_X \equiv_{\mathrm{cyc}} \Gamma_{X'}$. Then X and X' are T-equivalent.*

Proof. Let $\overline{X}$ and $\overline{X'}$ be the stabilizations of X and X'. Observe that the dual graph of $\overline{X}$ is obtained from Γ_X by removing some vertices of valence 2 (corresponding to the exceptional components of X) so that the two edges adjacent to every such vertex become a unique edge. Therefore there is a natural isomorphism $H_1(\Gamma_X) \cong H_1(\Gamma_{\overline{X}})$. Moreover, this isomorphism fits in a commutative diagram

$$\begin{array}{ccc} H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \mathrm{Div}^{\underline{0}} X^\nu \\ \cong \downarrow & & \uparrow \\ H_1(\Gamma_{\overline{X}}, \mathbb{Z}) & \xrightarrow{\eta_{\overline{X}}} & \mathrm{Div}^{\underline{0}} \overline{X}^\nu \end{array}$$

where the right vertical arrow is induced by the obvious injection $\overline{X}^\nu \hookrightarrow X^\nu$. The diagram immediately shows that the map $\eta_{\overline{X}}$ is determined by η_X. The converse is also true, in fact if $E \subset X$ is an exceptional component, and $\pi_E : \mathrm{Div}^{\underline{0}} X^\nu \to \mathrm{Div}^0 E = \mathrm{Div}^0 \mathbb{P}^1$ the projection ($\mathrm{Div}^0 E$ is a factor of $\mathrm{Div}^{\underline{0}} X^\nu$), then the map $\pi_E \circ \eta_X$ is uniquely determined up to an automorphism of X. The same observation applies to X', of course.

Now to prove the lemma, notice that X and X' have the same number of irreducible components, because their dual graphs are cyclically equivalent. Denote by $X^\nu_+ \subset X^\nu$, respectively $X'^\nu_+ \subset X'^\nu$, the union of all components of X^ν, respectively X'^ν, having positive genus. To show that $X^\nu \cong X'^\nu$ it suffices to show that $X^\nu_+ \cong X'^\nu_+$. This follows immediately from the fact that the normalizations of $\overline{X}$ and $\overline{X'}$ are isomorphic.

Finally, by the initial observation, the maps η_X and $\eta_{X'}$ are determined by those of $\overline{X}$ and $\overline{X'}$, and hence property (c) of Definition 3.1.3 holds for X and X', because it holds for their stabilizations. □

4.4. Torelli theorem: proof of the sufficient condition

By Corollary 1.3.3 and Remark 1.3.1 it suffices to prove the first part of Theorem 2.1.7, i.e. we can assume that X and X' are C1-equivalent curves free from separating nodes. By Proposition 3.2.1, C1-equivalence and T-equivalence coincide; so we can use the second concept, which is now more convenient. Indeed the proof consists in applying some well known (some quite deep) facts about ppSSAV, on which our definition of T-equivalence is based.

By [AN99], and by [Ale04, Sec. 5.5] (where a short description, ad-hoc for the present case, is given) $\bar{\mathfrak{t}}_g(X)$ is determined by a set of "combinatorial data" (partly known also to Mumford and Namikawa, see [Nam79, Chap. 18] and [Nam80, Chap. 9.D]). Let us recall them. Denote by $J(X^\nu)^t$ be the dual abelian variety of $J(X^\nu)$. Now let

$$\lambda_X : J(X^\nu) \xrightarrow{\cong} J(X^\nu)^t$$

be the isomorphism associated to the class of the theta divisor of X^ν.

Let $\mathcal{P}$ be the universal, or Poincaré, line bundle on $J(X^\nu) \times J(X^\nu)^t$. Recall that its set of k-rational points, $\mathcal{P}(k)$, defines a biextension, called the *Poincaré biextension*, of $J(X^\nu) \times J(X^\nu)^t$ by k^*; see [Mum68, Sect. 2 p. 311] or [Bre].

Then $\bar{\mathfrak{t}}_g(X)$ is uniquely determined by the following data:

(1) The free abelian group $H_1(\Gamma_X, \mathbb{Z})$.

(2) The Delaunay decomposition of the real vector space $H_1(\Gamma_X, \mathbb{R})$ associated to the lattice $H_1(\Gamma_X, \mathbb{Z})$, with respect to the Euclidean scalar product.

(3) The classifying morphism of the semiabelian variety $J(X)$, together with its dual. In our present situation, this is the datum of the group homomorphism $c_X : H_1(\Gamma_X, \mathbb{Z}) \to J(X^\nu)$ already described in 3.1.1, together with its dual

$$c_X^t : H_1(\Gamma_X, \mathbb{Z}) \xrightarrow{c_X} J(X^\nu) \xrightarrow{\lambda_X} J(X^\nu)^t.$$

(4) The equivalence class of a trivialization of the pull-back to $H_1(\Gamma_X, \mathbb{Z}) \times H_1(\Gamma_X, \mathbb{Z})$ of the inverse of the Poincaré biextension, i.e. the class of a map

$$\tau_X : H_1(\Gamma_X, \mathbb{Z}) \times H_1(\Gamma_X, \mathbb{Z}) \to (c_X^t \times c_X)^* \mathcal{P}^{-1}(k).$$

This is determined by composing

$$\eta_X \times \eta_X : H_1(\Gamma_X, \mathbb{Z}) \times H_1(\Gamma_X, \mathbb{Z}) \to \mathrm{Div}^{\underline{0}} X^\nu \times \mathrm{Div}^{\underline{0}} X^\nu$$

with the Deligne symbol (see [SGA, XVII] and [Ale04, Sec. 5.5]).

Let us show that such data are the same for our T-equivalent curves X and X'.

As the graphs Γ_X and $\Gamma_{X'}$ are cyclically equivalent, there is an isomorphism $\epsilon_H : H_1(\Gamma_X, \mathbb{Z}) \xrightarrow{\cong} H_1(\Gamma_{X'}, \mathbb{Z})$. It induces an isomorphism $\mathrm{Del}(\Gamma_X) \cong \mathrm{Del}(\Gamma_{X'})$ between the Delaunay decompositions of X and X' (see [CV09, Prop. 3.2.3(i)]). Therefore the data (1) and (2) are the same for X and X'.

Since $X^\nu = X'^\nu$, we have $J(X^\nu) = J(X'^\nu)$ and the principal polarizations, of course, coincide:

$$\lambda_X = \lambda_{X'} : J(X^\nu) \to J(X^\nu)^t.$$

The classifying morphism has been described in 3.1.1. From 3.1.3(c), we get the commutativity of the diagram

$$\begin{array}{ccccc} c_X : H_1(\Gamma_X, \mathbb{Z}) & \xrightarrow{\eta_X} & \mathrm{Div}^{\underline{0}} X^\nu & \twoheadrightarrow & J(X^\nu) \\ \cong \downarrow \epsilon_H & & \cong \downarrow \alpha & & \cong \downarrow \bar{\alpha} \\ c_{X'} : H_1(\Gamma_{X'}, \mathbb{Z}) & \xrightarrow{\eta_{X'}} & \mathrm{Div}^{\underline{0}} X^\nu & \twoheadrightarrow & J(X^\nu) \end{array}$$

where $\overline{\alpha} \in \mathrm{Aut}(J(X^\nu))$ is the automorphism induced by α (recall that $J(X^\nu) = \mathrm{Pic}^{\underline{0}} X^\nu$). It is clear that the automorphisms of $J(X^\nu)$ have no effect on the isomorphism class of the semiabelian variety corresponding to the classifying morphisms. This shows that data (3) are also the same for X and X'.

Let now $\mathcal{P}'(k)$ be the Poincaré biextension of X'; see (4). By what we said so far, it is clear that

$$(\epsilon_H \times \epsilon_H)^*(c^t_{X'} \times c_{X'})^*\mathcal{P}'^{-1}(k) \cong (c^t_X \times c_X)^*\mathcal{P}^{-1}(k).$$

Now, the class of the map τ_X (respectively $\tau_{X'}$) is constructed using the Deligne symbol which is canonically defined on the pull-back of $\mathcal{P}^{-1}(k)$ (respectively $\mathcal{P}'^{-1}(k)$) to $\mathrm{Div}^{\underline{0}} X^\nu \times \mathrm{Div}^{\underline{0}} X^\nu$. Therefore, using the above isomorphism and the commutative diagram of 3.1.3(c), we get

$$\tau_X = (\epsilon_H \times \epsilon_H)^*\tau_{X'} : H_1(\Gamma_X, \mathbb{Z}) \times H_1(\Gamma_X, \mathbb{Z}) \to (c^t_X \times c_X)^*\mathcal{P}^{-1}(k).$$

Therefore the data of part (4) are also the same for X and X'. We thus proved that the data defining $\bar{\mathrm{t}}_g(X)$ and $\bar{\mathrm{t}}_g(X')$ are the same, hence we are done. □

5. The fibers of the Torelli morphism

5.1. Injectivity locus and fiber cardinality of the Torelli morphism

Where, in $\overline{M}_g$, is the compactified Torelli morphism $\bar{\mathrm{t}}_g$ injective? At this point it is clear (as was already known to Namikawa, see [Nam80, Thm. 9.30(iv)]) that this is the case for irreducible curves; the question is thus really interesting for reducible curves. To give it a precise answer we introduce some terminology.

5.1.1. A connected graph is *3-edge connected* if it remains connected after removing any two of its edges. We need the following characterization (Corollary 2.3.4 of [CV09]). *A connected graph free from separating edges is 3-edge connected if and only if every C1-set has cardinality one.*

Note also that given two cyclically equivalent connected graphs, one is 3-edge connected if and only if the other one is. In graph theory, the definition of a 3-edge connected graph is usually given for graphs having at least two vertices. Here we do not make this assumption, so for us a graph with one vertex is always 3-edge connected.

We shall define "Torelli curves" to be those stable curves for which the Torelli map is injective; see Definition 5.1.3 and Theorem 5.1.5. We first illustrate a simple case.

Example 5.1.2. The following is the simplest example of C1-equivalent stable curves. Let $X^\nu = X'^\nu = C_1 \sqcup C_2$, where the C_i are smooth of genus $g_i \geq 1$. Let $p_i, q_i \in C_i$ be distinct points; now define

$$X = \frac{C_1 \sqcup C_2}{(p_1 = p_2, q_1 = q_2)} \quad \text{and} \quad X' = \frac{C_1 \sqcup C_2}{(p_1 = q_2, q_1 = p_2)}.$$

It is clear that X and X' are C1-equivalent.

Observe now that they are not isomorphic, unless one of them, C_1 say, has an automorphism switching p_1 with q_1.

Indeed, suppose that there exists $\alpha_1 \in \operatorname{Aut} C_1$ such that $\alpha_1(p_1) = q_1$ and $\alpha_1(q_1) = p_1$. Then the automorphism $\phi \in \operatorname{Aut} X^\nu$ which restricts to α_1 on C_1 and to the identity on C_2 descends to an isomorphism between X and X', since $\nu' \circ \phi(p_1) = \nu' \circ \phi(p_2)$ and $\nu' \circ \phi(q_1) = \nu' \circ \phi(q_2)$. This example, when α_1 as above exists, is a special case of Torelli curve, defined as follows.

Definition 5.1.3. A stable curve X such that $X_{\mathrm{sep}} = \emptyset$ is called a *Torelli curve* if for every C1-set S such that $\#S = h \geq 2$, conditions (1) and (2) below hold.

(1) For every $i = 1, \ldots, h-1$ there exists an automorphism $\alpha_i \in \operatorname{Aut}(Y_i)$ such that $\alpha_i(p_i) = q_i$ and $\alpha_i(q_i) = p_i$ (where $Y_1, \ldots, Y_h$ are the connected components of Y_S and $p_i, q_i \in Y_i$ are the two gluing points).
(2) There is an isomorphism as marked curves $(Y_i; p_i, q_i) \cong (Y_j; p_j, q_j)$ for every $i, j \leq h-1$; or else $h = 3$ and there exists $\alpha_h \in \operatorname{Aut}(Y_h)$ such that $\alpha_h(p_h) = q_h$ and $\alpha_h(q_h) = p_h$.

Example 5.1.4. If Γ_X is 3-edge connected then X is a Torelli curve, by 5.1.1.

Theorem 5.1.5. *Let X be a stable curve free from separating nodes. Then*

(1) $\#\bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X)) \leq \lceil (g-2)!/2 \rceil$. *Furthermore the bound is sharp, and can be attained for X a cycle curve equal to the union of $g-1$ elliptic curves, no two of them isomorphic.*
(2) $\bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X)) = \{X\}$ *if and only if X is a Torelli curve.*

Proof. By Theorem 2.1.7 the set $\bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X))$ is the C1-equivalence class of X. The bound on its cardinality follows from Lemma 5.1.8 below.

Now, let X be the union of $g-1$ smooth curves $C_1, \ldots, C_{g-1}$ of genus 1, so that the dual graph of X is a cycle of length $g-1$. Suppose that $C_i \ncong C_j$ for all $i \neq j$. The curve X has a unique C1-set, namely $S = X_{\mathrm{sing}}$, and each curve C_i contains exactly two points of G_S, which we call p_i and q_i. With the notation of 2.3.2, let $[(\sigma_S, \psi_S)]$ be the gluing data of X. Since each C_i has an automorphism exchanging p_i with q_i, varying the marking ψ_S does not change the isomorphism class of the curve X. On the other hand, any change in σ_S (with the exception of σ_S^{-1} of course) changes the isomorphism class of the curve, because no two C_i are isomorphic. Therefore, we conclude that the number of non-isomorphic curves that are C1-equivalent to X is equal to 1 if $g \leq 3$, and $(g-2)!/2$ if $g \geq 4$. Part (1) is proved.

For (2) it suffices to prove the following. Let X be connected with $X_{\mathrm{sep}} = \emptyset$; then X is a Torelli curve if and only if the only curve C1-equivalent to X is X itself.

Assume first that X is a Torelli curve. If Γ_X is 3-edge connected, then every C1-set has cardinality 1 by 5.1.1, therefore we conclude by Lemma 5.1.6. We can henceforth assume that Γ_X is not 3-edge connected.

Let $S \in \mathrm{Set}^1 X$ have cardinality $h \geq 2$ (it exists by 5.1.1). We claim that $\mathrm{Aut}\, X$ acts transitively on the gluing data of S, described in 2.3.2. We use the notation of Definition 5.1.3. If $h = 3$ and Y_i has an automorphism exchanging p_i with q_i for $i = 1, 2, 3$, then the claim trivially holds.

Next, assume that the first $h - 1$ marked components $(Y_i; p_i, q_i)$ are isomorphic and have an automorphism switching the gluing points p_i, q_i. Using the set-up of 2.3.2, the gluing data are given by an ordering of the components, which we can assume has Y_h as last element, and by a marking of each pair (p_i, q_i) for all $i = 1, \ldots, \gamma - 1$. Now $\mathrm{Aut}\, X$ acts transitively on the orderings of the components, by permuting $Y_1, \ldots, Y_{h-1}$, which are all isomorphic by isomorphisms preserving the gluing points. Moreover for $i = 1, \ldots, \gamma - 1$ each pair of points (p_i, q_i) is permuted by the automorphism α_i. The claim is proved. Of course, the claim implies that X is unique in its C1-equivalence class.

Conversely, let X be the unique curve in its C1-equivalence class. If every C1-set of X has cardinality 1 then Γ_X is 3-edge connected (by 5.1.1) and we are done.

So, let $S \in \mathrm{Set}^1 X$ be such that $\#S \geq 2$ and let us check that the conditions of Definition 5.1.3 hold. With no loss of generality, and using the same notation as before, we may order the connected components of Y_S so that q_i is glued to p_{i+1} and p_i is glued to q_{i-1} (with the cyclic convention, so that p_1 is glued to q_h). Assume that Y_h has no automorphism exchanging p_h with q_h; let us change the gluing data of X by switching p_h with q_h, and by leaving everything else unchanged. Then the corresponding curve is C1-equivalent to X, and hence it is isomorphic to X, by hypothesis. Therefore, the curve $W = \overline{X \smallsetminus Y_h}$ must admit an automorphism switching p_1 with q_{h-1} (the two points glued to q_h and p_h). Now it is easy to see, by induction on the number of components of W, that such an automorphism exists if and only if W is a union of $h - 1$ marked components, $(Y_i; p_i, q_i)$, all isomorphic to $(Y_1; p_1, q_1)$, and if Y_1 has an involution switching p_1, q_1. Therefore X is a Torelli curve.

If instead Y_i has an automorphism exchanging the two gluing points for every $i = 1, \ldots, h$, and no $h - 1$ among the Y_i are isomorphic, it is clear that for $h \geq 4$ there exist different orderings of the Y_i giving different C1-equivalent curves. Therefore we must have $h = 3$, hence X is a Torelli curve. □

The proof of the theorem used the following lemmas.

Lemma 5.1.6. *Let X be a connected nodal curve free from separating nodes. Then the cardinality of the C1-equivalence class of X is at most*

$$\prod_{S \in \mathrm{Set}^1 X} 2^{\#S-1}(\#S - 1)!.$$

Proof. By the discussion in 2.3.2, the number of curves that are C1-equivalent to X is bounded above by the product of the number of all gluing data for each C1-set of X. The C1-sets with $\#S = 1$ admit only one gluing data, so they do not contribute.

Let S be a C1-set of cardinality at least 2. Clearly there are $2^{\#S}$ possible markings ψ_S, and $(\#S - 1)!$ possible choices for the cyclic permutation σ_S. Furthermore, recall that each gluing data can be given by two such pairs (ψ_S, σ_S), namely the two conjugate pairs under the involution (2.8). This gives us a total of $2^{\#S-1}(\#S - 1)!$ gluing data. □

We shall repeatedly use the following elementary

Remark 5.1.7. Let E be a connected nodal curve of genus at most 1, free from separating nodes. For any two smooth points p, q of E, there exists an automorphism of E exchanging p and q.

Lemma 5.1.8. *Let X be a connected curve of genus $g \geq 2$ free from separating nodes; let e be the number of its exceptional components. Then the C1-equivalence class of X has cardinality at most*

$$\lceil (g-2+e)!/2 \rceil.$$

Proof. Throughout this proof, we denote by $\{Y\}_{\mathrm{C1}}$ the C1-equivalence class of a nodal curve Y. We will use induction on g.

We begin with the following claim. Let $\overline{X}$ be the stabilization of X. If $\Gamma_{\overline{X}}$ is 3-edge connected, then $\#\{X\}_{\mathrm{C1}} = 1$.

Indeed, there is a natural bijection between the C1-sets of X and those of $\overline{X}$, which we denote by $S \mapsto \overline{S}$. By assumption, for every C1-set $\overline{S}$ of $\overline{X}$ the partial normalization $\overline{Y}_{\overline{S}}$ of $\overline{X}$ at $\overline{S}$ is connected (since $\#\overline{S} = 1$). Now, for any $S \in \mathrm{Set}^1 X$, the partial normalization Y_S of X at S is equal to the disjoint union of $\overline{Y}_{\overline{S}}$ together with some copies of $\mathbb{P}^1$. Using this explicit description and 5.1.7 we find that all the possible gluing data $[(\sigma_S, \psi_S)]$ of S (see 2.3.2) give isomorphic curves, i.e. X is unique inside its C1-equivalence class. The claim is proved.

Now we start the induction argument. Let us treat the cases $g = 2, 3$.

Using the above claim, it is easy to see that to prove the lemma for $g = 2, 3$ we need only worry about curves X of genus 3, whose stabilization $\overline{X}$ is the union of two components C_1 and C_2 of genus 1, meeting at two points. If $e = 0$ then X is unique in its C1-equivalence class by using 5.1.7. If $e > 0$ then the curves C1-equivalent to X are obtained by inserting two chains of exceptional components between C_1 and C_2, one of length e_1 for every $0 \leq e_1 \leq \lfloor e/2 \rfloor$, and the other of length $e - e_1$. It is obvious that for different values of e_1 we get non-isomorphic curves, and that we get all of the curves C1-equivalent to X in this way. Therefore

$$\#\{X\}_{\mathrm{C1}} = 1 + \lfloor e/2 \rfloor \leq \lceil (e+1)!/2 \rceil.$$

Assume now $g \geq 4$ and let $S \in \mathrm{Set}^1 X$ be such that $\#S = h$. As usual, we write $Y_S = \bigsqcup_{i=1}^{h} Y_i$, with Y_i free from separating nodes and of genus $g_i := g_{Y_i}$. We order the connected components Y_i of Y_S in such a way that:

- $Y_1, \ldots, Y_f$ have genus at least 4;
- $Y_{f+1}, \ldots, Y_{f+k_3}$ have genus 3;
- $Y_{f+k_3+1}, \ldots, Y_{f+k_3+k_2}$ have genus 2;
- $Y_{f+k_3+k_2+1}, \ldots, Y_{f+k_3+k_2+k_1}$ have genus 1;
- $Y_{f+k_3+k_2+k_1+1}, \ldots, Y_h$ have genus 0 and therefore are isomorphic to $\mathbb{P}^1$.

Let e_i be the number of exceptional components of X contained in Y_i; then Y_i has at most $e_i + 2$ exceptional components. We have the obvious relations

$$e = \sum_{i=1}^{h} e_i = \sum_{g_i \geq 2} e_i + \sum_{g_i = 1} e_i + h - f - k_3 - k_2 - k_1, \qquad (*)$$

$$g - 1 = \sum_i g_i = \sum_{g_i \geq 2} g_i + k_1. \qquad (**)$$

Consider now the gluing data $[(\sigma_S, \psi_S)]$ associated to S (notation as in 2.3.2). Denote, as usual, by $\{p_i, q_i\}$ the two points of G_S contained in the component Y_i. Since all the components Y_i with $g_i \leq 1$ have an automorphism that exchanges p_i and q_i (by 5.1.7), if we compose the marking ψ_S with the involution of G_h that exchanges s_i with t_i (for all indices i such that $g_i \leq 1$) the resulting curve will be isomorphic to the starting one. Therefore, the number of possible non-isomorphic gluing data associated to S is bounded above by $(h-1)!2^{f+k_2+k_3-1}$; since $g \geq 4$ this number is an integer (if $f = k_2 = k_3 = 0$ then $h \geq 3$). We conclude that

$$\#\{X\}_{\mathrm{C1}} \leq (h-1)!2^{f+k_3+k_2-1} \prod_{i=1}^{h} \#\{Y_i\}_{\mathrm{C1}}.$$

The components Y_i of genus at most 1 are unique inside their C1-equivalence class. For the components Y_i of genus $g_i \geq 2$ we can apply the induction hypothesis (note that $2 \leq g_i < g$) and we get

$$\#\{Y_i\}_{\mathrm{C1}} \leq \lceil (g_i - 2 + e_i + 2)!/2 \rceil = (g_i + e_i)!/2.$$

By substituting into the previous formula, we get

$$\#\{X\}_{\mathrm{C1}} \leq (h-1)!2^{f+k_3+k_2-1} \prod_{i=1}^{f+k_3+k_2} \frac{(g_i + e_i)!}{2} = \frac{(h-1)! \prod_{g_i \geq 2} (g_i + e_i)!}{2}.$$

The number of (non-trivial) factors of the product $(h-1)! \prod_{g_i \geq 2} (g_i + e_i)!$ is equal to $h - 2 + \sum_{g_i \geq 2} (g_i + e_i - 1)$. Using the formulas $(*)$ and $(**)$, we get

$$h - 2 + \sum_{g_i \geq 2} (g_i + e_i - 1) = g - 3 + e - \sum_{g_i = 1} e_i \leq g - 3 + e.$$

Since the factorial $(g - 2 + e)!$ has $g - 3 + e$ factors, we conclude from the above inequalities that

$$\#\{X\}_{\mathrm{C1}} \leq \frac{(h-1)! \prod_{g_i \geq 2} (g_i + e_i)!}{2} \leq \frac{(g - 2 + e)!}{2},$$

as claimed. □

Corollary 5.1.9. $\bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X)) = \{X\}$ *for every* $X \in \overline{M}_g$ *with* $X_{\mathrm{sep}} = \emptyset$ *if and only if* $g \leq 4$.

Remark 5.1.10. Consider a Torelli curve X of genus at least 5 with dual graph not 3-edge connected. It is not hard to see that X is the specialization of curves for which the Torelli morphism is not injective. On the other hand we just proved that $\bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X)) = \{X\}$. Therefore the Torelli morphism, albeit injective at X, necessarily ramifies at X.

5.2. Dimension of the fibers

Let X be a stable curve of genus g; now we shall assume that X_{sep} is not empty and bound the dimension of the fiber of the Torelli map over X.

Recall the notation of (1.5); the normalization of X at X_{sep} is denoted $\widetilde{X}$. We denote by $\widetilde{\gamma}_0$ the number of connected components of $\widetilde{X}$ of arithmetic genus 0, by $\widetilde{\gamma}_1$ the number of those of arithmetic genus 1, and by $\widetilde{\gamma}_+$ the number of those having positive arithmetic genus; that is,

$$\widetilde{\gamma}_j := \#\{i : \widetilde{g}_i = j\}, \quad j = 0, 1, \quad \text{and} \quad \widetilde{\gamma}_+ := \#\{i : \widetilde{g}_i \geq 1\}.$$

Proposition 5.2.1. *Let X be a stable curve of genus $g \geq 2$. Then*

$$\dim \bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X)) = 2\widetilde{\gamma}_+ - \widetilde{\gamma}_1 - 2$$

(*i.e. the maximal dimension of an irreducible component of $\bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X))$ is equal to $2\widetilde{\gamma}_+ - \widetilde{\gamma}_1 - 2$*).

Proof. According to Theorem 2.1.7, $\bar{\mathrm{t}}_g(X)$ depends on (and determines) the C1-equivalence class of the stabilizations $\widetilde{\widetilde{X}}_i$ of the components of $\widetilde{X}$ such that $\widetilde{g}_i > 0$. The C1-equivalence class of $\widetilde{\widetilde{X}}_i$ determines $\widetilde{\widetilde{X}}_i$ up to a finite choice. In particular, note that $\widetilde{\gamma}_0$ and the number, call it e, of exceptional components of $\bigsqcup_{\widetilde{g}_i>0} \widetilde{X}_i$ is not determined by $\bar{\mathrm{t}}_g(X)$.

The dimension of the locus of curves in the fiber $\bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X))$ having the same topological type of X is equal to

$$2\#X_{\mathrm{sep}} - 3\widetilde{\gamma}_0 - \widetilde{\gamma}_1 - e. \tag{5.1}$$

Indeed, each separating node gives two free parameters, because we can arbitrarily choose the two branches of the node. The components $\widetilde{X}_i$ of arithmetic genus 0 reduce the number of parameters by 3 because they have a 3-dimensional automorphism group, similarly the components of arithmetic genus 1 reduce the number of parameters by 1. Finally, each exceptional component of $\bigsqcup_{\widetilde{g}_i>0} \widetilde{X}_i$ reduces the number of parameters by 1, because it contains at least one branch of one of the separating nodes and exactly two branches of non-separating nodes.

Formula (5.1) shows that the curves X' in the fiber $\bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X))$ whose topological type attains the maximal dimension are the ones for which $e' = 0$ (i.e. each positive genus component of $\widetilde{X}'$ is stable) and $\widetilde{\gamma}'_0 = 0$ (i.e. $\widetilde{X}'$ has no genus 0 component).

In particular, since $\widetilde{\gamma}'_+ = \widetilde{\gamma}'$, such a curve X' has $\#X'_{\mathrm{sep}} = \widetilde{\gamma}'_+ - 1$ separating nodes. Applying formula (5.1) to the curve X' we obtain $\dim \bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X)) \leq 2\widetilde{\gamma}'_+ - \widetilde{\gamma}'_1 - 2$. To conclude that equality holds we must check that the locus of curves X' is not empty. This

is easy: given $\bigsqcup_{\widetilde{g}_i>0} \widetilde{X}_i$ we can glue (in several ways) the stabilizations of the $\widetilde{X}_i$ so that they form a tree. This, by our results, yields curves in $\bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X))$. □

Corollary 5.2.2. *Let X be a stable curve of genus g. Then*

$$\begin{cases} \dim \bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X)) \le g-2 & \text{with equality iff } \widetilde{g}_i \le 2 \text{ for all } i, \\ \dim \bar{\mathrm{t}}_g^{-1}(\bar{\mathrm{t}}_g(X)) \ge \widetilde{\gamma}_+ - 2 & \text{with equality iff } \widetilde{g}_i \le 1 \text{ for all } i. \end{cases}$$

Proof. The first inequality follows from the proposition and

$$g = \widetilde{\gamma}_1 + \sum_{\widetilde{g}_i \ge 2} \widetilde{g}_i \ge \widetilde{\gamma}_1 + 2(\widetilde{\gamma}_+ - \widetilde{\gamma}_1) = 2\widetilde{\gamma}_+ - \widetilde{\gamma}_1,$$

with equality if and only if all $\widetilde{g}_i \le 2$ for all i.

The second inequality follows from $\widetilde{\gamma}_1 \le \widetilde{\gamma}_+$, with equality if and only $\widetilde{g}_i \le 1$. □

Using Theorem 5.1.5 and Corollary 5.2.2, one finds that for $g \ge 3$ the locus in $\overline{M}_g$ where $\bar{\mathrm{t}}_g$ has finite fibers is exactly the open subset of stable curves free from separating nodes; see [Nam80, Thm. 9.30(vi)] and [V03, Thm. 1.1] for the analogous results for the map $\bar{\mathrm{t}}_g^{\mathrm{Vor}}$. On the other hand $\bar{\mathrm{t}}_g$ is an isomorphism for $g = 2$; again see [Nam80, Thm. 9.30(v)].

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