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The gradient flow of Higgs pairs

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Abstract. We consider the gradient flow of the Yang–Mills–Higgs functional of Higgs pairs on a Hermitian vector bundle (E, H_0) over a Kähler surface (M, ω), and study the asymptotic behavior of the heat flow for Higgs pairs at infinity. The main result is that the gradient flow with initial condition (A_0, ϕ_0) converges, in an appropriate sense which takes into account bubbling phenomena, to a critical point (A_∞, ϕ_∞) of this functional. We also prove that the limiting Higgs pair (A_∞, ϕ_∞) can be extended smoothly to a vector bundle E_∞ over (M, ω), and the isomorphism class of the limiting Higgs bundle $(E_\infty, A_\infty, \phi_\infty)$ is given by the double dual of the graded Higgs sheaves associated to the Harder–Narasimhan–Seshadri filtration of the initial Higgs bundle (E, A_0, ϕ_0).

Keywords. Higgs bundles, Kähler surface, Harder–Narasimhan–Seshadri filtration

1. Introduction

Given a complex vector bundle E over a compact Kähler manifold (M, ω), suppose that there is a Hermitian structure H_0 on E. Let $\mathcal{A}_{H_0}$ denote the space of connections on E compatible with the metric H_0, and let $\mathcal{A}^{1,1}_{H_0}$ denote the space of unitary integrable connections on E. Given a Hermitian metric H_0 on a holomorphic bundle $(E, \overline{\partial}_E)$, there is a unique H_0-unitary connection A on E satisfying $D_A^{(0,1)} = \overline{\partial}_E$, where $D_A^{(0,1)}$ denotes the $(0, 1)$ part of D_A; this connection is also called the Chern connection on $(E, \overline{\partial}_E, H_0)$. We will sometimes denote it by $A = (\overline{\partial}_E, H_0)$. Conversely, given a unitary integrable connection A on (E, H_0) (i.e. one whose curvature F_A is of type $(1, 1)$), $D_A^{(0,1)} = \overline{\partial}_E$ defines a holomorphic structure on E, and $A = (\overline{\partial}_E, H_0)$.

The *Yang–Mills functional* is defined on $\mathcal{A}_{H_0}$ by

$$\mathrm{YM}(A) = \int_M |F_A|^2 \, dV_\omega,$$

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where dV_ω is the volume form of ω. We call A a *Yang–Mills connection* of E if A is a critical point of the Yang–Mills functional, i.e. it satisfies the *Yang–Mills equation*

$$D_A^* F_A = 0,$$

where D_A^* is the adjoint operator of covariant differentiation associated with the connection D_A.

In this paper, we are interested in a more general case. A pair $(A, \phi) \in \mathcal{A}_{H_0}^{1,1} \times \Omega^{1,0}(\mathrm{End}(E))$ is called a *Higgs pair* if $\overline{\partial}_A \phi = 0$ and $\phi \wedge \phi = 0$. Let $\mathcal{B}_{(E,H_0)}$ denote the space of all Higgs pairs on the Hermitian vector bundle (E, H_0). We consider the *Yang–Mills–Higgs functional* defined on $\mathcal{B}_{(E,H_0)}$ by

$$\mathrm{YMH}(A, \phi) = \int_M (|F_A + [\phi, \phi^*]|^2 + 2|\partial_A \phi|^2)\, dV_g. \tag{1.1}$$

A *Yang–Mills–Higgs pair* (A, ϕ) is a critical point of the Yang–Mills–Higgs functional. Equivalently, (A, ϕ) satisfies the *Yang–Mills–Higgs equations*

$$\begin{cases} D_A^* F_A + \sqrt{-1}(\partial_A \Lambda_\omega - \overline{\partial}_A \Lambda_\omega)[\phi, \phi^*] = 0, \\ [\sqrt{-1}\, \Lambda_\omega (F_A + [\phi, \phi^*]), \phi] = 0, \end{cases} \tag{1.2}$$

where the operator Λ_ω is contraction with ω, and ϕ^* denotes the dual of ϕ with respect to the given metric H_0.

By Chern–Weil theory, we have

$$\begin{aligned} \mathrm{YMH}(A, \phi) &= \int_M (|F_A + [\phi, \phi^*]|^2 + 2|\partial_A \phi|^2)\, \frac{\omega^n}{n!} \\ &= \int_M |\sqrt{-1}\, \Lambda_\omega (F_A + [\phi, \phi^*])|^2\, \frac{\omega^n}{n!} + 4\pi^2 \int_M (2c_2(E) - c_1(E) \wedge c_1(E)) \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &= \int_M |\sqrt{-1}\, \Lambda_\omega (F_A + [\phi, \phi^*]) - \lambda\, \mathrm{Id}_E\, |^2 \frac{\omega^n}{n!} + \lambda^2\, \mathrm{rank}(E) \int_M \frac{\omega^n}{n!} \\ &\quad + 4\pi^2 \int_M (2c_2(E) - c_1(E) \wedge c_1(E)) \wedge \frac{\omega^{n-2}}{(n-2)!}, \end{aligned}$$

where

$$\lambda = \frac{2\pi \int_M c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!}}{\mathrm{rank}(E) \int_M \frac{\omega^n}{n!}}.$$

From the above identity, we see that if (A, ϕ) satisfies the Hermitian-Einstein equation

$$\sqrt{-1}\, \Lambda_\omega (F_A + [\phi, \phi^*]) = \lambda\, \mathrm{Id}_E,$$

then it must satisfy the above Euler–Lagrange equation (1.2), in fact it is the absolute minimum of the above Yang–Mills–Higgs functional. Equivalently, if (A, ϕ) satisfies the above Hermitian-Einstein equation, then H_0 must be the Hermitian-Einstein metric on the Higgs bundle $(E, \overline{\partial}_A, \phi)$, studied by Hitchin [Hi] and Simpson [Si1]. In [Hi] and

[Si1], it is proved that a Higgs bundle admits the Hermitian-Einstein metric iff it is Higgs poly-stable.

The Yang–Mills flow was first suggested by Atiyah–Bott in [AB]. Donaldson [Do] used it to establish the connection between Hermitian-Yang–Mills connections and holomorphic stable bundles. He proved the global existence of the Yang–Mills flow in a holomorphic bundle over a Kähler surface, and proved the convergence of the flow at infinity in the case where the holomorphic bundle is stable. For the Higgs bundle, Simpson [Si1] proved the long time existence of the Hermitian Yang–Mills–Higgs flow and showed the convergence under the condition that the Higgs bundle is stable. Without the assumption of the stability of the bundles, the above flows may not converge at infinity. Daskalopoulos [Da] and Daskalopoulos and Wentworth [DW1] studied the asymptotic behavior of the Yang–Mills flow over Riemannian surfaces and Kähler surfaces, and showed that there is a relation between the Yang–Mills flow and the Harder–Narasimhan filtration of holomorphic bundles.

In [St1], Struwe studied the global existence and uniqueness of the Yang–Mills flow in vector bundles over compact Riemannian four-manifolds for a given initial connection with finite energy. For general vector bundles, the Yang–Mills heat flow may develop singularities in finite time.

In this paper, we study the evolution equations of the above Euler–Lagrange equations (1.4), i.e. the gradient flow of the Yang–Mills–Higgs functional of Higgs pairs. A regular solution is given by a family of $(A(x,t), \phi(x,t)) \in \mathcal{B}_{(E,H_0)}$ such that

$$\begin{cases} \dfrac{\partial A}{\partial t} = -D_A^* F_A - \sqrt{-1}(\partial_A \Lambda_\omega - \overline{\partial}_A \Lambda_\omega)[\phi, \phi^*], \\ \dfrac{\partial \phi}{\partial t} = -[\sqrt{-1}\,\Lambda_\omega(F_A + [\phi, \phi^*]), \phi]. \end{cases} \tag{1.3}$$

The above flow can be seen as a Higgs pairs version of the Yang–Mills flow. In this paper, we first show some basic properties of the above flow, including the energy inequality, Bochner-type inequality, monotonicity of certain quantities and a small action regularity theorem. We prove the global existence and uniqueness of the solution for the above gradient flow. Then we study the asymptotic behavior of a regular solution at infinity. To a Higgs bundle (E, A_0, ϕ_0), one can associate a filtration by ϕ_0-invariant holomorphic subsheaves, which will be called the Harder–Narasimhan filtration, whose successive quotients are Higgs semistable. The topological type of the pieces in the associated graded object is encoded into an R-tuple $\vec{\mu} = (\mu_1, \ldots, \mu_R)$ of rational numbers called the Harder–Narasimhan type of the Higgs bundle (E, A, ϕ). Let $(A(t), \phi(t))$ be a smooth solution of the gradient flow (1.3) with initial data (A_0, ϕ_0) over a compact Kähler surface. We prove that there exists a sequence $t_j \to \infty$ such that (A_{t_j}, ϕ_{t_j}) converges, modulo gauge transformations, to a Yang–Mills Higgs pair (A_∞, ϕ_∞) outside finite points, and (A_∞, ϕ_∞) can be extended smoothly to a vector bundle E_∞ over a Kähler surface. This limiting Higgs pair can also be called an Uhlenbeck limit. We also consider the Harder–Narasimhan type and the isomorphism class of the Uhlenbeck limits. To state the result precisely, let $\mathrm{Gr}_\omega^{\mathrm{hns}}(E, A_0, \phi_0)$ denote the Harder–Narasimhan–Seshadri filtration of the initial Higgs bundle (E, A_0, ϕ_0) with respect to the Kähler form ω, and let

$\mathrm{Gr}^{\mathrm{hns}}_{\omega}(E, A_0, \phi_0)^{**}$ be its double dual. Our result is that the Harder–Narasimhan type and the isomorphism class of the Uhlenbeck limits are independent of the subsequence and are determined solely by the initial data (A_0, ϕ_0). More precisely:

Main Theorem. *Let (E, H_0) be a Hermitian vector bundle on a compact Kähler surface (M, ω), and $(A(t), \phi(t))$ be a global smooth solution of the gradient flow (1.3) with smooth initial Higgs pair (A_0, ϕ_0). Then there exists a sequence $t_i \to \infty$ such that $(A, \phi)(x, t_i)$ converges, modulo gauge transformations, to a Yang–Mills–Higgs pair (A_∞, ϕ_∞) in the smooth topology outside a closed set $\Sigma^{\mathrm{an}} \subset M$, where Σ^{an} is a finite collection of points. The limiting Yang–Mills–Higgs pair (A_∞, ϕ_∞) can be extended smoothly by a continuous gauge transformation to a smooth Yang–Mills–Higgs pair on a Hermitian bundle (E_∞, H_∞) over M, and the extension $(E_\infty, H_\infty, A_\infty, \phi_\infty)$ has a holomorphic orthogonal splitting as a direct sum:*

$$(E_\infty, H_\infty, A_\infty, \phi_\infty) = \bigoplus_{i=1}^{l} (E^i_\infty, H^i_\infty, A^i_\infty, \phi^i_\infty),$$

where H^i_∞ is a Hermitian-Einstein metric on the Higgs bundle $(E^i_\infty, A^i_\infty, \phi^i_\infty)$. Moreover, the Harder–Narasimhan type of the Higgs bundle $(E_\infty, A_\infty, \phi_\infty)$ is the same as that of (E, A_0, ϕ_0), and $(E_\infty, A_\infty, \phi_\infty) \simeq \mathrm{Gr}^{\mathrm{hns}}_{\omega}(E, \overline{\partial}_{A_0}, \phi_0)^{**}$.

Our result can be seen as a Higgs bundle version of Theorem 1 in [DW1] (for Yang–Mills case). In discussing the HN type and the isomorphism class of the Uhlenbeck limits, we follow some ideas in [DW1]. Recently, Wilkin [Wi] studied the gradient flow of Higgs pairs over compact Riemann surfaces. In [DW2], Daskalopoulos and Wentworth also consider the blow-up locus of Yang–Mills flow on Kähler surfaces; they show that the blow-up locus is determined by the Harder–Narasimhan–Seshadri filtration of the initial holomorphic bundle. Their result should be true in the Higgs bundle case; we will further discuss this problem in the future.

This paper is organized as follows. In Section 2, we prove global existence and uniqueness of the solution for the gradient flow (1.3), and derive basic estimates and analytic preliminaries over general Kähler manifolds. In Section 3, we consider the convergence properties of a global solution of the gradient flow over a Kähler surface. In Section 4, we discuss the Harder–Narasimhan–Seshadri filtration of Higgs bundles. In Section 5, we focus on the HN type of Uhlenbeck limits. In the last section, we prove that the Uhlenbeck limits are holomorphically isomorphic to the double dual of the graded object of the Harder–Narasimhan–Seshadri filtration of the initial Higgs bundle. The main theorem follows from Theorems 3.11, 5.14 and 6.1.

2. Analytic preliminaries and basic estimates

2.1. Existence of the gradient flow

In this section, we prove the long-time existence of the gradient flow for Higgs pairs on a Hermitian bundle (E, H_0) over a Kähler manifold (M, ω). Here the idea is similar to that in [Do] and [H]. Let (A_0, ϕ_0) be an initial Higgs pair on (E, H_0). Then we consider the following heat flow for Hermitian metrics on the Higgs bundle (E, A_0, ϕ_0) with initial metric H_0:

$$H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\,\Lambda_\omega(F_H + [\phi_0, \phi_0^{*H}]) - \lambda\,\mathrm{Id}_E), \tag{2.1}$$

where F_H is the curvature form of the Chern connection A_H on E with respect to H, the operator Λ_ω is contraction with ω, and ϕ^{*H} is the adjoint of ϕ with respect to the Hermitian metric H, i.e $\langle\phi(X), Y\rangle_H = \langle X, \phi^{*H}(Y)\rangle_H$ for any $X, Y \in E$. In [Si1], Simpson proved that solutions to the above nonlinear heat equation exist for all time and depend continuously on the initial condition H_0.

Suppose $H(t)$ is a solution of the above heat equation, and let $h(t) = H_0^{-1}H(t)$. Then

$$\frac{\partial h}{\partial t} = -2\sqrt{-1}\,h\Lambda_\omega(F_{A_0} + \overline{\partial}_{A_0}(h^{-1}\partial_{A_0}h) + [\phi_0, h^{-1}\phi_0^{*H_0}h]) + 2\lambda h. \tag{2.2}$$

Denote the complex gauge group (resp. unitary gauge group) of the Hermitian vector bundle (E, H_0) by $\mathcal{G}^{\mathbb{C}}$ (resp. $\mathcal{G}$, where $\mathcal{G} = \{\sigma \in \mathcal{G}^{\mathbb{C}} \mid \sigma^{*H_0}\sigma = \mathrm{Id}\}$). The group $\mathcal{G}^{\mathbb{C}}$ acts on the space $\mathcal{A}^{1,1}_{H_0} \times \Omega^{1,0}(\mathrm{End}(E))$ as follows: for $\sigma \in \mathcal{G}^{\mathbb{C}}$,

$$\overline{\partial}_{\sigma(A)} = \sigma \circ \overline{\partial}_A \circ \sigma^{-1}, \qquad \partial_{\sigma(A)} = (\sigma^{*H_0})^{-1} \circ \partial_A \circ \sigma^{*H_0},$$
$$\sigma(\phi) = \sigma \circ \phi \circ \sigma^{-1}.$$

In the following, we denote ϕ^{*H_0} just by ϕ^*, and σ^{*H_0} by σ^* for simplicity.

The derivative of the $\mathcal{G}^{\mathbb{C}}$ action at (A, ϕ) is

$$\theta \mapsto (-\overline{\partial}_A\theta + \partial_A\theta^*, [\theta, \phi]).$$

On the other hand, one can check that

$$(\sqrt{-1}\,\Lambda_\omega(F_A + [\phi, \phi^*]))^* = \sqrt{-1}\,\Lambda_\omega(F_A + [\phi, \phi^*]).$$

From the gradient flow equations (1.3), we know that the gradient vector at (A, ϕ) lies in the tangent space to the orbit of the complex group $\mathcal{G}^{\mathbb{C}}$ at (A, ϕ).

Let $\sigma(t) \in \mathcal{G}^{\mathbb{C}}$ satisfy $\sigma^*(t)\sigma(t) = h(t)$. By direct calculation, we have

$$(\sigma^*)^{-1}\frac{\partial\sigma^*}{\partial t} + \frac{\partial\sigma}{\partial t}\sigma^{-1} = -2\sqrt{-1}\,\Lambda_\omega(F_{\sigma(A_0)} + [\sigma(\phi_0), (\sigma(\phi_0))^*]) + 2\lambda\,\mathrm{Id}_E.$$

Let $\tilde{A}(t) = \sigma(t)(A_0)$ and $\tilde{\phi}(t) = \sigma(t)(\phi_0)$. We get

$$\begin{aligned}
\frac{\partial\tilde{A}}{\partial t} &= -\overline{\partial}_{\tilde{A}}\left(\frac{\partial\sigma}{\partial t}\sigma^{-1}\right) + \partial_{\tilde{A}}\left((\sigma^*)^{-1}\frac{\partial\sigma^*}{\partial t}\right)\\
&= \frac{1}{2}(\partial_{\tilde{A}} - \overline{\partial}_{\tilde{A}})\left(\frac{\partial\sigma}{\partial t}\sigma^{-1} + (\sigma^*)^{-1}\frac{\partial\sigma^*}{\partial t}\right) - \frac{1}{2}(\partial_{\tilde{A}} + \overline{\partial}_{\tilde{A}})\left(\frac{\partial\sigma}{\partial t}\sigma^{-1} - (\sigma^*)^{-1}\frac{\partial\sigma^*}{\partial t}\right)\\
&= -\sqrt{-1}(\partial_{\tilde{A}} - \overline{\partial}_{\tilde{A}})\Lambda_\omega(F_{\tilde{A}} + [\tilde{\phi}, \tilde{\phi}^*]) - \frac{1}{2}(\partial_{\tilde{A}} + \overline{\partial}_{\tilde{A}})\left(\frac{\partial\sigma}{\partial t}\sigma^{-1} - (\sigma^*)^{-1}\frac{\partial\sigma^*}{\partial t}\right),
\end{aligned}$$

$$\frac{\partial\tilde{\phi}}{\partial t} = -\left[\phi, \frac{\partial\sigma}{\partial t}\sigma^{-1}\right] = [\tilde{\phi}, \sqrt{-1}\,\Lambda_\omega(F_{\tilde{A}} + [\tilde{\phi}, \tilde{\phi}^*])] - \frac{1}{2}\left[\tilde{\phi}, \frac{\partial\sigma}{\partial t}\sigma^{-1} - (\sigma^*)^{-1}\frac{\partial\sigma^*}{\partial t}\right].$$

Set

$$\alpha = -\frac{1}{2}\left(\frac{\partial\sigma}{\partial t}\sigma^{-1} - (\sigma^*)^{-1}\frac{\partial\sigma^*}{\partial t}\right).$$

It is obvious that $\alpha(t) \in \mathrm{Lie}(\mathcal{G})$. Now let $S(t) \in \mathcal{G}$ be the unique solution to the linear ODE

$$\frac{dS}{dt} = S\alpha, \qquad S(0) = I.$$

Then, let $A = S(\tilde{A})$ and $\phi = S(\tilde{\phi})$. It is easy to check that

$$\begin{aligned}\frac{\partial A}{\partial t} &= S\circ\left(\frac{\partial\tilde{A}}{\partial t} - D_{\tilde{A}}\left(S^{-1}\frac{\partial S}{\partial t}\right)\right)\circ S^{-1} = S\circ\left(\frac{\partial\tilde{A}}{\partial t} - D_{\tilde{A}}\alpha\right)\circ S^{-1}\\ &= -D_A^*F_A - \sqrt{-1}(\partial_A\Lambda_\omega - \overline{\partial}_A\Lambda_\omega)[\phi, \phi^*],\end{aligned}$$

where we have used the Kähler identity

$$\partial_A^* = \sqrt{-1}[\Lambda_\omega, \overline{\partial}_A], \qquad \overline{\partial}_A^* = -\sqrt{-1}[\Lambda_\omega, \partial_A].$$

It is clear that

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= S\circ\left(\frac{\partial\tilde{\phi}}{\partial t} - \left[\tilde{\phi}, S^{-1}\frac{\partial S}{\partial t}\right]\right)\circ S^{-1} = S\circ\left(\frac{\partial\tilde{\phi}}{\partial t} - [\tilde{\phi}, \alpha]\right)\circ S^{-1}\\ &= -[\sqrt{-1}\,\Lambda_\omega(F_A + [\phi, \phi^*]), \phi].\end{aligned}$$

So, we have

$$\begin{cases}\dfrac{\partial A}{\partial t} = -D_A^*F_A - \sqrt{-1}(\partial_A\Lambda_\omega - \overline{\partial}_A\Lambda_\omega)[\phi, \phi^*],\\ \dfrac{\partial\phi}{\partial t} = -[\sqrt{-1}\,\Lambda_\omega(F_A + [\phi, \phi^*]), \phi],\end{cases}$$

i.e. $(A(t), \phi(t))$ is a smooth solution of the gradient flow with initial Higgs pair (A_0, ϕ_0). On the other hand, it is obvious that $\phi(t)\wedge\phi(t) = 0$, and $\overline{\partial}_{A(t)}\phi(t) = 0$.

To prove the uniqueness, we suppose that $(A(t), \phi(t)) = S(t)\sigma(t)(A_0, \phi_0)$ is the smooth solution constructed above, and $(\tilde{A}(t), \tilde{\phi}(t))$ is another smooth solution of the gradient flow (1.3) with initial Higgs pair (A_0, ϕ_0). Let $g(t) \in \mathcal{G}^{\mathbb{C}}$ satisfy

$$\frac{\partial g}{\partial t}g^{-1} = (-\sqrt{-1}(F_{\tilde{A}} + [\tilde{\phi}, \tilde{\phi}^*]) + \lambda\,\mathrm{Id}_E), \qquad g(0) = \mathrm{Id}_E\,.$$

It is easy to check that $\frac{\partial}{\partial t}(g^{-1}(\tilde{A}, \tilde{\phi})) = 0$, so $(\tilde{A}(t), \tilde{\phi}(t)) = g(t)(A_0, \phi_0)$. Noting that solutions to Simpson's heat flow are unique ([Si1]), we have $g^*g = h$. Let $\tilde{S} = g\circ\sigma^{-1}$. Then

$$\tilde{S}^*\tilde{S} = \mathrm{Id}_E, \qquad \tilde{S}(0) = \mathrm{Id}_E,$$

and

$$\begin{aligned}\frac{\partial \tilde{S}}{\partial t} &= \frac{1}{2}\left(\frac{\partial \tilde{S}}{\partial t} - \tilde{S}\frac{\partial \tilde{S}^*}{\partial t}\tilde{S}\right)\\ &= \frac{1}{2}\left(\frac{\partial g}{\partial t}\sigma^{-1} - g\sigma^{-1}\frac{\partial \sigma}{\partial t}\sigma^{-1} + \tilde{S}(\sigma^*)^{-1}\frac{\partial \sigma^*}{\partial t}(\sigma^*)^{-1}g^*\tilde{S} - \tilde{S}(\sigma^*)^{-1}\frac{\partial g^*}{\partial t}\tilde{S}\right)\\ &= \frac{1}{2}\left(\frac{\partial g}{\partial t}g^{-1}\tilde{S} - \tilde{S}\frac{\partial \sigma}{\partial t}\sigma^{-1} + \tilde{S}(\sigma^*)^{-1}\frac{\partial \sigma^*}{\partial t} - (g^*)^{-1}\frac{\partial g^*}{\partial t}\tilde{S}\right)\\ &= \frac{1}{2}\tilde{S}\left(-\frac{\partial \sigma}{\partial t}\sigma^{-1} + (\sigma^*)^{-1}\frac{\partial \sigma^*}{\partial t}\right).\end{aligned}$$

By linear ODE theory, we have $\tilde{S} = S$. Then

$$(\tilde{A}(t), \tilde{\phi}(t)) = g(t)(A_0, \phi_0) = S(t)\sigma(t)(A_0, \phi_0) = (A(t), \phi(t)).$$

So we have proved the following theorem:

Theorem 2.1. *Let (E, H_0) be a Hermitian vector bundle over a closed Kähler manifold (M, ω). Given any Higgs pair (A_0, ϕ_0), the gradient flow (1.3) has a unique solution in the complex gauge orbit of (A_0, ϕ_0) with initial value (A_0, ϕ_0).*

Let H_1 and H_2 be two metrics on the bundle E. The distance $\sigma(H_1, H_2)$ is defined by

$$\sigma(H_1, H_2) = \mathrm{Tr}(H_1^{-1}H_2 + H_2^{-1}H_1 - 2\mathrm{Id}_E).$$

Proposition 2.1′. *Let $H(t)$ be a smooth solution of the heat flow (2.1). Then for any $T > 0$, $\Lambda_\omega(F_{A_H} + [\phi_0, \phi_0^{*H}])$ at time T depends continuously on the initial conditions in L^2 norm.*

Proof. Let $H_1(t)$ and $H_2(t)$ be two smooth solutions of the heat flow (2.1), and let $h_1(t) = H_1^{-1}(0)H_1(t)$, $h_2(t) = H_1^{-1}(0)H_2(t)$, $h_2(t) - h_1(t) = \overline{h}(t)$, $S(t) = h_1^{-1}(t)h_2(t) = H_1^{-1}(t)H_2(t)$. The proof of Prop. 6.3 in [Si1] shows that $\sup \sigma(H_1(t), H_2(t))$ is decreasing with time. Thus we have

$$\begin{aligned}&|S(t) - \mathrm{Id}_E|^2 + |S^{-1}(t) - \mathrm{Id}_E|^2 \le C(\sqrt{5\sigma/4} + \sigma/2)^2,\\ &|\overline{h}(t)|^2 \le C|h_1(t)|^2(\sqrt{5\sigma/4} + \sigma/2)^2,\\ &|h_2(t)|^2 \le C|h_1(t)|^2[(\sqrt{5\sigma/4} + \sigma/2)^2 + 1],\\ &|h_2^{-1}(t)|^2 \le C|h_1^{-1}(t)|^2[(\sqrt{5\sigma/4} + \sigma/2)^2 + 1],\end{aligned}$$

where C is a constant depending only on $\mathrm{rank}(E)$, $\sigma = \sup \sigma(H_1(0), H_2(0))$ and we may assume $\sigma < 1$. From equation (2.2), we have

$$\begin{aligned}\frac{\partial \overline{h}}{\partial t} = &-2\sqrt{-1}\,\Lambda_\omega\{\overline{\partial}_{A_0}\partial_{A_0}\overline{h} + \overline{h}F_{A_0} + \phi_0^{*H_0} \circ \overline{h} \circ \phi_0 + \overline{h} \circ \phi_0 h_2^{-1} \circ \phi_0^{*H_0} \circ h_2\\ &- h_1 \circ \phi_0 \circ h_1^{-1} \circ \overline{h} \circ h_2^{-1} \circ \phi_0^{*H_0} \circ h_2 + h_1 \circ \phi_0 \circ h_1^{-1} \circ \phi_0^{*H_0} \circ \overline{h}\} + 2\lambda\overline{h},\end{aligned}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\overline{h}(t)|^2 \leq C(t)|\overline{h}(t)|^2,$$

where A_0 is the Chern connection determined by $H_1(0)$, and $C(t)$ depends only on $h_1(t)$, A_0 and ϕ_0. Using the Kähler identity, we have

$$\begin{aligned}\sqrt{-1}\,\partial_{A_0}\Lambda_\omega(\overline{\partial}_{A_0}\partial_{A_0}\overline{h}) &= \overline{\partial}^*_{A_0}(\overline{\partial}_{A_0}\partial_{A_0}\overline{h}) + \sqrt{-1}\,\Lambda_\omega(\partial_{A_0}\overline{\partial}_{A_0}\partial_{A_0}\overline{h}) \\ &= \overline{\partial}^*_{A_0}(\overline{\partial}_{A_0}\partial_{A_0}\overline{h}) + \sqrt{-1}\,\Lambda_\omega(F_{A_0}\wedge\partial_{A_0}\overline{h} - \partial_{A_0}\overline{h}\wedge F_{A_0}),\end{aligned}$$

and

$$-\sqrt{-1}\,\overline{\partial}_{A_0}\Lambda_\omega(\overline{\partial}_{A_0}\partial_{A_0}\overline{h}) = \partial^*_{A_0}(\overline{\partial}_{A_0}\partial_{A_0}\overline{h}).$$

From the above inequalities, we conclude that

$$\frac{d}{dt}\|D_{A_0}\overline{h}(t)\|^2_{L^2} \leq C(t)(\|\overline{h}\|^2_{L^2} + \|D_{A_0}\overline{h}\|^2_{L^2} + \|\overline{\partial}_{A_0}\partial_{A_0}\overline{h}\|^2_{L^2}),$$

where $C(t)$ depends only on $h_1(t)$, A_0 and ϕ_0. By direct calculation, we have

$$\begin{aligned}&(-2\sqrt{-1}\,\overline{\partial}_{A_0}\partial_{A_0}\Lambda_\omega(\overline{\partial}_{A_0}\partial_{A_0}\overline{h}), \overline{\partial}_{A_0}\partial_{A_0}\overline{h})_{L^2} \\ &\quad= (-2\sqrt{-1}\,\overline{\partial}_{A_0}\Lambda_\omega(\partial_{A_0}\overline{\partial}_{A_0}\partial_{A_0}\overline{h}) - 2\overline{\partial}_{A_0}\overline{\partial}^*_{A_0}(\overline{\partial}_{A_0}\partial_{A_0}\overline{h}), \overline{\partial}_{A_0}\partial_{A_0}\overline{h})_{L^2} \\ &\quad= -2\|\overline{\partial}^*_{A_0}(\overline{\partial}_{A_0}\partial_{A_0}\overline{h})\|^2_{L^2} + (2\partial^*_{A_0}(\partial_{A_0}\overline{\partial}_{A_0}\partial_{A_0}\overline{h}), \overline{\partial}_{A_0}\partial_{A_0}\overline{h})_{L^2} \\ &\qquad - 2(\sqrt{-1}\,\Lambda_\omega(\overline{\partial}_{A_0}\partial_{A_0}\overline{\partial}_{A_0}\partial_{A_0}\overline{h}), \overline{\partial}_{A_0}\partial_{A_0}\overline{h})_{L^2} \\ &\quad= -2\|\overline{\partial}^*_{A_0}(\overline{\partial}_{A_0}\partial_{A_0}\overline{h})\|^2_{L^2} + 2\|F_{A_0}\wedge\partial_{A_0}\overline{h} - \partial_{A_0}\overline{h}\wedge F_{A_0}\|^2_{L^2} \\ &\qquad - 2(\sqrt{-1}\,\Lambda_\omega(F_{A_0}\wedge\overline{\partial}_{A_0}\partial_{A_0}\overline{h} - \overline{\partial}_{A_0}\partial_{A_0}\overline{h}\wedge F_{A_0}), \overline{\partial}_{A_0}\partial_{A_0}\overline{h})_{L^2}.\end{aligned}$$

From the above formula, we conclude that

$$\frac{d}{dt}\|\overline{\partial}_{A_0}\partial_{A_0}\overline{h}\|^2_{L^2} \leq C(t)((\|\overline{h}\|^2_{L^2} + \|D_{A_0}\overline{h}\|^2_{L^2} + \|\overline{\partial}_{A_0}\partial_{A_0}\overline{h}\|^2_{L^2})).$$

So, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\overline{h}\|^2_{L^2} + \|D_{A_0}\overline{h}\|^2_{L^2} + \|\overline{\partial}_{A_0}\partial_{A_0}\overline{h}\|^2_{L^2}) \\ &\qquad\qquad \leq C(t)(\|\overline{h}\|^2_{L^2} + \|D_{A_0}\overline{h}\|^2_{L^2} + \|\overline{\partial}_{A_0}\partial_{A_0}\overline{h}\|^2_{L^2}),\end{aligned}$$

where $C(t)$ depends only on $h_1(t)$, A_0 and ϕ_0. Then

$$\begin{aligned}&\|\Lambda_\omega(F_{A_{H_1}} + [\phi_0, \phi_0^{*H_1}]) - \Lambda_\omega(F_{A_{H_2}} + [\phi_0, \phi_0^{*H_2}])\|^2_{L^2}(T) \\ &\quad\leq B(T)(\|\overline{h}\|^2_{L^2} + \|D_{A_0}\overline{h}\|^2_{L^2} + \|\overline{\partial}_{A_0}\partial_{A_0}\overline{h}\|^2_{L^2})(T) \\ &\quad\leq B(T)\exp\left\{\int_0^T C(t)\,dt\right\}(\|\overline{h}\|^2_{L^2} + \|D_{A_0}\overline{h}\|^2_{L^2} + \|\overline{\partial}_{A_0}\partial_{A_0}\overline{h}\|^2_{L^2})(0),\end{aligned}$$

where $B(T)$ depends only on $h_1(T)$, A_0 and ϕ_0. This completes the proof. □

2.2. Basic estimates

Let $(A(t), \phi(t))$ be a regular solution of the gradient flow (1.3) in the space $\mathcal{B}$ of Higgs pairs, and let f be a real smooth function on M. Then we have

$$\begin{aligned}
&\frac{d}{dt}\int_M f^2(|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2)\,dV_g\\
&\quad=\int_M f^2\left(\frac{d}{dt}|F_A+[\phi,\phi^*]|^2+\frac{d}{dt}|\partial_A\phi|^2+\frac{d}{dt}|\overline{\partial}_A\phi^*|^2\right)dV_g\\
&\quad=2\,\mathrm{Re}\int_M f^2\left\{\left\langle F_A+[\phi,\phi^*],D_A\frac{dA}{dt}+\left[\frac{d\phi}{dt},\phi^*\right]+\left[\phi,\left(\frac{d\phi}{dt}\right)^*\right]\right\rangle\right.\\
&\qquad+\left\langle\partial_A\phi,\partial_A\frac{d\phi}{dt}+\frac{dA}{dt}\wedge\phi+\phi\wedge\frac{dA}{dt}\right\rangle\\
&\qquad\left.+\left\langle\overline{\partial}_A\phi^*,\overline{\partial}_A\frac{d\phi^*}{dt}+\frac{dA}{dt}\wedge\phi^*+\phi^*\wedge\frac{dA}{dt}\right\rangle\right\}dV_g.
\end{aligned}$$

On the other hand, one can check that

$$\begin{aligned}
\int_M f^2\left\langle\partial_A\phi,\partial_A\frac{d\phi}{dt}\right\rangle dV_g&=\int_M\left\langle\partial_A^*(f^2\partial_A\phi),\frac{d\phi}{dt}\right\rangle dV_g\\
&=\int_M\left\langle f^2\sqrt{-1}\,\Lambda_\omega(\overline{\partial}_A\partial_A\phi)+\sqrt{-1}\,\Lambda_\omega(\overline{\partial}(f^2)\wedge\partial_A\phi),\frac{d\phi}{dt}\right\rangle dV_g\\
&=\int_M\left\langle f^2\sqrt{-1}\,\Lambda_\omega(F_A\wedge\phi-\phi\wedge F_A)+\sqrt{-1}\,\Lambda_\omega(\overline{\partial}(f^2)\wedge\partial_A\phi),\frac{d\phi}{dt}\right\rangle dV_g;
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Re}\int_M\left\langle f^2\overline{\partial}_A\phi^*,\overline{\partial}_A\frac{d\phi^*}{dt}\right\rangle dV_g\\
&\qquad=\mathrm{Re}\int_M\left\langle f^2\sqrt{-1}\,\Lambda_\omega(F_A\wedge\phi-\phi\wedge F_A)+\sqrt{-1}\,\Lambda_\omega(\overline{\partial}(f^2)\wedge\partial_A\phi),\frac{d\phi}{dt}\right\rangle dV_g;
\end{aligned}$$

$$\int_M f^2\left\langle\partial_A\phi,\frac{dA}{dt}\wedge\phi+\phi\wedge\frac{dA}{dt}\right\rangle dV_g=\int_M f^2\left\langle\sqrt{-1}\,\Lambda_\omega(\partial_A[\phi,\phi^*]),\frac{dA}{dt}\right\rangle dV_g;$$

$$\int_M f^2\left\langle\overline{\partial}_A\phi^*,\frac{dA}{dt}\wedge\phi^*+\phi^*\wedge\frac{dA}{dt}\right\rangle dV_g=-\int_M f^2\left\langle\sqrt{-1}\Lambda_\omega(\overline{\partial}_A[\phi,\phi^*]),\frac{dA}{dt}\right\rangle dV_g;$$

and

$$\begin{aligned}
&\mathrm{Re}\int_M f^2\left\langle F_A+[\phi,\phi^*],\left[\frac{d\phi}{dt},\phi^*\right]+\left[\phi,\left(\frac{d\phi}{dt}\right)^*\right]\right\rangle dV_g\\
&=2\,\mathrm{Re}\int_M f^2\left\langle F_A+[\phi,\phi^*],\left[\frac{d\phi}{dt},\phi^*\right]\right\rangle dV_g\\
&=\mathrm{Re}\int_M f^2\left\langle\sqrt{-1}\,\Lambda_\omega\{\phi\wedge(F_A+[\phi,\phi^*])\}-\phi\circ\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]),\frac{d\phi}{dt}\right\rangle dV_g\\
&\quad-2\,\mathrm{Re}\int_M f^2\left\langle\sqrt{-1}\,\Lambda_\omega\{(F_A+[\phi,\phi^*])\wedge\phi\}-\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*])\circ\phi,\frac{d\phi}{dt}\right\rangle dV_g
\end{aligned}$$

$$
\begin{aligned}
&= 2\,\mathrm{Re}\int_M f^2\Big\langle \sqrt{-1}\,\Lambda_\omega\{\phi\wedge F_A\} - \phi\circ\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]),\, \frac{d\phi}{dt}\Big\rangle dV_g\\
&\quad - 2\,\mathrm{Re}\int_M f^2\Big\langle \sqrt{-1}\,\Lambda_\omega\{F_A\wedge\phi\} - \sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*])\circ\phi,\, \frac{d\phi}{dt}\Big\rangle dV_g;
\end{aligned}
$$

where we have used $\langle F_A+[\phi,\phi^*],[\phi,(d\phi/dt)^*]\rangle = \overline{\langle F_A+[\phi,\phi^*],[d\phi/dt,\phi^*]\rangle}$ and $\phi\wedge\phi=0$. Combining the above identities, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_M f^2(|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2)\,dV_g\\
&\quad= 2\,\mathrm{Re}\int_M f^2\Big\langle D_A^*F_A+\sqrt{-1}\,(\partial_A\Lambda_\omega-\overline{\partial}_A\Lambda_\omega)[\phi,\phi^*],\,\frac{dA}{dt}\Big\rangle dV_g\\
&\qquad + 4\,\mathrm{Re}\int_M f^2\Big\langle [\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]),\phi],\,\frac{d\phi}{dt}\Big\rangle dV_g\\
&\qquad + 2\,\mathrm{Re}\int_M \Big\langle \sqrt{-1}\,\Lambda_\omega((\overline{\partial}-\partial)(f^2)\wedge(F_A+[\phi,\phi^*])),\,\frac{dA}{dt}\Big\rangle dV_g\\
&\qquad - 2\,\mathrm{Re}\int_M \Big\langle (\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]))(\overline{\partial}-\partial)(f^2),\,\frac{dA}{dt}\Big\rangle dV_g\\
&\qquad + 4\,\mathrm{Re}\int_M \Big\langle \sqrt{-1}\,\Lambda_\omega(\overline{\partial}(f^2)\wedge\partial_A\phi,\,\frac{d\phi}{dt}\Big\rangle dV_g\\
&\quad= -2\int_M f^2\left(\left|\frac{\partial A}{\partial t}\right|^2+2\left|\frac{\partial\phi}{\partial t}\right|^2\right)dV_g + 4\,\mathrm{Re}\int_M\Big\langle \sqrt{-1}\,\Lambda_\omega(\overline{\partial}(f^2)\wedge\partial_A\phi),\,\frac{d\phi}{dt}\Big\rangle dV_g\\
&\qquad + 2\,\mathrm{Re}\int_M \Big\langle \sqrt{-1}\,\Lambda_\omega((\overline{\partial}-\partial)(f^2)\wedge(F_A+[\phi,\phi^*])),\,\frac{dA}{dt}\Big\rangle dV_g\\
&\qquad - 2\,\mathrm{Re}\int_M \Big\langle (\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]))(\overline{\partial}-\partial)(f^2),\,\frac{dA}{dt}\Big\rangle dV_g. \qquad (2.3)
\end{aligned}
$$

Setting $f\equiv 1$ on M, and integrating the above identity on $[0,t]$, we obtain the following lemma:

Lemma 2.2. *Let (A,ϕ) be a solution of the heat flow (1.3) with initial Higgs pair (A_0,ϕ_0). Then*

$$
\mathrm{YMH}(t)+2\int_0^t\int_M\left(\left|\frac{\partial A}{\partial t}\right|^2+2\left|\frac{\partial\phi}{\partial t}\right|^2\right)=\mathrm{YMH}(0). \qquad (2.4)
$$

Let f be a cut-off function with support inside $B_{2R}(x_0)$ and $f\equiv 1$ on $B_R(x_0)$ such that $0\le f\le 1$ and $|df|\le 2R^{-1}$. Set

$$
e(A,\phi)=|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2.
$$

From the identity (2.3), we have

$$\left|\frac{d}{dt}\int_M f^2 e(A,\phi)\,dV_g + 2\int_M f^2\left(\left|\frac{\partial A}{\partial t}\right|^2 + 2\left|\frac{\partial \phi}{\partial t}\right|^2\right)dV_g\right| \\ \leq \frac{C(n)}{R}\left(\int_M f^2 e(A,\phi)\,dV_g\right)^{1/2}\left(\int_M\left(\left|\frac{\partial A}{\partial t}\right|^2 + 2\left|\frac{\partial \phi}{\partial t}\right|^2\right)dV_g\right)^{1/2},$$

where $C(n)$ is a positive number depending only on the dimension of M. Integrating the above inequality and using Lemma 2.2, we obtain the following local energy estimate.

Lemma 2.2′ (local energy estimates). *Let (A,ϕ) be a solution of the heat flow (1.3) with initial Higgs pair (A_0,ϕ_0). For any x_0 with $B_{2R}(x_0) \subset M$ and for any two finite numbers s,τ, we have*

$$\int_{B_R(x_0)} e(A,\phi)(\cdot,s)\,dV_g \\ \leq \int_{B_{2R}(x_0)} e(A_1,A_2,\phi)(\cdot,\tau)\,dV_g + 2\int_{\min\{s,\tau\}}^{\max\{s,\tau\}}\int_M\left(\left|\frac{\partial A}{\partial t}\right|^2 + 2\left|\frac{\partial \phi}{\partial t}\right|^2\right)dV_g\,dt \\ + C\left(\frac{|s-\tau|}{R^2}\,\mathrm{YMH}(A_0,,\phi_0)\int_{\min\{s,\tau\}}^{\max\{s,\tau\}}\int_M\left(\left|\frac{\partial A}{\partial t}\right|^2 + 2\left|\frac{\partial \phi}{\partial t}\right|^2\right)dV_g\,dt\right)^{1/2}.$$

By choosing normal coordinates at the point under consideration, we have

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)|\phi|^2 &= 2|\nabla_A\phi|^2 + 2\,\mathrm{Re}\left\langle \nabla_{\overline{\partial}_\alpha}\nabla_{\partial_\alpha}\phi - \frac{\partial}{\partial t}\phi,\phi\right\rangle \\ &= 2|\nabla_A\phi|^2 + 2\,\mathrm{Re}\left\langle (\nabla_{\overline{\partial}_\alpha}\nabla_{\partial_\alpha}\phi_\beta)dz^\beta + \phi_\beta(\nabla_{\overline{\partial}_\alpha}\nabla_{\partial_\alpha}dz^\beta) - \frac{\partial}{\partial t}\phi,\phi\right\rangle \\ &= 2\,\mathrm{Re}\left\langle -[\sqrt{-1}\,\Lambda_\omega F_A,\phi] - \frac{\partial}{\partial t}\phi,\phi\right\rangle - 2\,\mathrm{Re}\,\langle\phi\sharp R,\phi\rangle + 2|\nabla_A\phi|^2 \\ &= 2\,\mathrm{Re}\,\langle[\sqrt{-1}\,\Lambda_\omega[\phi,\phi^*],\phi],\phi\rangle - 2\,\mathrm{Re}\,\langle\phi\sharp R,\phi\rangle + 2|\nabla_A\phi|^2 \\ &= 2|\Lambda_\omega[\phi,\phi^*]|^2 - 2\,\mathrm{Re}\,\langle\phi\sharp R,\phi\rangle + 2|\nabla_A\phi|^2. \end{aligned} \tag{2.5}$$

Since $\phi\wedge\phi = \phi_\alpha dz^\alpha \wedge \phi_\beta dz^\beta = 0$, we have $\phi_\alpha\phi_\beta = \phi_\beta\phi_\alpha$. Then

$$\begin{aligned}|[\phi,\phi^*]|^2 &= \sum_{\alpha,\beta}|[\phi_\alpha,\phi_\beta^*]|^\alpha = \mathrm{Tr}\,(\phi_\alpha\phi_\beta^* - \phi_\beta^*\phi_\alpha)(\phi_\beta\phi_\alpha^* - \phi_\alpha^*\phi_\beta) \\ &= \mathrm{Tr}\{\phi_\alpha\phi_\beta^*\phi_\beta\phi_\alpha^* + \phi_\beta^*\phi_\alpha\phi_\alpha^*\phi_\beta - \phi_\alpha\phi_\beta^*\phi_\alpha^*\phi_\beta - \phi_\beta^*\phi_\alpha\phi_\beta\phi_\alpha^*\} \\ &= \mathrm{Tr}\,(\phi_\alpha\phi_\alpha^* - \phi_\alpha^*\phi_\alpha)(\phi_\beta\phi_\beta^* - \phi_\beta^*\phi_\beta) = |\sqrt{-1}\,\Lambda_\omega[\phi,\phi^*]|^2.\end{aligned}$$

Since ϕ is in the complex group orbit of ϕ_0, we have a uniform bound for the eigenvalues of ϕ_α. Based on Lemma 2.7 in [Si2], we obtain

$$|[\phi_\alpha,\phi_\alpha^*]|^2 \geq C_1(|\phi_\alpha|^2+1)^2 - C_2(|\phi_\alpha|^2+1),$$

where C_1 and C_2 are positive constants depending only on ϕ_0. Then

$$|\sqrt{-1}\,\Lambda_\omega[\phi,\phi^*]|^2 \geq \sum_\alpha |[\phi_\alpha,\phi_\alpha^*]|^2 \geq \frac{C_1}{n}(|\phi|^2+1)^2 - C_2(|\phi|^2+1),$$

and

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\phi|^2 \geq 2|\nabla_A\phi|^2 + C_3(|\phi|^2+1)^2 - C_4(|\phi|^2+1),$$

where C_3 and C_4 are positive constants depending only on ϕ_0 and the geometry of (M,ω). From the above inequality and using the maximum principle in parabolic theory, we can deduce the uniform C^0 bound of ϕ.

Lemma 2.3. *Let (A,ϕ) be a solution of the heat flow (1.3) with initial Higgs pair (A_0,ϕ_0). Then*

$$|\phi(x,t)|^2 \leq \max\{\sup_M |\phi_0|^2, C_4/C_3\} \quad \forall (x,t)\in M\times[0,\infty).$$

For simplicity, we set

$$\theta = \Lambda_\omega(F_A + [\phi,\phi^*]).$$

Direct calculation shows that

$$\begin{aligned}\frac{\partial}{\partial t}\theta &= \Lambda_\omega\left(D_A\left(\frac{\partial A}{\partial t}\right) + \left[\frac{\partial\phi}{\partial t},\phi^*\right] + \left[\phi,\left(\frac{\partial\phi}{\partial t}\right)^*\right]\right)\\ &= \Delta_A\theta + \sqrt{-1}\,\Lambda_\omega([\phi,[\theta,\phi^*]] - [[\theta,\phi],\phi^*]).\end{aligned} \tag{2.6}$$

Here we have used the fact that $\theta^* = -\theta$, and we set $\Delta_A = -D_A^* D_A$.

Let $\mathfrak{u}(r)$ denote the Lie algebra of the unitary group $U(r)$. Given a smooth convex ad-invariant function φ, we can define a gauge-invariant functional Φ on the space of Higgs pairs as in [AB]:

$$\Phi(A,\phi) = \int_M \varphi(\Lambda_\omega(F_A + [\phi,\phi^*])).$$

Using the equality (2.6), we have

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(\theta) &= \varphi'_\theta(\Delta_A\theta + \sqrt{-1}\,\Lambda_\omega([\phi,[\theta,\phi^*]] - [[\theta,\phi],\phi^*]))\\ &= \langle\varphi'(\theta), \Delta_A\theta + \sqrt{-1}\,\Lambda_\omega([\phi,[\theta,\phi^*]] - [[\theta,\phi],\phi^*])\rangle,\end{aligned}$$

where $\varphi' : \mathfrak{u} \to \mathfrak{u}$ is the derivative of φ, and φ' is an equivariant map relative to the adjoint action of $U(r)$. It is easy to check that

$$\begin{aligned}\langle\varphi'(\theta), \sqrt{-1}\,\Lambda_\omega([\phi,[\theta,\phi^*]] - [[\theta,\phi],\phi^*])\rangle &= -\langle[\phi-\phi^*,\varphi'(\theta)],[\phi-\phi^*,\theta]\rangle\\ &= -\varphi''(\theta)([\phi-\phi^*,\theta],[\phi-\phi^*,\theta]).\end{aligned}$$

So, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\varphi(\theta) = -\varphi''(\theta)([\phi-\phi^*,\theta],[\phi-\phi^*,\theta]) - \varphi''(\theta)(\nabla_A\theta,\nabla_A\theta) \leq 0. \tag{2.7}$$

As a special case, we have

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\theta|^2 \geq 0. \tag{2.8}$$

From the above inequality, one can derive a uniform bound of $|\theta|$.

By direct calculation, we have

$$\begin{aligned}\frac{\partial}{\partial t}(|D_A\theta|^2 + 2|[\theta,\phi]|^2) &= 2\,\mathrm{Re}\left\langle \frac{\partial}{\partial t}(D_A\theta),\, D_A\theta\right\rangle + 4\,\mathrm{Re}\left\langle \frac{\partial}{\partial t}[\theta,\phi],\, [\theta,\phi]\right\rangle \\ &= 2\,\mathrm{Re}\left\langle \left[\frac{\partial A}{\partial t}, \theta\right] + D_A\left(\frac{\partial}{\partial t}\theta\right),\, D_A\theta\right\rangle + 4\,\mathrm{Re}\left\langle \left[\frac{\partial}{\partial t}\theta, \phi\right] + \left[\theta, \frac{\partial}{\partial t}\phi\right],\, [\theta,\phi]\right\rangle,\end{aligned}$$

and

$$\begin{aligned}\left\langle \left[\frac{\partial}{\partial t}\theta, \phi\right], [\theta,\phi]\right\rangle &= \langle [-D_A^* D_A\theta, \phi], [\theta,\phi]\rangle \\ &\quad + \langle [\Lambda_\omega[\phi, ([-\sqrt{-1}\,\theta,\phi])^*], \phi], [\theta,\phi]\rangle + \langle [\Lambda_\omega[[-\sqrt{-1}\,\theta,\phi],\phi^*],\phi],[\theta,\phi]\rangle.\end{aligned}$$

Setting $l = [\theta, \phi]$, we have

$$\begin{aligned}\langle [\Lambda_\omega[-\sqrt{-1}\,l, \phi^*], \phi], l\rangle &= -\langle [l_\alpha\phi_\alpha^* - \phi_\alpha^* l_\alpha, \phi], l\rangle \\ &= -\mathrm{Tr}\{((l_\alpha\phi_\alpha^* - \phi_\alpha^* l_\alpha)\phi_\beta - \phi_\beta(l_\alpha\phi_\alpha^* - \phi_\alpha^* l_\alpha))l_\beta^*\} \\ &= -\mathrm{Tr}\,(l_\alpha\phi_\alpha^* - \phi_\alpha^* l_\alpha)(l_\beta\phi_\beta^* - \phi_\beta^* l_\beta)^* = -|\sqrt{-1}\,\Lambda_\omega[l,\phi^*]|^2,\end{aligned}$$

and

$$\begin{aligned}|[\theta,\phi]|^2 &= \langle l, \theta\phi - \phi\theta\rangle = \mathrm{Tr}(l_\alpha(\theta\phi_\alpha)^* - l_\alpha(\phi_\alpha\theta)^*) \\ &= \mathrm{Tr}((l_\alpha\phi_\alpha^* - \phi_\alpha^* l_\alpha)\theta^*) = \langle \sqrt{-1}\,\Lambda_\omega[l,\phi^*], \theta\rangle \leq |\theta|\,|\sqrt{-1}\,\Lambda_\omega[l,\phi^*]|.\end{aligned}$$

Combining the above inequalities, and setting

$$I(t) = \int_M \{|D_A\theta|^2 + 2|[\theta,\phi]|^2\},$$

we have

$$\begin{aligned}\frac{d}{dt}I &= \int_M \frac{\partial}{\partial t}(|D_A\theta|^2 + 2|[\theta,\phi]|^2) \\ &\leq 16\int_M \{|\theta|\,|D_A\theta|^2 + (|\theta| + |\phi|^2)|[\theta,\phi]|^2 + |\phi|\,|[\theta,\phi]|\,|D_A^* D_A\theta|\} \\ &\quad - 2\int_M |D_A^* D_A\theta|^2 - 4\int_M |\sqrt{-1}\,\Lambda_\omega[l,\phi^*]|^2 \\ &\leq C_5 I - C_6 I^2,\end{aligned}$$

where C_5 and C_6 are positive constants. From the formula (2.4), we have $\int_0^\infty I(t)\,dt < \infty$. Using the above inequality and arguing as in [DK, Prop. 6.2.14], we can prove that

$$I(t) \to 0 \quad (t \to \infty). \tag{2.9}$$

Noting that the $\mathrm{End}(E)$-valued $(1,0)$-form ϕ can also be seen as a section of the bundle $\mathrm{End}(E) \otimes \bigwedge^{1,0}(M)$, and denoting the induced connection on the bundle $\mathrm{End}(E) \otimes \bigwedge^{1,0}(M)$ also by ∇_A for simplicity, we have

$$\begin{aligned}\int_M \langle \nabla_A\phi, \nabla_A\phi\rangle &= \int_M \langle \nabla_A^*\nabla_A\phi, \phi\rangle \\ &= \int_M \langle \sqrt{-1}\,\Lambda_\omega F_A \circ \phi - \phi \circ (\sqrt{-1}\,\lambda_\omega F_A \otimes \mathrm{Id}_{T^{1,0}M} + \mathrm{Id}_E \otimes \mathrm{Ric}_M), \phi\rangle,\end{aligned}$$

where Ric_M denotes the Ricci transformation of the Kähler manifold (M, ω). On the other hand, one can check that

$$\begin{aligned}\int_M \{\langle F_A, [\phi, \phi^*]\rangle + \langle [\phi, \phi^*], F_A\rangle + 2|\partial_A\phi|^2\} &= 2\,\mathrm{Re}\int_M \{\langle F_A, [\phi, \phi^*]\rangle + |\partial_A\phi|^2\} \\ &= 2\,\mathrm{Re}\int_M \langle [\sqrt{-1}\,\Lambda_\omega F_A, \phi], \phi\rangle.\end{aligned}$$

Then

$$\mathrm{YMH}(t) = 2\int_M |\nabla_A\phi|^2 + 2\int_M \langle \phi \sharp \mathrm{Ric}_M, \phi\rangle + \int_M \{|F_A|^2 + |[\phi, \phi^*]|^2\}. \tag{2.10}$$

Remark. From the above identity and Lemmas 2.2 and 2.3, we see that $\int_M |\nabla_A\phi|^2$ is also bounded uniformly along the heat flow (1.3).

Lemma 2.4. *Let (A, ϕ) be a smooth solution of the heat flow (1.3). Then*

$$\begin{aligned}&\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla_A\phi|^2 - 2|\nabla_A\nabla_A\phi|^2 \\ &\qquad \geq -C_7(|F_A| + |\mathrm{Rm}| + |\mathrm{Ric}_M| + |\phi|^2)|\nabla_A\phi|^2 - C_7|\phi|\,|\nabla\,\mathrm{Ric}|\,|\nabla_A\phi|, \end{aligned}\tag{2.11}$$

where C_7 is a constant depending only on the dimension m and Rm *is the Riemannian curvature of M.*

Proof. Choosing normal complex coordinates at the point considered, we have

$$\begin{aligned}\Delta|\nabla_A\phi|^2 &= 2\partial_\alpha\overline{\partial}_\alpha|\nabla_A\phi|^2 \\ &= 2|\nabla_A\nabla_A\phi|^2 + 2\,\mathrm{Re}\,\langle (\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}} + \nabla_{A,\overline{\alpha}}\nabla_{A,\alpha})\nabla_A\phi, \nabla_A\phi\rangle \\ &= 2|\nabla_A\nabla_A\phi|^2 + 2\,\mathrm{Re}\,\langle (\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}\phi)dz^\gamma, \nabla_A\phi\rangle \\ &\quad + 2\,\mathrm{Re}\,\langle (\nabla_{A,\overline{\alpha}}\nabla_{A,\alpha}\nabla_{A,\gamma}\phi)dz^\gamma + (\nabla_{A,\gamma})\phi\nabla_{A,\overline{\alpha}}\nabla_{A,\alpha}dz^\gamma, \nabla_A\phi\rangle.\end{aligned}$$

Using the Bianchi identity, we obtain

$$\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}\phi = \nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}((\nabla_{A,\gamma}\phi_\beta)dz^\beta + \phi_\beta\nabla_\gamma dz^\beta)$$

$$
\begin{aligned}
&= (\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}\phi_\beta)dz^\beta + (\nabla_{A,\alpha}\phi_\beta)\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}dz^\beta + \phi_\beta\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}dz^\beta\\
&= ([\nabla_{A,\gamma}F_{A,\overline{\alpha}\alpha},\phi_\beta] - [F_{A,\gamma\overline{\alpha}},\nabla_{A,\alpha}\phi_\beta])dz^\beta\\
&\quad + (\nabla_{A,\alpha}\phi_\beta)\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}dz^\beta + \phi_\beta\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}dz^\beta\\
&= -[\partial_A(\sqrt{-1}\,\Lambda_\omega F_A),\phi] - ([F_{A,\gamma\overline{\alpha}},\nabla_{A,\alpha}\phi_\beta])dz^\beta\\
&\quad + (\nabla_{A,\alpha}\phi_\beta)\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}dz^\beta + \phi_\beta\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}dz^\beta
\end{aligned}
$$

and

$$
\begin{aligned}
&\nabla_{A,\overline{\alpha}}\nabla_{A,\alpha}\nabla_{A,\gamma}\phi\\
&\quad = -[\partial_A(\sqrt{-1}\,\Lambda_\omega F_A),\phi] - ([F_{A,\gamma\overline{\alpha}},\nabla_{A,\alpha}\phi_\beta])dz^\beta - [\sqrt{-1}\,\Lambda_\omega F_A,\nabla_{A,\gamma}\phi]\\
&\qquad + (\nabla_{A,\gamma}\phi_\beta)\nabla_{A,\overline{\alpha}}\nabla_{A,\alpha}dz^\beta + (\nabla_{A,\alpha}\phi_\beta)\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}dz^\beta + \phi_\beta\nabla_{A,\overline{\alpha}}\nabla_{A,\alpha}\nabla_{A,\gamma}dz^\beta.
\end{aligned}
$$

On the other hand, using the equations (1.3), we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}|\nabla_A\phi|^2 = 2\,\mathrm{Re}\left\langle \nabla_A\left(\frac{\partial\phi}{\partial t}\right) + \left[\frac{\partial A}{\partial t},\phi\right], \nabla_A\phi\right\rangle\\
&= -2\,\mathrm{Re}\,\langle 2[\partial_A(\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*])),\phi] + [\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]),\nabla_A\phi],\nabla_A\phi\rangle.
\end{aligned}
$$

Then (2.11) follows from the above identities. This proves the lemma. □

Let (A,ϕ) be a smooth solution of the heat flow (1.3). We have

$$
\begin{aligned}
&\frac{\partial}{\partial t}(|F_A+[\phi,\phi^*]|^2 + 2|\partial_A\phi|^2)\\
&\quad = 2\,\mathrm{Re}\left\langle D_A\frac{\partial A}{\partial t} + \left[\frac{\phi}{\partial t},\phi^*\right] + \left[\phi,\left(\frac{\phi}{\partial t}\right)^*\right], F_A+[\phi,\phi^*]\right\rangle\\
&\qquad + 4\,\mathrm{Re}\left\langle \partial_A\frac{\phi}{\partial t} + \frac{A}{\partial t}\wedge\phi + \phi\wedge\frac{A}{\partial t},\partial_A\phi\right\rangle\\
&\quad = 2\,\mathrm{Re}\,\langle -\sqrt{-1}\,(\overline{\partial}_A\partial_A - \partial_A\overline{\partial}_A)(\Lambda_\omega(F_A+[\phi,\phi^*])), F_A+[\phi,\phi^*]\rangle\\
&\qquad - 4\,\mathrm{Re}\,\langle [[\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]),\phi],\phi^*], F_A+[\phi,\phi^*]\rangle\\
&\qquad - 4\,\mathrm{Re}\langle 2[\partial_A(\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*])),\phi] + [\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]),\partial_A\phi],\partial_A\phi\rangle.
\end{aligned}
$$

Choosing normal complex coordinates centered at the point under consideration, we have

$$
\begin{aligned}
&\Delta(2|\partial_A\phi|^2) = 4\partial_\alpha\overline{\partial}_\alpha|\partial_A\phi|^2 = 4|\nabla_A(\partial_A\phi)|^2 + 4\,\mathrm{Re}\,\langle(\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}} + \nabla_{A,\overline{\alpha}}\nabla_{A,\alpha}),\partial_A\phi\rangle\\
&\quad = 4|\nabla_A(\partial_A\phi)|^2 + 4\,\mathrm{Re}\,\langle(\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}\nabla_{A,\gamma}\phi_\beta + \nabla_{A,\overline{\alpha}}\nabla_{A,\alpha}\nabla_{A,\gamma}\phi_\beta)dz^\gamma\wedge dz^\beta,\partial_A\phi\rangle\\
&\qquad + 4\,\mathrm{Re}\,\langle(\nabla_{A,\gamma}\phi_\beta - \nabla_{A,\beta}\phi_\gamma)(\nabla_{\overline{\alpha}}\nabla_\alpha dz^\gamma)\wedge dz^\beta,\partial_A\phi\rangle\\
&\quad = 4|\nabla_A(\partial_A\phi)|^2 + 4\,\mathrm{Re}\,\langle(\nabla_{A,\gamma}\phi_\beta - \nabla_{A,\beta}\phi_\gamma)(\nabla_{\overline{\alpha}}\nabla_\alpha dz^\gamma)\wedge dz^\beta,\partial_A\phi\rangle\\
&\qquad - 8\,\mathrm{Re}\,\langle[F_{A;\gamma\overline{\alpha}},\nabla_{A,\alpha}\phi_\beta]dz^\gamma\wedge dz^\beta + [\partial_A(\sqrt{-1}\,\Lambda_\omega F_A),\phi],\partial_A\phi\rangle\\
&\qquad - 4\,\mathrm{Re}\,\langle[\sqrt{-1}\,\Lambda_\omega F_A,\partial_A\phi],\partial_A\phi\rangle.
\end{aligned}
$$

Using the condition $\phi\wedge\phi=0$, we have

$$\begin{aligned}
&\Delta(2|\partial_A\phi|^2)\\
&=4|\nabla_A(\partial_A\phi)|^2+4\,\mathrm{Re}\,\langle(\nabla_{A,\gamma}\phi_\beta-\nabla_{A,\beta}\phi_\gamma)(\nabla_{\overline{\alpha}}\nabla_\alpha dz^\gamma)\wedge dz^\beta,\,\partial_A\phi\rangle\\
&-8\,\mathrm{Re}\,\langle[F_{A;\gamma\overline{\alpha}}+[\phi_\gamma,\phi^*_{\overline{\alpha}}],\nabla_{A,\alpha}\phi_\beta]dz^\gamma\wedge dz^\beta+[\partial_A(\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*])),\phi],\,\partial_A\phi\rangle\\
&\quad-4\,\mathrm{Re}\,\langle[\sqrt{-1}\,\Lambda_\omega F_A,\partial_A\phi],\partial_A\phi\rangle-8\,\mathrm{Re}\,\langle[\phi^*_{\overline{\alpha}},[\partial_A\phi,\phi_\alpha]],\partial_A\phi\rangle\\
&\quad+8\,\mathrm{Re}\,\langle[[\phi_\gamma,\phi^*_{\overline{\alpha}}],\nabla_{A,\alpha}\phi_\beta-\nabla_{A,\beta}\phi_\alpha]dz^\gamma\wedge dz^\beta,\,\partial_A\phi\rangle.
\end{aligned}$$

By direct calculation, we have

$$\begin{aligned}
&\Delta|F_A+[\phi,\phi^*]|^2\\
&=2\partial_\alpha\overline{\partial}_\alpha|F_A+[\phi,\phi^*]|^2\\
&=2\,\mathrm{Re}\,\langle(\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}+\nabla_{A,\overline{\alpha}}\nabla_{A,\alpha})(F_A+[\phi,\phi^*]),\,F_A+[\phi,\phi^*]\rangle+2|\nabla_A(F_A+[\phi,\phi^*])|^2\\
&=2|\nabla_A(F_A+[\phi,\phi^*])|^2+2\,\mathrm{Re}\,\langle(\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}(F_A+[\phi,\phi^*])_{\beta\overline{\gamma}})dz^\beta\wedge d\overline{z}^\gamma,\,F_A+[\phi,\phi^*]\rangle\\
&\quad+2\,\mathrm{Re}\,\langle(\nabla_{A,\overline{\alpha}}\nabla_{A,\alpha}(F_A+[\phi,\phi^*])_{\beta\overline{\gamma}})dz^\beta\wedge d\overline{z}^\gamma,\,F_A+[\phi,\phi^*]\rangle\\
&\quad+2\,\mathrm{Re}\,\langle(F_A+[\phi,\phi^*])_{\beta\overline{\gamma}}(dz^\beta\wedge\nabla_\alpha\nabla_{\overline{\alpha}}d\overline{z}^\gamma+\nabla_{\overline{\alpha}}\nabla_\alpha dz^\beta\wedge d\overline{z}^\gamma),\,F_A+[\phi,\phi^*]\rangle.
\end{aligned}$$

Using the Bianchi identity, we obtain

$$\begin{aligned}
\nabla_{A,\alpha}\nabla_{A,\overline{\alpha}}&(F_A+[\phi,\phi^*])_{\beta\overline{\gamma}}\\
&=\nabla_{A,\alpha}(\nabla_{A,\overline{\gamma}}(F_{A,\beta\overline{\alpha}}+[\phi_\beta,\phi^*_{\overline{\alpha}}])+[\phi_\beta,\nabla_{A,\overline{\alpha}}\phi^*_{\overline{\gamma}}-\nabla_{A,\overline{\gamma}}\phi^*_{\overline{\alpha}}])\\
&=\nabla_{A,\overline{\gamma}}\nabla_{A,\alpha}(F_{A,\beta\overline{\alpha}}+[\phi_\beta,\phi^*_{\overline{\alpha}}])+[F_{A,\alpha\overline{\gamma}},F_{A,\beta\overline{\alpha}}+[\phi_\beta,\phi^*_{\overline{\alpha}}]]\\
&\quad+\nabla_\alpha([\phi_\beta,\nabla_{A,\overline{\alpha}}\phi^*_{\overline{\gamma}}-\nabla_{A,\overline{\gamma}}\phi^*_{\overline{\alpha}}])\\
&=\nabla_{A,\overline{\gamma}}\nabla_{A,\beta}(F_{A,\alpha\overline{\alpha}}+[\phi_\alpha,\phi^*_{\overline{\alpha}}])+[F_{A,\alpha\overline{\gamma}},F_{A,\beta\overline{\alpha}}+[\phi_\beta,\phi^*_{\overline{\alpha}}]]\\
&\quad+\nabla_{\overline{\gamma}}([\nabla_{A,\alpha}\phi_\beta-\nabla_{A,\beta}\phi_\alpha,\phi^*_{\overline{\alpha}}])+\nabla_{A,\alpha}([\phi_\beta,\nabla_{A,\overline{\alpha}}\phi^*_{\overline{\gamma}}-\nabla_{A,\overline{\gamma}}\phi^*_{\overline{\alpha}}])\\
&=\nabla_{A,\overline{\gamma}}\nabla_{A,\beta}(F_{A,\alpha\overline{\alpha}}+[\phi_\alpha,\phi^*_{\overline{\alpha}}])+[F_{A,\alpha\overline{\gamma}},F_{A,\beta\overline{\alpha}}+[\phi_\beta,\phi^*_{\overline{\alpha}}]]\\
&\quad+[\nabla_{A,\alpha}\phi_\beta-\nabla_{A,\beta}\phi_\alpha,\nabla_{\overline{\gamma}}\phi^*_{\overline{\alpha}}]+[\nabla_{A,\alpha}\phi_\beta,\nabla_{A,\overline{\alpha}}\phi^*_{\overline{\gamma}}-\nabla_{A,\overline{\gamma}}\phi^*_{\overline{\alpha}}]\\
&\quad+[\phi_\beta,[F_{A,\alpha\overline{\alpha}}+[\phi_\alpha,\phi^*_{\overline{\alpha}}],\phi^*_{\overline{\gamma}}]]-[\phi_\beta,[F_{A,\alpha\overline{\gamma}}+[\phi_\alpha,\phi^*_{\overline{\gamma}}],\phi^*_{\overline{\alpha}}]]\\
&\quad-[[F_{A,\alpha\overline{\gamma}}+[\phi_\alpha,\phi^*_{\overline{\gamma}}],\phi_\beta],\phi^*_{\overline{\alpha}}]+[[F_{A,\beta\overline{\gamma}}+[\phi_\beta,\phi^*_{\overline{\gamma}}],\phi_\alpha],\phi^*_{\overline{\alpha}}],
\end{aligned}$$

and similarly,

$$\begin{aligned}
\nabla_{A,\overline{\alpha}}\nabla_{A,\alpha}&(F_A+[\phi,\phi^*])_{\beta\overline{\gamma}}\\
&=\nabla_{A,\beta}\nabla_{A,\overline{\gamma}}(F_{A,\alpha\overline{\alpha}}+[\phi_\alpha,\phi^*_{\overline{\alpha}}])-[F_{A,\beta\overline{\alpha}},F_{A,\alpha\overline{\gamma}}+[\phi_\alpha,\phi^*_{\overline{\gamma}}]]\\
&\quad+[\nabla_{A,\alpha}\phi_\beta-\nabla_{A,\beta}\phi_\alpha,\nabla_{\overline{\alpha}}\phi^*_{\overline{\gamma}}]+[\nabla_{A,\beta}\phi_\alpha,\nabla_{A,\overline{\alpha}}\phi^*_{\overline{\gamma}}-\nabla_{A,\overline{\gamma}}\phi^*_{\overline{\alpha}}]\\
&\quad+[\phi_\alpha,[F_{A,\beta\overline{\alpha}}+[\phi_\beta,\phi^*_{\overline{\alpha}}],\phi^*_{\overline{\gamma}}]]-[\phi_\alpha,[F_{A,\beta\overline{\gamma}}+[\phi_\beta,\phi^*_{\overline{\gamma}}],\phi^*_{\overline{\alpha}}]]\\
&\quad-[[F_{A,\alpha\overline{\alpha}}+[\phi_\alpha,\phi^*_{\overline{\alpha}}],\phi_\beta],\phi^*_{\overline{\gamma}}]+[[F_{A,\beta\overline{\alpha}}+[\phi_\beta,\phi^*_{\overline{\alpha}}],\phi_\alpha],\phi^*_{\overline{\gamma}}].
\end{aligned}$$

Combining the above identities, we have the following lemma.

Lemma 2.5. *Let (A, ϕ) be a smooth solution of the heat flow (1.3). Then*

$$\left(\Delta - \frac{\partial}{\partial t}\right)(|F_A + [\phi, \phi^*]|^2 + 2|\partial_A \phi|^2) - 2|\nabla_A(F_A + [\phi, \phi^*])|^2 - 4|\nabla_A(\partial_A \phi)|^2$$
$$\geq -C_8(|F_A + [\phi, \phi^*]| + |\nabla_A \phi| + |\phi|^2 + |\mathrm{Rm}|)(|F_A + [\phi, \phi^*]|^2 + 2|\partial_A \phi|^2), \quad (2.12)$$

and

$$\left(\Delta - \frac{\partial}{\partial t}\right)(|F_A + [\phi, \phi^*]|^2) - 2|\nabla_A(F_A + [\phi, \phi^*])|^2$$
$$\geq -C_8(|F_A + [\phi, \phi^*]| + |\nabla_A \phi| + |\phi|^2 + |\mathrm{Rm}|)(|F_A + [\phi, \phi^*]|^2 + 2|\partial_A \phi|^2), \quad (2.13)$$

where C_8 is a constant depending only on the complex dimension m.

2.3. *Monotonicity inequality*

In this subsection, we derive a monotonicity inequality for the heat flow (1.3), which is an analogue to harmonic map heat flow [St2]. Similar arguments have been used in studying the Yang–Mills flow ([CS1], [CS2]) and the Yang–Mills–Higgs flow ([HT]).

Let (M, ω) be a Kähler manifold with complex dimension m. For any $x_0 \in M$, there exist complex normal coordinates $\{z_1, \dots, z_m\}$ in the geodesic ball $B_r(x_0)$ with center x_0 and radius $r \leq i(M)$ such that $x_0 = (0, \dots, 0)$ and for some constant $C(x_0)$,

$$|g_{i\bar{j}}(z) - \delta_{ij}| \leq C|z|^2, \qquad \left|\frac{\partial g_{i\bar{j}}}{\partial z_k}\right| \leq C|z|, \qquad \forall z \in B_r.$$

Here $i(M)$ is the infimum of the injectivity radius over $x \in M$, and $(g_{i\bar{j}})$ is the Kähler metric of M given by $\langle \partial/\partial z_i, \partial/\partial \bar{z}_j \rangle = g_{i\bar{j}}$.

Let $u = (x, t)$ be a point in $M \times \mathbb{R}$. For a fixed point $u_0 = (x_0, t_0) \in M \times \mathbb{R}_+$, we write

$$S_r(u_0) = M \times \{t = t_0 - r^2\},$$
$$T_r(u_0) = \{u = (x, t) : t_0 - 4r^2 < t < t_0 - r^2,\ x \in M\},$$
$$P_r(u_0) = B_r(x_0) \times [t_0 - r^2, t_0 + r^2].$$

For simplicity, we denote $S_r(0, 0)$, $T_r(0, 0)$ and $P_r(0, 0)$ by S_r, T_r and P_r respectively.

The fundamental solution of the (backward) heat equation with singularity at (z_0, t_0) is

$$G_{(z_0, t_0)}(z, t) = \frac{1}{(4\pi(t_0 - t))^m} \exp\left(-\frac{|z - z_0|^2}{4(t_0 - t)}\right), \qquad t < t_0.$$

For simplicity, we denote $G_{(0,0)}(x, t)$ by $G(x, t)$.

Assume that $(A(t), \phi(t))$ is a solution of the heat flow (1.3) in $M \times \mathbb{R}_+$. Let f be a smooth cut-off function, i.e. $|f| \leq 1$, $f \equiv 1$ on $B_{i(M)/2}$, $f \equiv 0$ outside $B_{i(M)}$ and $|\nabla f| \leq 2/i(M)$. For any $(x, t) \in M \times (0, \infty)$, we set

$$e(A, \phi)(x, t) = |F_A + [\phi, \phi^*]|^2 + 2|\partial_A \phi|^2.$$

For any $u_0 = (x_0, t_0) \in M \times [0, T]$, we set

$$\Phi(r; A, \phi) = r^2 \int_{T_r(u_0)} e(A, \phi) f^2 G_{u_0}\, dV_g\, dt,$$
$$\Psi(r; A, \phi) = r^4 \int_{S_r(u_0)} e(A, \phi) f^2 G_{u_0}\, dV_g.$$

Then we have

Theorem 2.6 (Monotonicity inequality). *Let (A, ϕ) be a regular solution of the heat flow equation (1.3) with initial value (A_0, ϕ_0). Then for $u_0 = (x_0, t_0) \in M \times [0, T]$, and for r_1 and r_2 with $0 < r_1 \leq r_2 \leq \min\{i(M), \sqrt{t_0}/2\}$, we have*

$$\Phi(r_1; A, \phi) \leq C_9 \exp(C_9(r_2 - r_1))\Phi(r_2; A, \phi) + C_9(r_2^2 - r_1^2)\,\mathrm{YMH}(A_0, \phi_0),$$
$$\Psi(r_1; A, \phi) \leq C_{10} \exp(C_{10}(r_2 - r_1))\Psi(r_2; A, \phi) + C_{10}(r_2^2 - r_1^2)\,\mathrm{YMH}(A_0, \phi_0)$$

where C_9, C_{10} are positive constants depending only on the geometry of M.

Proof. Choosing complex normal coordinates $\{z_1, \ldots, z_m\}$ in the geodesic ball $B_r(x_0)$ with $r \leq i(M)$, and setting $z = R\tilde{z}$, $t = R^2\tilde{t} + t_0$, we have

$$\begin{aligned}
\Phi(R; A, \phi) &= R^2 \int_{T_R(u_0)} e(A, \phi) f^2 G_{u_0}\, dV_g\, dt \\
&= R^2 \int_{t_0-4R^2}^{t_0-R^2} \int_{\mathbb{C}^m} e(A, \phi)(z, t) f^2(z) G_{u_0}(z, t) \det(g_{i\bar{j}})\, dz\, dt \\
&= R^4 \int_{-4}^{-1} \int_{\mathbb{C}^m} e(A, \phi)(R\tilde{z}, R^2\tilde{t} + t_0) f^2(R\tilde{z}) G(\tilde{z}, \tilde{t}) \det(g_{i\bar{j}})(R\tilde{z})\, d\tilde{z}\, d\tilde{t},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dR}\Phi(R; A, \phi) &= 4R^3 \int_{-4}^{-1} \int_{\mathbb{C}^m} e(A, \phi)(R\tilde{z}, R^2\tilde{t} + t_0) f^2(R\tilde{z}) G(\tilde{z}, \tilde{t}) \det(g_{i\bar{j}})(R\tilde{z})\, d\tilde{z}\, d\tilde{t} \\
&+ R^3 \int_{-4}^{-1} \int_{\mathbb{C}^m} 2\,\mathrm{Re}\left[\left(z^i \frac{\partial}{\partial z^i}\right) e(A, \phi)\right](R\tilde{z}, R^2\tilde{t} + t_0) f^2(R\tilde{z}) G(\tilde{z}, \tilde{t}) \det(g_{i\bar{j}})(R\tilde{z})\, d\tilde{z}\, d\tilde{t} \\
&+ R^4 \int_{-4}^{-1} \int_{\mathbb{C}^m} 2R\tilde{t}\left[\frac{\partial}{\partial t} e(A, \phi)\right](R\tilde{z}, R^2\tilde{t} + t_0) f^2(R\tilde{z}) G(\tilde{z}, \tilde{t}) \det(g_{i\bar{j}})(R\tilde{z})\, d\tilde{z}\, d\tilde{t} \\
&+ R^3 \int_{-4}^{-1} \int_{\mathbb{C}^m} 2e(A, \phi)(R\tilde{z}, R^2\tilde{t} + t_0)\,\mathrm{Re}\left[\left(z^i \frac{\partial}{\partial z^i}\right) f^2 \det(g_{i\bar{j}})\right](R\tilde{z}) G(\tilde{z}, \tilde{t})\, d\tilde{z}\, d\tilde{t} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Defining the connection $A' = A + \phi + \phi^*$, we have

$$F_{A'} = F_A + [\phi, \phi^*] + \partial_A \phi + \overline{\partial}_A \phi^*.$$

The Bianchi identity is

$$D_{A'} F_{A'} = D_A F_{A'} + [\phi + \phi^*, F_{A'}] = 0.$$

It follows that

$$\begin{aligned}
&I_2 = \\
&R^3 \int_{-4}^{-1} \int_{\mathbb{C}^m} 2\,\mathrm{Re}\left[\left(z^i \frac{\partial}{\partial z^i}\right) e(A,\phi)\right](R\tilde{z}, R^2\tilde{t} + t_0) f^2(R\tilde{z}) G(\tilde{z}, \tilde{t}) \det(g_{i\bar{j}})(R\tilde{z})\, d\tilde{z}\, d\tilde{t} \\
&= R \int_{T_R(u_0)} 2\,\mathrm{Re}\left[\left(z^i \frac{\partial}{\partial z^i}\right) e(A,\phi)\right] f^2 G_{u_0}\, dV_g dt \\
&= R 2\,\mathrm{Re} \int_{T_R(u_0)} \langle z^i \nabla_{A,\partial/\partial z^i} F_{A'} + \overline{z}^i \nabla_{A,\partial/\partial \overline{z}^i} F_{A'}, F_{A'} \rangle f^2 G_{u_0}\, dV_g\, dt
\end{aligned}$$

and

$$\begin{aligned}
&R 2\,\mathrm{Re} \int_{T_R(u_0)} \langle \overline{z}^i \nabla_{A,\partial/\partial \overline{z}^i} F_{A'}, F_{A'} \rangle f^2 G_{u_0}\, dV_g\, dt \\
&\quad = R 2\,\mathrm{Re} \int_{T_R(u_0)} \left\langle \overline{z}^i (\nabla_{A,\partial/\partial \overline{z}^i} F_{A'}) \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\right) dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*] \right\rangle f^2 G_{u_0}\, dV_g\, dt \\
&\qquad + R\,\mathrm{Re} \int_{T_R(u_0)} \left\langle \overline{z}^i (\nabla_{A,\partial/\partial \overline{z}^i} F_{A'}) \left(\frac{\partial}{\partial \overline{z}^j}, \frac{\partial}{\partial \overline{z}^k}\right) d\overline{z}^j \wedge d\overline{z}^k, \overline{\partial}_A \phi^* \right\rangle f^2 G_{u_0}\, dV_g\, dt \\
&\qquad + R\,\mathrm{Re} \int_{T_R(u_0)} \left\langle \overline{z}^i (\nabla_{A,\partial/\partial \overline{z}^i} F_{A'}) \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right) dz^j \wedge dz^k, \partial_A \phi \right\rangle f^2 G_{u_0}\, dV_g\, dt \\
&\quad =: a + b + c.
\end{aligned}$$

By the Bianchi identity, we have

$$\begin{aligned}
&(\nabla_{A,\partial/\partial \overline{z}^i} F_{A'}) \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\right) = (\nabla_{A,\partial/\partial z^j} F_{A'}) \left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k}\right) + (\nabla_{A,\partial/\partial \overline{z}^k} F_{A'}) \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^i}\right) \\
&\qquad - [\phi^*_{\bar{i}}, (F_A + [\phi, \phi^*])_{j\bar{k}}] + [\phi^*_{\bar{k}}, (F_A + [\phi, \phi^*])_{j\bar{i}}] + [\phi_j, (\overline{\partial}_{A,\bar{i}} \phi^*_{\bar{k}} - \overline{\partial}_{A,\bar{k}} \phi^*_{\bar{i}})], \\
&\left\langle \overline{z}^i (\nabla_{A,\partial/\partial z^j} F_{A'}) \left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k}\right) dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*] \right\rangle \\
&\qquad = \langle \overline{z}^i \partial_{A,\partial/\partial z^j} (\overline{\partial}_{A,\bar{i}} \phi^*_{\bar{k}} - \overline{\partial}_{A,\bar{k}} \phi^*_{\bar{i}}) dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*] \rangle \\
&\qquad = \langle \overline{z}^i ([F_{A,j\bar{i}}, \phi^*_{\bar{k}}] - [F_{A,j\bar{k}}, \phi^*_{\bar{i}}]) dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*] \rangle,
\end{aligned}$$

and

$$
\begin{aligned}
&\left\langle \overline{z}^i (\nabla_{A,\partial/\partial \overline{z}^k} F_{A'})\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^i}\right) dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*]\right\rangle \\
&\quad = \langle -\overline{z}^i \overline{\partial}_A((F_A + [\phi, \phi^*])_{j\overline{i}} dz^j) - \overline{z}^i \Gamma^{\overline{n}}_{\overline{i}\overline{k}} (F_A + [\phi, \phi^*])_{j\overline{n}} dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*]\rangle \\
&\quad = \langle -\overline{\partial}_A(\overline{z}^i (F_A + [\phi, \phi^*])_{j\overline{i}} dz^j) - \overline{z}^i \Gamma^{\overline{n}}_{\overline{i}\overline{k}} (F_A + [\phi, \phi^*])_{j\overline{n}} dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*]\rangle \\
&\qquad - |F_A + [\phi, \phi^*]|^2.
\end{aligned}
$$

Using $\phi^* \wedge \phi^* = 0$, we have

$$
\begin{aligned}
&\langle \overline{z}^i (-[\phi^*_{\overline{i}}, (F_A + [\phi, \phi^*])_{j\overline{k}}] + [\phi^*_{\overline{k}}, (F_A + [\phi, \phi^*])_{j\overline{i}}]) dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*]\rangle \\
&\qquad\qquad = \langle \overline{z}^i (-[\phi^*_{\overline{i}}, F_{A, j\overline{k}}] + [\phi^*_{\overline{k}}, F_{A, j\overline{i}}]) dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*]\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \overline{z}^i [\phi_j, (\overline{\partial}_{A,\overline{i}} \phi^*_{\overline{k}} - \overline{\partial}_{A,\overline{k}} \phi^*_{\overline{i}})] dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*]\rangle \\
&\quad = \langle -[\phi^*, z^i (\partial_{A,i}\phi_k - \partial_{A,k}\phi_i) dz^k]^*, -(F_A + [\phi, \phi^*])^*\rangle \\
&\quad = \langle F_A + [\phi, \phi^*], [\phi^*, z^i (\partial_{A,i}\phi_k - \partial_{A,k}\phi_i) dz^k]\rangle \\
&\quad = \langle -\sqrt{-1}\, \Lambda_\omega [F_A + [\phi, \phi^*], \phi] + [\sqrt{-1}\, \Lambda_\omega (F_A + [\phi, \phi^*]), \phi], z^i (\partial_{A,i}\phi_k - \partial_{A,k}\phi_i) dz^k\rangle.
\end{aligned}
$$

The above identities yield

$$
\begin{aligned}
a = R2\,\mathrm{Re} \int_{T_R(u_0)} &\{\langle \overline{\partial}(f^2 G_{u_0}) \wedge (\overline{z}^i (F_A + [\phi, \phi^*])_{j\overline{i}} dz^j), F_A + [\phi, \phi^*]\rangle \\
&- \langle f^2 G_{u_0} \overline{z}^i (F_A + [\phi, \phi^*])_{j\overline{i}} dz^j, \sqrt{-1}\, \partial_A \Lambda_\omega (F_A + [\phi, \phi^*]) - \sqrt{-1}\, \Lambda_\omega \partial_A [\phi, \phi^*]\rangle \\
&- f^2 G_{u_0} \langle \overline{z}^i \Gamma^{\overline{n}}_{\overline{i}\overline{k}} (F_A + [\phi, \phi^*])_{j\overline{n}} dz^j \wedge d\overline{z}^k, F_A + [\phi, \phi^*]\rangle \\
&- f^2 G_{u_0} |F_A + [\phi, \phi^*]|^2 - f^2 G_{u_0} \langle z^i (\partial_{A,i}\phi_k - \partial_{A,k}\phi_i) dz^k, \sqrt{-1}\, \Lambda_\omega [F_A + [\phi, \phi^*], \phi]\rangle \\
&+ f^2 G_{u_0} \langle z^i (\partial_{A,i}\phi_k - \partial_{A,k}\phi_i) dz^k, [\sqrt{-1}\, \Lambda_\omega (F_A + [\phi, \phi^*]), \phi]\rangle\}\, dV_g\, dt.
\end{aligned}
$$

Using the Bianchi identity and $\phi^* \wedge \phi^* = 0$ again, we have

$$
\begin{aligned}
b &= R\,\mathrm{Re} \int_{T_R(u_0)} \left\langle \overline{z}^i (\nabla_{A,\partial/\partial \overline{z}^i} F_{A'})\left(\frac{\partial}{\partial \overline{z}^j}, \frac{\partial}{\partial \overline{z}^k}\right) d\overline{z}^j \wedge d\overline{z}^k, \overline{\partial}_A \phi^*\right\rangle f^2 G_{u_0}\, dV_g\, dt \\
&= R2\,\mathrm{Re} \int_{T_R(u_0)} \left\langle \overline{z}^i (\nabla_{A,\partial/\partial \overline{z}^j} F_{A'})\left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k}\right) d\overline{z}^j \wedge d\overline{z}^k, \overline{\partial}_A \phi^*\right\rangle f^2 G_{u_0}\, dV_g\, dt \\
&= R2\,\mathrm{Re} \int_{T_R(u_0)} \{\langle \overline{z}^i \overline{\partial}_{A,\overline{j}} (\overline{\partial}_{A,\overline{i}} \phi^*_{\overline{k}} - \overline{\partial}_{A,\overline{k}} \phi^*_{\overline{i}}) d\overline{z}^j \wedge d\overline{z}^k, \overline{\partial}_A \phi^*\rangle f^2 G_{u_0} \\
&\quad - \langle \overline{z}^i \Gamma^{\overline{n}}_{\overline{i}\overline{j}} (\overline{\partial}_{A,\overline{n}} \phi^*_{\overline{k}} - \overline{\partial}_{A,\overline{k}} \phi^*_{\overline{n}}) d\overline{z}^j \wedge d\overline{z}^k, \overline{\partial}_A \phi^*\rangle f^2 G_{u_0}\}\, dV_g\, dt \\
&= R2\,\mathrm{Re} \int_{T_R(u_0)} \{\langle \overline{\partial}_A (f^2 G_{u_0} \overline{z}^i (\overline{\partial}_{A,\overline{i}} \phi^*_{\overline{k}} - \overline{\partial}_{A,\overline{k}} \phi^*_{\overline{i}}) d\overline{z}^k), \overline{\partial}_A \phi^*\rangle \\
&\quad - \langle \overline{\partial}(f^2 G_{u_0}) \wedge (\overline{z}^i (\overline{\partial}_{A,\overline{i}} \phi^*_{\overline{k}} - \overline{\partial}_{A,\overline{k}} \phi^*_{\overline{i}}) d\overline{z}^k), \overline{\partial}_A \phi^*\rangle \\
&\quad - 2|\overline{\partial}_A \phi^*| f^2 G_{u_0} - \langle \overline{z}^i \Gamma^{\overline{n}}_{\overline{i}\overline{j}} (\overline{\partial}_{A,\overline{n}} \phi^*_{\overline{k}} - \overline{\partial}_{A,\overline{k}} \phi^*_{\overline{n}}) d\overline{z}^j \wedge d\overline{z}^k, \overline{\partial}_A \phi^*\rangle f^2 G_{u_0}\}\, dV_g\, dt
\end{aligned}
$$

$$= R2\,\mathrm{Re}\int_{T_R(u_0)}\{f^2G_{u_0}\langle z^i(\partial_{A,i}\phi_k-\partial_{A,k}\phi_i)dz^k,\sqrt{-1}\,\Lambda_\omega[F_A,\phi]\rangle$$
$$-\langle\overline{\partial}(f^2G_{u_0})\wedge(\overline{z}^i(\overline{\partial}_{A,\overline{i}}\phi^*_{\overline{k}}-\overline{\partial}_{A,\overline{k}}\phi^*_{\overline{i}})d\overline{z}^k),\overline{\partial}_A\phi^*\rangle$$
$$-2|\overline{\partial}_A\phi^*|f^2G_{u_0}-\langle\overline{z}^i\Gamma^{\overline{n}}_{\overline{i}\overline{j}}(\overline{\partial}_{A,\overline{n}}\phi^*_{\overline{k}}-\overline{\partial}_{A,\overline{k}}\phi^*_{\overline{n}})d\overline{z}^j\wedge d\overline{z}^k,\overline{\partial}_A\phi^*\rangle f^2G_{u_0}\}\,dV_g\,dt$$

and

$$c = R\,\mathrm{Re}\int_{T_R(u_0)}\left\langle\overline{z}^i(\nabla_{A,\partial/\partial\overline{z}^i}F_{A'})\left(\frac{\partial}{\partial z^j},\frac{\partial}{\partial z^k}\right)dz^j\wedge dz^k,\partial_A\phi\right\rangle f^2G_{u_0}\,dV_g\,dt$$
$$= R\,\mathrm{Re}\int_{T_R(u_0)}\langle\overline{z}^i\overline{\partial}_{A,\overline{i}}(\partial_{A,j}\phi_k-\partial_{A,k}\phi_j)dz^j\wedge dz^k,\partial_A\phi\rangle f^2G_{u_0}\,dV_g\,dt$$
$$= -R2\,\mathrm{Re}\int_{T_R(u_0)}\langle\overline{z}^i[F_{A,j\overline{i}}dz^j,\phi],\partial_A\phi\rangle f^2G_{u_0}\,dV_g\,dt$$
$$= -R2\,\mathrm{Re}\int_{T_R(u_0)}\langle\overline{z}^i[(F_A+[\phi,\phi^*])_{j\overline{i}}dz^j,\phi],\partial_A\phi\rangle f^2G_{u_0}\,dV_g\,dt$$
$$= -R2\,\mathrm{Re}\int_{T_R(u_0)}\langle\overline{z}^i(F_A+[\phi,\phi^*])_{j\overline{i}}dz^j,\sqrt{-1}\,\Lambda_\omega\partial_A[\phi,\phi^*]\rangle f^2G_{u_0}\,dV_g\,dt.$$

Then

$$R2\,\mathrm{Re}\int_{T_R(u_0)}\langle\overline{z}^i\nabla_{A,\partial/\partial\overline{z}^i}F_{A'},F_{A'}\rangle f^2G_{u_0}\,dV_g\,dt = a+b+c$$
$$= R2\,\mathrm{Re}\int_{T_R(u_0)}\{\langle\overline{\partial}(f^2G_{u_0})\wedge(\overline{z}^i(F_A+[\phi,\phi^*])_{j\overline{i}}dz^j),F_A+[\phi,\phi^*]\rangle$$
$$-\langle f^2G_{u_0}\overline{z}^i(F_A+[\phi,\phi^*])_{j\overline{i}}dz^j,\sqrt{-1}\,\partial_A\Lambda_\omega(F_A+[\phi,\phi^*])\rangle$$
$$-f^2G_{u_0}\langle\overline{z}^i\Gamma^{\overline{n}}_{\overline{i}\overline{k}}(F_A+[\phi,\phi^*])_{j\overline{n}}dz^j\wedge d\overline{z}^k,F_A+[\phi,\phi^*]\rangle-f^2G_{u_0}|F_A+[\phi,\phi^*]|^2$$
$$+f^2G_{u_0}\langle z^i(\partial_{A,i}\phi_k-\partial_{A,k}\phi_i)dz^k,[\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]),\phi]\rangle$$
$$-\langle\overline{\partial}(f^2G_{u_0})\wedge(\overline{z}^i(\overline{\partial}_{A,\overline{i}}\phi^*_{\overline{k}}-\overline{\partial}_{A,\overline{k}}\phi^*_{\overline{i}})d\overline{z}^k),\overline{\partial}_A\phi^*\rangle$$
$$-2|\overline{\partial}_A\phi^*|f^2G_{u_0}-\langle\overline{z}^i\Gamma^{\overline{n}}_{\overline{i}\overline{j}}(\overline{\partial}_{A,\overline{n}}\phi^*_{\overline{k}}-\overline{\partial}_{A,\overline{k}}\phi^*_{\overline{n}})d\overline{z}^j\wedge d\overline{z}^k,\overline{\partial}_A\phi^*\rangle f^2G_{u_0}\}\,dV_gdt.$$

By a similar calculation, we have

$$R2\,\mathrm{Re}\int_{T_R(u_0)}\langle z^i\nabla_{A,\partial/\partial z^i}F_{A'},F_{A'}\rangle f^2G_{u_0}\,dV_g\,dt$$
$$= R2\,\mathrm{Re}\int_{T_R(u_0)}\{-\langle\partial(f^2G_{u_0})\wedge(z^i(F_A+[\phi,\phi^*])_{i\overline{j}}d\overline{z}^j),F_A+[\phi,\phi^*]\rangle$$
$$-\langle f^2G_{u_0}z^i(F_A+[\phi,\phi^*])_{i\overline{j}}d\overline{z}^j,\sqrt{-1}\,\overline{\partial}_A\Lambda_\omega(F_A+[\phi,\phi^*])\rangle$$
$$-f^2G_{u_0}\langle z^i\Gamma^n_{ij}(F_A+[\phi,\phi^*])_{n\overline{k}}dz^j\wedge d\overline{z}^k,F_A+[\phi,\phi^*]\rangle-f^2G_{u_0}|F_A+[\phi,\phi^*]|^2$$
$$+f^2G_{u_0}\langle z^i(\partial_{A,i}\phi_k-\partial_{A,k}\phi_i)dz^k,[\sqrt{-1}\,\Lambda_\omega(F_A+[\phi,\phi^*]),\phi]\rangle$$
$$-\langle\partial(f^2G_{u_0})\wedge(z^i(\partial_{A,i}\phi_k-\partial_{A,k}\phi_i)dz^k),\partial_A\phi\rangle$$
$$-2|\partial_A\phi|f^2G_{u_0}-\langle z^i\Gamma^n_{ij}(\partial_{A,n}\phi_k-\partial_{A,k}\phi_n)dz^j\wedge dz^k,\partial_A\phi\rangle f^2G_{u_0}\}\,dV_g\,dt.$$

Note that

$$|\Gamma_{ij}^{k}| \le C|Z|, \qquad \left| Z^{i} \frac{\partial}{\partial z^{i}} \det(g_{i\bar{j}}) \right| \le C|z|^{2} \det(g_{i\bar{j}}),$$

and

$$\frac{\partial}{\partial z^{i}} = \frac{G}{2(t-t_{0})} \bar{z}^{i}, \qquad \frac{\partial}{\partial \bar{z}^{i}} = \frac{G}{2(t-t_{0})} z^{i}.$$

By direct calculation, we have

$$I_{1} = 4R \int_{T_{R}(u_{0})} e(A,\phi) f^{2} G_{u_{0}} \, dV_{g} \, dt,$$

$$I_{4} \ge -CR \int_{T_{R}(u_{0})} e(A,\phi)(|z|^{2} f^{2} + |z| \, |\nabla f| \, |f|) G_{u_{0}} \, dV_{g} \, dt,$$

and

$$\begin{aligned}
I_{1}+I_{2} \ge\ & 2R \,\mathrm{Re} \int_{T_{R}(u_{0})} \Big\{ -C|F_{A}+[\phi,\phi^{*}]|^{2}(|\nabla f|\,|f|\,|z| G_{u_{0}} + f^{2} G_{u_{0}} |z|^{2}) \\
& - C|\partial_{A}\phi|^{2}(|\nabla f|\,|f|\,|z| G_{u_{0}} + f^{2} G_{u_{0}} |z|^{2}) \\
& + f^{2} G_{u_{0}} \Big\langle \bar{z}^{i} (F_{A}+[\phi,\phi^{*}])_{j\bar{i}} dz^{j} - z^{i} (F_{A}+[\phi,\phi^{*}])_{i\bar{j}} d\bar{z}^{j}, \frac{\partial A}{\partial t} \Big\rangle \\
& - 2 f^{2} G_{u_{0}} \Big\langle z^{i} (\partial_{A,i}\phi_{k} - \partial_{A,k}\phi_{i}) dz^{k}, \frac{\partial \phi}{\partial t} \Big\rangle \\
& + \langle f^{2} \bar{\partial} G_{u_{0}} \wedge \bar{z}^{i} (F_{A}+[\phi,\phi^{*}])_{j\bar{i}} dz^{j}, F_{A}+[\phi,\phi^{*}] \rangle \\
& - \langle f^{2} \partial G_{u_{0}} \wedge z^{i} (F_{A}+[\phi,\phi^{*}])_{i\bar{j}} d\bar{z}^{j}, F_{A}+[\phi,\phi^{*}] \rangle \\
& - \langle f^{2} \bar{\partial} G_{u_{0}} \wedge (\bar{z}^{i} (\bar{\partial}_{A,\bar{i}} \phi^{*}_{\bar{k}} - \bar{\partial}_{A,\bar{k}} \phi^{*}_{\bar{i}}) d\bar{z}^{k}), \bar{\partial}_{A} \phi^{*} \rangle \\
& - \langle f^{2} \partial G_{u_{0}} \wedge (z^{i} (\partial_{A,i}\phi_{k} - \partial_{A,k}\phi_{i}) dz^{k}), \partial_{A}\phi^{*} \rangle \Big\} \, dV_{g} \, dt \\
=\ & 2R \,\mathrm{Re} \int_{T_{R}(u_{0})} \Big\{ -C e(A,\phi)(|\nabla f|\,|f|\,|z| G_{u_{0}} + f^{2} G_{u_{0}} |z|^{2}) \\
& + f^{2} G_{u_{0}} \Big\langle \bar{z}^{i} (F_{A}+[\phi,\phi^{*}])_{j\bar{i}} dz^{j} - z^{i} (F_{A}+[\phi,\phi^{*}])_{i\bar{j}} d\bar{z}^{j}, \frac{\partial A}{\partial t} \Big\rangle \\
& - 2 f^{2} G_{u_{0}} \Big\langle z^{i} (\partial_{A,i}\phi_{k} - \partial_{A,k}\phi_{i}) dz^{k}, \frac{\partial \phi}{\partial t} \Big\rangle \\
& - f^{2} \frac{G_{u_{0}}}{2(t-t_{0})} \langle \bar{z}^{i} (F_{A}+[\phi,\phi^{*}])_{j\bar{i}} dz^{j}, \bar{z}^{k} g^{k\bar{n}} (F_{A}+[\phi,\phi^{*}])_{m\bar{n}} dz^{m} \rangle \\
& - f^{2} \frac{G_{u_{0}}}{2(t-t_{0})} \langle z^{i} (F_{A}+[\phi,\phi^{*}])_{i\bar{j}} d\bar{z}^{j}, z^{k} g^{m\bar{k}} (F_{A}+[\phi,\phi^{*}])_{m\bar{n}} d\bar{z}^{m} \rangle \\
& - f^{2} \frac{G_{u_{0}}}{2(t-t_{0})} \langle \bar{z}^{i} (\bar{\partial}_{A,\bar{i}} \phi^{*}_{\bar{k}} - \bar{\partial}_{A,\bar{k}} \phi^{*}_{\bar{i}}) d\bar{z}^{k}, \bar{z}^{j} g^{j\bar{m}} (\bar{\partial}_{A,\bar{m}} \phi^{*}_{\bar{n}} - \bar{\partial}_{A,\bar{n}} \phi^{*}_{\bar{m}}) d\bar{z}^{n}) \rangle \\
& - f^{2} \frac{G_{u_{0}}}{2(t-t_{0})} \langle z^{i} (\partial_{A,i}\phi_{k} - \partial_{A,k}\phi_{i}) dz^{k}, z^{j} g^{m\bar{j}} (\partial_{A,m}\phi_{n} - \partial_{A,n}\phi_{m}) dz^{n}) \rangle \Big\} \, dV_{g} \, dt.
\end{aligned}$$

Using the heat equation (1.3) and the Stokes formula, we have

$$\begin{aligned}
I_3 &= R^4 \int_{-4}^{-1} \int_{\mathbb{C}^m} 2R\tilde{t}\left[\frac{\partial}{\partial t} e(A,\phi)\right](R\tilde{z}, R^2\tilde{t}+t_0) f^2(R\tilde{z}) G(\tilde{z},\tilde{t}) \det(g_{i\bar{j}})(R\tilde{z})\, d\tilde{z}\, d\tilde{t} \\
&= 2R \int_{T_R(u_0)} (t-t_0)\left[\frac{\partial}{\partial t} e(A,\phi)\right] f^2 G_{u_0}\, dV_g\, dt \\
&= -4R \int_{T_R(u_0)} (t-t_0)\left[\left|\frac{\partial A}{\partial t}\right|^2 + 2\left|\frac{\partial \phi}{\partial t}\right|^2\right] f^2 G_{u_0}\, dV_g\, dt \\
&\quad + 16R \operatorname{Re} \int_{T_R(u_0)} (t-t_0)\left\langle \bar{\partial}_j (f^2 G_{u_0}) g^{i\bar{j}} (\partial_A \phi)_{ki} dz^k, \frac{\partial \phi}{\partial t}\right\rangle dV_g\, dt \\
&\quad + 4R \operatorname{Re} \int_{T_R(u_0)} (t-t_0)\left\langle \partial_j (f^2 G_{u_0}) g^{j\bar{i}} (F_A + [\phi,\phi^*])_{k\bar{i}} dz^k, \frac{\partial A}{\partial t}\right\rangle dV_g\, dt \\
&\quad - 4R \operatorname{Re} \int_{T_R(u_0)} (t-t_0)\left\langle \bar{\partial}_j (f^2 G_{u_0}) g^{i\bar{j}} (F_A + [\phi,\phi^*])_{i\bar{k}} d\bar{z}^k, \frac{\partial A}{\partial t}\right\rangle dV_g\, dt.
\end{aligned}$$

On the other hand, we know (e.g. [CSt]) that

$$R^{-1}|z|^2 G_{u_0} \le C(1+G_{u_0}), \qquad R^{-1}|t-t_0|^{-1}|z|^4 G_{u_0} \le C(1+G_{u_0}),$$

on $T_R(u_0)$. Moreover, we have $|\nabla f| G_{u_0} \le C$ due to the fact that $\nabla f = 0$ for $B_{i(M)/2}$. Combining these facts and the above inequalities, we have

$$\begin{aligned}
&\frac{d}{dR}\Phi(R;A,\phi) = I_1 + I_2 + I_3 + I_4 \\
&\ge R \int_{T_R(u_0)} \Bigg\{ 2|t-t_0| \left|\frac{\partial \phi}{\partial t} - \frac{z^k}{|t-t_0|} z^j g^{i\bar{k}} (\partial_{A,i}\phi_j - \partial_{A,j}\phi_i) dz^j)\right|^2 f^2 G_{u_0} \\
&\quad + |t-t_0| \left|\frac{\partial A}{\partial t} + \frac{1}{|t-t_0|} (\bar{z}^j g^{j\bar{i}} (F_A + [\phi,\phi^*])_{k\bar{i}} dz^k - z^i g^{j\bar{i}} (F_A + [\phi,\phi^*])_{j\bar{k}} d\bar{z}^k)\right|^2 f^2 G_{u_0} \\
&\quad - C e(A,\phi)\left(|\nabla f|\,|f|\,|z| G_{u_0} + f^2 G_{u_0}|z|^2 + f^2 G_{u_0}\frac{|z|^4}{|t-t_0|}\right) \\
&\quad - C|t-t_0|\,|\nabla f|^2 G_{u_0} |F_A + [\phi,\phi^*]|^2 \Bigg\} dV_g\, dt \\
&\ge -C\Phi(R;A,\phi) - CR\, \mathrm{YMH}(A_0,\phi_0).
\end{aligned}$$

By integrating the above inequality over R, we prove the claim for Φ. The claim for Ψ can be proved by a similar argument. □

3. Convergence properties of the heat flow

In the following part of this paper, we will always suppose that (M,ω) is a Kähler surface, i.e. the complex dimension m of (M,ω) is 2, and we will assume the volume of M with respect to ω is normalized to be 2π.

Theorem 3.1 (ϵ-regularity theorem). *Let (A,ϕ) be a solution of the gradient flow (1.3) over the Kähler surface (M,ω) with initial value (A_0,ϕ_0). There exist positive constants ϵ_0 and δ_0 such that if, for some R with $0<R<\min\{i(M),\sqrt{t_0}/2\}$, the inequality*

$$R^{2-2m}\int_{P_R(x_0,t_0)} e(A,\phi)\,dx\,dt \le \epsilon_0$$

holds, where $m=2$, then for any $\delta\in(0,\min\{1/4,\delta_0\})$ we have

$$\sup_{P_{\delta R}(x_0,t_0)}(|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2)\le 16(\delta R)^{-4},$$

$$\sup_{P_{\delta R}(x_0,t_0)}|\nabla_A\phi|^2\le C_{11},$$

where C_{11} is a positive constant depending only on δ, R, the initial data (A_0,ϕ_0) and the geometry of (M,ω).

Proof. For any $\delta\in(0,1/4]$, we define the function

$$f(r)=(2\delta R-r)^4\sup_{P_r(x_0,t_0)}(|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2).$$

Since $f(r)$ is continuous and $f(2\delta R)=0$, $f(r)$ attains its maximum at a certain $r_0\in[0,2\delta R)$. We can find $(x_1,t_1)\in P_{r_0}(x_0,t_0)$ such that

$$(|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2)(x_1,t_1)=\sup_{P_{r_0}(z_0)}(|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2).$$

We claim that

$$f(r_0)\le 16. \tag{3.1}$$

Otherwise, we have

$$\rho_0:=(2\delta R-r_0)f(r_0)^{-1/4}<\frac{2\delta R-r_0}{2}.$$

Rescaling the Riemannian metric $\tilde g=\rho_0^{-2}g$ and $t=t_1+\rho_0^2\tilde t$, we have

$$|F_A+[\phi,\phi^*]|_{\tilde g}^2=\rho_0^4|F_A+[\phi,\phi^*]|_g^2,\qquad |\partial_A\phi|_{\tilde g}^2=\rho_0^4|\partial_A\phi|_g^2.$$

Set

$$e_{\rho_0}(x,\tilde t):=|F_A+[\phi,\phi^*]|_{\tilde g}^2+2|\partial_A\phi|_{\tilde g}^2=\rho_0^4 e(A,\phi)(x,t_1+\rho_0^2\tilde t),$$
$$b_{\rho_0}(x,\tilde t):=|\nabla_A\phi|_{\tilde g}^2(x,t_1+\rho_0^2\tilde t)=\rho_0^4|\nabla_A\phi|_g^2(x,t_1+\rho_0^2\tilde t).$$

We have $e_{\rho_0}(x_1,0)=1$ and

$$\sup_{\tilde P_1(x_1,0)} e_{\rho_0}=\rho_0^4\sup_{P_{\rho_0}(x_1,t_1)}(|F_A+[\phi,\phi^*]|_g^2+2|\partial_A\phi|_g^2)$$
$$\le\rho_0^4\sup_{P_{2\delta R+r_0/2}(x_0,t_0)}(|F_A+[\phi,\phi^*]|_g^2+2|\partial_A\phi|_g^2)\le\rho_0^4\left(\frac{2\delta R-r_0}{2}\right)^{-4}f(r_0)=16.$$

From the Bochner type inequality (2.11), we have

$$\begin{aligned}\left(\frac{\partial}{\partial\tilde{t}}-\Delta_{\tilde{g}}\right)(b_{\rho_0}+\rho_0^4) &= \rho_0^6\left(\frac{\partial}{\partial t}-\Delta_g\right)(|\nabla_A\phi|_g^2+1)\\ &\le C_7\rho_0^6(|F_A|_g+|\mathrm{Rm}|_g+|\mathrm{Ric}|_g+|\phi|_g^2)|\nabla_A\phi|_g^2\\ &\quad + C_7\rho_0^6|\phi|_g|\nabla\,\mathrm{Ric}|_g|\nabla_A\phi|_g\\ &\le C(b_{\rho_0}+\rho_0^4) \quad \text{in } \tilde{P}_1(x_1,0),\end{aligned}$$

where C is a positive constant depending only on the initial data (A_0,ϕ_0) and the geometry of (M,ω). Then Moser's iteration yields the parabolic mean-value inequality

$$\begin{aligned}\sup_{\tilde{P}_{1/2}(x_1,0)}(b_{\rho_0}+\rho_0^4) &\le C'\int_{\tilde{P}_1(x_1,0)}(b_{\rho_0}+\rho_0^4)\,dV_{\tilde{g}}\,d\tilde{t}\\ &\le C'\rho_0^{2-2m}\int_{P_{\rho_0}(x_1,t_1)}(|\nabla_A\phi|_g^2+1)\,dV_g\,dt\\ &\le C_{13}\rho_0^{4-2m}=C_{13},\end{aligned} \tag{3.2}$$

where C_{13} is a positive constant depending only on the initial data (A_0,ϕ_0) and the geometry of (M,ω), and we have used the fact that $m=2$. Using the Bochner type estimate (2.12), we have

$$\begin{aligned}\left(\frac{\partial}{\partial\tilde{t}}-\Delta_{\tilde{g}}\right)e_{\rho_0} &= \rho_0^6\left(\frac{\partial}{\partial t}-\Delta_g\right)(|F_A+[\phi,\phi^*]|_g^2+2|\partial_A\phi|_g^2)\\ &\le C\rho_0^6(|F_A+[\phi,\phi_*]|_g+|\mathrm{Rm}|_g\\ &\quad +|\phi|_g^2+|\nabla_A\phi|_g)(|F_A+[\phi,\phi^*]|_g^2+2|\partial_A\phi|_g^2)\\ &\le C_{14}e_{\rho_0} \quad \text{in } \tilde{P}_{1/2}(x_1,0),\end{aligned}$$

where the positive constant C_{14} depends only on $i(M)$, the initial data (A_0,ϕ_0) and the geometry of (M,ω). Then Moser's iteration yields the parabolic mean-value inequality

$$\begin{aligned}1=e_{\rho_0}(x_1,0) &\le C'\int_{\tilde{P}_{1/2}(x_1,0)}e_{\rho_0}\,dV_{\tilde{g}}\,d\tilde{t}\\ &\le C'\rho_0^{2-2m}\int_{P_{\rho_0}(x_1,t_1)}(|F_A+[\phi,\phi^*]|_g^2+2|\partial_A\phi|_g^2)\,dV_g\,dt,\end{aligned} \tag{3.3}$$

where C' is a positive constant depending only the initial data (A_0,ϕ_0) and the geometry of (M,g).

We choose normal complex coordinates centered at x_1, and let $\varphi\in C_0^\infty(B_{3R/4}(x_1))$ be a cut-off function such that $\varphi\equiv 1$ on $B_{R/2}(x_1)$, $|\varphi|\le 1$ and $|\nabla\varphi|\le 8/R$. Taking $r_1=\rho_0$ and $r_2=\min\{1/4,\delta_0\}R$ in Theorem 2.3, we have

$$
\begin{aligned}
&\rho_0^{2-2m}\int_{P_{\rho_0}(x_1,t_1)}(|F_A+[\phi,\phi^*]|_g^2+2|\partial_A\phi|_g^2)\,dV_g\,dt\\
&\le C^*\rho_0^2\int_{P_{\rho_0}(x_1,t_1)}(|F_A+[\phi,\phi^*]|_g^2+2|\partial_A\phi|_g^2)G_{(x_1,t_1+2\rho_0^2)}\varphi^2\,dV_g\,dt\\
&\le C^*\rho_0^2\int_{T_{\rho_0}(x_1,t_1+2\rho_0^2)}e(A,\phi)G_{(x_1,t_1+2\rho_0^2)}\varphi^2\,dV_g\,dt\\
&\le \bar{C}R^2\int_{T_{\min\{1/4,\delta_0\}R}(x_1,t_1+2\rho_0^2)}e(A,\phi)G_{(x_1,t_1+2\rho_0^2)}\varphi^2\,dV_g\,dt\\
&+\bar{C}\delta_0^2R^2\,\mathrm{YMH}(A_1^0,A_2^0,\phi^0)\\
&\le \bar{C}R^{2-2m}\int_{P_R(x_0,t_0)}e(A,\phi)\,dV_g\,dt+\bar{C}\delta_0^2R^2\,\mathrm{YMH}(A_1^0,A_2^0,\phi^0)\\
&\le \tilde{C}(\epsilon_0+\delta_0),
\end{aligned}
\tag{3.4}
$$

where the above constants depend only the geometry of (M, g) and the initial data (A_0, ϕ_0).

Choosing ϵ_0 and δ_0 sufficiently small, we see that (3.3) contradicts (3.2). Thus, we have proved the claim (3.1). It implies

$$\sup_{P_{\delta R}(x_0,t_0)}(|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2)\le f(r_0)(\delta R)^{-4}\le 16(\delta R)^{-4}.$$

On the other hand, using the Bochner type inequality (2.11), inequality (3.4), and Moser's iteration again, we have

$$\sup_{P_{\delta R}(x_0,t_0)}(|\nabla_A\phi|^2+1)\le C_{15}\int_{P_{\min\{1/4,\delta_0\}R}(x_0,t_0)}(|\nabla_A\phi|^2+1)\,dV_g\,dt\le C_{12}$$

for any $\delta\in(0,\min\{1/4,\delta_0\})$, where C_{15} and C_{12} are positive constants depending only on δ, R, the initial data (A_0,ϕ_0) and the geometry of (M,ω). This proves Theorem 3.1.

□

Using the above ϵ-regularity theorem, we can analyze the asymptotic behavior of the gradient heat flow (1.3) on a Kähler surface. In fact, we obtain the following theorem.

Theorem 3.2. *Let $(A,\phi)(x,t)$ be a global smooth solution of the gradient heat flow (1.3) on a Kähler surface (M,ω) with smooth initial data. Then there exists a sequence $t_i\to\infty$ such that $(A,\phi)(x,t_i)$ converges, modulo gauge transformations, to a solution $(A,\phi)(\cdot,\infty)$ of the Yang–Mills–Higgs equation (1.2) in the smooth topology outside a closed set $\Sigma^{\mathrm{an}}\subset M$, where Σ^{an} is a finite collection of points.*

Proof. Let $(A(t),\phi(t))$ be a regular solution of the gradient flow (1.3) in the space $\mathcal{B}$ of Higgs pairs. By Lemma 2.2, we know that $\int_0^\infty\int_M(|\partial A/\partial t|^2+2|\partial\phi/\partial t|^2)\,dV_g\,dt$ is finite, so we can choose a sequence $t_k\to\infty$ such that

$$\int_{t_k-a}^{t_k+a}\int_M\left(\left|\frac{\partial A}{\partial t}\right|^2+2\left|\frac{\partial\phi}{\partial t}\right|^2\right)dV_g\,dt\to 0 \tag{3.5}$$

for any $a > 0$, and $\frac{\partial A}{\partial t}(\cdot, t_k)$ and $\frac{\partial \phi}{\partial t}(\cdot, t_k) \to 0$ strongly in L^2. Denote

$$\Sigma := \bigcap_{0<r<i(M)} \left\{ x \in M : \liminf_{k\to\infty} \int_{B_r(x)} e(A, \phi)(\cdot, t_k)\, dV_g \geq \epsilon_1 \right\},$$

where ϵ_1 is determined later.

Suppose $x_1 \in M \setminus \Sigma$. Then there exists an $r_1 > 0$ such that, for some subsequence of t_k, still denoted by t_k, we have

$$\int_{B_{r_1}(x_1)} e(A, \phi)(\cdot, t_k)\, dV_g < \epsilon_1.$$

For sufficiently large k, using the local energy estimate lemma 2.2′, we have

$$\begin{aligned} r_1^{-2} &\int_{P_{r_1/2}(x_1,t_k)} e(A, \phi)(\cdot, t_k)\, dV_g\, dt \\ &\leq \frac{1}{2} \int_{B_{r_1}(x_1)} e(A, \phi)(\cdot, t_k)\, dV_g + C \int_{t_k - r_1^2}^{t_k + r_1^2} \int_M \left(\left| \frac{\partial A}{\partial t} \right|^2 + 2 \left| \frac{\partial \phi}{\partial t} \right|^2 \right) dV_g\, dt \\ &\quad + C \left(\mathrm{YMH}(A_0, \phi_0) \int_{t_k - r_1^2}^{t_k + r_1^2} \int_M \left(\left| \frac{\partial A}{\partial t} \right|^2 + 2 \left| \frac{\partial \phi}{\partial t} \right|^2 \right) dV_g\, dt \right)^{1/2} \\ &\leq \epsilon_1, \end{aligned}$$

where we choose $\epsilon_1 = \epsilon_0$ and ϵ_0 is the constant of Theorem 3.1. Noting $|\phi|$ is bounded by a constant, by the ϵ-estimate in Theorem 3.1, we have

$$\sup_{P_{\delta r_1}(x_1,t_k)} (|F_A|^2 + |\nabla_A \phi|^2) \leq C_{16}, \tag{3.6}$$

where C_{16} is a positive constant depending only on δ, r_1, the initial data, and the geometry of (M, ω). Then, for any $x_1 \in M \setminus \Sigma$, there exists a sufficiently small number δ such that $B_{\delta r_1}(x_1) \subset M \setminus \Sigma$. This implies that the singular set Σ is closed.

Next we prove that Σ is a finite collection of points. For a given $\delta > 0$, due to the fact that Σ is closed, we may find a finite collection of geodesic balls $\{B_{r_i}(x_i)\}_{i\in\Gamma}$, $r_i \leq \delta$, such that $\{B_{r_i}(x_i)\}_{i\in\Gamma}$ is a cover of Σ, $x_i \in \Sigma$ for all $i \in \Gamma$, and $B_{r_i/2}(x_i) \cap B_{r_j/2}(x_j) = \emptyset$ for $i \neq j$. For k sufficiently large, we have

$$\int_{B_{r_i/2}} e(A, \phi)(\cdot, t_k)\, dV_g \geq \epsilon_1$$

for all $i \in \Gamma$. Summing over $i \in \Gamma$ we then get

$$\sum_{i\in\Gamma} 1 \leq \frac{1}{\epsilon_1} \mathrm{YMH}(A, \phi)(t_k) \leq \frac{1}{\epsilon_1} \mathrm{YMH}(A_0, \phi_0),$$

i.e. Σ is a finite collection of points, and the number of points is less than $(1/\epsilon_1)\, \mathrm{YMH}(A_0, \phi_0)$.

By using Uhlenbeck's Theorem ([U2, Theorem 3.6]) and the above estimates (3.6), there exists a subsequence $\{k'\}$ of $\{k\}$ and gauge transformations $\sigma(k')$ on E such that $(\sigma(k')(A(t_{k'})), \sigma(k')(\phi(t_{k'})))$ converges to a pair $(A, \phi)(\cdot, \infty)$ weakly in the $H^{1,2}$ topology on any compact subset outside Σ and $(A, \phi)(\cdot, \infty)$ is a solution of the Yang–Mills–Higgs equation (1.2) outside Σ. Using standard parabolic regularity techniques (similar to that in [HT, Proposition 6]), we can also prove that $(\sigma(k')(A(t_{k'})), \sigma(k')(\phi(t_{k'})))$ converges to $(A, \phi)(\cdot, \infty)$ in the C^∞ topology outside Σ. In fact, for $x_1 \in M \setminus \Sigma$, from the above, we see that there exists a small R_0 such that

$$\sup_{P_{R_0}(x_1,t_k)} (|F_A|^2 + |\nabla_A \phi|^2) \leq C$$

for sufficiently large k. We assume that

$$\sup_{P_{R_j}(x_1,t_k)} (|\nabla_A^j F_A|^2 + |\nabla_A^{j+1}\phi|^2) \leq C$$

for $j = 0, \ldots, l-1$. By an argument similar to the one used in the proof of Lemmas 2.4 and 2.5, we have

$$\left(\Delta - \frac{\partial}{\partial t}\right)(|\nabla_A^j F_A|^2 + |\nabla_A^{j+1}\phi|^2) \geq 2(|\nabla_A^{j+1} F_A|^2 + |\nabla_A^{j+2}\phi|^2) - C(|\nabla_A^j F_A|^2 + |\nabla_A^{j+1}\phi|^2)$$

in $P_{R_j}(x_1, t_k)$ with small R_j and $j = 0, \ldots, l$. Then Moser's parabolic estimate yields

$$\sup_{P_{\delta R_l}(x_1,t_k)} (|\nabla_A^l F_A|^2 + |\nabla_A^{l+1}\phi|^2) \leq C.$$

We can now use a standard diagonal process of gluing gauges ([DK, Th. 4.4.8]) and again passing to a subsequence to obtain smooth gauge transformations σ_i such that $(\sigma_i(A(t_i)), \sigma_i(\phi(t_i)))$ converge to a field (A_∞, ϕ_∞) in the smooth topology on compact subsets of $M \setminus \Sigma$. This proves the theorem. □

Remark. From the above theorem, we know that the limiting Higgs pair (A_∞, ϕ_∞) satisfies the Yang–Mills–Higgs equation (1.2) outside a finite collection of points Σ^{an}. On the other hand, from Lemma 2.3 and (2.10), we know that $|\phi_\infty|$ and $\int_M |\nabla_{A_\infty}\phi_\infty|^2$ are bounded. In the following, we will give a removable singularities theorem for the solution of the Yang–Mills–Higgs equation (1.2), from which we can deduce that the limiting Higgs pair extends smoothly to M by a continuous gauge transformation. For convenience, we concentrate on bundles over flat manifolds. For the regularity theory, the curvature of the base manifold itself is not particularly important. The restriction to a flat base manifold does not crucially affect our results.

Before discussing the removable singularities theorem, we state some standard a priori estimates in PDE which will be used repeatedly.

Theorem 3.3 ([HL]). *Assume that $n \geq 3$ and B_1 is the unit ball in $\mathbb{R}^n$. Suppose that $a^{ij} \in L^\infty(B_1)$ satisfies $\lambda_1|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \lambda_2|\xi|^2$ for any $x \in B_1$, $\xi \in \mathbb{R}^n$, for some* *positive constants* λ_1 *and* λ_2*. Assume that* $f_1 \in L^{n/2}(B_1)$ *and* $f_2 \in L^q(B_1)$ *for some* $q \in [2n/(n+2), n/2)$*. Suppose that* $u \in H^1(B_1)$ *is a subsolution in the following sense:*

$$\int_{B_1} \{a^{ij}\partial_i u \partial_j \varphi + f_1 u \varphi\} \le \int_{B_1} f_2 \varphi$$

for any $\varphi \in H_0^1(B_1)$ *and* $\varphi \ge 0$ *in* B_1*. Then* $u^+ \in L^p_{\mathrm{loc}}(B_1)$ *for* $1/p + 2/n = 1/q$*. Moreover, there exists a small positive constant* ϵ_2 *such that if* $\|f_1\|_{L^{n/2}(B_1)} < \epsilon_2$ *then*

$$\|u^+\|_{L^p(B_{1/2})} \le C(\|u^+\|_{L^2(B_1)} + \|f_2\|_{L^q(B_1)}),$$

where C *is a positive constant depending only on* $n, \lambda_1, \lambda_2, q, \epsilon_2$*.*

Theorem 3.4 ([M]). *Assume that* $b \in L^q(U)$*,* $q > n/2$*,* $u^\lambda \in W^{1,2}_{\mathrm{loc}}(U)$ *with* $1/2 < \lambda \le 1$*, and* $u \ge 0$ *satisfies the following subelliptic inequality in a weak sense:*

$$\Delta u + bu \ge 0.$$

Then u *is bounded on compact subdomains of* U*. Moreover, if* $B(x, r) \subset B(x_0, r_0) \subset U$*, then*

$$|u^\lambda(x)|^2 \le Cr^{-n} \int_{B(x_0,r_0)} |u^\lambda|^2,$$

where the constant C *depends on* n, q, λ *and* $r_0^{2/n-1/q}\|b\|_{L^q(B(x_0,r_0))}$*.*

In the following, we prove regularity of small energy solutions of the Yang–Mills–Higgs equation (1.2).

Theorem 3.5. *There exists a positive constant* ϵ_3 *such that if* (A, ϕ) *is a smooth solution of the Yang–Mills–Higgs equations* (1.2) *on* $B_2 \subset \mathbb{R}^4$*, satisfying* $\int_{B_2}\{|F_A|^2 + |\nabla_A\phi|^2 + |\phi|^4\} < \epsilon_3$*, then*

$$\|F_A + [\phi, \phi^*]\|_{C^0(B_1)} + \|\nabla_A\phi\|_{C^0(B_1)} \le C(\|F_A + [\phi, \phi^*]\|_{L^2(B_2)} + \|\nabla_A\phi\|_{L^2(B_2)}),$$

where C *is a positive constant depending only on* ϵ_3*.*

Proof. From (2.5), (2.11), (2.12) and (2.13), we have

$$\Delta|\phi|^2 \ge -C_4|\phi|^2, \tag{3.7}$$

$$\Delta|\nabla_A\phi|^2 - 2|\nabla_A\nabla_A\phi|^2 \ge -C_7(|F_A| + |\phi|^2)|\nabla_A\phi|^2, \tag{3.8}$$

$$\begin{aligned}\Delta(|F_A + [\phi, \phi^*]|^2 + 2|\partial_A\phi|^2) - 2|\nabla_A(F_A + [\phi, \phi^*])|^2 - 4|\nabla_A(\partial_A\phi)|^2 \\ \ge -C_8(|F_A| + |\nabla_A\phi| + |\phi|^2)(|F_A + [\phi, \phi^*]|^2 + 2|\partial_A\phi|^2),\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\Delta(|F_A + [\phi, \phi^*]|^2) - 2|\nabla_A(F_A + [\phi, \phi^*])|^2 \\ \ge -C_8(|F_A| + |\nabla_A\phi| + |\phi|^2)(|F_A + [\phi, \phi^*]|^2 + 2|\partial_A\phi|^2).\end{aligned} \tag{3.10}$$

It follows from (3.8) and (3.10) that

$$\Delta(|F_A+[\phi,\phi^*]|^2+|\nabla_A\phi|^2)-2|\nabla_A(F_A+[\phi,\phi^*])|^2-2|\nabla_A\nabla_A\phi|^2 \\ \geq -C_{17}(|F_A|+|\nabla_A\phi|+|\phi|^2)(|F_A+[\phi,\phi^*]|^2+|\nabla_A\phi|^2).$$

Using the Kato inequality, we have

$$\Delta|\nabla_A\phi|\geq -C_{18}(|F_A|+|\phi|^2)|\nabla_A\phi|,$$
$$\Delta\sqrt{|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2}\geq -C_{19}(|F_A|+|\nabla_A\phi|+|\phi|^2)\sqrt{|F_A+[\phi,\phi^*]|^2+2|\partial_A\phi|^2},$$

and

$$\Delta\sqrt{|F_A+[\phi,\phi^*]|^2+|\nabla_A\phi|^2} \\ \geq -C_{20}(|F_A|+|\nabla_A\phi|+|\phi|^2)\sqrt{|F_A+[\phi,\phi^*]|^2+|\nabla_A\phi|^2} \tag{3.11}$$

where C_{18}, C_{19} and C_{20} are positive constants depending only on the rank of the bundle E.

From (3.7) and the Sobolev inequality, we have

$$\big\||\phi|^2\big\|_{L^4(B_{1/2})}\leq C_{21}\big\||\phi|^2\big\|_{L^2(B_1)} \tag{3.12}$$

Applying (3.11) and Theorem 3.3 with $u=\sqrt{|F_A+[\phi,\phi^*]|^2+|\nabla_A\phi|^2}$, $f_1=-C_{20}(|F_A|+|\nabla_A\phi|+|\phi|^2)$, $f_2=0$, $p=4$, we see that if

$$\int_{B_1}\{|F_A|^2+|\nabla_A\phi|^2+|\phi|^4\}\leq\frac{1}{3}\left(\frac{\epsilon_2}{C_{20}}\right)^2,$$

i.e. $\|f_1\|_{L^2(B_1)}<\epsilon_2$, then

$$\left\|\sqrt{|F_A+[\phi,\phi^*]|^2+|\nabla_A\phi|^2}\right\|_{L^4(B_{1/2})} \\ \leq C_{\epsilon_2}\left\|\sqrt{|F_A+[\phi,\phi^*]|^2+|\nabla_A\phi|^2}\right\|_{L^2(B_1)}. \tag{3.13}$$

Choose $\epsilon_3=\frac{1}{3}(\epsilon_2/C_{20})^2$. If $\int_{B_2}\{|F_A|^2+|\nabla_A\phi|^2+|\phi|^4\}\leq\epsilon_3$ we can combine (3.12) and (3.13) to get the following inequality:

$$\|F_A\|_{L^4(B_{1/2})}+\|\nabla_A\phi\|_{L^4(B_{1/2})}+\|\,|\phi|^2\|_{L^4(B_{1/2})} \\ \leq 8\Big(\Big\|\sqrt{|F_A+[\phi,\phi^*]|^2+|\nabla_A\phi|^2}\Big\|_{L^4(B_{1/2})}+\big\||\phi|^2\big\|_{L^4(B_{1/2})}\Big) \\ \leq C_{22}\big(\big\||\phi|^2\big\|_{L^2(B_1)}+\|F_A\|_{L^2(B_1)}+\|\nabla_A\phi\|_{L^2(B_1)}\big),$$

where the constant C_{22} depends on ϵ_3.

Now, applying (3.11) and Theorem 3.4 with $u = \sqrt{|F_A + [\phi,\phi^*]|^2 + |\nabla_A\phi|^2}$, $b = C_{20}(|F_A| + |\nabla_A\phi| + |\phi|^2)$, $\lambda = 1$, $q = 4$, we have, for any $x \in B_1$,

$$
\begin{aligned}
(|F_A + [\phi,\phi^*]|^2 + |\nabla_A\phi|^2)(x) &\leq C_{23}\int_{B(x,1/2)}\{|F_A + [\phi,\phi^*]|^2 + |\nabla_A\phi|^2\} \\
&\leq C_{23}\int_{B_2}\{|F_A + [\phi,\phi^*]|^2 + |\nabla_A\phi|^2\}.
\end{aligned}
$$

Hence

$$
\|F_A + [\phi,\phi^*]\|_{C^0(B_1)} + \|\nabla_A\phi\|_{C^0(B_1)} \leq C_{24}(\|F_A + [\phi,\phi^*]\|_{L^2(B_2)} + \|\nabla_A\phi\|_{L^2(B_2)}).
$$

Since $\left\||\phi|^2\right\|_{L^2(B_1)}$, $\|F_A\|_{L^2(B_1)}$, $\|\nabla_A\phi\|_{L^2(B_1)}$ are all invariant under dilation, the size of the ball does not affect the constant C of Theorem 3.4, so C_{24} is a positive constant depending only on ϵ_3. □

Proposition 3.6. *There exists a positive constant ϵ_3 such that if (A,ϕ) is a smooth solution of the Yang–Mills–Higgs equations (1.2) on $B_2 \setminus \{0\}$, satisfying $\int_{B_2}\{|F_A|^2 + |\nabla_A\phi|^2 + |\phi|^4\} < \epsilon_3$, then for any $x \in B_1 \setminus \{0\}$,*

$$
|x|^2(|F_A + [\phi,\phi^*]| + |\nabla_A\phi|) \leq C(\|F_A + [\phi,\phi^*]\|_{L^2(B(0,2|x|))} + \|\nabla_A\phi\|_{L^2(B(0,2|x|))}), \tag{3.14}
$$

where C is a positive constant depending only on ϵ_3.

Proof. Choose ϵ_3 as in Theorem 3.5. For any $x_0 \in B_1 \setminus \{0\}$ we define

$$
\tilde{A}(x) := \frac{|x_0|}{2}A\left(x_0 + \frac{|x_0|}{2}x\right), \quad \tilde{\phi}(x) := \frac{|x_0|}{2}\phi\left(x_0 + \frac{|x_0|}{2}x\right).
$$

It is clear that $(\tilde{A},\tilde{\phi})$ is a smooth solution of the Yang–Mills–Higgs equations (1.2) on B_2, and $\int_{B_2}\{|F_{\tilde{A}}|^2 + |\nabla_{\tilde{A}}\tilde{\phi}|^2 + |\tilde{\phi}|^4\} < \epsilon_3$ by conformal invariance. From Theorem 3.5, we have

$$
\|F_{\tilde{A}} + [\tilde{\phi},\tilde{\phi}^*]\|_{C^0(B_1)} + \|\nabla_{\tilde{A}}\tilde{\phi}\|_{C^0(B_1)} \leq C_{24}(\|F_{\tilde{A}} + [\tilde{\phi},\tilde{\phi}^*]\|_{L^2(B_2)} + \|\nabla_{\tilde{A}}\tilde{\phi}\|_{L^2(B_2)}).
$$

Scaling back, we deduce the proposition. □

For further discussion, we first recall Uhlenbeck's result about the existence of a specific gauge, called *broken Hodge gauge*. In such a gauge the powerful regularity argument of elliptic theory can be applied to the Yang–Mills–Higgs equations (1.2).

Theorem 3.7 ([U1]). *Let A be a smooth connection form on $B_2 \setminus \{0\}$ with curvature F. Then there is a positive constant ϵ_4 such that if $|F(x)|\,|x|^2 \leq \epsilon_5 < \epsilon_4$ on $B_1 \setminus \{0\}$ then there exists a broken Hodge gauge on $B_1 \setminus \{0\}$ which satisfies the following properties. Set $U_i = \{x : 2^{-i} \leq |x| \leq 2^{-i+1}\}$, $S_i = \{x : |x| = 2^{-i}\}$, $i = 1, 2, \ldots$. Then the broken Hodge gauge is smooth on each U_i and agrees on S_i. Write*

$$
A^i = A|_{U_i}, \quad F^i = F|_{U_i}.
$$

Then $\{A^i\}$ and $\{F^i\}$ satisfy:

(1) $d^*A^i = 0$ *;*
(2) $A^i_\theta|_{S_i} = A^{i+1}_\theta|_{S_i}$*, where* $A_\theta|_{S_i}$ *are tangential components of* A*;*
(3) $d^*A^i_\theta|_{S_i} = d^*A^i_\theta|_{S_{i-1}} = 0$*;*
(4) $\int_{S_i} A^i_r = \int_{S_{i-1}} A^i_r = 0$*, where* A_r *is the radial component of* A*;*
(5) $\int_{S_0} |A^1|^2 \le b_1 \int_{S_0} |F^1|^2$*,* $\int_{U_i} |A^i|^4 \le b_2\epsilon_5^2 \int_{U_i} |F^i|^2$*,* $\int_{B_1} |A|^4 \le b_2\epsilon_5^2 \int_{B_1} |F|^2 \le b_2\epsilon_5^4$ *for some positive constants* b_1 *and* b_2 *whenever* ϵ_5 *is sufficiently small.*

Theorem 3.8 (removable singularity). *Let* (A, ϕ) *be a smooth solution of the Yang–Mills–Higgs equations* (1.2) *on* $B_2 \setminus \{0\}$ *with* $\|\phi\|_{L^\infty(B_2)}$ *and* $\int_{B_2}\{|F_A|^2 + |\nabla_A\phi|^2\}$ *finite. Then* (A, ϕ) *is gauge equivalent by a continuous gauge transformation to a smooth solution on* B_2.

Proof. By rescaling, we can assume that $\|\phi\|_{L^\infty(B_2)} < \epsilon_6$ and $\int_{B_2}\{|F_A|^2 + |\nabla_A\phi|^2 < \epsilon_6\}$ with ϵ_6 sufficiently small. By Proposition 3.6, for any $x \in B_1 \setminus \{0\}$,

$$|x|^2|F|(x) \le |x|^2|F + [\phi, \phi^*]|(x) + 2|x|^2|\phi|^2(x) \le C\epsilon_6.$$

By Theorem 3.7, the above inequality guarantees the existence of a broken Hodge gauge. In what follows, the broken Hodge gauge is used as a reference frame.

Integrating by parts on U_i gives

$$\int_{U_i} |F^i|^2 = \int_{U_i} \{\langle D_A^* F^i, A^i\rangle - \langle F^i, A^i \wedge A^i\rangle\} + \left(\int_{S_{i-1}} - \int_{S_i}\right) \mathrm{tr}(A^i_\theta \wedge (*F^i)_\theta).$$

Using equation (1.2) and the properties of the broken Hodge gauge, we get

$$\begin{aligned}\int_{U_i} \langle D_A^* F^i, A^i\rangle &\le C \int_{U_i} |A^i|\,|\nabla_A\phi|\,|\phi| \\ &\le C \int_{U_i} \{\epsilon_5^{-1}|A^i|^4 + \epsilon_5^{1/4}|\phi|^4 + (\epsilon_5^{1/2} + \epsilon_5^{1/4})|\nabla_A\phi|^2\} \\ &\le C \int_{U_i} \{b_2\epsilon_5|F^i|^2 + \epsilon_5^{1/4}|\phi|^4 + \epsilon_5^{1/4}|\nabla_A\phi|^2\},\end{aligned}$$

and

$$-\int_{U_i} \langle F^i, A^i \wedge A^i\rangle \le \int_{U_i} |F^i|\,|A^i|^2 \le \left(\int_{U_i} |F^i|^2\right)^{1/2}\left(\int_{U_i} |A^i|^4\right)^{1/2} \le \sqrt{b_2}\epsilon_5 \int_{U_i} |F^i|^2.$$

Summing over i and using Hölder's inequality gives

$$\int_{B_1} |F|^2 \le C\epsilon_5 \int_{B_1} |F|^2 + C\epsilon_5^{1/4} \int_{B_1} (|\phi|^4 + |\nabla_A\phi|^2) + \left(\int_{\partial B_1} |F^1|^2\right)^{1/2}\left(\int_{\partial B_1} |A^1|^2\right)^{1/2}.$$

Since the bound ϵ_5 could be sufficiently small, and noting the properties of the broken Hodge gauge, we get

$$\int_{B_1} |F|^2 \le C\epsilon_5^{1/4} \int_{B_1} (|\phi|^4 + |\nabla_A\phi|^2) + C \int_{\partial B_1} |F^1|^2.$$

By a rescaling argument, we have, for $0 < r \leq 1$,

$$\int_{B_r} |F|^2 \leq C\epsilon_5^{1/4} \int_{B_r} (|\phi|^4 + |\nabla_A \phi|^2) + Cr \int_{\partial B_r} |F^1|^2. \tag{3.15}$$

From (2.5), we have

$$\Delta|\phi|^2 = 2|\Lambda_\omega[\phi, \phi^*]|^2 + 2|\nabla_A \phi|^2.$$

Integrating the above equality on U_i, we have

$$\int_{U_i} |\nabla_A \phi|^2 = \frac{1}{2} \int_{U_i} \left\{ \Delta \frac{1}{2}|\phi|^2 + |\Lambda_\omega[\phi, \phi^*]|^2 \right\} \leq \frac{1}{2}\left(\int_{S_{i-1}} - \int_{S_i} \right) \frac{\partial}{\partial r}|\phi|^2 + C \int_{U_i} |\phi|^4.$$

From (3.14) and the bound of $|\phi|$, it is easy to check that

$$\int_{S_i} |\nabla_A \phi|\,|\phi| \to 0 \tag{3.16}$$

as $i \to \infty$. Summing over i, we get

$$\int_{B_1} |\nabla_A \phi|^2 \leq C \int_{B_1} |\phi|^4 + C \int_{\partial B_1} |\nabla_A \phi|^2 + C \int_{\partial B_1} |\nabla_A \phi|^2.$$

By a scaling argument again, we have, for any $0 < r \leq 1$,

$$\int_{B_r} |\nabla_A \phi|^2 \leq C \int_{B_r} |\phi|^4 + Cr \int_{\partial B_r} |\nabla_A \phi|^2 + Cr^{-1} \int_{\partial B_r} |\nabla_A \phi|^2. \tag{3.17}$$

Putting (3.15), (3.17) together, and using the bound of $|\phi|$, one checks that for any $0 < r \leq 1$,

$$\int_{B_r} (|F|^2 + |\nabla_A \phi|^2) \leq Cr \int_{\partial B_r} (|F|^2 + |\nabla_A \phi|^2) + Cr^2,$$

where $C > 0$ is a constant. Denote $f(r) := \int_{B_r} (|F|^2 + |\nabla_A \phi|^2)$. Then the above inequality implies that

$$f(r) \leq Crf'(r) + Cr^2.$$

Since $f'(r) \geq 0$, and we may choose $C > 1$, it follows that

$$\frac{d}{dr}\left(\frac{f(r)}{r^\alpha} + \frac{1}{2-\alpha} r^{2-\alpha} \right) \geq 0,$$

where $\alpha := 1/C$. From the above inequality, we have

$$f(r) \leq Cr^\alpha.$$

From the last inequality, it is easy to conclude that there exists $\beta > 2$ such that

$$\int_{B_1} (|F|^\beta + |\nabla_A \phi|^\beta) \leq C,$$

and hence

$$A \in W^{1,\beta}(B_1), \quad \phi \in W^{1,\beta}(B_1).$$

Since $\overline{\partial}_A\phi = 0$, we have $\Delta_A\phi = [\sqrt{-1}\,\Lambda_\omega F_A, \phi] - F_A \sharp \phi$. In the broken Hodge gauge, the Yang–Mills–Higgs equations (1.2) are uniformly elliptic systems. Above we have proved that $A \in W^{1,\beta}(B_1)$ and $\phi \in W^{1,\beta}(B_1)$ for some $\beta > 2$; we can then conclude that (A, ϕ) is smooth on B_1 by standard elliptic theory. This completes the proof of the theorem. □

Corollary 3.9. *The limiting Higgs pair (A_∞, ϕ_∞) of the gradient flow (1.3) can be extended smoothly by a continuous gauge transformation to a smooth solution of the Yang–Mills–Higgs equations (1.2) on M.*

Proof. From Theorem 3.2 and the remark, we know that $|\phi_\infty|$, $\int_M |F_{A_\infty} + [\phi_\infty, \phi_\infty^*]|^2$ and $\int_M |\nabla_{A_\infty}\phi_\infty|^2$ are bounded. So the statement follows by the removable singularities Theorem 3.8. □

Proposition 3.10. *Let (A, ϕ) be a Higgs pair on a Hermitian vector bundle (E, H) over a Kähler manifold M, and suppose that it is a critical point of the Yang–Mills–Higgs functional* (1.3) *(i.e. (A, ϕ) is a solution of the Yang–Mills–Higgs equations* (1.2)*). Then the Higgs bundle (E, H, A, ϕ) has a holomorphic orthogonal splitting*

$$(E, H, A, \phi) = \bigoplus_{i=1}^{l} (E^i, H^i, A^i, \phi^i),$$

where E_i are ϕ-invariant and H^i are Hermitian-Einstein metrics on the Higgs bundle (E^i, A^i, ϕ^i).

Proof. From the equation (1.2), we have

$$D_A\theta = 0, \quad [\theta, \phi] = 0,$$

where $\theta = \Lambda_\omega(F_A + [\phi, \phi^*])$. Since θ is parallel and $(\sqrt{-1}\,\theta)^* = \sqrt{-1}\,\theta$, we can decompose E according to the eigenvalues of $\sqrt{-1}\,\theta$. We obtain a holomorphic orthogonal decomposition

$$E = \bigoplus_{i=1}^{l} E^i,$$

and

$$\phi : E^i \to E^i.$$

Let H^i be the restrictions of H to E^i, ϕ^i be the restriction of ϕ to E^i, and $A^i = A|_{E^i}$. Then (A^i, ϕ^i) is a Higgs pair on (E^i, H_i) and satisfies

$$\sqrt{-1}\,\Lambda_\omega(F_{A^i} + [\phi^i, (\phi^i)^*]) = \lambda_i \,\mathrm{Id}_{E^i}.$$

So (A^i, ϕ^i) is a Hermitian-Einstein Higgs pair on (E^i, H^i), i.e. (E^i, H^i, A^i, ϕ^i) is a Hermitian-Einstein Higgs bundle. □

Combining Theorem 3.2, Corollary 3.9 and Proposition 3.10, we have the following theorem.

Theorem 3.11. *Let $(A,\phi)(x,t)$ be a global smooth solution of the gradient heat flow (1.3) on the Kähler surface (M,ω) with smooth initial data. Then there exists a sequence $t_i \to \infty$ such that $(A,\phi)(x,t_i)$ converges, modulo gauge transformations, to a solution (A_∞,ϕ_∞) of the Yang–Mills–Higgs equation (1.2) in the smooth topology outside a closed set $\Sigma^{\mathrm{an}} \subset M$, where Σ^{an} is a finite collection of points. We will call (A_∞,ϕ_∞) an Uhlenbeck limit of the gradient flow. Moreover, the limiting Higgs pair (A_∞,ϕ_∞) can be extended smoothly by a continuous gauge transformation to a smooth solution of the Yang–Mills–Higgs equations (1.2) on a Hermitian bundle (E_∞,H_∞) over M, and the extension $(E_\infty,H_\infty,A_\infty,\phi_\infty)$ has a holomorphic orthogonal splitting*

$$(E_\infty,H_\infty,A_\infty,\phi_\infty)=\bigoplus_{i=1}^{l}(E_\infty^i,H_\infty^i,A_\infty^i,\phi_\infty^i),$$

where H_∞^i is a Hermitian-Einstein metric on the Higgs bundle $(E_\infty^i,A_\infty^i,\phi_\infty^i)$.

Corollary 3.12. *Let (A_i,ϕ_i) be a sequence of Higgs pairs along the gradient heat flow (1.3) with Uhlenbeck limit (A_∞,ϕ_∞). Then:*

(1) *$\theta(A_i,\phi_i)\to\theta(A_\infty,\phi_\infty)$ in L^p for all $1\le p<\infty$, and $\lim_{t\to\infty}\int_M|\theta(A_t,\phi_t)|^2=\int_M|\theta(A_\infty,\phi_\infty)|^2$;*

(2) *$\|\theta(A_\infty,\phi_\infty)\|_{L^\infty}\le\|\theta(A_j,\phi_j)\|_{L^\infty}\le\|\theta(A_{t_0},\phi_{t_0})\|_{L^\infty}$ for $0\le t_0\le t_j$.*

Here $\theta(A,\phi)=\Lambda_\omega(F_A+[\phi,\phi^*])$.

Proof. (1) From Theorem 3.11, we know that (A_j,ϕ_j) converges to the Uhlenbeck limit in the smooth topology outside a finite collection of points; on the other hand, from (2.8), we know that $\|\theta(A_t,\phi_t)\|_{L^\infty}$ is decreasing in t, so it follows that $\theta(A_i,\phi_i)\to\theta(A_\infty,\phi_\infty)$ in L^p for all $1\le p<\infty$. The second part is a consequence of Lemma 2.2.

(2) Fix $t\ge 0$. We have

$$\|\theta(A_j,\phi_j)\|_{L^p}\le(\mathrm{Vol}(M))^{1/p}\|\theta(A_j,\phi_j)\|_{L^\infty}\le(\mathrm{Vol}(M))^{1/p}\|\theta(A_t,\phi_t)\|_{L^\infty}$$

for any $1\le p<\infty$ and j sufficiently large. Using the result in (1), we get $\|\theta(A_\infty,\phi_\infty)\|_{L^p}\le(\mathrm{Vol}(M))^{1/p}\|\theta(A_t,\phi_t)\|_{L^\infty}$. Letting $p\to\infty$, we conclude that $\|\theta(A_\infty,\phi_\infty)\|_{L^\infty}\le\|\theta(A_t,\phi_t)\|_{L^\infty}$. □

4. Harder–Narasimhan–Seshadri filtration of Higgs bundles

Given a Higgs bundle (E,A,ϕ) on a Kähler surface (M,ω), a *Higgs subsheaf* of (E,A,ϕ) is a coherent analytic subsheaf $V\subset(E,A)$ such that $\phi: V\to V\otimes\Omega_M^1$ (i.e. a ϕ-invariant coherent analytic subsheaf). The *ω-slope* $\mu(V)$ of a torsion-free sheaf $V\to M$ is defined by

$$\mu_\omega(V)=\frac{\deg_\omega(V)}{\mathrm{rank}(V)}=\frac{1}{\mathrm{rank}(V)}\int_M C_1(V)\wedge\frac{\omega^{n-1}}{(n-1)!}.$$

If V is a saturated subsheaf then outside a codimension 2 subset it is a subbundle. Since $\dim_{\mathbb{C}} M = 2$, the singular set of a saturated subsheaf is a locally finite collection of points.

A torsion-free Higgs sheaf (V, A, ϕ) is called *ω-stable* (resp. *ω-semistable*) if for all proper ϕ-invariant saturated subsheaves $F \subset V$, $\mu_\omega(F) < \mu_\omega(V)$ $(\mu_\omega(F) \leq \mu_\omega(V))$. When the Kähler form is understood we shall sometimes refer to (V, A, ϕ) as simply stable or semistable, and we will also omit subscripts and write $\mu(V)$.

In the following, we will give a description of the appropriate Higgs bundle versions of the Harder–Narasimhan filtration and the Harder–Narasimhan–Seshadri filtration; the proof is almost the same as in the holomorphic bundles case ([Ko, 7.15, 7.17, 7.18]), the only difference being that we always consider ϕ-invariant subsheaves instead of usual subsheaves. We omit the details here.

Proposition 4.1 *Let $(E, A, \phi) \to (M, \omega)$ be a Higgs bundle. There is a unique Higgs subsheaf V with torsion-free quotient E/V such that for every Higgs subsheaf $W \subset E$, we have:*

(1) $\mu(W) \leq \mu(V)$;
(2) $\mathrm{rank}(W) \leq \mathrm{rank}(V)$ if $\mu(W) = \mu(V)$.

Proposition 4.2 *Let $(E, A, \phi) \to (M, \omega)$ be a Higgs bundle. Then there is a filtration of E by ϕ-invariant coherent subsheaves*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E,$$

called the Harder–Narasimhan filtration *of the Higgs bundle (E, A, ϕ) (abbr. HN filtration), such that $Q_i = E_i/E_{i-1}$ is torsion-free and Higgs semistable. Moreover, $\mu(Q_i) > \mu(Q_{i+1})$, and the associated graded object $\mathrm{Gr}^{\mathrm{hn}}(E, A, \phi) = \bigoplus_{i=1}^{l} Q_i$ is uniquely determined by the isomorphism class of (E, A, ϕ).*

Proposition 4.3 *Let (V, ϕ) be a semistable Higgs sheaf over a Kähler surface (M, ω). Then there is a filtration of V by ϕ-invariant subsheaves*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_l = V,$$

called the Seshadri filtration *of (V, ϕ), such that V_i/V_{i-1} is torsion-free and Higgs stable. Moreover, $\mu(V_i/V_{i-1}) = \mu(V)$ for each i, and the associated graded object $\mathrm{Gr}^{\mathrm{s}}(V, \phi) = \bigoplus_{i=1}^{l} V_i/V_{i-1}$ is uniquely determined by the isomorphism class of (V, ϕ).*

Let (E, A, ϕ) be a Higgs bundle over a Kähler surface (M, ω). Then there is a double filtration, called the Harder–Narasimhan–Seshadri filtration *of the Higgs bundle (E, A, ϕ) (abbr. HNS filtration), with the following properties: if $\{E_i\}_{i=1}^{l}$ is the HN filtration of (E, A, ϕ), then*

$$E_{i-1} = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,l_i} = E_i$$

and the successive quotients $Q_{i,j} = E_{i,j}/E_{i,j-1}$ are Higgs stable torsion-free sheaves. Moreover, $\mu(Q_{i,j}) = \mu(Q_{i,j+1})$ and $\mu(Q_{i,j}) > \mu(Q_{i+1,j})$, and the associated graded *object*

$$\mathrm{Gr}^{\mathrm{hns}}(E, A, \phi) = \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{l_i} Q_{i,j}$$

is uniquely determined by the isomorphism class of (E, A, ϕ).

It will be convenient to denote the ϕ-invariant subsheaf E_i in the HN filtration by $\mathrm{F}_i^{\mathrm{hn}}(E, A, \phi)$, or by $\mathrm{F}_{i,\omega}^{\mathrm{hn}}(E, A, \phi)$, when we wish to emphasize the role of the Kähler structure.

Definition 4.4. For a Higgs bundle (E, A, ϕ) of rank R, construct a nonincreasing R-tuple of numbers

$$\vec{\mu}(E, A, \phi) = (\mu_1, \ldots, \mu_R)$$

from the HN filtration by setting $\mu_i = \mu(Q_j)$ for $\mathrm{rank}(E_{j-1}) + 1 \leq i \leq \mathrm{rank}(E_j)$. We call $\vec{\mu}(E, A, \phi)$ the *Harder–Narasimhan type* of (E, A, ϕ).

Remark. For a pair $\vec{\mu}, \vec{\lambda}$ of R-tuples satisfying $\sum_{i=1}^{R} \mu_i = \sum_{i=1}^{R} \lambda_i$, we define

$$\vec{\mu} \leq \vec{\lambda} \ \Leftrightarrow \ \sum_{i \leq k} \mu_i \leq \sum_{i \leq k} \lambda_i \quad \text{for all } k = 1, \ldots, R.$$

We next turn to the HN type of the Uhlenbeck limit. We can obtain the following:

Lemma 4.5. *Let* $(A_j, \phi_j) = g_j(A_0, \phi_0)$ *be a sequence of complex gauge equivalent Higgs pairs on a complex vector bundle* E *of rank* R *with Hermitian metric* H_0. *Let* S *be a coherent* ϕ_0*-invariant subsheaf of* (E, A_0). *Suppose that* $\sqrt{-1}\,\Lambda_\omega(F_{A_j} + [\phi_j, \phi_j^*]) \to \mathfrak{a}$ *in* L^1 *as* $j \to \infty$, *where* $\mathfrak{a} \in L^1(\sqrt{-1}\,\mathfrak{u}(E))$, *and that the eigenvalues* $\lambda_1 \geq \cdots \geq \lambda_R$ *of* $\mathfrak{a}$ *are constant almost everywhere. Then* $\deg(S) \leq \sum_{i \leq \mathrm{rank}(S)} \lambda_i$.

Proof. Since $\deg(S) \leq \deg(\mathrm{Sat}_E(S))$, we may assume that S is saturated. Let π_j denote the orthogonal projection onto $g_j(S)$ with respect to the Hermitian metric H_0. It is well known that π_j is an L_1^2 section of the smooth endomorphism bundle of E, and satisfies $\pi_j^2 = \pi_j = \pi_j^*$, $(\mathrm{Id} - \pi_j)\bar{\partial}_{A_j}\pi_j = 0$ and $(\mathrm{Id} - \pi_j)\phi_i\pi_j = 0$ (since $g_j(S)$ is ϕ_j-invariant). The usual degree formula applies (see [Si1, Lemma 3.2]), and one has

$$\deg(S) = \frac{1}{2\pi} \int_M \big(\mathrm{Tr}(\sqrt{-1}\,(F_{A_j} + [\phi_j, \phi_j^*])\pi_j) - |(\bar{\partial}_{A_j} + \phi)\pi_j|^2\big).$$

Then one can show the result by an argument similar to the one used in the proof of Proposition 2.21 in [DW1] for the Yang–Mills case. □

Using Corollary 3.12 and Lemma 4.5, and arguing as in [DW1, Proposition 2.21], we have:

Proposition 4.6. *Let* (A_i, ϕ_i) *be a sequence of Higgs pairs along the gradient heat flow* (1.3) *with Uhlenbeck limit* (A_∞, ϕ_∞). *Let* $\vec{\mu}_0 = (\mu_1, \cdots, \mu_R)$ *be the HN type of the Higgs bundle* (E, A_0, ϕ_0), *and let* $\vec{\lambda}_\infty = (\lambda_1, \ldots, \lambda_R)$ *be the type of the Higgs bundle* $(E_\infty, A_\infty, \phi_\infty)$. *Then* $\vec{\mu}_0 \leq \vec{\lambda}_\infty$.

5. The HN type of the Uhlenbeck limit

Let (A_t, ϕ_t) be a smooth solution of the gradient heat flow (1.3) over a Kähler surface with initial data (A_0, ϕ_0), and let (A_∞, ϕ_∞) be an Uhlenbeck limit. From Theorem 3.11, we know that (A_∞, ϕ_∞) is a smooth Yang–Mills–Higgs pair on a Hermitian bundle (E_∞, H_∞), so $\theta(A_\infty, \phi_\infty)$ is parallel, and the constant eigenvalues vector $\vec{\lambda}_\infty = (\lambda_1, \dots, \lambda_R)$ of $\sqrt{-1}\,\theta(A_\infty, \phi_\infty)$ is just the HN type of the Uhlenbeck limit Higgs bundle $(E_\infty, A_\infty, \phi_\infty)$. Let $\vec{\mu}$ be the HN type of the initial Higgs bundle (E, A_0, ϕ_0). We will prove that $\vec{\lambda}_\infty = \vec{\mu}$.

Let $\mathfrak{u}(R)$ denote the Lie algebra of the unitary group $U(R)$. Fix a real number $\alpha \geq 1$, and for any $\mathfrak{a} \in \mathfrak{u}(R)$, let $\varphi_\alpha(\mathfrak{a}) = \sum_{j=1}^{R} |\lambda_j|^\alpha$, where $\sqrt{-1}\,\lambda_j$ are the eigenvalues of $\mathfrak{a}$. It is easy to see that we can find a family $\varphi_{\alpha,\rho}$, $0 < \rho \leq 1$, of smooth convex ad-invariant functions such that $\varphi_{\alpha,\rho} \to \varphi_\alpha$ uniformly on compact subsets of $\mathfrak{u}(R)$ as $\rho \to 0$. Hence, from [AB, Prop. 12.16] it follows that φ_α is a convex function on $\mathfrak{u}(R)$. For a given real number N, define

$$\mathrm{HYM}_{\alpha,N}(A, \phi) = \int_M \varphi_\alpha(-\theta(A, \phi) + \sqrt{-1}\,N\,\mathrm{Id}_E)\,d\mathrm{vol}. \tag{5.1}$$

Also, we will set $\mathrm{HYM}_{\alpha,N}(\vec{\mu}) = \mathrm{HYM}_\alpha(\vec{\mu}+N) = 2\pi\varphi_\alpha(\sqrt{-1}\,(\vec{\mu}+N))$, where $\vec{\mu}+N = \mathrm{diag}(\mu_1 + N, \dots, \mu_R + N)$. We will need the following two lemmas, whose proofs can be found in [DW1, Lemma 2.23 and Prop. 2.24].

Lemma 5.1. *The functional* $\mathfrak{a} \mapsto (\int_M \varphi_\alpha(\mathfrak{a})\,d\mathrm{vol})^{1/\alpha}$ *defines a norm on* $L^\alpha(\mathfrak{u}(E))$ *which is equivalent to the* L^α *norm.*

Lemma 5.2. (1) *If* $\vec{\mu} \leq \vec{\lambda}$, *then* $\varphi_\alpha(\sqrt{-1}\,\vec{\mu}) \leq \varphi_\alpha(\sqrt{-1}\,\vec{\lambda})$ *for all* $\alpha \geq 1$.
(2) *Assume* $\mu_R \geq 0$ *and* $\lambda_R \geq 0$. *If* $\varphi_\alpha(\sqrt{-1}\,\vec{\mu}) = \varphi_\alpha(\sqrt{-1}\,\vec{\lambda})$ *for all* α *in some set* $S \subset [1, \infty)$ *possessing a limit point, then* $\vec{\mu} = \vec{\lambda}$.

Proposition 5.3. *Let* (A_t, ϕ_t) *be a solution of the gradient flow* (1.3) *and* (A_∞, ϕ_∞) *be a subsequential Uhlenbeck limit of* (A_t, ϕ_t). *Then for any* $\alpha \geq 1$ *and any* N, $t \mapsto \mathrm{HYM}_{\alpha,N}(A_t, \phi_t)$ *is nonincreasing, and* $\lim_{t\to\infty} \mathrm{HYM}_{\alpha,N}(A_t, \phi_t) = \mathrm{HYM}_{\alpha,N}(A_\infty, \phi_\infty)$.

Proof. From the above we can approximate φ_α by smooth convex ad-invariant functions $\varphi_{\alpha,\rho}$. On the other hand, from inequality (2.7), we know that the functional $t \mapsto \int_M \varphi_{\alpha,\rho}(\theta(A_t, \phi_t) - \sqrt{-1}\,N\,\mathrm{Id}_E)\,d\mathrm{vol}$ is nonincreasing along the flow, so $t \mapsto \mathrm{HYM}_{\alpha,N}(A_t, \phi_t)$ is also nonincreasing.

By Th. 3.11 and Cor. 3.12, we can choose a sequence $t_j \to \infty$ such that

$$\mathrm{HYM}_{\alpha,N}(A_{t_j}, \phi_{t_j}) \to \mathrm{HYM}_{\alpha,N}(A_\infty, \phi_\infty).$$

Then the convergence follows because $\mathrm{HYM}_{\alpha,N}(A_t, \phi_t)$ is nonincreasing in t. □

By Proposition 4.3, a Higgs bundle (E, A, ϕ) admits a filtration (i.e. HNS filtration) by saturated ϕ-invariant subsheaves E_i so that the successive quotients $Q_i = E_i/E_{i-1}$ are Higgs stable and torsion-free. For each i, we have an exact sequence of sheaves

$$0 \to Q_i \to Q_i^{**} \to T_i \to 0,$$

where Q_i^{**} is locally free and T_i is a torsion sheaf supported at finitely many points. Define the set Σ_i to be the support of T_i, and let $\Sigma^{\text{alg}} = \bigcup \Sigma_i$. We will refer to Σ^{alg} as the *singular set* of the filtration $\{E_i\}$. Arguing as in [DW1, Section 3, Lemma 3.3], we can consider $\overline{M}$ as a sequence of blow-ups at points, and construct a family of Kähler forms on $\overline{M}$.

Lemma 5.4. *Let $\pi : \overline{M} \to M$ be a sequence of monoidal transformations with exceptional set* $\mathbf{e}$*, where $\pi(\mathbf{e}) = \Sigma^{\text{alg}}$, and choose a Kähler form ω on M. Then there is a smooth, closed $(1,1)$-form η on $\overline{M}$ and a number ϵ_0 with the following properties:*

(1) *$\omega_\epsilon = \pi^*\omega + \epsilon\eta$ is a Kähler form on $\overline{M}$ for all $0 < \epsilon \leq \epsilon_0$;*
(2) *for any closed 2-form α on M, $\int_{\overline{M}} \pi^*\alpha \wedge \eta = 0$.*

In the following, we assume that the slope $\mu(E)$ of a sheaf E on X will be taken with respect to ω. For a sheaf $\overline{E}$ on $\overline{M}$, we denote by $\mu_\epsilon(\overline{E})$ the slope of $\overline{E}$ with respect to the metric ω_ϵ. Similarly, a subscript ϵ will indicate that the quantity in question is taken with respect to the metric ω_ϵ. We define $\mu_{\max}(E)$ to be the maximal slope of a ϕ-invariant subsheaf of E, and $\mu_{\min}(E)$ to be the minimal slope of a ϕ-invariant torsion-free quotient subsheaf of E.

Proposition 5.5. *Given a Higgs bundle $(\overline{\mathbf{E}}, \overline{\phi})$ over $\overline{M}$ and a Higgs sheaf $(\mathbf{E}, \phi)$ over M with $\pi_*\overline{\mathbf{E}} = \mathbf{E}$, $\phi(\pi_* X) = \pi_*\overline{\phi}(X)$ for any $X \in \overline{\mathbf{E}}$, and given $\delta > 0$, there is $\varrho_2 > 0$, depending upon $(\overline{E}, \overline{A}, \overline{\phi})$, such that for all $0 < \epsilon \leq \varrho_2$ we have the following inequalities:*

(1) *$\mu(E) - \delta \leq \mu_\epsilon(\overline{E}) \leq \mu(E) + \delta$;*
(2) *$\mu_{\max}(E) - \delta \leq \mu_{\max,\epsilon}(\overline{E}) \leq \mu_{\max}(E) + \delta$;*
(3) *$\mu_{\min}(E) - \delta \leq \mu_{\min,\epsilon}(\overline{E}) \leq \mu_{\min}(E) + \delta$.*

Proof. Parts (1) and (2) are essentially contained in [Bu1, Lemma 5], the only difference is that we always consider ϕ-invariant subsheaves instead of usual subsheaves. For (3), it is sufficient to prove $\mu_{\min}(E) = -\mu_{\max}(E^*)$. For any ϵ, consider an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

such that E_1 is a ϕ-invariant subsheaf, E_2 is torsion-free, and $\mu_{\min}(E) \geq \mu(E_2) - \epsilon$. Dualizing, we have an exact sequence

$$0 \to E_2^* \to E^* \to E_1^*.$$

We know that E_2^* is a ϕ^*-invariant subsheaf of E^* because E_1 is ϕ-invariant, where $[\phi^*(\theta)](X) = \theta(\phi(X))$ for any $\theta \in E^*$ and $X \in E$. Then we have

$$\mu_{\min}(E) \geq \mu(E_2) - \epsilon = -\mu(E_2^*) - \epsilon \geq -\mu_{\max}(E^*) - \epsilon.$$

We also consider an exact sequence

$$0 \to \varphi_1 \to E^* \to \varphi_2 \to 0$$

such that φ_1 is ϕ^*-invariant, φ_2 is torsion-free and $\mu_{\max}(E^*) \leq \mu(\phi_1) + \epsilon$. Dualizing, we have an exact sequence

$$0 \to \varphi_2^* \to E^{**} \to \varphi_1^*.$$

Considering E as a subsheaf of E^{**} under the natural injection, we define $F_1 = E \cap \varphi_2^*$, $F_2 = E/F_1$. Now F_1 is a ϕ-invariant subsheaf of E, since φ_2 is ϕ^*-invariant. On the other hand, it is easy to check that $\det(\varphi_2^*/F_1)$ is a trivial line bundle, i.e. $\det(\varphi_2^*) = \det(F_1)$. In particular, $\mu(F_1) = \mu(\varphi_2^*)$. Then

$$\begin{aligned}\mu_{\max}(E^*) \leq \mu(\phi_1) + \epsilon &= (\mathrm{rank}(E)\mu(E^*) - \mathrm{rank}(\phi_2)\mu(\phi_2)) + \epsilon \\ &= -(\mathrm{rank}(E)\mu(E^{**}) - \mathrm{rank}(\phi_2^*)\mu(\phi_2^*)) + \epsilon \\ &= -(\mathrm{rank}(E)\mu(E) - \mathrm{rank}(F_1)\mu(F_1)) + \epsilon \\ &\leq -\mu_{\min}(E) + \epsilon.\end{aligned}$$

Since ϵ is arbitrary, we have $\mu_{\max}(E) = -\mu_{\min}(E^*)$. □

An inductive argument repeatedly using Prop. 5.5 implies convergence of the HN type.

Corollary 5.6. *Let* $(\overline{\mathbf{E}}, \overline{\phi})$ *be a Higgs bundle over* $\overline{M}$ *and* $(\mathbf{E}, \phi)$ *be a Higgs sheaf over* M *with* $\pi_*\overline{\mathbf{E}} = \mathbf{E}$, $\phi(\pi_* X) = \pi_*\overline{\phi}(X)$ *for any* $X \in \overline{\mathbf{E}}$. *Let* $\vec{\mu}_\epsilon$ *denote the HN type of* $(\overline{\mathbf{E}}, \overline{\phi})$ *with respect to* ω_ϵ *and* $\vec{\mu}$ *the HN type of* $(\mathbf{E}, \phi)$ *with respect to* ω. *Then* $\vec{\mu}_\epsilon \to \vec{\mu}$ *as* $\epsilon \to 0$.

As a consequence, we have ([Bu2, Prop. 3.4(d)]):

Corollary 5.7. *Let* $(\overline{\mathbf{E}}, \overline{\phi})$ *be a Higgs bundle over* $\overline{M}$ *and* $(\mathbf{E}, \phi)$ *be a Higgs sheaf over* M *with* $\pi_*\overline{\mathbf{E}} = \mathbf{E}$, $\phi(\pi_* X) = \pi_*\overline{\phi}(X)$ *for any* $X \in \overline{\mathbf{E}}$. *If the Higgs sheaf* $(\mathbf{E}, \phi)$ *is* ω*-stable, then there is a number* $\varrho_2 > 0$, *depending upon* $(\overline{E}, \overline{A}, \overline{\phi})$, *such that the Higgs bundle* $(\overline{\mathbf{E}}, \overline{\phi})$ *is* ω_ϵ*-stable for all* $0 < \epsilon \leq \varrho_2$.

The following proposition was proved in [DW1, Proposition 3.7].

Proposition 5.8. *Let* $0 = E_0 \subset E_1 \subset \cdots \subset E_{l-1} \subset E_l = \mathbf{E}$ *be a filtration of a holomorphic vector bundle* $\mathbf{E} \to M$ *by saturated subsheaves* E_i, *and set* $Q_i = E_i/E_{i-1}$. *Then there is a monoidal transformation* $\pi : \overline{M} \to M$ *with exceptional set* $\mathbf{e}$ *and a filtration* $0 = \overline{E}_0 \subset \overline{E}_1 \subset \cdots \subset \overline{E}_{l-1} \subset \overline{E}_l = \overline{E} = \pi^*\mathbf{E}$ *such that each* $\overline{E}_i = \mathrm{Sat}_{\overline{E}}(\pi^* E_i)$ *is a subbundle of* $\overline{E}$. *If we set* $\overline{Q}_i = \overline{E}_i/\overline{E}_{i-1}$, *we also have exact sequences* $0 \to Q_i \to \pi_*\overline{Q}_i \to T_i \to 0$, *where* T_i *is a torsion sheaf supported at the singular set of* Q_i. *Moreover* $\pi(\mathbf{e}) = \Sigma^{\mathrm{alg}}$ *is the union of the singular sets of* Q_i; $\pi_*\overline{E}_i = E_i$; *and* $Q_i^{**} = (\pi_*\overline{Q}_i)^{**}$.

Let H be a smooth Hermitian metric on the holomorphic bundle $\mathbf{E}$, and let $F = \{F_i\}_{i=1}^l$ be a filtration of E by saturated subsheaves: $0 = F_0 \subset F_1 \subset \cdots \subset F_{l-1} \subset F_l = \mathbf{E}$. Associated to each F_i and the metric H we have the unitary projection π_i^H onto F_i. It is well known that π_i^H are bounded L_1^2 Hermitian endomorphisms. For convenience, we set $\pi_0^H = 0$. Given real numbers $\mu_1, \ldots, \mu_l$ and a filtration F, we define a bounded L_1^2 Hermitian endomorphism of $\mathbf{E}$ by $\Psi(F, (\mu_1, \ldots, \mu_l), H) = \sum_{i=1}^l \mu_i(\pi_i^H - \pi_{i-1}^H)$.

Given a Hermitian metric on a Higgs bundle $(\mathbf{E}, \phi)$, the *Harder–Narasimhan–Seshadri projection* $\Psi_\omega(\mathbf{E}, \phi, H)$ is the bounded L_1^2 Hermitian endomorphism defined above in the particular case where F is the HNS filtration $F_i = \mathrm{F}_i^{\mathrm{hns}}(E)$ and $\mu_i = \mu(F_i/F_{i-1})$.

Definition 5.9. Fix $\delta > 0$ and $1 \leq p \leq \infty$. An *L^p-δ-approximate critical Hermitian metric* on a Higgs bundle $(\mathbf{E}, \phi)$ is a smooth metric H such that

$$\|\sqrt{-1}\,\Lambda_\omega(F_{A_H} + [\phi, \phi^*]) - \Psi_\omega(\mathbf{E}, \phi, H)\|_{L^p(\omega)} \leq \delta,$$

where A_H is the Chern connection determined by $(\bar\partial_E, H)$.

For further considerations, we need the following lemma.

Lemma 5.10 ([DW1, Lemma 3.14]). *Let $\pi : \overline{M} \to M$ be a sequence of monoidal transformations with exceptional set* $\mathbf{e}$, *and let $\omega_\epsilon = \pi^*\omega + \epsilon\eta$ be the family of Kähler metrics defined in Lemma* 5.4. *Then there is a positive integer $\tilde m$ associated to $\overline{M}$ with the following property: given any p with $1 \leq p < 1 + 1/\tilde m$, any $\epsilon_1 > 0$, and any $\tilde p$ satisfying $p(1 - \tilde m(p-1))^{-1} < \tilde p \leq \infty$ there is a constant $C(\tilde p, \epsilon_1)$ depending only on $\tilde p$ and ϵ_1 such that $\|\Lambda_{\omega_\epsilon} f\|_{L^p(\omega_\epsilon)} \leq C(\tilde p, \epsilon_1)\|\Lambda_{\omega_{\epsilon_1}} f\|_{L^{\tilde p}(\omega_{\epsilon_1})}$ for all smooth $(1,1)$-forms f on $\overline{M}$ and all $0 < \epsilon \leq \epsilon_1$.*

Proposition 5.11. *Let $(\mathbf{E}, \phi)$ be a Higgs bundle on a smooth Kähler surface (M, ω), and $\mu_i = \mu_\omega(\mathrm{F}_i^{\mathrm{hns}}(E, \phi)/\mathrm{F}_{i-1}^{\mathrm{hns}}(E, \phi))$. Then there is a sequence of monoidal transformations determining a Kähler surface $\pi : \overline{M} \to M$, a number $p_0 > 1$, and a family of Kähler metrics ω_ϵ converging to $\pi^*\omega$ as $\epsilon \to 0$, such that the following holds: Let $\overline{F}$ be the filtration of $\overline{\mathbf{E}} = \pi^*\mathbf{E}$ given by $\{\mathrm{Sat}_{\overline{E}}(\pi^*\mathrm{F}_i^{\mathrm{hns}}(E, \phi))\}$. Then for any $\delta > 0$ and any $1 \leq p < p_0$ there are $\epsilon_1 > 0$ and a smooth Hermitian metric $\overline{H}$ on $\overline{\mathbf{E}}$ such that for all $0 < \epsilon \leq \epsilon_1$,*

$$\|\sqrt{-1}\,\Lambda_{\omega_\epsilon}(F_{A_{\overline{H}}} + [\overline\phi, \overline\phi^*]) - \Psi(\overline{F}, (\mu_1, \ldots, \mu_l), \overline{H})\|_{L^p(\omega_\epsilon)} \leq \delta,$$

where $A_{\overline{H}} = (\bar\partial_{\overline{E}}, \overline{H})$.

Proof. The case of rank 1 is trivial, since line bundles admit Hermitian-Einstein metrics. Suppose that $\mathrm{rank}(\mathbf{E}) > 1$, and consider the HNS filtration $\{\mathrm{F}_i^{\mathrm{hns}}(\mathbf{E}, \phi)\}$. For convenience, set $E_i = \mathrm{F}_i^{\mathrm{hns}}(\mathbf{E}, \phi)$, $Q_i = E_i/E_{i-1}$, and $\mu_i = \mu_\omega(Q_i)$. By Prop. 5.8 there is a resolution $\pi : \overline{M} \to M$ where the filtration $\overline{E}_i = \mathrm{Sat}_{\overline{E}}(\pi^* E_i)$ is a filtration of $\overline{\mathbf{E}} = \pi^*\mathbf{E}$ by bundles. Let $\overline\phi = \pi^*\phi$. Then $(\overline{\mathbf{E}}, \overline\phi)$ is a Higgs bundle over $\overline{M}$ and $\overline{E}_i$ is a $\overline\phi$-invariant subbundle for all i. It follows from Prop. 5.5 and Corollary 5.7 that for a given δ_1 we may assume ϵ_1 small enough so that $|\mu_\epsilon(\overline{Q_i}) - \mu_i| \leq \delta_1$ and Higgs bundles $(\overline{Q_i}, \overline\phi|_{\overline{Q_i}})$ are ω_ϵ-stable for all $0 < \epsilon \leq \epsilon_1$. By [Si1, Theorem 1], we have a Hermitian-Einstein metric $\overline{H}_i^\epsilon$ on the Higgs bundle $(\overline{Q_i}, \overline\phi|_{\overline{Q_i}})$ with respect to ω_ϵ. In particular,

$$\sqrt{-1}\,\Lambda_{\omega_\epsilon}(F_{A_{\overline{H}_i^\epsilon}} + [\overline\phi, \overline\phi^*]) - \mu_\epsilon(\overline{Q_i})\,\mathrm{Id}_{\overline{Q}_i} = 0. \tag{5.2}$$

We can also choose ϵ_1 small enough such that $|\Lambda_{\omega_\epsilon}(\omega_{\epsilon_1} - \omega_\epsilon)| < \delta$ for all $0 < \epsilon \leq \epsilon_1$. Associated to $\overline{M}$ there is an integer $\tilde m$ as in Lemma 5.10. We choose p_0 sufficiently close to 1 so that $p_0 < 1 + 1/(2\tilde{m})$. Then the conclusion of Lemma 5.10, along with (5.2), guarantees that for each $1 \le p < p_0$, each i, and each $0 < \epsilon \le \epsilon_1$, we have

$$\begin{aligned}
&\|\sqrt{-1}\,\Lambda_{\omega_\epsilon}(F_{A_{\overline{H}_i^{\epsilon_1}}} + [\overline{\phi}, \overline{\phi}^{*1}]) - \mu_i \operatorname{Id}_{\overline{Q}_i}\|_{L^p(\omega_\epsilon)} \\
&\quad \le \|\sqrt{-1}\,\Lambda_{\omega_\epsilon}\{F_{A_{\overline{H}_i^{\epsilon_1}}} + [\overline{\phi}, \overline{\phi}^{*1}] + (\sqrt{-1}/2)\omega_{\epsilon_1}\mu_{\epsilon_1}(\overline{Q}_i)\operatorname{Id}_{\overline{Q}_i}\}\|_{L^p(\omega_\epsilon)} \\
&\qquad + \left\|\tfrac{1}{2}\Lambda_{\omega_\epsilon}(\omega_\epsilon - \omega_{\epsilon_1})\mu_{\epsilon_1}(\overline{Q}_i)\operatorname{Id}_{\overline{Q}_i}\right\|_{L^p(\omega_\epsilon)} + \|(\mu_{\epsilon_1}(\overline{Q}_i) - \mu_i)\operatorname{Id}_{\overline{Q}_i}\|_{L^p(\omega_\epsilon)} \\
&\quad \le C\delta_1,
\end{aligned}$$

for a constant C independent of ϵ and δ_1. Let $\overline{H} = \bigoplus_{i=1}^{l} \overline{H}_i^{\epsilon_1}$; this is a smooth metric on $\overline{E}$, because the filtration $\{\overline{E}_i\}$ is by subbundles. Then $\overline{H}$ is the desired metric if we choose δ_1 sufficiently small compared to δ. □

Lemma 5.12. *Let (E, A_0, ϕ_0) be a Higgs bundle of HN type $\vec{\mu}_0$. There is $\alpha_0 > 1$ such that the following holds: given any $\delta > 0$, and any N, there is a Hermitian metric H on E such that*

$$\mathrm{HYM}_{\alpha,N}(A, \phi_0) \le \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta$$

for all $1 \le \alpha \le \alpha_0$, where the connection A is $(\overline{\partial}_{A_0}, H)$.

Proof. Let $\pi : \overline{M} \to M$ be a resolution of the HNS filtration guaranteed by Prop. 5.8, ω_ϵ the family of Kähler metrics on $\overline{M}$ from Lemma 5.4, and $\overline{\mathbf{E}} = \pi^*(E, \overline{\partial}_{A_0})$, $\overline{\phi} = \pi^*(\phi)$. As a direct consequence of Prop. 5.11, there is $\overline{\alpha}_0 = p_0 > 1$ such that the following holds: given any $\delta > 0$ there exists a smooth Hermitian metric $\overline{H}$ on $\overline{\mathbf{E}}$ and $\epsilon_1 > 0$ such that

$$\mathrm{HYM}^{\omega_\epsilon}_{\alpha,N}(A_{\overline{H}}, \overline{\phi}) \le \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta/2 \tag{5.3}$$

for all $1 \le \alpha \le \overline{\alpha}_0$, and all $0 < \epsilon \le \epsilon_1$. In order to obtain the desired metric on M, we follow the proof of [DW1, Lemma 4.2.], and use a cut-off argument. Let $x_0 \in \Sigma^{\mathrm{alg}}$, and choose a coordinate neighborhood U of x_0. We choose a holomorphic trivialization of $(E, \overline{\partial}_{A_0}) \to U$; this also gives a trivialization of $\overline{\mathbf{E}}$ on $\overline{U} = \pi^{-1}(U)$. Given $R > 0$ sufficiently small, we may choose a smooth function f_R on U with $0 \le f_R \le 1$, $f_R \equiv 0$ on a ball of radius $R/2$ centered at x_0, and $f_R \equiv 1$ outside a ball of radius R, and such that $|f_R'| \le CR^{-1}$ and $|f_R''| \le CR^{-2}$, where C is a positive constant independent of R. Define a metric H_R as follows: If $\overline{H}(e_i, e_j) = \lambda_i\delta_{ij}$ with respect to a holomorphic frame $\{e_i\}$, then $H_R(e_i, e_j) = (f_R\lambda_i + 1 - f_R)\delta_{ij}$. With this definition, H_R extends to a smooth metric on $E \to U$. Continuing this way for all points in Σ^{alg}, we obtain a smooth metric on $E \to M$, still denoted by H_R. Let $\overline{H}_R$ denote the pull-back metric on $\overline{\mathbf{E}} \to \overline{M}$. We know that $\overline{H}_R = \overline{H}$ outside the union U_R of the balls B_R, $\mathrm{vol}_\epsilon(U_R) = O(R^4)$, and H_R is standard with respect to the trivialization inside $U_{R/2}$. We define a gauge transformation σ such that

$$\sigma(e_i) = \left(\frac{f_R\lambda_i + 1 - f_R}{\lambda_i}\right)^{1/2} e_i$$

on $\pi^{-1}(U)$, and $\sigma \equiv \mathrm{Id}$ outside U_R. Then we have:

$$
\begin{aligned}
&| \,\mathrm{HYM}^{\omega_\epsilon}_{\alpha,N}(A_{\overline{H}_R},\overline{\phi}) - \mathrm{HYM}^{\omega_\epsilon}_{\alpha,N}(A_{\overline{H}},\overline{\phi})| \\
&\qquad \leq C\int_{\overline{M}} |\sigma\circ\theta(A_{\overline{H}_R},\overline{\phi})\circ\sigma^{-1} - \theta(A_{\overline{H}},\overline{\phi})|^{\alpha}_{\overline{H}}\, dv_{\omega_\epsilon} \\
&\qquad \leq \|\Lambda_{\omega_\epsilon}(F_{A_{\overline{H}_R}} - F_{A_{\overline{H}}})\|_{L^{\alpha}_{\omega_\epsilon}(\pi^{-1}(U_R\setminus U_{R/2}))} + \|\Lambda_{\omega_\epsilon}F_{A_{\overline{H}}}\|_{L^{\alpha}_{\omega_\epsilon}(\pi^{-1}(U_{R/2}))} \\
&\qquad\quad + \|\Lambda_{\omega_\epsilon}([\sigma\circ\overline{\phi}\circ\sigma^{-1},(\sigma\circ\overline{\phi}\circ\sigma^{-1})^{*\overline{H}}] - [\overline{\phi},\overline{\phi}^{*\overline{H}}])\|_{L^{\alpha}_{\omega_\epsilon}(\pi^{-1}(U_R))}.
\end{aligned}
$$

By the construction of $\overline{H}$ and σ, the second term and the third term on the right hand side tend to zero as $R \to 0$, uniformly in ϵ. Then we can choose R sufficiently small so that the last two terms are less than $\delta/8$. On the other hand, from Lemma 4.2 in [DW1], we have the following bound of the first term on the right hand side:

$$
\|\Lambda_{\omega_\epsilon}(F_{A_{\overline{H}_R}} - F_{A_{\overline{H}}})\|_{L^{\alpha}_{\omega_\epsilon}(\pi^{-1}(U_R\setminus U_{R/2}))} \leq C(1+R^{-2\alpha})R^4,
$$

where C is independent of R and ϵ. Now by (5.3), provided $1 < \alpha_0 < \min\{\overline{\alpha}_0, 2\}$, we may take R so small that, for all $\alpha \leq \alpha_0$,

$$
\mathrm{HYM}^{\omega}_{\alpha,N}(A_{H_R},\phi) = \lim_{\epsilon\to 0}\mathrm{HYM}^{\omega_\epsilon}_{\alpha,N}(A_{\overline{H}_R},\overline{\phi}) \leq \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta. \qquad \square
$$

Theorem 5.13. *Let α_0 be as in Lemma* 5.12*, and $\vec{\mu}_0$ be the Harder–Narasimhan type of the Higgs bundle (E, A_0, ϕ_0). Let (A_t, ϕ_t) be a smooth solution of the gradient flow* (1.3) *on the Hermitian vector bundle (E, H_0) with initial condition $(A_0, \phi_0) \in \mathcal{B}_{(E,H_0)}$. Then*

$$
\lim_{t\to\infty}\mathrm{HYM}_{\alpha,N}(A_t,\phi_t) = \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0)
$$

for all $1 \leq \alpha \leq \alpha_0$, and all N.

Proof. For fixed α, $1 \leq \alpha \leq \alpha_0$, and fixed N, we define $\delta_0 > 0$ by

$$
2\delta_0 + \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) = \min\{\mathrm{HYM}_{\alpha,N}(\vec{\mu}) : \mathrm{HYM}_{\alpha,N}(\vec{\mu}) > \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0)\},
$$

where $\vec{\mu}$ runs over all possible HN types of Higgs bundles on M with the rank of E.

Assume that the initial pair (A_0, ϕ_0) satisfies

$$
\mathrm{HYM}_{\alpha,N}(A_0,\phi_0) \leq \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta_0. \tag{5.4}
$$

Let (A_∞, ϕ_∞) be the Uhlenbeck limit along the flow with initial pair (A_0, ϕ_0). By Prop. 4.6 and Prop. 5.3, we obtain

$$
\mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) \leq \mathrm{HYM}_{\alpha,N}(A_\infty,\phi_\infty) \leq \mathrm{HYM}_{\alpha,N}(A_0,\phi_0) \leq \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta_0.
$$

Hence, we must have $\mathrm{HYM}_{\alpha,N}(A_\infty,\phi_\infty) = \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0)$. This shows that the result holds for the initial condition satisfying (5.4).

Let H be a Hermitian metric on the complex bundle E, and (A^H_t, ϕ^H_t) be the solution to the gradient heat flow (1.3) on the Hermitian vector bundle (E, H) with initial pair

$(A_0^H, \phi_0) \in \mathcal{B}_{(E,H_0)}$, where $A_0^H = (\overline{\partial}_{A_0}, H)$. As in [DW1, Lemma 4.3], we are going to prove that for any metric H and any initial data $(\overline{\partial}_{A_0}, \phi_0)$, there is $T \geq 0$ such that

$$\mathrm{HYM}_{\alpha,N}(A_t^H, \phi_t^H) < \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta$$

for all $t \geq T$. Without loss of generality, we can assume $0 < \delta \leq \delta_0/2$.

Let us denote by $\mathbf{H}_\delta$ the set of smooth Hermitian metrics on E with the property that the above inequality holds for some T. From Lemma 5.12 and the discussion above, $\mathbf{H}_\delta$ is nonempty. Let H^j be a sequence of smooth Hermitian metrics on E with each H^j in $\mathbf{H}_\delta$. Suppose $H^j \to K$ in the C^∞ topology, for some metric K. Since $H^j \in \mathbf{H}_\delta$, we have a sequence $T_j \geq 0$ such that

$$\mathrm{HYM}_{\alpha,N}(A_t^{H^j}, \phi_t^{H^j}) < \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta$$

for all $t \geq T_j$. Let ξ^j be gauge transformations satisfying $(\xi^j)^{*K}\xi^j = K^{-1}H^j$. We can suppose that ξ^j are uniformly bounded, because of the C^∞ convergence of H^j. Set $D_{\overline{A}_t^j} = \xi^j \circ D_{A_t^{H^j}} \circ (\xi^j)^{-1}$ and $\overline{\phi}_t^j = \xi^j \circ \phi_t^{H^j} \circ (\xi^j)^{-1}$. We see that $(\overline{A}_t^j, \overline{\phi}_t^j)$ are Higgs pairs on the Hermitian vector bundle (E, K), in particular $\overline{A}_t^j$ are integrable K-unitary connections. From Lemma 2.3, formula (2.8), formula (2.10), and the C^∞ convergence of H^j, we know that $|\overline{\phi}_t^j|_K$, $\|\lambda F_{\overline{A}_t^j}\|_{L^\infty(K)}$, $\|\nabla_{\overline{A}_t^j}\overline{\phi}^j\|_{L^2(K)}$ are uniformly bounded for all j and t. It follows from (2.9) that $\|D_{\overline{A}^j}\theta(\overline{A}^j, \overline{\phi})\|_{L^2(K)} \to 0$ as $t \to \infty$. Hence, it follows from the Uhlenbeck compactness ([DW1, Prop. 2.9, Corollary 2.12]), Theorem 3.11 and Corollary 3.12 that there exists a sequence $t_j \geq T_j$, Yang–Mills–Higgs pairs $(A_\infty^{(1)}, \phi_\infty^{(1)})$ and $(A_\infty^{(2)}, \phi_\infty^{(2)})$, and bubbling sets $Z_{(1)}^{\mathrm{an}}$ and $Z_{(2)}^{\mathrm{an}}$, such that $\overline{A}_{t_j}^j \to A_\infty^{(1)}$ weakly in $L_{1,\mathrm{loc}}^p(M \setminus Z_{(1)}^{\mathrm{an}})$, $\overline{\phi}_{t_j}^j \to \phi_\infty^{(1)}$ weakly in $L_{1,\mathrm{loc}}^2(M \setminus Z_{(1)}^{\mathrm{an}})$, and $(A_{t_j}^K, \phi_{t_j}^K) \to (A_\infty^{(2)}, \phi_\infty^{(2)})$ smoothly outside $Z_{(2)}^{\mathrm{an}}$. Moreover, $\theta(\overline{A}_{t_j}^j, \overline{\phi}_{t_j}^j) \to \theta(A_\infty^{(1)}, \phi_\infty^{(1)})$ and $\theta(A_{t_j}^K, \phi_{t_j}^K) \to \theta(A_\infty^{(2)}, \phi_\infty^{(2)})$ strongly in L^p, for all $1 \leq p < \infty$. Now, we will show that

$$(A_\infty^{(1)}, \phi_\infty^{(1)}) = (A_\infty^{(2)}, \phi_\infty^{(2)}).$$

Let $H^j(t)$ and $K(t)$ be solutions of the heat flow (2.1) with initial data H^j and K respectively. Write $h^j(t) = K^{-1}(t)H^j(t)$. It follows from [Si1, Prop. 6.3] that

$$\sup_M \sigma(K(t), H^j(t)) \to 0$$

as $j \to \infty$, uniformly in t, where σ is the usual C^0 distance on the space of Hermitian metrics on E. In particular, $\sup |h^j(t) - \mathrm{Id}_E| \to 0$ as $j \to \infty$. From Section 2, we know that

$$D_{A_t^{H^j}} = g^j(t) \circ D_{A_{H^j(t)}} \circ (g^j(t))^{-1}, \quad \phi_t^{H^j} = g^j(t) \circ \phi_0 \circ (g^j(t))^{-1},$$
$$D_{A_t^K} = \eta(t) \circ D_{A_{K(t)}} \circ (\eta(t))^{-1}, \quad \phi_t^K = \eta(t) \circ \phi_0 \circ (\eta(t))^{-1},$$

where $(g^j(t))^{*H^j} g^j(t) = (H^j)^{-1} H^j(t)$ and $(\eta(t))^{*K}\eta(t) = K^{-1}K(t)$. Pick gauge transformations $\beta^j(t)$ satisfying $(\beta^j(t))^{*K}\beta^j(t) = \eta(t)h^j(t)\eta^{-1}(t)$, and $\|\beta^j(t_j) - \mathrm{Id}_E\|_{L^p} \to 0$ as $j \to 0$. It is easy to check that there exists a K-unitary gauge transformation $S^j(t)$ (i.e. $(S^j(t))^{*K} S^j(t) = \mathrm{Id}_E$) such that $S^j \circ \xi^j \circ g^j = \beta^j \circ \eta$. So, modulo a unitary gauge transformation, we can suppose that

$$D_{\overline{A}_t^j} = \beta^j(t)\circ\eta(t)\circ D_{A_{H^j(t)}}\circ(\beta^j(t)\eta(t))^{-1}, \quad \overline{\phi}_t^j = \beta^j(t)\eta(t)\circ\phi_0\circ(\beta^j(t)\eta(t))^{-1}.$$

So we have

$$\begin{aligned}
\overline{\partial}_{\overline{A}_t^j} &= \overline{\partial}_{A_0} + \beta^j\circ(\overline{\partial}_{A_0}(\beta^j)^{-1}) + \beta^j\circ\eta(\overline{\partial}_{A_0}(\eta)^{-1})\circ(\beta^j)^{-1},\\
\overline{\partial}_{A_t^K} &= \overline{\partial}_{A_0} + \eta(\overline{\partial}_{A_0}(\eta)^{-1}),\\
\partial_{\overline{A}_t^j} &= \beta^j(t)\circ\eta(t)\circ(\partial_{A_{K(t)}} + (h^j(t))^{-1}\partial_{A_{K(t)}}h^j(t))\circ(\beta^j(t)\eta(t))^{-1}\\
&= \partial_{A_t^K} + [\beta^j(t)\circ\eta(t)\circ(h^j(t))^{-1}\eta^{-1}(t)]\circ\partial_{A_t^K}[\beta^j(t)\circ\eta(t)\circ(h^j(t))^{-1}\eta^{-1}(t)]^{-1}\\
&= \partial_{A_t^K} + [(\beta^j(t))^{*K}]^{-1}\circ\partial_{A_t^K}[(\beta^j(t))^{*K}],
\end{aligned}$$

and

$$\partial_{A_t^K}[(\beta^j(t))^{*K}] = (\beta^j(t))^{*K}\circ(\partial_{\overline{A}_t^j} - \partial_{A_t^K}).$$

Let $Z^{\mathrm{an}} = Z_1^{\mathrm{an}} \cup Z_2^{\mathrm{an}}$, and choose a smooth test form $f \in \Omega^1(\mathrm{End}(E))$, compactly supported on $M \setminus Z^{\mathrm{an}}$. From the convergence of $\overline{A}_{t_j}^j$ and $A_{t_j}^K$, it is easy to deduce that $(\overline{\partial}_{A_0}(\beta^j(t_j))^{-1})$ $\partial_{A_{t_j}^K}[(\beta^j(t_j))^{*K}]$ are uniformly bounded in L^p. Then we have

$$\begin{aligned}
(\overline{\partial}_{\overline{A}_{t_j}^j} - \overline{\partial}_{A_{t_j}^K}, f)_{L^2} &= (\beta^j\circ(\overline{\partial}_{A_0}(\beta^j)^{-1}) + \beta^j\circ\eta(\overline{\partial}_{A_0}(\eta)^{-1})\circ(\beta^j)^{-1} - \eta(\overline{\partial}_{A_0}(\eta)^{-1}), f)_{L^2}\\
&= ((\beta^j - \mathrm{Id}_E)\circ\eta(\overline{\partial}_{A_0}(\eta)^{-1})\circ(\beta^j)^{-1} + \eta(\overline{\partial}_{A_0}(\eta)^{-1})\circ[(\beta^j)^{-1} - \mathrm{Id}_E], f)_{L^2}\\
&\quad + ((\beta^j - \mathrm{Id}_E)\circ(\overline{\partial}_{A_0}(\beta^j)^{-1}), f)_{L^2} + ((\beta^j)^{-1} - \mathrm{Id}_E, (\overline{\partial}_{A_0})^* f)_{L^2}\\
&\to 0,
\end{aligned}$$

and

$$\begin{aligned}
(\partial_{\overline{A}_{t_j}^j} - \partial_{A_{t_j}^K}, f)_{L^2} &= ([(\beta^j(t_j))^{*K}]^{-1}\circ\partial_{A_{t_j}^K}[(\beta^j(t))^{*K}], f)_{L^2}\\
&= (([(\beta^j(t_j))^{*K}]^{-1} - \mathrm{Id}_E)\circ\partial_{A_{t_j}^K}[(\beta^j(t_j))^{*K}], f)_{L^2} + ([(\beta^j(t_j))^{*K}, (\partial_{A_{t_j}^K})^* f)_{L^2}\\
&= (([(\beta^j(t_j))^{*K}]^{-1} - \mathrm{Id}_E)\circ\partial_{A_{t_j}^K}[(\beta^j(t_j))^{*K}], f)_{L^2}\\
&\quad + ([(\beta^j(t_j))^{*K}, (\partial_{A_{t_j}^K} - \partial_{A_\infty^{(2)}})^* f)_{L^2} + ([(\beta^j(t_j))^{*K} - \mathrm{Id}_E, (\partial_{A_\infty^{(2)}})^* f)_{L^2}\\
&\to 0,
\end{aligned}$$

because $\sup|\beta^j(t_j) - \mathrm{Id}_E| \to 0$ and $A^K(t_j) \to A_\infty^{(2)}$ in $C^\infty(M \setminus Z^{\mathrm{an}})$. From the above, we have $\overline{A}_{t_j}^j - A_{t_j}^K \to 0$ weakly in $L^2_{\mathrm{loc}}(M \setminus Z^{\mathrm{an}})$. On the other hand it is easy to check that $\overline{\phi}_{t_j}^j - \phi_{t_j}^K \to 0$ weakly in $L^2_{\mathrm{loc}}(M \setminus Z^{\mathrm{an}})$. So, we have proved that $(A_\infty^{(1)}, \phi_\infty^{(1)}) = (A_\infty^{(2)}, \phi_\infty^{(2)})$.

Set $(A_\infty^{(1)}, \phi_\infty^{(1)}) = (A_\infty^{(2)}, \phi_\infty^{(2)}) = (A_\infty, \phi_\infty)$. Noting that $\theta(\overline{A}_{t_j}^j, \overline{\phi}_{t_j}^j) \to \theta(A_\infty, \phi_\infty)$ and $\theta(A_{t_j}^K, \phi_{t_j}^K) \to \theta(A_\infty, \phi_\infty)$ strongly in L^p, for all p, we have for j sufficiently large,

$$\begin{aligned}\mathrm{HYM}_{\alpha,N}(A_{t_j}^K, \phi_{t_j}^K) &\le \mathrm{HYM}_{\alpha,N}(A_\infty, \phi_\infty) + \delta \le \lim_{j\to\infty} \mathrm{HYM}_{\alpha,N}(\overline{A}_{t_j}^j, \overline{\phi}_{t_j}^j) + \delta \\ &\le \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + 2\delta \le \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta_0.\end{aligned}$$

It follows from the discussion above that $\lim_{t\to\infty} \mathrm{HYM}_{\alpha,N}(A_t^K, \phi_t^K) = \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0)$. Therefore $K \in \mathbf{H}_\delta$. By the continuous dependence of the flow on initial conditions (Prop. 2.1′), $\mathbf{H}_\delta$ is also open. Since the space of smooth metrics is connected, we conclude that every metric is in $\mathbf{H}_\delta$. Thus, we have $\lim_{t\to\infty} \mathrm{HYM}_{\alpha,N}(A_t^H, \phi_t^H) = \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0)$ for any metric H. □

Theorem 5.14. *Let (A_t, ϕ_t) be a smooth solution of the gradient flow (1.3) on the Hermitian vector bundle (E, H_0) with initial condition $(A_0, \phi_0) \in \mathcal{B}_{(E,H_0)}$, and (A_∞, ϕ_∞) be an Uhlenbeck limit. Let E_∞ denote the vector bundle obtained from (A_∞, ϕ_∞) as in Theorem* 3.11. *Then the Harder–Narasimhan type of $(E_\infty, A_\infty, \phi_\infty)$ is the same as that of (E_0, A_0, ϕ_0).*

Proof. Let $\vec{\mu}_0 = (\mu_1, \ldots, \mu_R)$ (resp. $\vec{\lambda}_\infty = (\lambda_1, \ldots, \lambda_R)$) be the HN type of (E_0, A_0, ϕ_0) (resp. $(E_\infty, A_\infty, \phi_\infty)$). By Prop. 5.3 and Theorem 5.13, $\varphi_\alpha(\vec{\mu}_0 + N) = \varphi_\alpha(\vec{\lambda}_\infty + N)$ for all $1 \le \alpha \le \alpha_0$ and all N. We may choose N sufficiently large so that $\mu_R + N > 0$ and $\lambda_R + N > 0$. From Lemma 5.2, we conclude that $\vec{\lambda}_\infty + N = \vec{\mu}_0 + N$, and so $\vec{\lambda}_\infty = \vec{\mu}_0$. □

6. Convergence to the graded object of the filtration

Let (A_0, ϕ_0) be a Higgs structure on a complex vector bundle E. Then there is a ϕ_0-invariant Harder–Narasimhan–Seshadri filtration $\{E_i\}_{i=1}^l$ of $(E, \overline{\partial}_{A_0})$, where E_i are saturated ϕ-invariant subsheaves and $Q_i = E_i/E_{i-1}$ are Higgs stable and torsion-free. The associated graded object $\mathrm{Gr}^{\mathrm{hns}}(E, A_0, \phi_0) = \bigoplus_{i=1}^l Q_i$ is uniquely determined by the isomorphism class of (A_0, ϕ_0). The quotients Q_i are not necessarily locally free. For each i, we have an exact sequence of sheaves

$$0 \to Q_i \to Q_i^{**} \to T_i \to 0,$$

where Q_i^{**} is locally free and T_i is a torsion sheaf supported at finitely many points. Define Σ_i to be the support of T_i, and let $\Sigma^{\mathrm{alg}} = \bigcup \Sigma_i$. We will refer to Σ^{alg} as the *singular set* of the filtration $\{E_i\}$. We denote the associated graded object by

$$\mathrm{Gr}^{\mathrm{hns}}(E, A_0, \phi_0)^{**} = \bigoplus_{i=1}^l Q_i^{**}.$$

Let $(A(t), \phi(t))$ be a smooth solution of the gradient heat flow (1.3) with initial data (A_0, ϕ_0), and let (A_∞, ϕ_∞) be an Uhlenbeck limit, i.e. there is a subsequence $t_j \to \infty$ such that $(A(t_j), \phi(t_j))$ converges, modulo gauge transformation, to (A_∞, ϕ_∞) in the smooth topology outside Σ^{an}, where Σ^{an} is a finite collection of points. In Theorem 3.11, we have proved that the Uhlenbeck limit can be extended smoothly to a smooth solution of the Yang–Mills–Higgs equation (1.2) on the Hermitian vector bundle (E_∞, H_∞), and the Higgs bundle $(E_\infty, A_\infty, \phi_\infty)$ has a holomorphic splitting as a direct sum of Higgs stable subbundles. The purpose of this section is to provide an algebraic description of the isomorphism class of the limiting Higgs bundle $(E_\infty, A_\infty, \phi_\infty)$. We will give a Higgs bundle version of Theorem 1 of [DW1] (for Yang–Mills case). The main theorem of this section is the following

Theorem 6.1. *$(E_\infty, A_\infty, \phi_\infty)$ is holomorphically isomorphic to $\mathrm{Gr}^{hns}(E, A_0, \phi_0)^{**}$ in the Higgs bundle sense.*

The proof follows from the argument in the proof of Theorem 5.1 in [DW1]. Denote $(A_{t_j}, \phi(t_j))$ by (A_j, ϕ_j), and let g_j be the complex gauge transformation such that $(A_j, \phi_j) = g_j(A_0, \phi_0)$. Set $\Sigma = \Sigma^{an} \cup \Sigma^{alg}$ and $\Omega = M \setminus \Sigma$. Let S be the maximal Higgs stable subsheaf of (E, A_0, ϕ_0). Now $S|_\Omega$ is a subbundle, and let $f_0 : S|_\Omega \to E$ be the ϕ_0-invariant holomorphic inclusion. Define the map $f_j : S|_\Omega \to E$ by $f_j = g_j \circ f_0$. It is easy to check that

$$\overline{\partial}_{A_0, A_j} f_j = 0, \qquad f_j \circ \phi_0 = \phi_j \circ f_j,$$

i.e. f_j is a ϕ-invariant holomorphic map. For simplicity, we will denote $\overline{\partial}_{A_0, A_j}$ by $\overline{\partial}_{0,j}$.

Lemma 6.2. *Up to a subsequence, f_j converges in $C^\infty(\Omega_0)$ to some nonzero ϕ-invariant holomorphic map f_∞ for any compact set $\Omega_0 \subset \Omega$.*

Proof. Let $\Omega_0 \subset \Omega$ be the complement of a union of balls around the points of Σ. By Theorem 3.11, we can assume that (A_j, ϕ_j) converges to (A_∞, ϕ_∞) in $C^\infty(\Omega_0)$. Replace f_j by $f_j / \|f_j\|_{L^2}$ and let $\beta_j = A_j - A_\infty$. Since f_j is holomorphic, we have

$$\begin{aligned} \Delta_{0,j} f_j &= \sqrt{-1}\, \Lambda_\omega(\partial_{0,j}\overline{\partial}_{0,j} - \overline{\partial}_{0,j}\partial_{0,j}) f_j = -\sqrt{-1}\, \Lambda_\omega(\partial_{0,j}\overline{\partial}_{0,j} + \overline{\partial}_{0,j}\partial_{0,j}) f_j \\ &= -\sqrt{-1}\, \Lambda_\omega(F_{A_j} f_j - f_j F_{A_0}), \end{aligned}$$

and

$$\overline{\partial}_{0,\infty} f_j = \overline{\partial}_{A_\infty} \circ f_j - f_j \circ \overline{\partial}_{A_0} = -\beta_j^{0,1} \circ f_j. \tag{6.1}$$

By elliptic theory, we can obtain uniform bounds on $|\nabla^k_{0,j} f_j|$ for any k. Since $(A_j, \phi_j) \to (A_\infty, \phi_\infty)$ in $C^\infty(\Omega_0)$, we can also obtain uniform bounds on $|\nabla^k_{0,\infty} f_j|$ for any k. Then we can choose a subsequence, which we also denote by $\{f_j\}$, such that f_j converges to a smooth map f_∞ in $C^\infty(\Omega_0)$. Using formula (6.1), it is easy to check that

$$\overline{\partial}_{A_0, A_\infty} f_\infty = 0, \qquad f_\infty \circ \phi_0 = \phi_\infty \circ f_\infty,$$

i.e. f_∞ is a ϕ-invariant holomorphic map. Since $\|f_j\|_{L^2} = 1$ for all j we have $f_\infty \neq 0$. $\square$

The proof of the following lemma is completely similar to the proof of [Ko, 7.11, 7.12] for holomorphic bundles and so we omit it.

Lemma 6.3. *Let* $(\mathbf{E}_1, \phi_1)$ *and* $(\mathbf{E}_2, \phi_2)$ *be semistable Higgs sheaves with* $\operatorname{rank}(\mathbf{E}_1) = \operatorname{rank}(\mathbf{E}_2)$ *and* $\deg(\mathbf{E}_1) = \deg(\mathbf{E}_2)$. *If* $(\mathbf{E}_1, \phi_1)$ *is stable, let* $f : \mathbf{E}_1 \to \mathbf{E}_2$ *be a sheaf homomorphism satisfying* $f \circ \phi_1 = \phi_2 \circ f$. *Then either* $f = 0$ *or* f *is an isomorphism.*

Proof of Theorem 6.1. We will prove the result by induction on the length of the HNS filtration. The inductive hypotheses on the bundle $Q \to \Omega$ are the following:

Inductive hypotheses. (1) $(D_j^Q, \phi_j^Q) \to (D_\infty^Q, \phi_\infty^Q)$ in $C^\infty(\Omega)$;

(2) $\overline{\partial}_j^Q = h_j \circ \overline{\partial}_0^Q \circ h_j^{-1}$ and $\phi_j^Q = h_j \circ \phi_0^Q \circ h_j^{-1}$ for some $h_j \in \mathcal{G}^{\mathbb{C}}(Q)$;

(3) $(Q, \overline{\partial}_0^Q, \phi_0^Q)$ and $(Q_\infty, \overline{\partial}_\infty^Q, \phi_\infty^Q)$ extend to M as reflexive Higgs sheaves with the same HN type.

We obtained a nonzero smooth ϕ-invariant holomorphic map $f_\infty : S|_\Omega \to E|_\Omega$. By Hartogs' theorem, f_∞ extends to $f_\infty : S^{**} \to E_\infty$ on M. If π_j denotes the projection to $f_j(S)$, then we can prove that $\pi_j \to \pi_\infty$ in $C^\infty(\Omega)$ because $f_j \to f_\infty$ in $C^\infty(\Omega)$. So $(f_\infty(S), A_\infty, \phi_\infty)$ is a ϕ_∞-invariant subbundle of E_∞ with the same rank and degree as S. By Theorem 5.14, we know that (E, A_0, ϕ_0) and $(E_\infty, A_\infty, \phi_\infty)$ have the same HN type, so $(f_\infty(S), A_\infty, \phi_\infty)$ is Higgs semistable. By Lemma 6.3, the nonzero holomorphic map f_∞ must be an isomorphism, i.e. $S^{**} \simeq f_\infty(S)$. Denote $S_\infty = f_\infty(S)$. Then $(S_\infty, A_\infty, \phi_\infty)$ is a ϕ_∞-invariant Higgs stable subbundle. Write $\mathrm{Gr}^{\mathrm{hns}}(E_\infty, A_\infty, \phi_\infty) = S_\infty \oplus Q_\infty$. Using Lemma 5.12 in [Da], we can choose a sequence of unitary gauge transformations u_j such that $\pi_j = u_j \tilde{\pi}_j u_j^{-1}$ where $\tilde{\pi}_j(E) = \pi_\infty(E)$ and $u_j \to \mathrm{Id}_E$ in $C^\infty(\Omega)$. Let $Q = E/S$, and consider the induced connections on Q,

$$
\begin{aligned}
D_j^Q &= u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ D_j \circ \pi_j^\perp \circ u_j \circ u_0^{-1},\\
\phi_j^Q &= u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ \phi_j \circ \pi_j^\perp \circ u_j \circ u_0^{-1} \in \Omega^{1,0}(\mathrm{End}(Q)),\\
h_j &= u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ g_j \in \mathcal{G}^{\mathbb{C}}(Q).
\end{aligned}
$$

Then

$$
\begin{aligned}
\overline{\partial}_j^Q &= u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ \overline{\partial}_j \circ \pi_j^\perp \circ u_j \circ u_0^{-1}\\
&= u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ g_j \circ \pi_0^\perp \circ \overline{\partial}_0 \circ \pi_0^\perp \circ g_j^{-1} \circ u_j \circ u_0^{-1}\\
&= h_j \circ \overline{\partial}_0^Q \circ h_j^{-1},\\
\phi_j^Q &= u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ g_j \circ \phi_0 \circ g_j^{-1} \circ \pi_j^\perp \circ u_j \circ u_0^{-1}\\
&= u_0 \circ u_j^{-1} \circ \pi_j^\perp \circ g_j \circ \pi_0^\perp \circ \phi_0 \circ \pi_0^\perp \circ g_j^{-1} \circ u_j \circ u_0^{-1}\\
&= h_j \circ \phi_0^Q \circ h_j^{-1},
\end{aligned}
$$

and

$$
\overline{\partial}_j^Q \phi_j^Q = \pi_0^\perp \circ (\overline{\partial}_0 \circ \phi_0 + \phi_0 \circ \overline{\partial}_0)\pi_0^\perp = 0,
$$

where we have used $h_j^{-1} = \pi_0^\perp \circ g_j^{-1} \circ u_j \circ u_0^{-1}$. On the other hand, by the definition, it is easy to check that $D_j^Q \to D_\infty^Q$ and $\phi_j^Q \to \phi_\infty^Q$ in $C^\infty(\Omega)$. The third statement again follows from Theorem 5.14. So, (Q, D_j^Q, ϕ_j^Q) satisfy the inductive hypotheses. By induction $Q_\infty \simeq \mathrm{Gr}^{\mathrm{hns}}(Q, \bar\partial_0^Q, \phi_0^Q)^{**} \simeq \bigoplus_{i=2}^l Q_i^{**}$, which completes the proof of Theorem 6.1. □

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