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Right coideal subalgebras of $U_q^+(\mathfrak{so}_{2n+1})$

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Abstract. We give a complete classification of right coideal subalgebras that contain all grouplike elements for the quantum group $U_q^+(\mathfrak{so}_{2n+1})$, provided that q is not a root of 1. If q has a finite multiplicative order $t > 4$, this classification remains valid for homogeneous right coideal subalgebras of the Frobenius–Lusztig kernel $u_q^+(\mathfrak{so}_{2n+1})$. In particular, the total number of right coideal subalgebras that contain the coradical equals $(2n)!!$, the order of the Weyl group defined by the root system of type B_n.

Keywords. Coideal subalgebra, Hopf algebra, PBW-basis

1. Introduction

In the present paper, we continue the classification of right coideal subalgebras in quantised enveloping algebras begun in [13]. We offer a complete classification of right coideal subalgebras that contain all grouplike elements for the multiparameter version of the quantum group $U_q^+(\mathfrak{so}_{2n+1})$, provided that the main parameter q is not a root of 1. If q has a finite multiplicative order $t > 4$, this classification remains valid for homogeneous right coideal subalgebras of the multiparameter version of the Frobenius–Lusztig kernel $u_q^+(\mathfrak{so}_{2n+1})$. The main result of the paper is the establishment of a bijection between all sequences $(\theta_1, \ldots, \theta_n)$ such that $0 \leq \theta_k \leq 2n - 2k + 1$, $1 \leq k \leq n$, and the set of all (homogeneous if $q^t = 1$, $t > 4$) right coideal subalgebras of $U_q^+(\mathfrak{so}_{2n+1})$, $q^t \neq 1$ (respectively of $u_q^+(\mathfrak{so}_{2n+1})$) that contain the coradical. (Recall that in a pointed Hopf algebra, the grouplike elements span the coradical.) In particular, there are $(2n)!!$ different right coideal subalgebras that contain the coradical. Interestingly, this number coincides with the order of the Weyl group for the root system of type B_n. In [13], we proved that the number of different right coideal subalgebras that contain the coradical of $U_q^+(\mathfrak{sl}_{n+1})$ equals $(n+1)!$, the order of the Weyl group for the root system of type A_n. Recently, B. Pogorelsky [16] proved that the quantum Borel algebra $U_q^+(\mathfrak{g})$ for the simple Lie algebra of type G_2 has 12 different right coideal subalgebras over the coradical. This

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number also coincides with the order of the Weyl group of type G_2. Although there is no theoretical explanation for why the Weyl group appears in these results, we state the following general hypothesis.

Conjecture. *Let $\mathfrak{g}$ be a simple Lie algebra defined by a finite root system R. The number of different right coideal subalgebras that contain the coradical in a quantum Borel algebra $U_q^+(\mathfrak{g})$ equals the order of the Weyl group defined by the root system R, provided that q is not a root of 1.*[1]

In Section 2, following [13], we introduce the main concepts of the paper and we formulate known results that are useful for classification. In the third section, we prove auxiliary relations in a multiparameter version of $U_q^+(\mathfrak{so}_{2n+1})$. In the fourth section, we note that the Weyl basis

$$\{u[k,m] \stackrel{df}{=} [\dots[x_k, x_{k+1}], \dots, x_m] \mid 1 \le k \le m \le 2n-k,\ x_{n+r} \stackrel{df}{=} x_{n-r+1}\}$$

of the Borel subalgebra $\mathfrak{so}_{2n+1}^+$ with skew bracket $[u,v] = uv - \chi^u(g_v)vu$ in place of the Lie operation is a set of PBW-generators for $U_q^+(\mathfrak{so}_{2n+1})$ and $u_q^+(\mathfrak{so}_{2n+1})$. By means of the shuffle representation, in Theorem 4.3, we prove an explicit formula for the coproduct of these PBW-generators, which is the key result for further considerations:

$$\Delta(u[k,m]) = u[k,m] \otimes 1 + g_{km} \otimes u[k,m] + \sum_{i=k}^{m-1} \tau_i (1-q^{-2}) g_{ki} u[i+1,m] \otimes u[k,i],$$

where $\tau_i = 1$ with only one exception, $\tau_n = q$, while g_{ki} are suitable grouplike elements. Interestingly, this coproduct formula differs from that in $U_{q^2}^+(\mathfrak{sl}_{2n+1})$ in just one term (see formula (3.3) in [11]).

In Section 5, we show that each homogeneous right coideal subalgebra in $U_q^+(\mathfrak{so}_{2n+1})$ or in $u_q^+(\mathfrak{so}_{2n+1})$ has PBW-generators of a special form, $\Phi^S(k,m)$, where S is a set of integers from the interval $[1, 2n]$. The polynomial $\Phi^S(k,m)$ is defined by induction on the number r of elements in $S \cap [k, m-1] = \{s_1, \dots, s_r\}$, $k \le s_1 < \cdots < s_r < m$, as follows:

$$\Phi^S(k,m) = u[k,m] - (1-q^{-2}) \sum_{i=1}^{r} \alpha_{km}^{s_i}\, \Phi^S(1+s_i, m) u[k, s_i],$$

where α_{km}^s are scalars, $\alpha_{km}^s = \tau_s p(u(1+s,m), u(k,s))^{-1}$. The existence of those generators implies that the set of all (homogeneous) right coideal subalgebras that contain the coradical is finite (Corollary 5.7).

In Sections 6 and 7, we single out special sets S, called (k,m)-regular sets. In Proposition 7.10, we establish a kind of duality for elements $\Phi^S(k,m)$ with regular S, which provides a powerful tool for investigating PBW-generators for right coideal subalgebras.

In Section 8, we define a root sequence $r(\mathbf{U}) = (\theta_1, \dots, \theta_n)$ in the following way. The number θ_i is the maximal m such that for some S the value of $\Phi^S(i,m)$ belongs to $\mathbf{U}$,

[1] *Note added in proof:* Recently this conjecture was proved by I. Heckenberger and H.-J. Schneider in "Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid", arXiv:0909.0293, 43 pp.

while the degree $x_i + x_{i+1} + \cdots + x_m$ of $\Phi^S(i, m)$ is not a sum of other nonzero degrees of elements from $\mathbf{U}$. In Theorem 8.2, we show that the root sequence uniquely defines the right coideal subalgebra $\mathbf{U}$ that contains the coradical.

In Section 9, we consider some important examples, including the right coideal subalgebra generated by $\Phi^S(k, m)$ with regular S. We also analyze in detail the simplest (but not trivial [2]) case, $n = 2$.

In Section 10, we associate a right coideal subalgebra $\mathbf{U}_\theta$ to each sequence of integers $\theta = (\theta_1, \ldots, \theta_n)$, $0 \le \theta_i \le 2n - 2i + 1$, so that $r(\mathbf{U}_\theta) = \theta$. First, by downward induction on k, we define sets

$$R_k \subseteq [k, 2n - k], \qquad T_k \subseteq [k, 2n - k + 1], \qquad 1 \le k \le 2n,$$

as follows. For $k > n$, we put $R_k = T_k = \emptyset$. Suppose that R_i, T_i, $k < i \le 2n$, are already defined. Denote by $\mathbf{P}$ the following binary predicate on the set of all ordered pairs $i \le j$:

$$\mathbf{P}(i, j) \rightleftharpoons j \in T_i \vee 2n - i + 1 \in T_{2n-j+1}.$$

If $\theta_k = 0$, then we set $R_k = T_k = \emptyset$. If $\theta_k \neq 0$, then by definition, R_k contains $\tilde{\theta}_k = k + \theta_k - 1$ and all m satisfying the following three properties:

(a) $k \le m < \tilde{\theta}_k$;
(b) $\neg \mathbf{P}\,(m + 1, \tilde{\theta}_k)$;
(c) $\forall r\ (k \le r < m)\ \mathbf{P}(r + 1, m) \Leftrightarrow \mathbf{P}(r + 1, \tilde{\theta}_k)$.

Further, we define an auxiliary set

$$T_k' = R_k \cup \bigcup_{s \in R_k} \{a \mid s < a \le 2n - k,\ \mathbf{P}(s + 1, a)\},$$

and we put

$$T_k = \begin{cases} T_k' & \text{if } (2n - R_k) \cap T_k' = \emptyset; \\ T_k' \cup \{2n - k + 1\} & \text{otherwise.} \end{cases}$$

Next, the subalgebra $\mathbf{U}_\theta$ is, by definition, generated over $\mathbf{k}[G]$ by values in $U_q^+(\mathfrak{so}_{2n+1})$ or in $u_q^+(\mathfrak{so}_{2n+1})$ of the polynomials $\Phi^{T_k}(k, m)$, $1 \le k \le m$, with $m \in R_k$.

Theorems 8.2 and 10.3 together show that all right coideal subalgebras over the coradical have the form $\mathbf{U}_\theta$.

In Section 11, we restate the main result in a slightly more general form. We consider homogeneous right coideal subalgebras $\mathbf{U}$ such that the intersection $\Omega = \mathbf{U} \cap G$ with the group G of all grouplike elements is a subgroup. In this case $\mathbf{U} = \mathbf{k}[\Omega]\mathbf{U}_\theta^1$, where $\mathbf{U}_\theta^1$ is the subalgebra generated by $g_a^{-1} a$ when $a = \Phi^{T_k}(k, m)$ runs through the above described generators of $\mathbf{U}_\theta$.

The present paper extends [13] by using similar methods in a parallel way. However, it is much more complicated technically. The proof of the explicit formula for comultiplication (Theorem 4.3) essentially depends on the shuffle representation given in Proposition 4.2, while the same formula for the case A_n was proved by a simple induction [11]. The elements $\Phi^S(k, m)$ that naturally appear as PBW-generators for right coideal subalgebras do not satisfy all necessary properties for further development. Therefore, in Section 7, we introduce and investigate the elements $\Phi^S(k, m)$ with so called (k, m)-regular sets S. In Proposition 7.10, we establish a powerful duality for such elements. Interestingly, as a consequence of the classification, we prove that every right coideal subalgebra over the coradical is generated as an algebra by elements $\Phi^S(k,m)$ with (k,m)-regular sets S (Corollary 10.4). The construction of $\mathbf{U}_\theta$ is more complicated and it has an important new element, a binary predicate defined on the ordered pairs of indices. In [13], we find, relatively easily, a differential subspace generated by $\Psi^S(k,m)$, since this element is linear in each variable that it depends on. However, the elements $\Phi^S(k,m)$ that appear in the present work are not linear in some variables. Therefore, we fail to find their partial derivatives in an appropriate form. Instead, in Theorem 9.8, using the root technique developed in Section 8, we find algebra generators of the right coideal subalgebra generated by $\Phi^S(k,m)$ with a (k,m)-regular set S.

2. Preliminaries

PBW-generators

Let A be an algebra over a field $\mathbf{k}$ and B its subalgebra with a fixed basis $\{g_j \mid j \in J\}$. A linearly ordered subset $V \subseteq A$ is said to be a *set of PBW-generators of A over B* if there exists a function $h : V \to \mathbb{Z}^+ \cup \infty$, called the *height function*, such that the set of all products

$$g_j v_1^{n_1} \cdots v_k^{n_k}, \tag{2.1}$$

where $j \in J$, $v_1 < \cdots < v_k \in V$, $n_i < h(v_i)$, $1 \leq i \leq k$, is a basis of A. The value $h(v)$ is called the *height* of v in V.

Skew brackets

Recall that a Hopf algebra H is referred to as a *character* Hopf algebra if the group G of all grouplike elements is commutative and H is generated over $\mathbf{k}[G]$ by skew primitive semi-invariants a_i, $i \in I$:

$$\Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i, \qquad g^{-1}a_i g = \chi^i(g)a_i, \qquad g, g_i \in G, \tag{2.2}$$

where χ^i, $i \in I$, are characters of the group G. By means of the Dedekind Lemma, it is easy to see that every character Hopf algebra is graded by the monoid G^* of characters generated by χ^i:

$$H = \sum_{\chi \in G^*} \oplus H^\chi, \qquad H^\chi = \{a \in H \mid g^{-1}ag = \chi(g)a,\ g \in G\}. \tag{2.3}$$

Let us associate a "quantum" variable x_i to a_i. For each word u in $X = \{x_i \mid i \in I\}$, we denote by g_u or $\mathrm{gr}(u)$ the element of G that arises from u by replacing each x_i with g_i. In the same way, χ^u denotes the character that arises from u by replacing each x_i with χ^i. We define a bilinear skew commutator on homogeneous linear combinations of words in a_i or in x_i, $i \in I$, by the formula

$$[u, v] = uv - \chi^u(g_v)vu, \tag{2.4}$$

where we use the notation $\chi^u(g_v) = p_{uv} = p(u, v)$. Of course, $p(u, v)$ is a bimultiplicative map:

$$p(u, vt) = p(u, v)p(u, t), \quad p(ut, v) = p(u, v)p(t, v). \tag{2.5}$$

The brackets satisfy the following Jacobi identity:

$$[[u, v], w] = [u, [v, w]] + p_{wv}^{-1}[[u, w], v] + (p_{vw} - p_{wv}^{-1})[u, w] \cdot v, \tag{2.6}$$

or equivalently, in a less symmetric form,

$$[[u, v], w] = [u, [v, w]] + p_{vw}[u, w] \cdot v - p_{uv} v \cdot [u, w]. \tag{2.7}$$

The Jacobi identity (2.6) implies the following conditional identity:

$$[[u, v], w] = [u, [v, w]] \quad \text{provided that} \quad [u, w] = 0. \tag{2.8}$$

By the evident induction on length, this conditional identity admits the following generalisation (see [13, Lemma 2.2]).

Lemma 2.1. *If $y_1, \ldots, y_m$ are homogeneous linear combinations of words such that $[y_i, y_j] = 0$, $1 \le i < j - 1 < m$, then the bracketed polynomial $[y_1 \ldots y_m]$ is independent of the arrangement of brackets:*

$$[y_1 \ldots y_m] = [[y_1 \ldots y_s], [y_{s+1} \ldots y_m]], \quad 1 \le s < m. \tag{2.9}$$

The brackets are related to the product by the following ad-identities:

$$[u \cdot v, w] = p_{vw}[u, w] \cdot v + u \cdot [v, w], \tag{2.10}$$

$$[u, v \cdot w] = [u, v] \cdot w + p_{uv} v \cdot [u, w]. \tag{2.11}$$

In particular, if $[u, w] = 0$, we have

$$[u \cdot v, w] = u \cdot [v, w]. \tag{2.12}$$

The antisymmetry identity transforms into the following two equalities:

$$[u, v] = -p_{uv}[v, u] + (1 - p_{uv} p_{vu}) u \cdot v, \tag{2.13}$$

$$[u, v] = -p_{vu}^{-1}[v, u] + (p_{vu}^{-1} - p_{uv}) v \cdot u. \tag{2.14}$$

In particular, if $p_{uv} p_{vu} = 1$, the "colour" antisymmetry, $[u, v] = -p_{uv}[v, u]$, holds.

The group G acts on the free algebra $\mathbf{k}\langle X\rangle$ by $g^{-1}ug = \chi^u(g)u$, where u is an arbitrary monomial in X. The skew group algebra $G\langle X\rangle$ has the natural Hopf algebra structure:

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad i \in I, \quad \Delta(g) = g \otimes g, \quad g \in G.$$

We fix a Hopf algebra homomorphism

$$\xi : G\langle X\rangle \to H, \quad \xi(x_i) = a_i, \quad \xi(g) = g, \quad i \in I,\ g \in G. \tag{2.15}$$

PBW-basis of a character Hopf algebra

The *constitution* of a word u in $G \cup X$ is a family $\{m_x \mid x \in X\}$ of nonnegative integers such that u has m_x occurrences of x. Certainly, almost all m_x in the constitution are zero. We fix an arbitrary complete order, $<$, on the set X. Normally, if $X = \{x_1, \dots, x_n\}$, we set $x_1 > \cdots > x_n$.

Let Γ^+ be the free additive (commutative) monoid generated by X. The monoid Γ^+ is completely ordered by declaring

$$m_1x_{i_1} + \cdots + m_kx_{i_k} > m_1'x_{i_1} + \cdots + m_k'x_{i_k} \tag{2.16}$$

if the leftmost nonzero number in $(m_1 - m_1', \dots, m_k - m_k')$ is positive, where $x_{i_1} > \cdots > x_{i_k}$ in X. We associate a formal degree $D(u) = \sum_{x\in X} m_x x \in \Gamma^+$ to a word u in $G \cup X$, where $\{m_x \mid x \in X\}$ is the constitution of u. Moreover, if $f = \sum \alpha_i u_i \in G\langle X\rangle$, $0 \neq \alpha_i \in \mathbf{k}$, then

$$D(f) = \max_i D(u_i). \tag{2.17}$$

On the set of all words in X, we fix the lexicographical order with priority from left to right, where a proper initial segment of a word is considered to be greater than the word itself.

A nonempty word u is called *standard* (or a *Lyndon* or *Lyndon–Shirshov* word) if $vw > wv$ for each decomposition $u = vw$ with nonempty v, w. A *nonassociative* word is a word in which brackets [,] are arranged to show how the multiplication applies. If $[u]$ denotes a nonassociative word, then u denotes the associative word obtained from $[u]$ by removing the brackets. The set of *standard nonassociative* words is the largest set *SL* that contains all variables x_i and has the following properties:

1) If $[u] = [[v][w]] \in SL$, then $[v], [w] \in SL$, and $v > w$ are standard.
2) If $[u] = [\,[[v_1][v_2]]\,[w]\,] \in SL$, then $v_2 \leq w$.

Every standard word has only one arrangement of brackets such that the resulting nonassociative word is standard (Shirshov theorem [19]). To find this arrangement, one may use the following inductive procedure:

Algorithm

The factors v, w of the nonassociative decomposition $[u] = [[v][w]]$ are the standard words such that $u = vw$ and v has minimal length ([20], see also [14]).

Definition 2.2. A *super-letter* is a polynomial that equals a nonassociative standard word, where the brackets are as in (2.4). A *super-word* is a word in super-letters.

By Shirshov's theorem, every standard word u defines only one super-letter: in what follows, we shall denote it by $[u]$. The order on the super-letters is defined in the natural way: $[u] > [v] \Leftrightarrow u > v$.

In what follows, we reserve the notation H for a character Hopf algebra that is homogeneous in each a_i (see (2.2) and (2.15)).

Definition 2.3. A super-letter $[u]$ is called *hard in H* provided its value in H is not a linear combination of values of super-words of the same degree (2.17) in super-letters smaller than $[u]$.

Definition 2.4. We say that the *height* of a hard super-letter $[u]$ in H equals $h = h([u])$ if h is the smallest number such that the following hold: first, p_{uu} is a primitive t-th root of 1 and either $h = t$ or $h = tl^r$, where $l = \operatorname{char}(\mathbf{k})$; and the value of $[u]^h$ in H is a linear combination of super-words of the same degree (2.17) in super-letters smaller than $[u]$. If no such number exists, then the height equals infinity.

Theorem 2.5 ([7, Theorem 2]). *The values of all hard super-letters in H with the above-defined height function form a set of PBW-generators for H over $\mathbf{k}[G]$.*

PBW-basis of a homogeneous right coideal subalgebra

A set T of PBW-generators for a homogeneous right coideal subalgebra $\mathbf{U}$, $\mathbf{k}[G] \subseteq \mathbf{U} \subseteq H$, can be obtained from the PBW-basis given in Theorem 2.5 in the following way (see [12, Theorem 1.1]).

Suppose that for a hard super-letter $[u]$ there exists a homogeneous element $c \in \mathbf{U}$ with leading term $[u]^s$ in the PBW-decomposition given in Theorem 2.5:

$$c = [u]^s + \sum_i \alpha_i W_i \in \mathbf{U}, \tag{2.18}$$

where W_i are the basis super-words starting with super-letters smaller than $[u]$. We fix one of the elements with the minimal s, and we denote it by c_u. Thus, for every hard super-letter $[u]$ in H, we have at most one element c_u. We define the height function by means of the following lemma.

Lemma 2.6 ([12, Lemma 4.3]). *In the representation* (2.18) *of c_u either $s = 1$, or $p(u,u)$ is a primitive t-th root of* 1 *and $s = t$, or* (*in the case of positive characteristic*) $s = t(\operatorname{char}\mathbf{k})^r$.

If the height of $[u]$ in H is infinite, then the height of c_u in $\mathbf{U}$ is also defined to be infinite. If the height of $[u]$ in H equals t, then, according to the above lemma, $s = 1$ (recall that in the PBW-decomposition (2.18) the exponent s must be less than the height of $[u]$). In this case, the height of c_u in $\mathbf{U}$ is defined to be t as well. If the characteristic l is positive and the height of $[u]$ in H equals tl^r, then we define the height of c_u in $\mathbf{U}$ to be equal to tl^r/s.

Proposition 2.7 ([12, Proposition 4.4]). *The set of all c_u with the above-defined height function is a set of PBW-generators for* $\mathbf{U}$ *over* $\mathbf{k}[G]$.

The reader is cautioned that the PBW-basis is not uniquely defined in the above process. Nevertheless, the set of leading terms of the PBW-generators is indeed uniquely defined.

Definition 2.8. The degree $sD(c_u) \in \Gamma^+$ of a PBW-generator c_u is said to be a $\mathbf{U}$-*root*. A $\mathbf{U}$-root $\gamma \in \Gamma^+$ is called *simple* if it is not the sum of two or more other $\mathbf{U}$-roots.

The set of $\mathbf{U}$-roots and the set of simple $\mathbf{U}$-roots are invariants for any right coideal subalgebra $\mathbf{U}$.

Shuffle representation

If the kernel of ξ defined in (2.15) is contained in the ideal $G\langle X\rangle^{(2)}$ generated by x_ix_j, $i, j \in I$, then there exists a Hopf algebra projection $\pi : H \to \mathbf{k}[G]$, $a_i \to 0$, $g_i \to g_i$. Hence, by the Radford theorem [18], we have a decomposition into a biproduct, $H = A\,\#\,\mathbf{k}[G]$, where A is a subalgebra generated by a_i, $i \in I$ (see [1, §1.5, §1.7]).

Definition 2.9. In what follows, Λ denotes the largest Hopf ideal in $G\langle X\rangle^{(2)}$. The ideal Λ is homogeneous in each $x_i \in X$ (see [11, Lemma 2.2]).

If $\operatorname{Ker}\xi = \Lambda$ or equivalently if A is a quantum symmetric algebra (a Nichols algebra [1, §1.3, Section 2]), then A has a shuffle representation as follows.

The algebra A has the structure of a *braided Hopf algebra* [21] with a braiding $\tau(u \otimes v) = p(v,u)^{-1} v \otimes u$. The braided coproduct Δ^b on A is connected with the coproduct on H in the following way:

$$\Delta^b(u) = \sum_{(u)} u^{(1)}\mathrm{gr}(u^{(2)})^{-1} \underline{\otimes}\, u^{(2)}. \tag{2.19}$$

The tensor space $T(V)$, $V = \sum x_i\mathbf{k}$, also has the structure of a braided Hopf algebra, which is the *quantum shuffle algebra* $Sh_\tau(V)$ with the coproduct

$$\Delta^b(u) = \sum_{i=0}^{m} (z_1 \dots z_i)\, \underline{\otimes}\, (z_{i+1} \dots z_m), \tag{2.20}$$

where $z_i \in X$, and $u = (z_1 \dots z_m)$ is the tensor $z_1 \otimes \cdots \otimes z_m$ considered as an element of $Sh_\tau(V)$. The shuffle product satisfies

$$(w)(x_i) = \sum_{uv=w} p(x_i, v)^{-1}(ux_iv), \qquad (x_i)(w) = \sum_{uv=w} p(u, x_i)^{-1}(ux_iv). \tag{2.21}$$

The map $a_i \to (x_i)$ defines an embedding of the braided Hopf algebra A into the braided Hopf algebra $Sh_\tau(V)$. This embedding is extremely useful for calculating the coproduct due to formulae (2.19) and (2.20).

Differential calculus

The free algebra $\mathbf{k}\langle X\rangle$ has a coordinate differential calculus

$$\partial_j(x_i) = \delta_i^j, \qquad \partial_i(uv) = \partial_i(u) \cdot v + \chi^u(g_i)u \cdot \partial_i(v). \tag{2.22}$$

The partial derivatives connect the calculus with the coproduct on $\mathbf{k}\langle X\rangle$ via

$$\Delta(u) \equiv u \otimes 1 + \sum_i g_i\partial_i(u) \otimes x_i \pmod{G\langle X\rangle \otimes \mathbf{k}\langle X\rangle^{(2)}}, \tag{2.23}$$

where $\mathbf{k}\langle X\rangle^{(2)}$ is the ideal generated by x_ix_j, $1 \le i, j \le n$.

Lemma 2.10. *Let $u \in \mathbf{k}\langle X\rangle$ be an element homogeneous in each x_i. If p_{uu} is a t-th primitive root of 1, then*

$$\partial_i(u^t) = p(u, x_i)^{t-1} \underbrace{[u, [u, \dots [u}_{t-1}, \partial_i(u)] \dots]]. \tag{2.24}$$

Proof. First, we note that the sequence $p_{uu}, p_{uu}^2, \dots, p_{uu}^{t-1}$ contains all t-th roots of 1 except 1 itself. All members in this sequence are different. Hence, we may write the polynomial equality

$$1 - x^t = (1 - x) \prod_{s=1}^{t-1} (1 - p_{uu}^s x). \tag{2.25}$$

Let us calculate the right-hand side of (2.24). We denote by L_u and R_u the operators of left and right multiplication by u, respectively. The right-hand side of (2.24) has the following operator representation:

$$p(u, x_i)^{t-1} \Big(\partial_i(u) \cdot \prod_{s=1}^{t-1} (L_u - Q p_{uu}^{s-1} R_u) \Big),$$

where $Q = p(u, \partial_i(u)) = p_{uu} p(u, x_i)^{-1}$. Consider the polynomial

$$f(\lambda) = \prod_{s=1}^{t-1} (1 - Q p_{uu}^{s-1} \lambda) \stackrel{df}{=} \sum_{k=0}^{t-1} \alpha_k \lambda^k.$$

Because the operators R_u and L_u commute, we may develop the multiplication in the operator product considering R_u and L_u as formal commutative variables:

$$\prod_{s=1}^{t-1} (L_u - Q p_{uu}^{s-1} R_u) = L_u^{t-1} f\left(\frac{R_u}{L_u} \right) = \sum_{k=0}^{t-1} \alpha_k L_u^{t-1-k} R_u^k.$$

Thus the right-hand side of (2.24) equals

$$p(u, x_i)^{t-1} \sum_{k=0}^{t-1} \alpha_k u^{t-1-k} \partial_i(u) u^k.$$

Further, because $Q = p_{uu} p(u, x_i)^{-1}$, the polynomial f has a representation

$$f(\lambda) = \prod_{s=1}^{t-1} (1 - p_{uu}^s \xi),$$

where $\xi = \lambda p(u, x_i)^{-1}$. Taking into account (2.25), we obtain

$$\begin{aligned} f(\lambda) &= \frac{1 - \xi^t}{1 - \xi} = \frac{1 - \lambda^t p(u, x_i)^{-t}}{1 - \lambda p(u, x_i)^{-1}} \\ &= 1 + \lambda p(u, x_i)^{-1} + \lambda^2 p(u, x_i)^{-2} + \cdots + \lambda^{t-1} p(u, x_i)^{1-t}; \end{aligned}$$

that is, $\alpha_k = p(u, x_i)^{-k}$, while the right-hand side of (2.24) takes the form

$$\sum_{k=0}^{t-1} p(u, x_i)^{t-1-k} u^{t-1-k} \partial_i(u) u^k. \tag{2.26}$$

At the same time the Leibniz formula (2.22) shows that $\partial_i(u^t)$ also equals (2.26). □

MS-criterion

The quantum symmetric algebra has several convenient characterisations. One of these characterisations says that the quantum symmetric algebra is the *optimal algebra* for the calculus defined by (2.22). In other words, the above-defined algebra A is a quantum symmetric algebra (or equivalently $\operatorname{Ker}\xi = \Lambda$) if and only if all constants in A are scalars.

For braidings of the Cartan type, this characterisation was proved by A. Milinski and H.-J. Schneider in [15] and then generalised to arbitrary (even not necessarily invertible) braidings by the author in [10, Theorem 4.11]. Moreover, if X is finite, then $\Lambda \cap \mathbf{k}\langle X\rangle$ (as well as any differential ideal in $\mathbf{k}\langle X\rangle$) is generated as a left ideal by constants from $\mathbf{k}\langle X\rangle^{(2)}$ (see [10, Corollary 7.8]). Thus, we may formulate the following criterion, which is useful for checking relations.

Lemma 2.11 (Milinski–Schneider criterion). *Suppose that* $\operatorname{Ker}\xi = \Lambda$. *If a polynomial* $f \in \mathbf{k}\langle X\rangle$ *is a constant in* A (*that is,* $\partial_i(f) \in \Lambda$, $i \in I$), *then there exists* $\alpha \in \mathbf{k}$ *such that* $f - \alpha = 0$ *in* A.

Of course, one can easily prove this criterion by means of (2.19), (2.20) and (2.23) using the above shuffle representation because (2.20) implies that all constants in the shuffle coalgebra are scalars.

Quantum Borel algebra

Let $C = \|a_{ij}\|$ be a generalised Cartan matrix, symmetrisable by $D = \operatorname{diag}(d_1, \ldots, d_n)$: $d_i a_{ij} = d_j a_{ji}$. We denote by $\mathfrak{g}$ the Kac–Moody algebra defined by C (see [5]). Suppose that parameters p_{ij} are related by

$$p_{ii} = q^{d_i}, \qquad p_{ij} p_{ji} = q^{d_i a_{ij}}, \qquad 1 \le i, j \le n. \tag{2.27}$$

Denote by g_j the linear transformation $g_j : x_i \to p_{ij} x_i$ of the linear space spanned by a set of variables $X = \{x_1, \ldots, x_n\}$. Let χ^i denote the character $\chi^i : g_j \to p_{ij}$ of the group G generated by g_i, $1 \le i \le n$. We consider each x_i as a "quantum variable" with parameters g_i, χ^i. As above, $G\langle X\rangle$ denotes the skew group algebra with commutation rules $x_i g_j = p_{ij} g_j x_i$, $1 \le i, j \le n$. This algebra has the structure of a character Hopf algebra

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \qquad \Delta(g_i) = g_i \otimes g_i. \tag{2.28}$$

In this case the multiparameter quantisation $U_q^+(\mathfrak{g})$ of the Borel subalgebra $\mathfrak{g}^+$ is a homomorphic image of $G\langle X\rangle$ defined by Serre relations with the skew bracket in place of the Lie operation:

$$[\dots[[x_i,\underbrace{x_j],x_j],\dots,x_j]}_{1-a_{ji}\text{ times}}=0,\qquad 1\le i\ne j\le n. \tag{2.29}$$

By [6, Theorem 6.1], the left-hand sides of these relations are skew-primitive elements in $G\langle X\rangle$. Therefore the ideal generated by these elements is a Hopf ideal, while $U_q^+(\mathfrak{g})$ has the natural structure of a character Hopf algebra.

Lemma 2.12 ([13, Corollary 3.2]). *If q is not a root of 1, and C is of finite type, then every subalgebra U of $U_q^+(\mathfrak{g})$ containing G is homogeneous with respect to each of the variables x_i.*

Definition 2.13. If the multiplicative order t of q is finite, then we define $u_q^+(\mathfrak{g})$ as $G\langle X\rangle/\Lambda$, where Λ is the largest Hopf ideal in $G\langle X\rangle^{(2)}$ (see Definition 2.9).

Because a skew-primitive element generates a Hopf ideal, Λ contains all skew-primitive elements of $G\langle X\rangle^{(2)}$. Hence relations (2.29) are still valid in $u_q^+(\mathfrak{g})$.

3. Relations in the quantum Borel algebra $U_q^+(\mathfrak{so}_{2n+1})$

In what follows, we fix a parameter q such that $q^4\ne 1$, $q^3\ne 1$. If C is a Cartan matrix of type B_n, relations (2.27) take the form

$$p_{nn}=q,\qquad p_{ii}=q^2,\qquad p_{i\,i+1}p_{i+1\,i}=q^{-2},\qquad 1\le i<n; \tag{3.1}$$
$$p_{ij}p_{ji}=1,\qquad j>i+1. \tag{3.2}$$

Starting with parameters p_{ij} satisfying these relations, we define the group G and the character Hopf algebra $G\langle X\rangle$ as in the above subsection. In this case the quantum Borel algebra $U_q^+(\mathfrak{so}_{2n+1})$ is a homomorphic image of $G\langle X\rangle$ subject to the following relations:

$$[x_i,[x_i,x_{i+1}]]=0,\qquad 1\le i<n;\qquad [x_i,x_j]=0,\qquad j>i+1; \tag{3.3}$$
$$[[x_i,x_{i+1}],x_{i+1}]=[[[x_{n-1},x_n],x_n],x_n]=0,\qquad 1\le i<n-1. \tag{3.4}$$

Here, we slightly modify the Serre relations (2.29) so that the left-hand side of each relation is a super-letter. This modification is possible due to the following general relation in $\mathbf{k}\langle X\rangle$ (see [9, Corollary 4.10]):

$$[\dots[[x_i,\underbrace{x_j],x_j],\dots x_j]}_{n}=\alpha\,\underbrace{[x_j,[x_j,\dots[x_j}_{n},x_i]\dots]],\qquad 0\ne\alpha\in\mathbf{k}, \tag{3.5}$$

provided that $p_{ij}p_{ji}=p_{jj}^{1-n}$.

Definition 3.1. The elements u, v are said to be *separated* if there exists an index j, $1 \le j \le n$, such that either $u \in \mathbf{k}\langle x_i \mid i < j\rangle$, $v \in \mathbf{k}\langle x_i \mid i > j\rangle$ or vice versa: $u \in \mathbf{k}\langle x_i \mid i > j\rangle$, $v \in \mathbf{k}\langle x_i \mid i < j\rangle$.

Lemma 3.2. *In the algebra $U_q^+(\mathfrak{so}_{2n+1})$, any two separated elements u, v, homogeneous in each $x_i \in X$, (skew) commute:* $[u, v] = [v, u] = 0$.

Proof. Relations (3.2) and conditional antisymmetry (2.13) show that $[x_i, x_j] = [x_j, x_i] = 0$ provided that $|i - j| > 1$. Now relations (2.10) and (2.11) allow one to perform an evident induction. □

Certainly, the subalgebra of $U_q^+(\mathfrak{so}_{2n+1})$ generated over $\mathbf{k}[g_1, \dots, g_{n-1}]$ by x_i, $1 \le i < n$, is the Hopf algebra $U_{q^2}^+(\mathfrak{sl}_n)$ defined by the Cartan matrix of type A_{n-1}. Let us replace just one parameter, $p_{nn} \leftarrow q^2$. Then the quantum Borel algebra $U_{q^2}^+(\mathfrak{sl}_{n+1})$ is a homomorphic image of $G'\langle X\rangle$ subject to the relations

$$[[x_i, x_{i+1}], x_{i+1}] = [x_i, [x_i, x_{i+1}]] = [x_i, x_j] = 0, \qquad j > i + 1. \tag{3.6}$$

Here, G' is the group generated by the transformations $g_1, \dots, g_{n-1}, g_n'$, where $g_n'(x_i) = g_n(x_i)$ for $i \ne n$ and $g_n'(x_n) = q^2 x_n$.

Lemma 3.3. *A relation $f = 0$, $f \in \mathbf{k}\langle X\rangle$, linear in x_n is valid in $U_q^+(\mathfrak{so}_{2n+1})$ if and only if it is valid in the above algebra $U_{q^2}^+(\mathfrak{sl}_{n+1})$.*

Proof. The element f, an element of a free algebra, belongs to the ideal generated by the defining relations that are independent of x_n or linear in x_n. All these relations are the same for $U_q^+(\mathfrak{so}_{2n+1})$ and for $U_{q^2}^+(\mathfrak{sl}_{n+1})$. □

Lemma 3.4. *If u is a standard word, then either $u = x_k x_{k+1} \dots x_m$, $k \le m \le n$, or $[u] = 0$ in $U_{q^2}^+(\mathfrak{sl}_{n+1})$. Here $[u]$ is a nonassociative word with the standard arrangement of brackets: see the Algorithm on page 1682.*

Proof. See the third statement of [9, Theorem A_n]. □

As a corollary of the above two lemmas, we can prove some relations in $U_q^+(\mathfrak{so}_{2n+1})$:

$$[[x_{k+1} x_k x_{k-1}], x_k] = 0, \qquad [[x_{k-1} x_k x_{k+1}], x_k] = 0, \qquad k < n. \tag{3.7}$$

Indeed, $x_{k-1} x_k x_{k+1} x_k$ is a standard word, and the standard arrangement of brackets is precisely $[[x_{k-1}, [x_k, x_{k+1}]], x_k]$. Hence, (2.8) together with Lemmas 3.3 and 3.4 implies the latter relation.

The former relation reduces to the latter by means of the replacement $x_i \leftarrow x_{n-i+1}$, $1 \le i \le n$, $k \leftarrow n - k + 1$. Note that the defining relations (3.6) are invariant under this replacement (see (3.5)), and we again use Lemmas 3.3 and 3.4.

Definition 3.5. In what follows, x_i, $n < i \le 2n$, denotes the generator x_{2n-i+1}. Moreover, $u(k, m)$, $1 \le k \le m \le 2n$, is the word $x_k x_{k+1} \dots x_m$, while $u(m, k)$ is the word $x_m x_{m-1} \dots x_k$. If $1 \le i \le 2n$, then $\psi(i)$ denotes the number $2n - i + 1$, so that $x_i = x_{\psi(i)}$. We shall frequently use the following properties of ψ: if $i < j$, then $\psi(i) > \psi(j)$; $\psi(\psi(i)) = i$; $\psi(i + 1) = \psi(i) - 1$.

Definition 3.6. If $k \leq i < m \leq 2n$, then we set

$$\sigma_k^m \stackrel{df}{=} p(u(k,m), u(k,m)), \tag{3.8}$$

$$\mu_k^{m,i} \stackrel{df}{=} p(u(k,i), u(i+1,m)) \cdot p(u(i+1,m), u(k,i)). \tag{3.9}$$

Of course, one can find μ's and σ's by means of (3.1), (3.2). It turns out that these coefficients depend only on q. More precisely,

$$\sigma_k^m = \begin{cases} q & \text{if } m = n \text{ or } k = n+1; \\ q^4 & \text{if } m = \psi(k); \\ q^2 & \text{otherwise.} \end{cases} \tag{3.10}$$

Indeed, the bimultiplicativity of $p(-,-)$ implies that $\sigma_k^m = \prod_{k \leq s,t \leq m} p_{st}$ is the product of all entries of the $(m-k+1) \times (m-k+1)$-matrix $\|p_{st}\|$. By (3.1) all coefficients on the main diagonal equal q^2 with only two possible exceptions, $p_{nn} = q$, $p_{n+1\,n+1} = q$. In particular, if $m < n$ or $k > n+1$, then for nondiagonal coefficients, we have $p_{st} p_{ts} = 1$ unless $|s-t| = 1$, while $p_{s\,s+1} p_{s+1\,s} = q^{-2}$. Hence, $\sigma_k^m = q^{2(m-k+1)} \cdot q^{-2(k-m)} = q^2$. If $m = n$ or $k = n+1$, then, by the same reasoning, we have $\sigma_k^m = q^{2(m-k)+1} \cdot q^{-2(k-m)} = q$. In the remaining case, $k \leq n < m$, we split the matrix into four submatrices as follows:

$$\sigma_k^m = \sigma_k^n \cdot \sigma_{n+1}^m \cdot \prod_{k \leq s \leq n,\, n+1 \leq t \leq m} p_{st} \cdot \prod_{n+1 \leq s \leq m,\, k \leq t \leq n} p_{st}. \tag{3.11}$$

According to Definition 3.5, we have $p_{st} = p_{\psi(s)\,t} = p_{s\,\psi(t)} = p_{\psi(s)\,\psi(t)}$. Therefore, the third and fourth factors in (3.11) equal, respectively,

$$\prod_{k \leq s \leq n,\, \psi(m) \leq t \leq n} p_{st}, \qquad \prod_{\psi(m) \leq s \leq n,\, k \leq t \leq n} p_{st}.$$

In particular, if $\psi(m) = k$, then all four factors in (3.11) coincide with $\sigma_k^n = q$. Hence, $\sigma_k^m = q^4$. If $\psi(m) \neq k$, say $\psi(m) > k$, then we split the rectangle $A = [k,n] \times [\psi(m), n]$ into the union of the square $B = [\psi(m), n] \times [\psi(m), n]$ and the rectangle $C = [k, \psi(m)-1] \times [\psi(m), n]$. Similarly, the rectangle $A^* = [\psi(m), n] \times [k,n]$ is the union of the same square and the rectangle $C^* = [\psi(m), n] \times [k, \psi(m)-1]$. Certainly, if $(s,t) \in C$, then $t - s > 1$ unless $t = \psi(m)-1$, $s = \psi(m)$. Hence, relations (3.2) imply

$$\prod_{(s,t) \in C} p_{st} p_{ts} = p_{\psi(m)-1\,\psi(m)} p_{\psi(m)\,\psi(m)-1} = q^{-2}.$$

At the same time $\prod_{(s,t) \in B} p_{st} = \sigma_{\psi(m)}^n = q$. Finally, (3.11) takes the form

$$\sigma_k^m = q \cdot q \cdot \Big(\prod_{(s,t) \in B} p_{st}\Big)^2 \cdot \prod_{(s,t) \in C} p_{st} p_{ts} = q^2,$$

which proves (3.10).

To find μ's we consider decomposition (3.11) with $n \leftarrow i$. Because $p(-,-)$ is a bimultiplicative map, the product of the last two factors is precisely $\mu_k^{m,i}$. In particular,

$$\mu_k^{m,i} = \sigma_k^m (\sigma_k^i \sigma_{i+1}^m)^{-1}. \tag{3.12}$$

This formula, with (3.10), allows one to find the μ's easily. More precisely, if $m < \psi(k)$, then

$$\mu_k^{m,i} = \begin{cases} q^{-4} & \text{if } m > n,\ i = \psi(m) - 1; \\ 1 & \text{if } i = n; \\ q^{-2} & \text{otherwise.} \end{cases} \tag{3.13}$$

If $m = \psi(k)$, that is, $x_m = x_k$, then

$$\mu_k^{m,i} = \begin{cases} q^2 & \text{if } i = n; \\ 1 & \text{otherwise.} \end{cases} \tag{3.14}$$

If $m > \psi(k)$, then the μ's satisfy $\mu_k^{m,i} = \mu_{\psi(m)}^{\psi(k),\, \psi(i)-1}$, hence one may use (3.13):

$$\mu_k^{m,i} = \begin{cases} q^{-4} & \text{if } k \leq n,\ i = \psi(k); \\ 1 & \text{if } i = n; \\ q^{-2} & \text{otherwise.} \end{cases} \tag{3.15}$$

We define the bracketing of $u(k,m)$, $k \leq m$, as follows:

$$u[k,m] = \begin{cases} [[[\ldots[x_k, x_{k+1}], \ldots], x_{m-1}], x_m] & \text{if } m < \psi(k); \\ [x_k, [x_{k+1}, [\ldots, [x_{m-1}, x_m]\ldots]]] & \text{if } m > \psi(k); \\ \beta[u[n+1,m], u[k,n]] & \text{if } m = \psi(k), \end{cases} \tag{3.16}$$

where $\beta = -p(u(n+1,m), u(k,n))^{-1}$ normalizes the coefficient of $u(k,m)$. The conditional identity (2.9) shows that the value of $u[k,m]$ in $U_q^+(\mathfrak{so}_{2n+1})$ is independent of the arrangement of brackets provided that $m \leq n$ or $k > n$.

In what follows, $\sim$ denotes projective equality: $a \sim b$ if and only if $a = \alpha b$, where $0 \neq \alpha \in \mathbf{k}$.

Lemma 3.7. *If $t \notin \{k-1, k\}$, $t < n$, then $[u[k,n], x_t] = [x_t, u[k,n]] = 0$.*

Proof. If $t \leq k-2$, then the equality follows from the second group of defining relations (3.3). Let $k < t < n$. By (2.8), we may write

$$[u[k,n], x_t] = \big[[u[k,t-2], u[t-1,n]], x_t\big] = \big[u[k,t-2], [u[t-1,n], x_t]\big].$$

By Lemma 3.4, the element $[u[t-1,n], x_t]$ equals zero in $U_{q^2}^+(\mathfrak{sl}_{n+1})$ because the word $u(t-1,n)x_t$ is standard, and the standard bracketing is precisely $[u[t-1,n], x_t]$. This element is linear in x_n. Hence, $[u[k,n], x_t] = 0$ in $U_q^+(\mathfrak{so}_{2n+1})$ due to Lemma 3.3. Because $p(u(k,n), x_t)p(x_t, u(k,n)) = p_{t\,t+1}p_{tt}p_{t\,t-1} \cdot p_{t+1\,t}p_{tt}p_{t-1\,t} = 1$, the antisymmetry identity (2.13) applies. □

Lemma 3.8. *If $t \notin \{\psi(m)-1, \psi(m)\}$, $t<n<m$, then*

$$[x_t, u[n+1,m]] = [u[n+1,m], x_t] = 0.$$

Proof. If $t \leq \psi(m)-2$, then the required relation follows from the second group of relations (3.3). Let $\psi(m) < t < n$. By Lemma 2.1, the value of $u[n+1,m]$ in $U_q^+(\mathfrak{so}_{2n+1})$ is independent of the arrangement of brackets. In particular, $u[n+1,m] = [[w, [x_{t+1}x_tx_{t-1}]], v]$, where $w = u[n+1, \psi(t)-2]$, $v = u[\psi(t)+2, m]$. Because $p_{t\,t+1}p_{tt}p_{t\,t-1} \cdot p_{t+1\,t}p_{tt}p_{t-1\,t} = 1$, the antisymmetry identity (2.13) and the first equality of (3.7) imply $[x_t, [x_{t+1}x_tx_{t-1}]] \sim [[x_{t+1}x_tx_{t-1}], x_t] = 0$. Note that $[x_t, w] = [w, x_t] = 0$, $[x_t, v] = [v, x_t] = 0$ according to the second group of defining relations (3.3). □

Lemma 3.9. *If $k \leq n < m < \psi(k)$, then the value in $U_q^+(\mathfrak{so}_{2n+1})$ of the bracketed word $[y_kx_{n+1}x_{n+2}\dots x_m]$, where $y_k = u[k,n]$, is independent of the arrangement of brackets.*

Proof. To apply (2.9), it suffices to check $[u[k,n], x_t] = 0$, $n+1 < t \leq m$. Because the application of ψ changes the order, we have $k < \psi(m) \leq \psi(t) < n$. Hence, taking into account $x_t = x_{\psi(t)}$, one may use Lemma 3.7. □

Lemma 3.10. *If $k \leq n < \psi(k) < m$, then the value in $U_q^+(\mathfrak{so}_{2n+1})$ of the bracketed word $[x_kx_{k+1}\dots x_ny_m]$, where $y_m = u[n+1,m]$, is independent of the arrangement of brackets.*

Proof. To apply (2.9), we need $[x_t, u[n+1,m]] = 0$, $k \leq t < n$. To obtain these equalities, one may use Lemma 3.8. □

Lemma 3.11. *If $m \neq \psi(k)$, $k \leq i < n < m$, then*

$$[u[k,i], u[n+1,m]] = [u[n+1,m], u[k,i]] = 0$$

unless $i = \psi(m)-1$.

Proof. We denote $u = u[k,i]$, $w = u[n+1,m]$. Relations (3.1), (3.2) imply $p_{uw}p_{wu} = 1$. Hence, by (2.13), we have $[u, w] = -p_{uw}[w, v]$.

If $\psi(m) < k$, then by Lemma 3.8, we have $[x_t, u[n+1,m]] = 0$, $k \leq t \leq i$. Hence, $[u[k,i], u[n+1,m]] = 0$.

Suppose that $\psi(m) > k$. If $i < \psi(m)-1$, then by the second group of defining relations (3.3), we have $[x_t, u[n+1,m]] = 0$, $k \leq t \leq i$. Hence, $[u[k,i], u[n+1,m]] = 0$.

Let $\psi(m) \leq i < n$. If we define $u_1 = u[k, \psi(m)-2]$, $u_2 = u[\psi(m)-1, i]$, then certainly $u = [u_1, u_2]$ unless $k = \psi(m)-1$, $u = u_2$. Because $[u_1, w] = 0$, the conditional Jacobi identity (2.8) implies that, in both cases, we only need to check $[u_2, w] = 0$.

Let us put $u_3 = [x_{\psi(m)-1}, x_{\psi(m)}]$, $u_4 = u[\psi(m)+1, i]$. Then $u_2 = [u_3, u_4]$ unless $i = \psi(m)$, $u_2 = u_3$. By Lemma 3.8, we have $[x_t, u[n+1,m]] = 0$ for all t, $\psi(m) < t < n$. Hence, $[u_4, w] = 0$. Now the Jacobi identity (2.6) with $u \leftarrow u_3$, $v \leftarrow u_4$ shows that it suffices to prove the equality $[u_3, w] = 0$.

Let us put $w_1 = u[n+1, m-2]$, $w_2 = [x_{m-1}, x_m]$. Then $w = [w_1, w_2]$ unless $m-2 = n$, $w = w_2$ (recall that we are considering the case $\psi(m) \le i < n$, in particular $\psi(m) \le n-1$, and hence $m \ge \psi(n-1) = n+2$). We now have $[u_3, w_1] = 0$. Therefore the Jacobi identity (2.6) with $u \leftarrow u_3$, $v \leftarrow w_1$, $w \leftarrow w_2$ shows that it is sufficient to obtain the equality $[u_3, w_2] = 0$, that is, $[[x_{t-1}, x_t], [x_{t+1}, x_t]] = 0$ with $t = \psi(m) < n$. Since $[[x_{t-1}, x_t], x_t] = 0$ is one of the defining relations, the conditional identity (2.8) implies $[[x_{t-1}, x_t], [x_{t+1}, x_t]] = [[x_{t-1}x_t x_{t+1}], x_t]$. It remains to apply the second equality of (3.7). □

Lemma 3.12. *If $m \neq \psi(k)$, $k \le n < i < m$, then*

$$[u[k, n], u[i+1, m]] = [u[i+1, m], u[k, n]] = 0$$

unless $i = \psi(k)$.

Proof. The proof is quite similar to the preceding one. It is based on Lemma 3.7 and the first equality of (3.7) in the same way as the proof of the above lemma is based on Lemma 3.8 and the second equality of (3.7). □

Corollary 3.13. *If $m \neq \psi(k)$, $k \le n < m$, then in $U_q^+(\mathfrak{so}_{2n+1})$ we have*

$$u[k, m] = [u[k, n], u[n+1, m]] = \beta[u[n+1, m], u[k, n]], \tag{3.17}$$

where $\beta = -p(u(n+1, m), u(k, n))^{-1}$.

Proof. Let us denote $u = u[k, n]$, $v = u[n+1, m]$. Equalities (3.13) and (3.15) with $i = n$ show that $p_{uv}p_{vu} = \mu_k^{m,n} = 1$ provided that $m \neq \psi(k)$. Hence, $[u, v] = uv - p_{uv}vu = -p_{uv}[v, u]$, which proves the second equality. To prove the first, we apply Lemma 3.9 if $m < \psi(k)$, and otherwise we apply Lemma 3.10. □

Proposition 3.14. *If $m \neq \psi(k)$, then in $U_q^+(\mathfrak{so}_{2n+1})$ for each i, $k \le i < m$, we have*

$$[u[k, i], u[i+1, m]] = u[k, m]$$

with only two possible exceptions, $i = \psi(m) - 1$ and $i = \psi(k)$.

Proof. If $m \le n$ or $k \ge n+1$, then the statement follows from (2.9). Thus, we may suppose that $m > n$.

If $i = n$, then Corollary 3.13 implies the required formula.

If $i > n$, then Corollary 3.13 yields $u[k, i] = [u[k, n], u[n+1, i]]$, while by Lemma 3.12 we have $[u[k, n], u[i+1, m]] = 0$. Hence, (2.8) implies

$$\big[[u[k, n], u[n+1, i]], u[i+1, m]\big] = \big[u[k, n], [u[n+1, i], u[i+1, m]]\big].$$

Now, (2.9) shows that $[u[n+1, i], u[i+1, m]] = [u[n, m]]$, and again Corollary 3.13 implies the required formula.

If $i < n$, then Corollary 3.13 yields $u[i+1, m] = [u[i+1, n], u[n+1, m]]$, while by Lemma 3.11 we have $[u[k, i], u[n+1, m]] = 0$. Hence, (2.8) implies

$$\big[[u[k, i], [u[i+1, n], u[n+1, m]]\big] = \big[[u[k, i], [u[i+1, n]], u[n+1, m]\big].$$

Now, (2.9) shows that $[u[k, i], u[i+1, n] = u[k, n]$, and again Corollary 3.13 implies the required formula. □

Proposition 3.15. *If* $m \neq \psi(k)$, $k \leq i < j < m$, $m \neq \psi(i)-1$, $j \neq \psi(k)$, *then* $[u[k,i], u[j+1,m]] = 0$. *If, additionally,* $i \neq \psi(j)-1$, *then* $[u[j+1,m], u[k,i]] = 0$.

Proof. If $m \leq n$ or $k > n$, then $u[k,i]$ and $u[j+1,m]$ are separated by x_j; hence, the statement follows from Lemma 3.2.

If $k \leq n < i$, then by Corollary 3.13, we have $u[k,i] = [a,b]$ with $a = u[k,n]$, $b = u[n+1,i]$. The second group of relations (3.3) implies $[b, u[j+1,m]] = 0$, while Lemma 3.12 implies $[a, u[j+1,m]] = 0$. Hence by (2.6) we obtain the required relation.

If $j < n \leq m$, then, again by Corollary 3.13, we have $u[j+1,m] = [a,b]$ with $a = u[j+1,n]$, $b = u[n+1,m]$. The second group of relations (3.3) implies $[u[k,i],a] = 0$, while Lemma 3.11 implies $[u[k,i],b] = 0$. Hence, by (2.6), we obtain the required relation.

Assume $i \leq n \leq j$. If $i > \psi(j)-1$, then, by taking into account Lemma 3.3, one may apply Lemma 3.12 with $n \leftarrow i$, $i \leftarrow j$. Similarly, if $i < \psi(j)-1$, one may apply Lemma 3.11 with $n \leftarrow \psi(j)-1$. Let $i = \psi(j)-1$. We may apply the case "$i > \psi(j)-1$", which was already considered, to the sequence $k \leq i < j' < m$ with $j' = j+1$, unless $j' = m$ or $j' = \psi(k)$. Thus, $[u[k,i], u[j+2,m]] = 0$ provided that $j+1 \neq m$, $j+1 \neq \psi(k)$. Lemma 2.1 implies

$$[u[k,i], x_i] = \big[u[k,i-2], [[x_{i-1}, x_i], x_i]\big] = 0, \tag{3.18}$$

because the inequality $i < j-1$ and the equality $i = \psi(j)-1$ imply $i < n$. Now, if $j+1 \neq m$, $j+1 \neq \psi(k)$, then using Lemma 2.1, we have

$$[u[k,i], u[j+1,m]] = \big[u[k,i], [x_i, u[j+2,m]]\big] \overset{(2.8)}{=} \big[[u[k,i], x_i], u[j+2,m]\big] \overset{(3.18)}{=} 0,$$

for $x_{j+1} = x_i$. The exceptional equality $j+1 = \psi(k)$ implies $k = \psi(j)-1 = i$. In this case, taking into account Lemma 2.1, we have

$$[x_i, u[j+1,m]] = [[x_i, [x_i, x_{i-1}]], u[j+3,m]] = 0.$$

The exceptional equality $j+1 = m$ implies $u[1+j,m] = x_m = x_i$, for $\psi(j+1) = i$. Hence, relation (3.18) applies. The equality $[u[k,i], u[j+1,m]] = 0$ is proven.

Assume $i \neq \psi(j)-1$. Definition (3.9) shows that

$$p(u(k,i), u(j+1,m)) \cdot p(u(j+1,m), u(k,i)) = \mu_k^{m,i} (\mu_k^{j,i})^{-1}.$$

Using (3.13) and (3.15), we shall prove that $\mu_k^{m,i} = \mu_k^{j,i}$. If $i = n$, then $\mu_k^{m,i} = \mu_k^{j,i} = 1$. Let $i \neq n$. If $m < \psi(k)$, then $\mu_k^{m,i} = q^{-2}$ because $i = \psi(m)-1$ is equivalent to $m = \psi(i)-1$. Similarly, $\mu_k^{j,i} = q^{-2}$ since $j \neq \psi(i)-1$ and $j \leq m < \psi(k)$.

If $m > \psi(k)$ and $i \neq \psi(k)$, then by (3.15), we have $\mu_k^{m,i} = q^{-2}$, while $\mu_k^{j,i} = q^{-2}$ in both cases: if $j < \psi(k)$ by (3.13), and if $j > \psi(k)$ by (3.15). Finally, if $i = \psi(k)$, then $j > i = \psi(k)$; hence, (3.15) implies $\mu_k^{m,i} = \mu_k^{j,i} = q^{-4}$.

To obtain $[u[j+1,m], u[k,i]] = 0$, apply (2.13). □

4. PBW-generators of the quantum Borel algebra

Proposition 4.1. *If $q^3 \neq 1$, $q^4 \neq 1$, then the values of the elements $u[k,m]$, $k \leq m < \psi(k)$, form a set of PBW-generators for the algebra $U_q^+(\mathfrak{so}_{2n+1})$ over $\mathbf{k}[G]$. All heights are infinite.*

Proof. By [9, Theorem B_n, p. 211] the set of PBW-generators (the values of hard super-letters; see Theorem 2.5) consists of $[u_{km}]$, $k \leq m \leq n$, and $[w_{ks}]$, $1 \leq k < s \leq n$, where $[u_{km}]$, $[w_{ks}]$ are precisely the words $u(k,m)$, $u(k,\psi(s))$ with the standard arrangement of brackets (see Algorithm p. 1682). By conditional identity (2.9) we have $[u_{km}] = u[k,m]$ in $U_q^+(\mathfrak{so}_{2n+1})$. According to [9, Lemma 7.8], the brackets in $[w_{ks}]$ are set by the following recurrence formulae:

$$\begin{array}{ll} [w_{ks}] = [x_k[w_{k+1\,s}]] & \text{if } 1 \leq k < s-1; \\ [w_{k\,k+1}] = [[w_{k\,k+2}]x_{k+1}] & \text{if } 1 \leq k < n, \end{array} \tag{4.1}$$

where, by definition, $w_{k\,n+1} = u(k,n)$. We shall check the equality $[w_{ks}] = u[k,\psi(s)]$ in $U_q^+(\mathfrak{so}_{2n+1})$.

If $k = n-1$ and $s = n$, then $w_{ks} = [[x_{n-1}, x_n], x_n] = u[n-1, n+2]$.

If $k < s-1$, then, by (2.8), we have

$$\big[x_k, [u[k+1,n], u[n+1,\psi(s)]]\big] = \big[u[k,n], u[n+1,\psi(s)]\big],$$

for $[x_k, x_t] = 0$, $n+1 \leq t \leq \psi(s)$. Thus, the evident induction applies because of (3.17).

If $s = k+1 < n$, then the second option of (4.1) is fulfilled. This allows us to apply the already proven equality for $[w_{k\,k+2}]$. □

If q is not a root of 1, then the fourth statement of [9, Theorem B_n, p. 211] shows that each skew-primitive element in $U_q^+(\mathfrak{so}_{2n+1})$ is proportional to either x_i, $1 \leq i \leq n$, or $1-g$, $g \in G$. In particular, $\xi(G\langle X\rangle^{(2)})$ has no nonzero skew-primitive elements. At the same time, due to the Heyneman–Radford theorem [4], [8, Corollary 5.3] every bi-ideal of a character Hopf algebra has a nonzero skew-primitive element. Therefore, $\operatorname{Ker}\xi = \Lambda$, while the subalgebra A generated by the values of x_i, $1 \leq i \leq n$, in $U_q^+(\mathfrak{so}_{2n+1})$ has the shuffle representation given in Section 2.

If the multiplicative order of q is finite, then by the definition of $H = u_q^+(\mathfrak{so}_{2n+1})$, we have $\operatorname{Ker}\xi = \Lambda$. Hence, the subalgebra A generated by the values of x_i, $1 \leq i \leq n$, in $u_q^+(\mathfrak{so}_{2n+1})$ also has the shuffle representation.

Recall that $(u(m,k))$ denotes the tensor $x_m \otimes x_{m-1} \otimes \cdots \otimes x_k$ considered as an element of $Sh_\tau(V)$.

Proposition 4.2. *Let $k \leq m \leq 2n$. In the shuffle representation, we have*

$$u[k,m] = \alpha_k^m \cdot (u(m,k)), \qquad \alpha_k^m \overset{df}{=} \varepsilon_k^m (q^2-1)^{m-k} \cdot \prod_{k \leq i < j \leq m} p_{ij}, \tag{4.2}$$

where

$$\varepsilon_k^m = \begin{cases} 1 & \text{if } m \leq n \text{ or } k > n; \\ q^{-1} & \text{if } k \leq n < m,\ m \neq \psi(k); \\ q^{-3} & \text{if } m = \psi(k). \end{cases} \tag{4.3}$$

Proof. We use induction on $m-k$. If $m=k$, the equality reduces to $x_k=(x_k)$.

(a) Consider first the case $m<\psi(k)$. By the inductive supposition, we have $u[k,m-1] = \alpha_k^{m-1}\cdot(w)$, $w=u(m-1,k)$. Using (2.21), we may write

$$u[k,m]=\alpha_k^{m-1}\{(w)(x_m)-p(w,x_m)\cdot(x_m)(w)\}$$
$$=\alpha_k^{m-1}\sum_{uv=w}\{p(x_m,v)^{-1}-p(w,x_m)p(u,x_m)^{-1}\}(ux_mv). \tag{4.4}$$

Because $w=uv$, we have $p(w,x_m)p(u,x_m)^{-1}=p(v,x_m)$.

If $m\le n$, then relations (3.2) imply $p(v,x_m)p(x_m,v)=1$ except when $v=w$. Hence, the sum (4.4) has just one term. The coefficient of $(x_mw)=(u(m,k))$ equals

$$\alpha_k^{m-1}p(w,x_m)(p(w,x_m)^{-1}p(x_m,w)^{-1}-1)=\alpha_k^{m-1}p(w,x_m)(q^2-1),$$

as required.

If $m=n+1$, then $p(v,x_m)p(x_m,v)=1$ still holds, with two exceptions: for $v=w$ and $v=u(n-1,k)$. In both cases, (ux_mv) equals $(u(m,k))$. Hence, the coefficient of $(u(m,k))$ in the sum (4.4) equals

$$p(x_n,u(k,n-1))^{-1}-p(u(k,n-1),x_n)+p(x_n,u(k,n))^{-1}-p(u(k,n),x_n)$$
$$=p(w,x_{n+1})\{p_{n\,n-1}^{-1}p_{n-1\,n}^{-1}p_{nn}^{-1}-p_{nn}^{-1}+p_{n\,n-1}^{-1}p_{nn}^{-1}p_{nn}^{-1}p_{n-1\,n}^{-1}-1\}.$$

Due to (3.1), (3.2) we obtain $\alpha_k^m=\alpha_k^{m-1}p(w,x_{n+1})(q^2-1)q^{-1}$, as required.

Suppose that $m>n+1$. In this case, by definition, $x_m=x_t$, where $t=\psi(m)<\psi(n+1)=n$. Let $v=u(s,k)$. If $s<t-1$, then v depends only on x_i, $i<t-1$, and relations (3.1), (3.2) imply $p(v,x_m)p(x_m,v)=1$. If $s>t$, $s\neq m-1$, then $p(v,x_m)p(x_m,v)=p_{t-1\,t}p_{tt}p_{t+1\,t}\cdot p_{t\,t-1}p_{tt}p_{t+1\,t}=1$. Hence, in (4.4), three terms remain: with $s=t-1$, $s=t$, and $s=m-1$. If $v=u(t-1,k)$ or $v=u(t,k)$, then (ux_mv) equals $(u(k,t)x_t^2u(t+1,m-1))$, while the coefficient of this tensor in (4.4) is

$$p(x_t,u(k,t-1))^{-1}-p(u(k,t-1),x_t)+p(x_t,u(k,t))^{-1}-p(u(k,t),x_t)$$
$$=p(u(k,t),x_t)\{p_{t\,t-1}^{-1}p_{t-1\,t}^{-1}p_{tt}^{-1}-p_{tt}^{-1}+p_{tt}^{-1}p_{t\,t-1}^{-1}p_{t-1\,t}^{-1}p_{tt}^{-1}-1\}=0.$$

Thus, in (4.4) only one term remains, with $v=u(m-1,k)$. This term has the required coefficient:

$$\alpha_k^m=\alpha_k^{m-1}(p(x_m,w)^{-1}-p(w,x_m))=\alpha_k^{m-1}p(w,x_m)(q^2-1).$$

(b) In perfect analogy, we consider the case $m>\psi(k)$. By the inductive supposition, we have $u[k+1,m]=\alpha_{k+1}^m\cdot(w)$, $w=u(m,k+1)$. Using (2.21), we may write

$$u[k,m]=\alpha_{k+1}^m\{(x_k)(w)-p(x_k,w)\cdot(w)(x_k)\}$$
$$=\alpha_{k+1}^m\sum_{uv=w}\{p(u,x_k)^{-1}-p(x_k,u)\}(ux_kv). \tag{4.5}$$

If $k > n$, then $p(u, x_k)p(x_k, u) = 1$ unless $u = w$. Hence, (4.5) has only one term, and the coefficient equals

$$\alpha_{k+1}^m p(x_k, w)(p(w, x_k)^{-1} p(x_k, w)^{-1} - 1) = \alpha_{k+1}^m p(x_k, w)(q^2 - 1),$$

as required.

If $k = n$, then $p(u, x_k)p(x_k, u) = 1$ with two exceptions, $u = w$ and $u = u(m, n+2)$. In both cases, $(ux_k v)$ equals $(u(m, k))$, while the coefficient takes the form

$$\begin{aligned} &p(w, x_n)^{-1} - p(x_n, w) + p(u(m, n+2), x_n)^{-1} - p(x_n, u(m, n+2)) \\ &\qquad = p(x_n, w)\{p_{n\,n-1}^{-1} p_{n-1\,n}^{-1} p_{nn}^{-2} - 1 + p_{n\,n-1}^{-1} p_{n-1\,n}^{-1} p_{nn}^{-1} - p_{nn}^{-1}\}. \end{aligned}$$

Due to relations (3.1), (3.2) we obtain $\alpha_n^m = \alpha_{n+1}^m p(x_n, w)(q^2 - 1)q^{-1}$, as required.

Suppose that $k < n$. In this case, $x_k = x_t$ with $m > t \overset{df}{=} \psi(k) > \psi(n) = n + 1$. Let $u = u(m, s)$. If $s > t$, then u depends only on x_i, $i < k - 1$, and relations (3.1), (3.2) imply $p(x_k, u)p(u, x_k) = 1$. If $s < t - 1$, $s \neq k + 1$, then $p(x_k, u)p(u, x_k) = p_{k-1\,k} p_{kk} p_{k+1\,k} \cdot p_{k\,k-1} p_{kk} p_{k+1\,k} = 1$. Hence, three terms remain in (4.5), with $s = t$, $s = t + 1$, and $s = k + 1$. If $u = u(m, t)$ or $u = u(m, t + 1)$, then $ux_k v = u(m, t + 1)x_k^2 u(t - 1, k)$, while the coefficient of the corresponding tensor is

$$\begin{aligned} &p(u(m, t + 1), x_k)^{-1} - p(x_k, u(m, t + 1)) + p(u(m, t), x_k)^{-1} - p(x_k, u(m, t)) \\ &\qquad = p(x_k, u(m, t + 1))\{p_{k-1\,k}^{-1} p_{k\,k-1}^{-1} - 1 + p_{kk}^{-1} p_{k-1\,k}^{-1} p_{k\,k-1}^{-1} - p_{kk}\} = 0. \end{aligned}$$

Thus, only one term remains in (4.4), and

$$\alpha_k^m = \alpha_{k+1}^m (p(w, x_k)^{-1} - p(x_k, w)) = \alpha_{k+1}^m p(x_k, w)(q^2 - 1).$$

(c) Let us consider the remaining case, $m = \psi(k)$. In this case, $x_m = x_k$. If $k = n$, $m = n + 1$, then $u[n, n + 1] = -p_{nn}^{-1}[x_n, x_n] = (1 - q^{-1})x_n^2$, while in the shuffle representation we have $(x_n)(x_n) = (1 + q^{-1})(x_n x_n)$. Hence, $u[n, n+1] = (1 - q^{-2})(x_{n+1} x_n)$, which is as required: $(1 - q^{-2}) = q^{-3} \cdot (q^2 - 1) \cdot p_{nn}$.

If $k < n$, we put $u = u[n + 1, m]$, $v = x_k$, $w = u[k + 1, n]$. By definition (3.16), we have $u[k, m] = \beta[u, [v, w]]$, where $\beta = -p(u(n + 1, m), u(k, n))^{-1}$; that is, $\beta = -p_{u,vw}^{-1}$. Because $u[n + 1, m] = [u[n + 1, m - 2], [x_{k+1}, x_k]]$, the conditional identity (2.8) implies $[u, v] = [u[n + 1, m - 2], [[x_{k+1}, x_k], x_k]] = 0$. Thus, $[[u, v], w] = 0$, and formula (2.7) yields

$$\beta^{-1} u[k, m] = p_{uv} x_k \cdot [u, w] - p_{vu}[u, w] \cdot x_k. \tag{4.6}$$

Formula (3.17) implies $\beta_1[u, w] = u[k + 1, m]$ with $\beta_1 = -p_{uw}^{-1}$. Hence case (b) allows us to find the shuffle representation $[u, w] = \alpha \cdot (z)$ with $z = u(m, k + 1)$ and $\alpha = -p_{uw}\alpha_{k+1}^m$. By (2.21), the shuffle representation of the right-hand side of (4.6) is

$$\alpha \sum_{sy = u(m, k+1)} (p_{uv} p(s, x_k)^{-1} - p_{vw} p(x_k, y)^{-1}) \cdot (s x_k y)$$

We have $\beta\alpha = -\beta p_{uw}\alpha_{k+1}^m = p_{uv}^{-1}\alpha_{k+1}^m$, and

$$p_{uv}p_{vu} = p_{k+1\,k}p_{kk}p_{k\,k+1}p_{kk} = q^2$$

because $k < n$. Therefore, we obtain

$$u[k,m] = \alpha_{k+1}^m \sum_{sy=u(m,k+1)} (p(s,x_k)^{-1} - q^{-2}p(x_k,s)) \cdot (sx_ky). \tag{4.7}$$

If $s \notin \{\emptyset, x_m, z = u(m,k+1)\}$, then $p(s,x_k)p(x_k,s) = p_{k+1\,k}p_{kk}p_{k\,k+1}p_{kk} = q^2$; that is, only three terms remain in (4.7). If $s = \emptyset$ or $s = x_m$, then $(sx_ky) = (x_kz)$ because $x_m = x_k$. Hence, the coefficient of (x_kz) in (4.7) equals $1 - q^{-2} + p_{kk}^{-1} - q^{-2}p_{kk} = 0$. Thus, in (4.7) only one term remains, with the coefficient

$$\alpha_{k+1}^m(p(z,x_k)^{-1} - q^{-2}p(x_k,z)) = \alpha_{k+1}^m p(x_k,z)q^{-2}(q^2-1) = \alpha_k^m$$

because $p(z,x_k)\cdot p(x_k,z) = p_{kk}p_{k+1\,k}p_{k+1\,k}\cdot p_{kk}p_{k\,k+1}p_{k\,k+1} = 1$. □

Theorem 4.3. *In $U_q^+(\mathfrak{so}_{2n+1})$ the coproduct on the elements $u[k,m]$, $k \le m \le 2n$, has the following explicit form:*

$$\begin{aligned}\Delta(u[k,m]) &= u[k,m]\otimes 1 + g_kg_{k+1}\cdots g_m \otimes u[k,m]\\ &\quad + \sum_{i=k}^{m-1} \tau_i(1-q^{-2})g_kg_{k+1}\ldots g_i\, u[i+1,m]\otimes u[k,i],\end{aligned} \tag{4.8}$$

where $\tau_i = 1$ for $i \neq n$ and $\tau_n = q$.

Proof. Formulae (4.2), (2.20), and (2.19) show that the coproduct has the form (4.8), where $\tau_i(1-q^{-2}) = \alpha_k^m(\alpha_k^i\alpha_{i+1}^m)^{-1}\chi^{u(i+1,m)}(g_kg_{k+1}\ldots g_i)$. We now have

$$\Big(\prod_{k\le a<b\le i} p_{ab} \prod_{i+1\le a<b\le m} p_{ab}\Big)^{-1} \prod_{k\le a<b\le m} p_{ab} = p(u(k,i),u(i+1),m).$$

Therefore the definition of μ_k^m given in (3.9) and the definition of α_k^m given in (4.2) imply $\tau_i(1-q^{-2}) = \varepsilon_k^m(\varepsilon_k^i\varepsilon_{i+1}^m)^{-1}(q^2-1)\mu_k^{m,i}$; that is, $\tau_i = \varepsilon_k^m(\varepsilon_k^i\varepsilon_{i+1}^m)^{-1}q^2\mu_k^{m,i}$. By (3.12), we have $\mu_k^{m,i} = \sigma_k^m(\sigma_k^i\sigma_{i+1}^m)^{-1}$. Using (3.10) and (4.3), we see that

$$\varepsilon_k^m\sigma_k^m = \begin{cases} q^2 & \text{if } m < n \text{ or } k > n+1;\\ q & \text{otherwise.}\end{cases} \tag{4.9}$$

Now, it is easy to check that the τ's have the following elegant form:

$$\tau_i = \varepsilon_k^m\sigma_k^m(\varepsilon_k^i\sigma_k^i)^{-1}(\varepsilon_{i+1}^m\sigma_{i+1}^m)^{-1}q^2 = \begin{cases} q & \text{if } i = n;\\ 1 & \text{otherwise.}\end{cases} \tag{4.10}$$

Interestingly, the coproduct formula differs from that in $U_{q^2}^+(\mathfrak{sl}_{2n+1})$ in just one term: see formula (3.3) in [11]. □

Now we are going to find PBW-generators for $u_q^+(\mathfrak{so}_{2n+1})$. To do this, we need more relations in $U_q^+(\mathfrak{so}_{2n+1})$.

Lemma 4.4. *If $k \le m < \psi(k)$, then in the algebra $U_q^+(\mathfrak{so}_{2n+1})$ we have*

$$\big[u[k,m],[u[k,m],u[k+1,m]]\big]=0. \tag{4.11}$$

Proof. Suppose first $m < \psi(k)-1$. In this case, both words $u(k,m)$ and $u(k+1,m)$ are standard. The standard arrangement of brackets for these words is defined by (4.1). However, in Proposition 4.1, we have seen that $[u(k,m)] = u[k,m]$, and hence $[u(k+1,m)] = u[k+1,m]$ in the algebra $U_q^+(\mathfrak{so}_{2n+1})$.

The word $w = u(k,m)u(k,m)u(k+1,m)$ is standard. The Algorithm on p. 1682 shows that the standard arrangement of brackets is precisely

$$\big[[u(k,m)],[[u(k,m)],[u(k+1,m)]]\big].$$

Hence, the value of the super-word $[w]$ in $U_q^+(\mathfrak{so}_{2n+1})$ equals the left-hand side of (4.11).

By Proposition 4.1, all hard super-letters in $U_q^+(\mathfrak{so}_{2n+1})$ are $[u(k,m)]$, $k \le m < \psi(k)$. Hence, $[w]$ is not hard. The multiple use of Definition 2.3 shows that the value of $[w]$ is a linear combination of the values of super-words in hard super-letters smaller than $[w]$. Because $U_q^+(\mathfrak{so}_{2n+1})$ is homogeneous, each of the super-words in that decomposition has two hard super-letters smaller than $[w]$ and of degree 1 in x_k (if a hard super-letter $[u(r,s)]$ is of degree 2 in x_k, then $r < k$ and $u(r,s) > w$). At the same time, all such hard super-letters are $[u(k,m+1)], [u(k,m+2)], \dots, [u(k,2n-k)]$. Each has degree 2 in x_{m+1} if $m \ge n$, and each has degree at least 1 if $m < n$. Hence, the super-word has degree at least 4 in x_{m+1} if $m \ge n$, and at least 1 if $m < n$. However, w is of degree 3 in x_{m+1} if $m \ge n$, and it is independent of x_{m+1} if $m < n$. Therefore, the decomposition is empty, and $[w] = 0$.

Let, then, $m = \psi(k)-1$. In this case, $u(k+1,m)$ is not standard, and we cannot apply the above arguments. Nevertheless, we shall prove similarly that $[u[k,2n-k],x_t] = 0$, $k < t \le n$, which will imply both $[u[k,2n-k],u[k+1,2n-k]] = 0$ and (4.11).

If $k+1 < t < n$, then Lemmas 3.7 and 3.8 imply

$$[u[k,n],x_t] = [u[n+1,2n-k],x_t] = 0.$$

Due to Corollary 3.13 we have $[u[k,2n-k],x_t] = 0$.

If $t = k+1$, we consider the word $v = u(k,2n-k)x_{k+1}$. It is standard, and the standard arrangement of brackets is $[v] = [[u(k,2n-k)]x_{k+1}]$. Therefore, the value of the super-letter $[v]$ equals $[u[k,2n-k],x_{k+1}]$. At the same time, $[v]$ does not belong to the set of PBW-generators; that is, it is not hard. The multiple use of Definition 2.3 shows that the value of $[v]$ is a linear combination of the values of super-words in hard super-letters smaller than $[v]$. Each of the super-words in that decomposition has a hard super-letter smaller than $[v]$ and of degree 1 in x_k. However, there are no such super-letters. Thus, the decomposition is empty, and $[v] = 0$.

Let $t = n$. If $k = n-1$, then $[u[k,2n-k],x_n] = [[[x_{n-1},x_n],x_n],x_n] = 0$ because of (3.4). If $k = n-2$, we consider the word $u = u(k,2n-k)x_n = x_{n-2}x_{n-1}x_nx_nx_{n-1}x_n$. It is standard, while the super-letter $[u]$ is not hard. Again, there is no hard super-letter smaller than $[u]$ and of degree 1 in x_{n-2}. Hence, $[u]=0$ in $U_q^+(\mathfrak{so}_{2n+1})$. The standard arrangement of brackets is $[[x_{n-2}x_{n-1}x_nx_n][x_{n-1}x_n]]$. Hence, we obtain

$$\big[[x_{n-2},[[x_{n-1},x_n],x_n]],[x_{n-1},x_n]\big]=0.$$

At the same time, $[x_{n-2},x_n]=0$ and $[[[x_{n-1},x_n],x_n],x_n]=0$ imply

$$\big[[x_{n-2},[[x_{n-1},x_n],x_n]],x_n\big]=0.$$

The conditional identity (2.8) yields

$$\big[[x_{n-2},[[x_{n-1},x_n],x_n]],[x_{n-1},x_n]\big]=\big[[[x_{n-2},[[x_{n-1},x_n],x_n]],x_{n-1}],x_n\big],$$

which is as required because $[u[n-2,n+2],x_n]=[[[x_{n-2},[[x_{n-1},x_n],x_n]],x_{n-1}],x_n]$.

Finally, suppose that $k<n-2$. Denote $u_1=u[k,n-3]$, $v_1=u[n+3,2n-k]$, $w_1=u[n-2,n+2]$. We have already proved that $[w_1,x_n]=0$. The second group of relations (3.3) implies $[u_1,x_n]=0$, $[v_1,x_n]=0$. At the same time, due to Proposition 3.14, we have $u[k,2n-k]=[u[k,n+2],v_1]$ and $u[k,n+2]=[u_1,w_1]$; that is, $u[k,2n-k]=[[u_1,w_1],v_1]$, which certainly implies the required relation $[u[k,2n-k],x_n]=0$. □

Proposition 4.5. *If the multiplicative order t of q is finite, $t>4$, then the values of $u[k,m]$, $k\le m<\psi(k)$, form a set of PBW-generators for $u_q^+(\mathfrak{so}_{2n+1})$ over* $\mathbf{k}[G]$. *The height h of $u[k,m]$ equals t if $m=n$ or t is odd. If $m\neq n$ and t is even, then $h=t/2$. In all cases, $u[k,m]^h=0$ in $u_q^+(\mathfrak{so}_{2n+1})$.*

Proof. First, we note that Definition 2.3 implies that a nonhard super-letter in $U_q^+(\mathfrak{so}_{2n+1})$ is still nonhard in $u_q^+(\mathfrak{so}_{2n+1})$. Hence, all hard super-letters in $u_q^+(\mathfrak{so}_{2n+1})$ are in the list $u[k,m]$, $k\le m<\psi(k)$. Next, if $u[k,m]$ is not hard in $u_q^+(\mathfrak{so}_{2n+1})$, then by the multiple use of Definition 2.3, the value of $u[k,m]$ is a linear combination of super-words in hard super-letters smaller than the given $u[k,m]$. Because $u_q^+(\mathfrak{so}_{2n+1})$ is homogeneous, each of the super-words in that decomposition has a hard super-letter smaller than $u[k,m]$ and of degree 1 in x_k. At the same time, all such hard super-letters are in the list $[u(k,m+1)],[u(k,m+2)],\dots,[u(k,2n-k)]$. Each of these super-letters has degree 2 in x_{m+1} if $m\ge n$, and at least 1 if $m<n$. Hence, the super-word has a degree of at least 2 if $m\ge n$, and at least 1 if $m<n$. However $u[k,m]$ is of degree 1 in x_{m+1} if $m\ge n$, and is independent of x_{m+1} if $m<n$. Therefore the decomposition is empty, and $u[k,m]=0$. We obtain a contradiction with Proposition 4.2 because $(u(m,k))\neq 0$ in the shuffle algebra.

For short we denote $u=u[k,m]$. Equation (3.10) implies $p_{uu}=q$ if $m=n$ and $p_{uu}=q^2$ otherwise (recall that now $m<\psi(k)$). By Definition 2.4 the minimal possible value for the height is precisely the h given in the proposition. It remains to show that $u^h=0$ in $u_q^+(\mathfrak{so}_{2n+1})$. By Lemma 2.11, it suffices to prove that $\partial_i(u^h)=0$, $1\le i\le n$. Lemma 2.10 yields

$$\partial_i(u^h)=p(u,x_i)^{h-1}\underbrace{[u,[u,\dots[u}_{h-1},\partial_i(u)]\dots]].$$

The coproduct formula (4.8) with (2.23) implies

$$\partial_i(u) = \begin{cases} (1-q^{-2})\tau_k u[k+1,m] & \text{if } i \in \{k, \psi(k)\}, k < m; \\ 0 & \text{if } i \notin \{k, \psi(k)\}; \\ 1 & \text{if } i \in \{k, \psi(k)\}, k = m. \end{cases} \tag{4.12}$$

At the same time, Lemma 4.4 provides the relation $[u, [u, u[k+1, m]]] = 0$ in $U_q^+(\mathfrak{so}_{2n+1})$, and hence in $u_q^+(\mathfrak{so}_{2n+1})$ as well. Because always $h > 2$, we obtain the required equalities $\partial_i(u^h) = 0$, $1 \le i \le n$. □

Remark. To prove (4.8), we have used the shuffle representation. Therefore, if q has a finite multiplicative order, then (4.8) is proved only for $u_q^+(\mathfrak{so}_{2n+1})$. However, we have seen that the kernel of the natural homomorphism $U_q^+(\mathfrak{so}_{2n+1}) \to u_q^+(\mathfrak{so}_{2n+1})$ is generated by the elements $u[k,m]^h$, $k \le m < \psi(k)$. The degree of $u[k,m]^h$ in a given x_i is either zero or greater than 2. At the same time, all tensors in (4.8) have degree at most 2 in each variable. Therefore, (4.8), and hence (4.12), are also valid in $U_q^+(\mathfrak{so}_{2n+1})$ provided that q has a finite multiplicative order $t > 4$.

5. PBW-generators for right coideal subalgebras

In what follows A_{k+1}, $k < n$, denotes the subalgebra of $U_q^+(\mathfrak{so}_{2n+1})$ or $u_q^+(\mathfrak{so}_{2n+1})$ generated by x_i, $k < i \le n$, and correspondingly A is the subalgebra generated by all x_i, $1 \le i \le n$. Of course, $\mathbf{k}[g_{k+1}, \dots, g_n]A_{k+1}$ may be identified with $U_q^+(\mathfrak{so}_{2(n-k)+1})$ or $u_q^+(\mathfrak{so}_{2(n-k)+1})$.

Suppose that a homogeneous element $f \in \mathbf{k}\langle X\rangle$ is linear in the maximal letter x_k, $1 \le k \le n$, that it depends on: $\deg_k(f) = 1$, $\deg_i(f) = 0$, $i < k$. Then, in the decomposition of $a = \xi(f)$ in the PBW-basis defined in Proposition 4.1 or Proposition 4.5, each summand has only one PBW-generator that depends on x_k because $U_q^+(\mathfrak{so}_{2n+1})$ and $u_q^+(\mathfrak{so}_{2n+1})$ are homogeneous in each x_i. Moreover, this PBW-generator, considered as a super-letter, starts with x_k. Hence, it is the maximal super-letter of the summand. In particular, this super-letter is located at the end of the basis super-word; that is, the PBW-decomposition takes the form

$$a = \sum_{i=k}^{2n-k} F_i u[k,i], \qquad F_i \in A_{k+1}. \tag{5.1}$$

Definition 5.1. The set $\mathrm{Sp}(a)$ of all i such that $F_i \neq 0$ in (5.1) is called the *spectrum* of a.

Let S be a set of integers from the interval $[1, 2n]$. We define a polynomial $\Phi^S(k,m)$, $1 \le k \le m \le 2n$, by induction on the number r of elements in $S \cap [k, m-1] = \{s_1, \dots, s_r\}$, $k \le s_1 < \dots < s_r < m$, as follows:

$$\Phi^S(k,m) = u[k,m] - (1-q^{-2}) \sum_{i=1}^{r} \alpha_{km}^{s_i} \, \Phi^S(1+s_i, m) u[k, s_i] \tag{5.2}$$

where $\alpha_{km}^s = \tau_s p(u(1+s,m), u(k,s))^{-1}$, while the τ's were defined in (4.10).

We represent the element $\Phi^S(k,m)$ schematically as a sequence of black and white points labelled by the numbers $k-1,\ k,\ k+1,\ldots,\ m-1,\ m$, where the first point is always white, and the last one is always black. An intermediate point labelled by i is black if and only if $i\in S$:

$$\begin{array}{cccccccccc} {\scriptstyle k-1} & {\scriptstyle k} & {\scriptstyle k+1} & {\scriptstyle k+2} & {\scriptstyle k+3} & & {\scriptstyle m-2} & {\scriptstyle m-1} & {\scriptstyle m} \\ \circ & \circ & \circ & \bullet & \circ & \cdots & \bullet & \circ & \bullet \end{array} \tag{5.3}$$

Sometimes, if $k\le n<m$, it is more convenient to represent the element $\Phi^S(k,m)$ in two lines, putting the points labelled by indices $i,\ \psi(i)$ that define the same variable $x_i=x_{\psi(i)}$ in one column:

$$\begin{array}{ccccccccc} & & & \overset{m}{\bullet} & \cdots & \bullet & \overset{\psi(i)}{\circ} & \cdots & \overset{n+1}{\bullet} \\ \overset{k-1}{\circ} & \circ & \cdots & \overset{\psi(m)}{\circ} & \cdots & \bullet & \overset{i}{\bullet} & \cdots & \overset{n}{\circ} \end{array} \tag{5.4}$$

To illustrate the notion of a regular set, we need a *shifted representation* that arises from (5.4) by shifting the upper line to the left one step and copying the coloured point labelled by n, if any, to the vacant position (so that this point appears twice in the shifted scheme):

$$\begin{array}{cccccccccc} & & & \overset{m}{\bullet} & \cdots & \circ & \overset{n+i}{\circ} & \cdots & \overset{n+1}{\bullet} & \overset{n}{\circ}\Leftarrow \\ \overset{k-1}{\circ} & \circ & \cdots & \overset{\psi(m)-1}{\bullet} & \cdots & \bullet & \overset{n-i}{\bullet} & \cdots & \overset{n-1}{\circ} & \overset{n}{\circ} \end{array} \tag{5.5}$$

If $k\le m<\psi(k)$, then definition (5.2) shows that the spectrum of $\Phi^S(k,m)$ is contained in $S\cup\{m\}$, while its leading term is $u[k,m]$. However, if $m\ge\psi(k)$, then (5.2) does not provide sufficient information even for the immediate conclusion that $\Phi^S(k,m)\ne 0$. In particular some of the factors $\Phi^S(1+s_i,m)$ in (5.2) may be zero even if $k\le m<\psi(k)$. Hence, *a priori* the spectrum of $\Phi^S(k,m)$, $k\le m<\psi(k)$, may be a proper subset of $S\cup\{m\}$.

Let π_{kl}, $1\le k\le l<\psi(k)$, denote a natural projection of $U_q^+(\mathfrak{so}_{2n+1})$ or $u_q^+(\mathfrak{so}_{2n+1})$ onto $\mathbf{k}u[k,l]$ with respect to the PBW-basis defined in Proposition 4.1 or 4.5 respectively.

Lemma 5.2. *If $a\in A_{k+1}$, then $\pi_{kl}(au[k,i])=0$, $k\le i<\psi(k)$, unless $a\in\mathbf{k}$, $i=l$.*

Proof. The PBW-decomposition $\tilde a$ of a in the basis defined in Proposition 4.1 or 4.5 involves only PBW-generators that belong to A_{k+1}. They are all smaller than $u[k,i]$. Hence, the PBW-decomposition of $au[k,i]$ is $\tilde a u[k,i]$. We have $\pi_{kl}(\tilde a u[k,i])\ne 0$ only if $\tilde a\in\mathbf{k}$, $i=l$. □

Lemma 5.3. *If $a\in A_{k+1}$, $k\le l<\psi(k)$, then*

$$\Delta(au[k,i])\cdot(\mathrm{id}\otimes\pi_{kl})=\begin{cases}0 & \text{if } i<l;\\ ag_{kl}\otimes u[k,l] & \text{if } i=l;\\ \tau_l(1-q^{-2})a\,g_{kl}u[l+1,i]\otimes u[k,l] & \text{if } i>l,\end{cases} \tag{5.6}$$

where, by definition, $g_{kl}=g(u[k,l])=g_kg_{k+1}\ldots g_l$.

Proof. By (4.8), we have $\Delta(au[k,i])=\sum_{(a),\,j}a^{(1)}\alpha_jg_{kj}u[j+1,i]\otimes a^{(2)}u[k,j]$ for suitable $\alpha_j\in\mathbf{k}$. By the above lemma, we obtain $\pi_{kl}(a^{(2)}u[k,j])=0$ unless $a^{(2)}\in\mathbf{k}$, $i=l$. It remains to apply the explicit formula (4.8). □

Lemma 5.4. *If $k \le l < m < \psi(k)$, then*

$$\Delta(\Phi^S(k,m)) \cdot (\mathrm{id} \otimes \pi_{kl}) = \begin{cases} 0 & \text{if } l \in S; \\ \tau_l(1-q^{-2}) g_{kl} \Phi^S(1+l,m) \otimes u[k,l] & \text{if } l \notin S. \end{cases}$$

Proof. Let us apply $\Delta(\mathrm{id} \otimes \pi_{kl})$ to (5.2). Because $a_i \stackrel{df}{=} \Phi^S(1+s_i,m) \in A_{k+1}$, we may use Lemma 5.3. We now have $a_i g_{kl} = \chi^{a_i}(g_{kl}) g_{kl} a_i$, $\chi^{a_i}(g_{kl}) = p(u(1+s_i,m), u(k,l))$. Thus, if $s_i > l$, then $\alpha_{km}^{s_i} \chi^{a_i}(g_{kl}) = \alpha_{1+l\,m}^{s_i}$, while if $s_i = l$, then $\alpha_{km}^{l} \chi^{a_l}(g_{kl}) = \tau_l$. Now, (5.6) implies the required relation. □

Lemma 5.5. *Let $k \le l < m < \psi(k)$ and $a \in A_{k+1}$ be a nonzero homogeneous element with $D(a) = D(u(1+l,m))$. Denote by ν_a any homogeneous projection $\nu_a : U_q^+(\mathfrak{so}_{2n+1}) \to a\mathbf{k}$. If $D(b) = D(u(1+i,m))$, then*

$$\Delta(bu[k,i]) \cdot (\mathrm{id} \otimes \nu_a) = \begin{cases} 0 & \text{if } l < i < m; \\ g_a u[k,l] \otimes a & \text{if } i = l,\ b = a; \\ g_a b' u[k,i] \otimes a & \text{if } i < l. \end{cases}$$

Proof. All right-hand components of the tensors in (4.8) depend on x_k except the first summand. Because ν_a kills all elements with a positive degree in x_k, we have

$$\Delta(bu[k,i]) \cdot (\mathrm{id} \otimes \nu_a) = \sum_{(b)} b^{(1)} u[k,i] \otimes \nu_a(b^{(2)}). \tag{5.7}$$

If $l < i < m$, then $D(b^{(2)}) \le D(b) < D(a)$. Hence, $\nu_a(b^{(2)}) = 0$.
If $b = a$, $i = s$, then $D(b^{(2)}) = D(a)$ only if $b^{(1)} = g_a$, $b^{(2)} = a$.
If $i < l$, then (5.7) provides the third option given in the lemma. □

Proposition 5.6. *If a right coideal subalgebra $\mathbf{U} \supseteq \mathbf{k}[G]$ of $U_q^+(\mathfrak{so}_{2n+1})$ or $u_q^+(\mathfrak{so}_{2n+1})$ contains a homogeneous element $c \in A$ with the leading term $u[k,m]$, $k \le m < \psi(k)$, then $\Phi^S(k,m) \in \mathbf{U}$ for a suitable subset S of the spectrum of c.*

Proof. Every summand of the decomposition of c in the PBW-basis defined in Proposition 4.1 or 4.5 has only one PBW-generator that depends on x_k because $U_q^+(\mathfrak{so}_{2n+1})$ and $u_q^+(\mathfrak{so}_{2n+1})$ are homogeneous in each x_i. Moreover, this PBW-generator, considered as a super-letter, starts with x_k, and hence it is the maximal super-letter of the summand. The maximal super-letter is located at the end of the basis super-word; that is, the PBW-decomposition takes the form

$$c = u[k,m] + \sum_{i=k}^{m-1} F_i u[k,i], \qquad F_i \in A_{k+1},\ k \le i < m. \tag{5.8}$$

By definition, i belongs to the spectrum $\mathrm{Sp}(a)$ of a if and only if $F_i \neq 0$. We may rewrite this representation in the following way:

$$\Phi^{S_t}(k,m) + \sum_{i \in \mathrm{Sp}(a),\, i<t} F_i u[k,i] \in \mathbf{U}, \tag{5.9}$$

where $t = m$, and, by definition, $S_m = \emptyset$. We shall prove that relation (5.9) with a given t, $k < t \leq m$, $S_t \subseteq \mathrm{Sp}(a)$, and $t \leq \inf S_t$ implies a relation of the same type with $t \leftarrow l$, $S_l = S_t \cup \{l\}$, where l, as above, is the maximal i in (5.9) such that $F_i \neq 0$. Because certainly $l < t$, by downward induction this will imply (5.9) with $t = k$, $S = S_k \subseteq \mathrm{Sp}(a)$:

$$\Phi^S(k, m) \in \mathbf{U}. \tag{5.10}$$

Let us apply $\Delta \cdot (\mathrm{id} \otimes \pi_{kl})$ to (5.9), where π_{kl} is the projection onto $\mathbf{k}u[k, l]$, and l is the maximal i in (5.9) with $F_i \neq 0$. By Lemma 5.3, we have $\Delta(F_i u[k, i]) \cdot (\mathrm{id} \otimes \pi_{kl}) = 0$ if $i < l$, while $\Delta(F_l u[k, l]) \cdot (\mathrm{id} \otimes \pi_{kl}) = F_l g_{kl} \otimes [k, l]$. Lemma 5.4 implies $\Delta(\Phi^{S_t}(k, m)) \cdot (\mathrm{id} \otimes \pi_{kl}) = \tau_l(1 - q^{-2}) g_{kl} \Phi^{S_t}(1 + l, m) \otimes u[k, l]$. Because U is a right coideal subalgebra that contains all grouplike elements, we get

$$F_l + \chi^{F_l}(g_{kl})^{-1} \tau_l (1 - q^{-2}) \Phi^{S_t}(1 + l, m) = v \in \mathbf{U}. \tag{5.11}$$

We further consider any homogeneous projection ν_a with $a = F_l$. Let us apply $\Delta \cdot (\mathrm{id} \otimes \nu_a)$ to (5.9). As $l < \inf S_t$, Lemma 5.5 and definition (5.2) imply $\Delta(\Phi^{S_t}(k, m)) \cdot (\mathrm{id} \otimes \nu_a) = 0$. Lemma 5.5 also shows that $\Delta(F_l u[k, l]) \cdot (\mathrm{id} \otimes \nu_a) = g_a u[k, l] \otimes a$, while $\Delta(F_i u[k, i]) \cdot (\mathrm{id} \otimes \nu_a) = g_a A_i' u[k, i] \otimes a$, $i < l$. Hence, we arrive at the relation

$$u[k, l] + \sum_{i \in \mathrm{Sp}(a),\, i < l} F_i' u[k, i] = w \in \mathbf{U}. \tag{5.12}$$

Relations (5.11), (5.12) imply

$$F_l u[k, l] = vw - \sum_{i \in \mathrm{Sp}(a),\, i < l} v F_i' u[k, i] - \chi^{F_l}(g_{kl})^{-1} \tau_l (1 - q^{-2}) \Phi^{S_t}(1 + l, m) \cdot u[k, l].$$

This equality allows one to replace $F_l u[k, l]$ in (5.9). According to definition (5.2) we have $\Phi^{S_t}(k, m) - \chi^{F_l}(g_{kl})^{-1} \tau_l (1 - q^{-2}) \Phi^{S_t}(1 + l, m) \cdot u[k, l] = \Phi^{S_t \cup \{l\}}(k, m)$; therefore we obtain the required relation

$$\Phi^{S_l}(k, m) + \sum_{i \in \mathrm{Sp}(a),\, i < l} (F_i - v F_i') u[k, i] \in \mathbf{U}. \qquad \square$$

Corollary 5.7. *If the main parameter q is not a root of 1, then every right coideal subalgebra of $U_q^+(\mathfrak{so}_{2n+1})$ that contains the coradical has a set of PBW-generators of the form $\Phi^S(k, m)$. In particular, there exist only a finite number of right coideal subalgebras of $U_q^+(\mathfrak{so}_{2n+1})$ that contain the coradical. If q has a finite multiplicative order $t > 4$, then this is the case for the right coideal subalgebras of $u_q^+(\mathfrak{so}_{2n+1})$ homogeneous in each $x_i \in X$.*

Proof. If $\mathbf{U}$ is a right coideal subalgebra of $U_q^+(\mathfrak{so}_{2n+1})$ that contains $\mathbf{k}[G]$, then, by Lemma 2.12, it is homogeneous in each x_i. By Propositions 4.1 and 2.7, $\mathbf{U}$ has PBW-generators of the form (2.18):

$$c_u = u^s + \sum \alpha_i W_i \in \mathbf{U}, \quad u = u[k, m],\ k \leq m \leq \psi(k). \tag{5.13}$$

By (3.10), we have $p_{uu} = \sigma_k^m = q^2$ if $m \neq n$, and $p_{uu} = q$ otherwise. Thus, if q is not a root of 1, Lemma 2.6 shows that in (5.13) the exponent s equals 1, while all heights of the c_u's in $\mathbf{U}$ are infinite.

If q has a finite multiplicative order $t > 4$, then $u[k,m]^h = 0$ in $u_q^+(\mathfrak{so}_{2n+1})$, where h is the multiplicative order of p_{uu} (see Proposition 4.5). By Lemma 2.6, in (5.13), we have $s \in \{1, h, hl^r\}$. Because $u[k,m]^h = u[k,m]^{hl^r} = 0$, the exponent s in (5.13) equals 1, while the height of c_u in $\mathbf{U}$ equals h.

Because $\mathbf{U}$ is homogeneous with respect to each $x_i \in X$, the PBW-generators of $\mathbf{U}$ in both cases have the form

$$c_u = u[k,m] + \sum \alpha_i W_i, \qquad k \leq m \leq \psi(k), \tag{5.14}$$

where W_i are the basis super-words starting with super-letters smaller than $u[k,m]$, $D(W_i) = D(u[k,m]) = x_k + x_{k+1} + \cdots + x_m$. By Proposition 5.6, we have $\Phi^S(k,m) \in \mathbf{U}$. The leading term of $\Phi^S(k,m)$ equals $u[k,m]$; see definition (5.2). Hence, we may replace c_u with $\Phi^S(k,m)$ in the set of PBW-generators. The number of possible elements $\Phi^S(k,m)$ is finite. Hence, the total number of possible sets of PBW-generators of the form $\Phi^S(k,m)$ is also finite. □

6. Elements $\Phi^{[k,m-1]}(k,m)$

In this section, we are going to prove the following relation in $U_q^+(\mathfrak{so}_{2n+1})$:

$$\Phi^{[k,m-1]}(k,m) = (-1)^{m-k} \Big(\prod_{m \geq i > j \geq k} p_{ij}^{-1} \Big) \cdot u[\psi(m), \psi(k)], \tag{6.1}$$

where, as above, $\psi(i) = 2n - i + 1$. The main idea of the proof is to use the Milinski–Schneider criterion (Lemma 2.11). To do this, we need to find the partial derivatives of both sides. In what follows, ∂_i, $1 \leq i \leq 2n$, denotes the partial derivation with respect to x_i; see (2.22). In particular $\partial_i = \partial_{\psi(i)}$. The coproduct formula (4.8) with (2.23) implies

$$\partial_i(u[k,m]) = \begin{cases} (1-q^{-2})\tau_k u[k+1,m] & \text{if } x_i = x_k,\ k < m; \\ 0 & \text{if } x_i \neq x_k; \\ 1 & \text{if } x_i = x_k, k = m. \end{cases} \tag{6.2}$$

This equality allows us to easily find the derivatives of the right-hand side. By induction on $m-k$ we shall prove a similar formula

$$\partial_i(\Phi^{[k,m-1]}(k,m)) = \begin{cases} \beta_k^m \Phi^{[k,m-2]}(k,m-1) & \text{if } x_i = x_m,\ k < m; \\ 0 & \text{if } x_i \neq x_m; \\ 1 & \text{if } x_i = x_m, k = m, \end{cases} \tag{6.3}$$

where $\beta_k^m = -(1-q^{-2})\alpha_{km}^{m-1} = -(1-q^{-2})\tau_{m-1} p(x_m, u(k,m-1))^{-1}$. To simplify the notation, we remark that $\Phi^{[k,m-1]}(k,m) = \Phi^S(k,m)$ for each S that contains the interval $[k,m-1]$. In particular, in the above formula, $\Phi^{[k,m-2]}(k,m-1) = \Phi^{[k,m-1]}(k,m-1)$.

If $x_i \neq x_m$, $x_i \neq x_k$, then (6.2) and the inductive supposition applied to definition (5.2) imply $\partial_i(\Phi^{[k,m-1]}(k,m)) = 0$.

If $x_i = x_k \neq x_m$, then $\partial_i = \partial_k$. Taking into account definition (5.2) we have

$$\partial_k(\Phi^{[k,m-1]}(k,m)) = \partial_k\Big(u[k,m] - (1-q^{-2})\sum_{i=k}^{m-1} \alpha_{km}^i \Phi^{[k,m-1]}(1+i,m)u[k,i]\Big),$$

where $\alpha_{km}^i = \tau_i p(u(1+i,m), u(k,i))^{-1}$, while the τ's have been defined in (4.10). By the inductive supposition, the skew differential Leibniz formula (2.22), and (6.2), the above displayed expression equals

$$\begin{aligned}&(1-q^{-2})\tau_k(u[k+1,m] - \tau_k^{-1}\alpha_{km}^k p(u(1+k,m),x_k)\Phi^{[k,m-1]}(1+k,m)\\ &- (1-q^{-2})\sum_{i=k+1}^{m-1} \alpha_{km}^i\, p(u(1+i,m),x_k)\Phi^{[k,m-1]}(1+i,m)u[k+1,i]).\end{aligned} \tag{6.4}$$

Because obviously, $\alpha_{km}^k p(u(1+k,m),x_k) = \tau_k$, $\alpha_{km}^i\, p(u(1+i,m),x_k) = \alpha_{k+1\,m}^i$, definition (5.2) shows that the above expression is zero.

If $x_i = x_m \neq x_k$, then $\partial_i = \partial_m$. Again, by definition (5.2), the inductive supposition, the skew differential Leibniz formula (2.22), and (6.2), we have

$$\begin{aligned}\partial_m(\Phi^{[k,m-1]}(k,m)) = &-(1-q^{-2})\sum_{i=k}^{m-2} \alpha_{km}^i\, \beta_{1+i}^m \Phi^{[k,m-2]}(1+i,m-1)u[k,i]\\ &- (1-q^{-2})\alpha_{km}^{m-1}u[k,m-1].\end{aligned} \tag{6.5}$$

By definition, $-(1-q^{-2})\alpha_{km}^{m-1} = \beta_k^m$. At the same time

$$\begin{aligned}&\alpha_{km}^i\beta_{1+i}^m = \tau_i p(u(1+i,m),u(k,i))^{-1}\cdot\{-(1-q^{-2})\tau_{m-1}p(x_m,u(1+i,m-1))^{-1}\}\\ &= -(1-q^{-2})\tau_{m-1}p(x_m,u(k,m-1))^{-1}\cdot\tau_i p(u(1+i,m-1),u(k,i))^{-1} = \beta_k^m\cdot\alpha_{k\,m-1}^i.\end{aligned}$$

Thus, according to (5.2), the right-hand side of (6.5) equals $\beta_k^m\Phi^{[k,m-2]}(k,m-1)$, as required.

Finally, if $x_i = x_m = x_k$, $k \neq m$, that is, $m = \psi(k)$, then due to the skew differential Leibniz formula (2.22), the derivative $\partial_i(\Phi^{[k,m-1]}(k,m))$ equals the sum of the expression (6.4) with the right-hand side of (6.5). Note that (6.4) is still zero, while the right-hand side of (6.5) still equals $\beta_k^m\Phi^{[k,m-2]}(k,m-1)$. Formula (6.3) is completely proved.

We are now ready to prove (6.1) by induction on $m-k$. If $m=k$, both sides equal x_k. If $k < m$, then the derivatives ∂_i of both sides are zero for all i except $i = m$ and $i = \psi(m)$. Due to (6.2), the derivative ∂_m applied to the right-hand side of (6.1) equals

$$(-1)^{m-k}\Big(\prod_{m\geq i>j\geq k} p_{ij}^{-1}\Big)(1-q^{-2})\tau_{\psi(m)}\cdot u[\psi(m)+1,\psi(k)]. \tag{6.6}$$

Because $\psi(m) = n$ if and only if $m-1 = n$, formula (4.10) yields $\tau_{\psi(m)} = \tau_{m-1}$. At the same time, (6.3) and the inductive supposition imply

$$\partial_m(\Phi^{[k,m-1]}(k,m)) = \beta_k^m(-1)^{m-1-k}\Big(\prod_{m>i>j\geq k} p_{ij}^{-1}\Big)u[\psi(m)+1,\psi(k)]. \tag{6.7}$$

By definition, we have

$$\beta_k^m = -(1-q^{-2})\tau_{m-1}p(x_m,u(k,m-1))^{-1} = -(1-q^{-2})\tau_{m-1}\prod_{m>j\geq k} p_{mj}^{-1}.$$

Thus, (6.6) coincides with (6.7), and, due to the MS-criterion, (6.1) is proved.

Remark. To prove (6.1), we used the MS-criterion. Therefore, if q has a finite multiplicative order t, relation (6.1) is proved only for $u_q^+(\mathfrak{so}_{2n+1})$. However, we have seen in Proposition 4.5 that if $t > 4$, then the kernel of the natural homomorphism $U_q^+(\mathfrak{so}_{2n+1}) \to u_q^+(\mathfrak{so}_{2n+1})$ is generated by the elements $u[k,m]^h$, $h \geq 3$. At the same time, all polynomials in (6.1) have degree at most 2 in each variable. Therefore, (6.1) is valid in $U_q^+(\mathfrak{so}_{2n+1})$ provided that $t > 4$.

7. (k,m)-regular sets

Definition 7.1. Let $1 \leq k \leq n < m \leq 2n$. A set S is said to be *white* (k,m)*-regular* if for every i, $k-1 \leq i < m$, such that $k \leq \psi(i) \leq m+1$, either i or $\psi(i)-1$ does not belong to $S \cup \{k-1, m\}$.

A set S is said to be *black* (k,m)*-regular* if for every i, $k \leq i \leq m$, such that $k \leq \psi(i) \leq m+1$, either i or $\psi(i)-1$ belongs to $S \setminus \{k-1, m\}$.

If $m \leq n$ or $k > n$ (or equivalently if $u[k,m]$ is of degree ≤ 1 in x_n), then, by definition, each set S is both white and black (k,m)-regular.

A set S is said to be (k,m)*-regular* if it is either black or white (k,m)-regular.

If $k \leq n < m$ and S is white (k,m)-regular, then $n \notin S$, for $\psi(n)-1 = n$. If additionally $m < \psi(k)$, then taking $i = \psi(m)-1$, we obtain $\psi(i)-1 = m$. Hence, the definition implies $\psi(m)-1 \notin S$. We see that if $m < \psi(k)$, $k \leq n < m$, then S is white (k,m)-regular if and only if the shifted scheme of $\Phi^S(k,m)$ given in (5.5) has no black columns:

$$\begin{array}{ccccccccccc} & & \overset{m}{\bullet} & \cdots & \bullet & \overset{n+i}{\circ} & \circ & \cdots & \overset{n}{\circ} \Leftarrow \\ \overset{k-1}{\circ} & \cdots & \overset{\psi(m)-1}{\circ} & \cdots & \circ & \overset{n-i}{\bullet} & \circ & \cdots & \overset{n}{\circ} \end{array} \tag{7.1}$$

In the same way, if $m > \psi(k)$, then for $i = \psi(k)$, we obtain $\psi(i)-1 = k-1$, and hence $\psi(k) \notin S$. That is, if $m > \psi(k)$, $k \leq n < m$, then S is white (k,m)-regular if and only if the shifted scheme (5.5) has no black columns and the leftmost complete column is white:

$$\begin{array}{ccccccccccc} \overset{m}{\bullet} & \cdots & \overset{\psi(k)}{\circ} & \cdots & \bullet & \overset{n+i}{\circ} & \circ & \cdots & \overset{n}{\circ} \Leftarrow \\ & & \overset{k-1}{\circ} & \cdots & \circ & \overset{n-i}{\bullet} & \circ & \cdots & \overset{n}{\circ} \end{array} \tag{7.2}$$

Similarly, if $k \le n < m$ and S is black (k, m)-regular, then $n \in S$. If additionally $m < \psi(k)$, then taking $i = \psi(m) - 1$ we obtain $\psi(i) - 1 = m$, and hence $\psi(m) - 1 \in S$. We see that if $m < \psi(k)$ and $k \le n < m$, then S is black (k, m)-regular if and only if the shifted scheme (5.5) has no white columns and the leftmost complete column is black:

$$\begin{array}{ccccccccc} & & \overset{m}{\bullet} & \cdots & \bullet & \overset{n+i}{\circ} & \bullet & \cdots & \overset{n}{\bullet} \Leftarrow \\ \overset{k-1}{\circ} & \cdots & \overset{\psi(m)-1}{\bullet} & \cdots & \bullet & \overset{n-i}{\bullet} & \circ & \cdots & \overset{n}{\bullet} \end{array} \tag{7.3}$$

If $m > \psi(k)$, then for $i = \psi(k)$ we get $\psi(i) - 1 = k - 1$, hence $\psi(k) \in S$. That is, if $m > \psi(k)$, $k \le n < m$, then S is black (k, m)-regular if and only if the shifted scheme (5.5) has no white columns:

$$\begin{array}{ccccccccc} \overset{m}{\bullet} & \cdots & \overset{\psi(k)}{\bullet} & \cdots & \bullet & \overset{n+i}{\circ} & \bullet & \cdots & \overset{n}{\bullet} \Leftarrow \\ & & \overset{k-1}{\circ} & \cdots & \circ & \overset{n-i}{\bullet} & \bullet & \cdots & \overset{n}{\bullet} \end{array} \tag{7.4}$$

At the same time, we should stress that if $m = \psi(k)$, then no set is (k, m)-regular. Indeed, for $i = k - 1$, we have $\psi(i) - 1 = m$. Hence, both $i, \psi(i) - 1$ belong to $S \cup \{k - 1, m\}$, and therefore S is not white $(k, \psi(k))$-regular. If we take $i = m$, then $\psi(i) - 1 = k - 1$, and neither i nor $\psi(i) - 1$ belongs to $S \setminus \{k - 1, m\}$. Thus, S is not black $(k, \psi(k))$-regular either.

Let $S \cap [k, m - 1] = \{s_1, \ldots, s_r\}$, $s_1 < \cdots < s_r$. We denote $u_i = u[1 + s_i, s_{i+1}]$, $0 \le i \le r$, where we formally put $s_0 = k - 1$, $s_{r+1} = m$, while $u[k, m]$ has been defined in (3.16).

Lemma 7.2. *If S is white (k, m)-regular, then the values in $U_q^+(\mathfrak{so}_{2n+1})$ of the bracketed words $[u_r u_{r-1} \ldots u_1 u_0]$ and $[u_0 u_1 \ldots u_{r-1} u_r]$ are independent of the arrangement of brackets.*

Proof. Let $0 \le i < j - 1$, $j \le r$. Assume $k \le n < m$. The points s_i and $\psi(1 + s_i)$ form a column in the shifted scheme (7.1) or (7.2) since $s_i + \psi(1 + s_i) = 2n$. Hence, $\psi(1 + s_i) = \psi(s_i) - 1$ is not a black point. In particular $s_{j+1} \ne \psi(1 + s_i)$, $s_j \ne \psi(1 + s_i)$. Similarly, the points s_{i+1} and $\psi(s_{i+1}) - 1$ form a column in the shifted scheme, and hence $s_{j+1} \ne \psi(s_{i+1}) - 1$, $s_j \ne \psi(s_{i+1}) - 1$.

We now have $1 + s_i \le s_{i+1} < s_j < s_{j+1}$, $s_{j+1} \ne \psi(1 + s_i)$, $s_{j+1} \ne \psi(s_{i+1}) - 1$, $s_j \ne \psi(1 + s_i)$, and $s_j \ne \psi(s_{i+1}) - 1$. Therefore, Proposition 3.15 with $k \leftarrow 1 + s_i$, $i \leftarrow s_{i+1}$, $j \leftarrow s_j$, $m \leftarrow s_{j+1}$ implies $[u_i, u_j] = [u_j, u_i] = 0$. If $m \le n$ or $k > n$, then u_i and u_j are separated. Hence, we still have $[u_i, u_j] = [u_j, u_i] = 0$ due to Lemma 3.2. It remains to apply Lemma 2.1. □

Lemma 7.3. *If S is white (k, m)-regular, then $[u_0 u_1 \ldots u_r] = u[k, m]$.*

Proof. We use induction on r. If $r = 0$, the equality is clear. In the general case, the inductive supposition yields $[u_0 u_1 \ldots u_{r-1}] = u[k, s_r]$ because S is white (k, s_r)-regular. By Proposition 3.14, $[u[k, s_r], u_r] = u[k, m]$ unless $s_r = \psi(m) - 1$ or $s_r = \psi(k)$. However, the white (k, m)-regularity implies that $\psi(m) - 1$, $\psi(k)$ are not black points. □

Lemma 7.4. *If S is white (k, m)-regular, then in the above notation we have*

$$\Phi^S(k, m) = (-1)^r \prod_{r \geq i > j \geq 0} p(u_i, u_j)^{-1} \cdot [u_r u_{r-1} \dots u_0]. \tag{7.5}$$

Proof. To prove the equality, it suffices to check the recurrence relations (5.2) for the right-hand side. We shall use induction on r. If $r = 0$, there is nothing to prove. By Lemma 7.3, we have $u[k, m] = [u_0 u_1 \dots u_{r-1} u_r]$. The inductive supposition for the white (k, m)-regular set $S \setminus \{s_1\}$ takes the form

$$\begin{aligned}(-1)^{r-1} p(u_1, u_0) &\prod_{r \geq i > j \geq 0} p(u_i, u_j)^{-1} \cdot [u_r u_{r-1} \dots u_2 [u_0 u_1]] = [u_0 u_1 u_2 \dots u_r] \\ - (1 - q^{-2}) \sum_{l=2}^{r} \alpha_{k,m}^{s_l} (-1)^{r-l} &\prod_{r \geq i > j \geq l} p(u_i, u_j)^{-1} \cdot [u_r u_{r-1} \dots u_l] \cdot [[u_0 u_1] u_2 \dots u_{l-1}].\end{aligned} \tag{7.6}$$

By definition, $p(u_0, u_1) p(u_1, u_0) = \mu_k^{s_2, s_1}$ (see Definition 3.6), while by (3.13) and (3.15), we have $\mu_k^{s_2, s_1} = q^{-2}$ because the regularity condition implies $s_1 \neq n$, $s_1 \neq \psi(s_2) - 1$, $s_1 \neq \psi(k)$. Hence, by (2.13), we may write

$$p(u_1, u_0)[u_0, u_1] = -[u_1, u_0] + (1 - q^{-2}) u_1 \cdot u_0.$$

The above implies

$$\begin{aligned}&p(u_1, u_0)[u_r u_{r-1} \dots u_2 [u_0 u_1]] \\ &\qquad = -[u_r u_{r-1} \dots u_2 u_1 u_0] + (1 - q^{-2})[[u_r u_{r-1} \dots u_2], u_1 \cdot u_0].\end{aligned}$$

Because $[u_i, u_0] = 0$, $i \geq 2$, the ad-identity (2.11) yields

$$[[u_r u_{r-1} \dots u_2], u_1 \cdot u_0] = [u_r u_{r-1} \dots u_2 u_1] \cdot u_0.$$

Thus, the left-hand side of (7.6) reduces to

$$(-1)^r \prod_{r \geq i > j \geq 0} p(u_i, u_j)^{-1} \cdot [u_r u_{r-1} \dots u_2 u_1 u_0] + \mathfrak{A},$$

where

$$\mathfrak{A} = (1 - q^{-2})(-1)^{r-1} \prod_{r \geq i > 0} p(u_i, u_0)^{-1} \prod_{r \geq i > j \geq 1} p(u_i, u_j)^{-1} \cdot [u_r u_{r-1} \dots u_1] \cdot u_0.$$

At the same time, $\mathfrak{A}$ coincides up to a sign with the missing summand of the right-hand side of (7.6) corresponding to $l = 1$ because

$$\alpha_{k,m}^{s_1} = \tau_{s_1} p(u_r u_{r-1} \dots u_1, u_0)^{-1} = \prod_{r \geq i > 0} p(u_i, u_0)^{-1}. \qquad \square$$

Corollary 7.5. *If S is white (k,m)-regular, $s\in S\cup\{n\}$, $k\le s<m$, then*

$$\Phi^S(k,m)=-p_{ab}^{-1}[\Phi^S(1+s,m),\Phi^S(k,s)],$$

where $a=u(1+s,m)$, $b=u(k,s)$.

Proof. Let $s=s_t$, $1\le t\le r$. By Lemma 7.2, the value of the bracketed word $[u_ru_{r-1}\dots u_0]$ is independent of the arrangement of brackets. Therefore, we have $[u_ru_{r-1}\dots u_0]=[[u_ru_{r-1}\dots u_t],[u_{t-1}\dots u_0]]$. It remains to apply Lemma 7.4.

Let $k\le s=n<m$. Because n is always white in a white regular set, we can find j such that $s_j<n<s_{j+1}$. We denote $u_j'=u[1+s_j,n]$ and $u_j''=u[n+1,s_{j+1}]$. The points s_j and $\psi(1+s_j)$ form a column in the shifted scheme (7.1) or (7.2). Hence, $\psi(1+s_j)$ is a white point. In particular, $s_{j+1}\neq\psi(1+s_j)$. Thus, by Corollary 3.13 with $k\leftarrow 1+s_j$, $m\leftarrow s_{j+1}$, we have $u_j=[u_j',u_j'']=-p(u_j'',u_j')^{-1}[u_j'',u_j']$.

Note that the value of the bracketed word

$$[u_ru_{r-1}\dots u_{j+1}u_j''u_j'u_{j-1}\dots u_0] \tag{7.7}$$

is independent of the arrangement of brackets. Indeed, Lemma 3.12 with $k\leftarrow 1+s_j$, $i\leftarrow s_i$, $m\leftarrow s_{i+1}$ states $[u_i,u_j']=0$, $i>j$, unless $s_{i+1}=\psi(1+s_j)$ or $s_i=\psi(1+s_j)$. However, the points s_j and $\psi(1+s_j)$ form a column in the shifted scheme (7.1) or (7.2). Hence, $\psi(1+s_j)$ is not a black point. In particular $s_{i+1}\neq\psi(1+s_j)$ and $s_i\neq\psi(1+s_j)$.

At the same time, if $i<j-1$, then u_j' and u_i are separated by u_{j-1} (Definition 3.1); hence, Lemma 3.2 implies $[u_j',u_i]=0$.

In perfect analogy, we obtain $[u_j'',u_i]=0$, $i<j$, and $[u_i,u_j'']=0$, $i>j+1$. Thus, Lemma 2.1 implies that (7.7) is independent of the arrangement of brackets. In particular,

$$[u_ru_{r-1}\dots u_{j+1}u_j''u_j'u_{j-1}\dots u_0]=[[u_ru_{r-1}\dots u_{j+1}u_j''],[u_j'u_{j-1}\dots u_0]].$$

It remains to apply Lemma 7.4. □

Lemma 7.6. *If $k\le t<m$, $t\notin S$, then*

$$\Phi^{S\cup\{t\}}(k,m)-\Phi^S(k,m)=(q^{-2}-1)p_{ab}^{-1}\tau_t\Phi^S(1+t,m)\Phi^S(k,t), \tag{7.8}$$

where $a=u(1+t,m)$, $b=u(k,t)$.

Proof. We use induction on $m-k$. If $m=k$, there is nothing to prove. By definition (5.2), we have

$$\begin{aligned}\Phi^{S\cup\{t\}}(k,m)-\Phi^S(k,m)=&-(1-q^{-2})\{\tau_tp_{ab}^{-1}\Phi^S(1+t,m)u[k,t]\\&+\sum_{s_i<t}\tau_{s_i}p_{u_iv_i}^{-1}(\Phi^{S\cup\{t\}}(1+s_i,m)-\Phi^S(1+s_i,m))u[k,s_i]\},\end{aligned}$$

where $u_i=u(1+s_i,m)$, $v_i=u(k,s_i)$. By the inductive supposition the above equals

$$\begin{aligned}&(q^{-2}-1)p_{ab}^{-1}\tau_t\Phi^S(1+t,m)\\&\quad\cdot\Big\{u[k,t]-(1-q^{-2})\sum_{s_i<t}\tau_{s_i}p_{u_iv_i}^{-1}p_{ab_i}^{-1}p_{ab}\Phi^S(1+s_i,t)u[k,s_i]\Big\},\end{aligned}$$

where $b_i = u(1+s_i,t)$. It remains to note that

$$p_{u_i v_i}^{-1} p_{ab_i}^{-1} p_{ab} = p(u(1+s_i,t), u(k,s_i))^{-1}$$

and to use definition (5.2). □

Corollary 7.7. *If $S \cup \{t\}$ is white (k,m)-regular, $t \notin S$, $k \le t < m$, then*

$$\Phi^S(k,m) \sim [\Phi^S(k,t), \Phi^S(1+t,m)]. \tag{7.9}$$

Proof. We denote $A = \Phi^S(k,t)$, $B = \Phi^S(1+t,m)$. By Corollary 7.5 we have $\Phi^{S\cup\{t\}}(k,m) = -p_{ab}^{-1}[B,A]$. At the same time, $t \neq n$ (for $S\cup\{t\}$ is white (k,m)-regular), and hence, by Lemma 7.6, we get $\Phi^{S\cup\{t\}}(k,m) - \Phi^S(k,m) = (q^{-2}-1)p_{ab}^{-1}BA$. These two equalities imply

$$\begin{aligned}\Phi^S(k,m) &= -p_{ab}^{-1}[B,A] - (q^{-2}-1)p_{ab}^{-1}BA\\ &= p_{ab}^{-1}(-BA + p_{BA}AB - (q^{-2}-1)BA)\\ &= p_{ab}^{-1}p_{BA}(AB - q^{-2}p_{BA}^{-1}BA).\end{aligned} \tag{7.10}$$

By definition (3.6), we know that $p_{AB}p_{BA} = \mu_k^{m,t}$. In this case schemes (7.1) and (7.2) related to the white regular set $S \cup \{t\}$ show that $t \neq \psi(m)-1$, $t \neq n$, $t \neq \psi(k)$, $m \neq \psi(k)$ because t, m are black points. Hence, formulae (3.13), (3.15) imply $\mu_k^{m,t} = q^{-2}$. Thus, we get $p_{AB}p_{BA} = q^{-2}$; that is, $q^{-2}p_{BA}^{-1} = p_{AB}$. Now, (7.10) reduces to (7.9). □

Lemma 7.8. *A set S is white (k,m)-regular if and only if $\overline{\psi(S)-1}$ is black regular with respect to $(\psi(m),\psi(k))$. Here, $\psi(S)-1$ denotes $\{\psi(s)-1 \mid s \in S\}$, while the bar denotes the complement with respect to the interval $[\psi(m),\psi(k)-1]$.*

Proof. Let us replace the parameter i with $j = \psi(i)-1$ in the definition of regularity. Because ψ changes the order, we see that $k-1 \le i < m$ is equivalent to $\psi(k)+1 \ge \psi(i) > \psi(m)$, that is, $\psi(k) \ge j \ge \psi(m)$. Similarly, the condition $k \le \psi(i) \le m+1$ is equivalent to $\psi(k) \ge i \ge \psi(m)-1$. Because $\psi(j) = i+1$, we obtain $\psi(k)+1 \ge \psi(j) \ge \psi(m)$.

The condition $i \notin S \cup \{k-1, m\}$ is equivalent to $j \notin (\psi(S)-1) \cup \{\psi(m)-1, \psi(k)\}$, which, in turn, is equivalent to $j \in (\overline{\psi(S)-1}) \setminus \{\psi(m)-1, \psi(k)\}$. In the same way, $\psi(i)-1 \notin S \cup \{k-1, m\}$ is equivalent to $\psi(j)-1 \in (\overline{\psi(S)-1}) \setminus \{\psi(m)-1, \psi(k)\}$. □

Lemma 7.9. *A set S is black (k,m)-regular if and only if $\overline{\psi(S)-1}$ is white $(\psi(m),\psi(k))$-regular.*

Proof. This follows from the above lemma under the substitutions $k \leftarrow \psi(m)$, $m \leftarrow \psi(k)$, $S \leftarrow \overline{\psi(S)-1}$. □

Alternatively, one may easily check Lemmas 7.8 and 7.9 by means of the scheme interpretation (7.1–7.4). Indeed, the shifted representation for $\Phi^T(\psi(m),\psi(k))$, $T = \overline{\psi(S)-1}$ arises from one for $\Phi^S(k,m)$ by changing the colour of all points and switching the rows.

Proposition 7.10. *If S is black (k,m)-regular, then*

$$\Phi^S(k,m) = (-1)^{m-k} q^{-2r} \Big(\prod_{m \geq i > j \geq k} p_{ij}^{-1} \Big) \cdot \Phi^T(\psi(m), \psi(k)),$$

where $T = \overline{\psi(S) - 1}$ is a white $(\psi(m), \psi(k))$-regular set with r elements, and, as above, $\psi(S) - 1$ denotes $\{\psi(s) - 1 \mid s \in S\}$, while the bar denotes the complement with respect to the interval $[\psi(m), \psi(k) - 1]$.

Proof. We use double induction on r and on $m - k$. If $m = k$, then the equality reduces to $x_k = x_{\psi(k)}$. If for given k, m we have $r = 0$, then S contains the interval $[k, m-1]$ and the equality reduces to (6.1).

Suppose that $r > 0$. We fix $t \in T$. By the inductive supposition on r, we obtain

$$\Phi^{S \cup \{\psi(t)-1\}}(k,m) = (-1)^{m-k} q^{-2(r-1)} \Big(\prod_{m \geq i > j \geq k} p_{ij}^{-1} \Big) \cdot \Phi^{T \setminus \{t\}}(\psi(m), \psi(k)). \quad (7.11)$$

We have $t \notin \psi(S) - 1$, and hence $\psi(t) - 1 \notin S$. In particular $\psi(t) - 1 \neq n$, and $\tau_{\psi(t)-1} = 1$; see (4.10). Thus, relation (7.8) with $t \leftarrow \psi(t) - 1$ implies

$$\Phi^S(k,m) = \Phi^{S \cup \{\psi(t)-1\}}(k,m) + (1 - q^{-2}) p_{ab}^{-1} a \cdot b, \quad (7.12)$$

where $a = \Phi^S(\psi(t), m)$, $b = \Phi^S(k, \psi(t) - 1)$. The inductive supposition on $m - k$ yields

$$a = (-1)^{m-\psi(t)} q^{-2r_1} \Big(\prod_{m \geq i > j \geq \psi(t)} p_{ij}^{-1} \Big) \cdot \Phi^T(\psi(m), t),$$
$$b = (-1)^{\psi(t)-1-k} q^{-2r_2} \Big(\prod_{\psi(t) > i > j \geq k} p_{ij}^{-1} \Big) \cdot \Phi^T(1 + t, \psi(k)),$$

where r_1 is the number of elements in $T \cap [\psi(m), t-1]$, and r_2 is the number of elements in $T \cap [1 + t, \psi(k) - 1]$. Obviously, $r_1 + r_2 = r - 1$. Therefore,

$$p_{ab}^{-1} ab = (-1)^{m-k-1} q^{-2(r-1)} \Big(\prod_{m \geq i > j \geq k} p_{ij}^{-1} \Big) \cdot cd, \quad (7.13)$$

where $c = \Phi^T(\psi(m), t)$, $d = \Phi^T(1 + t, \psi(k))$. Now, (7.12) and (7.11) imply

$$\Phi^S(k,m) = (-1)^{m-k} q^{-2(r-1)} \Big(\prod_{m \geq i > j \geq k} p_{ij}^{-1} \Big) \cdot \{\Phi^{T \setminus \{t\}}(\psi(m), \psi(k)) - (1 - q^{-2}) cd\}. \quad (7.14)$$

We now have $t \neq n$ because T is white regular. Hence, relation (7.8) with $S \leftarrow T \setminus \{t\}$, $t \leftarrow t$, $k \leftarrow \psi(m)$, $m \leftarrow \psi(k)$ implies

$$\Phi^{T \setminus \{t\}}(\psi(m), \psi(k)) = \Phi^T(\psi(m), \psi(k)) + (1 - q^{-2}) p_{dc}^{-1} dc,$$

and the expression in braces in (7.14) reduces to

$$\Phi^T(\psi(m), \psi(k)) + (1 - q^{-2}) p_{dc}^{-1}[d, c]. \tag{7.15}$$

At the same time, Corollary 7.5 with $S \leftarrow T$, $s \leftarrow t$, $k \leftarrow \psi(m)$, $m \leftarrow \psi(k)$ shows that $p_{dc}^{-1}[d, c] = -\Phi^T(\psi(m), \psi(k))$. This equality shows that (7.15) is equal to

$$\Phi^T(\psi(m), \psi(k)) - (1 - q^{-2})\Phi^T(\psi(m), \psi(k)) = q^{-2}\Phi^T(\psi(m), \psi(k)).$$

To obtain the required relation, it remains to replace the expression in braces in (7.14) with $q^{-2}\Phi^T(\psi(m), \psi(k))$. □

Corollary 7.11. *If S is (k, m)-regular, then $\Phi^S(k, m) \sim \Phi^T(\psi(m), \psi(k))$ for a suitable $(\psi(m), \psi(k))$-regular set T.*

Proof. If S is black (k, m)-regular, we apply Proposition 7.10. If S is white (k, m)-regular, we may still apply Proposition 7.10 with $S \leftarrow T$, $T \leftarrow S$ by Lemma 7.9. □

Corollary 7.12. *Let S be (k, m)-regular. If $m > \psi(k)$, then the leading term of $\Phi^S(k, m)$ is proportional to $u[\psi(m), \psi(k)]$. In particular always $\Phi^S(k, m) \neq 0$.*

Proof. If $m < \psi(k)$, then definition (5.2) shows that the leading term of $\Phi^S(k, m)$ in the PBW-decomposition is $u[k, m]$; hence, $\Phi^S(k, m) \neq 0$.

If $m > \psi(k)$, then Proposition 7.10 (with $T \leftarrow S$, $S \leftarrow T$ provided that S is white regular) shows that $\Phi^S(k, m)$ is proportional to $\Phi^T(\psi(m), \psi(k)) \neq 0$ because $\psi(k) < \psi(\psi(m)) = m$. □

Corollary 7.13. *If S is black (k, m)-regular and $t \notin S \setminus \{n\}$, $k \leq t < m$, then*

$$\Phi^S(k, m) \sim [\Phi^S(k, t), \Phi^S(1 + t, m)].$$

Proof. If $t \notin S \setminus \{n\}$, then $\psi(t) - 1 \in T \cup \{n\}$, where $T = \overline{\psi(S) - 1}$. By Proposition 7.10 we have $\Phi^S(k, m) \sim \Phi^T(\psi(m), \psi(k))$. Corollary 7.5 yields

$$\Phi^T(\psi(m), \psi(k)) \sim [\Phi^T(\psi(t), \psi(k)), \Phi^T(\psi(m), \psi(t) - 1)].$$

Because t is a white point or $t = n$, the set S is black (k, t)-regular and black $(1 + t, m)$-regular; see the shifted schemes (7.3), (7.4). Hence, Proposition 7.10 implies $\Phi^S(k, t) \sim \Phi^T(\psi(t), \psi(k))$, $\Phi^S(1 + t, m) \sim \Phi^T(\psi(m), \psi(t) - 1)$. □

Corollary 7.14. *If $S \setminus \{s\}$ is black (k, m)-regular, $s \in S$, $k \leq s < m$, then*

$$\Phi^S(k, m) \sim [\Phi^S(1 + s, m), \Phi^S(k, s)]. \tag{7.16}$$

Proof. This follows from Lemma 7.7 and Proposition 7.10 in a similar way. □

8. Root sequence

Our next goal is to show that the total number of right coideal subalgebras containing $\mathbf{k}[G]$ is less than or equal to $(2n)!! = 2^n \cdot n!$.

In what follows we shall denote by $[k:m]$, $k \leq m \leq 2n$, the element $x_k + x_{k+1} + \cdots + x_m$, considered as an element of the group Γ^+. Of course, $[k:m] = [\psi(m):\psi(k)]$. If $k \leq m < \psi(k)$, then $[k:m]$ is a $U_q^+(\mathfrak{so}_{2n+1})$-root because $u[k,m]$ is a PBW-generator for $U_q^+(\mathfrak{so}_{2n+1})$. The simple $U_q^+(\mathfrak{so}_{2n+1})$-roots are precisely the generators $x_k = [k:k]$, $1 \leq k \leq n$. To put it another way, the $U_q^+(\mathfrak{so}_{2n+1})$-roots form the positive part R^+ of the classical root system of type B_n, provided that we formally replace symbols x_i with α_i (the Weyl basis for R, see [3, Chapter IV, §6, Theorem 7]).

We fix the notation $\mathbf{U}$ for a (homogeneous if $q^t = 1$, $t > 4$) right coideal subalgebra of $U_q^+(\mathfrak{so}_{2n+1})$, $q^t \neq 1$ (respectively, of $u_q^+(\mathfrak{so}_{2n+1})$) that contains G. The $\mathbf{U}$-roots form a subset $D(\mathbf{U})$ of R^+. In this section we will see, in particular, that $D(\mathbf{U})$ uniquely defines $\mathbf{U}$.

Definition 8.1. Let γ_k be a simple $\mathbf{U}$-root of the form $[k:m]$, $k \leq m < \psi(k)$, with m maximal. We denote by θ_k the number $m - k + 1$, which equals the length of γ_k. If there are no simple $\mathbf{U}$-roots of the form $[k:m]$, $k \leq m < \psi(k)$, we put $\theta_k = 0$. The sequence $r(\mathbf{U}) = (\theta_1, \dots, \theta_n)$ satisfies $0 \leq \theta_k \leq 2n - 2k + 1$ and is uniquely defined by $\mathbf{U}$. We shall call $r(\mathbf{U})$ a *root sequence of* $\mathbf{U}$, or just an *r-sequence of* $\mathbf{U}$. We define $\tilde{\theta}_k$ to be $k + \theta_k - 1$, the maximal value of m for the simple $\mathbf{U}$-roots of the form $[k:m]$ with fixed k.

Theorem 8.2. *For each sequence $\theta = (\theta_1, \dots, \theta_n)$ such that $0 \leq \theta_k \leq 2n - 2k + 1$, $1 \leq k \leq n$, there exists at most one (homogeneous if $q^t = 1$, $t > 4$) right coideal subalgebra $\mathbf{U} \supseteq G$ of $U_q^+(\mathfrak{so}_{2n+1})$, $q^t \neq 1$ (respectively, of $u_q^+(\mathfrak{so}_{2n+1})$) with $r(\mathbf{U}) = \theta$.*

This will result from the following lemmas.

Lemma 8.3. *If $[k:m]$ is a simple $\mathbf{U}$-root, then there exists only one element $a \in \mathbf{U}$ of the form $a = \Phi^S(k,m)$.*

Proof. Suppose that $a = \Phi^S(k,m)$ and $b = \Phi^{S'}(k,m)$ are two different elements in $\mathbf{U}$. Then $a - b$ is not a PBW-generator for $\mathbf{U}$ because its leading term, with respect to the PBW-decomposition given in Proposition 4.1, is not equal to $u[k,m]$. Hence, the nonzero homogeneous element $a - b$ is a polynomial in the PBW-generators of $\mathbf{U}$. Thus, $[k:m]$, being the degree of $a - b$, is a sum of $\mathbf{U}$-roots, which is a contradiction. □

Lemma 8.4. *Let $\Phi^S(k,m) \in \mathbf{U}$, $k \leq m < \psi(k)$. Suppose that $\Phi^{S'}(k,m) \notin \mathbf{U}$ for all subsets $S' \subset S$. If $j \notin S$, $k \leq j < m$, then $\Phi^S(1+j,m) \in \mathbf{U}$. If $j \in S$, $k \leq j < m$, then $\Phi^{S''}(k,j) \in \mathbf{U}$ with some $S'' \subseteq S \cap [k,j]$. In particular $[k:j]$ is a $\mathbf{U}$-root.*

Proof. If in (5.2) we have $\Phi^S(1+s_i,m) = 0$, then the spectrum $\mathrm{Sp}(a)$ of $a = \Phi^S(k,m)$ is a proper subset of $S \cup \{m\}$. By Proposition 5.6, there exists a subset $S' \subseteq \mathrm{Sp}(a) \subset S$ such that $\Phi^{S'}(k,m) \in \mathbf{U}$. This contradiction implies that $\Phi^S(1+j,m) \neq 0$ for all $j \in S \cap [k, m-1]$.

If $j \notin S$, then Lemma 5.4 implies $\Phi^S(1+j,m) \in \mathbf{U}$.

If $j \in S$, then we apply $\Delta \cdot (\mathrm{id} \otimes \nu_a)$ with $a = \Phi^S(1+j, m) \neq 0$ as defined in Lemma 5.5 to both sides of (5.2). Lemma 5.5 shows that the value of $\Delta(\Phi^S(1+i, m)u[k,i]) \cdot (\mathrm{id} \otimes \nu_a)$ has the following three options: if $j < i < m$, it is zero; if $i = j$, it is $g_a u[k, j] \otimes a$; if $i < r$, it is $g_a b_i' u[k,i]$, $b_i' \in A_{k+1}$. Because $\Delta(u[k,m]) \cdot (\mathrm{id} \otimes \nu_a) = 0$ due to (4.8), we obtain

$$b = u[k, j] + \sum_{i<j,\, i \in S} b_i' u[k,i] \in \mathbf{U}, \qquad b_i' \in A_{k+1}.$$

By definition this relation means that $[k : j]$ is a $\mathbf{U}$-root, while Proposition 5.6 implies $\Phi^{S''}(k, j) \in \mathbf{U}$ with $S'' \subseteq \mathrm{Sp}(b) \subseteq S \cap [k, j]$. □

Lemma 8.5. *If* $[k : m]$ *is a simple* $\mathbf{U}$*-root,* $k \leq m < \psi(k)$*, then the minimal* S *such that* $\Phi^S(k, m) \in \mathbf{U}$ *equals* $\{j \mid k \leq j < m,\ [k : j]$ *is a* $\mathbf{U}$*-root*$\}$*, and it is a* (k, m)*-regular set* (*see Definition* 7.1).

Proof. Suppose that S is not (k, m)-regular; we then have $k \leq n < m$.

If n is a white point, $n \notin S$, then by Lemma 8.4, we have $\Phi^S(1+n, m) \in \mathbf{U}$. Hence $[n+1 : m] = [\psi(m) : n]$ is a $\mathbf{U}$-root due to Corollary 7.12. Because S is not white (k, m)-regular, in the shifted scheme (7.2) we can find a black column, say $n + i \in S \cup \{m\}$, $n - i \in S$. By Lemma 8.4 applied to $\Phi^S(n+1, m)$, $[n+1 : n+i]$ is a $\mathbf{U}$-root, while the same lemma applied to $\Phi^S(k, m)$ shows that $[k : n-i]$ is also a $\mathbf{U}$-root. Now,

$$[k : m] = [k : n] + [n+1 : m] = [k : n-i] + [n+1 : n+i] + [n+1 : m]$$

is a sum of $\mathbf{U}$-roots, which is a contradiction.

If n is a black point, $n \in S$, then by Lemma 8.4, we have $\Phi^{S''}(k, n) \in \mathbf{U}$, and $[k : n]$ is a $\mathbf{U}$-root. Because S is not black (k, m)-regular, we can find i, $1 \leq i \leq m - n$, such that $n + i \notin S \setminus \{m\}$, $n - i \notin S$ (see (7.3)). We have $n - i \notin S''$ because $S'' \subseteq S$. Hence Lemma 8.4 applied to $\Phi^{S''}(k, n)$ implies that $[1+n-i : n] = [n+1 : n+i]$ is a $\mathbf{U}$-root. The same lemma applied to $\Phi^S(k, m)$ shows that $\Phi^S(1+n+i, m) \in \mathbf{U}$. Hence, due to Corollary 7.12, the element $[1+n+i : m] = [\psi(m) : n-i]$ is also a $\mathbf{U}$-root. We now have a similar contradiction:

$$[k : m] = [k : n] + [n+1 : m] = [k : n] + [n+1 : n+i] + [1+n+i : m].$$

Due to Lemma 8.4 it remains to show that if $[k : j]$ is a $\mathbf{U}$-root, then $j \in S$. Suppose that $j \notin S$. Then Lemma 8.4 implies $a = \Phi^S(1+j, m) \in \mathbf{U}$.

If S is $(1+j, m)$-regular, or $1 + j < \psi(m)$, then $a \neq 0$ and $[1+j : m]$ is a $\mathbf{U}$-root (see Corollary 7.12). This is a contradiction, for $[k : m] = [k : j] + [1+j : m]$.

Suppose, finally, that S is not $(1+j, m)$-regular and $1 + j \geq \psi(m)$. Because S is indeed (k, m)-regular, these conditions hold only in two cases: $j = \psi(m) - 1$, or $n \notin S$, $\psi(j) - 1 \in S$; see the shifted scheme representations (7.2), (7.4).

In the former case, by Lemma 8.4, either $\Phi^S(1+n, m) \in \mathbf{U}$ (if $n \notin S$), or $\Phi^{S''}(k, m) \in \mathbf{U}$ and $\Phi^S(1+j, m) \in \mathbf{U}$ because $j \notin S'' \subseteq S$ (if $n \in S$). Therefore, $[n+1 : m] = [\psi(m), n] = [j+1 : n]$ is a $\mathbf{U}$-root due to Corollary 7.12. We have a contradiction $[k : m] = [k : \psi(m) - 1] + [\psi(m), n] + [n+1 : m]$.

In the latter case, similarly, $\Phi^S(1+n, m) \in \mathbf{U}$ and $\Phi^{S''}(1+j, n) \in \mathbf{U}$. Hence, Corollary 7.12 implies that $[n+1 : m]$, $[1+j : n]$ are $\mathbf{U}$-roots. Again we have a contradiction: $[k : m] = [k : j] + [1+j, n] + [n+1 : m]$. □

Lemma 8.6. *If* $[k : m] = \sum_{i=1}^{r+1}[l_i : m_i]$, $k \le m \le 2n$, $l_i \le m_i < \psi(l_i)$, *then it is possible to replace some of the pairs* (l_i, m_i) *with* $(\psi(m_i), \psi(l_i))$ *so that the given decomposition takes the form*

$$[k : m] = [1+k_0 : k_1] + [1+k_1 : k_2] + \cdots + [1+k_r : m] \tag{8.1}$$

with $k-1 = k_0 < k_1 < k_2 < \cdots < k_r < m = k_{r+1}$.

Proof. We use induction on $m-k$. Either x_k or x_m is the maximal letter among $\{x_j \mid k \le j \le m\}$. Hence, there exists at least one i such that, respectively, $l_i = k$ or $l_i = \psi(m)$. In the former case, we may put $k_1 = m_i$ and apply the inductive supposition to $[m_i+1 : m]$. In the latter case, we put $k_r = \psi(m_i) - 1$. Then $[k_r+1 : m] = [\psi(m_i) : \psi(l_i)]$ and one may apply the inductive supposition to $[k : k_r]$. □

Lemma 8.7. *If* $[k : m]$, $k \le m \ne \psi(k)$, *is a sum of* $\mathbf{U}$*-roots, then* $[k : m]$ *itself is a* $\mathbf{U}$*-root.*

Proof. Without loss of generality, we may suppose that $m < \psi(k)$ because $[k : m] = [\psi(m) : \psi(k)]$. By Lemma 8.6, we have a decomposition (8.1), where $[1+k_i : k_{i+1}]$, $0 \le i < r$, are $\mathbf{U}$-roots. By increasing r if necessary, we may suppose that all roots $[1+k_i : k_{i+1}]$, $0 \le i < r$, are simple.

If $k_{i+1} < \psi(1+k_i)$, then by Proposition 5.6 we find a set $S_i \subseteq [1+k_i, k_{i+1}-1]$ such that $\Phi^{S_i}(1+k_i, k_{i+1}) \in \mathbf{U}$. Moreover, by Lemma 8.5, the set S_i may be taken to be $(1+k_i, k_{i+1})$-regular.

If $k_{i+1} > \psi(1+k_i)$, then of course $\psi(1+k_i) < \psi(\psi(k_{i+1}))$, and again by Proposition 5.6 and Lemma 8.5, we find a $(\psi(k_{i+1}), \psi(1+k_i))$-regular set $T_i \subseteq [\psi(k_{i+1}), \psi(1+k_i) - 1]$ such that $\Phi^{T_i}(\psi(k_{i+1}), \psi(1+k_i)) \in \mathbf{U}$. By Corollary 7.11 with $S \leftarrow T_i$, we have $\Phi^{T_i}(\psi(k_{i+1}), \psi(1+k_i)) \sim \Phi^{S_i}(1+k_i, k_{i+1})$, where S_i is $(1+k_i, k_{i+1})$-regular. Thus, in all cases

$$f_i \stackrel{df}{=} \Phi^{S_i}(1+k_i, k_{i+1}) \in \mathbf{U}, \qquad S_i \subseteq [1+k_i, k_{i+1}-1], \tag{8.2}$$

with regular S_i (we stress that this is a restriction on S_i only if $1+k_i \le n < k_{i+1}$).

By Definition 2.8, we must construct an element $c \in \mathbf{U}$ with the leading super-word $u[k, m]$. First we shall prove that for $r = 1$, the element $c = [f_0, f_1]$ is such an element even if $[1+k_i : k_{i+1}]$ are not necessarily simple roots, but S_i, $i = 0, 1$, are still regular sets.

There is the following natural reduction process for the decomposition of a linear combination of super-words in the PBW-basis given in Theorem 2.5 and Propositions 4.1, 4.5. Let W be a super-word. First, according to [7, Lemma 7], we decompose the super-word W into a linear combination of smaller monotonous super-words. Then, we replace each nonhard super-letter with the decomposition of its value that exists by Definition 2.3, and again we decompose the arising super-words into linear combinations of smaller monotonous super-words, and so on, until we obtain a linear combination of monotonous super-words in hard super-letters. If these super-words are not restricted, we may apply Definition 2.4 and repeat the process until we obtain only monotonous restricted words in hard super-letters.

This process shows that if a super-word W starts with a super-letter smaller than $u[k,m]$, then so do all the super-words in the PBW-decomposition of W. Using this remark we shall prove the following auxiliary statement.

If $k \leq i < j < m < \psi(k)$, $m \neq \psi(i) - 1$, *then all super-words in the PBW-decomposition of* $[u[k,i], \Phi^S(1+j,m)]$ *start with super-letters smaller than* $u[k,m]$.

Indeed, by definition (5.2) we have

$$\Phi^S(1+j,m) = u[1+j,m] + \sum_{m>s\geq 1+j} \gamma_s \Phi^S(1+s,m)\cdot u[1+j,s], \qquad \gamma_s \in \mathbf{k}.$$

We now use induction on $m-j$. By Proposition 3.15 we have $[u[k,i], u[1+j,m]] = 0$, for the inequalities $\psi(k) > m > j$ imply $j \neq \psi(k)$. We denote $u = u[k,i]$, $v = \Phi^S(1+s,m)$, $w = u[1+j,s]$. Relation (2.11) reads $[u, v\cdot w] = [u,v]\cdot w + p_{uv} v\cdot[u,w]$. By the inductive supposition, all super-words in the PBW-decomposition of $[u,v]$ start with super-letters smaller than $u[k,m]$, and consequently so do those for $[u,v]\cdot w$. The element v depends only on x_i, $i > k$, and therefore so do all super-letters in the PBW-decomposition of v, while the starting super-letters of $v\cdot[u,w]$ are still less than $u[k,m]$. Thus, all super-words in the PBW-decomposition of $[u[k,i], \Phi^S(1+j,m)]$ start with super-letters smaller than $u[k,m]$. The auxiliary statement is proved.

We now have

$$\begin{aligned}
[f_1, f_2] &= [\Phi^{S_0}(k,k_1), \Phi^{S_1}(1+k_1,m)] \\
&= \Big[u[k,k_1] + \sum_{k_1>s\geq k} \gamma_s \Phi^{S_0}(1+s,k_1)\cdot u[k,s], \\
&\qquad\qquad u[1+k_1,m] + \sum_{m>l\geq 1+k_1} \beta_l \Phi^{S_1}(1+l,m)\cdot u[1+j,l]\Big] \\
&= u[k,m] + \sum_{m>l\geq 1+k_1} \beta_l [u[k,k_1], \Phi^{S_1}(1+l,m)\cdot u[1+k_1,l]] \\
&\quad + \sum_{k_1>s\geq k} \gamma_s [\Phi^{S_0}(1+s,k_1)\cdot u[k,s], f_2].
\end{aligned}$$

We see that each element in the latter sum has a nontrivial left factor that depends only on x_i, $i > k$, which is is either $\Phi^{S_0}(1+s,k_1)$ or f_2. Hence, all super-words in the PBW-decomposition of that element start with super-letters smaller than $u[k,m]$. To check the former sum, we denote $u = u[k,k_1]$, $v = \Phi^{S_1}(1+l,m)$, $w = u[1+k_1,l]$. By (2.11) the general element in the sum is proportional to $[u, v\cdot w] = [u,v]\cdot w + p_{uv} v\cdot[u,w]$. By the above auxiliary statement with $i \leftarrow k_1$, $j \leftarrow l$, all super-words in the PBW-decomposition of $[u,v]$ start with super-letters smaller than $u[k,m]$, and hence so do those for $[u,v]\cdot w$. The element v depends only on x_i, $i > k$. Therefore, the starting super-letters in the PBW-decomposition of $v \cdot [u, w]$ are also smaller than $u[k, m]$. Thus, the leading term of $[f_0, f_1]$ is indeed $u[k, m]$. The case $r = 1$ is completed.

Consider the general case. Denote by t the index such that $1 + k_t \le n \le k_{t+1}$, if any. Recall that S_t is either white or black $(1 + k_t, k_{t+1})$-regular, while each S_i, $i \ne t$, is both white and black $(1 + k_i, k_{i+1})$-regular because its degree in x_n is less than or equal to 1. We shall consider four options for the regular set S_t given in (7.1–7.4) separately.

1. $k_{t+1} < \psi(1 + k_t)$, *and* S_t *is white regular*. Let $S = \bigcup_{i=0}^{t} S_i \cup \{k_i \mid 0 < i < t\}$. The set S is white (k, k_{t+1})-regular because all complete columns in the shifted scheme (7.1) for $\Phi^S(k, k_{t+1})$ coincide with ones for $\Phi^{S_t}(k_t, k_{t+1})$. By Lemma 7.4, we have

$$\Phi^S(k, k_{t+1}) \sim [f_t f_{t-1} \dots f_0]$$

with an arbitrary arrangement of brackets on the right-hand side. In the same way consider the set $S' = \bigcup_{i=t+1}^{r} S_i \cup \{k_i \mid t+1 < i < r\}$. This set is white $(1 + k_{t+1}, m)$-regular because the shifted scheme (7.1) for $\Phi^{S'}(1 + k_{t+1}, m)$ has no complete columns at all. Lemma 7.4 yields

$$\Phi^{S'}(1 + k_{t+1}, m) \sim [f_r f_{r-1} \dots f_{t+1}].$$

Now we may apply the case $r = 1$ with $S_0 \leftarrow S$, $S_1 \leftarrow S'$, $t_1 \leftarrow k_{t+1}$. Thus, the leading super-word of the element

$$c = [[f_t f_{t-1} \dots f_0], [f_r f_{r-1} \dots f_{t+1}]] \tag{8.3}$$

equals $u[k, m]$, and obviously $c \in \mathbf{U}$ since $f_i \in \mathbf{U}$, $0 \le i \le r$.

2. $k_{t+1} > \psi(1 + k_t)$, *and* S_t *is white regular*. In perfect analogy we consider the sets $S = \bigcup_{i=0}^{t-1} S_i \cup \{k_i \mid 0 < i < t-1\}$ and $S' = \bigcup_{i=t}^{r} S_i \cup \{k_i \mid t < i < r\}$. By the case $r = 1$ under the substitutions $S_0 \leftarrow S$, $S_1 \leftarrow S'$, $t_1 \leftarrow k_t$, we see that the required element is

$$c = [[f_{t-1} f_{t-2} \dots f_1 f_0], [f_r f_{r-1} \dots f_{t+1} f_t]]. \tag{8.4}$$

3. $k_{t+1} < \psi(1 + k_t)$, *and* S_t *is black regular*. Let $S = \bigcup_{i=0}^{t} S_i$. The set S is black (k, k_{t+1})-regular because all complete columns in the shifted scheme (7.3) for $\Phi^S(k, k_{t+1})$ coincide with ones for $\Phi^{S_t}(k_t, k_{t+1})$. None of the points $k_1, \dots, k_r$ belongs to S (see (8.2)). Therefore, by multiple use of Corollary 7.13, we have

$$\Phi^S(k, k_{t+1}) \sim [f_0 f_1 \dots f_t]$$

with an arbitrary arrangement of brackets on the right-hand side. In the same way, consider the set $S' = \bigcup_{i=t+1}^{r} S_i$. It is black $(1 + k_{t+1}, m)$-regular because the shifted scheme (7.3) for $\Phi^{S'}(1 + k_{t+1}, m)$ has no complete columns at all. The multiple use of Corollary 7.13 yields

$$\Phi^{S'}(1 + k_{t+1}, m) \sim [f_{t+1} f_{t+2} \dots f_r].$$

Now, we may find c using the case $r = 1$ with $S_0 \leftarrow S$, $S_1 \leftarrow S'$, $t_1 \leftarrow k_{t+1}$:

$$c = [[f_0 f_1 \dots f_t], [f_{t+1} f_{t+2} \dots f_r]]. \tag{8.5}$$

4. $k_{t+1} > \psi(1+k_t)$, *and* S_t *is black regular*. In perfect analogy we consider the sets $S = \bigcup_{i=0}^{t-1} S_i$ and $S' = \bigcup_{i=t}^{r} S_i$. By the case $r = 1$ under the substitutions $S_0 \leftarrow S$, $S_1 \leftarrow S'$, $t_1 \leftarrow k_t$, we see that the required element is

$$c = [[f_0 f_1 \dots f_{t-1}], [f_t f_{t+1} \dots f_r]]. \tag{8.6}$$

The proof is complete. □

Lemma 8.8. *If* $[k : m]$, $k \le m < \psi(k)$, *is a simple* **U**-*root*, $k \le j < m$, *then* $[k : j]$ *is a* **U**-*root if and only if* $[1 + j : m]$ *is not a sum of* **U**-*roots.*

Proof. If $[k, j]$ is a **U**-root, then $[1 + j : m]$ is not a sum of **U**-roots because $[k : m] = [k : j] + [1 + j : m]$ is a simple **U**-root.

We note, first, that the converse statement is valid if the minimal S with $\Phi^S(k, m) \in \mathbf{U}$ is $(1 + j, m)$-regular. Indeed, in this case, $\Phi^S(1 + j, m) \ne 0$ due to Corollary 7.12. By Lemma 8.5, the element $[k : j]$ is a **U**-root if and only if $j \in S$. If $j \notin S$, then by Lemma 8.4, we have $a = \Phi^S(1 + j, m) \in \mathbf{U}$. Hence, the nonzero homogeneous element a is a polynomial in PBW-generators of **U**. Thus, $[1 + j : m]$, being the degree of a, is a sum of **U**-roots (by Lemma 8.7, it is even a **U**-root because the regularity hypothesis implies $\psi(1 + j) \ne m$).

Suppose, next, that S is not $(1+j, m)$-regular and $j \notin S$. In this case, $1+j \le n < m$. Moreover, $m \ge \psi(1 + j)$, because otherwise all complete columns in the shifted scheme (7.1)–(7.4) of $\Phi^S(1 + j, m)$ coincide with those of $\Phi^S(k, m)$. Obviously, in general, only the leftmost complete column for $\Phi^S(1 + j, m)$ may be different from a complete column for $\Phi^S(k, m)$. Hence, we have only the following three options: 1) $\psi(1 + j) = m$; 2) $\psi(1 + j) \in S$, while $n \notin S$; 3) $\psi(1 + j) \notin S$, while $n \in S$.

1) In the shifted scheme of $\Phi^S(k, m)$, the point $j = \psi(m) - 1$ has the same colour as n (see (7.1), (7.3)); that is, n is a white point. At the same time, because S is always $(n + 1, m)$-regular, we already know that n is white if and only if $[n + 1 : m]$ is a **U**-root. Thus, $[n+1 : m]$ is a **U**-root, while $[1+j : m] = [1+j : n]+[n+1 : m] = 2[n+1 : m]$ is a sum of two **U**-roots.

2) In the second case, S is certainly $(n + 1, m)$-regular. Hence, $n \notin S$ implies that $[n + 1 : m]$ is a **U**-root. By Lemma 8.4, we have $\Phi^{S''}(k, \psi(1 + j)) \in \mathbf{U}$ with $S'' \subseteq S$, for $\psi(1 + j) \in S$. In particular, we still have $n \notin S$. Hence the same lemma again implies $a = \Phi^S(n + 1, \psi(1 + j)) \in \mathbf{U}$. By Corollary 7.12, the leading super-word of a equals $u[1 + j, n]$; that is, $[1 + j : n]$ is a **U**-root. Now, $[1 + j : m] = [1 + j : n] + [n + 1 : m]$ is a sum of two roots, as required.

3) By Lemma 8.4, we have $\Phi^{S''}(k, n) \in \mathbf{U}$ with $S'' \subseteq S$ since $n \in S$. In particular, we still have $j \notin S''$. Hence the same lemma implies that $[1 + j : n]$ is a **U**-root. Because $\psi(1 + j) \notin S$, and obviously S is $(\psi(1 + j), m)$-regular, we already know that $[1 + \psi(1 + j) : m] = [\psi(j) : m]$ is a **U**-root. Now $[1 + j : m] = [1 + j : n] + [n + 1 : \psi(1 + j)] + [\psi(j) : m]$ is a sum of **U**-roots because $[n + 1 : \psi(1 + j)] = [1 + j : n]$. □

Lemma 8.9. *A* (*homogeneous*) *right coideal subalgebra* **U** *that contains* $\mathbf{k}[G]$ *is uniquely defined by the set of all its simple roots.*

Proof. Two subalgebras with the same PBW-basis obviously coincide; hence, it suffices to find a PBW-basis of $\mathbf{U}$ that depends only on the set of simple $\mathbf{U}$-roots. We note first that the set of all $\mathbf{U}$-roots is uniquely defined by the set of simple $\mathbf{U}$-roots. Indeed, if $[k:m]$ is a $\mathbf{U}$-root, then it is a sum of simple $\mathbf{U}$-roots. By Lemma 8.6 there exists a sequence $k-1=k_0<k_1<\cdots<k_r<m=k_{r+1}$ such that $[1+k_i:k_{i+1}]$, $0\le i\le r$, are simple $\mathbf{U}$-roots. Conversely, if there exists a sequence $k-1=k_0<k_1<\cdots<k_r=m+1$ such that $[1+k_i:k_{i+1}]$, $0\le i<r$, are simple $\mathbf{U}$-roots, then by Lemma 8.7, the element $[k:m]$ is a $\mathbf{U}$-root. Of course, the decomposition of $[k:m]$ into a sum of simple $\mathbf{U}$-roots is not unique in general. However, for the construction of the PBW-basis, we may fix a decomposition for each nonsimple $\mathbf{U}$-root from the very beginning.

Now, if $[k:m]$ is a simple $\mathbf{U}$-root, Lemmas 8.3 and 8.5 show that the element $\Phi^S(k,m)\in\mathbf{U}$ is uniquely defined by the set of simple $\mathbf{U}$-roots. We include this element in the PBW-basis of $\mathbf{U}$. If $[k:m]$ is a nonsimple $\mathbf{U}$-root with a fixed decomposition into a sum of simple $\mathbf{U}$-roots, then we include in the PBW-basis the element c defined in one of the formulae (8.3)–(8.6) depending on the type of decomposition. □

Lemma 8.10. *If for (homogeneous) right coideal subalgebras* $\mathbf{U}$, $\mathbf{U}'$ *containing* $\mathbf{k}[G]$ *we have* $r(\mathbf{U})=r(\mathbf{U}')$, *then* $\mathbf{U}=\mathbf{U}'$.

Proof. By Lemma 8.9, it suffices to show that the r-sequence uniquely defines the set of all simple roots. We use downward induction on k defined by a simple $\mathbf{U}$-root $[k:m]$. If $k=n$, then the only possible $\gamma=[n:n]$ is a simple $\mathbf{U}$-root if and only if $\theta_n=1$. Let $k<n$. By definition, simple $\mathbf{U}$-roots of the form $[k:m]$, $m>\tilde{\theta}_k$, do not exist, while $[k:\tilde{\theta}_k]$ is a simple $\mathbf{U}$-root. If $m<\tilde{\theta}_k$, then by Lemma 8.8, the element $[k:m]$ is a $\mathbf{U}$-root if and only if $[m+1:\tilde{\theta}_k]$ is not a sum of $\mathbf{U}$-roots starting with a number greater than k. By the inductive supposition, the r-sequence defines all roots starting with a number greater than k. Hence, by Lemma 8.8, the r-sequence also defines the set of all $\mathbf{U}$-roots of the form $[k:m]$, $m<\tilde{\theta}_k$. Thus, the r-sequence defines the set of all $\mathbf{U}$-roots and the set of all simple $\mathbf{U}$-roots. □

9. Examples

In this section, we find the simple roots for fundamental examples of right coideal subalgebras. We keep all the notation of the above section.

Example 9.1. Let $\mathbf{U}(k,m)$ be the right coideal subalgebra generated over $\mathbf{k}[G]$ by a single element $u[k,m]$, $k\le m\le\psi(k)$. By (4.8), the right coideal generated by $u[k,m]$ is spanned by the elements $g_{ki}u[i+1,m]$. Hence, $\mathbf{U}(k,m)$, as an algebra, is generated over $\mathbf{k}[G]$ by the elements $u[i,m]$, $k\le i\le m$. Accordingly, the additive monoid of degrees of homogeneous elements from $\mathbf{U}(k,m)$ is generated by $[i:m]$, $k\le i\le m$. In this monoid, the indecomposable elements (by definition, they are simple $\mathbf{U}(k,m)$-roots) are precisely $[i:m]$, $k\le i\le m$, $i\ne\psi(m)$. The length of $[i:m]$ equals $m-i+1$. However, if $i>\psi(m)$, then the maximal letter among x_j, $i\le j\le m$, is $x_{\psi(m)}$ because $[i:m]=[\psi(m):\psi(i)]$, with $\psi(m)\le\psi(i)<\psi(\psi(m))$. Hence, the maximal length of a simple root starting with $\psi(m)$ equals $m-(\psi(m)+1)+1=2(m-n)-1$, while there are no simple roots of the form $[k':m']$, $k'\leq m'<\psi(k')$, with $k'>\psi(m)$. Thus because of Definition 8.1, we have

$$\theta_i=\begin{cases} m-i+1 & \text{if } k\leq i<\psi(m);\\ 2(m-n)-1 & \text{if } k\leq i=\psi(m)\leq n;\\ 0 & \text{otherwise.}\end{cases} \tag{9.1}$$

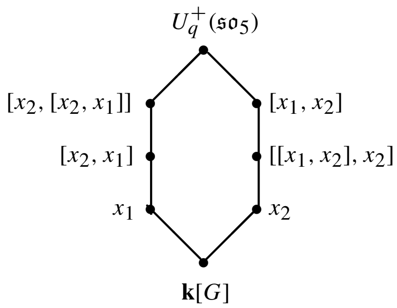
The set $\{u[i,m]\mid k\leq i\leq m,\ i\neq\psi(m)\}$ is a set of PBW-generators for $\mathbf{U}(k,m)$ over $\mathbf{k}[G]$.

Example 9.2. Let us analyse in detail the simplest (but not trivial [2]) case $n=2$. Consider the six elements $w_1=u[1,3]=[[x_1,x_2],x_2]$, $w_2=u[2,4]=[x_2,[x_2,x_1]]$, $w_3=u[1,2]=[x_1,x_2]$, $w_4=u[3,4]=[x_2,x_1]$, $w_5=x_1$, $w_6=x_2$. We denote by U_j, $1\leq j\leq 6$, the right coideal subalgebra generated by w_j and $\mathbf{k}[G]$.

By (9.1), we have $r(U_1)=(3,1)$. Indeed, in this case, $k=1$, $m=3$, $\psi(m)=2$; hence, $\theta_1=m-1+1=3$ according to the first option of (9.1), while $\theta_2=2(m-n)-1=1$ by the second option of (9.1).

In the same way, $r(U_2)=(3,0)$ because in this case $k=2$, $m=4$, $\psi(m)=1$; hence $\theta_1=2(m-2)-1=3$ according to the second option, while $\theta_2=0$ due to the third option.

In perfect analogy, we have $r(U_3)=(2,1)$, $r(U_4)=(2,0)$, $r(U_5)=(1,0)$, $r(U_6)=(0,1)$. We see that all six right coideal subalgebras are different. There are two more (improper) right coideal subalgebras $U_7=U_q^+(\mathfrak{so}_5)$, $U_8=\mathbf{k}[G]$ with the r-sequences $(1,1)$ and $(0,0)$ respectively. Thus, we have found all $(2n)!!=8$ possible right coideal subalgebras in $U_q^+(\mathfrak{so}_5)$ containing G, and they form the following lattice:



We note that in [17], B. Pogorelsky found a similar lattice for the quantum groups $U_q(\mathfrak{g})$, $u_q(\mathfrak{g})$, where $\mathfrak{g}$ is the simple Lie algebra of type G_2.

Our next goal is to generalise formula (9.1) to an arbitrary right coideal subalgebra $\mathbf{U}^S(k,m)$ generated over $\mathbf{k}[G]$ (as a right coideal subalgebra) by a single element $\Phi^S(k,m)$ with a (k,m)-regular set S.

Proposition 9.3. *If S is (k,m)-regular, then the coproduct of $\Phi^S(k,m)$ has a decomposition*

$$\Delta(\Phi^S(k,m)) = \sum a^{(1)} \otimes a^{(2)}, \tag{9.2}$$

where the degrees of the left components of tensors belong to the additive monoid Σ generated by all $[1+t:s]$ with t being a white point ($t=k-1$, or $t \notin S$, $k \leq t < m$) and s being a black point ($s \in S \cap [k, m-1]$, or $s = m$).

Proof. Let S be white (k,m)-regular. Lemma 7.4 shows that $\Phi^S(k,m)$ is a linear combination of products (in different orders) of $u_i = u[1+s_i, s_{i+1}]$, $0 \leq i \leq r$. Hence, by (4.8), the coproduct is a linear combination of products of the tensors

$$u_i \otimes 1, \quad f_i \otimes u_i, \quad h_i u[1+t_i, s_{i+1}] \otimes u[1+s_i, t_i], \tag{9.3}$$

where $s_i < t_i < s_{t+1}$, $f_i = \mathrm{gr}(u_i)$, $h_i = \mathrm{gr}(u[1+s_i, t_i])$. The degrees of the left components of these tensors, except $u_i \otimes 1$, $i > 0$, belong to Σ. We stress that in each product there is exactly one tensor of (9.3) related to a given i.

We denote by Σ' the additive monoid generated by all $[1+t:s]$, where $t \notin S$, $k \leq t < m$, while s is a black point. By induction on the number r of elements in $S \cap [k, m-1]$, we shall prove that there exists a decomposition (9.2) such that for each i either $D(a^{(1)}) \in \Sigma'$ or $D(a^{(1)}) = [k:s] + \alpha$, where s is a black point and $\alpha \in \Sigma'$.

If $r = 0$, then $\Phi^S(k,m) = u[k,m]$, and the statement follows from (4.8).

If $r > 0$, then Corollary 7.5 implies that $\Phi^S(k,m) \sim [\Phi^S(1+s_1, m), u[k, s_1]]$. By the inductive supposition, we have $\Delta(\Phi^S(1+s_1,m)) = \sum b^{(1)} \otimes b^{(2)}$, where either $D(b^{(1)}) = \alpha \in \Sigma_1'$ or $D(b^{(1)}) = [1+s_1:s] + \alpha$, $\alpha \in \Sigma_1'$, with s being a black point in the scheme of $\Phi^S(1+s_1, m)$; see (5.3). Here, Σ_1' is the Σ' related to $\Phi^S(1+s_1,m)$: the additive monoid generated by all $[1+t:s]$, where $t \notin S$, $s_1 < t < m$, and s is a black point. Certainly, $\Sigma_1' \subseteq \Sigma'$ because in the scheme of $\Phi^S(1+s_1,m)$, there is only one point, s_1, that has a colour different from the one it has in the scheme of $\Phi^S(k,m)$.

By (4.8), the coproduct of $u_0 = u[k, s_1]$ is a linear combination of the tensors (9.3) with $i = 0$. The degree of the left components of the tensors of

$$[b^{(1)} \otimes b^{(2)}, h_0 u[1+t_0, s_1] \otimes u[k, t_0]]$$

equals either $[1+t_0:s_1] + \alpha$ or $[1+t_0:s_1] + [1+s_1:s] + \alpha = [1+t_0:s] + \alpha$. In both cases, the degree belongs to Σ' because t_0 is a white point in both schemes, and $t_0 \neq k-1$.

In the same way, the degree of the left components of the tensors of $[b^{(1)} \otimes b^{(2)}, u_0 \otimes 1]$ equals either $[k:s_1] + \alpha$ or $[k:s_1] + [1+s_1:s] + \alpha = [k:s] + \alpha$. In both cases, the degree has the required form.

It remains to consider the skew commutator

$$[b^{(1)} \otimes b^{(2)}, f_0 \otimes u_0] = b^{(1)} f_0 \otimes b^{(2)} u_0 - p(b^{(1)} b^{(2)}, u_0) f_0 b^{(1)} \otimes u_0 b^{(2)}.$$

The degree of the left components of these tensors equals $D(b^{(1)})$. We shall prove that one of the following three options is valid: $[b^{(1)} \otimes b^{(2)}, f_0 \otimes u_0] = 0$, or $D(b^{(1)}) \in \Sigma'$, or $D(b^{(1)}) = [k:s] + \alpha$, $\alpha \in \Sigma'$ with s black.

The comments on (9.3) show that there exists a sequence of elements $(t_i \mid 0 \le i \le r)$ such that $s_i \le t_i \le s_{i+1}$, and

$$D(b^{(1)}) = \sum_{i=1}^{r}[1+t_i : s_{i+1}], \qquad D(b^{(2)}) = \sum_{i=1}^{r}[1+s_i : t_i], \tag{9.4}$$

where, formally, $[1+s_i : s_i] = [1+s_{i+1} : s_{i+1}] = 0$. We consider the following two cases separately.

Case 1. $t_1 > s_1$. Due to the first equality of (9.4), the degree of $b^{(1)}$ in x_{1+s_1} is less than or equal to 1. At the same time the equality $D(b^{(1)}) = [1+s_1 : s] + \alpha$ shows that this degree equals 1, and the x_{1+s_1}-th component of α is zero. Hence, there exists $i \ge 2$ such that $t_i < \psi(1+s_1) \le s_{i+1}$. However, $\psi(1+s_1) = \psi(s_1) - 1$ is a white point because S is white (k,m)-regular. In particular, $\psi(1+s_1) \ne s_{i+1}$; that is, $\psi(1+s_1) < s_{i+1}$. Now, the nonempty interval $[1+\psi(1+s_1) : s_{i+1}] = [\psi(s_1) : s_{i+1}]$ must be covered by $\alpha \in \Sigma_1'$. This is possible only if α has a summand $\alpha_1 = [\psi(s_1) : s_j]$, $j \ge i+1$, because the degree of $\Phi^S(1+s_1, m)$ in each x_l, $\psi(s_1) \le l \le m$, equals 1, while the $x_{\psi(s_1)-1}$-th component of α is zero (recall that $x_{\psi(s_1)-1} = x_{1+s_1}$). Thus, we have $\alpha - \alpha_1 \in \Sigma_1'$.

If $\psi(s_j) > k$, or equivalently $s_j < \psi(k)$, then $\psi(s_j) - 1$ is a white point because $\psi(1+s_1) < s_{i+1} \le s_j$ implies $s_1 > \psi(s_j) - 1$. We have

$$\alpha_1 + [1+s_1 : s] = [\psi(s_1) : s_j] + [1+s_1 : s] = [\psi(s_j) : s] \in \Sigma'.$$

Hence, $D(b^{(1)}) = (\alpha_1 + [1+s_1 : s]) + (\alpha - \alpha_1) \in \Sigma'$, as required.

If $\psi(s_j) < k$, or equivalently $s_j > \psi(k)$, then $\psi(k)$ is a white point (see (7.2)). Hence, $[\psi(s_j) : k-1] = [1+\psi(k) : s_j] \in \Sigma'$, while

$$\alpha_1 + [1+s_1 : s] = [\psi(s_j) : k-1] + [k : s] = [\psi(s_j) : s] \in [k : s] + \Sigma',$$

and $D(b^{(1)}) = (\alpha_1 + [1+s_1 : s]) + (\alpha - \alpha_1) \in [k : s] + \Sigma'$.

Of course, $s_j \ne \psi(k)$ because S is white (k,m)-regular (see (7.2)).

Case 2. $t_1 = s_1$. Assume first that the sequence $(t_i \mid 1 < i \le r)$ does not contain the point $\psi(s_1) - 1 = \psi(1+s_1)$. We have seen (see comments regarding (9.3)) that $b^{(2)}$ is the product of the elements $u[1+s_i, t_i]$, $i > 0$, in some order. For $i = 1$ the tensor $u_1 \otimes 1$ does enter the construction of $b^{(1)} \otimes b^{(2)}$ (recall that now $t_1 = s_1$). By Proposition 3.15 with $i \leftarrow s_1$, $j \leftarrow s_i$, $m \leftarrow t_i$ we have $[u[1+s_i, t_i], u_0] = 0$, $i > 1$, because now $t_i \ne \psi(s_1) - 1$ and $s_i \ne \psi(k)$ (see (7.2)). Hence, the ad-identity (2.10) implies $[b^{(2)}, u_0] = 0$; that is, $b^{(2)}u_0 = p(b^{(2)}, u_0)u_0 b^{(2)}$. Because $f_0 = \mathrm{gr}(u_0)$, we have

$$\begin{aligned}(b^{(1)} \otimes b^{(2)})(f_0 \otimes u_0) &= b^{(1)} f_0 \otimes b^{(2)} u_0 \\ &= p(b^{(1)}, u_0) f_0 b^{(1)} \otimes p(b^{(2)}, u_0) u_0 b^{(2)} = p(b^{(1)} b^{(2)}, u_0)(f_0 \otimes u_0)(b^{(1)} \otimes b^{(2)}).\end{aligned}$$

In more compact form, this equality is $[b^{(1)} \otimes b^{(2)}, f_0 \otimes u_0] = 0$, which is one of the desired options.

Suppose next that $\psi(s_1) - 1 = t_i$ for a suitable i, $1 < i \leq r$. By the first equality of (9.4) the degree of $b^{(1)}$ in $x_{1+s_i} = x_{t_i}$ equals 1, while the equality $D(b^{(1)}) = [1 + s_1 : s] + \alpha$ implies that the x_{1+s_1}-th component of α is zero. At the same time, $t_i \neq s_{i+1}$ because t_i and s_1 are in the same column of the shifted scheme (7.1), (7.2). Hence, again by the first equality of (9.4), the nonempty interval $[1 + t_i : s_{i+1}] = [\psi(s_1) : s_{i+1}]$ must be covered by $\alpha \in \Sigma'$. This is possible only if α has a summand $\alpha_1 = [\psi(s_1) : s_j]$, $j \geq i + 1$, because the degree of $\Phi^S(1 + s_1, m)$ in each x_l, $\psi(s_1) \leq l \leq m$, equals 1, while the $x_{\psi(s_1)-1}$-th component of α is zero (recall that $x_{\psi(s_1)-1} = x_{1+s_1}$). Thus, we have $\alpha - \alpha_1 \in \Sigma'_1$.

If $\psi(s_j) > k$, or equivalently $s_j < \psi(k)$, then $\psi(s_j) - 1$ is a white point because $\psi(1 + s_1) < s_{i+1} \leq s_j$ implies $s_1 > \psi(s_j) - 1$. We now have

$$\alpha_1 + [1 + s_1 : s] = [\psi(s_1) : s_j] + [1 + s_1 : s] = [\psi(s_j) : s] \in \Sigma'.$$

Hence, $D(b^{(1)}) = (\alpha_1 + [1 + s_1 : s]) + (\alpha - \alpha_1) \in \Sigma'$, as required.

If $\psi(s_j) < k$, or equivalently $s_j > \psi(k)$, then $\psi(k)$ is a white point (see (7.2)). Hence $[\psi(s_j) : k - 1] = [1 + \psi(k) : s_j] \in \Sigma'$, while

$$\alpha_1 + [1 + s_1 : s] = [\psi(s_j) : k - 1] + [k : s] = [\psi(s_j) : s] \in [k : s] + \Sigma',$$

and $D(b^{(1)}) = (\alpha_1 + [1 + s_1 : s]) + (\alpha - \alpha_1) \in [k : s] + \Sigma'$. Of course $s_j \neq \psi(k)$ because S is white (k, m)-regular (see (7.2)). The proof for a white regular set S is completed.

If S is black (k, m)-regular, then by Proposition 7.10 we have $\Phi^S(k, m) \sim \Phi^T(\psi(m), \psi(k))$, where $T = \overline{\psi(S) - 1}$ is a white $(\psi(m), \psi(k))$-regular set. If t, s are, respectively, white and black points for $\Phi^S(k, m)$, then so are $\psi(s) - 1$ and $\psi(t) - 1$ with respect to $\Phi^T(\psi(m), \psi(k))$. We have

$$[1 + t : s] = [\psi(s) : \psi(1 + t)] = [1 + (\psi(s) - 1) : \psi(t) - 1].$$

Hence, $\Phi^S(k, m)$ and $\Phi^T(\psi(m), \psi(k))$ define the same additive monoid Σ. It remains to apply the already proven statement to $\Phi^T(\psi(m), \psi(k))$. □

Corollary 9.4. *If S is (k, m)-regular, then all $\mathbf{U}^S(k, m)$-roots belong to the monoid Σ defined in the above proposition.*

Proof. We recall that the coassociativity of the coproduct implies that the left components of the tensor (9.2) span a right coideal. Hence, $\mathbf{U}^S(k, m)$ as an algebra is generated by the $a^{(1)}$'s and by $\mathbf{k}[G]$. Hence, the degrees of all homogeneous elements from $\mathbf{U}^S(k, m)$ belong to Σ. In particular, all $\mathbf{U}^S(k, m)$-roots, being the degrees of PBW-generators, belong to Σ as well. □

Lemma 9.5. *Let S be a white (k, m)-regular set. An element $[1 + t : s]$, $t < s$, with t white and s black is indecomposable in Σ if and only if one of the following conditions is fulfilled:*

(a) *$\psi(1 + t)$ is not black (it is white or does not appear in the scheme at all).*
(b) *In the shifted scheme, all columns between t and s are white-black or black-white (in particular, all are complete and $n \notin [t, s]$).*

Proof. If none of the conditions is fulfilled, then $\psi(1+t)$ is a black point, and there exists j, $t \le j \le s$, such that both j and $\psi(1+j)$ are white points in the scheme (the white regular shifted scheme has no black-black columns). Certainly, $j \neq t$, $j \neq s$. We have

$$[1+t:j] = [\psi(j):\psi(1+t)] = [1+\psi(1+j):\psi(1+t)] \in \Sigma.$$

Thus, $[1+t:s] = [1+t:j] + [1+j:s]$ is a nontrivial decomposition in Σ.

Conversely, assume that $[1+t:s]$ is decomposable in Σ:

$$[1+t:s] = \sum_{i=1}^{r} [1+l_i:s_i]. \tag{9.5}$$

Without loss of generality we may suppose that $s_i \le \psi(1+l_i)$ since $[1+l_i:s_i] = [\psi(s_i):\psi(1+l_i)]$. Moreover, if $s_i = \psi(1+l_i)$, then $[1+l_i:n] = [1+n:s_i] \in \Sigma$ because n is a white point (S is white regular). This equality allows one to replace $[1+l_i:s_i]$ with $2[1+n:s_i]$ in (9.5). Thus, we may suppose that $s_i \le \psi(1+l_i)$ for all i in (9.5).

By Lemma 8.6, we find a sequence $t = t_0 < t_1 < \cdots < t_r < s = t_{r+1}$ such that for each i, either t_i is white and t_{i+1} black, or $\psi(1+t_{i+1})$ is white and $\psi(1+t_i)$ black. In the former case, we associate the sign "+" to the index i, while in the latter case we mark it "−". It is clear that in the sequence of indices $0, 1, 2, \ldots, r$, no pair of neighbours have the same sign.

Now, if $\psi(1+t)$ is not a black point, then 0 is marked "+". Hence, 1 is marked "−". In particular, $\psi(1+t_1)$ is black point. However, t_1 is also black. This combination is impossible because S is white regular.

Assume that in the shifted scheme, all columns between t and s are white-black or black-white. If t_1 is a white point, then both $t_0 = t$ and t_1 are white, while both $\psi(1+t_1)$ and $\psi(1+t_0)$ are black; that is, no sign can be associated to index 0. Hence, t_1 is a black point, while $\psi(1+t_1)$ must be white. In this case, 1 cannot be marked "−", so it is marked "+". However t_1 is then a white point, which is a contradiction. □

Lemma 9.6. *Let S be a black (k, m)-regular set. An element $[1+t:s]$, $t < s$, with t white and s black is indecomposable in Σ if and only if one of the following conditions is fulfilled:*

(a) *$\psi(1+s)$ is not white (it is black or does not appear in the scheme at all).*
(b) *In the shifted scheme, all columns between t and s are white-black or black-white (in particular, all are complete and $n \notin [t, s]$).*

Proof. This follows from Lemma 9.5 by means of Lemma 7.9 and Proposition 7.10. □

Lemma 9.7. *Let S be a (k, m)-regular set. An element $\alpha = [a:b]$ is a simple $\mathbf{U}^S(k, m)$-root if and only if $\alpha \in \Sigma$ and it is indecomposable in Σ (in particular $\alpha = [1+t:s]$, $t < s$, with t white and s black determined in Lemmas 9.5, 9.6).*

Proof. Without loss of generality, we may suppose that $k \le m < \psi(k)$ due to Proposition 7.10. We have already mentioned that all $\mathbf{U}^S(k, m)$-roots belong to Σ (see Corollary 9.4).

Certainly, $[k:m]$ is a $\mathbf{U}^S(k,m)$-root, for $\Phi^S(k,m)\in\mathbf{U}^S(k,m)$. Because $\psi(k-1)-1=\psi(k)>m$, the point $\psi(k-1)-1$ does not appear in the scheme of $\Phi^S(k,m)$. If S is black (k,m)-regular, then $\psi(m)-1$ is a black point (see (7.3)). Hence, Lemmas 9.5 and 9.6 show that, in both cases, $[k:m]$ is indecomposable in Σ. Thus, $[k:m]$ is a simple $\mathbf{U}^S(k,m)$-root.

If s is a black point, then $[1+s:m]\notin\Sigma$ (otherwise $[k:m]$ would be decomposable in Σ). In particular, $[1+s\ :\ m]$ is not a sum of $\mathbf{U}^S(k,m)$-roots. By Lemma 8.8, the element $[k:s]$ is an $\mathbf{U}^S(k,m)$-root (in particular, Lemma 8.5 implies that S equals the minimal set S' such that $\Phi^{S'}(k,m)\in\mathbf{U}^S(k,m)$). If additionally $[k:s]$ is indecomposable in Σ, then it is a simple $\mathbf{U}^S(k,m)$-root.

If t,s are, respectively, white and black points, $k\le t<s$, then by Lemma 8.4, we have $\Phi^{S''}(k,s)\in\mathbf{U}^S(k,m)$ for a suitable (minimal) set $S''\subseteq S$. Because t is still a white point for $\Phi^{S''}(k,s)$, the same lemma applied to $\Phi^{S''}(k,s)$ implies $\Phi^{S''}(1+t,s)\in\mathbf{U}^S(k,m)$.

Let α be indecomposable in Σ. Because by definition, Σ is an additive monoid generated by elements of the form $[1+t:s]$ with t white and s black, all indecomposable elements have a similar form: $\alpha=[1+t:s]$. First, if $[1+t:s]$ has property (b) of Lemma 9.5 or Lemma 9.6, then $n\notin[t,s]$. Hence S'' (as well as any other set) is white and black $(1+t,s)$-regular. By Corollary 7.12 we have $\Phi^{S''}(1+t,s)\neq 0$, hence $[1+t:s]$ is a $\mathbf{U}^S(k,m)$-root. This root is simple because it is indecomposable in Σ.

Next, if $[1+t:s]$ has property (a) of Lemma 9.5 or Lemma 9.6, then so does $[k:s]$; that is, $[k:s]$ is indecomposable in Σ. In particular $[k:t]\notin\Sigma$, and hence $[k:t]$ is not a $\mathbf{U}^S(k,m)$-root. By the application of Lemma 8.8 to the simple $\mathbf{U}^S(k,m)$-root $[k:s]$, we see that $[1+j:s]$ is a sum of $\mathbf{U}^S(k,m)$-roots. Because $[1+j:s]$ is indecomposable in Σ and all roots belong to Σ, the sum has just one summand; that is, $[1+j:s]$ is a simple $\mathbf{U}^S(k,m)$-root.

Conversely, if α is a simple $\mathbf{U}^S(k,m)$-root, then by Corollary 9.4, we have $\alpha\in\Sigma$. In particular, α is a sum of elements indecomposable in Σ. However, we have already proved that each element indecomposable in Σ is a $\mathbf{U}^S(k,m)$-root. Thus, the sum has only one summand; that is, α is indecomposable in Σ. □

Theorem 9.8. *Let S be a white [black] (k,m)-regular set. The right coideal subalgebra $\mathbf{U}^S(k,m)$ coincides with the subalgebra $\mathfrak{A}$ generated over $\mathbf{k}[G]$ by all elements $\Phi^S(1+t,s)$, where $t<s$ are, respectively, white and black points that satisfy one of the conditions of Lemma 9.5 [Lemma 9.6].*

Proof. Of course, we should show that $\Phi^S(1+t,s)\in\mathbf{U}^S(k,m)$. First, let us suppose that $s<\psi(1+t)$. We denote by S' a minimal set such that $\Phi^{S'}(1+t,s)\in\mathbf{U}^S(k,m)$ (see Lemmas 8.5, 9.7).

If $a\in S\cap[1+t,s-1]$, then, by definition, $[1+t:a]\in\Sigma$. Hence, $[1+t:a]$ is a sum of $\mathbf{U}^S(k,m)$-roots. Lemma 8.7 applied to $[1+t:s]$ shows that $[1+t:a]$ itself is a $\mathbf{U}^S(k,m)$-root (note that $a\neq\psi(1+t)$ because $a<s<\psi(1+t)$). Thus, Lemma 8.5 applied to $[1+t:s]$ shows that $a\in S'$; that is, $S\cap[1+t,s-1]\subseteq S'$.

If $b\in S'$, then by Lemma 8.5 applied to $[1+t:s]$, the element $[1+t:b]$ is a $\mathbf{U}^S(k,m)$-root. In particular, $[1+t:b]\in\Sigma$. If $b\notin S$, then by definition, $[1+b:s]\in\Sigma$, and we get a contradiction $[1+t:s] = [1+t:b] + [1+b:s]$. Thus, $b \in S$; that is, $S' = S \cap [1+t, s-1]$, and $\Phi^S(1+t, s) = \Phi^{S'}(1+t, s) \in \mathbf{U}^S(k, m)$.

If $s > \psi(1+t)$, then by Proposition 7.10, we have $\Phi^S(1+t, s) \sim \Phi^T(\psi(s), \psi(1+t))$. Certainly $\psi(1+t) < \psi(\psi(s))$. Therefore, we may apply the case already considered: $\Phi^T(\psi(s), \psi(1+t)) \in \mathbf{U}^T(\psi(m), \psi(k)) = \mathbf{U}^S(k, m)$.

If $[a:b]$ is a nonsimple $\mathbf{U}^S(k, m)$-root, then it has a decomposition into a sum of simple roots of the form $[1+t:s]$. The element c defined in each of the formulae (8.3)–(8.6) belongs to the subalgebra $\mathfrak{A}$ generated by all $\Phi^S(1+t, s)$. Hence, $\mathbf{U}^S(k, m)$ has PBW-generators from $\mathfrak{A}$; that is, $\mathbf{U}^S(k, m) = \mathfrak{A}$. □

The theorem just proved allows one to easily find the root sequence for $\mathbf{U}^S(k, m)$ with regular S. By Corollary 7.11, it suffices to consider the case $k \leq m < \psi(k)$.

Proposition 9.9. *Let S be a white (k, m)-regular set, $k \leq m < \psi(k)$. The root sequence $(\theta_i, 1 \leq i \leq n)$ for $\mathbf{U}^S(k, m)$ has the following form in terms of the shifted scheme of $\Phi^S(k, m)$:*

$$\theta_i = \begin{cases} 0 & \text{if } i-1 \text{ is not white;} \\ \psi(i) - a_i & \text{if } i-1 \text{ is white and } \psi(i) \text{ is black;} \\ b_i - i + 1 & \text{if } i-1 \text{ is white and } \psi(i) \text{ is not black,} \end{cases} \tag{9.6}$$

where a_i is the minimal number such that $(a_i, \psi(a_i) - 1)$ is a white-white column, while b_i, $i \leq b_i < \psi(i)$, is the maximal black point, if any; otherwise, $b_i = i - 1$ (hence $\theta_i = b_i - i + 1 = 0$).

Proof. An element $\alpha = [1+t:s]$ given in Lemma 9.7 defines a simple $\mathbf{U}^S(k, m)$-root starting with i if either $i = 1+t$ & $s < \psi(1+t)$ or $s = \psi(i)$ & $s > \psi(1+t)$.

If $i-1$ is not a white point, then of course $i \neq 1+t$; hence $s = \psi(i)$. The column $(s, i-1) = (\psi(i), i-1)$ is not black-black because S is white-regular, and therefore it is incomplete; that is, $t = i-1$ does not appear in the scheme, which is a contradiction. Thus, there are no simple $\mathbf{U}^S(k, m)$-roots starting with i, and $\theta_i = 0$.

Assume $i-1$ is white and $\psi(i)$ is black. In this case, $[1+n:\psi(i)]$ satisfies condition (a) of Lemma 9.5. Hence, $[i:n] = [\psi(n):\psi(i)] = [1+n:\psi(i)]$ is a simple $\mathbf{U}^S(k, m)$-root starting with i. In particular, $\theta_i > n - i$.

If $i = 1+t$, $s < \psi(1+t)$, then $[1+t:s]$ does not satisfy condition (a) of Lemma 9.5 because $\psi(1+t) = \psi(i)$ is black. If $[1+t:s]$ satisfies condition (b), then its length is less than $n - i$.

If $s = \psi(i)$, $s > \psi(1+t)$, then $[1+t:s]$ satisfies condition (a) of Lemma 9.5 if and only if $(t, \psi(t+1))$ is a white-white column. In this case, its length equals $s-(1+t)+1 = \psi(i) - t$. This length has the maximal value if t is minimal: $t = a_i$.

Assume $i-1$ is white and $\psi(i)$ is not black. In this case, $s \neq \psi(i)$. Hence, $i = 1+t$, and s is a black point such that $s < \psi(1+t) = \psi(i)$. The length of $[1+t:s]$ equals $s - t = s - i + 1$. This is maximal if s is the maximal black point such that $i \leq s < \psi(i)$; that is, $s = b_i$. If all points in the interval $[i, \psi(i) - 1]$ are white, then there are no simple $\mathbf{U}^S(k, m)$-roots starting with i. Hence, we still have $\theta_i = b_i - i + 1 = 0$. □

Proposition 9.10. *Let S be a black (k, m)-regular set, $k \le m < \psi(k)$. The root sequence $(\theta_i, 1 \le i \le n)$ for $\mathbf{U}^S(k, m)$ has the following form in terms of the shifted scheme of $\Phi^S(k, m)$:*

$$\theta_i = \begin{cases} 0 & \text{if } i-1 \text{ is not white and } \psi(i) \text{ is not black;} \\ \psi(i) - d_i & \text{if } i-1 \text{ is not white and } \psi(i) \text{ is black;} \\ \psi(i) - c_i & \text{if } i-1 \text{ is white,} \end{cases} \tag{9.7}$$

where c_i is the minimal number such that $(c_i, \psi(c_i) - 1)$ is a black-black column, while d_i, $i \le d_i < \psi(i)$, is the minimal white point, if any; otherwise $d_i = \psi(i)$ (hence $\theta_i = \psi(i) - d_i = 0$).

Proof. This follows from Lemma 9.6 just as the above proposition follows from Lemma 9.5. □

Example 9.11. Consider the right coideal subalgebra $U(w)$ generated over $\mathbf{k}[G]$ by the element $w = [[x_3, [x_3x_2x_1]], x_2]$ with $n = 3$ (recall that the value of $[x_3x_2x_1]$ in $U_q^+(\mathfrak{so}_7)$ is independent of the arrangement of brackets; see (2.8)). By definition (3.16), we have $[x_3, [x_3, [x_2, x_1]]] = u[3, 6]$, while Lemma 7.4 implies $[u[3, 6], x_2] \sim \Phi^{\{2\}}(2, 6)$. Here, $\{2\}$ is a white $(2, 6)$-regular set; however, $6 > \psi(2) = 5$. By Proposition 7.10, we have $\Phi^{\{2\}}(2, 6) \sim \Phi^{\{1,2,3\}}(1, 5)$. Because $5 < \psi(1) = 6$ and $\{1, 2, 3\}$ is a black $(1, 5)$-regular set, to find the root sequence for $U(w) = \mathbf{U}^{\{1,2,3\}}(1, 5)$, we may apply Proposition 9.10. The shifted scheme

$$\begin{array}{ccccc} & \overset{5}{\bullet} & \overset{4}{\circ} & \overset{3}{\bullet} & \Leftarrow \\ \overset{0}{\circ} & \overset{1}{\bullet} & \overset{2}{\bullet} & \overset{3}{\bullet} & \end{array} \tag{9.8}$$

shows that $c_1 = 1$, $c_2 = 3$, $c_3 = 3$, while $d_1 = 4$, $d_2 = 4$, $d_3 = \psi(3) = 4$. If $i = 1$, then $i - 1 = 0$ is a white point, and by the third option of (9.7) we have $\theta_1 = \psi(1) - c_1 = 5$. If $i = 2$, then $i - 1 = 1$ and $\psi(i) = 5$ are black points. Hence, the second option of (9.7) applies: $\theta_2 = \psi(2) - d_2 = 5 - 4 = 1$. If $i = 3$, then $i - 1 = 2$ is a black point, while $\psi(i) = 4$ is white; that is, according to the first option of (9.7) we have $\theta_3 = 0$. Thus, $\theta(U(w)) = (5, 1, 0)$.

10. Construction

Our next goal is to construct a right coideal subalgebra with a given root sequence

$$\theta = (\theta_1, \dots, \theta_n) \quad \text{such that} \quad 0 \le \theta_k \le 2n - 2k + 1, \ 1 \le k \le n. \tag{10.1}$$

We shall require the following auxiliary objects.

Definition 10.1. By downward induction on k, we define sets $R_k \subseteq [k, \psi(k) - 1]$, $T_k \subseteq [k, \psi(k)]$, $1 \le k \le 2n$, associated to a given sequence (10.1) as follows.

For $k > n$ we put $R_k = T_k = \emptyset$.

Suppose that R_i, T_i, $k < i \leq 2n$, are already defined. Let $\mathbf{P}$ be the following binary predicate on the set of all ordered pairs $i \leq j$:

$$\mathbf{P}(i, j) \rightleftharpoons j \in T_i \vee \psi(i) \in T_{\psi(j)}. \tag{10.2}$$

Of course, for the time being the predicate is defined only on pairs (i, j) such that $k < i \leq j < \psi(k)$. We note that $\mathbf{P}(i, j) = \mathbf{P}(\psi(j), \psi(i))$. Also, it is useful to note that for given i and j one of the conditions $j \in T_i$ or $\psi(i) \in T_{\psi(j)}$ is false because $T_s \subseteq [s, \psi(s)]$ for all s, and $T_s = \emptyset$ for $s > n$, except for $j = \psi(i)$ when these conditions coincide. In particular, we see that if $j < \psi(i)$, then $\mathbf{P}(i, j)$ is equivalent to $j \in T_i$.

If $\theta_k = 0$, then we set $R_k = T_k = \emptyset$. If $\theta_k \neq 0$, then by definition, R_k contains $\tilde{\theta}_k = k + \theta_k - 1$ and all m satisfying the following three conditions:

$$\begin{aligned}
&\text{(a) } k \leq m < \tilde{\theta}_k;\\
&\text{(b) } \neg\mathbf{P}\,(m+1, \tilde{\theta}_k);\\
&\text{(c) } \forall r\ (k \leq r < m)\ \ \mathbf{P}\,(r+1, m) \Leftrightarrow \mathbf{P}\,(r+1, \tilde{\theta}_k).
\end{aligned} \tag{10.3}$$

Further, we define an auxiliary set

$$T_k' = R_k \cup \bigcup_{s \in R_k} \{a \mid s < a < \psi(k),\, \mathbf{P}(s+1, a)\}, \tag{10.4}$$

and finally,

$$T_k = \begin{cases} T_k' & \text{if } \psi(R_k + 1) \cap T_k' = \emptyset;\\ T_k' \cup \{\psi(k)\} & \text{otherwise.}\end{cases} \tag{10.5}$$

For example, the first step of the construction is as follows. If $\theta_n = 0$, we certainly have $R_n = T_n = \emptyset$. Because by definition $\theta_n \leq 2n - 2n + 1 = 1$, there exists only one additional option $\theta_n = 1$. In this case $\tilde{\theta}_n = n$ and $R_n = \{n\}$, while $T_n' = R_n$. We have $\psi(R_n + 1) \cap T_n' = \{n\} \neq \emptyset$. Hence, $T_n = \{n, \psi(n)\} = \{n, n+1\}$.

Example 10.2. Assume $n = 3$, $\theta = (5, 1, 0)$. Because $\theta_3 = 0$, by definition we have $R_k = T_k = \emptyset$, $k \geq 3$.

Let $k = 2$. We have $\theta_2 = 1 \neq 0$; hence, $\tilde{\theta}_2 = 2 + \theta_2 - 1 = 2 \in R_2$. Certainly, there are no points m that satisfy $k = 2 \leq m < \tilde{\theta}_2 = 2$; that is, $R_2 = \{2\}$. Now (10.4) yields

$$T_2' = \{2\} \cup \bigcup_{s \in \{2\}} \{a \mid 2 = s < a < \psi(2) = 5,\ \mathbf{P}(3, a)\} = \{2\}.$$

We have $\psi(R_2 + 1) \cap T_2' = \{4\} \cap \{2\} = \emptyset$, hence $T_2 = \{2\}$.

To find R_1, it is convenient to tabulate the already known values of the predicate $\mathbf{P}$.

Values of $\mathbf{P}$

$i \setminus j$	2	3	4	5
2	T	F	F	F
3		F	F	F
4			F	F
5				T

We have $\theta_1 \neq 0$; that is, $\tilde{\theta}_1 = 1 + 5 - 1 = 5 \in R_1$.

There exist four points m that satisfy $k = 1 \leq m < \tilde{\theta}_1 = 5$; they are 1, 2, 3, and 4. Point $m = 4$ does not satisfy (b) because $\mathbf{P}(5, 5)$ is true. Hence, $4 \notin R_1$. Points $m = 1, 2$, and 3 satisfy (b) because in the last column of the table, there is only one "T"; this corresponds to $m + 1 = 5$.

Let us check condition (c) for $m = 1$. The interval $1 = k \leq r < m = 1$ is empty. Therefore, the equivalence (c) is true (elements of the empty set satisfy all conditions, even $r \neq r$). Thus, $1 \in R_1$.

In terms of the table of the values of $\mathbf{P}$, condition (c) means that the column corresponding to $j = m$ equals a subcolumn corresponding to $j = \tilde{\theta}_1 = 5$. This is indeed the case for $m = 3$, but not for $m = 2$. Thus $R_1 = \{1, 3, 5\}$.

To find T_1' we only need to check the two remaining points: $a = 2, 4$. From the table, we see that $\mathbf{P}(x, 4)$ is always false; hence, $4 \notin T_1'$. At the same time, $\mathbf{P}(s + 1, 2)$ is true for $s = 1 \in R_1$. Hence, $2 \in T_1'$.

Finally, $\psi(R_1 + 1) \cap T_1' = \{5, 3, 1\} \cap \{1, 2, 3, 5\} \neq \emptyset$; hence, $T_1 = \{1, 2, 3, 5, 6\}$.

Thus, for $\theta = (5, 1, 0)$ we have $R_3 = T_3 = \emptyset$, $R_2 = T_2 = \{2\}$, $R_1 = \{1, 3, 5\}$, and $T_1 = \{1, 2, 3, 5, 6\}$.

Theorem 10.3. *For each sequence $\theta = (\theta_1, \ldots, \theta_n)$ such that*

$$0 \leq \theta_k \leq 2n - 2k + 1, \qquad 1 \leq k \leq n,$$

there exists a homogeneous right coideal subalgebra $\mathbf{U} \supseteq \mathbf{k}[G]$ with $r(\mathbf{U}) = \theta$. In what follows, we shall denote this subalgebra by $\mathbf{U}_\theta$.

Proof. We denote by $\mathbf{U}$ the subalgebra generated over $\mathbf{k}[G]$ by the values in $U_q^+(\mathfrak{so}_{2n+1})$ or in $u_q^+(\mathfrak{so}_{2n+1})$ of the elements

$$\Phi^S(k, m), \quad 1 \leq k \leq m, \quad \text{with } m \in R_k,\ S = T_k. \tag{10.6}$$

(For example, if $\theta = (5, 1, 0)$, then the generators are x_1, x_2, $[x_3x_2x_1]$, $[[x_3[x_3x_2x_1]], x_2]$.) We shall prove that $\mathbf{U}$ is a right coideal subalgebra with $r(\mathbf{U}) = \theta$. To this end, we need to check some properties of R_k, T_k, and $\mathbf{P}$.

Claim 1. $\mathbf{P}(k, m)$ *is true if and only if there exists a sequence*

$$k - 1 = k_0 < k_1 < \cdots < k_r < m = k_{r+1} \tag{10.7}$$

such that for each i, $0 \leq i \leq r$, either $k_{i+1} \in R_{1+k_i}$ or $\psi(1 + k_i) \in R_{\psi(k_{i+1})}$.

We use induction on $m - k$. If $m = k$, then the condition $k \in T_k$ is equivalent to $k \in T_k'$ because $k \neq \psi(k)$. Formula (10.4) implies, in turn, that $k \in T_k'$ is equivalent to $k \in R_k$. Thus, $\mathbf{P}(k, k)$ is equivalent to $k \in R_k \vee \psi(k) \in R_{\psi(k)}$; that is, we have a sequence (10.7) with $r = 0$.

Assume first $m \in T_k$. If $m \in R_k$, we put $k_1 = m + 1$, $r = 1$.

If $m \notin R_k$, $m \neq \psi(k)$, then by definition $m \in T_k'$; that is, by (10.4) there exists $s \in R_k$, $s < m$, such that $\mathbf{P}(s + 1, m)$ is true. By the inductive supposition applied to

$(s+1,m)$, there exists a sequence (10.7) with $k_0=s$. One may extend it on the left with $k-1<k<s$ as $s\in R_k$.

If $m=\psi(k)$, then by definition, $\psi(s_1+1)\in T_k'$ for a suitable $s_1\in R_k$. Of course, $\mathbf{P}(k,\psi(s_1+1))$ is true. Hence, the above case with $m\leftarrow\psi(s_1+1)$ yields a sequence (10.7) with $k_{r+1}=\psi(s_1+1)$. We may extend it on the right with $\psi(s_1+1)<\psi(k)=m$ because $s_1=\psi(1+\psi(s_1+1))\in R_{\psi(\psi(k))}=R_k$.

Next, we assume $\psi(k)\in T_{\psi(m)}$. Because $\psi(k)-\psi(m)=m-k$, we may apply the above case with $k\leftarrow\psi(m)$, $m\leftarrow\psi(k)$. Hence, there exists a sequence (10.7) with $k_0=\psi(m)-1$, $k_{r+1}=\psi(k)$. Let us denote $k_i'=\psi(k_i)-1$, $0\le i\le r+1$. We have

$$k-1=k_{r+1}'<k_r'<\cdots<k_1'<k_0'=m. \tag{10.8}$$

In this case, $k_i'\in R_{1+k_{i+1}'}$ is equivalent to $\psi(1+k_i)\in R_{\psi(k_{i+1})}$, while $\psi(1+k_{i+1}')\in R_{\psi(k_i')}$ is equivalent to $k_{i+1}\in R_{1+k_i}$.

Conversely, suppose that we have a sequence (10.7). Without loss of generality, we may suppose that $m\le\psi(k)$; otherwise, we turn to (10.8). The inductive supposition implies that $\mathbf{P}(1+k_1,m)$ is true. Moreover, $k_1\in R_k$. Indeed, otherwise $\psi(k)\in R_{\psi(k_1)}\subseteq[\psi(k_1),k_1-1]$. In particular $\psi(k)<k_1$, and hence $k>\psi(k_1)$. However, $k_1\le m\le\psi(k)$ implies $\psi(k_1)\ge k$. Now, if $m\ne\psi(k)$, then definition (10.4) with $s\leftarrow k_1$, $a\leftarrow m$ implies $m\in T_k'$.

Let $m=\psi(k)$. In this case, considering the sequence (10.8) as above, we have $k_r'\in R_k$. By definition, $k_r'=\psi(k_r)-1$. Hence, $k_r\in\psi(R_k+1)$. At the same time, definition (10.4) shows that $k_r\in T_k'$ because the inductive supposition implies that $\mathbf{P}(1+k_1,k_r)$ is true provided that $r>1$, while if $r=1$, then $k_r=k_1\in R_k$. Thus, definition (10.5) implies $m=\psi(k)\in T_k$.

Claim 2. *If* $\mathbf{P}(k,s)$ *and* $\mathbf{P}(s+1,m)$, *then* $\mathbf{P}(k,m)$.

Indeed, one may extend the sequence (10.7) corresponding to the pair (k,s) by the sequence corresponding to $(s+1,m)$.

Claim 3. *If* $\mathbf{P}(k,m)$, *then for each* s, $k\le s<m$, *either* $\mathbf{P}(k,s)$ *or* $\mathbf{P}(s+1,m)$.

We use induction on $m-k$. Without loss of generality, we may suppose that $m\le\psi(k)$ because $\mathbf{P}(k,m)$ is equivalent to $\mathbf{P}(\psi(m),\psi(k))$. By Claim 1, there exists a sequence (10.7) with $k_0=k-1$, $k_{r+1}=m$. The same claim implies $\mathbf{P}(1+k_1,m)$ provided that $r\ge 1$.

Because $k\le s<m$, there exists i, $1\le i\le r$, such that $k_i<s\le k_{i+1}$. If $i\ge 1$, then the inductive supposition applied to $(1+k_1,m)$ implies that either $\mathbf{P}(1+k_1,s)$ or $\mathbf{P}(s+1,m)$ holds. In the latter case, we have obtained the required condition. If $\mathbf{P}(1+k_1,s)$ is true, then Claim 2 implies $\mathbf{P}(k,s)$ because $\mathbf{P}(k,k_1)$ is true according to Claim 1.

Thus, it remains to check the case $i=0$; that is, $k\le s\le k_1$. In this case, $k_1\in R_k$. Indeed, otherwise $\psi(k)\in R_{\psi(k_1)}\subseteq[\psi(k_1),k_1-1]$. In particular, $\psi(k)<k_1$, and hence $k>\psi(k_1)$. However, $k_1\le m\le\psi(k)$ implies $\psi(k_1)\ge k$.

Claim 2 with $s\leftarrow 1+k_1$, $k\leftarrow s+1$ states that $\mathbf{P}(s+1,k_1)$ and $\mathbf{P}(1+k_1,m)$ imply $\mathbf{P}(s+1,m)$. Hence, it is sufficient to show that either $\mathbf{P}(k,s)$ or $\mathbf{P}(s+1,k_1)$ is true. If

$s=k_1$, then of course $s=k_1\in R_k$ yields $\mathbf{P}(k,s)$. Therefore, we may replace m with k_1 and suppose further that $m\in R_k$, $i=0$. In this case, condition (10.3)(c) with $r\leftarrow s$ is "$\mathbf{P}(s+1,m)\Leftrightarrow\mathbf{P}(s+1,\tilde{\theta}_k)$." Therefore we only need to consider one case, $m=\tilde{\theta}_k$.

Let us suppose that for some s, $k\le s<\tilde{\theta}_k$, we have $\neg\mathbf{P}(k,s)$ and $\neg\mathbf{P}(s+1,\tilde{\theta}_k)$. By induction on s, in addition to the induction on $m-k$, we shall show that these conditions are inconsistent (more precisely, they imply $s\in R_k$, which contradicts $\neg\mathbf{P}(k,s)$; see definition (10.4)).

Definition (10.3) with $m=k$ shows that $k\in R_k$ if and only if $\neg\mathbf{P}(k+1,\tilde{\theta}_k)$. Since in our case, $\neg\mathbf{P}(s+1,\tilde{\theta}_k)$, we have $s\in R_k$ provided that $s=k$.

Let $s>k$. Conditions (10.3)(a) and (10.3)(b) with $m\leftarrow s$ are valid. Suppose that (10.3)(c) fails. In this case, we may find a number t, $k\le t<s$, such that $\neg(\mathbf{P}(t+1,s)\Leftrightarrow\mathbf{P}(t+1,\tilde{\theta}_k))$.

If $\mathbf{P}(t+1,s)$, but $\neg\mathbf{P}(t+1,\tilde{\theta}_k)$, then by the inductive supposition (induction on s), either $\mathbf{P}(k,t)$ or $\mathbf{P}(t+1,\tilde{\theta}_k)$; that is, $\mathbf{P}(k,t)$ is true. Claim 2 implies $\mathbf{P}(k,s)$, which is a contradiction.

If $\mathbf{P}(t+1,\tilde{\theta}_k)$, but $\neg\mathbf{P}(t+1,s)$, then the inductive supposition (of the induction on $m-k$) with $k\leftarrow t+1$, $m\leftarrow\tilde{\theta}_k$ shows that either $\mathbf{P}(t+1,s)$ or $\mathbf{P}(s+1,\tilde{\theta}_k)$; that is, $\mathbf{P}(s+1,\tilde{\theta}_k)$, which is again a contradiction.

Thus, s satisfies all conditions (10.3)(a)–10.3(c); hence, $s\in R_k$.

Claim 4. *If $k\le m<\tilde{\theta}_k$, then $m\in T_k$ if and only if $\neg\mathbf{P}(m+1,\tilde{\theta}_k)$.*

First, recall that condition $m\in T_k$ is equivalent to $\mathbf{P}(k,m)$ because by definition, $\tilde{\theta}_k<\psi(k)$.

According to Claim 3, one of the conditions $\mathbf{P}(k,m)$ or $\mathbf{P}(m+1,\tilde{\theta}_k)$ always holds. If both conditions are valid, then, because of Claim 1, we find a sequence (10.7) with $k_0=k-1$, $k_{r+1}=m$, such that $k_{i+1}\in R_{1+k_i}\vee\psi(1+k_i)\in R_{\psi(k_{i+1})}$, $0\le i\le r$. By (10.3)(b), we have $m\notin R_k$, and of course $\psi(k)\notin R_{\psi(m)}$, for $m\le\tilde{\theta}_k<\psi(k)$. Hence, $r>1$.

Again by the first claim, we obtain $\mathbf{P}(1+k_1,m)$. Because $k_1\le m<\psi(k)$, we have $\psi(k)\notin R_{\psi(k_1)}$. Hence, $k_1\in R_k$. Therefore, k_1 satisfies condition (10.3)(b), which is $\neg\mathbf{P}(1+k_1,\tilde{\theta}_k)$. However, Claim 2 shows that the conditions $\mathbf{P}(1+k_1,m)$ and $\mathbf{P}(m+1,\tilde{\theta}_k)$ imply $\mathbf{P}(1+k_1,\tilde{\theta}_k)$; this is a contradiction, which proves the claim.

Claim 5. *The set T_k is (k,m)-regular for all $m\in R_k$.*

We may suppose that $k\le n<m$ because otherwise we have nothing to prove.

First, assume that n is a white point, that is, $n\notin T_k$, while scheme (7.1) has a black column, say $n-i\in T_k$, $n+i\in T_k$, $i>0$. Condition $n+i\in T_k$ implies $\mathbf{P}(k,n+i)$. Hence, by Claim 3 with $m\leftarrow n+i$, $s\leftarrow n$, we have $\mathbf{P}(k,n)\vee\mathbf{P}(n+1,n+i)$. However, $n\notin T_k$ implies $\neg\mathbf{P}(k,n)$ as $\psi(k)\notin T_{\psi(n)}=T_{n+1}=\emptyset$. Hence $\mathbf{P}(n+1,n+i)$ is true. We see that $\mathbf{P}(n+1,n+i)=\mathbf{P}(\psi(n+i),\psi(n+1))=\mathbf{P}(n-i+1,n)$ is also true. Because $n-i\in T_k$ implies $\mathbf{P}(k,n-i)$, Claim 2 with $s\leftarrow n-i$, $m\leftarrow n$ shows that $\mathbf{P}(k,n)$ is true. However, $n\notin T_k$ implies $\neg\mathbf{P}(k,n)$, which is a contradiction.

Then, let n be a black point, that is, $n\in T_k$, while scheme (7.3) has a white column, say $n-i\notin T_k$, $n+i\notin T_k$, $i>0$. Condition $n-i\notin T_k$ implies $\neg\mathbf{P}(k,n-i)$, since

$T_{\psi(n-i)} = T_{n+i+1} = \emptyset$. By Claim 3 with $m \leftarrow n$, $s \leftarrow n-i$, we have $\mathbf{P}(n-i+1, n)$, because $n \in T_k$ implies $\mathbf{P}(k, n)$. Hence, $\mathbf{P}(n-i+1, n) = \mathbf{P}(\psi(n), \psi(n-i+1)) = \mathbf{P}(n+1, n+i)$ is also true. At the same time, Claim 4 with $m \leftarrow n+i$ implies $\mathbf{P}(n+i+1, \tilde{\theta}_k)$, while Claim 2 with $k \leftarrow n+1$, $s \leftarrow n+i$ implies $\mathbf{P}(n+1, \tilde{\theta}_k)$. Again, Claim 4 with $m \leftarrow n$ shows that $n \notin T_k$, which is a contradiction.

Next, it remains to show that if $n \in T_k$, then the leftmost complete column of (7.3) is black; that is, $\psi(m) - 1 \in T_k$. Assume $\psi(m) - 1 \notin T_k$. We then have $\neg\mathbf{P}(k, \psi(m) - 1)$ since $T_{\psi(\psi(m)-1)} = T_{m+1} = \emptyset$. Claim 3 with $s \leftarrow \psi(m)-1$, $m \leftarrow n$ implies $\mathbf{P}(\psi(m), n)$, while Claim 4 with $m \leftarrow n$ implies $\neg\mathbf{P}(n+1, \tilde{\theta}_k)$. We see that point $r = n < m$ does not satisfy condition (10.3)(c), because $\mathbf{P}(n+1, m) = \mathbf{P}(\psi(n), m) = \mathbf{P}(\psi(m), n)$ is true, while $\mathbf{P}(n+1, \tilde{\theta}_k)$ is false. Thus $m \notin R_k$, which is a contradiction.

Claim 6. *Let $\tilde{\mathbf{U}}$ be the subalgebra generated by all right coideals $\mathbf{U}^{T_k}(k, m)$, $m \in R_k$. If $1 \le a \le b \le 2n$, $b \ne \psi(a)$, then $\mathbf{P}(a, b)$ is true if and only if $[a : b]$ is a $\tilde{\mathbf{U}}$-root. In particular, the set of all $\tilde{\mathbf{U}}$-roots is $\{[k : m] \mid m \in T'_k\}$.*

Certainly, $\tilde{\mathbf{U}}$ is a right coideal subalgebra that contains $\mathbf{k}[G]$. By Theorem 9.8, it is generated over $\mathbf{k}[G]$ by elements $\Phi^{T_k}(1+t, s)$, where $t < s$ are, respectively, white and black points for $\Phi^{T_k}(k, m)$; that is, $t = k-1$ or $t \notin T_k$, and $s = m$ or $s \in T_k$. In particular, $\mathbf{P}(k, s)$ is true, while $\mathbf{P}(k, t)$ is false ($\psi(k) \notin [t, \psi(t)] \supseteq T_{\psi(t)}$ since $k \le t < s < \psi(k)$). Hence, by Claim 3 with $s \leftarrow t$, we have $\mathbf{P}(1+t, s)$.

If $\gamma = [a : b]$, $a \le b < \psi(a)$, is a $\tilde{\mathbf{U}}$-root, then, by definition, in $\tilde{\mathbf{U}}$ there exists a homogeneous element $c_u \in \tilde{\mathbf{U}}$ of the form (5.14) of degree γ. Because $\tilde{\mathbf{U}}$ is generated by $\Phi^{T_k}(1+t, s)$, the degree γ is a sum of degrees $[1+t : s]$ of the generators. In particular, $\gamma = \sum_i [a_i : b_i]$, where $\mathbf{P}(a_i, b_i)$ are true and $b_i \ne \psi(a_i)$. By Lemma 8.6, we may modify the decomposition of γ so that

$$\gamma = [k_0 - 1 : k_1] + [1 + k_1 : k_2] + \cdots [1 + k_r : k_{r+1}],$$

where $a - 1 = k_0 < k_1 < \cdots < k_r < b = k_{r+1}$, and for each i, $0 \le i \le r$, we still have $\mathbf{P}(1 + k_i, k_{i+1})$ true. Now, Claim 2 implies $\mathbf{P}(a, b)$. Hence, $b \in T'_a$, for $a \le b < \psi(a)$.

Conversely, if $m \in T'_k$, then by Claim 1, we have a sequence $k - 1 = k_0 < k_1 < \cdots < k_r < m = k_{r+1}$ such that for each i, $0 \le i \le r$, either $k_{i+1} \in R_{1+k_i}$ or $\psi(1+k_i) \in R_{\psi(k_{i+1})}$. By definition, $\tilde{\mathbf{U}}$ contains elements $\Phi^{T_{a_i}}(a_i, b_i)$, where $a_i = 1 + k_i$, $b_i = k_{i+1}$ provided that $k_{i+1} < \psi(1 + k_i)$, and $a_i = \psi(k_{i+1})$, $b_i = \psi(1 + k_i)$ provided that $k_{i+1} > \psi(1 + k_i)$. Hence, $[a_i : b_i] = [1 + k_i : k_{i+1}]$ are $\tilde{\mathbf{U}}$-roots. In particular, $[k : m]$ is a sum of $\tilde{\mathbf{U}}$-roots. By Lemma 8.7, the element $[k : m]$ itself is a $\tilde{\mathbf{U}}$-root.

Claim 7. *The set of all simple $\tilde{\mathbf{U}}$-roots is $\{[k : m] \mid m \in R_k\}$. In particular $r(\tilde{\mathbf{U}}) = \theta$.*

If $\gamma = [k : m]$, $k \le m < \psi(k)$, is a simple $\tilde{\mathbf{U}}$-root, then, due to the above claim, $\mathbf{P}(k, m)$ is true. Hence, according to Claim 1, we may find a sequence (10.7). In this case, $\gamma = [k : k_1] + [1 + k_1 : k_2] + \cdots + [1 + k_r : m]$ is a sum of $\tilde{\mathbf{U}}$-roots, because $\mathbf{P}(1 + k_i, k_{i+1})$ is true by definition (10.2); this is a contradiction for the simple root γ, unless $r = 0$. Thus, $m = k_1 \in R_k$ because $\psi(k) \notin [m, \psi(m)] \supseteq R_{\psi(m)}$.

Conversely, let $m \in R_k$. Then, by definition (10.5), we have $m \in T_k$. Claim 6 implies that $[k : m]$ is a $\tilde{\mathbf{U}}$-root. If it is not simple, then it is a sum of two or more $\tilde{\mathbf{U}}$-roots,

$[k:m]=[k:k_1]+[1+k_1:k_2]+\cdots+[1+k_r:m]$, where, due to Claim 6, $\mathbf{P}(1+k_i,k_{i+1})$, $0\leq i\leq r$, are true. Claim 2 implies that $\mathbf{P}(1+k_1,m)$ is also true. Definition (10.3)(c) with $r\leftarrow k_1$ implies $\mathbf{P}(1+k_1,\tilde{\theta}_k)$. Now, Claim 4 provides a contradiction, $k_1\notin T_k$ (recall that $\mathbf{P}(k,k_1)$ implies $k_1\in T_k$ because $k\leq k_1\leq m<\psi(k)$).

Claim 8. *$\tilde{\mathbf{U}}$ is generated as an algebra by $\mathbf{k}[G]$ and $\Phi^{T_k}(k,m)$, $m\in R_k$; that is, $\tilde{\mathbf{U}}=\mathbf{U}$.*

It suffices to note that $\mathbf{U}$ contains a set of PBW-generators for $\tilde{\mathbf{U}}$ over $\mathbf{k}[G]$. If $[k:m]$ is a $\tilde{\mathbf{U}}$-root, then it is a sum of simple $\tilde{\mathbf{U}}$-roots, $[k:m]=\sum[k_i:m_i]$, $m_i\in R_{k_i}$. The elements $f_i=\Phi^{T_{k_i}}(k_i,m_i)$, by definition, belong to $\mathbf{U}$. The PBW-generator corresponding to the root $[k:m]$ can be taken to be a polynomial in f_i determined in one of the formulae (8.3)–(8.6) depending on the type of decomposition of $[k:m]$ into a sum of simple roots.

Theorem 10.3 is completely proved. □

Corollary 10.4. *Every (homogeneous if $q^t=1$, $t>4$) right coideal subalgebra $\mathbf{U}$ of $U_q^+(\mathfrak{so}_{2n+1})$, $q^t\neq 1$ (respectively, of $u_q^+(\mathfrak{so}_{2n+1})$) that contains G is generated as an algebra by G and a set of elements $\Phi^S(k,m)$ with (k,m)-regular sets S.*

Proof. Theorems 8.2 and 10.3 imply that $\mathbf{U}$ has the form $\mathbf{U}_\theta$, where θ is the root sequence. At the same time, definition (10.6) shows that $\mathbf{U}_\theta$, as an algebra, is generated by G and elements $\Phi^{T_k}(k,m)$, $m\in R_k$. It remains to apply Claim 5. □

11. Right coideal subalgebras that do not contain the coradical

In this brief section, we restate the main result in a slightly more general form. We consider homogeneous right coideal subalgebras in $U_q^+(\mathfrak{so}_{2n+1})$ (respectively, in $u_q^+(\mathfrak{so}_{2n+1})$) that do not contain G, but whose intersection with G is a subgroup. We recall that for every submonoid $\Omega\subseteq G$, the set of all linear combinations $\mathbf{k}[\Omega]$ is a right coideal subalgebra. Conversely, if $U_0\subseteq\mathbf{k}[G]$ is a right coideal subalgebra, then $U_0=\mathbf{k}[\Omega]$ for $\Omega=U_0\cap G$ because $a=\sum_i\alpha_i g_i\in U_0$ implies $\Delta(a)=\sum_i\alpha_i g_i\otimes g_i\in U_0\otimes\mathbf{k}[G]$; that is, $\alpha_i g_i\in U_0$.

Definition 11.1. For a sequence $\theta=(\theta_1,\ldots,\theta_n)$ such that $0\leq\theta_k\leq 2n-2k+1$, $1\leq k\leq n$, we define $\mathbf{U}_\theta^1$ to be the subalgebra with 1 generated by $g_{km}^{-1}\Phi^S(k,m)$, where $g_{km}=g(u(k,m))$ and $m\in R_k$, $S=T_k$; see Theorem 10.3.

Lemma 11.2. *The subalgebra $\mathbf{U}_\theta^1$ is a homogeneous right coideal, and $\mathbf{U}_\theta^1\cap G=\{1\}$.*

Proof. The subalgebra $\mathbf{U}_\theta^1$ is homogeneous because it is generated by homogeneous elements. Its zero homogeneous component equals $\mathbf{k}$ because among the generators only one, the unity, has degree zero.

We denote by B_θ the $\mathbf{k}$-subalgebra generated by $\Phi^S(k,m)$ with $m\in R_k$, $S=T_k$. The algebra $\mathbf{U}_\theta^1$ is spanned by all elements of the form $g_a^{-1}a$, $a\in B_\theta$. Because $\mathbf{U}_\theta$ is a right coideal, for any homogeneous $a\in B_\theta$, we have $\Delta(a)=\sum g(a^{(2)})a^{(1)}\otimes a^{(2)}$ where $a^{(1)}\in B_\theta$, $g_a=g(a^{(1)})g(a^{(2)})$. Therefore, $\Delta(g_a^{-1}a)=\sum g(a^{(1)})^{-1}a^{(1)}\otimes g_a^{-1}a^{(2)}$ with $g(a^{(1)})^{-1}a^{(1)}\in\mathbf{U}_\theta^1$. □

Lemma 11.3. *If Ω is a submonoid of G, then $\mathbf{k}[\Omega]\mathbf{U}_\theta^1$ is a homogeneous right coideal subalgebra, and $\mathbf{k}[\Omega]\mathbf{U}_\theta^1 \cap G = \Omega$. Moreover $\mathbf{k}[\Omega]\mathbf{U}_\theta^1 = \mathbf{k}[\Omega']\mathbf{U}_{\theta'}^1$ if and only if $\Omega = \Omega'$ and $\theta = \theta'$.*

Proof. The subalgebra $\mathbf{k}[\Omega]\mathbf{U}_\theta^1$ is homogeneous because it is generated by homogeneous elements. Its zero homogeneous component equals $\mathbf{k}[\Omega]$. Hence $\mathbf{k}[\Omega]\mathbf{U}_\theta^1 \cap G = \Omega$. By the above lemma, we have

$$\Delta(\mathbf{k}[\Omega]\mathbf{U}_\theta^1) \subseteq (\mathbf{k}[\Omega] \otimes \mathbf{k}[\Omega]) \cdot (\mathbf{U}_\theta^1 \otimes U_q^+(\mathfrak{so}_{2n+1})).$$

Hence, $\mathbf{k}[\Omega]\mathbf{U}_\theta^1$ is a right coideal subalgebra. Finally, the equality $\mathbf{k}[\Omega]\mathbf{U}_\theta^1 = \mathbf{k}[\Omega']\mathbf{U}_{\theta'}^1$ implies both the equality of zero homogeneous components, $\mathbf{k}[\Omega] = \mathbf{k}[\Omega']$, and $\mathbf{U}_\theta = \mathbf{k}[G]\mathbf{U}_\theta^1 = \mathbf{k}[G]\mathbf{U}_{\theta'}^1 = \mathbf{U}_{\theta'}$. Hence $\theta = \theta'$ by Theorem 10.3. □

Theorem 11.4. *If U is a homogeneous right coideal subalgebra of $U_q^+(\mathfrak{so}_{2n+1})$ (resp. of $u_q^+(\mathfrak{so}_{2n+1})$) such that $\Omega \overset{df}{=} U \cap G$ is a group, then $U = \mathbf{k}[\Omega]\mathbf{U}_\theta^1$ for some θ.*

Proof. Let $u = \sum h_i a_i \in U$ be a homogeneous element of degree $\gamma \in \Gamma^+$ with different $h_i \in G$, and $a_i \in A$, where A is the $\mathbf{k}$-subalgebra generated by x_i, $1 \le i \le n$. We denote by π_γ the natural projection on the homogeneous component of degree γ. Moreover, π_g, $g \in G$, is a projection on the subspace $\mathbf{k}g$. We have $\Delta(u) \cdot (\pi_\gamma \otimes \pi_{h_i}) = h_i a_i \otimes h_i$. Thus, $h_i a_i \in U$.

By Theorems 10.3 and 8.2, we have $\mathbf{k}[G]U = \mathbf{U}_\theta$ for some θ. If $u = ha \in U$, $h \in G$, $a \in A$, then $\Delta(u) \cdot (\pi_{hg_a} \otimes \pi_\gamma) = hg_a \otimes ha$. Therefore, $hg_a \in U \cap G = \Omega$; that is, $u = \omega g_a^{-1} a$, $\omega \in \Omega$. Because Ω is a subgroup, we obtain $g_a^{-1} a \in U$. It remains to note that all elements $g_a^{-1} a$ such that $ha \in U$ span the algebra $\mathbf{U}_\theta^1$. □

If $U \cap G$ is not a group, then U may have a more complicated structure; see [13, Example 6.4].

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