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Cobham's theorem for substitutions

Abstract. The seminal theorem of Cobham has given rise during the last 40 years to a lot of work about non-standard numeration systems and has been extended to many contexts. In this paper, as a result of fifteen years of improvements, we obtain a complete and general version for the so-called substitutive sequences.

Let α and β be two multiplicatively independent Perron numbers. Then a sequence $x \in A^{\mathbb{N}}$, where A is a finite alphabet, is both α-substitutive and β-substitutive if and only if x is ultimately periodic.

1. Introduction

The seminal theorem of Cobham has given rise during the last 40 years to a lot of work about non-standard numeration systems and has been extended to many contexts. The original Cobham's theorem is concerned with integer base numeration systems. In this paper, as a result of fifteen years of improvements, we obtain a complete and general version for the so-called substitutive sequences.

A set $E \subset \mathbb{N}$ is *p-recognizable* for some $p \in \mathbb{N} \setminus \{0, 1\}$ if the language consisting of the p-ary expansions of elements of E is recognizable by a finite automaton. It is obvious that E is recognizable if and only if it is p^k-recognizable. In 1969, A. Cobham obtained the following remarkable theorem.

Cobham's theorem ([Cob69]). *Let $p, q \geq 2$ be two multiplicatively independent integers (i.e., $p^k \neq q^\ell$ for all integers $k, \ell > 0$). A set $E \subset \mathbb{N}$ is both p-recognizable and q-recognizable if and only if E is a finite union of arithmetic progressions.*

It is interesting to recall what S. Eilenberg wrote in his book [Eil74]: *The proof is correct, long and hard. It is a challenge to find a more reasonable proof of this fine theorem.* In this direction, G. Hansel proposed a simpler presentation in [Han82]; also one can see [Per90] or the dedicated chapter in [AS03] for an expository presentation, with a mistake corrected in [RW06].

In [Cob72], Cobham made precise the structure of these p-recognizable sets: they are exactly the images by letter-to-letter morphisms of constant-length substitution fixed

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points. He also defined the notion of p-automatic sequences: The n-th term of the sequence is a mapping of the last reached state of the automaton when its input is the digits of n in some given base p numeration system. Clearly $E \subset \mathbb{N}$ is p-recognizable if and only if its characteristic sequence is p-automatic. Automata provide a nice and easy description of p-recognizable sets whereas substitutions afford an algorithm to produce such sets. From this a lot of other characterizations have been given, the first and major being in terms of:

(1) p-definable sets (see [B–V94] for a survey);
(2) p-kernel [Eil74];
(3) (when p is prime) algebraic series over $\mathbb{F}_p(X)$ [Chr79, C–R80].

This opened a wide range of further work we briefly describe below.

The Cobham theorem has been generalized to recognizable subsets of $\mathbb{N}^d$ by A. L. Semenov [Sem77] for base p numeration systems (also called later *standard numeration systems*) to give the so-called Cobham–Semenov theorem (see [B–V94] for a nice survey). Then the efforts were concentrated to simplify Cobham–Semenov's theorem and to generalize it to non-standard numeration systems given by linear recurrence relations like the Fibonacci one. Alternative and satisfactory proofs (in terms of simplification) have been proposed, among them a very interesting logical proof due to C. Michaux and R. Villemaire [MV93, MV96] (see also [Bès97, Muc03]), using p-definable sets in the formalism of first order logic in some arithmetic models like the Presburger arithmetic $\langle \mathbb{N}, + \rangle$ (see [Bès01] for a survey on these methods). Another recent proof makes use of ergodic measures [Dur08] and the fact that p-recognizable subsets of $\mathbb{N}^d$ are characterized by multidimensional substitutions [ČG86, Sal87]. This last characterization is an extension of Cobham's result of 1972.

The first result obtained for non-standard numeration systems is due to S. Fabre [Fab94]. He considered subsets of $\mathbb{N}$ that are both p-recognizable and U-recognizable where U is a non-standard numeration system associated with some restricted class of Pisot numbers. Then, V. Bruyère and F. Point [PB97] proved such a result for subsets of $\mathbb{N}^d$ under less restrictive (and more natural) assumptions on the Pisot numbers involved. Later A. Bès [Bès00] succeeded in generalizing Cobham–Semenov's theorem to subsets of $\mathbb{N}^d$ recognizable by automata in two non-standard numeration systems which are associated with the minimal polynomials of multiplicatively independent Pisot numbers. Up to now it is the best generalization obtained for $d \geq 2$.

Prior to this result, for $d = 1$, Cobham's theorem was extended to a much wider class of non-standard numeration systems in [Dur98c] where no "Pisot conditions" are needed, and later in [DR09] to abstract numeration systems (defined in [Rig00]). This last result includes all previously known such results in dimension 1. It is important to notice that the proofs in [Dur98c, DR09] used substitution fixed points while the papers [Bès00, B–V94, MV93, MV96, Muc03, PB97] used a first order logic approach.

With every substitution there is associated an integer square matrix with non-negative entries. It is well-known (see [HJ90] for instance) that such a matrix has a real eigenvalue α which is greater than or equal to the modulus of all other eigenvalues. It is usually called the dominant eigenvalue of M. This allows us to define the notion of α-substitutive

sequences. From the characterization given in [Cob72] it is easy to deduce that the characteristic sequences of p-recognizable subsets of $\mathbb{N}$ are p-substitutive sequences. This suggested to G. Hansel the following result (see [AM95]) that is the main result of this paper.

Theorem 1. *Let α and β be two multiplicatively independent Perron numbers. Let A be a finite alphabet and x be a sequence of $A^{\mathbb{N}}$. Then x is both α-substitutive and β-substitutive if and only if x is ultimately periodic.*

Partial answers have been given in [Fab94, Dur98b, Dur02a] with conditions on the substitutions.

Observe that the main result in [Dur98c] on numeration systems is a consequence of Cobham's theorem for primitive substitutions established in [Dur98b] thanks to a result of Fabre [Fab95] that characterizes the characteristic sequences of recognizable sets of integers in non-standard numeration systems in terms of fixed points of substitutions.

Let us mention further generalizations of Cobham's theorem. We could weaken the assumption that a sequence is p- and q-automatic assuming that a p-automatic and a q-automatic sequence share the same language (the set of finite words occurring in the sequence), and asking if a Cobham's theorem type result still holds in this context. It does, as proven in [Fag97]. This was generalized to primitive substitutions in [Dur98c]. Translated to the framework of dynamical systems, this result means that if two subshifts, one generated by a p-automatic sequence and the other by a q-automatic one (with p and q multiplicatively independent), have a common topological factor then it contains a unique minimal (and also uniquely ergodic) subshift, and moreover it is periodic. This point of view is developed in [Dur02a] and provides a new proof of Cobham's theorem. It uses the ergodic measures of the subshifts and the values they take on cylinder sets. Moreover this way to tackle the problem also works in higher dimensions. For example, it can be used to get a new proof of Cobham–Semenov's theorem (see [Dur08]) using multidimensional subshifts. Moreover as subsets of $\mathbb{N}^d$ can be seen as tilings of $\mathbb{R}^d_+$ (which are self-similar, see [Sol97] for the definition) it is not surprising that there are generalizations to self-similar tiling dynamical systems [CD08].

The notion of recognizable subsets of $\mathbb{R}$ or $\mathbb{R}^d$, in standard numeration systems, can be easily defined. The authors of [BB07, BBB08, BBL09] obtained a very nice generalization of Cobham's theorem. In dimension one the same result (but in a different setting) has been obtained in [AB10a] independently.

The case of the ring of Gaussian integers with the numeration systems $((-a + i)^n)_n$, $a \in \mathbb{N} \setminus \{0\}$ (see [KS75]) has been investigated in [HS03]. They obtained a very partial result and faced the problem of proving that $\frac{\ln a}{\ln b}$, $\frac{\tan^{-1} a}{2\pi} \frac{\ln a}{\ln b} - \frac{\tan^{-1} b}{2\pi}$, 1 are rationally independent, which seems to be a difficult number theoretic problem. The generalization remains open.

In [AS92], the notion of p-regular sequences is defined. They take values in a ring and are defined using the notion of k-kernel. When the ring is finite they are k-automatic sequences. J. Bell [Bel07] generalized Cobham's theorem to this context.

G. Christol [Chr79] gave a famous and very concrete description of the elements of $\mathbb{F}_q((t))$ that are algebraic over $\mathbb{F}_q(t)$ ($q = p^n$ with p a prime number); it shows

that being an algebraic power series is equivalent to the sequence of coefficients being p-automatic. K. Kedlaya [Ked06] generalized this theorem to the so-called generalized power series of Hahn ($\mathbb{F}_q((t^{\mathbb{Q}}))$) in terms of quasi-automatic functions. Then, B. Adamczewski and J. Bell [AB08] proved an extension of Cobham's theorem to quasi-automatic functions and put it together with Kedlaya's result to derive an analogue of the main result in [C–R80] asserting that a sequence of coefficients represents two algebraic power series in distinct characteristics if and only if these power series are rational functions.

For more details on all these developments we refer to [AB10b, DR10].

2. Words, morphisms, substitutions and numeration systems

In this section we recall classical definitions and notation.

2.1. Words and sequences

An *alphabet* A is a finite set of elements called *letters*. A *word* over A is an element of the free monoid generated by A, denoted by A^*. Let $x = x_0x_1 \cdots x_{n-1}$ (with $x_i \in A$, $0 \le i \le n-1$) be a word; its *length* is n and is denoted by $|x|$. The *empty word* is denoted by ϵ, $|\epsilon| = 0$. The set of non-empty words over A is denoted by A^+. The elements of $A^{\mathbb{N}}$ are called *sequences*. If $x = x_0x_1 \cdots$ is a sequence (with $x_i \in A$, $i \in \mathbb{N}$) and $I = [k, l]$ an interval of $\mathbb{N}$ we set $x_I = x_kx_{k+1} \cdots x_l$ and we say that x_I is a *factor* of x. If $k = 0$, we say that x_I is a *prefix* of x. The set of factors of length n of x is written $\mathcal{L}_n(x)$ and the set of factors of x, or the *language* of x, is denoted $\mathcal{L}(x)$. The *occurrences* in x of a word u are the integers i such that $x_{[i,i+|u|-1]} = u$. If u has an occurrence in x, we also say that u *appears* in x. When x is a word, we use the same terminology with similar definitions.

The sequence x is *ultimately periodic* if there exist a word u and a non-empty word v such that $x = uv^{\omega}$, where $v^{\omega} = vvv \cdots$. It is *periodic* if u is the empty word. A word u is *recurrent* in x if it appears in x infinitely many times. The set of recurrent words of x is denoted by $\mathcal{L}_{\text{rec}}(x)$. A sequence x is *uniformly recurrent* if every factor u of x appears infinitely often in x and the greatest difference of two successive occurrences of u is bounded.

2.2. Morphisms and matrices

Let A and B be two alphabets. Let τ be a *morphism* from A^* to B^*. Such a map induces by concatenation a morphism from A^* to B^*. When $\tau(A) = B$, we say τ is a *coding*. Thus, codings are onto. If $\tau(A)$ is included in B^+, it induces by concatenation a map from $A^{\mathbb{N}}$ to $B^{\mathbb{N}}$. These two maps are also called τ. With the morphism τ there is naturally associated a matrix $M_{\tau} = (m_{i,j})_{i \in B, j \in A}$ where $m_{i,j}$ is the number of occurrences of i in the word $\tau(j)$.

It is well-known that any non-negative square matrix M has a real eigenvalue α which is real and greater than or equal to the modulus of any other eigenvalue. We call α the *dominating eigenvalue* of M. Moreover α is a *Perron number*: it is an algebraic real number > 1 strictly dominating the modulus of all its algebraic conjugates (see for instance

[LM95]). The matrix M is called *primitive* if it has a power such that all its entries are positive. In this case the dominating eigenvalue is unique, positive and it is a simple root of the characteristic polynomial. This is Perron's Theorem.

2.3. Substitutions and substitutive sequences

A *substitution* is a morphism $\tau : A^* \to A^*$. If there exist a letter $a \in A$ and a word $u \in A^+$ such that $\sigma(a) = au$ and $\lim_{n \to +\infty} |\sigma^n(a)| = +\infty$, then σ is said to be *prolongable on a*.

Since for all $n \in \mathbb{N}$, $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$ and because $|\sigma^n(a)|$ tends to infinity with n, the sequence $(\sigma^n(aaa \cdots))_{n \geq 0}$ converges (for the usual product topology on $A^{\mathbb{N}}$) to a sequence denoted by $\sigma^\omega(a)$. The morphism σ being continuous for the product topology, $\sigma^\omega(a)$ is a fixed point of σ: $\sigma(\sigma^\omega(a)) = \sigma^\omega(a)$. A sequence obtained in this way by iterating a prolongable substitution is said to be *purely substitutive* (with respect to σ). If $x \in A^{\mathbb{N}}$ is purely substitutive and if $\phi : A^* \to B^*$ is a coding, then the sequence $y = \phi(x)$ is said to be *substitutive*.

Whenever the matrix associated with τ is primitive we say that τ is a *primitive substitution*. We say τ is a *growing* substitution if $\lim_{n \to +\infty} |\tau^n(b)| = +\infty$ for all $b \in A$. It is *erasing* if there exists $b \in A$ such that $\tau(b)$ is the empty word.

Definition 2. Let A be a finite alphabet. A sequence $x \in B^{\mathbb{N}}$ is said to be α-*substitutive* (with respect to σ) if $\sigma : A^* \to A^*$ is a substitution prolongable on the letter a such that:

(1) all letters of A have an occurrence in $\sigma^\omega(a)$;
(2) α is the dominating eigenvalue of the incidence matrix of σ;
(3) there exists a coding $\phi : A^* \to B^*$ with $x = \phi(\sigma^\omega(a))$.

If moreover σ is primitive, then $\phi(\sigma^\omega(a))$ is said to be a *primitive α-substitutive infinite sequence* (*with respect to σ*).

The condition (1) is important. Indeed, consider the substitution τ defined by $\tau(a) = aaab$, $\tau(b) = bc$ and $\tau(c) = cb$. It has three fixed points $\tau^\omega(a)$, $\tau^\omega(b)$ and $\tau^\omega(c)$. The sequence $\tau^\omega(a)$ is 3-substitutive and we do not want to say that $\tau^\omega(b)$ and $\tau^\omega(c)$ are 3-substitutive. With our definition they are 2-substitutive.

2.4. Growth type and erasures

The following well known result (see Chapter III.7 in [SS78]) will be very useful in what follows.

Proposition 3. *Let $\sigma : A^* \to A^*$ be a substitution. For all $a \in A$, one of the following two situations occurs:*

(1) *there exists $N \in \mathbb{N}$ such that for all $n > N$, $|\sigma^n(a)| = 0$, or*

(2) *there exist $d(a) \in \mathbb{N}$ and real numbers $c(a), \theta(a)$ such that*

$$\lim_{n \to +\infty} \frac{|\sigma^n(a)|}{c(a)\, n^{d(a)}\, \theta(a)^n} = 1.$$

Moreover, in the situation (2), *for all $i \in \{0, \dots, d(a)\}$ there exists a letter $b \in A$ appearing in $\sigma^j(a)$ for some $j \in \mathbb{N}$ satisfying*

$$\lim_{n \to +\infty} \frac{|\sigma^n(b)|}{c(b)\, n^i\, \theta(a)^n} = 1.$$

This justifies the following definition.

Definition 4. Let $\sigma : A^* \to A^*$ be a non-erasing substitution. For all $a \in A$, the couple $(d(a), \theta(a))$ defined in Proposition 3 is called the *growth type* of a and $\theta(a)$ is called the *growth rate* of a (with respect to σ). The growth type of a word is the maximal growth type of its letters. If (d, θ) and (e, β) are two growth types we say that (d, θ) is *less than* (e, β) (written $(d, \theta) < (e, \beta)$) whenever $\theta < \beta$, or $\theta = \beta$ and $d < e$.

We say that $a \in A$ is a *growing letter* if $(d(a), \theta(a)) > (0, 1)$ or equivalently, if $\lim_{n \to +\infty} |\sigma^n(a)| = +\infty$.

We set $\Theta := \max\{\theta(a) \mid a \in A\}$, $D := \max\{d(a) \mid a \in A, \theta(a) = \Theta\}$ and $A_{\max} := \{a \in A \mid \theta(a) = \Theta, d(a) = D\}$. The dominating eigenvalue of M is Θ. Consequently, any sequence which is substitutive with respect to σ is Θ-substitutive. We will say that the letters of $A_{\max}$ are *of maximal growth* and that (D, Θ) is the *growth type* of σ. Observe that if $\Theta = 1$, then in view of the last part of Proposition 3, there exists at least one non-growing letter (of growth type $(0, 1)$). In other words, if a letter has a polynomially bounded growth, then there exists at least one non-growing letter. Consequently, σ is growing (i.e., all its letters are growing) if and only if $\theta(a) > 1$ for all $a \in A$. We observe that for all $k \geq 1$, the growth type of σ^k is (D, Θ^k).

The following theorem allows us to suppose that the substitutions we deal with are non-erasing. We refer to [CN03, Théorème 4] for the proof, even if property (3) is not stated in that paper but is clear from the proof. Other references with other proofs are in [Cob68, Pan83, AS03, Hon09].

Theorem 5. *Let $\sigma : A^* \to A^*$ be a substitution prolongable on the letter a and $\phi : A^* \to C^*$ be a morphism such that $\phi(\sigma^\omega(a))$ belongs to $A^{\mathbb{N}}$. Then $\phi(\sigma^\omega(a)) = \psi(\tau^\omega(b))$ where:*

(1) *$\psi : B^* \to C^*$ is a coding;*
(2) *$\tau : B^* \to B^*$ is a non-erasing substitution prolongable on b;*
(3) *there exist $\gamma : A \to B^*$ and k such that*

$$\gamma \circ \sigma^k = \tau \circ \gamma \quad \text{and} \quad \psi \circ \gamma = \phi.$$

From the classical theory of non-negative matrices (see [HJ90]) we deduce the following corollary.

Corollary 6. *The image under a non-erasing morphism of an α-substitutive sequence is an α^k-substitutive sequence with respect to a non-erasing substitution for some k.*

Proof. Let $\sigma : A^* \to A^*$ be a substitution prolongable on the letter a, and α the dominating eigenvalue of its incidence matrix. Let $y = \sigma^\omega(a)$ and $\phi : A^* \to C^*$ be a non-erasing morphism. We can suppose all letters of C appear in some words of $\phi(A)$. It suffices to show that $\phi(y)$ is α^k-substitutive for some k.

From Theorem 5 we can suppose $\phi(y) = \psi(\tau^\omega(b))$ where ψ and τ satisfy (1)–(3) of this theorem. It suffices to show that the dominating eigenvalue of the incidence matrix M_τ of τ is α^k (where k comes from (3)). Let M_ϕ, M_ψ and M_σ be the incidence matrices of the corresponding morphisms. Let v be an eigenvector of M_σ for the eigenvalue α. Then $\alpha^k M_\gamma v = M_\tau M_\gamma v$. As ϕ is non-erasing and $M_\psi M_\gamma = M_\phi$, $M_\gamma v$ is a non-zero vector. Hence it is an eigenvector of M_τ for the eigenvalue α^k.

Conversely, let β be the dominating eigenvalue of M_τ, and w be a corresponding left eigenvector. Then $w M_\gamma M_\sigma^k = \beta w M_\gamma$. As ψ is a coding, there exists w' with non-negative coordinates such that $w = w' M_\psi$. Thus, $w' M_\phi M_\sigma^k = \beta w' M_\phi$. Since all letters of C appear in some words of $\phi(A)$, $w' M_\phi$ is a non-negative vector different from 0. This concludes the proof.   □

Once Theorem 1 is proven, it will show that, in this corollary, the non-erasing assumption cannot be removed. For example, we can consider σ defined by $\sigma(a) = ab$, $\sigma(b) = bac$ and $\sigma(c) = ccc$, and, the (erasing) morphism ϕ defined by $\phi(a) = a$, $\phi(b) = b$ and $\phi(c)$ is the empty word. Then $\sigma^\omega(a)$ is 3-substitutive while $\phi(\sigma^\omega(a))$ is a non-ultimately periodic 2-substitutive sequence.

3. Proof of Theorem 1

3.1. Already known results used to prove the conjecture

The proofs of most of the generalizations of Cobham's theorem are divided into two parts.

(i) Dealing with a subset X of integers, we have to prove that X is syndetic. Equivalently, dealing with an infinite sequence x, we have to prove that the letters occurring infinitely many times in x appear with bounded gaps.

(ii) In the second part of the proof, the ultimate periodicity of X or x has to be demonstrated.

The first part is already known in the framework of substitutions.

Theorem 7 ([DR09, Theorem 17]). *Let $\alpha, \beta \in]1, +\infty[$ be two multiplicatively independent real numbers. If a sequence x is both α-substitutive and β-substitutive then the words having infinitely many occurrences in x appear in x with bounded gaps.*

The second part is proved in the context of "good" substitutions in [Dur02b] and was proven previously in many other papers (see [DR10] for a survey).

Definition 8. Let $\sigma : A^* \to A^*$ be a substitution whose dominating eigenvalue is α. If there exists $B \subseteq A$ such that $\sigma(b) \in B^*$ for all $b \in B$, then the substitution $\tau : B^* \to B^*$ defined by $\tau(b) = \sigma(b)$ for all $b \in B$ is a *sub-substitution* of σ. The substitution σ is *good* if it is growing and has a primitive sub-substitution whose dominating eigenvalue is α.

Not all non-erasing substitutions have a primitive sub-substitution, for example $0 \mapsto 010$, $1 \mapsto 2$ and $2 \mapsto 1$. But all have a power that has at least a primitive sub-substitution ([Dur02b]). But, even up to some power, some are not good. For example the substitution σ defined by $0 \mapsto 0100$, $1 \mapsto 12$ and $2 \mapsto 21$ has a unique primitive sub-substitution ($\tau : 1 \mapsto 12$ and $2 \mapsto 21$) but is not good because the dominant eigenvalue of σ is 3 and it is 2 for τ.

We recall that in the (non-erasing) purely substitutive context the expected extension of Cobham's theorem is known.

Theorem 9 ([Dur02b, Corollary 19]). *Let $\sigma : A^* \to A^*$ and $\tau : A^* \to A^*$ be two non-erasing growing substitutions prolongable on $a \in A$ with respective dominating eigenvalues α and β. Suppose that all letters of A appear in $\sigma^\omega(a)$ and in $\tau^\omega(a)$ and that α and β are multiplicatively independent. If $x = \sigma^\omega(a) = \tau^\omega(a)$, then x is ultimately periodic.*

3.2. Concatenation of return words

Let A be a finite alphabet. Let $x \in A^{\mathbb{N}}$ and $u \in \mathcal{L}(x)$. A *return word to u* (*for x*) is a word w such that wu belongs to $\mathcal{L}(x)$, u is a prefix of wu, and u has exactly two occurrences in wu. The set of return words is denoted by $\mathcal{R}_x(u)$. A sequence $x \in A^{\mathbb{N}}$ is *linearly recurrent* (*with constant L*) if x is uniformly recurrent and for all $w \in \mathcal{R}_x(u)$ we have $|w| \leq L|u|$.

Lemma 10 ([DHS99, Theorem 24]). *Let $x \in A^{\mathbb{N}}$ be a non-periodic linearly recurrent sequence* (*with constant L*). *Then:*

(1) *for all $w \in \mathcal{R}_x(u)$ we have $|u|/L \leq |w|$;*
(2) *there exists a constant K such that for all $u \in \mathcal{L}(x)$ we have $\#\mathcal{R}_x(u) \leq K$.*

The following lemma will be, in some sense, the last decisive argument to prove Theorem 1. When W is a set of words, W^* stands for the set of all concatenations of elements of W.

Lemma 11. *Let x be a non-periodic linearly recurrent sequence* (*with constant L*). *Then there exists a constant K such that for all $u \in \mathcal{L}(x)$ and all $l \in \mathbb{N}$,*

$$\# \bigcup_{0 \leq n \leq l} (\mathcal{R}_x(u)^* \cap \mathcal{L}_n(x)) \leq (1 + K)^{lL/|u|}.$$

Proof. Let K be the constant given by Lemma 10. Let $l, n \in \mathbb{N}$ with $n \leq l$, $u \in \mathcal{L}(x)$ and $w \in \mathcal{R}_x(u)^* \cap \mathcal{L}_n(x)$. As the distance between two occurrences of u in w is at least

$|u|/L$, in w there are at most $Ll/|u|$ occurrences of u. Hence w is a concatenation of exactly $Ll/|u|$ words belonging to $\mathcal{R}_x(u) \cup \{\epsilon\}$. This concludes the proof. □

The following proposition allows us to apply this lemma to primitive substitutive sequences.

Theorem 12 ([Dur98a, Theorem 4.5]). *All primitive substitutive sequences are linearly recurrent.*

Note that it is shown in [Dur96] through the Chacon example that uniformly recurrent substitutive sequences are primitive substitutive. In [Dur10] the generality of the case of the Chacon substitution is explained. The same result has been obtained in [DL06] but with a significantly longer proof.

3.3. Reduction of the problem

Let us make two obvious but important remarks. Let $\alpha, \beta > 1$. When a sequence is α-substitutive with respect to a substitution σ, it is also α^k-substitutive with respect to σ^k. Moreover, for all positive integers l and k, α and β are multiplicatively independent if and only if α^l and β^k are multiplicatively independent. Hence due to the statement we want to prove, we can suppose that σ has the properties some σ^k would have (changing σ to σ^k if needed). Hence we can always suppose that:

(1) σ is non-erasing (Corollary 6);
(2) σ is growing or there exists a growing letter $a \in A$ such that $\sigma(a) = vau$ (or uav) with $u \in B^* \setminus \{\epsilon\}$ where B is the set of non-growing letters ([Pan84, Théorème 4.1]);
(3) σ has a primitive sub-substitution (see [Dur02b]).

3.4. Technical lemmata and proof of Theorem 1

Before proving the main result of this paper we need to establish some lemmata.

Lemma 13. *Let x be a substitutive sequence with respect to a growing substitution. If each word in $\mathcal{L}_{\mathrm{rec}}(x)$ appears with bounded gaps in x then there exists a primitive substitutive sequence y such that $\mathcal{L}_{\mathrm{rec}}(x) = \mathcal{L}(y)$.*

Proof. Let $\sigma : A^* \to A^*$ be a growing substitution prolongable on the letter a and $\phi : A^* \to B^*$ be a coding such that $x = \phi(\sigma^\omega(a))$. We can suppose that all letters of A occur in some $\sigma^n(a)$, $n \in \mathbb{N}$. Proposition 15 in [Dur02a] asserts that growing substitutions have at least one primitive sub-substitution. Let $\tau : C^* \to C^*$ be a primitive sub-substitution of σ. It is necessarily a growing substitution and it is easy to check that each word $\sigma^n(b)$, $b \in C$, $n \in \mathbb{N}$, is recurrent in $\sigma^\omega(a)$. There exist $c \in C$ and a positive integer l such that c is a prefix of $\sigma^l(c)$. Thus $y = \phi((\sigma^l)^\omega(c))$ exists, is primitive substitutive and $\mathcal{L}(y) \subset \mathcal{L}_{\mathrm{rec}}(x)$. As primitive substitutive sequences are uniformly recurrent, from the hypothesis we deduce that $\mathcal{L}_{\mathrm{rec}}(x) = \mathcal{L}(y)$. □

Lemma 14. *Let x be a sequence and u be a word such that if $u = v^k$ then $k = 1$. Suppose that each word in $\mathcal{L}_{\mathrm{rec}}(x)$ appears in x with bounded gaps and that $\mathcal{L}_{\mathrm{rec}}(x)$ contains $\{u^n \mid n \in \mathbb{N}\}$. Then x is ultimately periodic and $\mathcal{L}_{\mathrm{rec}}(x) = \mathcal{L}(uuu \cdots)$.*

Proof. Let u be such that if $u = v^k$ then $k = 1$. Let us reproduce arguments already used in [Dur02b, Theorem 18] in order to conclude that x is ultimately periodic. As u belongs to $\mathcal{L}_{\mathrm{rec}}(x)$, it appears with bounded gaps in x. Thus, the set $\mathcal{R}_x(u)$ of return words to u is finite. There exists an integer N such that all the words $wu \in \mathcal{L}(x_N x_{N+1} \cdots)$, $w \in \mathcal{R}_x(u)$, appear infinitely many times in x. Hence these words appear with bounded gaps in x. We set $t = x_N x_{N+1} \cdots$. We can suppose that u is a prefix of t. Then t is a concatenation of return words to u. Let w be a return word to u such that wu belongs to $\mathcal{L}(t)$. It appears in some power of u: $wu = su^k p$ where $k \geq 1$, s is a suffix of u with $|s| < |u|$, and p a prefix of u with $|p| < |u|$. Hence p is also a suffix of u and there exists p' such that $u = pp' = p'p$. If p and p' are non-empty then $u = v^k$ for some v and $k \geq 2$ (see [Lot83, Proposition 1.3.2]). This is in contradiction with our assumption. Consequently, p or p' is empty. Doing the same with s we will deduce that necessarily $w = u$. It follows that $t = u^\omega$, x is ultimately periodic and $\mathcal{L}_{\mathrm{rec}} = \mathcal{L}(t) = \mathcal{L}(uuu \cdots)$. $\qquad\square$

We have seen that Theorem 1 was proven in [Dur02b] in the context of good substitutions. The following lemma, together with Theorem 7, is the main argument to treat, in Theorem 1, the case where one of the two substitutive sequences is not good.

Lemma 15. *Let x be purely substitutive with respect to $\sigma : A^* \to A^*$ satisfying (1) and (2) of Section 3.3, having a dominating eigenvalue α strictly greater than 1 and such that all letters of A occur in x. Suppose there exists a coding $\phi : A^* \to B^*$ such that all words belonging to $\mathcal{L}_{\mathrm{rec}}(\phi(x))$ appear with bounded gaps in $\phi(x)$. Then either all letters of A have the same growth rate (with respect to σ) or $\phi(x)$ is ultimately periodic.*

Proof. Suppose σ is prolongable on a', $x = \sigma^\omega(a')$ and $y = \phi(x)$. Suppose σ has at least two growth rates and let us show $\phi(x)$ is ultimately periodic. The growth type of σ is (d, α) for some d.

We consider two cases. Suppose σ is not a growing substitution. Then, from assumption (2) in Section 3.3, there exists a growing letter $c \in A$ such that $\sigma(c) = vcu$ (or ucv) with $u \in B^* \setminus \{\epsilon\}$ where B is the set of non-growing letters. It is convenient to notice that $\sigma^n(c) = \sigma^{n-1}(v)cu\sigma(u) \cdots \sigma^{n-1}(u)$ or $\sigma^n(c) = \sigma^{n-1}(u) \cdots \sigma(u)uc\sigma^{n-1}(v)$ and there exist two distinct positive integers i and j such that $\sigma^i(u) = \sigma^j(u)$.

Let us show that c necessarily belongs to $\mathcal{L}_{\mathrm{rec}}(x)$ whenever x is not ultimately periodic.

Suppose $c = a'$ and $\sigma(c) = vcu$. If v is not the empty word then $\sigma(c) = cv'cu$ and c is clearly recurrent in x. If v is the empty word then $\sigma^n(c) = cu\sigma(u) \cdots \sigma^{n-1}(u)$. Consequently, because $\sigma^i(u) = \sigma^j(u)$, x is ultimately periodic. Suppose $c = a'$ and $\sigma(c) = ucv$. Then a' is a non-growing letter. This contradicts our assumptions. Suppose $c \neq a'$. Then $\sigma^m(a') = a'sct$ for some m, and c is clearly recurrent.

The letter c, and consequently $\sigma^n(c)$, $n \in \mathbb{N}$, having infinitely many occurrences in x, it is also the case for the words $u^{(n)} = (\sigma^i(u)\sigma^{i+1}(u) \cdots \sigma^{j-1}(u))^n$, $n \in \mathbb{N}$. Then all

words in $\{(\phi(u^{(1)}))^n \mid n \in \mathbb{N}\}$ appear infinitely many times in $\phi(x)$. Lemma 14 implies that $\phi(x)$ is ultimately periodic.

Suppose now that σ is a growing substitution. Then all letters of A have an exponential growth rate γ with $\gamma > 1$. From the assumptions, the numbers

$$\beta = \max\{\theta(a) \mid \theta(a) < \alpha, \, a \in A\} \quad \text{and} \quad f = \max\{d(a) \mid \theta(a) = \beta, \, a \in A\}$$

exist and (f, β) is a growth type of some letter of A with $1 < \beta < \alpha$.

There necessarily exist two letters occurring infinitely many times in x, one having growth type (f, β) and the other having an exponential growth rate α. Hence, there exists a non-empty word awb, appearing infinitely many times in x, where w has growth type (f, β), and a and b have an exponential growth rate α. Once we observe that for all letters c with exponential growth rate α there exist letters c', c'' with exponential growth rate α such that $\sigma(c) = u'c'u = vc''v'$ where u and v (possibly empty) have exponential growth rates strictly less than α. Then, by a recurrence starting with awb, it is easy to prove that there exist two sequences of letters $(a_n)_n$ and $(b_n)_n$ and two sequences of words $(u_n)_n$ and $(v_n)_n$ such that for all $n \in \mathbb{N}$:

(1) $a_n u_n \sigma(u_{n-1}) \cdots \sigma^{n-1}(u_1) \sigma^n(w) \sigma^{n-1}(v_1) \cdots \sigma(v_{n-1}) v_n b_n$ appears infinitely many times in x;
(2) a_n has growth type (d_n, α);
(3) b_n has growth type (e_n, α);
(4) u_n has growth type (f_n, β_n) with $\beta_n < \alpha$ and $|u_n| < \max_{c \in A} |\sigma(c)|$;
(5) v_n has growth type (g_n, γ_n) with $\gamma_n < \alpha$ and $|v_n| < \max_{c \in A} |\sigma(c)|$.

Consequently, for all n and k the word $W(n, k) = \sigma^k(a_n) U(n, k) \sigma^k(b_n)$, where

$$U(n, k) = \sigma^k(u_n) \cdots \sigma^{k+n-1}(u_1) \sigma^{k+n}(w) \sigma^{k+n-1}(v_1) \cdots \sigma^k(v_n),$$

appears infinitely many times in x. There exist a strictly increasing sequence (m_i) and two letters a and b such that

$$a_{m_i} = a'' \quad \text{and} \quad b_{m_i} = b'' \quad \text{for all } i. \tag{3.1}$$

From Lemma 13, there exists a linearly recurrent sequence z such that

(P1) $\mathcal{L}(z) = \mathcal{L}_{\text{rec}}(y)$.

If z is periodic, we complete the proof with Lemma 14. Hence we suppose z is not periodic. From Lemmata 10 and 11 there exist some constants K, d, e such that:

(P2) for all $u \in \mathcal{L}_{\text{rec}}(y)$ and $w \in \mathcal{R}_y(u) \cap \mathcal{L}_{\text{rec}}(y)$ we have $|u|/K \le |w| \le K|u|$;
(P3) for all $u \in \mathcal{L}_{\text{rec}}(y)$ we have $\#\mathcal{R}_y(u) \cap \mathcal{L}_{\text{rec}}(y) \le K$;
(P4) for all $u \in \mathcal{L}_{\text{rec}}(y)$ and all $l \in \mathbb{N}$,

$$\# \bigcup_{0 \le n \le l} (\mathcal{R}_y(u)^* \cap \mathcal{L}_n(y) \cap \mathcal{L}_{\text{rec}}(y)) \le (1 + K)^{lK/|u|};$$

(P5) for all n and k,

$$|U(n, k)| \leq K((n + k)^{f+1} \beta^{n+k}),$$

$$\frac{1}{K} k^d \alpha^k \leq |\sigma^k(a'')| \leq K k^d \alpha^k, \qquad \frac{1}{K} k^e \alpha^k \leq |\sigma^k(b'')| \leq K k^e \alpha^k.$$

From (3.1) and the previous inequalities, there exists k such that $|\sigma^k(b'')| \leq |\sigma^k(a'')|$ (the other case can be treated in the same way), $2(1 + K) \leq |\sigma^k(b'')|$ and

$$|U(m_i, k)| \leq |\sigma^k(b'')| \quad \text{for all } 1 \leq i \leq (1 + K)^{2(K+3)(K+1)K} + 1. \qquad (3.2)$$

From (P1) and (P2), for all j, all words in $\mathcal{L}_{\mathrm{rec}}(y) \cap \mathcal{L}_j(y)$ appear in all words of $\mathcal{L}_{\mathrm{rec}}(y) \cap \mathcal{L}_{(K+1)j}(y)$. Let u be a prefix of $\phi(\sigma^k(b''))$ such that

$$\frac{|\sigma^k(b'')|}{K + 1} - 1 \leq |u| \leq \frac{|\sigma^k(a'')|}{K + 1}.$$

Then u is non-empty and occurs in $\phi(\sigma^k(a''))$. We can decompose $\phi(\sigma^k(a''))$ and $\phi(\sigma^k(b''))$ in such a way that $\phi(\sigma^k(a'')) = Q_p u Q_s$ and $\phi(\sigma^k(b'')) = u R_s$ with

$$|Q_s| \leq (K + 1)|u|.$$

Observe that u, a'', b'' and Q_s do not depend on i. Moreover, for all i belonging to $[1, (1 + K)^{2(K+3)(K+1)K} + 1]$, the word $u Q_s \phi(U(m_i, k))$ belongs to $\mathcal{L}_{\mathrm{rec}}(y)$, is a concatenation of return words to u and satisfies

$$|u Q_s \phi(U(m_i, k))| \leq (K + 3)|\sigma^k(b'')|.$$

Moreover,

$$\# \bigcup_{n=0}^{(K+3)|\sigma^k(b'')|} \mathcal{R}_u(y)^* \cap \mathcal{L}_n(y) \leq (1 + K)^{(K+3)|\sigma^k(b'')|K/|u|}$$

$$\leq (1 + K)^{2(K+3)(K+1)K}. \qquad (3.3)$$

But observe that $(|U(m_i, k)|)_{0 \leq i \leq 1+(1+K)^{2(K+3)(K+1)K}}$ being strictly increasing, the words $u Q_s \phi(U(m_i, k))$, $0 \leq i \leq 1 + (1 + K)^{2(K+3)(K+1)K}$, are all distinct (and belong to $\bigcup_{0 \leq n \leq (K+3)|\sigma^k(b'')|} \mathcal{R}_u(y)^* \cap \mathcal{L}_n(y)$). This is in contradiction with (3.3). $\qquad \square$

Proof of Theorem 1. The sufficiency is proven in [Dur02b, Proposition 7]. The necessity is proven in [Dur02b, Theorem 18] in the context of good substitutions. We conclude by reducing the problem to better assumptions (Subsection 3.3), then using Theorem 7 and Lemma 15 (after we observe that substitutions having a primitive sub-substitution and that are not good necessarily have at least two growth rates). $\qquad \square$

3.5. For morphisms instead of codings

We start with an example. Let $\sigma : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be defined by $\sigma(0) = 01$ and $\sigma(1) = 0$. The sequence $x = \sigma^\omega(0)$ is $(1 + \sqrt{5})/2$-substitutive. Now let $\tau : \{a, 0, 1\}^* \rightarrow \{a, 0, 1\}^*$ be defined by $\tau(a) = a0a$, $\sigma(0) = a01$ and $\sigma(1) = a0a$. The sequence $y = \sigma^\omega(a)$ is 3-substitutive. But $x = \phi(y)$ where $\phi : \{a, 0, 1\}^* \rightarrow \{0, 1\}^*$ is defined by $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(a)$ is the empty word. Hence a Cobham-like theorem does not hold for erasing coding morphisms instead of coding. But thanks to Corollary 6 (and Theorem 5) it holds for non-erasing morphisms.

Theorem 16. *Let $\alpha > 1$ and $\beta > 1$ be two multiplicatively independent Perron numbers, and $\phi : A^* \rightarrow C^*$ and $\psi : B^* \rightarrow C^*$, defined on finite alphabets, be non-erasing morphisms. Suppose $x \in A^{\mathbb{N}}$ is α-substitutive and $y \in B^{\mathbb{N}}$ is β-substitutive. If $\phi(x) = \psi(y)$ then $\phi(x)$ is ultimately periodic.*

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