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David Ben-Zvi · Thomas Nevins

$\mathcal{D}$-bundles and integrable hierarchies

Abstract. We study the geometry of $\mathcal{D}$-*bundles*—locally projective $\mathcal{D}$-modules—on algebraic curves, and apply them to the study of integrable hierarchies, specifically the multicomponent Kadomtsev–Petviashvili (KP) and spin Calogero–Moser (CM) hierarchies. We show that KP hierarchies have a geometric description as flows on moduli spaces of $\mathcal{D}$-bundles; in particular, we prove that the local structure of $\mathcal{D}$-bundles is captured by the full Sato Grassmannian. The rational, trigonometric, and elliptic solutions of KP are therefore captured by $\mathcal{D}$-bundles on cubic curves E, that is, irreducible (smooth, nodal, or cuspidal) curves of arithmetic genus 1. We develop a Fourier–Mukai transform describing $\mathcal{D}$-modules on cubic curves E in terms of (complexes of) sheaves on a twisted cotangent bundle $E^\natural$ over E. We then apply this transform to classify $\mathcal{D}$-bundles on cubic curves, identifying their moduli spaces with phase spaces of general CM particle systems (realized through the geometry of spectral curves in $E^\natural$). Moreover, it is immediate from the geometric construction that the flows of the KP and CM hierarchies are thereby identified and that the poles of the KP solutions are identified with the positions of the CM particles. This provides a geometric explanation of a much-explored, puzzling phenomenon of the theory of integrable systems: the poles of meromorphic solutions to KP soliton equations move according to CM particle systems.

Contents

D. Ben-Zvi: Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA;
e-mail: benzvi@math.utexas.edu

T. Nevins: Department of Mathematics, University of Illinois at Urbana-Champaign,
Urbana, IL 61801, USA; e-mail: nevins@uiuc.edu

1. Introduction

We study the geometry of $\mathcal{D}$-*bundles*—locally projective $\mathcal{D}$-modules—on algebraic curves, and apply them to the study of integrable hierarchies, specifically the multicomponent Kadomtsev–Petviashvili (KP) and spin Calogero–Moser (CM) hierarchies. We show that KP hierarchies have a geometric description as flows on moduli spaces of $\mathcal{D}$-bundles; in particular, we prove that the local structure of $\mathcal{D}$-bundles is captured by the full Sato Grassmannian. The rational, trigonometric, and elliptic solutions of KP are therefore captured by $\mathcal{D}$-bundles on cubic curves E, that is, irreducible (smooth, nodal, or cuspidal) curves of arithmetic genus 1. We develop a Fourier–Mukai transform describing $\mathcal{D}$-modules on cubic curves E in terms of (complexes of) sheaves on a twisted cotangent bundle $E^\natural$ over E. We then apply this transform to classify $\mathcal{D}$-bundles on cubic curves, identifying their moduli spaces with phase spaces of general CM particle systems (realized through the geometry of spectral curves in $E^\natural$). Moreover, it is immediate from the geometric construction that the flows of the KP and CM hierarchies are thereby identified and that the poles of the KP solutions are identified with the positions of the CM particles. This provides a geometric explanation of a much explored, puzzling phenomenon of the theory of integrable systems: the poles of meromorphic solutions to KP soliton equations move according to CM particle systems.

1.1. A brief history of the KP/CM correspondence

We begin with a rough sketch of the history of the problem in integrable systems that motivated the present work—see the review articles [Be, GW] for more complete histories and bibliographies. Also, an exposition of the present work appears in [BN2], together with a leisurely historical discussion and overview.

In the seminal work [AMM], Airault, McKean, and Moser wrote down rational, trigonometric, and elliptic solutions of the Korteweg–deVries equation and discovered that the motion of their poles is governed by the Calogero–Moser classical many-body systems of particles on the line, cylinder, and torus (respectively) with inverse-square potentials. Krichever [Kr1, Kr2] and the Chudnovskys [CC] extended this correspondence to the meromorphic solutions of the KP equation, where it becomes an isomorphism between the phase spaces of generic rational (decaying at infinity), trigonometric, and elliptic KP solitons and the corresponding Calogero–Moser systems. In fact, Krichever wrote the elliptic CM systems in Lax form with elliptic spectral parameter and showed that the generic elliptic KP solutions are algebro-geometric solutions associated to line bundles on the corresponding branched covers of elliptic curves. This approach is extended in [Kr3], where the elliptic CM systems are considered as part of a general Hamiltonian theory of Lax operators on algebraic curves, and [AKV], where this field analog of the elliptic CM system is related to the KP hierarchy. A detailed algebro-geometric study of the CM spectral curves, as geometric phase spaces for the elliptic KP/CM systems, was undertaken by Treibich and Verdier [TV1, TV2] and lead, in particular, to a complete classification of elliptic solutions of the Korteweg–deVries (KdV) equation.

The rational KP/CM correspondence was explored and deepened by Shiota [Shi] and, especially, by Wilson [W2] (see [W4] for a review). Shiota identified all the higher flows of the KP hierarchy on generic rational solutions with the higher Hamiltonians of the rational CM particles. Wilson extended the correspondence away from generic solutions by allowing collisions of Calogero–Moser particles. In [W1], Wilson had identified the completed phase space of the rational KP hierarchy with an *adelic Grassmannian* (which appears independently in the work of Cannings and Holland [CH1] classifying ideals in the Weyl algebra of differential operators on the affine line). In [W2], Wilson gives an explicit formula that defines a point of the adelic Grassmannian from the linear algebra data describing the rational CM space. He then proves by direct calculations that this map extends continuously to the completed phase spaces and takes the CM flows to the KP flows.

The emergent relation between CM spaces and the Weyl algebra was explored in the papers [BW1, BW2] by Berest and Wilson and extended in [BGK1, BGK2] by Baranovsky, Ginzburg, and Kuznetsov. Inspired by ideas of Le Bruyn [LB], these authors used noncommutative geometry to identify the rational CM space (and its spin versions) with the isomorphism classes of ideals in (and generally torsion-free modules over) the Weyl algebra, and thus with Hilbert schemes of points on a noncommutative surface (see also [NS]). This is closely related to the study of noncommutative instantons on $\mathbb{C}^2$ (see, for example, [NeS, KKO, BrNe]).

An independent development of great significance for the present project was the work of Nakayashiki and Rothstein [N1, N2, Ro1] and especially [Ro2]. In these works, the Fourier–Mukai transform on Jacobians of smooth curves is applied to construct $\mathcal{D}$-modules and, through Sato's $\mathcal{D}$-module description of the KP hierarchy, to describe the Krichever construction of KP solutions from line bundles on the curve.

In this paper, we provide a direct relation between arbitrary meromorphic (rational, trigonometric, and elliptic) solutions of KP (and its multicomponent generalization) and the noncommutative geometry of modules over differential operators. We show that this geometric description of the KP hierarchy is directly identified, through a Fourier–Mukai transform, with the geometric description of the completed (spin generalizations of the) Calogero–Moser systems. This provides a uniform conceptual description of the KP/CM correspondence in its most general setting.

We now describe the contents of the present paper in detail.

1.2. Background

In Section 2.1, we include a brief reminder on our backdrop, the family of Weierstrass cubic curves. These fall into three types, which parallel the three flavors of many-body systems: smooth elliptic curves (the elliptic case), the projective line with one node (trigonometric case), and the projective line with one cusp (rational case). All the constructions of this paper work over the universal family of cubic curves; however, for readability, we usually work over individual cubic curves. Over each such curve E, we consider a ruled surface $\overline{E}^\natural$, with a section E_∞ whose complement is the (unique nontrivial) affine bundle $E^\natural$ over E. In the elliptic case, $E^\natural$ can be described alternatively as the universal additive extension of E, as the space of line bundles with flat connections, or as the home of the Weierstrass ζ-function.

In Section 2.2, we discuss the spin CM particle systems associated to cubic curves following [BN3] (see also the overview [Ne2]). These are Hamiltonian systems describing a system of n particles living in (the smooth part of) E and completed so as to allow collisions of particles. The usual spin CM particles carry spins in an auxiliary vector space $\mathbb{C}^k$. We consider a more general version, in which the spins take value in a length k torsion sheaf T on (the smooth locus of) E; the usual setting corresponds to T being a length k skyscraper at the marked point of E. We present the general spin CM systems in "geometric action-angle" variables, that is, in terms of line bundles on spectral curves. More precisely, in [BN3] it is shown (following in the spinless case the works [Kr2, TV2, DW, GN, Ne1], among others) that the phase spaces of the generalized spin CM systems are identified with spaces of torsion-free sheaves supported on curves in $\overline{E}^\natural$ (the CM spectral sheaves). The relation between the particle and spectral curve descriptions is given by a Fourier transform. Specifically, the positions of CM particles are recovered from the spectral curve description as the finite support of the Fourier–Mukai transform of the projection to E of the spectral sheaf. The CM Hamiltonian flows are explicitly identified with a hierarchy of flows that preserve the curve and "tweak" the sheaves along the intersection with the curve $E_\infty \subset \overline{E}^\natural$.

In fact, we may tweak sheaves by arbitrary meromorphic endomorphisms near the curve E_∞, giving rise to a natural Lie algebroid acting on CM spectral sheaves, the CM algebroid. For generic framings—for example, in the spinless case $T = \mathcal{O}_b$)—these endomorphisms consist of several copies of the Lie algebra of Laurent series, so we get several commuting hierarchies of flows (whose labeling depends on a choice of coordinate on the spectral curve). In general, the algebroid is isomorphic to a sum of several twisted loop algebras $L\mathfrak{gl}_k$ corresponding to the rank of a sheaf on the components of its support. To give names to our flows, we may "Higgs" our spectral sheaves by picking a distinguished such endomorphism, breaking down the symmetry to give a commuting family of flows. Higgsed spectral sheaves carry a family of flows labeled by natural numbers, corresponding to tweaking the sheaf by powers of the endomorphism. We refer to this enhancement as the "Higgsed CM hierarchy."

In Section 2.3, we review the multicomponent generalizations of the KP hierarchy, including their definition as flows on formal matrix-valued microdifferential Lax operators, and Sato's description as flows on the big cell of an infinite-dimensional Grassmannian. In particular, we are interested in Sato's reinterpretation of KP wave operators in terms of the free modules they generate over the ring $\mathcal{D}$ of differential operators with Taylor series coefficients (that is, as $\mathcal{D}$-modules on the disc). Sato thereby identifies the big cell of the Grassmannian with free $\mathcal{D}$-modules embedded in the algebra $\mathcal{E}$ of microdifferential operators with Taylor coefficients.

1.3. $\mathcal{D}$-bundles

In Section 3, we introduce $\mathcal{D}$-bundles on smooth algebraic curves X (and their natural extension to the case of cubic curves). A $\mathcal{D}$-bundle is a torsion-free module over the sheaf of differential operators on a curve. Examples include locally free $\mathcal{D}$-modules ($V \otimes \mathcal{D}_X$ for a vector bundle V on X), but also ideals in the Weyl algebra $\mathcal{D}_{\mathbb{A}^1}$, and have been extensively studied in [CH1, BW2, BGK1, BGK2, BN1], among others. It is convenient to consider $\mathcal{D}$-bundles from the point of view of noncommutative geometry, as torsion-free coherent sheaves on a noncommutative affine bundle $T^*_\hbar X$ over X. For the point of view of moduli problems, it is clearly better to consider torsion-free sheaves on a noncommutative *ruled surface* $\overline{T^*_\hbar X} = T^*_\hbar X \cup X$, which are the *framed $\mathcal{D}$-bundles*. For a vector bundle V on X, a V-framed $\mathcal{D}$-bundle is a torsion-free sheaf on $\overline{T^*_\hbar X}$ whose restriction to the curve X at infinity is identified with V. The resulting moduli spaces give deformations of Hilbert schemes of points on T^*X (in rank 1) and of more general moduli of framed torsion-free sheaves; they may be considered algebraic analogs of spaces of noncommutative instantons (see, e.g., [KKO, BrNe]). See Section 3 for precise algebraic definitions in terms of $\mathcal{D}$-modules equipped with normalized filtrations.

We state our central result on $\mathcal{D}$-bundles on cubic curves (Theorem 3.4) in Section 3.2 but defer the proof to Section 5, after we have studied the Fourier–Mukai transform. The theorem identifies the moduli spaces of $\mathcal{D}$-bundles with the phase spaces of the corresponding Calogero–Moser systems, via a generalized Fourier–Mukai transform. Fix a

cubic curve E and a semistable degree zero vector bundle V on E, which corresponds under the Fourier–Mukai transform to a torsion sheaf $V^\vee$ supported on the smooth locus of E.

Theorem 1.1 (Theorem 5.1). *The moduli stack* $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V)$ *of V-framed $\mathcal{D}$-bundles on a cubic curve is isomorphic to the $V^\vee$-framed spin Calogero–Moser phase space* $\mathfrak{CM}_n(E, V^\vee)$. *The isomorphism identifies the cusps of a $\mathcal{D}$-bundle with the positions of the corresponding Calogero–Moser particles.*

Note that the usual spin Calogero–Moser spaces are obtained in the case of trivial framing $V = \mathcal{O}_E^k$, $V^\vee = \mathcal{O}_b^k$.

The rank 1 case of this theorem recovers the identification of the space of ideals in the Weyl algebra with the rational CM space [BW2]. It also refines the separation of variables of [GNR], giving a birational identification between (spinless) Calogero–Moser spaces and Hilbert schemes of points, by identifying the full CM spaces with deformed Hilbert schemes of noncommutative points. Also, our description of this identification as a Fourier–Mukai transform establishes the speculation of [GNR] that separation of variables is a T-duality.

In Section 3.3, we relate $\mathcal{D}$-bundles on arbitrary curves to Wilson's *adelic Grassmannian* [W2]. We show that, in parallel to the situation for framed torsion-free sheaves on a ruled surface, framed $\mathcal{D}$-bundles on any curve have a canonical trivialization away from a finite subset of the curve, the *cusps* of the $\mathcal{D}$-bundle (named following [CH2, BN1]). Wilson's adelic Grassmannian can be identified, following Cannings and Holland [CH2] (see also [BW1, BGK1, BN1]), with the moduli space of (*unframed*) $\mathcal{D}$-bundles with generic trivialization (in Wilson's spinless settings, these are rank 1 $\mathcal{D}$-bundles, while we consider ones of arbitrary rank). Thus the canonical trivialization off cusps defines maps from our moduli spaces of framed $\mathcal{D}$-bundles to the adelic Grassmannians. These maps give a set theoretic decomposition of the adelic Grassmannians into finite-dimensional moduli spaces. The relevant "topologies," i.e. notions of *families* of $\mathcal{D}$-modules, are very different, so that the different moduli of $\mathcal{D}$-bundles become connected in the adelic Grassmannian. The results of Section 3.3 generalize those of [W2, BW1, BW2, BGK1, BGK2] on adelic Grassmannians and $\mathcal{D}$-modules while giving precise algebraic meaning to Wilson's set-theoretic decomposition of the rational rank 1 Grassmannian into Calogero–Moser spaces. Using the Cannings–Holland–inspired interpretation of $\mathcal{D}$-bundles on X as coherent sheaves on cuspidal curves normalized by X, we thus also obtain a construction of solutions of multicomponent KP hierarchies (orbits in the Sato Grassmannian) from $\mathcal{D}$-bundles.

1.4. $\mathcal{D}$-bundles and KP hierarchies

In Section 3.4, we relate the local structure of $\mathcal{D}$-bundles to the KP hierarchy. We generalize Sato's description of the big cell of the Grassmannian to a description of the full Grassmannian by replacing the free $\mathcal{D}$-modules in his construction by $\mathcal{D}$-bundles:

Theorem 1.2 (Theorem 6.5). *The rank n Sato Grassmannian is isomorphic to the moduli space of rank n $\mathcal{D}$-bundles M on the disc equipped with an isomorphism $M \otimes_{\mathcal{D}} \mathcal{E} \to \mathcal{E}^n$ of the microlocalization with the free rank n module over microdifferential operators $\mathcal{E}$ on the disc.*

Equivalently, the Grassmannian parametrizes $\mathcal{D}$-*lattices*, which are torsion-free finite rank $\mathcal{D}$-submodules of $\mathcal{E}^n$ which generate $\mathcal{E}^n$.[1] This construction is interpreted as a noncommutative version of the Krichever construction, where we replace line bundles on a curve trivialized near a point by $\mathcal{D}$-line bundles on the disc with a microlocal trivialization and obtain in this way the full Sato Grassmannian.

The $\mathcal{D}$-bundle description of the Sato Grassmannian also gives rise to a geometric reformulation of KP Lax operators, the *micro-opers*, introduced in Section 3.5. Micro-opers are $\mathcal{D}$-bundles equipped with a microlocal endomorphism, which may be considered as a flat connection, satisfying a strong form of Griffiths transversality. A micro-oper on a curve X canonically determines (and is determined by) a matrix Lax operator away from the cusps of the underlying $\mathcal{D}$-bundle. In other words, the cusps of a $\mathcal{D}$-bundle provide a natural geometric description of the poles of matrix KP Lax operators. Thus, micro-opers are perfectly suited for the study of meromorphic solutions of multicomponent KP hierarchies, which are our primary motivation. Micro-opers are the analogs for the setting of (multicomponent) KP equations of the *opers* of Beilinson–Drinfeld [BD3], or more precisely of the *affine opers* of [BF], for the setting of Drinfeld–Sokolov (generalized KdV) equations.

Micro-opers carry a hierarchy of flows that express the KP hierarchy on Lax operators on the disc. The flows are simply given by modifying (the transition functions of) the underlying $\mathcal{D}$-bundle by powers of the microlocal endomorphism. These flows are part of a natural Lie algebroid on the space of $\mathcal{D}$-bundles, the KP algebroid. This algebroid consists of microlocal deformations of $\mathcal{D}$-bundles, that is, deformations coming from endomorphisms of their microlocalizations, acting by deforming "transition functions along the curve at infinity." Thus a micro-oper structure on a $\mathcal{D}$-bundle can be considered as a choice of element of this Lie algebroid, with fixed polar part at E_∞.

It is immediate from the construction of the Fourier–Mukai identification between $\mathcal{D}$-bundles and CM spectral sheaves that the corresponding Lie algebroids are identified. Informally, the Fourier–Mukai transform identifies the (commutative and noncommutative) ruled surfaces on which CM spectral sheaves and $\mathcal{D}$-bundles live, and respects the sections at infinity. Since both hierarchies are given by modifying sheaves along the respective sections at infinity, the CM and KP flows are intertwined by the Fourier–Mukai transform.

Theorem 1.3 (Theorem 3.25). *Let* $\mathbf{F} : \mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V) \to \mathfrak{CM}_n(E, V^\vee)$ *denote the Fourier–Mukai isomorphism of framed $\mathcal{D}$-bundles and framed CM spectral sheaves.*

(1) $\mathbf{F}$ *identifies the KP Lie algebroid with the CM Lie algebroid.*

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[1] In fact, in Section 6 we prove a precise algebraic statement showing that Sato's Grassmannian represents an appropriate functor of flat families of $\mathcal{D}$-modules.

(2) **F** *lifts to an isomorphism of the moduli stack of V-framed micro-opers with the moduli stack of $V^\vee$-framed Higgsed CM spectral sheaves, identifying the multicomponent KP hierarchy on micro-opers with the Higgsed CM hierarchy.*

In the rank one case this theorem gives a strong form of the correspondence between CM particles and meromorphic KP solutions:

Corollary 1.4 (Rank one reformulation). *The completed phase spaces of the rational, trigonometric, and elliptic (spinless) Calogero–Moser systems are identified with the moduli spaces of rational, trigonometric, and elliptic (rank one) KP Lax operators (taken up to change of coordinate in ∂^{-1}). This isomorphism identifies poles of Lax operators with positions of Calogero–Moser particles and identifies the KP and CM hierarchies.*

To paraphrase, the tweaking flows on CM spectral sheaves are simply the expression in action-angle coordinates of the matrix KP hierarchy. The location of the CM particles and the poles of the matrix Lax operator are both described by the cusps of a $\mathcal{D}$-bundle, and all multicomponent KP flows are described spectrally by different tweakings of spectral sheaves near the curve E_∞. For generic framings, in particular in the rank 1 case, these flows form a commuting family that is identified with Laurent series on the spectral curve near its intersection(s) with E_∞.[2] In other situations, the possible flows form a noncommuting family, although by choosing a micro-oper or Higgs structure we pick out abelian subalgebras of flows. This choice parallels the well-known dependence of matrix KP or Drinfeld–Sokolov hierarchies on choices of Heisenberg subalgebras of loop groups.

This theorem demystifies and generalizes the results of [AMM, CC, Kr1, Kr2, TV1, TV2, Shi, W2, BBKT, T1] on the relation between CM particles (and their collisions) and the meromorphic solutions of KP. In particular, the theorem extends the results of Wilson [W2] in two directions: over the family of cubic curves (that is, to the trigonometric and elliptic cases), and to higher rank (multicomponent/spin setting, as predicted in [W2, W4] in the rational trivially framed case). In particular, in the higher rank case we have the choice of framing V, with the trivial case $V = \mathcal{O}^k$ corresponding to the ordinary spin CM system, but other framings correspond to integrable systems with very different geometry and dynamics—in fact, generic framings V give rise to a much simpler (abelian) hierarchy.

It is interesting to note that we now have (even in the rank 1 case) two independent relations between $\mathcal{D}$-bundles and KP solutions—our construction involving micro-opers and the original construction of Wilson [W2] and [CH2], whereby the adelic Grassmannian on a curve X parametrizes Krichever data for algebro-geometric solutions of KP associated to X together with all of its cuspidal quotients (curves obtained by adding cusps to X). While there is no general relation between the two constructions, in genus zero (where Wilson was working) they are identified by the geometric Fourier transform on $\mathbb{A}^1$ (the natural auto-equivalence of $\mathcal{D}$-modules on $\mathbb{A}^1$ exchanging multiplication and differentiation). In other words, the geometric Fourier transform induces a self-map of

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[2] In other words, the choice of micro-oper structure or Higgsing is unique up to formal changes of coordinates in ∂^{-1} or equivalently on the spectral curve.

the appropriate moduli of $\mathcal{D}$-bundles on $\mathbb{A}^1$ that exchanges the KP algebroid (microlocal deformations of $\mathcal{D}$-bundles, near "$\partial^{-1} = 0$") with the algebroid deforming $\mathcal{D}$-bundles near $z^{-1} = 0$. This gives a simple geometric explanation of (and multicomponent generalization of) the bispectral involution on rational solutions of KP, studied in detail in [W1, BW1] (see Section 3.6.2). We plan to apply this point of view further in the less-explored setting of difference modules.

1.5. Fourier transform and moduli of $\mathcal{D}$-bundles

In Section 4, we recall the Fourier–Mukai autoequivalence of the bounded derived category of coherent sheaves on a cubic curve [BuK]. In Section 4.3, we extend to singular cubic curves the Fourier–Mukai transform for $\mathcal{D}$-modules, discovered by Laumon [La] and Rothstein [Ro2] for abelian varieties (and extended by Polishchuk and Rothstein [PRo] to general D-algebras). Namely, the Fourier–Mukai transform on a cubic curve identifies $\mathcal{D}$-modules with (complexes of) coherent sheaves on the surface $E^\natural \to E$, the "twisted log cotangent bundle" of E:

Theorem 1.5 (Theorem 4.12). *The bounded derived category of coherent $\mathcal{D}_{\log}$-modules on a cubic curve E is equivalent to the bounded derived category of coherent sheaves on $E^\natural$.*

We also show, following [PRo], that this Fourier transform "compactifies," identifying the derived category of modules over the Rees algebra of $\mathcal{D}$ (coherent sheaves on the non-commutative ruled surface $\overline{T_\hbar^* E}$) with that of coherent sheaves on the ruled surface $\overline{E}^\natural$, and respects microlocalization (restriction near E_∞).

In Section 5, the technical heart of the paper, we apply the Fourier–Mukai transform to prove Theorem 1.1, describing the moduli spaces of $\mathcal{D}$-bundles on E which are framed by a semistable vector bundle V on E of degree zero. Let $V^\vee$ denote the torsion coherent sheaf on E Fourier dual to V. We show that V-framed $\mathcal{D}$-bundles are sent by the Fourier transform to coherent sheaves on $\overline{E}^\natural$ (rather than complexes), of pure 1-dimensional support, whose restriction to the curve $E_\infty \subset \overline{E}^\natural$ is identified with $V^\vee$. More precisely, we establish an equivalence of the corresponding moduli stacks. In the cuspidal case, the moduli spaces of spectral sheaves are the rational spin Calogero–Moser spaces, which are identified with certain Nakajima quiver varieties. In this case, our result recovers the theorems of [BW2] classifying ideals in the Weyl algebra (and more general framed $\mathcal{D}$-bundles on $\mathbb{A}^1$) in terms of quiver data.

In Section 6, we prove Theorem 1.2, establishing that the Sato Grassmannian represents the functor of flat families of $\mathcal{D}$-*lattices* on the disc, which are finitely-generated $\mathcal{D}$-submodules of the ring of microdifferential operators on the disc. This extends Sato's description of the big cell and refines it from a set-theoretic to a scheme-theoretic statement (i.e., to families).

Finally, in Section 7, we carefully explain some technical aspects of $\mathcal{D}$-bundles on (possibly singular) cubic curves. More precisely, we discuss the sheaf $\mathcal{D}_{\log}$ of "logarithmic" differential operators on a cubic curve E, which is generated by the invariant vector field for the group structure on the smooth part of E. We also discuss filtered $\mathcal{D}_{\log}$-modules and their properties.

1.6. Further directions

In [BN6], we extend the picture developed in this paper by replacing differential operators by difference operators. We describe the nonabelian Toda lattice hierarchies in terms of difference modules, in particular realizing meromorphic (rational, trigonometric, and elliptic) solutions in terms of difference modules on cubic curves. A modified version of the Fourier–Mukai transform then identifies the moduli spaces of difference modules with spaces of spectral sheaves on a ruled surface over the corresponding cubic curves. These spectral sheaves realize the (generalized spin) Ruijsenaars–Schneiders (RS) relativistic many-body systems, which are deformations of the corresponding CM systems. This gives a general geometric picture of the Toda/RS correspondence [KrZ]. This point of view also has applications to the difference version of bispectrality.

In [BN5], we study the factorization (or "vertex algebra space") structure [BD2] on the adelic Grassmannian. This structure is shown to encapsulate both (infinitesimally) the $\mathcal{W}_{1+\infty}$-symmetry of the KP hierarchy and (globally) the Bäcklund transformations.

In [BN4], we develop a simple algebraic description of $\mathcal{D}$-bundles on arbitrary curves, which generalizes the quiver description of Calogero–Moser spaces (the $\mathbb{A}^1$ case). We use Koszul duality to describe moduli spaces of $\mathcal{D}$-modules on curves as twisted cotangent bundles to moduli of complexes of sheaves on the curve, extending the relation between vector bundles with connections and Higgs bundles. The resulting picture of $\mathcal{D}$-bundles has the advantage of being concrete, local, and functorial.

1.7. Contents

We will now describe the contents of the paper section by section. Section 2 recollects some necessary background material. Section 3 is the heart of the paper, containing the main results on $\mathcal{D}$-bundles and their application to the KP/CM correspondence. Sections 4 through 7 contain the technical tools used in Section 3. Specifically, Section 4 discusses the main tool, the extended Fourier–Mukai transform for $\mathcal{D}$-modules on cubic curves. Section 5 applies this transform to prove an isomorphism of moduli stacks of $\mathcal{D}$-bundles and spectral sheaves. Section 6 provides the proof for the $\mathcal{D}$-bundle description of the Sato Grassmannian. Finally Section 7 is an appendix containing needed material about differential operators and D-algebras on cubic curves.

2. Background material

In this section, we review some features of the geometry of cubic curves, describe the spin Calogero–Moser systems, and review the multicomponent KP hierarchies.

2.1. Background on cubic curves

2.1.1. One-dimensional groups and cubic curves. Connected one-dimensional complex groups **G** fall into three classes: the additive group $\mathbb{C}$, the multiplicative group $\mathbb{C}^\times$, and the one-parameter family of elliptic curves. These cases fall under the monikers *rational,*

trigonometric, and *elliptic* according to the type of functions on the universal cover $\mathbb{C}$ which correspond to meromorphic functions on **G**.

A parallel classification applies to reduced and irreducible cubic plane curves, which we will also call (slightly abusively) *Weierstrass cubic curves*: reduced and irreducible complex projective curves E of arithmetic genus one with a nonsingular marked point $b \in E$. (See [FM2] for a detailed discussion of cubic curves and bundles on them.) The map $q \mapsto \mathcal{O}(q - b)$ defines an isomorphism from the smooth locus of E to the Jacobian $\mathrm{Pic}^0(E)$; in particular, the smooth locus, which we denote by **G**, is equipped with a structure of one-dimensional group, with b as the identity element. The three types are then:

Rational: E is a cuspidal cubic, and is isomorphic to the curve $y^2 = x^3$. Its normalization $\mathbb{P}^1 \to E$ collapses $2 \cdot \infty$ to a cusp on E, and defines a group structure $\mathbb{C} = \mathbf{G} \subset E$ on the smooth locus.

Trigonometric: E is a nodal cubic, and is isomorphic to the curve $y^2 = x^2(x - 1)$. Its normalization $\mathbb{P}^1 \to E$ identifies two points 0 and ∞ to a node on E, and defines a group structure $\mathbb{C}^{\times} = \mathbf{G} \subset E$ on the smooth locus.

Elliptic: E is a smooth elliptic curve (in particular a group), and may be described by an equation of the form $y^2 = x^3 + ax + b$ with $\Delta = -16(4a^3 + 27b^2) \neq 0$.

Let ∞ denote the singular point in the rational and trigonometric cases. Then E is identified with its own compactified Jacobian, the moduli space of torsion free sheaves of rank one and degree zero on E. The singular point corresponds to the unique rank 1 degree 0 torsion-free sheaf which is not locally free, namely the modification $\mathfrak{m}_{\infty}(b)$ of the ideal sheaf of ∞.

2.1.2. Differential operators on cubic curves. The group variety **G** acts on $E \cong \overline{\mathrm{Jac}}(E)$, and admits a unique nonzero invariant vector field ∂ on E up to a scalar. If E is smooth this is the usual translation-invariant vector field, which is constant in a global analytic coordinate. Writing the singular cubics in terms of their normalization $\mathbb{P}^1$, a choice of ∂ is represented by $\partial/\partial z$ in the cuspidal case (vanishing to order 2 at $\infty \in \mathbb{P}^1$) and by $z\frac{\partial}{\partial z}$ in the nodal case (vanishing to order one at $0, \infty$).

We will abuse notation to denote the sheaf $\mathcal{O}_E \cdot \partial$ (i.e. the action algebroid of **G**) by $\mathcal{T}_E$. Note that in the nodal case this is the log tangent bundle of E; the dual sheaf will be denoted by Ω_E. The total space of Ω_E (which is isomorphic to $E \times \mathbb{C}$) will be denoted T^*E. Both $\mathcal{T}_E$ and Ω_E are trivial line bundles on E; however, they are not trivial over families of cubic curves, i.e. one cannot fix a ∂ (or differential) uniformly for all cubic curves, hence we will maintain this distinction when necessary. A cubic curve with a choice of ∂ gives a *differential curve* (curve with choice of global vector field, in analogy with differential fields). In fact the constructions of this paper will not require a differential structure (we only use the canonical line $\mathbb{C}\partial$).

The sheaf $\mathcal{T}_E$ acts by derivations of $\mathcal{O}_E$, hence it embeds in the sheaf of (Grothendieck) differential operators on E.

Definition 2.1. $\mathcal{D}_{\log}$ is the sheaf of $\mathcal{O}_E$-subalgebras of $\mathcal{D}_E$ generated by $\mathbb{C}\partial$.

We will usually abuse notation and terminology by denoting $\mathcal{D}_E$ the sheaf of differential operators generated by $\mathcal{T}_E$ and $\mathcal{O}_E$, and referring to it as the sheaf of *log differential operators* on E (which it is in the smooth and nodal cases). See Section 7 for more on $\mathcal{D}_{\log}$.

2.1.3. The surface $E^\natural$. In this section we discuss a surface $E^\natural$ which figures prominently in the study of both Calogero–Moser systems and the Fourier–Mukai transform.

We first consider the smooth case. Thus fix an elliptic curve E. Let A denote the *Atiyah bundle* on E, that is, the unique nontrivial extension of $\mathcal{O}_E$ by itself (up to isomorphism),

$$0 \to \mathcal{O} \to A \to \mathcal{O} \to 0. \tag{2.1}$$

Definition 2.2. We set $\overline{E}^\natural = \mathbb{P}(A)$. The algebraic surface $E^\natural$ is the complement of the section $E_\infty = \mathbb{P}(\mathcal{O}) \cong E$ of the projectivization of the Atiyah bundle, $E^\natural = \mathbb{P}(A) \setminus E_\infty$.

The resulting surface $E^\natural$ (which is a Stein manifold, but not an affine algebraic manifold, when E is an elliptic curve) is the unique (up to isomorphism) nontrivial torsor over $\mathcal{O}_E$. In classical terms, the surface $E^\natural$ may be viewed as the receptacle for the Weierstrass ζ-function of E (see [BN3] for a discussion).

To fix A and $E^\natural$ canonically we set $E^\natural = \operatorname{Conn} \mathcal{O}(b)$, the sheaf of connections on the line bundle $\mathcal{O}(b)$; this is a *twisted cotangent bundle* of E [BB] (i.e. an affine bundle for $\Omega_E \simeq \mathcal{O}_E$ with compatible symplectic structure). Let $\mathcal{A}$ denote the pushforward of $\mathcal{O}_{E^\natural}$ to E, i.e. the algebra of functions on the fibers of E. Thus $A = \mathcal{A}_{\leq 1}$, the subsheaf of affine functions on $E^\natural$, is isomorphic to the Atiyah bundle A. The sheaf A is canonically an extension of $\mathcal{T}_E$ (which is isomorphic to $\mathcal{O}_E$) by $\mathcal{O}_E$, and is isomorphic as $\mathcal{O}_E$-module to $A = \mathcal{D}_{\leq 1}(\mathcal{O}_E(b))$, the sheaf of differential operators of order at most one acting on the line bundle $\mathcal{O}(b)$. Concretely, the sheaf A lies in between

$$\mathcal{O}_E \oplus \mathcal{T}_E(-b) \subset A \subset \mathcal{O}_E \oplus \mathcal{T}_E(b),$$

and A is generated (in the canonical local coordinate near b) by $\mathcal{O}_E \oplus \mathcal{T}_E(-b)$ and the section $\partial - 1/z$. It is useful to note that the fiber $F_b \subset E^\natural$ over the basepoint b is canonically identified with the cotangent fiber to E at b.

It is also well-known that the twisted cotangent bundle $E^\natural$ is canonically identified with the universal additive extension of E, which is identified with the moduli space of line bundles with flat connection on E. This isomorphism is uniquely characterized as an isomorphism of torsors over the cotangent bundle preserving basepoints in the fiber over the identity b (for flat connections the basepoint is the trivial connection).

2.1.4. $E^\natural$ for singular cubics. The definition and properties of $E^\natural$ extend naturally to general cubics E. For any Weierstrass cubic E we have

$$\operatorname{Ext}^1_E(\mathcal{O}, \mathcal{O}) = H^1(E, \mathcal{O}) \cong \mathbb{C}.$$

So E has a unique nontrivial extension A of $\mathcal{O}_E \simeq \mathcal{T}_E$ by $\mathcal{O}_E$, up to isomorphism. (Recall that $\mathcal{T}_E$ is the subsheaf of the tangent sheaf generated by the **G**-action.) We again fix

$A = \mathcal{D}_{\leq 1}(\mathcal{O}_E(b))$ (here as elsewhere $\mathcal{D}$ denotes the sheaf of log differential operators). Let $\overline{E}^{\natural} \stackrel{\text{def}}{=} \underline{\text{Proj}}(\text{Sym}^{\bullet} A)$ denote the associated ruled surface, and $p : \overline{E}^{\natural} \to E$ denote the projection map.

The quotient map $A \twoheadrightarrow \mathcal{O}_E$ defines a section $s : E \to \overline{E}^{\natural}$; we write $E_{\infty} = s(E)$ and refer to it as the *section at infinity*. Note that every other section of $\overline{E}^{\natural}$ has nonempty intersection with E_{∞} since the sequence (2.1) is nonsplit. However, unlike the case of smooth E, if E is singular there are curves in $\overline{E}^{\natural}$ that fail to intersect E_{∞}: for example, the normalization of E embeds in $E^{\natural}$ since the extension A splits when pulled back to $\mathbb{P}^1$. The surface $E^{\natural} \stackrel{\text{def}}{=} \overline{E}^{\natural} \smallsetminus E_{\infty}$ is called the *twisted (log) cotangent bundle* of E; it is the nontrivial torsor over Ω_E given by the nonzero class (up to scale) in $H^1(\Omega_E)$. Canonically, it is given by the space of log-connections (liftings of $\mathcal{T}_E$) on the line bundle $\mathcal{O}(b)$. (See [BN3] for the relation of $E^{\natural}$ to the Weierstrass ζ-function of E.) In the context of the Fourier–Mukai transform, we will identify $E^{\natural}$ in Section 4.4 with the moduli space of rank one torsion-free sheaves on E with log connection. Again this isomorphism is uniquely characterized as an isomorphism of torsors over the (log) cotangent bundle preserving basepoints in the fiber over the identity b.

2.2. Calogero–Moser systems and spectral curves

In this section we discuss the spin Calogero–Moser system [GH], summarizing the detailed treatment in [BN3] (to which we refer for details and more complete references). See also [BN2] and [Ne2] for reviews of the usual (spinless) complex Calogero–Moser system following [KKS, W2, Ne1] (see [BBT] for a general introduction to integrable systems, including CM and KP).

As in the previous section, we let $\mathbf{G}$ denote a one-dimensional complex group and E the corresponding cubic curve. The k-spin n-particle Calogero–Moser system is a Hamiltonian dynamical system describing n identical particles moving on $\mathbf{G}$, each equipped with a vector in the auxiliary k-dimensional vector space $\mathbb{C}^k$; this vector is known as the *spin* of the particle. The system is then naturally described in terms of the positions $q_i \neq q_j$ in $\mathbf{G}$ of n distinct particles, momenta $p_i \in \mathbb{C}$ of the particles, spin vectors $v_j \in \mathbb{C}^k$, and a collection of covectors $u_i \in (\mathbb{C}^k)^*$. Let $f_{ij} = u_i(v_j) \in \mathbb{C}$ be the contraction of the ith covector with the jth vector. The Hamiltonian for the spin Calogero–Moser system is given by

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i<j} f_{ij} f_{ji} U(q_i - q_j).$$

Here the potential energy function U on $\mathbf{G}$ has a double pole at the origin, and is given (as a function on $\mathbb{C}$, the universal cover of $\mathbf{G}$) by

$$\mathbf{Rat} : U(q) = \frac{1}{q^2}, \quad \mathbf{Trig} : U(q) = \frac{1}{\sin^2(q)}, \quad \mathbf{Ell} : U(q) = \wp(q)$$

where $\wp(q)$ is the Weierstrass $\wp$-function attached to the elliptic curve E. The usual (spinless) Calogero–Moser particle system is recovered in the case $k = 1$ with all $f_{ij} = 1$.

The (spin) Calogero–Moser systems have a variety of different group-theoretic and geometric descriptions, all of which have the feature that they incorporate collisions of

the particles, in other words a locus where the q_i are no longer distinct. Moreover the Hamiltonian $H = H_2$ is but one of a family H_i of Poisson commuting Hamiltonian functions on these completed phase spaces. These are conveniently realized by writing the system in Lax form (with spectral parameter) or as a Hitchin system, and taking (residues of) traces of powers of the Lax operator (respectively, Higgs field). These descriptions are explained in detail in [BN3]; in this paper, we rely on the description of CM particles by *spectral sheaves*, which are geometric action-angle variables for the system, and which we will now review.

2.2.1. Calogero–Moser spectral sheaves. Moduli spaces of spectral sheaves (specifically, of line bundles on curves in a Poisson surface) give a wide class of examples of integrable systems (see e.g. [DM, Hu]). The prototypical example of such a setting is the (GL_n) Hitchin system on the moduli space $T^* \mathrm{Bun}_n(X)$ of Higgs bundles on a curve X, which can be described as a moduli of torsion free sheaves on curves in T^*X finite of degree n over X. We similarly realize the spin CM systems in terms of spectral curves on the twisted cotangent bundle $E^\natural$ of E, or rather its completion $\overline{E}^\natural$. Specifically, we consider sheaves on $\overline{E}^\natural$ with pure 1-dimensional support, with a framing condition along the curve $E_\infty = \overline{E}^\natural \setminus E^\natural$: the restriction of the sheaf to E_∞ is identified with a k-fold skyscraper $\mathcal{O}_b^{\oplus k}$ at the basepoint.

We highlight two aspects of this geometric translation: the positions of the particles and the role of the spins. We encode the positions $q_i \in \mathbf{G}$ of the Calogero–Moser particles as follows: n distinct points in $\mathbf{G}$ define a rank n vector bundle $W = \bigoplus \mathcal{O}(q_i - b)$ on E, which is in fact semistable of degree zero. Conversely, a generic degree 0 semistable bundle on E is of the form $W = \bigoplus \mathcal{O}(q_i - b)$ for n distinct points $q_i \in \mathbf{G} \subset E$ (determined up to permutation). More generally, there is an equivalence between semistable degree zero vector bundles W on an elliptic curve E and length n torsion coherent sheaves $W^\vee$ on the smooth locus $\mathbf{G}$ of E: $W^\vee$ is given by the Fourier–Mukai transform of W (see Section 4 and [FM1]). The same holds for singular cubics E if we additionally require that the pullback of W to the normalization of E is a trivial vector bundle. In the generic case we have $W^\vee = \bigoplus \mathcal{O}_{q_i}$. Thus the space of such bundles W provides a partial completion of the configuration space of points in $\mathbf{G}$.

The identification between spin CM particles and spectral sheaves has the feature of identifying the auxiliary space $\mathbb{C}^k$ in which the spins live with the space of sections of the restriction of the spectral sheaves to E_∞, which we have normalized to be $\mathbb{C}^k = \Gamma(\mathcal{O}_b^{\oplus k})$. This leads to a natural generalization of the spin CM system, in which we allow the spins to live in a general length k coherent sheaf T on E: we simply consider sheaves whose restriction to E_∞ is identified with T.

Definition 2.3. Fix a finite length coherent sheaf T on $\mathbf{G} \subset E$. A T-*framed CM spectral sheaf* is a pair $(\mathcal{F}, \phi)$ consisting of a coherent sheaf $\mathcal{F}$ on $\overline{E}^\natural$ of pure dimension one, together with an isomorphism $\phi : \mathcal{F}|_{E_\infty} \to T$, satisfying two normalization conditions:

 (i) $W = p_* \mathcal{F}(-E_\infty)$ is a semistable vector bundle of degree 0; if E is singular, we also require that the pullback of W to the normalization of E is a trivial vector bundle.
 (ii) $\deg(p_* \mathcal{F}(kE_\infty)) = (k+1) \deg(T)$ for all $k \geq -1$.

The T-*Calogero–Moser space* $\mathfrak{CM}(E, T) = \bigcup \mathfrak{CM}_n(E, T)$ is the moduli space of T-framed CM spectral sheaves $(\mathcal{F}, \phi)$ (for which the rank of the vector bundle W is n).

For the (usual) spin CM system described above, one chooses $T = \mathcal{O}_b^k$, and we will call objects of $\mathfrak{CM}_n^k(E) = \mathfrak{CM}_n(E, \mathcal{O}_b^k)$ simply *CM spectral sheaves*.

Remark 2.4. The normalization conditions (i) and (ii) are open conditions on coherent sheaves of pure dimension one. Condition (i) on the vector bundle W was discussed above. Regarding condition (ii), note that we have the following:

Lemma 2.5. *For a coherent sheaf $\mathcal{F}$ of pure dimension one on $\overline{E}^\natural$ such that $\mathcal{F}|_{E_\infty} \cong T$, $\mathcal{F}$ satisfies condition* (ii) *in Definition* 2.3 *if and only if*

$$\deg(p_* \mathcal{F}(k E_\infty)) = (k + 1) \deg(T) \quad \textit{for all } k \gg 0.$$

We will see later (Section 4) that the full phase space $\mathfrak{CM}_n(E, T)$ is identified by an extended Fourier–Mukai transform with a moduli space of objects (framed $\mathcal{D}$-bundles) on E whose singularities are precisely at the points q_i, and which are framed by the semistable vector bundle $T^\vee$ on E Fourier dual to T. This moduli space carries a natural integrable system, the multicomponent KP system.

2.2.2. Flows on spectral sheaves. In [BN3] we discuss under the name "tweaking" a simple construction of flows on moduli spaces of sheaves from germs of meromorphic functions. This is a generalization of (the infinitesimal version of) the action of the Picard group on sheaves by tensor product. Specifically, let $\mathcal{F}$ be a sheaf on a variety Y and $\Sigma \subset Y$ the support of $\mathcal{F}$. First note that any class in $c \in H^1(\Sigma, \mathcal{O})$ defines a first order deformation of any sheaf $\mathcal{F}$—for example, we consider c as a sheaf on Σ over the dual numbers and tensor with $\mathcal{F}$ to define a deformation class in $\text{Ext}^1(\mathcal{F}, \mathcal{F})$. If f is the germ near a point $s \in \Sigma$ of a meromorphic function on Σ, it defines a class $[f] \in H^1(\Sigma, \mathcal{O})$ (as the connecting map coming from the inclusion of $\mathcal{O}$ into meromorphic germs at s), which in turn defines first order deformations as above. If we wish to deform sheaves framed along a divisor $D \subset \Sigma$ (i.e. deform sheaves fixing their restriction to D) we project f to a class in $H^1(\Sigma, \mathcal{O}(-D))$ covering $[f]$. In [BN3] the flows of all meromorphic GL_n Hitchin systems are uniformly described in this way.

The spin CM flows are defined in [BN3] in this fashion (following [TV2] in the rank one case). Specifically, the choice of a global vector field ∂ on E (equivalently, a trivialization of the tangent space $\mathcal{T}_b$ at b) defines a canonical function $\underline{t}$ on the completion of $E^\natural$ along the fiber F_b with first order pole at E_∞. Namely $\underline{t} = \underline{k} + \pi^* \zeta$, where $\underline{k} : E^\natural \to \mathbb{P}^1$ is the composition of the canonical map from $E^\natural = \text{Conn} \, \mathcal{O}(b)$ to $T^* E$ with pole along the zero fiber, and ζ is (the Laurent expansion at b of) the Weierstrass ζ-function. The ring of polynomials in $\underline{t}$ is canonically identified (independently of the choice of ∂) with $\text{Sym}^\bullet \mathcal{T}_b = \mathbb{C}[\partial]$. We thus have the following definition from [BN3]:

Definition 2.6. The *Calogero–Moser hierarchy* is the action of the polynomial ring $\text{Sym}^\bullet \mathcal{T}_b = \mathbb{C}[\partial]$ on $\mathfrak{CM}_n^k(E)$ by tweaking of spectral sheaves by powers of $\underline{t}$ at b_∞.

Proposition 2.7. *The CM flows are the flows associated to the CM Hamiltonians: the vector field given by the action of ∂^i is given by Poisson bracket with the ith CM Hamiltonian H_i.*

2.2.3. The Lie algebroid of tweakings. Rather than privileging the meromorphic germ $\underline{t}$, we may consider more general deformations of CM spectral sheaves. Let $\mathcal{A}_\mathcal{E}$ denote the sheaf of functions on the punctured formal neighborhood of $E_\infty \subset \overline{E}^\natural$, i.e. the sheaf of Laurent series along E_∞. For a spectral sheaf $\mathcal{F}$ we may consider its "microlocalization" $\mathcal{F}_\mathcal{E} = \mathcal{F} \otimes_{\mathcal{O}_{\overline{E}^\natural}} \mathcal{A}_\mathcal{E}$, i.e. the localization of $\mathcal{F}$ to this deleted formal neighborhood of E_∞. Since $\mathcal{F}$ has pure 1-dimensional support, it embeds (as $\mathcal{O}_{\overline{E}^\natural}$-module) into $\mathcal{F}_\mathcal{E}$. Given any endomorphism $\xi \in \operatorname{End}_{\mathcal{A}_\mathcal{E}}(\mathcal{F}_\mathcal{E})$, we obtain a canonical first order deformation of $\mathcal{F}$, deforming $\mathcal{F} \subset \mathcal{F}_\mathcal{E}$ to $(1+\epsilon\xi)\mathcal{F} \subset \mathcal{F}_\mathcal{E}$ (over the dual numbers). More formally, construct a first order deformation of $\mathcal{F}$, $[\xi] \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$, as the image (under a connecting homomorphism) of the operation of restricting sections of $\mathcal{F}$ to $\mathcal{F}_\mathcal{E}$:

$$\{s \mapsto \xi(s_\mathcal{E}) \bmod \mathcal{F}\} \in \operatorname{Hom}(\mathcal{F}, \mathcal{F}_\mathcal{E}/\mathcal{F}) \to \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}).$$

If we project instead to $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F}(-E_\infty))$ we likewise obtain deformations of framed spectral sheaves.

Lemma 2.8. *The sheaf* $\underline{\operatorname{End}}_\mathcal{E}$ *over* $\mathfrak{CM}_n(E, T)$ *of endomorphisms of the microlocalization of the universal sheaf (i.e. the sheaf whose fiber at $\mathcal{F}$ is* $\operatorname{End}_{\mathcal{A}_\mathcal{E}}(\mathcal{F}_\mathcal{E})$) *has the structure of Lie algebroid, the* CM algebroid, *given by the tweaking action on spectral sheaves. The anchor map sends an endomorphism to the corresponding* Ext^1 *class defined above.*

This algebroid is the spectral sheaf analog of the (transitive) Lie algebroid on the moduli stack $\operatorname{Bun}_G(X)$ of G-bundles on a curve associated to the loop algebra $L\mathfrak{g}$ at a point $x \in X$, whose fiber at a bundle $\mathcal{P}$ is the adjoint twist of the loop algebra, $(L\mathfrak{g})_\mathcal{P}$.

This Lie algebroid structure becomes very simple for the typical case of *rank one* spectral sheaves. Namely if $\mathcal{F}$ is a rank one torsion-free sheaf on its support, then its endomorphisms are simply given by functions on the support: $\operatorname{End}(\mathcal{F}_\mathcal{E})$ is the direct sum of Laurent series on each component of the support of $\mathcal{F}_\mathcal{E}$, i.e. each branch of the spectral curve passing through E_∞. In this case the action of the algebroid is simply given by tweaking $\mathcal{F}$ by meromorphic function germs, and the resulting flows may be easily written explicitly as Hitchin Hamiltonian flows following [BN3]. This is guaranteed if $T = \bigoplus \mathcal{O}_{x_i}$ is a direct sum of skyscrapers at *distinct* points $x_i \in E$, for example in the standard spinless case $T = \mathcal{O}_b$.

For general spectral sheaves this produces a noncommutative family of flows on the moduli space. Namely, the algebroid at $\mathcal{F}$ is (noncanonically) isomorphic to a sum

$$\operatorname{End}(\mathcal{F}_\mathcal{E}) \simeq \bigoplus L\mathfrak{gl}_{k_i}$$

of loop algebras, where for each component Σ_i of the spectral curve $\operatorname{Supp} \mathcal{F}$ near E_∞, k_i is the rank of $\mathcal{F}$ on Σ_i.

To put the resulting flows in the more familiar form of a hierarchy of flows labeled by natural numbers, we can instead look at spectral sheaves together with a "Higgs" structure, breaking the symmetry down to an abelian family of flows (in analogy with the abelian Lie algebroid on the moduli space of Higgs bundles, giving the Hitchin integrable system):

Definition 2.9. Fix $\partial \in \Gamma(\mathcal{T})$. A *Higgsed T-framed CM spectral sheaf* is a pair $(\mathcal{F}, \xi)$ consisting of a T-framed CM spectral sheaf $\mathcal{F}$ and a germ ξ of a section of $\operatorname{End}(\mathcal{F})(E_\infty)$ at $b_\infty \in E_\infty$, with $\xi|_{E_\infty} = \partial$ as a section of $\mathcal{O}_{\overline{E}^\natural}(E_\infty)|_{E_\infty} = \mathcal{T}_E$.

We may then define the Higgsed CM hierarchy on enhanced spectral sheaves as the action of the ring $\mathbb{C}[\partial]$, where $P(\partial)$ acts on $(\mathcal{F}, \xi)$ by deforming $\mathcal{F}$ by $P(\xi)$. Note that the CM flows defined above are the restriction of the CM hierarchy to the locus of sheaves Higgsed by $\underline{t}$.

2.3. The KP hierarchy

Recall that the *Kadomtsev–Petviashvili* (or *KP*) *equation* is the following partial differential equation for a function $u = u(t, x, y)$:[3]

$$\frac{3}{4} u_{xx} = \left(u_y - \frac{1}{4}(6uu_t + u_{ttt}) \right)_t, \tag{2.2}$$

which first arose in connection with the study of shallow water waves. Overviews of this equation and its algebro-geometric significance, as well as bibliographies, may be found in [Mul, DJM, Ar]. (Our understanding of soliton equations is deeply indebted to the seminal article [SW].) In this section we review the multicomponent KP hierarchies and their formulation using the Sato Grassmannian.

2.3.1. Microdifferential operators.

We begin by recalling the basic algebraic objects, the algebras of differential and microdifferential operators over the ring of formal power series: for the rest of this section, $\mathcal{D} \subset \mathcal{E}$ will denote the algebras of differential operators and of formal microdifferential operators (also known as pseudodifferential symbols) with coefficients in $\mathbb{C}[[t]]$. Specifically, an element of $\mathcal{E}$ is a Laurent series

$$M = \sum_{N \ll \infty} a_N \partial^N, \qquad a_i \in \mathbb{C}[[t]], \tag{2.3}$$

in the formal inverse ∂^{-1} of the derivation $\partial = \partial_t$ of $\mathbb{C}[[t]]$. Such an element M lies in $\mathcal{D}$ if $a_N = 0$ for $N < 0$.

The symbol ∂^{-1} does not, of course, make sense as an operator on $\mathbb{C}[[t]]$; however, it is possible to give the set $\mathcal{E}$ of microdifferential operators an algebra product. The composition in $\mathcal{E}$ is determined by the Leibniz rule,

$$\partial^n \cdot f = \sum_{i \geq 0} \binom{n}{i} f^{(i)} \partial^{n-i}, \tag{2.4}$$

where $\binom{n}{i}$ is defined for $n < 0$ by taking

$$\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 2 \cdot 1}.$$

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[3] For compatibility with our later choice of notation, we have permuted the usual labeling of variables in this equation.

This product structure makes $\mathcal{E}$ into a filtered algebra, with the kth term in the filtration[4] given by

$$\mathcal{E}_k = \{a_k \partial^k + a_{k-1}\partial^{k-1} + \dots\}.$$

The induced product on $\mathcal{D}$ is the usual one, and $\mathcal{D}$ becomes a filtered subalgebra (again, with the filtration induced from $\mathcal{E}$ agreeing with the usual order filtration on differential operators).

2.3.2. The multicomponent KP hierarchy. We now review the Lax formulation of the (multicomponent) KP hierarchy; for more information, see [KV, P].

The algebra $\mathbf{v}_n = \mathfrak{gl}_n(\mathcal{E})$ of matrix-valued microdifferential operators consists of series as in (2.3), but with $a_N \in \mathfrak{gl}_n[\![t]\!]$ with its usual multiplication. We also have a sub-algebra $\mathbf{v}_n^{\mathbb{C}} = \mathfrak{gl}_n((\partial^{-1})) \subset \mathcal{E}$ of *constant coefficient* matrix microdifferential operators, and an abelian subalgebra $\Gamma = \mathbb{C}((\partial^{-1})) \subset \mathbf{v}_n^{\mathbb{C}}$ of constant coefficient scalar operators.

Definition 2.10. (1) A *matrix KP Lax operator* is a microdifferential operator of the form

$$L = \mathrm{Id}\,\partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \dots \in \mathfrak{gl}_n(\mathcal{E}),$$

where $u_i \in \mathfrak{gl}_n[\![t]\!]$ for all i. We let $\mathcal{L}_n$ denote the set of all such $n \times n$ matrix KP Lax operators.

(2) A *matrix KP wave operator* is a microdifferential operator of the form

$$W = \mathrm{Id} + w_1 \partial^{-1} + w_2 \partial^{-2} + \dots, \tag{2.5}$$

where $w_i \in \mathfrak{gl}_n[\![t]\!]$ for all i. We refer to the group (in fact group scheme) of all matrix KP wave operators as the *n-component Volterra group* and denote it by $\mathbf{V}_n$.

We let $\mathbf{V}_n^{\mathbb{C}} \subset \mathbf{V}_n$ denote the multiplicative group of constant coefficient matrix microdifferential operators, that is, expressions of the form 2.5 with $w_i \in \mathfrak{gl}_n(\mathbb{C})$. We also let Γ_+ denote the abelian Lie algebra $\mathbb{C}[\partial] \subset \mathfrak{gl}_n(\mathcal{E})$.

Lemma 2.11. *The set $\mathcal{L}_n$ of matrix KP Lax operators forms an infinite-dimensional affine space. The n-component Volterra group scheme $\mathbf{V}_n$ acts transitively (on the left) on $\mathcal{L}_n$ by $(W, L) \mapsto WLW^{-1}$. Under this action, the stabilizer of ∂ is the sub-group-scheme $\mathbf{V}_n^{\mathbb{C}}$, and thus $\mathcal{L}_n$ is naturally identified with $\mathbf{V}_n^{\mathbb{C}} \backslash \mathbf{V}_n$ as a scheme.*

Definition 2.12. The *multicomponent KP hierarchy* is the collection of compatible evolution equations on a Lax operator L defined as follows:

$$\frac{\partial L}{\partial t_n} = [L, (L^n)_+], \tag{2.6}$$

where $(M)_+ = \sum_{N \geq 0} a_N \partial^N \in \mathfrak{gl}_n(\mathcal{D}) \subset \mathfrak{gl}_n(\mathcal{E})$ denotes the differential part of a matrix microdifferential operator M as in (2.3). That is, we let the operator $L = L(t, t_1, t_2, \dots)$ depend on the infinitely many time variables t_n and then require that the dependence of L on t_n (i.e. its "evolution along the nth time") satisfies (2.6).

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[4] We use subscripts to denote the filtration degree in $\mathcal{E}$; while this is nonstandard, we hope it will prevent confusion between filtration degree and the superscripts we will use to denote the rank of a free module.

The KP hierarchy can be written as an action of the abelian Lie algebra $\Gamma_+ = \mathbb{C}[\partial]$ on $\mathcal{L}_n$, i.e. a collection of commuting vector fields ∂_n on the affine space $\mathcal{L}_n$ corresponding to the action of $\partial^n \in \Gamma_+$. We define the vector field ∂_n on the affine space $\mathcal{L}_n$ by taking its value at L to be the commutator $[L, (L_n)_+]$. A solution L of the equations (2.6) of the KP hierarchy is then just an operator $L(t, t_1, t_2, \dots)$ that gives (formal) integral curves of all these vector fields simultaneously. Note that the first KP time t_1 is naturally identified with translation along the original variable t. In the case $n = 1$, the compatibility of the equations (2.6) in $x = t_2$ and $y = t_3$ (i.e. the fact that the corresponding vector fields on the space of Lax operators commute) implies that $u = u_1$ satisfies equation (2.2).

2.3.3. Sato Grassmannian. Sato's formulation of the KP hierarchy begins with the introduction of an infinite-dimensional Grassmannian.

Consider the vector space $\mathbb{C}((z^{-1}))^n$ of n-component Laurent series in a parameter z^{-1}. This is a topological vector space when equipped with a basis of open neighborhoods of 0 given by the subspaces $z^k \mathbb{C}[\![z^{-1}]\!]^n$ for $k \in \mathbb{Z}$.[5] The space $\mathbb{C}((z^{-1}))^n$ is the direct sum of the linearly compact vector space $z^{-1}\mathbb{C}[\![z^{-1}]\!]^n$ and the discrete vector space $\mathbb{C}[z]^n$.

A *c-lattice* (or simply *lattice*) in $\mathbb{C}((z^{-1}))^n$ is a compact open subspace; one can check that, equivalently, a vector subspace $B \subset \mathbb{C}((z^{-1}))^n$ is a c-lattice if there exist integers k and ℓ such that $(\mathbb{C}[\![z^{-1}]\!]z^k)^n \subseteq B \subseteq (\mathbb{C}[\![z^{-1}]\!]z^\ell)^n$. A *d-lattice* in $\mathbb{C}((z^{-1}))^n$ is a discrete subspace that is complementary to a *c*-lattice.

Definition 2.13. The *Sato Grassmannian* $\mathrm{GR}_n = \mathrm{GR}(\mathbb{C}((z^{-1}))^n)$ is the set of *d*-lattices in $\mathbb{C}((z^{-1}))^n$.

Equivalently, one has the following well-known description.

Lemma 2.14. *The Sato Grassmannian parametrizes subspaces $B \subset \mathbb{C}((z^{-1}))^n$ whose projections on $\mathbb{C}((z^{-1}))^n/\mathbb{C}[\![z^{-1}]\!]^n$ have finite-dimensional kernel and cokernel.*

The index of the projection map in the lemma, also known as the *index of the subspace B*, gives a numerical invariant of subspaces B. The *big cell* $\mathrm{GR}_n^\circ \subset \mathrm{GR}_n$ consists of subspaces B which project *isomorphically* onto $\mathbb{C}((z^{-1}))^n/z^{-1}\mathbb{C}[\![z^{-1}]\!]^n \cong \mathbb{C}[z]^n$; such *d*-lattices are said to be *generic*.

The Sato Grassmannian can be given the structure of an infinite-dimensional scheme: see Section 6.1. With this structure, each connected component consists of exactly the subspaces of index k for a fixed k; we call the index k component Gr_n^k.

2.3.4. KP flows via the Sato Grassmannian. Sato's purpose in introducing the Grassmannian GR_n (see [S]) was as follows.

Given a matrix KP wave operator W, one obtains a free right $\mathcal{D}$-submodule $W \cdot \mathcal{D}^n \subset \mathcal{E}^n$; since $W \cdot \mathcal{E}_{-1}^n = \mathcal{E}_{-1}^n$, the submodule $W \cdot \mathcal{D}^n$ is a generic $\mathcal{D}$-lattice, and thus $W \cdot (\mathcal{D}^n/\mathcal{D}^n t)$ is a generic *d*-lattice.

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[5] More precisely, this makes $\mathbb{C}((z^{-1}))^n$ into a *Tate vector space*, or locally linearly compact space [BD1, D]. This is a topological vector space which can be written as a direct sum of a discrete vector space and a linearly compact vector space (the topological dual to a discrete vector space).

The identification $\mathbb{C}((z^{-1}))^n = \mathcal{E}^n/\mathcal{E}^n t$ gives rise to an action of the group $GL_n(\mathcal{E})$ (on the left) and of the Lie algebra $\mathfrak{gl}_n(\mathcal{E})$ on $\mathbb{C}((z^{-1}))^n$ and therefore, one may check, on the Sato Grassmannian GR_n. In particular, we obtain actions on GR_n of the Volterra group $\mathbf{V}_n$ and its subgroup $\mathbf{V}_n^{\mathbb{C}}$ of constant coefficient negative microdifferential operators, as well as an action by vector fields of the Lie algebra Γ_n. The abelian subalgebra Γ thereby gives rise to an infinite family of commuting vector fields on GR_n. Sato's approach to the KP hierarchy is then encapsulated in the following theorem (see [Mul]):

Theorem 2.15 (Sato). (1) *The group* $\mathbf{V}_n$ *preserves the big cell* GR_n°, *on which it acts simply transitively. Thus, every generic d-lattice is of the form* $W \cdot (\mathcal{D}^n/\mathcal{D}^n t)$ *for a unique* $W \in \mathbf{V}_n$.

(2) *The isomorphism*

$$\mathbf{V}_n^{\mathbb{C}}\backslash\mathrm{GR}_n^{\circ} \xleftarrow{\sim} \mathbf{V}_n^{\mathbb{C}}\backslash\mathbf{V}_n \xrightarrow{\sim} \mathcal{L}_n, \qquad W \cdot (\mathcal{D}^n/\mathcal{D}^n t) \leftrightarrow W(\mathrm{Id}\,\partial)W^{-1}$$

between the quotient of the big cell and the space of Lax operators identifies the infinitesimal action of $\partial^n \in \mathbb{C}[\partial] \subset \Gamma$ *on* $\mathbf{V}_n^{\mathbb{C}}\backslash\mathrm{GR}_n^{\circ}$ *with the nth KP flow* $\partial/\partial t_n$ *on Lax operators.*

3. $\mathcal{D}$-bundles

In this section we study $\mathcal{D}$-bundles and use them to relate the KP and CM integrable systems. In Section 3.1 we define $\mathcal{D}$-bundles and describe their main properties. Rank one $\mathcal{D}$-bundles are, roughly, nothing more than right ideals in the algebra of differential operators on a curve (with the caveat that they can only be embedded in this algebra locally on the curve). In Section 3.2 we describe the identification between moduli spaces of $\mathcal{D}$-bundles and moduli spaces of CM spectral sheaves. In Section 3.3 we relate the moduli spaces of $\mathcal{D}$-bundles to Wilson's adelic Grassmannian. In Section 3.4 we give a $\mathcal{D}$-bundle description of the Sato Grassmannian, which we use in Section 3.5 to interpret KP Lax operators as enhanced $\mathcal{D}$-bundles, the *micro-opers*. Finally in Section 3.25 we explain the compatibility between the KP hierarchy on $\mathcal{D}$-bundles on cubic curves and the corresponding CM integrable systems.

3.1. $\mathcal{D}$-bundles on curves

Let X denote either a smooth quasiprojective curve or a Weierstrass cubic curve. We denote by $\mathcal{T}_X$ the tangent sheaf, T^*X the cotangent bundle, $\mathcal{D} = \mathcal{D}_X$ the sheaf of differential operators on X, with the convention that in the singular cubic case these notations are defined as in Section 2.1 (i.e. as "log" versions).

Definition 3.1 (see [BD2]). A $\mathcal{D}$-*bundle* M on X is a locally projective coherent right $\mathcal{D}_X$-module—if X is singular, we require in addition that M be isomorphic to $\mathcal{D}_X^n$ (for some n) in a neighborhood of the singular locus.

On an affine curve, one obtains a large number of examples of $\mathcal{D}$-bundles of rank one by taking finitely generated right ideals of $\mathcal{D}(X)$. A typical rank 1 $\mathcal{D}$-bundle M on a curve X is not locally free, but only *generically* locally free: away from finitely many

points of X, M is locally isomorphic to $\mathcal{D}$—note the similarity to the behavior of the ideal sheaf of a collection of points on an algebraic surface. For example, taking $X = \mathbb{A}^1$ and thus $\mathcal{D}(X) = \mathbb{C}[x, \partial]$, the right ideal I in $\mathcal{D}(\mathbb{A}^1)$ generated by x^2 and $1 - x\partial$ is projective (indeed, the ring $\mathcal{D}(\mathbb{A}^1)$ is hereditary), but there is no $f \in \mathbb{C}[x]$ such that M_f is a free $\mathcal{D}(X)$-module. See [CH2, BW2, BN1, BN2] for more discussion.

Fix a coherent torsion sheaf $V^\vee$ on E supported on the smooth locus of E; under the Fourier–Mukai transform (see Section 4), this determines a vector bundle V on E.

Definition 3.2. A V-*framed $\mathcal{D}$-bundle* is a $\mathcal{D}$-bundle M equipped with a $\mathcal{D}$-module filtration $\{M_k\}$ and an isomorphism

$$\phi : \bigoplus_{k \geq k'} \mathrm{gr}_k(M) \to V \otimes \mathrm{gr}_{\geq k'}(\mathcal{D})$$

of $\mathrm{gr}(\mathcal{D})$-modules for some k' with the following properties:

(i) If E is singular, there is an open neighborhood U of ∞ such that $M|_U$ is isomorphic to $V \otimes \mathcal{D}_{\log}|_U$ compatibly with the isomorphism ϕ.
(ii) The canonical filtration $\{M_k\}$ of M (see Definition 7.6) satisfies

$$\mathrm{rk}(M_k) = \begin{cases} \mathrm{rk}(V)(k + 1) & \text{for } k \geq 0, \\ 0 & \text{for } k < 0. \end{cases}$$

An isomorphism of V-framed $\mathcal{D}$-bundles $(M, \{M_k\}, \phi)$ and $(M', \{M'_k\}, \phi')$ is a $\mathcal{D}$-module isomorphism $\psi : M \to M'$ such that $\psi(M_k) = M'_k$ for all $k \gg 0$ and $\phi' \circ \mathrm{gr}(\psi) = \phi$ (when k' is large enough so that both sides are defined).

Remark 3.3. Conditions (i) and (ii) of the definition are not necessary for a reasonable theory of $\mathcal{D}$-bundles; however, they are essential for applications to multicomponent KP. Condition (i) simply tells us, in light of the relationship (which we will explain shortly) between singularities of $\mathcal{D}$-bundles and singularities of meromorphic KP solutions, that the KP solution is regular "at infinity." If we replace filtered modules over $\mathcal{D}$ by filtered modules over $\mathrm{Sym}(\mathcal{T}_E \oplus \mathcal{O}_E)$, then condition (ii) becomes the condition that the vector bundle on $S = \mathbb{P}(\mathcal{T}_E \oplus \mathcal{O}_E)$ corresponding to the graded module $\mathcal{R}(M)$ is trivial upon restriction to a generic fiber of the projection map $S \to E$—in particular, it is an open condition.

We let $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, V)$ denote the moduli stack of V-framed $\mathcal{D}$-bundles (for a precise definition see Section 5.1). For a projective curve X, $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, V)$ is an algebraic stack.

3.2. Moduli of $\mathcal{D}$-bundles and Calogero–Moser spectral sheaves

We now specialize to the case of a cubic curve $X = E$. Suppose V is a semistable vector bundle of degree zero (with trivial pullback to the normalization, in case E is singular), and let $V^\vee$ denote the finite length sheaf on $\mathbf{G} \subset E$ defined as the Fourier–Mukai transform of V (see Section 4). We will prove the following theorem (Theorem 5.1):

Theorem 3.4. *There is an isomorphism of stacks*

$$F : \mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V) \to \mathfrak{CM}(E, V^\vee),$$

given by the $\mathcal{D}$-module Fourier–Mukai transform (Theorem 4.12). In particular there is an isomorphism between trivially framed rank k $\mathcal{D}$-bundles $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, \mathcal{O}^k)$ and the union over n of the k-spin n-particle Calogero–Moser spaces $\mathfrak{CM}_n^k(E)$.

Note that the following invariant of $\mathcal{D}$-bundles appears implicitly in the theorem. Let M be a V-framed $\mathcal{D}$-bundle on E; the framing induces a canonical inclusion $\mathrm{gr}(M) \hookrightarrow V \otimes \mathrm{gr}(\mathcal{D})$ of finite colength (where we have used the canonical filtration on M, see Definition 7.6).

Definition 3.5. The *local second Chern class $c_2(M)$* is the numerical invariant

$$c_2(M) = \mathrm{length}(V \otimes \mathrm{gr}(\mathcal{D})/\mathrm{gr}(M)).$$

This local second Chern class is the noncommutative analog of the numerical invariant $\mathrm{length}(\mathcal{F}^{**}/\mathcal{F})$ for a torsion-free coherent sheaf $\mathcal{F}$ on a nonsingular surface, which, if the surface is projective, is exactly $c_2(\mathcal{F}) - c_2(\mathcal{F}^{**})$: thus, it is the part of the second Chern class that measures "how far $\mathcal{F}$ is from being a vector bundle." With this definition, the component of $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V)$ parametrizing $\mathcal{D}$-bundles with $c_2 = n$ corresponds to $\mathfrak{CM}_n(E, V^\vee)$.

Let M denote a V-framed $\mathcal{D}$-bundle and $\mathcal{F}$ the corresponding $V^\vee$-framed spectral sheaf. Let $W = \pi_* \mathcal{F}(-E_\infty)$ denote the associated semistable degree 0 vector bundle on E (part (i) of Definition 2.3), so that its Fourier transform $W^\vee$ is the torsion sheaf on E measuring the position of the Calogero–Moser particles.

Proposition 3.6. *The V-framed $\mathcal{D}$-bundle M is canonically identified with $V \otimes \mathcal{D}$ away from the location of the corresponding CM particles. More precisely, there is a short exact sequence*

$$0 \to M \to \widetilde{V} \otimes \mathcal{D} \to Q \to 0,$$

where the vector bundle $\widetilde{V}$ is identified with V away from the support of the torsion sheaf $W^\vee$ and $\mathrm{supp}(W^\vee) = \mathrm{supp}(Q)$.

Proof. Give M the canonical filtration. There is then a canonical injective homomorphism $M_0 = \mathrm{gr}_0(M) \to \mathrm{gr}_0 V \otimes \mathcal{D} = V$. Let $D = \mathrm{supp}(V/M_0)$ and $U = X \setminus D$; it is immediate that U is the largest open set over which $M \cong V \otimes \mathcal{D}$ and that the framing of M determines an inclusion $M \hookrightarrow (V \otimes \mathcal{D})|_U$. Since M is finitely generated, one may choose a coherent sheaf $\widetilde{V}$ so that $V \subseteq \widetilde{V} \subset V|_U$ and $M \subseteq \widetilde{V} \otimes \mathcal{D}$.

It remains to check that $D = \mathrm{supp}(W^\vee)$. For this, we use the exact sequence (5.2) of Proposition 5.8; it is then immediate from (5.3) and the discussion following it that, in the notation used there, $D = \mathrm{supp}(F^0(Q_k)) = \mathrm{supp}(W^\vee)$. $\qquad\square$

Remark 3.7. Proposition 3.6 is an analog of an easy statement about vector bundles (or torsion-free sheaves) on a ruled surface. Namely, the condition that the vector bundle be trivial on a fiber is open in the base (since the trivial bundle is open in bundles on $\mathbb{P}^1$), and over that open set, the trivialization is fixed by choosing a trivialization of the bundle along a section.

It is not hard to see that a V-framed $\mathcal{D}$-bundle on any nonsingular curve is canonically identified with $V \otimes \mathcal{D}$ outside of a finite subscheme, the *cusps* of the $\mathcal{D}$-bundle [BN1], where the local c_2 is supported; moreover, this is true for families of $\mathcal{D}$-bundles when properly formulated. This follows for example from the local description of $\mathcal{D}$-bundles (Proposition 3.16): the canonical trivialization is defined wherever the local data of the $\mathcal{D}$-bundle is in the big cell of the Sato Grassmannian. In fact, there is a more precise version of the proposition that holds on any nonsingular curve: namely, if we choose $\widetilde{V}$ minimal, then Q will have length $n = c_2(M)$. This exact sequence will prove its value in [BN4].

3.3. The adelic Grassmannian

In this section we let X denote an arbitrary smooth projective curve or a Weierstrass cubic.

Definition 3.8. The *rank k adelic Grassmannian* $\mathrm{Gr}_k^{\mathrm{ad}}(X)$ is the set of isomorphism classes of (unframed) $\mathcal{D}$-bundles M equipped with a generic trivialization $M \otimes K(X) \cong \mathcal{D}_X^k \otimes K(X)$. When $k = 1$ we denote $\mathrm{Gr}_1^{\mathrm{ad}}(X)$ by $\mathrm{Gr}^{\mathrm{ad}}(X)$.

Remark 3.9. The adelic Grassmannian is not, in any reasonable way, a variety or, more generally, ind-scheme. However, it does have a reasonable algebro-geometric structure if we keep track of the poles of the trivialization. More precisely, for a finite subset $\mathbf{x} = (x_1, \ldots, x_r) \in X^r$ the adelic Grassmannian $\mathrm{Gr}_k^{\mathrm{ad}}(X, \mathbf{x})$ at $\mathbf{x}$ is the moduli of $\mathcal{D}$-bundles, M equipped with an isomorphism $M|_{X \setminus \mathbf{x}} \to \mathcal{D}_X^k|_{X \setminus \mathbf{x}}$. The set $\mathrm{Gr}_k^{\mathrm{ad}}(X, \mathbf{x})$ is the set of points of an ind-scheme of ind-finite type over X^r, and the full adelic Grassmannian is the inductive limit of these ind-schemes under the (unfiltered) directed system of the X^r under the action of permutations and diagonal maps. The directed system of ind-schemes itself is what is known as a *factorization space* [BD2]—see [BN5].

Remark 3.10. The adelic Grassmannian we have defined is not quite the same as the adelic Grassmannian of Wilson [W2]. However, the difference is (mostly) harmless: Wilson considered the case in which $X = \mathbb{A}^1$ (or, more precisely, $X = \mathbb{P}^1$ with the canonical marked point ∞). The triviality of $\mathrm{Pic}(X)$ in this case (up to a twist at the marked point ∞) shows that Wilson's adelic Grassmannian is the quotient of our $\mathrm{Gr}^{\mathrm{ad}}(\mathbb{P}^1)_\infty$ by the action of multiplication by $K(\mathbb{P}^1)^\times = \mathbb{C}(z)^\times$ on the framing, together with modification at ∞ ($\mathcal{O} \mapsto \mathcal{O}(k\infty)$). We prefer this definition since it is representable by a factorization ind-scheme.

Fixing $\infty \in X$, let $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, V)_\infty \subset \mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, V)$ denote the open subspace of $\mathcal{D}$-bundles which are locally free at ∞ and let $\mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty \subset \mathrm{Gr}_k^{\mathrm{ad}}(X)$ be the subset of $\mathcal{D}$-bundles with a generic trivialization defined at ∞ (which has a structure of $\mathbb{C}$-space). For example if X is a singular cubic, we may take for ∞ the singular point, where all our $\mathcal{D}$-bundles are already required to be locally free.

Let $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, k)_\infty \to \mathrm{Bun}_k(X)$ denote the moduli stack for pairs (V, M) consisting of a rank k vector bundle V on X (of any degree) trivialized at ∞ and a V-framed $\mathcal{D}$-bundle M on X. Let $\mathcal{K}(X)_\infty$ denote the rational functions on X regular at ∞. We now state an algebraic description of Wilson's decomposition of the (rank one rational) adelic Grassmannian into Calogero–Moser spaces (which we interpret as moduli spaces of $\mathcal{D}$-bundles):

Corollary 3.11. (1) *There is a morphism of spaces* $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, \mathcal{O}^k)_\infty \to \mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty$ *sending a framed $\mathcal{D}$-bundle to its canonical trivialization on an open $\infty \in U \subset X$.*

(2) *Let* $GL_k(\mathcal{K}(X)_\infty)$ *act on* $\mathrm{Gr}_k^{\mathrm{ad}}(X)$ *by changing the trivialization. Then there is a bijection on the level of field-valued points between* $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, k)_\infty$ *and* $\mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty / GL_k(\mathcal{K}(X)_\infty)$.

Proof. The first assertion is a consequence of Proposition 3.6 in light of Remark 3.7.

Given a vector bundle V and a V-framed $\mathcal{D}$-bundle M as in the definition of $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, k)_\infty$, choose a trivialization of V in a neighborhood of infinity compatible with the given trivialization at ∞. By Proposition 3.6 and Remark 3.7, we obtain an object of $\mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty$. In fact, this is easily seen to give a morphism from a $GL_k(\mathcal{K}(X)_\infty)$-torsor over $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, k)_\infty$ to $\mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty$.

To get an inverse on the level of field-valued points, we proceed as follows. Let $\xi = \mathrm{Spec}(K)$. Given an object $M \cong \mathcal{D}_U^k$ of $\mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty$ for some nonempty open subset U of X that contains ∞, we may give M the filtration induced from that of $\mathcal{D}_U^k$—this makes M into a V-framed $\mathcal{D}$-bundle where $V \otimes T^\ell = \mathrm{gr}_\ell(M)$ for $\ell \gg 0$, and moreover V is trivialized near ∞ by the canonical trivialization of $\mathrm{gr}_0(\mathcal{D}^k)$ there. We thus get a function $\mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty(\xi) \to \mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, k)_\infty(\xi)$. This function is certainly invariant under the action of $GL_k(\mathcal{K}(X)_\infty)$, so induces a function

$$\mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty / GL_k(\mathcal{K}(X)_\infty)(\xi) \xrightarrow{p} \mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, k)_\infty(\xi).$$

These functions are easily checked to be inverses to each other. $\square$

The adelic Grassmannian was originally studied by Wilson as a parameter space for certain algebro-geometric (finite gap) solutions of the KP hierarchy. Namely, $\mathrm{Gr}^{\mathrm{ad}}(X)$ parametrizes all Krichever data for the curve X and its *cusp quotients* $X \to Y$, curves whose normalization is identified with X and for which the normalization map is a bijection. To see this we use the identification of $\mathcal{D}$-bundles with torsion-free sheaves on cusp quotients discovered by Cannings and Holland [CH2] (this identification is given by identifying the structure of the de Rham cohomology of $\mathcal{D}$-bundles with generic trivialization, as explained in [BD2]). The following proposition is due to [CH2] in rank one; a more general result, implying in particular the general case, appears in [BN1].

Proposition 3.12 ([CH2, BN1]). $\mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty$ *is isomorphic to the direct limit over cusp quotients* $X \to Y$ *smooth at* ∞ *of the set of isomorphism classes of rank k torsion-free $\mathcal{O}_Y$-modules equipped with a generic trivialization defined at* ∞.

Note that the same statement holds if we remove the restrictions on ∞.

This interpretation of $\mathcal{D}$-bundles gives rise to a construction of solutions to the multi-component KP hierarchy (orbits of the action of $\mathbb{C}[z]$ on $\mathrm{GR}^k / GL_k(\mathbb{C}[[z^{-1}]])$) from $\mathcal{D}$-bundles on X, smooth at ∞, considered as defining Krichever data on cusp quotients Y of X:

Corollary 3.13. *Fix a local coordinate z^{-1} on X at ∞. Then there is a canonical Krichever map $\mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty \to \mathrm{GR}^k$ from the adelic Grassmannian to the Sato Grassmannian, sending a rank k torsion-free sheaf $\mathcal{V}$ on a curve Y with trivialization near ∞ to the subspace of $\mathbb{C}((z^{-1}))^k$ defined by sections of $\mathcal{V}$ on $Y \setminus \infty$. Quotienting by change of trivialization gives a composite*

$$\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(X, k)_\infty \to \mathrm{Gr}_k^{\mathrm{ad}}(X)_\infty / GL_k(\mathcal{K}(X)_\infty) \to \mathrm{GR}^k / GL_k(\mathbb{C}[[z^{-1}]]).$$

It is also easy to describe the corresponding flows directly in terms of $\mathcal{D}$-bundles: the flows modify a $\mathcal{D}$-bundle M inside its localization $M \otimes \mathbb{C}((z^{-1}))$ by multiplication by powers of z. Below we will introduce a "Fourier dual" relation between $\mathcal{D}$-bundles on cubic curves and KP solutions, where we modify M inside its microlocalization by powers of a vector field ∂.

3.4. Local $\mathcal{D}$-bundles and the Sato Grassmannian

In this section we study $\mathcal{D}$-bundles on the disc $D = \mathrm{Spec}\,\mathbb{C}[[t]]$: so, we let $\mathcal{D}$ and $\mathcal{E}$ denote the rings of differential and microdifferential operators with coefficients in $\mathbb{C}[[t]]$, as in Section 2.3.[6] We introduce $\mathcal{D}$-lattices and state our generalization of Sato's $\mathcal{D}$-module description of the Sato Grassmannian.

3.4.1. $\mathcal{D}$-lattices. Consider the fiber $\mathcal{E}|_{t=0} \overset{\text{def}}{=} \mathcal{E}/\mathcal{E} \cdot t$ of the right $\mathbb{C}[[t]]$-module $\mathcal{E}$. Under the identification $z \leftrightarrow \partial$, we see that this fiber is isomorphic to the vector space $\mathbb{C}((z^{-1}))$; its realization as $\mathcal{E}/\mathcal{E}t$, however, gives it a natural structure of left $\mathcal{E}$-module. Likewise, the fiber $\mathcal{E}^n|_{t=0}$ is identified with the vector space $\mathbb{C}((z^{-1}))^n$. Taking the fibers at zero of the right $\mathbb{C}[[t]]$-submodules $\mathcal{D}^n$ and $\mathcal{E}_{-1}^n$ of $\mathcal{E}^n$, we obtain the subspaces $\mathbb{C}[z]^n$ and $z^{-1}\mathbb{C}[[z^{-1}]]^n$, respectively, of $\mathbb{C}((z^{-1}))^n$.

Generalizing the example of $\mathcal{D}^n \subset \mathcal{E}^n$, we have the following.

Definition 3.14. A *$\mathcal{D}$-lattice* in $\mathcal{E}^n$ is a finitely generated right $\mathcal{D}$-submodule $M \subset \mathcal{E}^n$ such that $M \cdot \mathcal{E} = \mathcal{E}^n$ (equivalently, the natural map $M \otimes_{\mathcal{D}} \mathcal{E} \to \mathcal{E}^n$ is an isomorphism).

A $\mathcal{D}$-lattice is said to be *generic* if it is transversal to $\mathcal{E}_{-1}^n$, that is, $\mathcal{E}^n = M \oplus \mathcal{E}_{-1}^n$.

A $\mathcal{D}$-lattice $M \subset \mathcal{E}^n$ is a $\mathcal{D}$-bundle on the disc with a microlocal trivialization (i.e. near $\partial = \infty$). Namely, M inherits the structure of $\mathcal{O}^n$-framed $\mathcal{D}$-bundle from its embedding in $\mathcal{E}^n$. Thus we may reformulate the notion of $\mathcal{D}$-lattice as an $\mathcal{O}^n$-framed $\mathcal{D}$-bundle on the disc, equipped with a *filtered* isomorphism $M \otimes_{\mathcal{D}} \mathcal{E} \to \mathcal{E}^n$.

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[6] It is important to note that we use the disc rather than the *formal* disc $\widehat{D} = \mathrm{Spf}(\mathbb{C}[[t]])$, on which the theory of $\mathcal{D}$-modules is trivial.

Lemma 3.15. *Let $M \subset \mathcal{E}^n$ be a finitely generated $\mathcal{D}$-submodule. Then M is a $\mathcal{D}$-lattice if and only if the following conditions are satisfied:*

(1) *There exists an integer k such that $M \to \mathcal{E}^n/\mathcal{E}_k^n$ is surjective.*
(2) *There exists an integer ℓ such that $M \to \mathcal{E}^n/\mathcal{E}_\ell^n$ is injective with cokernel a finitely generated projective $\mathbb{C}[\![z]\!]$-module.*

The proof uses standard techniques. In fact we will later adopt this equivalent reformulation, since it is easier to work with in families than Definition 3.14: see Section 6.

We also have the following characterization of generic $\mathcal{D}$-lattices:

Proposition 3.16. *A $\mathcal{D}$-lattice $M \subset \mathcal{E}^n$ is generic if and only if M (with its induced filtration, Section 7.2) is isomorphic to $\mathcal{D}^n$ (with its standard filtration) as filtered $\mathcal{D}$-modules.*

Proof. M is generic if and only if, with the induced filtration, the induced homomorphism

$$\psi : \mathrm{gr}(M) \to \mathrm{gr}(\mathcal{E}^n/\mathcal{E}_{-1}^n) \cong \mathrm{gr}(\mathcal{D}^n)$$

of $\mathbb{C}[\![t]\!]$-modules is an isomorphism. Observe that ψ has a kernel if and only if $M_{-1} \neq 0$. Furthermore, if $\ker(\psi) = 0$, then ψ is an isomorphism if and only if $\mathrm{gr}(M) \cong M_0 \otimes \mathrm{gr}(\mathcal{D})$. A standard argument shows that this last equality holds if and only if the induced $\mathcal{D}$-module map $M_0 \otimes \mathcal{D} \to M$ is an isomorphism of filtered $\mathcal{D}$-modules. $\square$

3.4.2. Sato Grassmannian as moduli space. We now state our general Sato theorem (restated and proved below as Theorem 6.5), relating $\mathcal{D}$-lattices to the Grassmannian.

To any $\mathcal{D}$-lattice $M \subset \mathcal{E}^n$ we may assign its fiber M/Mt at 0. This is a subspace (with no additional module structure in general) of $\mathcal{E}^n/\mathcal{E}^n t = \mathbb{C}(\!(z^{-1})\!)^n$, and in fact always gives a d-lattice: see Proposition 6.6.

We then have the following theorem:

Theorem 3.17 (see Theorem 6.5). *The Sato Grassmannian GR_n is isomorphic to the moduli space for $\mathcal{D}$-lattices, under the map taking a $\mathcal{D}$-lattice $M \subset \mathcal{E}^n$ to the d-lattice $M/Mt \subset \mathcal{E}^n/\mathcal{E}^n = \mathbb{C}(\!(z^{-1})\!)^n$. Under this map, generic $\mathcal{D}$-lattices are identified with d-lattices in the big cell.*

The set-theoretic identification of the set of *generic* (i.e. free) $\mathcal{D}$-lattices and the set of *generic d-lattices* (the big cell GR_n°) was proven by Sato; see [Mul]. We will postpone the proof until Section 6.

3.4.3. Interpretation: noncommutative Krichever data. Recall that a (rank one) *Krichever datum* (defining an algebro-geometric solution for KP [Mul]) associated to a curve X with a marked smooth point x and local coordinate z at x is a torsion-free sheaf $\mathcal{L}$ on X, equipped with a trivialization near x. Equivalently we have a torsion-free sheaf $\mathcal{L}$ on the affine curve $X \setminus x$ equipped with an isomorphism $\mathcal{L} \otimes_{\mathcal{O}_{X \setminus x}} \mathbb{C}(\!(z)\!)$ (allowing us to glue $\mathcal{L}$ on the punctured curve to the trivial bundle on the disc at x). Theorem 3.17 claims that the (rank one) Sato Grassmannian is identified with the moduli space of (rank one) $\mathcal{D}$-bundles on the disc, equipped with a trivialization near infinity (i.e. of the associated

filtered $\mathcal{E}$-module). Thus the entire Grassmannian parametrizes a kind of Krichever data for the noncommutative $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{D})$. The latter is characterized as the space on which coherent sheaves are filtered $\mathcal{D}$-modules (if we adopt the definition of noncommutative variety as Grothendieck category or as differential graded category), and is obtained by gluing the noncommutative affine bundle $\operatorname{Spec} \mathcal{D}$ to a "disc bundle at infinity" $\operatorname{Spec} \mathcal{E}_-$ along $\operatorname{Spec} \mathcal{E}$. Classical Krichever solutions correspond to commutative subrings of $\mathcal{D}$ (necessarily the affine rings of curves), in other words to maps from this noncommutative curve to ordinary curves. Thus classical algebro-geometric solutions correspond to $\mathcal{D}$-lattices which are pulled back from torsion-free sheaves $\mathcal{L}$ on curves X under maps $\mathbb{P}(\mathcal{D}) \to X$.

3.5. Lax operators and micro-opers

As we have seen, the $\mathcal{D}$-lattices provide a $\mathcal{D}$-module interpretation for points in the Sato Grassmannian, the big cell of which parametrizes KP wave operators. In this section we introduce *micro-opers*, which give a similar interpretation for points in the quotient $\mathbf{V}_n^{\mathbb{C}} \backslash \mathrm{GR}_n$ and thus also for KP Lax operators. They are less rigid structures than $\mathcal{D}$-lattices and hence turn out to be better suited for our goal of "globalizing" the Sato dictionary and geometrically interpreting meromorphic Lax operators on a (cubic) curve.

Given a right $\mathcal{D}$-module M, we write $M_{\mathcal{E}} = M \otimes_{\mathcal{D}} \mathcal{E}$.

Definition 3.18. A V-framed *micro-oper* on a differential curve (Section 2.1) (X, ∂) is a V-framed $\mathcal{D}$-bundle M on X equipped with a left $\mathcal{E}$-module endomorphism ∂_M of $M_{\mathcal{E}} = M \otimes_{\mathcal{D}} \mathcal{E}$ whose principal symbol is $\mathrm{Id}_V \otimes \partial$ with respect to the induced filtration of $M_{\mathcal{E}}$. In other words, ∂_M has degree 1 with respect to the filtration and induces the isomorphism

$$(\mathrm{gr}(\partial_M) = \mathrm{Id}_V \otimes \partial) : (\mathrm{gr}_n M_{\mathcal{E}} \simeq V \otimes \mathrm{gr}_n \mathcal{E}) \to (\mathrm{gr}_{n+1} M_{\mathcal{E}} \simeq V \otimes \mathrm{gr}_{n+1} \mathcal{E}).$$

By a *rank n local micro-oper* we denote an $\mathcal{O}^n$-framed micro-oper on the disc. A local micro-oper is *generic* if the underlying filtered $\mathcal{D}$-module is trivial, i.e. isomorphic to $\mathcal{D}^n$ with its filtration. Local micro-opers are parametrized by a quotient of the Sato Grassmannian:

Proposition 3.19. *The parameter space (moduli stack) for rank n local micro-opers is the quotient $\mathbf{V}_n^{\mathbb{C}} \backslash \mathrm{GR}_n$ of the Sato Grassmannian by negative constant coefficient operators.*

Proof. A $\mathcal{D}$-lattice $M \hookrightarrow \mathcal{E}^n$ defines a micro-oper structure on M by remembering only the natural filtration, framing and ∂ action on $\mathcal{M}_E = \mathcal{E}^n$. This defines a map from GR^n to micro-opers. Conversely, given a micro-oper structure on M we may pick a $\mathcal{D}$-lattice structure (i.e. an isomorphism $M \otimes_{\mathcal{D}} \mathcal{E} \to \mathcal{E}^n$) on M compatible with filtration and framing. Since ∂_M is acting by right $\mathcal{E}$-module endomorphisms of $M_{\mathcal{E}}$, the lattice structure identifies ∂_M with an endomorphism of $\mathcal{E}^n$ given by left multiplication by an operator of the form $\partial \otimes \mathrm{Id} + a_0 + a_1 \partial^{-1} + \cdots$, where a_i are matrices over formal power series. Changing the $\mathcal{D}$-lattice structure by the left action of $\mathrm{Id} + \mathfrak{gl}_n(\mathcal{E}_{-1})$ on $\mathcal{E}^n$, we can conjugate the image of ∂_M to $\partial \otimes \mathrm{Id}$. Moreover, we can do so uniquely up to the centralizer of

$\partial \otimes$ Id in $\mathfrak{gl}_n(\mathcal{E}_<)$, namely $\mathbf{V}_n^{\mathbb{C}}$. This identifies the $\mathcal{D}$-lattice structures on M inducing the given micro-oper structure on M with a $\mathbf{V}_n^{\mathbb{C}}$-orbit on GR^n, as desired.

The above proof extends immediately to S-families of micro-opers (note that $\mathcal{D}$-lattice structures on M will exist only locally on the parameter scheme S), proving the stronger assertion of the proposition. $\qquad\square$

Corollary 3.20. *The space of generic local micro-opers is isomorphic to the affine space of matrix KP Lax operators.*

Proof. The genericity condition precisely characterizes micro-opers in the image of $\mathbf{V}_n^{\mathbb{C}} \backslash \mathrm{GR}_n^\circ$, since $\mathcal{D}$-lattice structures on such a framed $\mathcal{D}$-module are automatically in the big cell by Proposition 3.16. Note also that $\mathcal{D}^n$ has no automorphisms as a framed $\mathcal{D}$-module. Thus a trivial framed $\mathcal{D}$-module has a unique trivialization. Therefore ∂_M defines a right $\mathcal{E}$-module endomorphism of $\mathcal{E}^n$, hence an element L of $\mathfrak{gl}_n(\mathcal{E})$ acting from the left. The micro-oper conditions guarantee that this is indeed a matrix KP Lax operator, i.e. has the form $\partial \otimes$ Id plus lower order terms. $\qquad\square$

On a global differential curve (X, ∂) we have the following "twisted" analog of a matrix Lax operator:

Definition 3.21. A *V-twisted Lax operator* over an open set $U \subseteq X$ is a filtration-preserving right $\mathcal{E}$-module endomorphism

$$L : V \otimes \mathcal{E} \to V \otimes \mathcal{E}(1)$$

with principal symbol Id $\otimes \partial$.

Note that "filtration-preserving" in this case means that

$$L(V \otimes \mathcal{E}_k) \subset (V \otimes \mathcal{E}(1))_k = V \otimes \mathcal{E}_{k+1}$$

for all k. An easy argument shows that, in the case $V = \mathcal{O}^n$, such L are exactly the matrix Lax operators (more precisely, their initial values for fixed KP times).

Let us fix a point $x \in X$ at which $\partial \neq 0$ and use ∂ to identify $\widehat{\mathcal{O}}_{X,x}$ with $\mathbb{C}[[t]]$. Then the restriction of any micro-oper on X near x, together with a trivialization of V near x, defines a micro-oper on the disc, i.e. a point of $\mathbf{V}_n^{\mathbb{C}} \backslash \mathrm{GR}^n$. If this local micro-oper is generic then this is equivalent to the data of a matrix Lax operator on the disc. In fact, by Proposition 3.6 there is an open set U on which $M_{\mathcal{E}}$ is canonically identified with $V \otimes \mathcal{E}$, so that the $\mathcal{E}$-module endomorphism ∂_M gives rise to a (V-twisted) matrix microdifferential operator of degree one and principal symbol the identity:

Corollary 3.22. *Let (M, ∂_M) denote a V-framed micro-oper, and $U \subset X$ the open subset where M is trivial as a V-framed $\mathcal{D}$-bundle (Proposition 3.6). Then the micro-oper $M|_U$ determines, and is determined by, a V-twisted Lax operator.*

3.5.1. Micro-opers and opers. We see that micro-opers on a curve are global geometric analogs of matrix Lax operators on the disc. Thus they are the matrix KP analogs of opers, introduced by Beilinson and Drinfeld [BD3] following Drinfeld and Sokolov [DS] (henceforth we only consider GL_n opers). Opers are special connections defined on any smooth curve, while opers on the disc are identified with KdV Lax operators, i.e. nth order differential operators $L = \partial^n + u_1\partial^{n-1} + \cdots + u_n \in \mathcal{D}$.

More precisely, micro-opers are the analogs of affine opers, introduced in [BF] for the geometric study of Drinfeld–Sokolov hierarchies. A (GL_n-)affine oper on a curve X is a vector bundle V on $X \times \mathbb{P}^1$, equipped with a connection ∇ along X on the bundle of sections $V_{\mathbb{A}^1}$ of V on $\mathbb{A}^1 \times X$, and a flag $V_{\infty}^{\cdot}$ on the fiber V_{∞} of V at $X \times \infty$. The connection is required to have a first order pole at ∞ and to satisfy a strict form of Griffiths transversality with respect to the flag at ∞. The open set of *generic* affine opers, for which V is trivial along $\mathbb{P}^1$, is identified with opers. Affine opers on the disc are identified with (a quotient of) the loop Grassmannian for GL_n, while opers form the corresponding big cell.

The GL_n loop Grassmannian is embedded in the Sato Grassmannian, reflecting the inclusion of the KdV (and Gelfand–Dickey) hierarchies in KP. This corresponds to the identification of affine opers with special micro-opers. Namely, affine opers (on any curve differential (X, ∂)) are identified with $\mathcal{O}$-framed micro-opers (M, ∂_M) for which ∂_M^n preserves the submodule $M \subset M_{\mathcal{E}}$. The identification preserves big cells: M is locally free if and only if the corresponding affine oper is generic. To define the vector bundle V on $X \times \mathbb{P}^1$ we consider $M_{\mathcal{E}}$ as an $\mathcal{O}_X((z^{-1}))$-module via the endomorphism $z = \partial_M^n$. The extension to $\mathbb{P}^1 \times X$ and the flag at ∞ are constructed from the filtration, while the affine oper connection comes from the (left version of the) right $\mathcal{D}$-module structure on $M_{\mathcal{E}}$. (See [BN2] for a more leisurely discussion.)

3.6. Flows of the KP and CM hierarchies

The flows of the multicomponent KP hierarchies on the Sato Grassmannian and the space of Lax operators have natural formulations in terms of $\mathcal{D}$-lattices and micro-opers, respectively. First recall that the KP flows on GR_n are given by the action of the subalgebra $\Gamma_+ = \mathbb{C}[\partial]\,\mathrm{Id} \subset \mathfrak{gl}_n(\mathcal{E})$ of constant coefficient scalar differential operators. The action of this Lie algebra has the following simple description on $\mathcal{D}$-lattices. Given a polynomial $P(\partial) \in \mathbb{C}[\partial]$, its infinitesimal action on a $\mathcal{D}$-lattice $M \subset \mathcal{E}^n$ is given by translating the submodule M by the action of the (right $\mathcal{D}$-module) endomorphism of $\mathcal{E}^n$ given by left multiplication by $P(\partial)$. This means we deform elements $m \in M \subset \mathcal{E}^n$ to $m + \epsilon P(\partial) \cdot m$. On the level of tangent spaces, write $B = M/Mt$ for the corresponding d-lattice, and make the identification

$$\mathrm{Hom}(B, \mathbb{C}((z^{-1}))^n/B) = \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{E}^n/M).$$

Then the tangent space to GR_n at M has the form

$$T_M\,GR_n = \mathrm{Hom}(B, \mathbb{C}((z^{-1}))^n/B).$$

The vector field on GR_n given by $P(\partial)$ then has value at M equal to the composite

$$M \hookrightarrow \mathcal{E}^n \xrightarrow{P(\partial)\cdot -} \mathcal{E}^n \twoheadrightarrow \mathcal{E}^n/M;$$

this is typically nontrivial since M is a right, but not left, $\mathcal{D}$-submodule of $\mathcal{E}^n$.

The action of Γ_+ descends from GR_n to its quotient $\mathbf{V}_n^{\mathbb{C}} \backslash \mathrm{GR}_n$ (whose big cell parametrizes Lax operators). By Proposition 3.19 this is the parameter space for local micro-opers. It is easy to describe this action directly on micro-opers:

Proposition 3.23. *The matrix KP flow associated to a polynomial $P(\partial) \in \mathbb{C}[\partial]$ on a local micro-oper $(M, \partial_M : M_{\mathcal{E}} \to M_{\mathcal{E}})$ is given by translating the $\mathcal{D}$-module $M \subset M_{\mathcal{E}}$ by the action of the (right $\mathcal{D}$-module) endomorphism $P(\partial_M)$ (that is, we deform elements $m \in M \subset M_{\mathcal{E}}$ to $m + \epsilon P(\partial_M) \cdot m$).*

Micro-opers on a global differential curve (X, ∂) (i.e. meromorphic twisted Lax operators) also carry an action of the abelian Lie algebra $\mathbb{C}[\partial]$, defined just as in the local setting.[7] Namely, the action of $P(\partial) \in \mathbb{C}[\partial]$ on a micro-oper (M, ∂_M) is simply given by translating the $\mathcal{D}$-submodule $M \subset M_{\mathcal{E}}$ by the action of the (right $\mathcal{D}$-module) endomorphism $P(\partial_M)$, and preserving the filtration, framing and endomorphism ∂_M on $M_{\mathcal{E}}$. In particular, these flows act on the associated rational Lax operator of Corollary 3.22 by the standard multicomponent KP flows. More precisely, if we trivialize $V|_U$ then P acts on the Lax operator $L_M \in \mathfrak{gl}_n(\mathcal{E})$ by commutator with $P(\partial)$.

3.6.1. The KP algebroid. We may also describe the KP flows on micro-opers in terms of deformations of the underlying $\mathcal{D}$-bundles. More precisely, we introduce, by analogy with Section 2.2.3, a Lie algebroid on $\mathcal{D}$-bundles, which describes all deformations of a $\mathcal{D}$-bundle M coming from its microlocal endomorphisms (such as ∂_M), acting by moving M inside $M_{\mathcal{E}}$:

Lemma 3.24. *The sheaf $\underline{\mathrm{End}}_{\mathcal{E}}$ over $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V)$ of endomorphisms of the microlocalization of the universal sheaf (i.e. the sheaf whose fiber at M is $\mathrm{End}_{\mathcal{E}}(M_{\mathcal{E}})$) has the structure of Lie algebroid, the* KP *algebroid, whose action on a $\mathcal{D}$-bundle M deforms $M \subset M_{\mathcal{E}}$ by left multiplication.*

On the other hand, from Definition 2.6 we have an action of $\mathbb{C}[\partial]$ on $V^\vee$-framed CM spectral sheaves $\mathfrak{CM}_n(E, V^\vee)$. More generally by Lemma 2.8 we have a natural algebroid describing all deformations of a CM spectral sheaf along the curve E_∞. The definitions of the CM and KP algebroids are precisely analogous, as are the structures of Higgsed spectral sheaf and micro-oper. It is then an immediate consequence of the Fourier–Mukai construction of the isomorphism between $\mathcal{D}$-bundles and spectral sheaves (Theorem 3.4) (and of the compatibility of the Fourier–Mukai transform with microlocalization [PRo]) that the corresponding algebroids and hierarchies are identified:

---

[7] Note that we do not in fact need the full structure of differential curve, only the choice of ∂ up to scalar, which is canonical in the cubic case.

Theorem 3.25. *Let* $\mathbf{F} : \mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V) \to \mathfrak{CM}_n(E, V^\vee)$ *denote the Fourier–Mukai isomorphism of framed $\mathcal{D}$-bundles and framed CM spectral sheaves.*

(1) $\mathbf{F}$ *identifies the KP and CM algebroids.*
(2) $\mathbf{F}$ *lifts to an isomorphism of the moduli stack of V-framed micro-opers with the moduli stack of $V^\vee$-framed Higgsed CM spectral sheaves, identifying the multicomponent KP hierarchy on micro-opers with the Higgsed CM hierarchy (*i.e. *intertwining the two $\mathbb{C}[\partial]$-actions*).

We defer the proof to Section 5.4, after the relevant facts about the Fourier–Mukai transform have been established.

3.6.2. Bispectrality. We have now defined two relations between $\mathcal{D}$-bundles and KP solutions: the construction of rational Lax operators on X from micro-opers on X, and the construction of Krichever data from the adelic Grassmannian, i.e. $\mathcal{D}$-bundles with generic trivialization. Alternatively, we have defined two Lie algebroids on the moduli $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V)$ of $\mathcal{D}$-bundles, describing deformations of a $\mathcal{D}$-bundle M by endomorphisms of $M \otimes_{\mathcal{D}} \mathcal{E}$ and of $M \otimes_{\mathcal{D}} \mathcal{D}((z^{-1}))$. For $\mathcal{D}$-bundles on a general curve, there is no obvious relation between the two constructions. In particular note that the constructions of KP solutions land in different copies of the Sato Grassmannian: Krichever data give rise to subspaces of $\mathbb{C}((z^{-1}))$, where z^{-1} is a local coordinate at a point $\infty \in X$, while micro-opers give rise to subspaces of $\mathbb{C}((\partial^{-1}))$, where ∂ is a nonvanishing vector field on X.

It is natural to expect that the relation between the two constructions is a sort of Fourier transform, exchanging microlocal and local trivializations. This can be made precise in the rational case $X = \mathbb{P}^1$, using the geometric Fourier transform.[8] This is an autoequivalence of the category of $\mathcal{D}$-modules on $\mathbb{A}^1$, and exchanges $\mathbb{C}((z^{-1}))$ and $\mathbb{C}((\partial^{-1}))$. Let us then consider the moduli problem for $\mathcal{D}$-bundles on $\mathbb{P}^1$, with framing (i.e. trivialization at $\partial^{-1} = 0$) and trivialization at $z^{-1} = 0$, i.e. the noncommutative version of sheaves on the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, framed along $\mathbb{P}^1 \times \infty \cup \infty \times \mathbb{P}^1$. This moduli stack acquires an automorphism from the geometric Fourier transform, interchanging the KP algebroid (deformations at $\partial^{-1} = 0$) with the algebroid of deformations at $z^{-1} = 0$ (given by endomorphisms of $\mathcal{D}$-bundles restricted to $\mathrm{Spec}\, \mathbb{C}((z^{-1}))$). In other words, the automorphism exchanges the spectral parameter ∂ and the differential parameter z, and is easily seen to give (on the level of points) the bispectral involution [W1, W3] in the rank one case. Specifically this is a geometric reformulation of the picture of Berest and Wilson [BW1], and the two algebroids in the rank one case are the two commutative subalgebras of the automorphisms of the Weyl algebra studied in [BW1].

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[8] Note that the categories of $\mathcal{D}$-modules on the rational cubic and on $\mathbb{P}^1$ are equivalent, and $\mathcal{D}$-bundles on $\mathbb{P}^1$ trivial at ∞ and $\mathcal{D}$-bundles on the cuspidal cubic curve are canonically identified [SS, BN1].

4. Fourier transform for cubic curves

In this section we explain the Fourier–Mukai autoequivalence of the derived category of a cubic curve, extend it to give an analog of the theorem of Laumon and Rothstein concerning $\mathcal{D}$-modules on abelian varieties, and give some fundamental calculations for the Fourier–Mukai transform of torsion-free sheaves.

4.1. Fourier equivalence for Weierstrass cubics

Fix a cubic curve E as before. In this section we describe the Fourier–Mukai autoequivalence of the derived category of coherent sheaves on E.

Recall that the generalized Jacobian $\mathrm{Jac}(E)$ (the group of line bundles of degree 0 on E) is isomorphic to the smooth locus $\mathbf{G} \subset E$ via the map $e \mapsto \mathcal{O}(e - b)$. The compactified Jacobian of E, denoted by $\overline{\mathrm{Jac}}(E)$, is the moduli space for rank 1, torsion-free sheaves on E of degree (equivalently, Euler characteristic) 0; it is isomorphic to E, with the additional point (when E is singular) coming from the unique rank 1 torsion-free sheaf of degree 0 that is not locally free (which is isomorphic, in a neighborhood of the singular point, with the ideal of the singular point).

Let $\Delta \subset E \times E$ denote the diagonally embedded copy of E. We define a sheaf $\mathcal{P}^\vee$ on $E \times \overline{\mathrm{Jac}}(E) = E \times E$ by $\mathcal{P}^\vee = I_\Delta \otimes (\mathcal{O}(b) \boxtimes \mathcal{O}(b))$. $\mathcal{P}^\vee$ is flat over both factors: indeed, we have a short exact sequence

$$0 \to \mathcal{P}^\vee \to \mathcal{O}(b) \boxtimes \mathcal{O}(b) \to \mathcal{O}_\Delta(2b) \to 0 \tag{4.1}$$

with the second and third terms flat over both factors, implying that the kernel $\mathcal{P}^\vee$ is as well.

Definition 4.1. Let $\pi_i : E \times E \to E$ denote projection on the ith factor. The *Fourier functor F* from the coherent derived category $D^b_{\mathrm{coh}}(E)$ to itself is defined by

$$F(M) = \mathbf{R}\pi_{2*}(\mathcal{P}^\vee \overset{\mathbf{L}}{\otimes} \pi_1^* M).$$

Note that $\mathbf{L}\pi_i^* = \pi_i^*$ since the projection is flat. Note also that our curve E is Gorenstein, so the dualizing sheaf $\omega = \omega_E$ is a line bundle—in fact, for our cubic curves, $\omega_E \simeq \mathcal{O}_E$.

To the sheaf $\mathcal{P}^\vee$ we may also associate its *derived dual*, the object $(\mathcal{P}^\vee)^\vee = \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{P}^\vee, \mathcal{O}_{E \times E})$ of the derived category of $E \times E$. We write $\mathcal{P} = (\mathcal{P}^\vee)^\vee$; this notation is consistent, since by taking the derived dual twice one obtains a complex quasi-isomorphic to the original complex $\mathcal{P}^\vee$. In fact, $\mathcal{P}$ is a sheaf that fits in an exact sequence

$$0 \to \mathcal{O}(-b) \boxtimes \mathcal{O}(-b) \to \mathcal{P} \to \mathcal{O}_\Delta(-2b) \to 0 \tag{4.2}$$

and, like $\mathcal{P}^\vee$, $\mathcal{P}$ is a flat family of torsion-free sheaves of rank 1 and degree 0 over both factors. Moreover, let $(-1) : E \to E$ denote the involution on E induced by the inverse for the group structure of $\mathbf{G}$. Then

$$\mathcal{P} = (\mathrm{id} \times (-1))^* \mathcal{P}^\vee. \tag{4.3}$$

Consider the functor

$$\overline{F}(N) = \mathbf{R}\pi_{1*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \pi_2^* N)[1]. \tag{4.4}$$

Theorem 4.2 (see [BuK, Theorem 2.12]). *The Fourier functor* $F : D^b_{\text{coh}}(E) \to D^b_{\text{coh}}(E)$ *is an exact equivalence of triangulated categories, with quasi-inverse given by* (4.4).

Remark 4.3. In [BN2, Theorem 5.2(1)], we announced a proof of this theorem (a proof using Bridgeland's criterion appeared in an early uncirculated draft of the present paper). The theorem cited in [BuK] is actually considerably more general, but a special case of that general theorem gives the fact we need.

Notation 4.4. We let $F^{-1} = \overline{F}[-1]$.

Corollary 4.5. *The Fourier functor* $F : D^b_{\text{qcoh}}(E) \to D^b_{\text{qcoh}}(E)$ *is an exact equivalence of triangulated categories.*

Proof. A complex M of quasicoherent sheaves is the colimit of coherent complexes M_i. By Remark 2.2 and Lemma 4.1 of [BoN], we then have

$$\overline{F} \circ F(M) = \varinjlim \overline{F}F(M_i) \simeq M \text{ and } F \circ \overline{F}(M) = \varinjlim F\overline{F}(M_i) \simeq M$$

as desired. □

Corollary 4.6. *We have*

$$\mathbf{R}(p_{13})_*(p_{12}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}^\vee) \simeq \Delta_*\mathcal{O}_E[-1] \text{ and } \mathbf{R}(p_{13})_*(p_{12}^*\mathcal{P}^\vee \otimes p_{23}^*\mathcal{P}) \simeq \Delta_*\mathcal{O}_E[-1] \tag{4.5}$$

in the derived category of coherent $\mathcal{O}_{E \times E}$*-modules.*

Proof. This is immediate from Corollary 4.5 and [To]. □

4.2. Fourier transform of torsion-free sheaves

Our goal in this subsection is to prove analogs, for singular Weierstrass cubics, of the usual characterizations of torsion-free sheaves on an elliptic curve whose Fourier transforms are again sheaves (that is, have cohomology in only a single degree). The reader who is familiar with the standard techniques for these problems may consult the statements of Propositions 4.7 and 4.8 for the expected facts; note, however, that while the proof of Proposition 4.8 is standard, the proof of Proposition 4.7 is *not* entirely standard.

Recall that, if M is a torsion-free coherent sheaf on E, the *slope* of M is

$$\mu(M) \overset{\text{def}}{=} \frac{\deg(M)}{\text{rk}(M)}. \tag{4.6}$$

On a Weierstrass cubic curve E, the slope of M satisfies $\mu(M) = \chi(M)/\text{rk}(M)$ where $\chi(M)$ is the Euler characteristic of M; see Section 0.2 of [FMW] for this and other basic facts about torsion-free sheaves and semistability on Weierstrass cubics.

Recall the *Harder–Narasimhan filtration* of a torsion-free coherent sheaf M on E: this is the unique decreasing filtration

$$F_\infty(M) = 0 \subset F_{\mu_1}(M) \subset \cdots \subset F_{\mu_r}(M) = M \qquad (4.7)$$

(that is, $\mu_1 > \cdots > \mu_r$) such that each term $\text{gr}_{\mu_i}(M) = F_{\mu_i}(M)/F_{\mu_{i-1}}(M)$ of the associated graded sheaf is torsion-free and semistable of slope μ_i.

Proposition 4.7. *Suppose M is a torsion-free coherent sheaf on E. In the notation of (4.7), $F(M)$ and $\overline{F}(M)$ are concentrated in cohomological degree 0 if and only if $\mu_r > 0$.*

Proof. We give the proof only for $\overline{F}(M)$; the proof for $F(M)$ is nearly identical in light of equation (4.3).

Use the exact sequence

$$0 \to \mathcal{P}^\vee = I_\Delta \otimes (\mathcal{O}(b) \boxtimes \mathcal{O}(b)) \to \mathcal{O}(b) \boxtimes \mathcal{O}(b) \to \mathcal{O}_\Delta(2b) \to 0$$

on $E \times E$. Tensoring with $p_1^* M$ and applying $\mathbf{R}p_{2*}$ gives an exact sequence

$$0 \to \overline{F}^0(M) \to \mathcal{O}(b) \otimes H^0(M(b)) \to M(2b) \to \overline{F}^1(M) \to \mathcal{O}(b) \otimes H^1(M(b)) \to 0$$

on E. So, $\overline{F}(M)$ is a sheaf in degree 0 if and only if

(a) $H^1(M(b)) = 0$,
(b) $M(b)$ is globally generated.

First, then, suppose that $\mu_r > 0$; we will prove that M satisfies (a) and (b). An inductive argument using the long exact cohomology sequence shows that if $H^1(T(b)) = 0$ for all torsion-free semistable T of slope $\mu(T) > 0$ then also $H^1(M(b)) = 0$. But for such T, Serre duality gives $H^1(T(b))^* = \text{Hom}(T(b), \mathcal{O}) = \text{Hom}(T, \mathcal{O}(-b)) = 0$ by the semistability of T. This proves (a).

For (b), we suppose for the moment that

$$T(b) \text{ is globally generated for all semistable } T \text{ with } \mu(T) > 0 \qquad (4.8)$$

and let Q_μ denote $M/F_\mu(M)$. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{O} \otimes H^0(\text{gr}_{\mu_i}(M)(b)) & \to & \mathcal{O} \otimes H^0(Q_{\mu_{i-1}}(b)) & \to & \mathcal{O} \otimes H^0(Q_{\mu_i}(b)) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{gr}_{\mu_i}(M)(b) & \to & Q_{\mu_{i-1}}(b) & \to & Q_{\mu_i}(b) & \to & 0
\end{array}
$$

Part (a) above shows that the top row is exact. By (4.8), the left-hand vertical arrow is surjective. Hence $M(b)/F_{\mu_{i-1}}(M)(b)$ is globally generated if $M(b)/F_{\mu_i}(M)(b)$ is, and an induction then proves part (b). So it suffices to prove (4.8).

To prove (4.8), suppose that T is torsion-free and semistable and $\mu(T) > 0$. If $T(b)$ is not globally generated, then there is some $p \in E$ such that

$$H^0(T(b)) \to H^0(T(b) \otimes \mathcal{O}_E/m_p) = T(b) \otimes \mathcal{O}_E/m_p$$

is not surjective. It follows that there is a quotient $T(b) \xrightarrow{\phi} \mathcal{O}_E/m_p$ such that

$$H^0(\phi) : H^0(T(b)) \to H^0(\mathcal{O}_E/m_p) \cong \mathbb{C}$$

is zero. The long exact cohomology sequence then implies that $\operatorname{Hom}(\ker(\phi), \mathcal{O})^* \cong H^1(\ker(\phi)) \neq 0$. Choosing a nonzero map $\ker(\phi) \xrightarrow{a} \mathcal{O}$, we get a pushout diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\phi) & \longrightarrow & T(b) & \xrightarrow{\phi} & \mathcal{O}_E/m_p & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow{\scriptstyle \simeq} & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \xi & \longrightarrow & \mathcal{O}_E/m_p & \longrightarrow & 0
\end{array}
$$

Letting $\operatorname{tors}(\xi)$ denote the (possibly zero) torsion subsheaf of ξ, we get a nonzero map $T(b) \to \xi/\operatorname{tors}(\xi)$, where $\xi/\operatorname{tors}(\xi)$ is torsion-free of slope at most 1. But this contradicts semistability of $T(b)$ (since $\mu(T(b)) > 1$), hence it contradicts semistability of T. This proves (b).

It remains to show that if M satisfies (a) and (b) then $\mu_r > 0$. So suppose that M satisfies (a) and (b). It then follows from the long exact cohomology sequence that

(a') $H^1(\operatorname{gr}_{\mu_r}(M)(b)) = 0$,
(b') $\operatorname{gr}_{\mu_r}(M)(b)$ is globally generated

hold as well.

Suppose that $\mu_r < 0$. Write $T = \operatorname{gr}_{\mu_r}(M)$. By paragraph 0.2 of [FMW],

$$\deg(T(b)) = \chi(T(b)) = \operatorname{rk}(T) + \deg(T) \leq \operatorname{rk}(T).$$

Since $H^1(T(b)) = 0$ by (a'), we get $h^0(T(b)) \leq \operatorname{rk}(T)$. By (b'), we have a surjective map $\mathcal{O} \otimes H^0(T(b)) \to T(b)$ of torsion-free sheaves, necessarily of the same rank. This map is, consequently, an isomorphism. But then

$$H^1(T(b)) = H^1(\mathcal{O} \otimes H^0(T(b))) \neq 0,$$

contradicting (a'). So $\mu_r > 0$ after all. This completes the proof. $\qquad\square$

Proposition 4.8. *Let M be a torsion-free coherent sheaf on E. Then $\overline{F}^0(M) = 0$ if and only if $\mu_1 \leq 0$ (in the notation of* (4.7)*).*

The proof is standard.

4.3. *Fourier equivalence for $\mathcal{D}_{\log}$-modules*

In general, given a quasicoherent sheaf of rings $\mathcal{A}$ on E, let $\mathcal{A}$-mod (resp. mod-$\mathcal{A}$) denote the category of left (resp. right) $\mathcal{A}$-modules that are quasicoherent as $\mathcal{O}_E$-modules.

Recall the sheaf of rings $\mathcal{D} = \mathcal{D}_{\log}$ of differential operators on a cubic curve E (for a singular curve these are the log differential operators defined in Section 2.1.2 and studied in Section 7). We wish to calculate the Fourier transform of $\mathcal{D}$ as an $\mathcal{O}_E$-bimodule (in fact as an algebra object in the derived category). Consider $\mathcal{D}$ as a quasicoherent sheaf on $E \times E$, and let $\mathcal{A} = (p_{14})_*(\mathcal{P} \otimes p_{23}^*\mathcal{D} \otimes \mathcal{P}^\vee)$ be its bimodule Fourier transform; this is a special D-algebra by Corollary 7.3 and [PRo, Props. 6.2 and 6.3]. We also let $\mathcal{R}(\mathcal{D})$ and $\mathcal{R}(\mathcal{A})$ denote the *Rees algebras* of $\mathcal{D}$ and $\mathcal{A}$, that is, the graded algebras given by

$$\mathcal{R}(\mathcal{D}) = \bigoplus_{k \geq 0} \mathcal{D}^k \cdot t^k \quad \text{and} \quad \mathcal{R}(\mathcal{A}) = \bigoplus_{k \geq 0} \mathcal{A}_k \cdot t^k$$

where t is a formal variable that keeps track of the grading. Note that $\mathcal{R}(\mathcal{D})/t \cdot \mathcal{R}(\mathcal{D}) = \operatorname{gr}(\mathcal{D})$ and similarly for $\mathcal{A}$. For a quasicoherent sheaf of graded rings $\mathcal{R}$ we let $\mathcal{R}$-mod denote the category of finitely generated *graded* $\mathcal{R}$-modules that are quasicoherent as $\mathcal{O}_E$-modules.

Theorem 4.9.

(1) *Let $(\mathcal{S}_\mathcal{D}, \mathcal{S}_\mathcal{A})$ be one of the following pairs:*

    (a) $(\mathcal{S}_\mathcal{D}, \mathcal{S}_\mathcal{A}) = (\mathcal{D}, \mathcal{A})$.

    (b) $(\mathcal{S}_\mathcal{D}, \mathcal{S}_\mathcal{A}) = (\mathcal{R}(\mathcal{D}), \mathcal{R}(\mathcal{A}))$.

    (c) $(\mathcal{S}_\mathcal{D}, \mathcal{S}_\mathcal{A}) = (\mathcal{R}(\mathcal{D})/t^k\mathcal{R}(\mathcal{D}), \mathcal{R}(\mathcal{A})/t^k\mathcal{R}(\mathcal{A}))$ *for some $k > 0$.*

    Then the Fourier functor F of Corollary 4.5 may be refined to a functor

$$F : D^b(\mathcal{S}_\mathcal{D}\text{-mod}) \to D^b(\mathcal{S}_\mathcal{A}\text{-mod})$$

    that is an exact equivalence of triangulated categories.

(2) *Fix $k > 0$. Then the change-of-rings functors*

    $\mathcal{R}(\mathcal{D})\text{-mod} \to (\mathcal{R}(\mathcal{D})/t^k\mathcal{R}(\mathcal{D}))\text{-mod}$ *and* $\mathcal{R}(\mathcal{A})\text{-mod} \to (\mathcal{R}(\mathcal{A})/t^k\mathcal{R}(\mathcal{A}))\text{-mod}$

    given by $\mathcal{N} \mapsto \mathcal{N}/t^k\mathcal{N}$ induce triangulated functors of the derived categories; moreover, the following diagram commutes:

$$
\begin{array}{ccc}
D^b(\mathcal{R}(\mathcal{D})\text{-mod}) & \xrightarrow{\quad F \quad} & D^b(\mathcal{R}(\mathcal{A})\text{-mod}) \\
\downarrow & & \downarrow \\
D^b((\mathcal{R}(\mathcal{D})/t^k\mathcal{R}(\mathcal{D}))\text{-mod}) & \xrightarrow{\quad F \quad} & D^b((\mathcal{R}(\mathcal{A})/t^k\mathcal{R}(\mathcal{A}))\text{-mod})
\end{array}
$$

Proof. This follows from [PRo, Theorem 6.5] by Corollaries 4.6 and 7.3.   □

4.4. The twisted log cotangent bundle

In this section we describe the Fourier transform algebra $\mathcal{A}$ of $\mathcal{D}$ from Theorem 4.9, and relate it to the twisted log cotangent bundle $E^\natural$ of E.

Let $\mu : \mathbf{G} \times E$ denote the action of the group $\mathbf{G}$ on E. For a torsion sheaf T on $\mathbf{G}$, we define the convolution action $D^b(E) \to D^b(E)$, $\mathcal{F} \mapsto T * \mathcal{F}$, by

$$T * \mathcal{F} = \mu_*(T \boxtimes \mathcal{F}).$$

(Note that μ is an affine morphism and is proper on the support of $T \boxtimes \mathcal{F}$ for any $\mathcal{F}$.)

Lemma 4.10. *The Fourier transform $F(T * \mathcal{F})$ of convolution is canonically identified with the tensor product $F(T) \otimes F(\mathcal{F})$.*

The proof of the lemma is identical to the analogous convolution statement for abelian varieties [Muk], once we note the character property of the Poincaré sheaf, namely the canonical isomorphism on $\mathbf{G} \times E \times E$ of $(\mu \times 1)^*\mathcal{P}$ and $\mathcal{P}_{12} \otimes \mathcal{P}_{23}$ (where $\mathcal{P}_{12}$ denotes the restriction of $\mathcal{P}$ to $\mathbf{G} \times E$, pulled back to $\mathbf{G} \times E \times E$).

Next we note that the sheaf $\mathcal{D}$ of log differential operators on E is the D-algebra generated by the action of the group $\mathbf{G}$ on E (or specifically from that of its enveloping algebra). Let U^k denote the kth filtered piece of the enveloping algebra of $\mathbf{G}$, which is (as $\mathcal{O}_{\mathbf{G}}$-module) the $\mathbb{C}$-dual of functions on the kth order neighborhood of the identity in $\mathbf{G}$ (i.e. $U^k = \Omega_E((k+1)b)/\Omega_E$ via the residue pairing). We obtain the following description of $\mathcal{D}$ as a bimodule (as for an arbitrary group action on a variety):

Lemma 4.11. *Let p_1, p_2 denote the projections of $\mathbf{G} \times E$ on the factors and μ the multiplication map to E. Then as an $\mathcal{O}_E$-bimodule (sheaf on $E \times E$), $\mathcal{D}^k$ for any k is identified with $(\mu \times p_2)_* p_1^* U^k$.*

We will now identify $\mathcal{A}$ in terms of the twisted cotangent bundle $E^\natural$ of E (Section 2.1.3). Recall that A denotes the Atiyah extension, and $\overline{E}^\natural \overset{\text{def}}{=} \underline{\mathrm{Proj}}(\mathrm{Sym}^\bullet A)$ the associated ruled surface. We let $p : \overline{E}^\natural \to E$ denote the projection map, and $E_\infty \subset \overline{E}^\natural$ the section at infinity. Note that every other section of $\overline{E}^\natural$ has nonempty intersection with E_∞ since the Atiyah sequence is nonsplit.

Theorem 4.12. (1) *The bimodule Fourier transform $\mathcal{A}_k$ of $\mathcal{D}^k$ is scheme-theoretically supported on the diagonal and is canonically identified with the Fourier transform $F(U^k)$ on E.*
(2) *The commutative algebra $\mathcal{A}$ is canonically isomorphic (as filtered $\mathcal{O}_E$-algebra) to $p_*\mathcal{O}_{E^\natural}$ (functions on the twisted cotangent bundle), inducing isomorphisms $\overline{E}^\natural = \underline{\mathrm{Proj}}(\mathcal{R}(\mathcal{A}))$ and $E^\natural = \underline{\mathrm{Spec}}(\mathcal{A})$.*
(3) *The Fourier transform induces an exact equivalence of the bounded derived category of coherent $\mathcal{D}$-modules and the bounded derived category of coherent sheaves on $E^\natural$.*

Proof. It follows from the above lemmas that

$$F(\mathcal{D}^k \otimes M) = F(U^k * M) = F(U^k) \otimes M$$

for any complex M, from which the description as a bimodule follows. Since the algebra $\mathcal{A}$ is completely determined by the bimodule $\mathcal{A}_1$ with the inclusion $\mathcal{O}_E \subset \mathcal{A}_1$ as an enveloping algebroid, it follows that it is commutative (this is immediate from the fact that the associated graded algebra is isomorphic to $\mathrm{Sym}^\bullet \mathcal{O}_E$). Next note that the Fourier transform of the nonsplit extension $U^1 = \Omega_E(2b)/\Omega_E$ of $\mathcal{T}_b$ by $\mathcal{O}_b$ is isomorphic to the Atiyah extension A of $\mathcal{T}_E$ by $\mathcal{O}_E$. It follows that $\mathcal{A}$ is isomorphic to $p_*\mathcal{O}_{E^\natural}$ compatibly with the inclusions of $\mathcal{O}_E$ in $\mathcal{A}$ and $p_*\mathcal{O}_{E^\natural}$. The isomorphisms on $\underline{\mathrm{Spec}}$ and $\underline{\mathrm{Proj}}$ follow. In order to fix these isomorphisms canonically, we note that the fiber of $\underline{\mathrm{Spec}}(\mathcal{A})$ over $b \in E$ has a canonical basepoint. This point d is characterized by the statement

that the Fourier–Mukai transform of the skyscraper at d (as $\mathcal{A}$-module) is the trivial line bundle $\mathcal{O}_E$ (which is guaranteed by the isomorphism $\mathcal{O}_d = \mathcal{O}_b$ as plain $\mathcal{O}_E$-modules) with its canonical $\mathcal{D}$-module structure. On the other hand from the definition of A we find that the fiber $F_b = p^{-1}(b) \subset E^\natural$ has a canonical identification with the cotangent fiber to E at b. There is now a unique isomorphism $\mathrm{Spec}(\mathcal{A}) \to E^\natural$ identifying these basepoints, fixing the algebra isomorphism above uniquely. The final assertion follows from Theorem 4.9, noting that the coherence condition for $\mathcal{D}$-modules and $\mathcal{A}$-modules is identified by the Fourier transform. $\qquad\square$

4.5. Microlocalization

We have, in addition to the algebras discussed above, *microlocalizations* of the algebras $\mathcal{D}$ and $\mathcal{A}$: these are obtained by inverting elements that have invertible principal symbol and completing with respect to the given filtration. If we microlocalize $\mathcal{D}$, we obtain the sheaf $\mathcal{E}$ of microdifferential operators associated to $\mathcal{D}$. Similarly, if we microlocalize $\mathcal{A}$ we obtain a sheaf of filtered algebras $\mathcal{A}_\mathcal{E}$ that is the sheaf of "functions on the punctured formal neighborhood of E_∞"—that is, if we take the structure sheaf of the formal completion of $\overline{E}^\natural$ along E_∞ and invert a local defining function for the closed subscheme E_∞ in this formal completion, the resulting sheaf of functions is exactly $\mathcal{A}_\mathcal{E}$.

By [AVV], the microlocalizations are obtained as follows. Writing $\mathcal{R}$ for the Rees ring of either algebra, we form the graded rings $\mathrm{gr}_{(n)} = \mathcal{R}/t^n\mathcal{R}$. Each of these is a nilpotent extension of $\mathrm{gr} = \mathcal{R}/t\mathcal{R}$, and so any local lift to $\mathrm{gr}_{(n)}$ of an invertible element of gr is invertible. Our ring $\mathrm{gr}_{(n)}$ is generated by a single element, which we denote by ∂. We lift this locally and invert to obtain localized rings $(\mathrm{gr}_{(n)})_\partial$ which form an inverse system of graded rings. We take the inverse limit and "reverse the formation of the Rees algebra" to obtain a filtered ring, the microlocalization, whose Rees algebra is this inverse limit. Since the graded rings at each stage are quasicoherent sheaves, one can make sense of the Fourier transform of the microlocalization (which is not itself quasicoherent) as the "de-Reesed inverse limit" of the Fourier-transformed inverse system.

Corollary 4.13. *We have $F(\mathcal{E}) = \mathcal{A}_\mathcal{E}$ and $F^{-1}(\mathcal{A}_\mathcal{E}) = \mathcal{E}$.*

Proof. This was essentially proven in the pre-publication (arXiv) version of [PRo], but we sketch the proof for completeness.

The Fourier transform of the localization $F\big((\mathrm{gr}_{(n)}(\mathcal{D}))_\partial\big)$ is a nilpotent extension of $F(\mathrm{gr}(\mathcal{D})_\partial) = \mathrm{gr}(\mathcal{A})_\partial$, and consequently any local lift of the element ∂ to $F\big((\mathrm{gr}_{(n)}(\mathcal{D}))_\partial\big)$ is invertible. The universal property of localizations then gives us an induced homomorphism

$$(\mathcal{R}(\mathcal{A})/t^k\mathcal{R}(\mathcal{A}))_\partial = F(\mathcal{R}(\mathcal{D})/t^k\mathcal{R}(\mathcal{D}))_\partial \to F\big((\mathcal{R}(\mathcal{D})/t^k\mathcal{R}(\mathcal{D}))_\partial\big).$$

A standard argument using the filtration shows that this is an isomorphism, and consequently we obtain an identification of the two inverse systems. The result now follows from the construction of the microlocalization in [AVV]. $\qquad\square$

In fact, we will also want slightly more: given a filtered $\mathcal{D}$-module M or an $\mathcal{A}$-module $\mathcal{F}$, we may similarly form an inverse system of graded modules $\mathrm{gr}_{(n)}(M)$ (or similarly for $\mathcal{F}$)

over the system of graded rings $\mathrm{gr}_{(n)}(\mathcal{R})$ used in Corollary 4.13, invert a local lift of ∂, and take the inverse limit. If M or $\mathcal{F}$ is equipped with a good filtration, this procedure gives us $M_{\mathcal{E}}$ or $\mathcal{F}_{\mathcal{E}}$, respectively. We can thus make sense of $F(M_{\mathcal{E}})$ or $F^{-1}(\mathcal{F}_{\mathcal{E}})$. With this in mind, we have the following.

Corollary 4.14. *Suppose* (*for simplicity*) *that M is a $\mathcal{D}$-module with good filtration whose Fourier dual is the $\mathcal{A}$-module F* (*in some cohomological degree*) *with induced good filtration. Then the microlocalizations $M_{\mathcal{E}}$ and $\mathcal{F}_{\mathcal{E}}$ satisfy $F(M_{\mathcal{E}}) = \mathcal{F}_{\mathcal{E}}$ and $F^{-1}(\mathcal{F}_{\mathcal{E}}) = M_{\mathcal{E}}$.*

Proof. The proof follows the same argument as in Corollary 4.13. □

5. Isomorphism theorem for moduli spaces of $\mathcal{D}$-bundles

In this section we give our principal application of the Fourier–Mukai transform.

5.1. Moduli stacks and isomorphism theorem

We begin by defining the moduli stacks for $V^{\vee}$-framed spectral sheaves and V-framed $\mathcal{D}$-bundles.

As above, let $V^{\vee}$ denote a nonzero coherent torsion sheaf on E supported on the smooth locus **G**. We will let $\mathfrak{CM}(E, V^{\vee})$ denote the *moduli stack of $V^{\vee}$-framed spectral sheaves* on $\overline{E}^{\natural}$: its objects are pairs $(\mathcal{F}, \psi)$ consisting of

(1) a finitely presented S-flat family $\mathcal{F}$ of sheaves of pure dimension 1 on $\overline{E}^{\natural}$ parametrized by a scheme S,
(2) an isomorphism $\psi : \mathcal{F}|_{E_{\infty} \times S} \to V^{\vee}$,

such that the restriction to $\overline{E}^{\natural} \times \{s\}$ is a $V^{\vee}$-framed spectral sheaf on $\overline{E}^{\natural}$ for each $s \in S$ (Definition 2.3). An isomorphism from $(\mathcal{F}, \psi)$ to $(\mathcal{F}', \psi')$ consists of an isomorphism $\mathcal{F} \to \mathcal{F}'$ of coherent sheaves on $\overline{E}^{\natural} \times S$ that is compatible with the framings ψ, ψ'.

Standard arguments prove that $\mathfrak{CM}(E, V^{\vee})$ is an algebraic stack, locally of finite type over $\mathbb{C}$. Note that $\mathfrak{CM}(E, V^{\vee})$ will have many components, since we have not fixed one discrete invariant of $\mathcal{F}$, namely $\mathrm{rk}(p_* \mathcal{F}(-E_{\infty}))$.

Let V denote the semistable vector bundle of degree 0 on E that is the Fourier transform of $V^{\vee}$. Let $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V)$ denote the *moduli stack of V-framed $\mathcal{D}$-bundles on E*: its objects are triples $(M, \{M_k\}, \phi)$ consisting of

(1) a finitely generated S-flat family M of locally projective $\mathcal{D}_S$-modules on $E \times S$,
(2) a $\mathcal{D}$-module filtration $\{M_k\}$ of M such that M_k is a vector bundle over $E \times S$ for $k \gg 0$,
(3) an isomorphism $\phi : \bigoplus_{k \geq N} \mathrm{gr}_k(M) \to \bigoplus_{k \geq N} \mathrm{gr}_k(\mathcal{D})_S \otimes V$ as $\mathrm{gr}(\mathcal{D})_S$-modules for $N \gg 0$,

such that the restriction to $E \times \{s\}$ is a V-framed $\mathcal{D}$-bundle for each $s \in S$. We will refer to such a triple as an *S-flat family of V-framed $\mathcal{D}$-bundles* on E, or, when S is understood, simply as a V-framed $\mathcal{D}$-bundle.

An isomorphism between two such triples $(M, \{M_k\}, \phi)$ and $(M', \{M'_k\}, \phi')$ is an isomorphism $\psi : M \to M'$ of $\mathcal{D}_S$-modules such that $\psi(M_k) = M'_k$ for all $k \gg 0$ and $\operatorname{gr} \psi \circ \phi|_{\operatorname{gr}_k(M)} = \phi' \circ \operatorname{gr} \psi|_{\operatorname{gr}_k(M)}$ for all $k \gg 0$.

The principal result of this section is the following isomorphism of moduli stacks.

Theorem 5.1. *The Fourier functor F induces an isomorphism of stacks*

$$F : \operatorname{Bun}_{\mathbb{P}(\mathcal{D})}(E, V) \to \mathfrak{CM}(E, V^\vee).$$

Remark 5.2. This is not, in fact, the most general such theorem possible, in the following sense. One may enlarge the class of $\mathcal{D}$-bundles and the class of spectral sheaves in such a way that the Fourier–Mukai transform induces an isomorphism between the moduli stacks for these more general classes of objects. Since these more general classes do not seem to be useful for studying solutions of the KP or CM systems, we confine ourselves to formulating the general theorem. The proof uses the same techniques as we use for Theorem 5.1.

Theorem 5.3. *Given a coherent sheaf $\mathcal{F}$ on $\overline{E}^\natural$ of pure dimension one, define the quotient sheaf $\mathcal{F}_{\leq 0}$ as in Definition 5.9. Then the Fourier–Mukai transform induces an isomorphism of the moduli stacks for the following objects:*

(1) *Pairs $(\mathcal{F}, \phi)$ consisting of a coherent sheaf $\mathcal{F}$ of pure dimension one on $\overline{E}^\natural$ and an isomorphism $\phi : \mathcal{F}|_{E_\infty} \to V^\vee$ that satisfies:*

   (a) *$\mathcal{F}$ has no nonzero subsheaves $\mathcal{G} \subset \mathcal{F}$ with both $\mathcal{F}|_{E_\infty} = 0$ and $\mu(p_*\mathcal{G}) > 0$.*
   (b) *The quotient sheaf $\mathcal{F}_{\leq 0}$ of $\mathcal{F}$ is zero.*

(2) *Pairs (M, ψ) consisting of a torsion-free $\mathcal{D}$-module M equipped with a good filtration and an isomorphism*

$$\psi : \bigoplus_{k \geq k'} \operatorname{gr}_k(M) \to V \otimes \bigoplus_{k \geq k'} \operatorname{gr}_k(\mathcal{D})$$

for some k' sufficiently large.

5.2. Framed $\mathcal{D}$-bundles and framed spectral sheaves

We begin by proving Theorem 5.1 at the level of points: the Fourier transform identifies V-framed $\mathcal{D}$-bundles (Definition 3.2) with $V^\vee$-framed spectral sheaves (Definition 2.3). We continue to fix a coherent torsion sheaf $V^\vee$ on E that is supported on $\mathbf{G}$ and $V = F^{-1}(V^\vee)$.

We first recall a basic result on sheaves on projective bundles (see Sections II.5 and III.8 of [Ha]). Suppose $F = \bigoplus_{k \geq n} F_k$ is a finitely generated graded $\mathcal{R}(\mathcal{A})$-module, and let $\widetilde{F}$ denote the corresponding sheaf on $\overline{E}^\natural = \underline{\operatorname{Proj}}(\mathcal{R}(\mathcal{A}))$. Conversely, given a coherent sheaf $\mathcal{F}$ on $\overline{E}^\natural$, let $\Gamma \mathcal{F} = \bigoplus_{k \geq 0} \Gamma_k \mathcal{F} \overset{\text{def}}{=} \bigoplus p_* \mathcal{F}(kE_\infty)$ denote the associated $\mathcal{R}(\mathcal{A})$-module.

Proposition 5.4. *Suppose S is a noetherian scheme, and let $\mathcal{A}_S = \mathcal{A} \boxtimes \mathcal{O}_S$ on $E \times S$. Then:*

(1) *If F is a finitely generated $\mathcal{R}(\mathcal{A}_S)$-module, then there is a map $F \to \Gamma \widetilde{F}$ of graded $\mathcal{R}(\mathcal{A}_S)$-modules that is functorial in F and is an isomorphism in sufficiently high degrees.*

(2) *If $\mathcal{F}$ is a coherent sheaf on $\overline{E}^\natural \times S$, then there is an isomorphism $\widetilde{\Gamma \mathcal{F}} \to \mathcal{F}$ of coherent sheaves that is functorial in $\mathcal{F}$.*

(3) *The functor $N \mapsto \widetilde{N}$ is exact and commutes with tensor product over $\mathcal{O}_S$.*

(4) *The functor Γ takes exact sequences of coherent sheaves on $\overline{E}^\natural \times S$ to complexes of graded $\mathcal{R}(\mathcal{A}_S)$-modules that are exact in all sufficiently high graded degrees.*

Suppose that M is a V-framed $\mathcal{D}$-bundle; in particular, it comes equipped with a canonical structure of filtered $\mathcal{D}$-module (see Definition 7.6). Let $\mathcal{R}(M)$ denote the Rees module of M.

We will need the following basic fact relating M and $\mathcal{R}(M)$:

Proposition 5.5.

(1) *Suppose that M is a V-framed $(\mathcal{D})_S$-bundle on E for some scheme S. Then the Rees module $\mathcal{R}(M)$ is an S-flat family of torsion-free graded $\mathcal{R}(\mathcal{D})$-modules, and the framing becomes an isomorphism $\mathcal{R}(M)/t\mathcal{R}(M) \xrightarrow{\phi} \mathrm{gr}(\mathcal{D})_S \otimes V$ in high degree as $\mathrm{gr}(\mathcal{D})_S$-modules.*

(2) *Conversely, if $N = \bigoplus N_k$ is an S-flat family of torsion-free graded $\mathcal{R}(\mathcal{D})$-modules equipped with a V-framing $N/tN \xrightarrow{\phi} \mathrm{gr}(\mathcal{D})_S \otimes V$, then $\varinjlim N_k$ together with the induced filtration, filtered $\mathcal{D}$-module structure and framing is an S-flat family of V-framed $\mathcal{D}$-bundles.*

These constructions give an equivalence between the groupoid of V-framed $\mathcal{D}$-bundles and the groupoid of V-framed torsion-free $\mathcal{R}(\mathcal{D})$-modules.

Proof. See [LvO]. $\square$

We will first check that M, together with its filtration, transforms to a filtered sheaf on $\overline{E}^\natural$.

Lemma 5.6. *Suppose that M is a V-framed $\mathcal{D}$-bundle. Then:*

(1) *$F(\mathcal{R}(M))$ is a sheaf in cohomological degree 1.*

(2) *The element $t \in \mathcal{R}(\mathcal{A})$ is a non-zero-divisor on $F(\mathcal{R}(M))$.*

Proof. Suppose (M, M_k, ϕ) is a V-framed $\mathcal{D}$-bundle. Part (i) of Proposition 7.8 implies that, for m sufficiently large, the canonical filtration Θ on M satisfies $\Theta_k(M) = M_k$ for all $k \geq m$. By parts (a) and (i) of Proposition 7.8, we find that

$$\Theta_k(M) \subseteq \Theta_k(M_\mathcal{E})/\Theta_{-1}(M_\mathcal{E})$$

for all k. Part (iii) of Proposition 7.8 implies that $\Theta_k(M_\mathcal{E})/\Theta_{-1}(M_\mathcal{E})$ is an iterated extension of copies of V; since $F^0(V) = 0$, we also have $F^0(M_k) = 0$.

Since $F^0(V) = 0$, we find that $F(M_k) = F^1(M_k) \to F^1(M_{k+1}) = F(M_{k+1})$ is injective for all k in the great range. This implies (2). $\square$

Corollary 5.7. *The exact sequence*

$$0 \to t\mathcal{R}(M) \to \mathcal{R}(M) \to \mathrm{gr}(\mathcal{D}) \otimes V \to 0$$

of graded $\mathcal{R}(\mathcal{D})$-modules in sufficiently high degree has as its Fourier transform a short exact sequence

$$0 \to tF(\mathcal{R}(M)) \to F(\mathcal{R}(M)) \to \mathrm{gr}(\mathcal{A}) \otimes V^\vee \to 0 \tag{5.1}$$

of graded $\mathcal{R}(\mathcal{A})$-modules in cohomological degree 1.

Proof. It follows from Lemma 5.6 that the exact sequence

$$0 \to t\mathcal{R}(M) \to \mathcal{R}(M) \to \mathcal{R}(M)/t\mathcal{R}(M) = \mathrm{gr}(\mathcal{D}) \otimes V \to 0$$

is taken by the Fourier functor to an exact sequence

$$0 \to tF(\mathcal{R}(M)) \to F(\mathcal{R}(M)) \to F(\mathcal{R}(M)/t\mathcal{R}(M)) = F(\mathrm{gr}(\mathcal{D}) \otimes V) \to 0$$

of sheaves in cohomological degree 1. So it suffices to check that $F(\mathrm{gr}(\mathcal{D}) \otimes V) = \mathrm{gr}(\mathcal{A}) \otimes V^\vee$ as $\mathrm{gr}(\mathcal{A})$-modules. This is immediate from part (2) of Theorem 4.9. □

Write $N_k = F(M_k)$ and $N = F(\mathcal{R}(\mathcal{M}))$, and let $\mathcal{F} = \widetilde{F(\mathcal{R}(M))} = \widetilde{N}$ (see 5.4). Taking the sequence of coherent sheaves on $\overline{E}^\natural$ associated to the sequence (5.1) gives an exact sequence

$$0 \to \mathcal{F}(-E_\infty) \to \mathcal{F} \to V^\vee \to 0$$

on $\overline{E}^\natural$; in other words, $\mathcal{F}$ is a $V^\vee$-framed coherent sheaf on $\overline{E}^\natural$.

Proposition 5.8. *If M is a V-framed $\mathcal{D}$-bundle, then $\mathcal{F} = \widetilde{F(\mathcal{R}(M))}$ (with its induced $V^\vee$-framing) is a $V^\vee$-framed CM spectral sheaf.*

Proof. Observe first that $\mathcal{F}$ has one-dimensional support: indeed, since $M_k \subset M_{\mathcal{E},k}/M_{\mathcal{E},-1}$ is torsion-free of negative degree, a standard Fourier–Mukai computation shows that $F(M_k)$ has support equal to E, and thus $\mathrm{supp}(\mathcal{F})$ has dimension at least one.

Suppose $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ of dimension 0. By part (1) of Proposition 5.4 and part (2) of Lemma 5.6, the natural map $p_*\mathcal{F}(kE_\infty) \to p_*\mathcal{F}((k+1)E_\infty)$ is injective for k sufficiently large. On the other hand, if the support of $\mathcal{F}'$ had nontrivial intersection with E_∞, the natural maps $p_*\mathcal{F}'(kE_\infty) \to p_*\mathcal{F}'((k+1)E_\infty)$ would fail to be injective for all k. Since $p_*\mathcal{F}'(kE_\infty) \subseteq p_*\mathcal{F}(kE_\infty)$ for all k, it follows that $\mathrm{supp}(\mathcal{F}') \cap E_\infty = \emptyset$. Then the maps $p_*\mathcal{F}'(kE_\infty) \to p_*\mathcal{F}'((k+1)E_\infty)$ are isomorphisms for all k, implying that $\bigcup_{k\geq 0} p_*\mathcal{F}'(kE_\infty)$ is a finite-length $\mathcal{O}_E$-submodule of $F(M)$. This transforms under F^{-1} to an $\mathcal{O}$-coherent $\mathcal{D}$-submodule of M, a contradiction since M is torsion-free over $\mathcal{D}$. Thus $\mathcal{F}$ is of pure dimension 1.

It remains to prove the normalization conditions (i) and (ii) in Definition 2.3. By hypothesis, $\deg(M_k) = -n$ and $\mathrm{rk}(M_k) = (k+1)\,\mathrm{rk}(V)$ for all k sufficiently large; a standard Fourier–Mukai computation then shows that $F(M_k) = N_k$ has rank n and degree $(k+1)\,\mathrm{rk}(V)$ for $k \gg 0$. Condition (i) then follows from Lemma 2.5.

To verify condition (ii), we use the exact sequence

$$0 \to M_k \to M_{\mathcal{E},k}/M_{\mathcal{E},-1} \to Q_k \to 0 \tag{5.2}$$

for $k \gg 0$ in which, by Definition 3.2 and Proposition 7.7, Q_k is a torsion $\mathcal{O}_E$-module supported on the smooth locus of E. Applying F, we get an exact sequence

$$0 \to F^0(Q_k) \to N_k \to N_k/N_{-1} \to 0. \tag{5.3}$$

For $k \gg 0$, moreover, we have $N_k = p_*\mathcal{F}(kE_\infty)$ and $N_k/N_{-1} \cong p_*(\mathcal{F}(kE_\infty)/\mathcal{F}(-E_\infty))$ by Theorem 4.9. It follows that for $k \gg 0$, $p_*\mathcal{F}(-E_\infty) = F(Q_k)$, a vector bundle W of the kind required by condition (i) of Definition 2.3. This proves the proposition. □

We next study the transform of a CM spectral sheaf.

Definition 5.9. Let

$$F_{\leq 0} \stackrel{\text{def}}{=} \frac{(\Gamma\mathcal{F}/\operatorname{tors}(\Gamma\mathcal{F}))}{F_{>0}(\Gamma\mathcal{F}/\operatorname{tors}(\Gamma\mathcal{F}))}.$$

The quotient sheaf $\mathcal{F} \to \mathcal{F}_{\leq 0}$ is defined by $\mathcal{F}_{\leq 0} = \widetilde{F_{\leq 0}}$.

Remark 5.10. Note that, by Proposition 5.4, we have $F_{\leq 0} \to \Gamma\mathcal{F}_{\leq 0}$ which is an isomorphism in high degree.

Proposition 5.11. *Every CM spectral sheaf $(\mathcal{F}, \phi)$ has the following two properties:*

(1) *$\mathcal{F}$ has no nonzero subsheaves $\mathcal{G} \subset \mathcal{F}$ satisfying both $\mathcal{G}|_{E_\infty} = 0$ and $\mu(p_*\mathcal{G}) > 0$.*
(2) *The quotient sheaf $\mathcal{F}_{\leq 0}$ of $\mathcal{F}$ is zero.*

Proof. Suppose that $\mathcal{G} \subset \mathcal{F}$ is a subsheaf satisfying $\mathcal{G}|_{E_\infty} = 0$. Then $p_*\mathcal{G} = p_*\mathcal{G}(-E_\infty) \subset p_*\mathcal{F}(-E_\infty)$, implying, since the last term is a semistable vector bundle of degree 0, that $\deg(p_*\mathcal{G}) \leq 0$. This proves statement (1) above. For statement (2), we proceed as follows. For every k, we have a map $\phi_k : F_{-1} \to (F_k)_{\leq 0}$ from a semistable vector bundle of degree 0 to a sheaf whose Harder–Narasimhan subquotients all have nonpositive degrees; a standard argument shows that the image is a semistable torsion-free sheaf of degree 0 with torsion-free quotient. From the short exact sequence $0 \to F_{-1} \to F_k \to Q_k \to 0$ for all $k \geq 0$, where Q_k is a torsion $\mathcal{O}$-module, it then follows that the induced map from Q_k to $(F_k)_{\leq 0}/\operatorname{Im}(\phi_k)$ is zero. Since F_k surjects onto $(F_k)_{\leq 0}$, however, it follows that ϕ_k must already be surjective, i.e. $(F_k)_{\leq 0}$ is torsion-free semistable of degree 0, and $F_{-1} \to (F_k)_{\leq 0}$ is surjective. The filtration on $(F_k)_{\leq 0}$ induced from the one on F_{-1} by part (ii) of [FM1, Lemma 1.2.5] then has subquotients that are line bundles of degree 0.

Suppose, then, that $\mathcal{F}_{\leq 0}$ is nonzero. Then, for all k sufficiently large, $p_*\mathcal{F}_{\leq 0}(kE_\infty) = (F_k)_{\leq 0}$. We claim that for all k sufficiently large, $p_*\mathcal{F}_{\leq 0}(kE_\infty) = p_*\mathcal{F}_{\leq 0}((k+1)E_\infty)$: if not, then, since for k large $\operatorname{rk}(p_*\mathcal{F}_{\leq 0}(kE_\infty)) = \operatorname{rk}(p_*\mathcal{F}_{\leq 0}((k+1)E_\infty))$, we find that $\deg(F_k)_{\leq 0}$ is a strictly increasing function of k, a contradicition since it is bounded above. It follows that, for k sufficiently large, the terms in the graded $\mathcal{A}$-module $\bigoplus_m (F_m)_{\leq 0}$ stabilize, and thus that there is a morphism—the multiplication map—of the form $\operatorname{At} \otimes (F_k)_{\leq 0} \to (F_k)_{\leq 0}$ that is the identity map on the subsheaf $\mathcal{O} \otimes (F_k)_{\leq 0}$—in other words, the extension given by tensoring the Atiyah extension with $(F_k)_{\leq 0}$ is split. But we

have already seen that $(F_k)_{\leq 0}$ is a successive extension of line bundles on E, and is in particular a vector bundle on E. Hence, taking the trace on $\mathrm{Ext}^1((F_k)_{\leq 0}, (F_k)_{\leq 0})$ gives a splitting (up to scale) of the inclusion map

$$\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}) \to \mathrm{Ext}^1((F_k)_{\leq 0}, (F_k)_{\leq 0})$$

that tensors an extension of $\mathcal{O}$ by $\mathcal{O}$ with the vector bundle $(F_k)_{\leq 0}$. In particular, the Atiyah extension *cannot* split when tensored with $(F_k)_{\leq 0}$ if the latter is nonzero. This completes the proof. $\qquad\square$

Proposition 5.12. *Suppose $\mathcal{F}$ is a $V^\vee$-framed spectral sheaf on $\overline{E}^\natural$. Then $F^{-1}(\Gamma \mathcal{F})$ corresponds, under the equivalence of Proposition 5.5, to a V-framed $\mathcal{D}$-bundle on E.*

Proof. Since $\mathcal{F}$ is of pure dimension 1 and $\mathcal{F}|_{E_\infty} = V^\vee$, $\mathcal{F}$ has no local sections supported set-theoretically on E_∞. Thus the map $\mathcal{F}(kE_\infty) \to \mathcal{F}((k+1)E_\infty)$ is injective for all k, and the action of t on $\Gamma \mathcal{F}$ is regular. Taking the direct image of

$$0 \to \bigoplus \mathcal{F}(kE_\infty) \to \bigoplus \mathcal{F}((k+1)E_\infty) \to \bigoplus \mathcal{F}((k+1)E_\infty)|_{E_\infty} = V^\vee(kE_\infty) \to 0$$

thus gives an exact sequence

$$0 \to t\Gamma \mathcal{F} \to \Gamma \mathcal{F} \to \Gamma \mathcal{F}/t\Gamma \mathcal{F} = \mathrm{gr}(\mathcal{A}) \otimes V^\vee \to 0 \tag{5.4}$$

in high degree.

Consider the torsion-free quotient $\Gamma_k \mathcal{F}/\mathrm{tors}(\Gamma_k \mathcal{F})$. By Proposition 5.11, for $k \gg 0$ all terms in its Harder–Narasimhan filtration have positive slope; applying Proposition 4.7 and using the exact sequence

$$0 \to \mathrm{tors}(\Gamma_k \mathcal{F}) \to \Gamma_k \mathcal{F} \to \Gamma_k \mathcal{F}/\mathrm{tors}(\Gamma_k \mathcal{F}) \to 0, \tag{5.5}$$

we find that $F^{-1}(\Gamma_k \mathcal{F})$ is a sheaf in cohomological degree 0 for $k \gg 0$. It then follows that the same is true for every term of (5.4) for $k \gg 0$, and we obtain an exact sequence

$$0 \to t F^{-1}(\Gamma \mathcal{F}) \to F^{-1}(\Gamma \mathcal{F}) \to \mathrm{gr}(\mathcal{D}) \otimes V \to 0$$

in high degrees. In particular, $M = \bigcup F^{-1}(\Gamma_k \mathcal{F})$ is a filtered $\mathcal{D}$-module equipped with an isomorphism $\mathrm{gr}(M) = \mathrm{gr}(\mathcal{D}) \otimes V$ of $\mathrm{gr}(\mathcal{D})$-modules in high degree.

We wish to prove next that M is torsion-free; so suppose not. The V-framing implies that the torsion submodule $\mathrm{tors}(M)$ is contained in $M_k = F^{-1}(\Gamma_k \mathcal{F})$ for some k sufficiently large. In particular, $\mathrm{tors}(M)$ is an $\mathcal{O}$-coherent $\mathcal{D}$-submodule of M. Since $F(M) = p_*(\mathcal{F}|_{E^\natural})$ is a sheaf concentrated in cohomological degree 1, we find that $F(\mathrm{tors}(M))$ is an $\mathcal{O}$-coherent sheaf concentrated in degree 1 that gives a subsheaf $\mathcal{G} = F(\mathcal{R}(\widetilde{\mathrm{tors}(M)}))$ of $\mathcal{F}$ with the property (since $\mathrm{tors}(M)_k = \mathrm{tors}(M)_{k+1}$ for $k \gg 0$) that $\mathcal{G}|_{E_\infty} = 0$. By Proposition 5.4, $p_*(\mathcal{G}) = F(\mathrm{tors}(M)_k)$ for $k \gg 0$; this sheaf is torsion-free since it is the direct image of a sheaf of pure dimension 1 that has zero intersection with E_∞. Because $F^{-1}(p_* \mathcal{G}) = \mathrm{tors}(M)_k$ in degree 0, Proposition 4.7 then implies that $p_* \mathcal{G}$ has positive slope, contradicting part (1) of Proposition 5.11. So $\mathrm{tors}(M) = 0$.

By condition (ii) of Definition 2.3, we get $\mathrm{rk}(M_k) = (k+1)\,\mathrm{rk}(V)$ for all $k \gg 0$. Since M is V-framed and $\mathrm{gr}(M)$ is torsion-free, we conclude that M is normalized if and only if $M_{-1} = 0$. So, suppose that $M_{-1} \neq 0$; it would then follow that $\mathrm{rk}(M_k) < \mathrm{rk}(V)$ for all $k < 0$. Let k be the smallest integer such that $M_k \neq 0$, and consider the $\mathcal{D}$-submodule N of M generated by M_k. This is a $\mathcal{D}$-bundle of rank $\mathrm{rk}(M_k)$. Then, under the induced filtration from M, $\mathrm{rk}(\mathrm{gr}_\ell(N)) = \mathrm{rk}(M_k)$ for all $\ell \geq k$. Thus, shifting the filtration by k, we find that $N(k)$ is a normalized $\mathcal{D}$-bundle. Let $\mathcal{G}$ be the Fourier transform of N. It follows from our earlier argument that $\deg(p_*(\mathcal{G}((-1+k)E_\infty))) = 0$, hence $\deg(p_*(\mathcal{G}(-E_\infty))) = (-k)\,\mathrm{rk}(M_k)$. Since $k < 0$ and $p_*\mathcal{G}(-E_\infty)$ is a subsheaf of $\Gamma_{-1}(\mathcal{F})$, this contradicts semistability of $\Gamma_{-1}(\mathcal{F})$. So M satisfies condition (ii) of Definition 3.2.

To prove that condition (i) of Definition 3.2 holds, we use the exact sequence

$$0 \to \Gamma_{-1}\mathcal{F} \to \Gamma_k\mathcal{F} \to p_*(\mathcal{F}(kE_\infty)/\mathcal{F}(-E_\infty)) \to 0.$$

Using Corollary 4.14, this transforms under Fourier–Mukai to the exact sequence

$$M_k \to M_{\mathcal{E},k}/M_{\mathcal{E},-1} \to F(\Gamma_{-1}\mathcal{F}) \to 0$$

for $k \gg 0$. By condition (i) of Definition 2.3, $F(\Gamma_{-1}\mathcal{F})$ is a coherent $\mathcal{O}_E$-torsion sheaf supported on the smooth locus of E, from which it follows that $M_k \cong M_{\mathcal{E},k}/M_{\mathcal{E},-1}$ as filtered sheaves in a neighborhood of ∞. The desired conclusion is then immediate. $\qquad\square$

We may summarize the results of Propositions 5.8 and 5.12 as follows.

Corollary 5.13. *The construction $M \mapsto F(\widetilde{\mathcal{R}(M)})$ gives a bijective correspondence between V-framed $\mathcal{D}$-bundles on E and $V^\vee$-framed spectral sheaves on $\overline{E}^\natural$.*

5.3. Proof of Theorem 5.1

Now that we have proven the isomorphism for $\mathbb{C}$-points of the stacks (Corollary 5.13), what remains is simply to extend this bijection to families and check that it is functorial and compatible with base change.

By standard limit arguments, we may assume that all parameter schemes S are noetherian; we will do so without comment below.

The proof will require the following facts from [Br].

Lemma 5.14. *Let S and T be schemes.*

(1) *Let $g : T \to S$ be a morphism and fix $\mathcal{E} \in D(S \times E)$ of finite Tor-dimension over S. Then there exists an isomorphism*

$$F \circ \mathbf{L}(g \times 1_E)^*\mathcal{E} \simeq \mathbf{L}(g \times 1_E)^* \circ F(\mathcal{E}).$$

(2) *Let $\mathcal{E}$ be a coherent sheaf on $S \times E$ that is flat over S. Suppose that for each $s \in S$, $F(\mathcal{E}_s)$ is a sheaf (in some cohomological degree). Then $F(\mathcal{E})$ is a sheaf on $S \times E$ that is flat over S.*

The isomorphism of Theorem 5.1 will follow from a certain collection of technical facts (Lemmas 5.15 through 5.20).

Given a V-framed $\mathcal{D}$-bundle $(M, \{M_k\}, \phi)$ on $E \times S$, we obtain an object $F(\mathcal{R}(M))$ of the derived category of $\mathcal{R}(\mathcal{A})$-modules.

Lemma 5.15. $F(\mathcal{R}(M))$ *is an S-flat sheaf of $\mathcal{R}(\mathcal{A})$-modules concentrated in degree* 1.

Proof. By assumption, $\mathcal{R}(M)$ is S-flat. By part (2) of Lemma 5.14 and part (1) of Lemma 5.6, the result follows. □

It then follows from part (3) of Proposition 5.4 that $\mathcal{F} = F(\widetilde{\mathcal{R}(M)})$ is an S-flat family of coherent sheaves on $\overline{E}^\natural$. We next prove:

Lemma 5.16. $F(\widetilde{\mathcal{R}(M)})|_{\overline{E}_s^\natural} = F(\widetilde{\mathcal{R}(M}|_{\overline{E}_s^\natural}))$ *for all $s \in S$.*

Proof. By Lemma 5.15 and part (1) of Lemma 5.14, we have $F(\mathcal{R}(M)_s) = F(\mathcal{R}(M))_s$ for all $s \in S$. The lemma then follows by part (3) of Proposition 5.4. □

Lemma 5.17. *The Fourier transform of the exact sequence*

$$0 \to t\mathcal{R}(M) \to \mathcal{R}(M) \to \mathrm{gr}(\mathcal{D}) \otimes V \to 0 \tag{5.6}$$

in high graded degrees becomes a short exact sequence

$$0 \to \mathcal{F}(-E_\infty) \to \mathcal{F} \to s_* V^\vee \to 0 \tag{5.7}$$

of sheaves on $\overline{E}^\natural$ (where $s : E \to \overline{E}^\natural$ is the section at infinity).

In particular, it then follows by Corollary 5.13 that $(\mathcal{F}, F(\phi))$ is an S-flat family of $V^\vee$-framed spectral sheaves.

Proof of Lemma 5.17. The Fourier transforms of the terms in (5.6) are all sheaves in cohomological degree 1 by Lemma 5.15. Hence, by the long exact cohomology sequence, F applied to (5.6) is an exact sequence of sheaves in degree 1 in high graded degrees. We get an exact sequence

$$0 \to F(\mathcal{R}(M))(-1) \to F(\mathcal{R}(M)) \to \mathrm{gr}(\mathcal{A}) \otimes V^\vee \to 0$$

as a result. Part (3) of Proposition 5.4 then proves the lemma. □

It is clear from the construction that this takes isomorphisms of V-framed $\mathcal{D}$-bundles to isomorphisms of $V^\vee$-framed spectral sheaves. Moreover, by Lemma 5.15 and part (1) of Lemma 5.14, this construction commutes with pull back along morphisms $S' \to S$. Hence it gives a morphism of stacks $F : \mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V) \to \mathfrak{CM}(E, V^\vee)$.

In the other direction, we start with an S-flat family of $V^\vee$-framed spectral sheaves $(\mathcal{F}, \psi)$. We will prove:

Lemma 5.18. *Applying* Γ *to*

$$0 \to \mathcal{F}(-E_\infty) \to \mathcal{F} \xrightarrow{\phi} V^\vee \to 0 \tag{5.8}$$

gives an exact sequence

$$0 \to \Gamma\mathcal{F}(-1) \to \Gamma\mathcal{F} \to \mathrm{gr}(\mathcal{A}) \otimes V^\vee \to 0 \tag{5.9}$$

of S-flat graded $\mathcal{R}(\mathcal{A})$-modules in all sufficiently high graded degrees. Moreover, for any morphism $S' \to S$ we have $\Gamma_k(\mathcal{F}_{S'}) = (\Gamma_k \mathcal{F})_{S'}$ for all k sufficiently large.

Proof. The exactness is immediate from part (4) of Proposition 5.4; indeed, the sequence is exact in graded degree k by construction whenever

$$\mathbf{R}^1 p_* \mathcal{F}((k-1)E_\infty) = 0.$$

For flatness over S, we begin with the observation that it is enough to check this locally over E. In particular, we may restrict attention to an open set of E over which $\overline{E}^\natural$ is a trivial $\mathbb{P}^1$-bundle and choose a Čech covering of $\mathbb{P}^1$ by two open sets. We then obtain a two-term Čech complex $C^\bullet$ over (an open set of) $\overline{E}^\natural \times S$; since $\mathcal{F}$ is S-flat, so are the terms of the complex $\mathcal{F}(kE_\infty) \otimes C^\bullet$. Thus the terms of

$$p_* \mathcal{F}(kE_\infty) \otimes C^0 \to p_* \mathcal{F}(kE_\infty) \otimes C^1 \tag{5.10}$$

are also S-flat. Choosing k sufficiently large that $\mathbf{R}^1 p_* \mathcal{F}(kE_\infty) = 0$, we find that $\Gamma_k \mathcal{F} = p_* \mathcal{F}(kE_\infty)$ is the kernel of the surjective map (5.10) of S-flat sheaves, hence is itself S-flat. Moreover, if $S' \to S$ is any morphism, the pullback of (5.10) to S' has $\Gamma_k(\mathcal{F}_{S'})$ as its kernel (by S-flatness of the terms of the exact sequence), completing the proof. $\square$

We next prove:

Lemma 5.19. *In all sufficiently high degrees, the Fourier transform of (5.9) is an exact sequence*

$$0 \to F^{-1}(\Gamma(\mathcal{F}))(-1) \to F^{-1}(\Gamma\mathcal{F}) \to \mathrm{gr}(\mathcal{D}) \otimes V \to 0 \tag{5.11}$$

of S-flat families of $\mathcal{R}(\mathcal{D})$-modules concentrated in cohomological degree 0.

By Lemma 5.18, Lemma 5.19, and part (1) of Lemma 5.14, the formation of the exact sequence (5.11) then commutes with arbitrary base changes $S' \to S$.

Proof of Lemma 5.19. By Lemma 5.18, part (2) of Lemma 5.14, and Corollary 5.13, $F^{-1}(\Gamma\mathcal{F})$ is, in sufficiently high graded degrees, an S-flat sheaf in cohomological degree 0. Since this applies to F^{-1} of all terms in (5.9), it follows from the long exact cohomology sequence that F^{-1} applied to (5.9) is an exact sequence of the form (5.11) in all sufficiently high graded degrees. $\square$

We obtain, as an immediate consequence, the following analog of Proposition 5.5:

Lemma 5.20. *The natural map $F^{-1}(\iota_k) : F^{-1}(\Gamma_k \mathcal{F}) \to F^{-1}(\Gamma_{k+1}\mathcal{F})$ is injective for all k sufficiently large and satisfies $F^{-1}(\iota_k)_{S'} = F^{-1}(\iota_{k,S'})$ for every $S' \to S$.*

It follows that $M = \bigcup_k F^{-1}(\Gamma_k \mathcal{F})$ is a $\mathcal{D}$-module satisfying

(1) $M_{S'} = \bigcup_k F^{-1}(\Gamma_k(\mathcal{F}_{S'}))$ for every $S' \to S$,
(2) $M_k = F^{-1}(\Gamma \mathcal{F}_k)$ defines a good filtration on M,
(3) $\bigoplus_{k \geq n} \mathrm{gr}_k(M) = \bigoplus_{k \geq n} \mathrm{gr}(\mathcal{D}) \otimes V$ as $\mathrm{gr}(\mathcal{D})$-modules for some n sufficiently large.

Combining these observations with Corollary 5.13, we find that $(M, \{M_k\}, F^{-1}(\psi))$ is an S-flat family of V-framed $\mathcal{D}$-bundles on E.

It is again clear from the constructions that the map $(\mathcal{F}, \psi) \mapsto (M, \{M_k\}, F^{-1}(\psi))$ takes isomorphisms of $V^{\vee}$-framed spectral sheaves to isomorphisms of V-framed $\mathcal{D}$-bundles. We have proven above the compatibility of this construction with base change, so we obtain a morphism of stacks $F^{-1} : \mathfrak{CM}(E, V^{\vee}) \to \mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V)$.

To see that these functors give isomorphisms of stacks, let $(\mathcal{F}, \psi) = F(M, \{M_k\}, \phi)$. Applying Γ to $\mathcal{F}$ we obtain an exact sequence of $\mathcal{R}(\mathcal{A})$-modules of the form (5.9) in which, by Proposition 5.4, the left-hand and middle terms are isomorphic to $F(\mathcal{R}(M))(-1)$ and $F(\mathcal{R}(M))$ respectively in high degrees. Applying F^{-1} to this exact sequence we then obtain, by the construction and Theorem 4.2, an exact sequence equipped with a canonical isomorphism to (5.6) in high degrees; this reconstructs $(M, \{M_k\}, \phi)$ if we forget the terms M_k in the filtration for small k, and hence is isomorphic to $(M, \{M_k\}, \phi)$ as a family of framed $\mathcal{D}$-bundles. Thus $F^{-1} \circ F \simeq 1$.

Similarly, starting from $(\mathcal{F}, \psi)$ we apply Γ to the sequence (5.8) to obtain (5.9); applying F^{-1} to this we obtain a short exact sequence of the form (5.11) and thereby a flat family of V-framed $\mathcal{D}$-bundles $(M, \{M_i\}, \phi)$. Applying F to these data then returns the complex (5.9) in sufficiently high degrees by construction and Theorem 4.2. Finally, taking the associated coherent sheaves on $\overline{E}^{\natural} \times S$ gives us (5.8) by Proposition 5.4. Thus $F \circ F^{-1} \simeq 1$.

This completes the proof of Theorem 5.1. □

5.4. Compatibility of KP and CM hierarchies

We now complete the proof of the compatibility between KP and CM hierarchies, stated in Theorem 3.25.

Proof of Theorem 3.25. We first establish that $\mathcal{F}$ identifies the KP algebroid $\underline{\mathrm{End}}_{\mathcal{E}}$ on $\mathrm{Bun}_{\mathbb{P}(\mathcal{D})}(E, V)$ with the CM algebroid, also denoted $\underline{\mathrm{End}}_{\mathcal{E}}$, on $\mathfrak{CM}(E, V^{\vee})$. Let $M, \mathcal{F}$ denote a $\mathcal{D}$-bundle and its Fourier transform spectral sheaf. By Corollaries 4.13 and 4.14, the Fourier–Mukai transform is compatible with microlocalization, sending $M_{\mathcal{E}}$ to $\mathcal{F}_{\mathcal{E}}$. It follows that endomorphisms of each, with their Lie bracket, are also identified. The definitions of the respective algebroid structures (i.e. the actions deforming M, $\mathcal{F}$ inside their microlocalization by multiplication) are then also clearly identified. Finally, the choice of a micro-oper structure ∂_M acting on $M_{\mathcal{E}}$ corresponds to the choice of endomorphism ξ of $\mathcal{F}_{\mathcal{E}}$, and the restriction on the symbol of ∂_M is precisely the condition making ξ a Higgsing of $\mathcal{F}$. Since the KP flows on micro-opers and CM hierarchy on Higgsed spectral sheaves are given by the action of the corresponding algebroids, the compatibility of hierarchies is established. □

6. $\mathcal{D}$-lattices and d-lattices

Our goal in this section is to prove an extension of Sato's $\mathcal{D}$-module description of the big cell GR_n°. Namely, we will show that the entire Sato Grassmannian is a moduli space for certain $\mathcal{D}$-submodules of $\mathcal{E}^n$, the $\mathcal{D}$-lattices.

6.1. d-lattices

We begin by recalling the scheme structure on GR_n, following [AMP]. In [AMP], it is proven that GR_n is an infinite-dimensional scheme that represents a certain functor which we now describe.

Let z be a formal parameter. Recall (Section 3.4) that a vector subspace $B \subset \mathbb{C}((z^{-1}))^n$ is called a *c-lattice* if there exist integers k and ℓ such that $(\mathbb{C}[\![z^{-1}]\!]z^k)^n \subseteq B \subseteq (\mathbb{C}[\![z^{-1}]\!]z^\ell)^n$.

For a scheme S, a *discrete sub-$\mathcal{O}_S$-module* (or *d-lattice*) $L \subset \mathcal{O}_S((z^{-1}))^n$ is a quasi-coherent $\mathcal{O}_S$-module L equipped with an injection of $\mathcal{O}_S$-modules in $\mathcal{O}_S((z^{-1}))^n$ such that

(1) For every morphism $S' \xrightarrow{f} S$, the base-changed (composite) morphism

$$f^*L \to f^*(\mathcal{O}_S((z^{-1}))^n) \to \mathcal{O}_{S'}((z^{-1}))^n$$

   is injective.
(2) For every $s \in S$, there exists an open neighborhood U_s of s and a c-lattice $B \subset \mathbb{C}((z^{-1}))^n$ such that $L|_{U_s} \cap \mathcal{O}_{U_s} \widehat{\otimes} B \subset \mathcal{O}_{U_s}((z^{-1}))^n$ is locally free of finite type over $\mathcal{O}_{U_s}$, and $L|_{U_s} \oplus \mathcal{O}_{U_s} \widehat{\otimes} B \to \mathcal{O}_{U_s}((z^{-1}))^n$ is surjective.

The functor $\mathrm{GR}_n : \underline{\mathrm{Sch}}^{\mathrm{op}} \to \underline{\mathrm{Sets}}$ takes, as its value on a scheme S, the set

$$\mathrm{GR}_n(S) = \{\text{d-lattices } L \subset \mathcal{O}_S((z^{-1}))^n\}.$$

Theorem 6.1 ([BS, AMP]). *The functor* GR_n *is represented by a scheme.*

The scheme GR_n is called the (*rank n*) *Sato Grassmannian*.

It is convenient for us to use a slightly different, but equivalent, definition of a d-lattice.

Lemma 6.2. *A quasicoherent $\mathcal{O}_S$-module L with an injective map $L \to \mathcal{O}_S((z^{-1}))^n$ is a d-lattice if and only if the following hold:*

(1) *For every $s \in S$ there is an open neighborhood U_s of s and an integer k such that the map $L \to \mathcal{O}_S((z^{-1}))^n/(\mathcal{O}_S[\![z^{-1}]\!]z^k)^n$ is surjective with kernel a locally free $\mathcal{O}_S$-module of finite type.*
(2) *For every $s \in S$ there is an open neighborhood U_s of s and an integer ℓ such that the map $L \to \mathcal{O}_S((z^{-1}))^n/(\mathcal{O}_S[\![z^{-1}]\!]z^\ell)^n$ is injective with cokernel a locally free $\mathcal{O}_S$-module of finite type.*

Proof. First, we suppose L is a d-lattice. By the definition, for each $s \in S$ there is an open set U_s containing s and a c-lattice $B \subset \mathbb{C}((z^{-1}))^n$ giving an exact sequence

$$0 \to L|_{U_s} \cap \mathcal{O}_{U_s} \widehat{\otimes} B \to L|_{U_s} \to \mathcal{O}_{U_s}((z^{-1}))^n / \mathcal{O}_{U_s} \widehat{\otimes} B \to 0$$

over U_s with kernel locally free of finite type. Choosing some k such that $B \subseteq (\mathbb{C}[\![z^{-1}]\!]z^k)^n$, we get a surjective map $f : L|_{U_s} \to \mathcal{O}_{U_s}((z^{-1}))^n / (\mathcal{O}_{U_s}[\![z^{-1}]\!]z^k)^n$ whose kernel sits in an exact sequence

$$0 \to L|_{U_s} \cap \mathcal{O}_{U_s} \widehat{\otimes} B \to \ker(f) \to \mathcal{O}_S \widehat{\otimes} ((\mathbb{C}[\![z^{-1}]\!]z^k)^n / B) \to 0.$$

This proves that (1) holds for L.

We next prove that there exists ℓ such that $L|_{U_s} \cap (\mathcal{O}_S[\![x^{-1}]\!]x^\ell)^n = 0$. To see that, start with U_s and B such that $F = L|_{U_s} \cap (\mathcal{O}_{U_s} \widehat{\otimes} B)$ is finitely generated and locally free over $\mathcal{O}_{U_s}$. Choosing U_s smaller and B larger if necessary, we may assume that $U_s = \mathrm{Spec}(R)$ is an affine scheme, that $B = (\mathbb{C}[\![z^{-1}]\!]z^k)^n$, and that F is a free $\mathcal{O}_{U_s}$-module of rank N. So, suppose that $F = R^N$ and let $P \subset R$ be the prime of R corresponding to the point $s \in S$. Let $\phi : R^N \to R((z^{-1}))^n$ be the corresponding map of R-modules; this map factors through $(R[\![z^{-1}]\!]z^k)^n \subset R((z^{-1}))^n$ by construction. By part (1) of the definition of d-lattice, the induced map $\phi_{R_P/P} : (R_P/P)^N \to (R_P/P)((z^{-1}))^n$ is injective. Since $K = R_P/P$ is a field, for any ℓ sufficiently negative the map $\phi_K : K^N \to (K[\![z^{-1}]\!]z^k)^n / (K[\![z^{-1}]\!]z^\ell)^n$ is injective. It follows from Nakayama's Lemma that, letting $r = k - \ell - N$, there is an R-homomorphism $\psi : R^r \to (R[\![z^{-1}]\!]z^k)^n / (R[\![z^{-1}]\!]z^\ell)^n$ such that the localized sum

$$R_P^N \oplus R_P^r \xrightarrow{\phi + \psi} (R_P[\![z^{-1}]\!]z^k)^n / (R_P[\![z^{-1}]\!]z^\ell)^n$$

is surjective, hence also an isomorphism. Since the cokernel of $\phi + \psi$ is finitely generated over R, it follows that there is some element $f \in R \setminus P$ such that

$$(\phi + \psi) \otimes R_f : R_f^N \oplus R_f^r \to (R_f[\![z^{-1}]\!]z^k)^n / (R_f[\![z^{-1}]\!]z^\ell)^n \tag{6.1}$$

is an isomorphism. Consequently, over $\mathrm{Spec}(R_f)$, the map

$$L|_{\mathrm{Spec}(R_f)} \to \mathcal{O}_{\mathrm{Spec}(R_f)}((z^{-1}))^n / (\mathcal{O}_S[\![z^{-1}]\!]z^\ell)^n$$

is injective with cokernel a locally free $\mathcal{O}_{\mathrm{Spec}(R_f)}$-module of finite type (isomorphic to $\mathcal{O}_{\mathrm{Spec}(R_f)}^r$), proving that (2) holds.

For the converse, part (1) of the definition of d-lattice is automatic from statement (2) in the lemma, and part (2) of that definition is immediate from statement (1). $\square$

6.2. $\mathcal{D}$-lattices

We now turn to the description of the $\mathcal{D}$-modules that interest us. For the remainder of the subsection, we let $\mathcal{D} = \mathbb{C}[\![x]\!][\partial]$ where $\partial = \partial/\partial x$. For a scheme S, we let $\mathcal{D}_S =$

$\mathcal{O}_S[[x]][\partial]$; that is, $\mathcal{D}_S$ is the sheaf on S whose sections on an open set $U \subset S$ are given by $\mathcal{D}_S(U) = \mathcal{O}_S(U)[[x]][\partial]$. Similarly, we define a sheaf $\mathcal{E}_S = \mathcal{O}_S[[x]]((\partial^{-1}))$ in the same way. Note that $\mathcal{D}_S$ is a quasicoherent sheaf on the formal scheme $S \times \mathrm{Spf}(\mathbb{C}[[x]])$. Note also that $\mathcal{E}_S$ is not quasicoherent over $S \times \mathrm{Spf}(\mathbb{C}[[x]])$. Nevertheless, it is clear from the construction that, if $S' \xrightarrow{f} S$ is a morphism of schemes, there is a homomorphism of sheaves of algebras $f^{-1}\mathcal{E}_S \to \mathcal{E}_{S'}$ on S'.

Note. We use subscripts rather than superscripts to denote the filtration on the sheaf $\mathcal{E}$; this is nonstandard in the theory of $\mathcal{D}$-modules, but makes our formulas more readable (especially in this section).

For a scheme S, a $\mathcal{D}$-*lattice* $M \subset \mathcal{E}_S^n$ is a quasicoherent sheaf on $S \times \mathrm{Spf}(\mathbb{C}[[x]])$ with a structure of right $\mathcal{D}_S$-module that is finitely presented as a $\mathcal{D}_S$-module and comes equipped with an injective $\mathcal{D}_S$-module homomorphism $M \to \mathcal{E}_S^n$ that satisfies the following:

(1) For all $s \in S$, there exist an open neighborhood U_s of s and an integer k such that $M|_{U_s} \to \mathcal{E}_{U_s}^n / \mathcal{E}_{U_s,k}^n$ is surjective, with kernel a finitely generated, locally projective $\mathcal{O}_{U_s}[[x]]$-module.
(2) For all $s \in S$, there exist an open neighborhood U_s of s and an integer ℓ such that $M|_{U_s} \to \mathcal{E}_{U_s}^n / \mathcal{E}_{U_s,\ell}^n$ is injective, with cokernel a finitely generated, locally projective $\mathcal{O}_{U_s}[[x]]$-module.

This definition is useful since, on $\mathbb{C}$-points, framed $\mathcal{D}$-bundles give $\mathcal{D}$-lattices:

Proposition 6.3. *Let M be an $\mathcal{O}^n$-framed $\mathcal{D}$-bundle. Choose an isomorphism $M_{\mathcal{E}} \cong \mathcal{E}^n$ compatible with the framing. Then:*

(1) *For any k sufficiently large, the map $M \to \mathcal{E}^n / \mathcal{E}_k^n$ is surjective, with kernel a finitely generated projective $\mathbb{C}[[x]]$-module.*
(2) *For any ℓ sufficiently small, the map $M \to \mathcal{E}^n / \mathcal{E}_\ell^n$ is injective, with cokernel a finitely generated projective $\mathbb{C}[[x]]$-module.*

Proof. Since we have chosen an isomorphism compatible with the framing, the canonical filtration on $\mathcal{E}^n$ induced from that on M (Proposition 7.8) agrees with the standard one on $\mathcal{E}^n$. Corollary 7.7 implies that $\mathrm{gr}\, M \subset \mathrm{gr}\, \mathcal{E}^n$ is finitely generated and torsion-free over $\mathrm{gr}\, \mathcal{D}$ and that the inclusion is an isomorphism in all sufficiently large graded degrees. An inductive argument proves that $M/M_k \cong \mathcal{E}^n / \mathcal{E}_k^n$ for all $k \gg 0$. Now each M_k is finitely generated over $\mathbb{C}[[x]]$ by assumption and torsion-free over $\mathbb{C}[[x]]$ since M is torsion-free over $\mathcal{D}$. Hence each M_k is projective over $\mathbb{C}[[x]]$. This proves (1).

For (2), observe that Proposition 7.8 yields $M \cap \mathcal{E}_\ell^n = 0$ for $\ell \ll 0$, so $M \to \mathcal{E}^n / \mathcal{E}_\ell^n$ is injective. It follows from Corollary 7.7 that $\mathcal{E}^n / (\mathcal{E}_\ell^n + M)$ is finitely generated over $\mathbb{C}[[x]]$, so to prove that it is projective, it suffices to prove that it is torsion-free.

First, suppose that $\mathcal{E}^n / M$ has nonzero $\mathbb{C}[[x]]$-torsion. Then it has a $\mathcal{D}$-submodule N that is finitely generated over $\mathcal{D}$ and consists of $\mathbb{C}[[x]]$-torsion. Taking the induced filtration on $\mathcal{E}^n / M$ and thus on N, we get a filtration of N with $\mathrm{gr}_\ell(N) \subset \mathrm{gr}_\ell(\mathcal{E}^n)$ for $\ell \ll 0$. But this is impossible (since the right-hand module is $\mathbb{C}[[x]]$-torsion-free), so the filtration

of N must be bounded below. However, by Corollary 7.7 the filtration is also bounded above, so N is a finitely generated torsion module over $\mathbb{C}[\![x]\!]$ equipped with a $\mathcal{D}$-module structure, a contradiction. Thus $\mathcal{E}^n/M$ is $\mathbb{C}[\![x]\!]$-torsion-free.

Now, suppose that $\mathcal{E}^n/(\mathcal{E}^n_\ell + M)$ has $\mathbb{C}[\![x]\!]$-torsion submodule T_ℓ. For $\ell \ll 0$, we have an exact sequence

$$0 \to \mathrm{gr}_\ell\, \mathcal{E}^n = \mathbb{C}[\![x]\!]^n \to \mathcal{E}^n/(M + \mathcal{E}^n_{\ell-1}) \to \mathcal{E}^n/(M + \mathcal{E}^n_\ell) \to 0,$$

so in particular the natural map $T_{\ell-1} \to T_\ell$ is injective. But T_ℓ is finitely generated over $\mathbb{C}[\![x]\!]$ for each ℓ; in particular, it is of finite length. Now $\varprojlim \mathcal{E}^n/(M + \mathcal{E}^n_\ell) = \mathcal{E}/M$ is $\mathbb{C}[\![x]\!]$-torsion-free, so $\varprojlim T_\ell = 0$. Since each T_ℓ is of finite length, we get $T_\ell = 0$ for all sufficiently negative ℓ. This completes the proof. $\qquad\square$

We remark that $\mathcal{D}$-lattices pull back well:

Lemma 6.4. *If $S' \xrightarrow{f} S$ is a morphism and $M \subset \mathcal{E}^n_S$ is a $\mathcal{D}$-lattice, then $f^*M \to f^*\mathcal{E}^n_S \to \mathcal{E}^n_{S'}$ makes f^*M a $\mathcal{D}$-lattice.*

We thus obtain a functor $\mathrm{GR}_{\mathcal{D},n} : \underline{\mathrm{Sch}}^{\mathrm{op}} \to \underline{\mathrm{Sets}}$ by setting

$$\mathrm{GR}_{\mathcal{D},n}(S) = \big\{ \mathcal{D}\text{-lattices } M \subset \mathcal{E}^n_S \big\}.$$

The main theorem of this section is the following:

Theorem 6.5. *The scheme GR_n represents the moduli functor $\mathrm{GR}_{\mathcal{D},n}$ of $\mathcal{D}$-lattices.*

6.3. Proof of Theorem 6.5

We begin the proof of Theorem 6.5 by describing a natural transformation between the two moduli functors.

Given a $\mathcal{D}$-lattice $M \subset \mathcal{E}^n_S$, we get a map

$$M \otimes \mathbb{C}[\![x]\!]/(x) \to \mathcal{E}^n_S \otimes \mathbb{C}[\![x]\!]/(x) \cong \mathcal{O}_S(\!(z^{-1})\!)^n,$$

where we have identified ∂^{-1} with z^{-1}.

Proposition 6.6. *This construction defines a natural transformation of functors $\mathrm{GR}_{\mathcal{D},n} \to \mathrm{GR}_n$.*

Proof. Suppose that $M \subset \mathcal{E}^n_S$ is a $\mathcal{D}$-lattice. Over an open set $U \subset S$, if we have $M|_U \twoheadrightarrow \mathcal{E}^n_U/\mathcal{E}^n_{U,k}$ surjective with locally free kernel K, then we get a surjective map $M|_U \otimes \mathbb{C}[\![x]\!]/(x) \to (\mathcal{E}^n_U/\mathcal{E}^n_U x)/(\mathcal{E}^n_{U,k}/\mathcal{E}^n_{U,k} x) = \mathcal{O}_U(\!(z^{-1})\!)^n/(\mathcal{O}_U[\![z^{-1}]\!]z^k)^n$ with kernel $K \otimes \mathbb{C}[\![x]\!]/(x)$, a locally free $\mathcal{O}_U$-module.

Similarly, over an open set $U \subset S$, if we take $\ell \ll 0$, we get

$$0 \to M|_U \to \mathcal{E}^n_U/\mathcal{E}^n_{U,\ell} \to W \to 0$$

exact with W finitely generated and locally free over $\mathcal{O}_U[\![x]\!]$. Tensoring with $\mathbb{C}[\![x]\!]/(x)$, we get

$$0 \to (M|_U)/(M|_U x) \to \mathcal{O}_U((z^{-1}))^n/(\mathcal{O}_U[\![z^{-1}]\!]z^k)^n \to W \otimes \mathbb{C}[\![x]\!]/(x) \to 0$$

exact, with $W \otimes \mathbb{C}[\![x]\!]/(x)$ locally free over $\mathcal{O}_U$. So $M/Mx \to \mathcal{E}^n/\mathcal{E}^n x$ is a d-lattice. We thus get a function $\mathrm{GR}_{\mathcal{D},n}(S) \to \mathrm{GR}_n(S)$ for every scheme S. Furthermore, since tensor product over $\mathbb{C}[\![x]\!]$ commutes with pullback, these functions commute with the pullback along morphisms $S' \to S$, and thus define a natural transformation, as desired. $\qquad\square$

We want to prove that this natural transformation is actually an isomorphism. In particular, we first show that it is surjective: given a d-lattice $L \subset \mathcal{O}_S((z^{-1}))^n$, we want to produce a $\mathcal{D}$-lattice $M \subset \mathcal{E}_S^n$ whose fiber at $x = 0$ is L. Since both our functors are sheaves in the Zariski topology, it suffices to do this over an open set U_s containing s for each point s of S. We begin with the construction for those L that are *transverse* to some $(\mathcal{O}_S[\![z^{-1}]\!]z^k)^n$.

Lemma 6.7. *If $L \subset \mathcal{O}_S((z^{-1}))^n$ is a d-lattice satisfying $L \oplus (\mathcal{O}_S[\![z^{-1}]\!]z^k)^n = \mathcal{O}_S((z^{-1}))^n$, then there is an injection of $\mathcal{D}_S$-modules $\mathcal{D}_S^n \xrightarrow{\iota} \mathcal{E}_S^n$ satisfying $\iota(\mathcal{D}_S^n) \oplus \mathcal{E}_{S,k}^n = \mathcal{E}_S^n$ and $\iota(\mathcal{D}_S^n \otimes \mathbb{C}[\![x]\!]/(x)) = L$. Moreover, the image $\iota(\mathcal{D}_S^n)$ consists of all sections θ of $\mathcal{E}_S^n$ such that $\theta \cdot D|_{x=0} \in L$ for every section D of $\mathcal{D}_S$.*

Proof. The first part follows from Theorem 7.4 of [Mul] and Theorem 6.2 of [LM]: indeed, those results construct such a $\mathcal{D}$-module embedding of $\mathcal{D}^n$ in $\mathcal{E}^n$ in the case in which the base scheme is $\mathbb{C}$, but the proofs, which produce an element Ψ of $GL_n(\mathcal{E}_<)$ such that $\iota(m) = \Psi \cdot m$, work over any base ring.

For the second part, let

$$N = \{\theta \in \mathcal{E}_S^n \mid (\theta \cdot D)|_{x=0} \in L \text{ for all } D \in \mathcal{D}_S\}. \tag{6.2}$$

Observe that $\iota(\mathcal{D}_S^n) \subseteq N$. Since $\iota(\mathcal{D}_S^n) = \Phi \cdot \mathcal{D}_S^n$ for some invertible $\Phi \in GL_n(\mathcal{E}_S)$, after multiplying by Φ^{-1} we may assume that $\iota(\mathcal{D}_S^n) = \mathcal{D}_S^n$ and that $L = \mathcal{O}_S[z]$. Then any $\theta \in N$ satisfies $\theta \cdot \mathcal{O}_S[z]^n \subset \mathcal{O}_S[z]^n$, which, by Proposition 6.1 of [LM], implies that $\theta \in \mathcal{D}^n$ (again, the proof there is written for the base $\mathbb{C}$ but the proof works equally well over any base ring). $\qquad\square$

Next, suppose that $L \subset \mathcal{O}_S((z^{-1}))^n$ is any d-lattice. Fix $s \in S$. By restricting attention to an open neighborhood of s in S, we may assume, by Lemma 6.2, that there are k and ℓ such that we have exact sequences

$$0 \to V \to L \xrightarrow{\alpha} \mathcal{O}_S((z^{-1}))^n/(\mathcal{O}_S[\![z^{-1}]\!]z^k)^n \to 0$$

and

$$0 \to L \to \mathcal{O}_S((z^{-1}))^n/(\mathcal{O}_S[\![z^{-1}]\!]z^\ell)^n \to W \to 0$$

with V, W vector bundles over $\mathcal{O}_S$. Choosing a splitting α^{-1} of α, we get an $\mathcal{O}_S$-submodule $L_- = \mathrm{Im}(\alpha^{-1}) \subset L$ such that $L_- \xrightarrow{\alpha} \mathcal{O}_S((z^{-1}))^n/(\mathcal{O}_S[\![z^{-1}]\!]z^k)^n$ is an isomorphism and $L/L_- \cong V$. Similarly, shrinking S further if necessary, we may choose a

splitting $W \xrightarrow{\beta} \mathcal{O}_S((z^{-1}))^n$ of $\mathcal{O}_S((z^{-1}))^n \to \mathcal{O}_S((z^{-1}))^n/(\mathcal{O}_S[\![z^{-1}]\!]z^\ell)^n \to W$. We then have $L \oplus \beta(W) \cong \mathcal{O}_S((z^{-1}))^n/(\mathcal{O}_S[\![z^{-1}]\!]z^\ell)^n$. We let $L_+ = L + \beta(W) \subset \mathcal{O}_S((z^{-1}))^n$.

We now have $L_- \subset L \subset L_+$ with $L_- \oplus (\mathcal{O}_S[\![z^{-1}]\!]z^k)^n = \mathcal{O}_S((z^{-1}))^n$ and $L_+ \oplus (\mathcal{O}_S[\![z^{-1}]\!]z^\ell)^n = \mathcal{O}_S((z^{-1}))^n$. By Lemma 6.7, we have injective $\mathcal{D}_S$-homomorphisms $\mathcal{D}_S^n \xrightarrow{\iota_-} \mathcal{E}_S^n$ and $\mathcal{D}_S^n \xrightarrow{\iota_+} \mathcal{E}_S^n$ such that $\iota_-(\mathcal{D}_S^n)|_{x=0} = L_-$ and $\iota_+(\mathcal{D}_S^n)|_{x=0} = L_+$. Lemma 6.7 also implies that $\iota_-(\mathcal{D}_S^n) \subset \iota_+(\mathcal{D}_S^n)$. By construction, moreover, $\iota_+(\mathcal{D}_S^n)/\iota_-(\mathcal{D}_S^n) \cong \mathcal{E}_{S,k}^n/\mathcal{E}_{S,\ell}^n$ as $\mathcal{O}_S[\![x]\!]$-modules.

Proposition 6.8. *There is a finitely presented $\mathcal{D}_S$-module $M \subset \mathcal{E}_S^n$ satisfying:*

(1) $\iota_-(\mathcal{D}_S^n) \subset M \subset \iota_+(\mathcal{D}_S^n)$.
(2) $M \otimes \mathbb{C}[\![x]\!]/(x) = L$.
(3) *M consists of all $\theta \in \mathcal{E}_S^n$ such that $\theta \cdot D|_{x=0} \in L$ for all sections D of $\mathcal{D}_S$.*

Proof. Consider the quotient $\mathcal{D}_S$-module $Q = \iota_-(\mathcal{D}_S^n)/\iota_+(\mathcal{D}_S^n)$; by construction, this is a finitely generated locally free $\mathcal{O}_S[\![x]\!]$-module, hence, working locally over S, choosing a basis, and changing to a left $\mathcal{D}_S$-module, Q corresponds to a vector bundle with flat connection. The usual construction of flat sections with power series coefficients shows that there is a bijection between vector subbundles of Q/Qx and right $\mathcal{D}_S$-submodules of Q. In particular, choosing the vector subbundle L/L_- of L_+/L_-, there is a lift to a $\mathcal{D}_S$-module M, $\iota_-(\mathcal{D}_S^n) \subset M \subset \iota_+(\mathcal{D}_S^n)$, with $M/Mx = L$.

For part (3), define a module N as in (6.2), and note that Lemma 6.7 implies that we have $\iota_-(\mathcal{D}_S^n) \subseteq M \subseteq N \subseteq \iota_+(\mathcal{D}_S^n)$. Then N/M defines a $\mathcal{D}_S$-submodule of $Q' = \iota_+(\mathcal{D}_S^n)/M$ all of whose sections θ satisfy $(\theta \cdot D)|_{x=0} = 0 \in Q'/Q'x$; the standard results for flat connections then imply that $N/M = 0$, as desired. $\square$

It follows that the natural transformation $\mathrm{GR}_{\mathcal{D},n} \to \mathrm{GR}_n$ is Zariski-locally surjective, hence surjective.

Finally, suppose that $M' \subset \mathcal{E}_S^n$ is a $\mathcal{D}$-lattice with $M \otimes \mathbb{C}[\![x]\!]/(x) = L$. Let $M \subset \mathcal{E}_S^n$ be the $\mathcal{D}$-lattice lifting L that was constructed by Proposition 6.8. By part (3) of Proposition 6.8, $M' \subseteq M$. Since M and M' are $\mathcal{D}$-lattices, the quotient M/M' is a finitely generated $\mathcal{O}_S[\![x]\!]$-module: this follows by applying part (2) of the Definition of $\mathcal{D}$-lattice and considering $M/M' \subset \mathcal{E}^n/\mathcal{E}_\ell^n$ for ℓ sufficiently small, to find that M/M' is a submodule of a finitely generated $\mathcal{O}_S[\![x]\!]$-module, hence is itself finitely generated (the ring $\mathcal{O}_S[\![x]\!]$ is Noetherian).

Tensoring the exact sequence

$$M' \to M \to M/M' \to 0$$

with $\mathbb{C}[\![x]\!]/(x)$ and noting that the map from M' to M becomes the identity map of L, we find that $(M/M') \otimes \mathbb{C}[\![x]\!]/(x) = 0$. It follows from Nakayama's Lemma (Theorem 2.2 of Matsumura [M]) that there is an element a of $\mathcal{O}_S[\![x]\!]$ such that $a(M/M') = 0$ and $a \cong 1 \mod (x)$. But the completeness of $\mathcal{O}_S[\![x]\!]$ with respect to (x) then implies that a is a unit, and thus $M/M' = 0$. So the natural transformation $\mathrm{GR}_{\mathcal{D},n} \to \mathrm{GR}_n$ is injective. This completes the proof of Theorem 6.5. $\square$

7. Appendix: D-algebras and log differential operators

In this appendix, we explain the construction of the sheaves $\mathcal{D}_{\log}$ on singular cubics that we use in the body of the paper. We also explain some facts about filtrations on $\mathcal{D}$-modules and microlocalization that are needed for our main theorems.

7.1. D-algebras

Let X denote a reduced and irreducible quasiprojective complex variety. A *differential $\mathcal{O}_X$-bimodule* is a quasicoherent sheaf M of $\mathcal{O}_X$-bimodules, such that, if we give M the filtration defined by $M_{-1} = 0$ and $M_i = \{m \in M \mid [r, m] \in M_{i-1} \text{ for all } r \in \mathcal{O}_X\}$, then $M = \bigcup_i M_i$.

Definition 7.1 (1.1.4 of [BB]). A *D-algebra* on X is a quasicoherent sheaf $\mathcal{D}$ of associative algebras on X equipped with an algebra homomorphism $\mathcal{O}_X \to \mathcal{D}$ making $\mathcal{D}$ a differential $\mathcal{O}_X$-bimodule. As above, a D-algebra $\mathcal{D}$ comes with a canonical filtration; to be consistent with the usual notation for the sheaf of rings of differential operators, we write $\mathcal{D}^i$ for the ith term in the filtration.

A D-algebra $\mathcal{A}$ on X is called a *special D-algebra* (Section 6.2 of [PRo]) if the terms $\mathrm{gr}_i(\mathcal{A})$ in the associated graded bimodule for the standard bimodule filtration are of the form $\mathrm{gr}_i(\mathcal{A}) = \Delta_* \mathcal{F}_i$ for some *free $\mathcal{O}_X$-modules* $\mathcal{F}_i$ (i.e. the terms in the associated graded bimodule are trivial vector bundles).

The standard example of a D-algebra on a scheme X is the sheaf of rings of differential operators [BB].

Lemma 7.2. *Let X be a reduced, irreducible complex variety, and $U \subseteq X$ a nonempty open subvariety. Let $\mathcal{D}_X$ denote the sheaf of rings of differential operators on X. Then:*

(1) $\mathcal{D}_X(X) \subseteq \mathcal{D}_X(U)$.
(2) $\mathcal{D}_X^k(X) = \mathcal{D}_X(X) \cap \mathcal{D}_X^k(U)$.
(3) *Suppose $\mathcal{A} \subseteq \mathcal{D}_X$ is a subalgebra containing $\mathcal{O}_X$. Then $\mathcal{A}$ is a D-algebra with canonical filtration given by $\mathcal{A}_k = \mathcal{A} \cap \mathcal{D}_X^k$.*

The proofs follow standard arguments.

Recall (Section 2.1.2) the definition of the sheaf $\mathcal{D}_{\log}$ of log differential operators on a cubic curve, generated by functions and the **G**-invariant vector fields.

Corollary 7.3. *Let E be a Weierstrass cubic. Then:*

(1) $\mathcal{D}_{\log}^k = \mathcal{D}_{\log} \cap \mathcal{D}_\mathbf{G}^k$.
(2) $\mathcal{D}_{\log}$ *is a special D-algebra with associated graded algebra isomorphic to* $\mathrm{Sym}^\bullet \mathcal{O}_E$.

Proof. (1) $\mathcal{D}_{\log}$ is a D-algebra by part (3) of Lemma 7.2. We then have

$$\mathcal{D}_{\log}^k = \mathcal{D}_{\log} \cap D_X^k \quad \text{by part (3) of Lemma 7.2}$$
$$= \mathcal{D}_{\log} \cap D_\mathbf{G}^k \quad \text{by part (2) of Lemma 7.2.}$$

(2) By part (1) above, we have

$$\mathrm{gr}(\mathcal{D}_{\log}) \subseteq \mathrm{gr}(\mathcal{D}_{\mathbf{G}}) = \mathrm{Sym}^{\bullet} \mathcal{O}_{\mathbf{G}}.$$

Since $\mathcal{D}_{\log}$ is generated over $\mathcal{O}_E$ by the vector field X, $\mathrm{gr}(\mathcal{D}_{\log})$ is generated over $\mathcal{O}_E$ by the image of the vector field X in $\mathcal{D}_{\mathbf{G}}^1 / \mathcal{D}_{\mathbf{G}}^0$. The $\mathcal{O}_E$-submodule generated by this element is exactly $\mathrm{Sym}^{\bullet} \mathcal{O}_E \subset \mathrm{Sym}^{\bullet} \mathcal{O}_{\mathbf{G}}$, as desired. $\square$

7.2. Microlocalization and torsion-free $\mathcal{D}$-modules

For the remainder of this section, we fix a reduced and irreducible quasiprojective complex curve X and a D-algebra $\mathcal{D}$ on X.

We will call the D-algebra $\mathcal{D}$ *symmetric* if it satisfies $\mathrm{gr}(\mathcal{D}) = \mathrm{Sym}^{\bullet}(B)$ as algebras for some line bundle B on X.

We will write $\mathcal{E}$ for the microlocalization of $\mathcal{D}$; for background on the procedure of microlocalization and its properties, see for example [Ka, vdE, AVV, Sch, Sp]. Note that, by our assumption that $\mathcal{D}$ is symmetric, the associated graded ring $\mathrm{gr}(\mathcal{E})$ for the natural filtration is isomorphic to $\bigoplus_{k \in \mathbb{Z}} B^k$ as a graded $\mathcal{O}_X$-algebra.

By a *finitely generated* (*left or right*) $\mathcal{D}$-module on X we mean a $\mathcal{D}$-module that is quasicoherent as a sheaf of $\mathcal{O}_X$-modules and locally finitely generated over $\mathcal{D}$.

A symmetric D-algebra is, in particular, a sheaf of noetherian domains by Proposition 2.6.1 of [Bj]. It follows by Theorem 1.15 of [MR] that, for any affine open set $U \subseteq X$, the ring $\mathcal{D}(U)$ has a quotient skew field, which we denote by $Q(\mathcal{D})$ (it does not depend on U since X is irreducible). Consequently, we may define a *torsion-free $\mathcal{D}$-module* to be a (left or right) $\mathcal{D}$-module M such that for every open set U of X and all nonzero $s \in H^0(U, \mathcal{D})$ and nonzero $m \in H^0(U, M)$ we have $s \cdot m \neq 0$. If M is a finitely generated and torsion-free right $\mathcal{D}$-module and U is an affine open set of X, then $H^0(U, M) \subset H^0(U, M) \otimes Q(\mathcal{D}) \cong Q(\mathcal{D})^r$ for some integer r which we refer to as the *rank* of M (and similarly for left $\mathcal{D}$-modules); note that this does not depend on the choice of U. Moreover, in this case there is some vector bundle W on X such that M embeds in the $\mathcal{D}$-module $\mathcal{D} \otimes_{\mathcal{O}_X} W$. We then have the following standard fact about the microlocalization of a finitely generated torsion-free $\mathcal{D}$-module.

Lemma 7.4. *If M is a torsion-free and finitely generated left* (*or right*) *$\mathcal{D}$-module then the natural map $M \to \mathcal{E} \otimes M$* (*respectively $M \to M_{\mathcal{E}} = M \otimes \mathcal{E}$*) *is injective.*

Recall that a *good filtration* of a $\mathcal{D}$-module M is a filtration $M = \bigcup F_i(M)$ that makes M a filtered $\mathcal{D}$-module and such that $\mathrm{gr}_F(M)$ is a finitely generated $\mathrm{gr}(\mathcal{D})$-module. We often write M_i in place of $F_i(M)$ when there can be no confusion.

Suppose that M is a $\mathcal{D}$-module with good filtration $F_{\bullet}$. Then $M_{\mathcal{E}} = M \otimes_{\mathcal{D}} \mathcal{E}$ has a canonical sequence of filtrations induced from $F_{\bullet}$: set $F_k(M_{\mathcal{E}}, \ell) = F_{\ell}(M) \cdot \mathcal{E}_{k-\ell}$; this is the kth term in the ℓth filtration. Note that for some choices of ℓ it may not be the case that $\bigcup_{k \geq 0} F_k(M_{\mathcal{E}}, \ell) = M_{\mathcal{E}}$.

We have the following easy lemma.

Lemma 7.5. *Suppose that M is a $\mathcal{D}$-module with good filtration $F_\bullet$.*

(1) *$F_\bullet(M_\mathcal{E}, \ell)$ makes $M_\mathcal{E}$ into a filtered $\mathcal{E}$-module for each ℓ.*
(2) *For each ℓ and k we have $F_k(M_\mathcal{E}, \ell - 1) \subseteq F_k(M_\mathcal{E}, \ell)$.*
(3) *There exists $\ell \gg 0$ such that for all $n \geq 0$ and all k, $F_k(M_\mathcal{E}, \ell + n) = F_k(M_\mathcal{E}, \ell)$. Moreover, for any such ℓ, $\bigcup_k F_k(M_\mathcal{E}, \ell) = M_\mathcal{E}$.*

Definition 7.6. We will refer to the filtration $F_\bullet(M_\mathcal{E}, \ell)$ for sufficiently large ℓ as in part (3) of Lemma 7.5 as the *induced filtration* or *canonical filtration* of $M_\mathcal{E}$, and will simply write $F_\bullet(M_\mathcal{E})$ or even $M_{\mathcal{E},\bullet}$ for its terms. We also call the induced filtration $M_k = M \cap M_{\mathcal{E},k}$ on M the *canonical filtration* on M.

We then have the following:

Corollary 7.7. *Suppose that M is a $\mathcal{D}$-module equipped with a good filtration $F_\bullet$ such that $\bigoplus_{k \geq n} \mathrm{gr}_k^F(M)$ is a torsion-free $\mathrm{gr}(\mathcal{D})$-module for some n. Then:*

(1) *$\mathrm{gr}_k(M_\mathcal{E}) = \mathrm{gr}_k^F(M)$ for all k sufficiently large.*
(2) *$F_\bullet(M_\mathcal{E})$ is a filtration of $M_\mathcal{E}$ with torsion-free associated graded module.*

Proof. For $k \geq \ell \gg 0$, we have $F_k(M_\mathcal{E}, \ell) = F_k(M_\mathcal{E}, k) = F_k(M) \cdot \mathcal{E}_0$. Also, for $k \geq \ell$, we have $F_{k-1}(M_\mathcal{E}, \ell) = F_{k-1}(M_\mathcal{E}, k) = F_k(M) \cdot \mathcal{E}_{-1}$. Since $F_k(M) \cdot \mathcal{E}_0 = F_k(M) + F_k(M) \cdot \mathcal{E}_{-1}$, the map $F_k(M)/F_{k-1}(M) \to F_k(M_\mathcal{E})/F_{k-1}(M_\mathcal{E})$ becomes

$$\frac{F_k(M)}{F_{k-1}(M)} \to \frac{F_k(M)}{F_k(M) \cdot \mathcal{E}_{-1} \cap M} \to \frac{F_k(M) + F_k(M) \cdot \mathcal{E}_{-1}}{F_k(M) \cdot \mathcal{E}_{-1}},$$

which is evidently surjective. To prove that it is injective, it suffices, by our assumption, to prove that its restriction to some nonempty open set $U \subset X$ is injective.

By choosing U sufficiently small and n sufficiently large, we may assume that $\mathrm{gr}_n(M) = W$ is a vector bundle, that the symbol map $F_n(M) \to \mathrm{gr}_n(M) = W$ is a split surjection, and, choosing a splitting $W \to F_n(M)$ as $\mathcal{O}_X$-modules, that the natural map $W \otimes \mathcal{D} \to F_n(M)$ is an isomorphism on associated graded modules in degrees greater than or equal to n. It follows by a standard inductive argument that the induced map of filtered $\mathcal{E}$-modules $W \otimes \mathcal{E} \to M_\mathcal{E}$ is an isomorphism. This proves (1). Part (2) is then immediate. □

Let M be a finitely generated, torsion-free right $\mathcal{D}$-module equipped with a good filtration such that $\mathrm{gr}(M)$ is a torsion-free $\mathrm{gr}(\mathcal{D})$-module in high degree—we will call M a $\mathcal{D}$-*module with torsion-free filtration* for simplicity.

Proposition 7.8. *Suppose M is a $\mathcal{D}$-module with torsion-free filtration on X. Then the canonical filtration on $M_\mathcal{E}$ induced by the given filtration on M is the unique $\mathcal{E}$-module filtration on $M_\mathcal{E}$ that satisfies:*

(a) *The induced map $M \to M_\mathcal{E}$ is a homomorphism of filtered $\mathcal{D}$-modules.*
(b) *The induced map $\mathrm{gr}_k(M) \to \mathrm{gr}_k(M_\mathcal{E})$ is an isomorphism of $\mathcal{O}_X$-modules for $k \gg 0$.*

Furthermore, this filtration has the properties:

(i) $M_k = M \cap M_{\mathcal{E},k}$ for all $k \gg 0$.

(ii) $M_{\mathcal{E},k} = \bigcup_{n \geq 0} M_{k+n} \cdot \mathcal{E}_{-n}$.

(iii) $\mathrm{gr}(M_{\mathcal{E}}) \cong V \otimes \mathrm{gr}(\mathcal{E})$ as graded $\mathrm{gr}(\mathcal{E})$-modules.

Proof. The canonical filtration satisfies (a) and (b) by construction and Corollary 7.7. If $\Theta_\bullet$ is any other filtration of $M_{\mathcal{E}}$ satisfying (a) and (b), then $\Theta_k(M_{\mathcal{E}}) = \Theta_k(M_{\mathcal{E}})\mathcal{E}_0 \subseteq F_k(M)\mathcal{E}_0 = F_k(M_{\mathcal{E}})$ for k sufficiently large. Since $\Theta_{k-n}(M_{\mathcal{E}}) = \Theta_k(M_{\mathcal{E}})\mathcal{E}_{n-}$ and $F_{k-n}(M_{\mathcal{E}}) = F_k(M_{\mathcal{E}})\mathcal{E}_{-n}$ for all $n \geq 0$, we find that $F_k(M_{\mathcal{E}}) \subseteq \Theta_k(M_{\mathcal{E}})$ for all k.

The induced map $\mathrm{gr}_k^F(M_{\mathcal{E}}) \to \mathrm{gr}_k^\Theta(M_{\mathcal{E}})$ is an isomorphism for k large by Corollary 7.7 and assumption (b), hence for all k since it is the restriction of a homomorphism of torsion-free graded $\mathrm{gr}(\mathcal{E})$-modules. It is immediate that $F_{k-1}(M_{\mathcal{E}}) = F_k(M_{\mathcal{E}}) \cap \Theta_{k-1}(M_{\mathcal{E}})$ for all k. Suppose, by way of inductive hypothesis, that $F_{k+n}(M_{\mathcal{E}}) \cap \Theta_{k-1}(M_{\mathcal{E}}) = F_{k-1}(M_{\mathcal{E}})$. Then

$$F_{k+n+1} \cap \Theta_{k-1} = (F_{k+n+1} \cap \Theta_{k+n}) \cap \Theta_{k-1} = F_{k+n} \cap \Theta_{k-1} = F_{k-1}.$$

It follows that $F_{k+n} \cap \Theta_k = F_k$ for all k. Since $M_{\mathcal{E}} = \bigcup_n F_{k+n}(M_{\mathcal{E}})$, we get $\Theta_k(M_{\mathcal{E}}) = F_k(M_{\mathcal{E}})$, proving the uniqueness statement.

Properties (ii) and (iii) are clear from the constructions. To prove (i), let $F_k'(M) = M \cap F_k(M_{\mathcal{E}})$. Then $F_k \subseteq F_k'$ for all k. The map $\mathrm{gr}_k^{F'}(M) \to \mathrm{gr}_k(M_{\mathcal{E}})$ is injective by construction, hence by (b) is an isomorphism for k sufficiently large. It follows that the map $\mathrm{gr}_k^F(M) \to \mathrm{gr}_k^{F'}(M)$ is an isomorphism for k sufficiently large, say $k \geq k_0$. It follows that $F_k \cap F_{k-1}' = F_{k-1}$ for $k \geq k_0$. An inductive argument just like that for F and Θ above then proves that $F_k = F_k'$ for all $k \geq k_0$, as desired. $\square$

Any finitely generated torsion-free $\mathcal{D}$-module can be equipped with a torsion-free filtration locally on X, and consequently we have:

Lemma 7.9. *If M is a finitely generated torsion-free $\mathcal{D}$-module, then M has no nonzero $\mathcal{O}$-coherent $\mathcal{D}$-submodules.*

In fact, more is true: every finitely generated torsion-free $\mathcal{D}$-module can be equipped with a torsion-free filtration globally on X, and in rank 1 this filtration is essentially unique.

Proposition 7.10. *Suppose that M is a finitely generated, torsion-free right $\mathcal{D}$-module on X.*

(1) *Let $n = \mathrm{rk}(M)$. Then M admits a filtration such that $\mathrm{gr}(M)$ is a torsion-free $\mathrm{gr}(\mathcal{D})$-module and $\mathrm{rk}(\mathrm{gr}_k(M)) = n$ for $k \geq 0$ and 0 for $k < 0$.*

(2) *Suppose that, in addition, M has rank $n = 1$ and that F^1 and F^2 are two $\mathcal{D}$-module filtrations of M such that for some n_0 sufficiently large, $\bigoplus_{k \geq n_0} \mathrm{gr}_k(M, F^i)$ is a torsion-free $\mathrm{gr}(\mathcal{D})$-module for $i = 1, 2$. Then there exists $l \in \mathbb{Z}$ such that for all $k \geq \min\{n_0, n_0 + l\}$, $F_{k+l}^1 = F_k^2$.*

Proof. (1) Since M is torsion-free, there is a nonempty open subset $U \subseteq X$ such that $M|_U \cong \mathcal{D}_U^n$. Let $D = X \setminus U$. Since M is finitely generated, there exists ℓ such that the inclusion $M \hookrightarrow \mathcal{D}_U^n$ factors through $\mathcal{O}(\ell D) \otimes \mathcal{D}^n$; the map $M \hookrightarrow \mathcal{O}(\ell D) \otimes \mathcal{D}^n$ is also an isomorphism over U. The induced filtration on M then has the desired property.

(2) The question is local on X, so we may restrict to an affine open subset U of X. The claim then follows from Lemma 3.2 of [NS]. $\square$

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