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A Note on Hilbert C^*-Modules Associated with a Foliation

By

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Introduction

Recently, M. Hilsum and G. Skandalis proved the stability property of foliation C^*-algebras ([4]). They constructed Hilbert C^*-modules $E_{W_1}^{W_2}$ for two transversal submanifolds W_1, W_2 in a foliated manifold M, and then reduced the stability of foliation C^*-algebras to that of Hilbert C^*-modules ([5] Th. 2). In the course of this reduction, they proved the relation, $\mathcal{K}(E_T^W) \cong C_r^*(G_W^W)$, with T a faithful transversal submanifold ($\mathcal{K}(E_T^W)$ denotes the C^*-algebra of 'compact' operators in E_T^W, [5], Def. 4). In this note, along the lines of their proof, we show that this relation is generalized to $\mathcal{K}(E_T^{W_1}, E_T^{W_2}) \cong E_{W_1}^{W_2}$.

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Notation. For a vector bundle E over a Manifold X, we denote the set of continuous sections of E over X with compact support by $C_c(X, E)$.

§ 1. Preliminaries (cf. [2], [3], [6])

Here we gather some elementary facts of foliation C^*-algebras. All of them are, more or less, direct consequences of definitions and their proofs are omitted.

Let $(M, \mathcal{F})$ be a $C^{\infty,0}$(C^∞ along leaves and C^0 along transversal

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direction) foliation and suppose that its holonomy groupoid G is Hausdorff. A submanifold W in M is said to be transversal to $\mathcal{F}$ (denoted by $W \pitchfork \mathcal{F}$) if for each point $x \in W$, there is a foliated neighborhood of x, $\Omega \cong \boldsymbol{R}^q \times \boldsymbol{R}^p$ ($\boldsymbol{R}^q$ and $\boldsymbol{R}^p$ are transversal and tangential coordinates respectively), such that $W \cap \Omega \cong \{(t, u) \in \boldsymbol{R}^q \times \boldsymbol{R}^p\ ;\ t^{k+1} = t^{k+2} = \cdots = t^q = 0\}$ ($k = \operatorname{codim} W$). We denote the set of such W's by $\mathcal{T}$. Note that every open subset of M is always transversal to $\mathcal{F}$. For $T_1, T_2 \in \mathcal{T}$, we set $G_{T_1}^{T_2} = \{\gamma \in G\ ;\ r(\gamma) \in T_2 \text{ and } s(\gamma) \in T_1\}$ which is, if not empty, a $C^{\infty,0}$ submanifold of G with the dimension equal to $\dim T_1 + \dim T_2 - \operatorname{codim} \mathcal{F}$.

Let $\mathcal{G}$ be the $C^{\infty,0}$ foliation in G induced from $\mathcal{F}$ ([2], p. 112). Recall that for $\gamma \in G$, the leaf through γ is given by $\{\gamma' \in G\ ;\ r(\gamma')$ and $r(\gamma)$ are in the same leaf of $\mathcal{F}\}$.

Lemma 1.1. *Let $G_{T_1}^{T_2}$ and $\mathcal{G}$ as above. We have*

$$G_{T_1}^{T_2} \pitchfork \mathcal{G}.$$

By this lemma, $\mathcal{G}$ defines a foliation $\mathcal{G}_{T_1}^{T_2}$ in $G_{T_1}^{T_2}$. A leaf of $\mathcal{G}_{T_1}^{T_2}$ is a connected component of $G_{T_1}^{T_2} \cap \mathcal{L}$ for some leaf $\mathcal{L}$ of $\mathcal{G}$. Set $\mathcal{E}_{T_1}^{T_2} = C_c(G_{T_1}^{T_2}, \Delta^{\frac{1}{2}}(T\mathcal{G}_{T_1}^{T_2}))$, where $\Delta^{\frac{1}{2}}(T\mathcal{G}_{T_1}^{T_2})$ is the half-density bundle of $T\mathcal{G}_{T_1}^{T_2}$, the tangent bundle of $\mathcal{G}_{T_1}^{T_2}$ ([1], Def. 3.1). If $G_{T_1}^{T_2} = \phi$, $\mathcal{E}_{T_1}^{T_2} = 0$, by definition. Note that, for an n-dimensional real vector bundle E over a manifold X, the α-density bundle $\Delta^\alpha(E)$ of E ($\alpha \in \boldsymbol{R}$) is a complex line bundle over X and an element ϕ in a fibre $\Delta^\alpha(E)_x$, $x \in X$, is a function which associates a complex number $\phi(eg) = |\det(g)|^\alpha \phi(e)$ to each frame $eg = (\sum_j g_{j1} e_j, \ldots, \sum_j g_{jn} e_j)$ where e is some fixed frame $e = (e_1, \ldots, e_n)$ at E_x and $g = (g_{jk}) \in GL(n, \boldsymbol{R})$.

If E is the tangent bundle TX, then every (Lebesgue) measurable section μ of 1-density bundle gives rise to a measure on X, which is denoted as $\int \mu(\delta x)$, $x \in X$. Recall that, given a local coordinate $(x^1, \ldots, x^n)$, the measure $\int \mu(\delta x)$ is expressed as

$$\mu\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right) |dx^1 \wedge \cdots \wedge dx^n|.$$

Remark 1.2. In general, $G_{T_1}^{T_2}$ is not required to support the whole of T_1 and T_2. However, if $G_{T_1}^{T_2} \neq \phi$, then we have $G_{T_1}^{T_2} = G_{T_1'}^{T_2'}$

where $T_1'=s(G_{T_1}^{T_2})$ and $T_2'=r(G_{T_1}^{T_2})$. We write $T_1 \succ T_2$ when $T_2=T_2'$. This relation satisfies the transitive law.

Lemma 1.3. *For $T\in\mathscr{T}$, let B_T (resp. B^T) be a vector bundle over $G_T=G_T^M$ (resp. $G^T=G_M^T$) defined by $B_T=\bigcup_{\gamma\in G_T} T_\gamma G_T^{r(\gamma)}$ (resp. $B^T=\bigcup_{\gamma\in G^T} T_\gamma G_{s(\gamma)}^T$). We provide B_T (resp. B^T) with $C^{\infty,0}$-bundle structure in a canonical manner. Then, for $T_1, T_2\in\mathscr{T}$,*

$$\Delta^{\frac{1}{2}}(T\mathscr{G}_{T_1}^{T_2})=\Delta^{\frac{1}{2}}(B^{T_2})\otimes\Delta^{\frac{1}{2}}(B_{T_1})$$

(cf. [3], p. 40).

By this lemma, we can regard an element ϕ in $\mathscr{E}_{T_1}^{T_2}$ as a map which associates a complex number $\phi(\delta^{T_2}\gamma, \delta_{T_1}\gamma)$ to each $\gamma\in G_{T_2}^{T_1}$ and a pair of frames $(\delta^{T_2}\gamma, \delta_{T_1}\gamma)$, where $\delta^{T_2}\gamma$ (resp. $\delta_{T_1}\gamma$) is a frame at $T_\gamma G_{s(\gamma)}^{T_2}$ (resp. at $T_\gamma G_{T_1}^{r(\gamma)}$).

Definition 1.4. Let $T_1, T_2, T_3\in\mathscr{T}$. For $\phi_1\in\mathscr{E}_{T_1}^{T_2}$ and $\phi_2\in\mathscr{E}_{T_2}^{T_3}$, we define $\phi_2*\phi_1\in\mathscr{E}_{T_1}^{T_3}$ by

$$(1.1)\qquad (\phi_2*\phi_1)(\delta^{T_3}\gamma, \delta_{T_1}\gamma) =\int_{\gamma'\in G_{T_2}^{r(\gamma)}}\phi_2((\delta^{T_3}\gamma)\gamma^{-1}\gamma', \delta_{T_2}\gamma')\phi_1((\delta_{T_2}\gamma')^{-1}\gamma, \gamma'^{-1}\delta_{T_1}\gamma)$$

and $\phi_1^*\in\mathscr{E}_{T_2}^{T_1}$ by

$$(1.2)\qquad \phi_1^*(\delta^{T_1}\gamma, \delta_{T_2}\gamma)=\overline{\phi((\delta_{T_2}\gamma)^{-1}, (\delta^{T_1}\gamma)^{-1})}.$$

Here the notation in the right-hand side of (1.1) is as follows: If $G_{T_2}^{T_3}\cdot G_{T_1}^{T_2}=\{\gamma_2\cdot\gamma_1\,;\,\gamma_2\in G_{T_2}^{T_3}, \gamma_1\in G_{T_1}^{T_2}$ and (γ_2, γ_1) is composable$\}$ is empty, we define $\phi_2*\phi_1$ to be zero. To explain the opposite case, let $\delta^{T_3}\gamma$ be a frame at $T_\gamma G_{s(\gamma)}^{T_3}$. Then the right translation $(\delta^{T_3}\gamma)\cdot\gamma^{-1}\gamma'$ of $\delta^{T_3}\gamma$ by $\gamma^{-1}\gamma'$ is a frame at $T_{\gamma'}G_{s(\gamma')}^{T_3}$, and hence we can evaluate ϕ_2 at $((\delta^{T_3}\gamma)\cdot\gamma^{-1}\gamma', \delta_{T_2}\gamma')$ for a frame $\delta_{T_2}\gamma'$ at $T_{\gamma'}G_{T_2}^{r(\gamma')}$. Next, the map $\gamma'\longmapsto\gamma'^{-1}$ defines a diffeomorphism of $G_{T_2}^{r(\gamma')}$ into $G_{r(\gamma')}^{T_2}$ and the induced map between tangent bundles transforms $\delta_{T_2}\gamma'$ into a frame $(\delta_{T_2}\gamma')^{-1}$ at $G_{r(\gamma')}^{T_2}$. Then the right translation $(\delta_{T_2}\gamma')^{-1}\cdot\gamma$ of $(\delta_{T_2}\gamma')^{-1}$ by γ is a frame at $T_{\gamma'^{-1}\gamma}G_{s(\gamma'^{-1}\gamma)}^{T_2}$ and we can evaluate ϕ_1 at $((\delta_{T_2}\gamma')^{-1}\cdot\gamma,$

$\gamma'^{-1}\cdot\delta_{T_1}\gamma)$ if $\delta_{T_1}\gamma$ is a frame at $T_\gamma G_{T_1}^{r(\gamma)}$ (because the left translation $\gamma'^{-1}\cdot\delta_{T_1}\gamma$ of $\delta_{T_1}\gamma$ by γ'^{-1} is a frame at $T_{\gamma'^{-1}\gamma}G_{T_1}^{r(\gamma'^{-1}\gamma)}$). Now, for fixed $\delta^{T_3}\gamma$ and $\delta_{T_1}\gamma$, the map, $\gamma'\longmapsto\phi_2((\delta^{T_3}\gamma)\cdot\gamma^{-1}\gamma',\delta_{T_2}\gamma')\phi_1((\delta_{T_2}\gamma')^{-1}\cdot\gamma,\gamma'^{-1}\cdot\delta_{T_1}\gamma)$ is an element in $C_c(G_{T_2}^{r(\gamma)}, \Delta^1(TG_{T_2}^{r(\gamma)}))$, and therefore we can integrate it over $G_{T_2}^{r(\gamma)}$, obtaining a complex number (=the right-hand side of (1.1)).

The meaning of the right-hand side of (1.2) is as explained above (bar denotes the complex conjugation).

(1.1) is an intrinsic form of convolution algebra (without any reference to a specific measure). Now we rewrite (1.1) into a more familiar form of convolution algebra. Let $\mathscr{F}_T$ $(T\in\mathscr{T})$ be the foliation in T induced from $\mathscr{F}$ (as before, a leaf of $\mathscr{F}_T$ is a connected component of $T\cap\mathscr{L}$ for some leaf $\mathscr{L}$ of $\mathscr{F}$). For a nowhere vanishing positive $C^{\infty,0}$ section D_2 (resp. D_1) of Δ^1 $(T\mathscr{F}_{T_2})$ (resp. $\Delta^1(T\mathscr{F}_{T_1})$), we define a $C^{\infty,0}$ section $\nu_{T_1}^{D_2}$ (resp. $\nu_{D_1}^{T_2}$) of $\Delta^1(B^{T_2})$ (resp. $\Delta^1(B_{T_1})$) as the pull back of D_2 (resp. D_1) by $s|_{G_{T1}^{T2}}$ (resp. $r|_{G_{T1}^{T2}}$). Then using a function $f_1\in C_c(G_{T_1}^{T_2})$, $\phi_1\in\mathscr{E}_{T_1}^{T_2}$ is represented as

$$\phi_1(\delta^{T_2}\gamma,\ \delta_{T_1}\gamma)=f_1(\gamma)\nu_{D_1}^{D_2}(\delta^{T_2}\gamma)^{\frac{1}{2}}\nu_{T_1}^{T_2}(\delta_{T_1}\gamma)^{\frac{1}{2}}. \tag{1.3}$$

Similarly, given $\nu_{T_2}^{D_3}$ and $\nu_{D_2}^{T_3}$, $\phi_2\in\mathscr{E}_{T_2}^{T_3}$ is represented by a function f_2 in $C_c(G_{T_2}^{T_3})$. In this situation, $\phi_2*\phi_1$ is represented by a function $f\in C_c(G_{T_1}^{T_3})$ (relative to $\nu_{T_1}^{D_3}$ and $\nu_{D_1}^{T_3}$), where f is given by

$$f(\gamma)=\int_{\gamma'\in G_{T_2}^{r(\gamma)}}\nu_{D_2}^{T_3}(\delta_{T_2}\gamma')f_2(\gamma')f_1(\gamma'^{-1}\gamma). \tag{1.4}$$

This is the usual form of convolution algebra.

Through the above identification of $\mathscr{E}_{T_1}^{T_2}$ with $C_c(G_{T_1}^{T_2})$, we can talk about the inductive limit topology of uniform convergence on compact sets for $\mathscr{E}_{T_1}^{T_2}$ (i.e., $\phi_n\longrightarrow\phi$ in $\mathscr{E}_{T_1}^{T_2}$ if supp ϕ and $\bigcup_n$ supp ϕ_n are contained in some compact set K of $G_{T_1}^{T_2}$ and ϕ_n converges to ϕ uniformly). For example, the operations defined by (1.1) and (1.2) are continuous with respect to the inductive limit topology of uniform convergence on compact sets.

Lemma 1.5. *The operation defined by* (1.1) *is associative and satisfies* $(\phi_2 * \phi_1)^* = \phi_1^* * \phi_2^*$.

Definition 1.6. Let $T_1, T_2 \in \mathscr{T}$. $\mathscr{E}_{T_1}^{T_2}$ is a right $\mathscr{E}_{T_1}^{T_1}$-module by the convolution. Furthermore, following [4], we provide $\mathscr{E}_{T_1}^{T_2}$ with a structure of pre-Hilbert $\mathscr{E}_{T_1}^{T_1}$-module by the inner product

$$\langle \phi, \psi \rangle = \phi^* * \psi \in \mathscr{E}_{T_1}^{T_1} \quad \text{for } \phi, \psi \in \mathscr{E}_{T_1}^{T_2}. \tag{1.5}$$

Since the reduced groupoid C^*-algebra $C_r^*(G_{T_1}^{T_1})$ of $G_{T_1}^{T_1}$ is a completion of $\mathscr{E}_{T_1}^{T_1}$ with respect to a C^*-norm $\| \ \|_{C^*}$, we can complete $\mathscr{E}_{T_1}^{T_2}$ with respect to the norm $\phi = \| \langle \phi, \phi \rangle \|_{C^*}^{\frac{1}{2}}$, $\phi \in \mathscr{E}_{T_1}^{T_2}$ to obtain a Hilbert $C_r^*(G_{T_1}^{T_1})$-module which we call $E_{T_1}^{T_2}$.

Definition 1.7. For $T_1, T_2 \in \mathscr{T}$, and a measure dx on T_1 in the Lebesgue measure class, set $\mathscr{H}^{T_2}(T_1, dx) = C_c(G_{T_1}^{T_2}, \Delta^{\frac{1}{2}}(B^{T_2}))$ and define a positive definite inner product in $\mathscr{H}^{T_2}(T_1, dx)$ by

$$(\xi, \eta) = \int_{T_1} dx \int_{\gamma \in G_x^{T_2}} \overline{\xi(\delta^{T_2}\gamma)}\, \eta(\delta\gamma). \tag{1.6}$$

For the meaning of $\int_{\gamma \in G_x^{T_2}} \xi(\delta^{T_2}\gamma)\eta(\delta\gamma)$, see the explanation above Ramark 1.2. We denote the completion of $\mathscr{H}^{T_2}(T_1, dx)$ (relative to the above inner product) by $H^{T_2}(T_1, dx)$. $H^{T_2}(T_1, dx)$ is a Hilbert space.

Lemma 1.8. *For* $\phi \in \mathscr{E}_{T_1}^{T_2}$ *and* $\xi \in \mathscr{H}^{T_1}(T, dx)$, *let* $\phi * \xi$ *be an element in* $\mathscr{H}^{T_2}(T, dx)$ *defined by*

$$(1.7) \qquad (\phi * \xi)(\delta^{T_2}\gamma) = \int_{\gamma' \in G_{T_1}^{r(\gamma)}} \phi((\delta^{T_2}\gamma) \cdot \gamma^{-1}\gamma', \delta_{T_1}\gamma')\, \xi((\delta_{T_1}\gamma')^{-1} \cdot \gamma)$$

for $\gamma \in G_{T_1}^{T_2}$.

Then the map $\xi \longmapsto \phi * \xi$ *gives rise to a bounded linear operator* $R_T(\phi)$ *of* $H^{T_1}(T, dx)$ *into* $H^{T_2}(T, dx)$. *Furthermore, the bilinear map defined by* $\mathscr{E}_{T_1}^{T_2} \times H^{T_1}(T, dx) \ni (\phi, \xi) \longmapsto R_T(\phi)\xi \in H^{T_2}(T, dx)$ *is jointly continuous if one equips* $\mathscr{E}_{T_1}^{T_2}$ *with the inductive limit topology of uniform convergence on compact sets and* $H^{T_j}(T, dx)$ $(j=1, 2)$ *with the* *norm topology.*

Lemma 1.9. *Let* $\phi_1 \in \mathscr{E}_{T_1}^{T_2}$, $\phi_2 \in \mathscr{E}_{T_2}^{T_3}$, $\xi_1 \in \mathscr{H}^{T_1}(T, dx)$, *and* $\xi_2 \in \mathscr{H}^{T_2}(T, dx)$. *Then we have*

(i) $(\phi_2 * \phi_1) * \xi_1 = \phi_2 * (\phi_1 * \xi_1)$,

(ii) $(\phi_1 * \xi_1, \xi_2) = (\xi_1, \phi_1^* * \xi_2)$.

Remark 1.10. In the same way as in (1.6)~(1.7), we construct a Hilbert space $H^T(x)$ from $C_c(G_x^T, \Delta^{\frac{1}{2}}(B^T))$ ($T \in \mathscr{T}$ and $x \in M$) and a bounded linear operator $R_x(\phi)$ of $H^{T_1}(x)$ into $H^{T_2}(x)$ for $\phi \in \mathscr{E}_{T_1}^{T_2}$. Furthermore, corresponding to Lemma 1.9, we have

(i) $R_x(\phi_2 * \phi_1) = R_x(\phi_2) R_x(\phi_1)$,

(ii) $R_x(\phi)^* = R_x(\phi^*)$.

Lemma 1.11. *Let* $T, T_1, T_2 \in \mathscr{T}$. *Given a Lebesgue measure* dx *on* T, *there are decompositions of Hilbert spaces*

$$H^{T_j}(T, dx) \cong \int_T^{\oplus} H^{T_j}(x)\, dx \quad (j=1, 2), \tag{1.8}$$

under which $R_T(\phi)$ ($\phi \in \mathscr{E}_{T_1}^{T_2}$) *is decomposed as*

$$R_T(\phi) \cong \int_T^{\oplus} R_x(\phi)\, dx. \tag{1.9}$$

Lemma 1.12. *Let* $\gamma \in G$ *with* $x = s(\gamma)$, $y = r(\gamma)$. *Then for any* $T \in \mathscr{T}$, *the right translation by* γ *gives rise to a unitary mapping* $U(\gamma)$ *from* $H^T(x)$ *onto* $H^T(y)$. *Furthermore, for* $\phi \in \mathscr{E}_{T_1}^{T_2}$, *the following diagram commutes.*

$$\begin{array}{ccc} H^{T_1}(x) & \xrightarrow{R_x(\phi)} & H^{T_2}(x) \\ \downarrow U(\gamma) & & \downarrow U(\gamma) \\ H^{T_1}(y) & \xrightarrow[R_y(\phi)]{} & H^{T_2}(y) \end{array} \quad . \tag{1.10}$$

§2. Regular Representation of Hilbert C^*-Modules E_T^W

In this section, we prove the relation $\mathscr{K}(E_T^{T_1}, E_T^{T_2}) \cong E_{T_1}^{T_2}$ (see 2.5), using the regular representation R_T cf $E_{T_1}^{T_2}$ (cf. [4]).

Lemma 2.1. *Let* $\phi \in \mathscr{E}_{T_1}^{T_2}$. Then the norm of ϕ in $E_{T_1}^{T_2}$ is given by

$$\sup_{x\in T_1} \| R_x(\phi) \|.$$

Here $\| R_x(\phi) \|$ is the operator norm of $R_x(\phi)$.

Proof. Since the reduced C^*-norm $\| \ \|_{C^*}$ is given by $\|\phi\|_{C^*} = \sup_{x\in T_1} \|R_x(\phi)\|$ ([2], [3], [6]), this is an immediate consequence of definition of the norm in $E_{T_1}^{T_2}$, $\|\phi\| = \|\langle\phi, \phi\rangle\|_{C^*}^{\frac{1}{2}}$.

Corollary 2.2. For $\phi \in \mathscr{E}_{T_1}^{T_2}$ and $T \in \mathscr{T}$, we have

$$\|R_T(\phi)\| \leqq \|\phi\|.$$

Proof. This is a consequence of Lemma 1.11, 1.12, and 2.1.

In view of Lemma 2.1 (resp. Corollary 2.2) we can extend R_x (resp. R_T) to $E_{T_1}^{T_2}$ by continuity.

Lemma 2.3. *Let* $T, T_1, T_2 \in \mathscr{T}$. If $T \succ T_1$ *(see Remark* 1.2), *then we have* $\|R_T(\phi)\| = \|\phi\|$ *for every* $\phi \in E_{T_1}^{T_2}$.

Proof. Let $\phi \in \mathscr{E}_{T_1}^{T_2}$. We claim that the function $x \longmapsto \|R_x(\phi)\|$ on T is lower semi-continuous. To see this, let ξ be an element in $C_c(G_T^{T_1}, \Delta^{\frac{1}{2}}(B^{T_1}))$ and denote by ξ_x the restriction of ξ to $G_x^{T_1}$. Then both of the functions on T, $x \longmapsto \|R_x(\phi)\xi_x\|$ and $x \longmapsto \|\xi_x\|$ are continuous and therefore

$$f_\xi(x) = \begin{cases} \|R_x(\phi)\xi_x\| / \|\xi_x\| & \text{if } \|\xi_x\| \neq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

is a lower semi-continuous function cf $x \in T$. Since for each $x \in T$, $\{\xi_x ; \xi \in C_c(G_T^{T_1}, \Delta^{\frac{1}{2}}(B^{T_1}))\}$ is dense in $H_x^{T_1}$, we have

$$\|R_x(\phi)\| = \sup_\xi \{f_\xi(x)\}.$$

Hence $x \longmapsto \|R_x(\phi)\|$ is lower semi-continuous as a supremum of lower semi-continuous functions.

Now we claim that $\|R_T(\phi)\| = \sup_{x\in T} \|R_x(\phi)\|$. Since $\|R_T(\phi)\| \leqq$

$\sup_{x\in T}\|R_x(\phi)\|$ (see 1.11), we need to prove the opposite inequality. By Lemma 1.11, we have

$$\|R_T(\phi)\|=\mu\text{-ess. sup }\{\|R_x(\phi)\|\ ;\ x\in T\}. \tag{2.2}$$

Take any $x_0\in T$. Since $\|R_x(\phi)\|$ is a lower semi-continuous function of $x\in T$, for any $\varepsilon>0$, we can find an open neighborhood U of x_0 such that $\inf\{\|R_x(\phi)\|\ ;\ x\in U\}\geqq\|R_{x_0}(\phi)\|-\varepsilon$. Then μ-ess. sup $\{\|R_x(\phi)\|\ ;\ x\in U\}\geqq\|R_{x_0}(\phi)\|-\varepsilon$, because $\mu(U)>0$. Thus we have $\|R_T(\phi)\|=\sup\{\|R_x(\phi)\|\ ;\ x\in T\}$ and the assertion of Lemma follows from Lemma 2.1 and Lemma 1.12.

Lemma 2.4. *Let $T, T_1, T_2\in\mathscr{T}$ and suppose that $T_1\prec T$, $T_2\prec T$. Then $\{\phi_1*\phi_2\ ;\ \phi_1\in\mathscr{E}_T^{T_1}, \phi_2\in\mathscr{E}_{T_2}^{T}\}$ is total in $\mathscr{E}_{T_2}^{T_1}$ with respect to the inductive limit topology of uniform convergence on compact sets.*

Proof. Take a nowhere vanishing positive $C^{\infty,0}$ density D (resp. D_1, D_2) along leaves in T (resp. T_1, T_2) and represent elements of $\mathscr{E}$'s by functions as explained after Definition 1.4. For $f_1\in C_c(G_T^{T_1})$ and $f_2\in C_c(G_{T_2}^{T})$,

$$\gamma\longmapsto\int_{\gamma'\in G_T^{r(\gamma)}}\nu_D^{T_1}(\delta_T\gamma')f_1(\gamma')\ f_2(\gamma'^{-1}\gamma) \tag{2.3}$$

is an element in $C_c(G_{T_2}^{T_1})$, and the question is whether there are sufficiently many functions of this form. By partition of unity in $G_{T_2}^{T_1}$, it suffices to show that each function in $C_c(G_{T_2}^{T_1})$ with support contained in a foliated coordinate neighborhood is approximated by a linear combination of functions of the form of (2.3). Let $q=\text{codim }\mathscr{F}$ and set $k=q-\dim T$, $k_j=q-\dim T_j$ $(j=1, 2)$. Locally the convolution of (2.3) is given by

$$(t, u_1, u_2)\longmapsto\int_{\boldsymbol{R}^k}du\ f_1(t, u_1, u)\ f_2(t, u, u_2) \tag{2.4}$$

for $(t, u_1, u_2)\in\boldsymbol{R}^q\times\boldsymbol{R}^{k_1}\times\boldsymbol{R}^{k_2}$, where du is a C^∞-measure on $\boldsymbol{R}^k$. Since any $\gamma\in G_{T_2}^{T_1}$ is expressed as $\gamma=\gamma_1\gamma_2$ with $\gamma_1\in G_T^{T_1}$ and $\gamma_2\in G_{T_2}^{T}$ (here we have used the assumption), the vector space generated by functions of this form contains $C_c(\boldsymbol{R}^q)\otimes C_c(\boldsymbol{R}^{k_1})\otimes C_c(\boldsymbol{R}^{k_2})$ and therefore is dense in $C_c(\boldsymbol{R}^q\times\boldsymbol{R}^{k_1}\times\boldsymbol{R}^{k_2})$ by Stone-Weierstrass approximation theorem. This completes the proof of Lemma.

As in [5], given a C^*-algebra A and two Hilbert A-module E_1, E_2, we denote the set of 'compact' operators from E_1 into E_2 by $\mathscr{K}(E_1, E_2)$. Recall that if we set $\{\theta_{x_2,x_1}$; $x_1\in E_1$, $x_2\in E_2$, and θ_{x_2,x_1} is a bounded linear mapping from E_1 into E_2 defined by $\theta_{x_2,x_1}(y_1) = x_2\langle x_1, y_1\rangle$, $y_1\in E_1\}$, then $\mathscr{K}(E_1, E_2)$ is the closure of the linear hull of this set relative to the operator norm.

Theorem 2.5. *Let* T, T_1, $T_2\in\mathscr{T}$ *and suppose that* $T\succ T_1$ *and* $T\succ T_2$ *(see Remark* 1.2). *Then we have* $\mathscr{K}(E_T^{T_1}, E_T^{T_2}) = E_{T_1}^{T_2}$.

Proof. First we imbed $\mathscr{E}_{T_1}^{T_2}$ into $\mathscr{L}(E_T^{T_1}, E_T^{T_2})$, the space cf 'intertwining' operators ([5] Def. 3). Let $\phi\in\mathscr{E}_{T_1}^{T_2}$ and $\phi_1\in\mathscr{E}_T^{T_1}$. Then, by Lemma 2.1, Lemma 2.3, and Remark 1.10,

$$\begin{aligned}\|\phi*\phi_1\| &= \sup_{x\in T} \|R_x(\phi*\phi_1)\| \\ &= \sup_{x\in T} \|R_x(\phi) R_x(\phi_1)\| \\ &\leqq (\sup_{x\in T} \|R_x(\phi)\|)(\sup_{x\in T} \|R_x(\phi_1)\|) \\ &= \|\phi\| \, \|\phi_1\|.\end{aligned}$$

So $\phi_1 \longmapsto \phi*\phi_1$, $\phi_1\in\mathscr{E}_T^{T_1}$ gives rise to a bounded linear operator $j(\phi)$ cf $E_T^{T_1}$ into $E_T^{T_2}$. Since $\langle j(\phi)\phi_1, \phi_2\rangle = \langle\phi_1, j(\phi^*)\phi_2\rangle$ ($\phi_1\in E_T^{T_1}$, $\phi_2\in E_T^{T_2}$), $j(\phi)$ is in $\mathscr{L}(E_T^{T_1}, E_T^{T_2})$. In particular when $T_1=T_2$, $\mathscr{L}(E_T^{T_1}, E_T^{T_2})$ is a C^*-algebra (cf. [5] Lemma 2) and j becomes a $*$-homomorphism cf C^*-algebras, $E_{T_1}^{T_1}=C_r^*(G_{T_1}^{T_1}) \longrightarrow \mathscr{L}(E_T^{T_1}, E_T^{T_1})$. Furthermore if $j(\phi)=0$ for some $\phi\in E_{T_1}^{T_1}$, then, for each $\phi_1\in\mathscr{E}_T^{T_1}$, $j(\phi)\phi_1=0$ and therefore $R_T(\phi)R_T(\phi_1)=R_T(j(\phi)\phi_1)=0$. Since $R_T(\mathscr{E}_T^{T_1})\mathscr{H}^T(T, dx)$ is total in $\mathscr{H}^{T_1}(T, dx)$ (essentially due to the same argument as in the proof of Lemma 2.4), we conclude that $R_T(\phi)=0$. By Lemma 2.3 this implies that $\phi=0$. In other words, j is an isomorphism between C^*-algebras, and so we have

$$\|j(\phi)\| = \|\phi\| \quad \text{for all} \quad \phi\in E_{T_1}^{T_1}. \tag{2.5}$$

Now returning to the original case, if $\phi\in\mathscr{E}_{T_1}^{T_2}$, then

$$\|j(\phi)\|^2 = \|j(\phi)^* j(\phi)\| \quad \text{(cf. [7] Prop. 2.5)}$$

$$
\begin{aligned}
&= \|j(\phi^* * \phi)\| \\
&= \|\phi^* * \phi\| \\
&= \|R_T(\phi^* * \phi)\| \quad \text{(by Lemma 2.3)} \\
&= \|R_T(\phi)^* R_T(\phi)\| \quad \text{(by Remark 1.10)} \\
&= \|R_T(\phi)\|^2 \\
&= \|\phi\|^2 \quad \text{(by Lemma 2.3)}.
\end{aligned}
$$

Thus j defines an isometric imbedding of $E_{T_1}^{T_2}$ into $\mathscr{L}(E_T^{T_1}, E_T^{T_2})$.

Finally we claim that $j(E_{T_1}^{T_2}) = \mathscr{K}(E_T^{T_1}, E_T^{T_2})$. For $\phi_1 \in E_T^{T_1}$ and $\phi_2 \in E_T^{T_2}$, define an operator θ_{ϕ_2, ϕ_1} in $\mathscr{L}(E_T^{T_1}, E_T^{T_2})$ by

$$\theta_{\phi_2, \phi_1}\psi_1 = \phi_2 * \langle \phi_1, \psi_1 \rangle = \phi_2 * \phi_1^* * \psi_1 \tag{2.6}$$

for $\psi_1 \in E_T^{T_1}$. Since $\{\theta_{\phi_2, \phi_1} ; \phi_1 \in E_T^{T_1}, \phi_2 \in E_T^{T_2}\}$ is total in $\mathscr{K}(E_T^{T_1}, E_T^{T_2})$ by definition, the above claim follows from Lemma 2.4. This completes the proof of Theorem.

Remark 2.6. If one takes $T_1 = T_2 = W$, then Theorem 2.5 reduces to the relation $\mathscr{K}(E_T^W, E_T^W) \cong C_r^*(G_W^W)$ because $E_W^W = C_r^*(G_W^W)$.

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