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Scattering for 1D cubic NLS and singular vortex dynamics

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Abstract. We study the stability of self-similar solutions of the binormal flow, which is a model for the dynamics of vortex filaments in fluids and super-fluids. These particular solutions $\chi_a(t, x)$ form a family of evolving regular curves in $\mathbb{R}^3$ that develop a singularity in finite time, indexed by a parameter $a > 0$. We consider curves that are small regular perturbations of $\chi_a(t_0, x)$ for a fixed time t_0. In particular, their curvature is not vanishing at infinity, so we are not in the context of known results of local existence for the binormal flow. Nevertheless, we construct solutions of the binormal flow with these initial data. Moreover, these solutions become also singular in finite time. Our approach uses the Hasimoto transform, which leads us to study the long-time behavior of a 1D cubic NLS equation with time-depending coefficients and small regular perturbations of the constant solution as initial data. We prove asymptotic completeness for this equation in appropriate function spaces.

Keywords. Vortex filaments, binormal flow, selfsimilar solutions, Schrödinger equations, scattering

Contents

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1. Introduction

In this work we complete the stability properties obtained in our previous paper [3] of selfsimilar solutions of the binormal flow of curves

$$\chi_t = \chi_x \wedge \chi_{xx}. \tag{1}$$

Here $\chi = \chi(t,x) \in \mathbb{R}^3$, x denotes the arclength parameter and t the time variable. Using the Frenet frame, the above equation can be written as

$$\chi_t = cb,$$

where c is the curvature of the curve and b its binormal. This geometric flow was proposed by Da Rios in 1906 [7] as an approximation of the evolution of a vortex filament in a 3-D incompressible inviscid fluid. Simple explicit and relevant examples of solutions of (1) are the straight lines, that remain stationary, the circles, that move in the orthogonal direction of the plane where they are contained and with velocity the inverse of the radius, and the helices that, besides exhibiting the same rigid motion of the circles, rotate with a constant velocity around their axis as a corkskrew. We refer the reader to [1], [4] and [19] for an analysis and discussion about the limitations of this model and to [18] for a survey about Da Rios' work.

Selfsimilar solutions with respect to scaling of (1) are easily found by first fixing the ansatz

$$\chi(t,x) = \sqrt{t}\, G(x/\sqrt{t}), \tag{2}$$

and then solving the corresponding ordinary differential equation. In geometric terms the solutions are determined by a curve with the properties

$$c(x) = a, \qquad \tau(x) = x/2,$$

for a parameter $a > 0$. Denoting by G_a the corresponding curve and T_a its unit tangent, it is rather easy to see that $T_a(x)$ has a limit $A_a^{\pm}$ as x goes to $\pm\infty$, so that G_a approaches asymptotically two lines. In the neighborhood of $x = 0$ the curve is similar to a circle of radius $1/a$ and for large s it has a helical shape of increasing pitch. Notice that equation (1) is reversible in time. So if at time $t = 1$ the filament is given by $\chi_a(1,x) = G_a(x)$ the evolution $\chi_a(t,x)$ for $0 < t < 1$ is given by (2). From this expression we see that the two lines at infinity remain fixed. However, the helices transport the "energy" from infinity towards the origin so that the overall effect is an increase of the curvature, which becomes $a/\sqrt{t}$. The final configuration at time $t = 0$ is given by the two lines determined by $A_a^{\pm}$. That these two lines are different is not so straightforward. It was proved in [13] that

$$\sin(\theta/2) = e^{-a^2/2},$$

where θ is the angle between the vectors A_a^+ and $-A_a^-$. As a consequence starting with G_a, a real analytic curve at $t = 1$, a corner is created at time $t = 0$. This particular solution is studied numerically in [9]. One of the conclusions of that paper is that the process of concentration around the origin is very stable. Moreover the similarity between the numerical solutions and those that appear experimentally in a colored fluid traversing a delta wing is quite remarkable: see Figure 1.1 in [9].

The stability results proved in [3] are based on a transformation due to Hasimoto [14]. He defines the so-called "filament function" ψ of a regular solution of (1) that has strictly positive curvature at all points. The precise expression is

$$\psi(t,x) = c(t,x)\exp\left\{ i\int_0^x \tau(t,x')\,dx' \right\}.$$

Then it is proved in [14] that ψ solves the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + \frac{1}{2}(|\psi|^2 - A(t))\psi = 0, \tag{3}$$

with

$$A(t) = \left(\pm 2\frac{c_{xx} - c\,\tau^2}{c} + c^2 \right)(t,0).$$

Notice that in (3), the nonlinear term appears with the focusing sign. The opposite case, the defocusing one, can be obtained in a similar way by assuming that the tangent vector χ_s has a constant hyperbolic length instead of the constant euclidean length as in (1). The equation has to be changed accordingly; see [3] and [8] for the details.

The particular selfsimilar solution $\chi_a(t,x)$ of (1) has as curvature and torsion

$$c_a(t,x) = a/\sqrt{t}, \qquad \tau_a(t,x) = x/2t,$$

so its filament function is

$$\psi_a(t,x) = a e^{ix^2/4t}/\sqrt{t}.$$

This function is a solution of (3) if

$$A(t) = a^2/t.$$

Notice that neither $\psi_a(t)$ nor any of its derivatives are in L^2, and that $\psi_a(0) = a e^{i\pi/4}\delta_{x=0}$. This initial data is too singular for the available theory ([20], [10], [6], [2]). Therefore one might think that this particular solution is not related to any natural energy. However, this is not the case, as can be proved by considering the pseudo-conformal transformation. Given a solution ψ of[1]

$$i\psi_t + \psi_{xx} \pm (|\psi|^2 - a^2/t)\psi = 0, \tag{4}$$

we define a new unknown v as

$$\psi(t,x) = \mathcal{T}v(t,x) = \frac{e^{ix^2/4t}}{\sqrt{t}}\overline{v}(1/t, x/t). \tag{5}$$

Then v solves

$$iv_t + v_{xx} \pm \frac{1}{t}(|v|^2 - a^2)v = 0, \tag{6}$$

[1] For simplicity we omit the $1/2$ factor in (3), which can be removed by a scaling argument.

and $v_a = a$ is a particular solution corresponding to ψ_a. A natural quantity associated to (6) is the normalized energy

$$E(v)(t) = \frac{1}{2}\int |v_x(t)|^2\,dx \mp \frac{1}{4t}\int (|v(t)|^2 - a^2)^2\,dx.$$

An immediate calculation gives

$$\partial_t E(v)(t) \mp \frac{1}{4t^2}\int (|v|^2 - a^2)^2\,dx = 0,$$

and in particular $E(v_a) = 0$.

The first stability result we give in [3] is the proof of the existence for small a of a modified wave operator for solutions of (4) that at time $t = 1$ are close to the constant $v_a = a$. Namely, we prove that if we fix an asymptotic state u_+ small in $L^1 \cap L^2$, then there is a a unique solution of (4) for $t > 1$ that behaves as time approaches infinity as

$$v_1(t,x) = a + e^{\pm ia^2 \log t} e^{it\partial_x^2} u_+(x).$$

Here $e^{it\partial_x^2}$ denotes the free propagator. Therefore the free dynamics has to be modified by the long-range factor $e^{\pm ia^2 \log t}$, due to the nonintegrability of the coefficient $1/t$ that appears in (6). This is similar to the framework of long range wave operators for cubic 1-d NLS ([17], [5], [15]). Here the situation is different since the L^∞-norm of the functions we are working with is not decaying as t goes to infinity, being just bounded. A link could also be made with the asymptotic results for the Gross–Pitaevskii equation around the constant solution ([11], [12]), but still our situation is not the same, and we treat the linearized equation in a different way.

The condition $u_+ \in L^1$ will be relaxed in this article to the weaker one that $\hat{u}_+(\xi)$ times positive powers of $|\xi|$ is bounded in a neighborhood of the origin. As we shall see, this latter assumption is the one that naturally appears when proving the asymptotic completeness of (6). Moreover, we shall prove in Theorem A.1 of Appendix A the existence of the modified wave operator by assuming this weaker property.

Once the solution v is constructed we recover ψ from (5). The result proved in [3] is that given u_+ as before, there exists a unique solution $\psi(t,x)$ of (4) such that ψ behaves like ψ_1 as t goes to zero, with

$$\psi_1(t,x) = a\frac{e^{ix^2/4t}}{\sqrt{t}} + \frac{e^{\pm ia^2 \log t}}{\sqrt{4\pi i}}\hat{\overline{u}}_+(-x/2).$$

The precise statement about the behavior of $\psi - \psi_1$ can be found in Corollary 1.2 of [3]. However, it is important to point out two facts. Firstly, the rate of convergence is $\|\psi - \psi_1\|_{L^2} < Ct^{1/4}$. And secondly, although the singular term $ae^{ix^2/4t}/\sqrt{t}$ has a limit, the correction does not. As a consequence neither ψ_1 nor ψ have a trace at $t = 0$, no matter how good u_+ is. Notice also that the condition about the boundedness of $\hat{u}_+$ is understood here as that the perturbation of the singular solution ψ_a has to be bounded close to the point where the singularity is created.

The next result in [3] is the construction of solutions of (1) that are close to χ_a. This is done by integrating the Frenet system using the filament function given by ψ. The role played by the euclidean geometry is crucial at this step, because by construction the binormal vector has unit euclidean length. Therefore to conclude the existence of a trace for $\chi(t)$ at $t = 0$ it is enough that the curvature, given by $|\psi(t,x)|$, is integrable at time zero. Although this is obtained by quite general u_+, even though there is not a trace for ψ at $t = 0$ as we already said, the question of the existence of a corner is much more delicate. In order to get it, it is necessary to improve the rate of convergence of $\psi - \psi_1$. This is done by assuming that $|\xi|^{-2}\hat{u}_+(\xi)$ is locally in L^2 (see Theorem 1.5 in [3]).

Our main result in this paper is the asymptotic completeness for solutions of (6) that at time $t = 1$ are close to the constant a. In order to give the precise statement we have to make several transformations of (6). First of all we write

$$v = w + a, \tag{7}$$

so that w has to be a solution of

$$iw_t + w_{xx} = \mp\frac{1}{t}(|a+w|^2 - a^2)(a+w). \tag{8}$$

The right hand side of the above equation has two linear terms. One is $\mp\frac{a^2}{t}w$ that is resonant, and it is the one that creates the logarithmic correction of the phase. The other one is similar, but involves $\bar{w}$ and therefore it is not resonant. Then, we define u as

$$u(t,x) = w(t,x)e^{\mp ia^2\log t}. \tag{9}$$

As a consequence u has to solve

$$iu_t = \left(iw_t \pm \frac{a^2}{t}w\right)e^{\mp ia^2\log t} = \left(-w_{xx} \mp \frac{|w|^2w + a(w^2+2|w|^2)}{t} \mp \frac{a^2}{t}\overline{w}\right)e^{\mp ia^2\log t},$$

so

$$iu_t + u_{xx} \pm \frac{a^2}{t^{1\pm 2ia^2}}\overline{u} + \frac{F(u)}{t} = 0, \tag{10}$$

with $F(u)$ given by

$$F(u) = F(we^{\mp ia^2\log t}) = \pm(|w|^2w + a(w^2+2|w|^2))e^{\mp ia^2\log t}. \tag{11}$$

As we see, F involves just quadratic and cubic terms in u.

Also, we need to introduce some auxiliary function spaces. For fixed γ and t_0 we define the space $X^{\gamma}_{t_0}$ of functions $f(x)$ such that the norm

$$\|f\|_{X^{\gamma}_{t_0}} = \frac{1}{t_0^{1/4}}\|f\|_{L^2} + \frac{t_0^{\gamma}}{\sqrt{t_0}}\left\||\xi|^{2\gamma}\hat{f}(\xi)\right\|_{L^{\infty}(\xi^2\le 1)} \tag{12}$$

is finite, and $Y^{\gamma}_{t_0}$ the space of functions $g(t,x)$ such that the norm

$$\|g\|_{Y^{\gamma}_{t_0}} = \sup_{t\ge t_0}\left(\frac{1}{t_0^{1/4}}\|g(t)\|_{L^2} + \left(\frac{t_0}{t}\right)^{a^2}\frac{t_0^{\gamma}}{\sqrt{t_0}}\left\||\xi|^{2\gamma}\hat{g}(t,\xi)\right\|_{L^{\infty}(\xi^2\le 1)}\right) \tag{13}$$

is finite.

We have the following result.

Theorem 1.1. *Let $0 \leq \gamma < 1/4$, $0 < a$ and let $u(1)$ be a function in X_1^γ small with respect to a. Then there exists a unique global solution $u \in Z^\gamma = Y_1^\gamma \cap L^4((1,\infty), L^\infty)$ of equation (10) with initial data $u(1)$ at time $t = 1$, and*

$$\|u\|_{Z^\gamma} \leq C(a)\|u(1)\|_{X_1^\gamma}.$$

Moreover, this solution scatters in L^2: there exists $f_+ \in L^2$ for which

$$\|u(t) - e^{i(t-1)\partial_x^2} f_+\|_{L^2} \leq \frac{C(a,\delta)}{t^{1/4-(\gamma+\delta)}}\|u(1)\|_{X_1^\gamma} \xrightarrow[t\to\infty]{} 0,$$

for any $0 < \delta < 1/4 - \gamma$. Finally, the asymptotic state f_+ satisfies for all $\xi^2 \leq 1$ the estimate

$$|\xi|^{2(\gamma+\delta)}|\hat{f}_+(\xi)| \leq C(a,\delta)\|u(1)\|_{X_1^\gamma}.$$

To obtain the theorem, we first study the linearized equation

$$iu_t + u_{xx} \pm \frac{a^2}{t^{1\pm 2ia^2}}\overline{u} = 0, \tag{14}$$

with initial data $u(t_0, x)$ at time $t_0 \geq 1$. We prove that $u(t)$ behaves for large times like a free Schrödinger evolution. The only difference is that the Fourier zero-mode of $u(t)$ can become singular. Then, by perturbative methods, we deduce the asymptotic completeness for the nonlinear equation (10). The main part of our proof uses Fourier analysis and exploits particularly the nonresonant structure of $\overline{u}$ in (14). This is done by oscillatory integral techniques and simple integration by parts arguments (see in particular Lemma 2.5 below).

As we see, even if at time $t = 1$ we are assuming that $\hat{u}(1)$ remains bounded in a neighborhood of the origin, we cannot prove a similar property for the asymptotic state f_+. This is not just a technical question. In Appendix B2 we shall prove that if $xu(1)$ is in L^2, so that

$$\phi(t) = \int_{-\infty}^{\infty} u(t,x)\,dx$$

is well defined for all $t > 1$, then under some conditions on $u(1)$,

$$|\phi(t)| \geq C \log t.$$

This property is rather easy to obtain, at least at a formal level, for the linearized equation

$$iw_t + w_{xx} = \mp\frac{a^2}{t}(w + \bar{w}). \tag{15}$$

In fact, set $y(t) = \Re\int_{-\infty}^{\infty} w(t,x)\,dx$ and $z(t) = \Im\int_{-\infty}^{\infty} w(t,x)\,dx$; then

$$iy'(t) - z'(t) = \mp 2\frac{a^2}{t}y(t).$$

Hence $y(t) = y(1)$ and $z(t) = z(1) \pm 2a^2 y(1)\log t$.

Our next step is to understand the above result in terms of the filament function $\psi(t, x)$. From (5), (7), and (9) we have, for $0 < t \leq 1$,

$$\psi(t, x) = a\frac{e^{ix^2/4t}}{\sqrt{t}} + e^{\pm ia^2 \log t}\, \mathcal{T}u(t, x). \tag{16}$$

Therefore

$$\psi(1, x) = ae^{ix^2} + \psi_1(x)$$

with $\psi_1(x) = e^{ix^2} u(1, x)$. For simplicity we will impose $\psi_1 \in L^1 \cap L^2$ to fulfil the hypothesis $|\xi|^{2\gamma}\widehat{u}(1, \xi) \in L^\infty(\xi^2 \leq 1) \cap L^2$ needed in Theorem 1.1 with $\gamma = 0$. This will imply the existence of an $f_+ \in L^2$ such that $u(t)$ behaves like $e^{i(t-1)\partial_x^2} f_+$. Now, on the one hand, the pseudo-conformal transform of $e^{i(t-1)\partial_x^2} f_+$ is the free evolution of $\frac{1}{\sqrt{4\pi i}}\, \widehat{e^{i\partial_x^2}\, \overline{f_+}}\left(-\frac{\cdot}{2}\right)$. On the other hand, $\mathcal{T}$ is an isometry of L^2. As a consequence we obtain from Theorem 1.1 the following scattering result.

Theorem 1.2. *Let $0 < a$ and let ψ_1 be a small function in $L^1 \cap L^2$ with respect to a. Then there exists a unique solution ψ of equation (4) for $0 < t \leq 1$ with*

$$\psi(1, x) = ae^{ix^2/4} + \psi_1(x),$$

such that $\psi(t, x) - a\frac{e^{ix^2/4t}}{\sqrt{t}} \in L^\infty((0, 1), L^2) \cap L^4((0, 1), L^\infty)$. Moreover, there exists $\psi_+ \in L^2$ such that

$$\left\| \psi(t, x) - a\frac{e^{ix^2/4t}}{\sqrt{t}} - e^{\pm ia^2 \log t} e^{it\partial_x^2} \psi_+(x) \right\|_{L^2} \leq C(a, \delta)\, t^{1/4-\delta} \|\psi_1\|_{L^1 \cap L^2}$$

for any $0 < \delta < 1/4$, and for $|x| \leq 2$ we have

$$|x|^{2\delta} |\psi_+(x)| \leq C(a, \delta)\, \|\psi_1\|_{L^1 \cap L^2}.$$

As we shall see in Corollary 3.5, if u_1 is regular in terms of Sobolev spaces, so is the solution $u(t)$ given in Theorem 1.1. So in particular $u(t)$ is uniformly bounded in terms of the size of u_1. Then from (16) we conclude that if u_1 is small enough with respect to a then $a/2\sqrt{t} \leq |\psi(t, x)| \leq 3a/2\sqrt{t}$, and therefore $|\psi(t, x)|$ becomes singular as t goes to zero. Hence we can use the Frenet system to construct a regular solution $\chi(t, x)$ of (1) for $0 < t \leq 1$, and the corresponding Frenet frame, that will also become singular as t approaches to zero (see for instance [16] or the Appendix of [3]). Notice also that this argument works in both settings, focusing and defocusing. Moreover, due to the fact that in the focusing situation the binormal has unit euclidean length, and that the curvature is integrable in time, we can define $\chi_0(x)$ as

$$\chi_0(x) = \chi(1, x) - \int_0^1 c(\tau, x) b(\tau, x)\, d\tau. \tag{17}$$

As a conclusion we have the following result.

Theorem 1.3. *Let $a > 0$ and $\chi_1(x)$ a regular curve with curvature and torsion c_1 and τ_1. Define*

$$\psi_1(x) = c_1(x)e^{i\int_0^x \tau_1(x')\,dx'}, \qquad u_1(x) = e^{-ix^2/4}\psi_1(x) - a,$$

and assume that $u_1 \in L^1 \cap H^3$ is small with respect to a. Then there exists a unique regular solution $\chi(t,x)$ of (1) for $0 < t \leq 1$ with $\chi(1,x) = \chi_1(x)$. Moreover, its curvature and torsion c and τ satisfy

$$\left|c(t,x) - \frac{a}{\sqrt{t}}\right| \leq \frac{C(u_1)}{\sqrt{t}}, \qquad \left|\tau(t,x) - \frac{x}{2t}\right| \leq \frac{C(u_1)}{t}, \tag{18}$$

and by defining $\chi_0(x)$ as in (17) we have

$$|\chi(t,x) - \chi_0(x)| \leq C(u_1)\sqrt{t}.$$

Remark 1.4. The bounds of the curvature and torsion given in (18) follow from their definition

$$c(t,x) = |\psi(t,x)|, \qquad \tau(t,x) = \Im\frac{\partial_x\psi(t,x)}{\psi(t,x)},$$

and from the rate of decay obtained in Corollary 3.5 below. The same calculations can be found in §3.2 of [3], therefore they will be omitted here.

Remark 1.5. As we said before, by Theorem 1.5 in [3], if a is small enough and if ψ_+ is small and regular enough with $|x|^{-2}\psi_+$ locally integrable, then $\chi_0(x)$ has a corner at the origin $x = 0$.

Remark 1.6. The use of the Frenet frame can be avoided. In fact, once a solution of (4) is obtained, a slight modification of Theorem 3.1 of [16] can be used to construct a solution for (1) for $0 < t \leq 1$, with a trace χ_0 in the focusing case defined as in (17). This is because $|\psi|^2 - a^2/t$ is in $L^2((\epsilon,1), L^\infty)$ for any positive ϵ. In this case $|\psi|$ becomes unbounded in the Strichartz norm $L^4((0,1), L^\infty)$, and therefore the corresponding frame will become also singular as t approaches zero, as does the Frenet frame.

The paper is organized as follows. In §2 we study the asymptotic completeness of the linear equation (14). Then in §3 we deduce Theorem 1.1 by perturbative methods. As already mentioned, Appendix A contains the proof of a new version of the existence of the wave operator of (10) that fits better the hypothesis needed to obtain the asymptotic completeness of Theorem 1.1. Finally in Appendix B we prove the growth of the zero Fourier mode for the solutions of the linear and the nonlinear equations, (14) and (10), a property that we think is interesting in itself.

2. Scattering for the linear equation

In this section we consider only the linear equation (14):

$$iu_t + u_{xx} \pm \frac{a^2}{t^{1\pm 2ia^2}}\overline{u} = 0,$$

with initial data $u(t_0,x)$ at time $t_0 \geq 1$. We start in §2.1 with the proof of some a priori estimates on the Fourier modes of $u(t)$, which will allow us in §2.2 to get a satisfactory global existence result. Then in §2.3 we prove the asymptotic completeness for (14), again with the help of the properties pointed out in §2.1. Finally, in §2.4 we obtain a regularity result for the asymptotic state and we prove a posteriori that $u \in L^4((t_0, \infty), L^\infty)$.

2.1. A priori controls

Lemma 2.1. *If u solves equation (14) then for $0 < t_0 \le t$,*

$$|\hat{u}(t,\xi)| \le \frac{t^{a^2}}{t_0^{a^2}}(|\hat{u}(t_0,\xi)| + |\hat{u}(t_0,-\xi)|). \tag{19}$$

In particular,

$$\|u(t)\|_{\dot{H}^k} \le \frac{t^{a^2}}{t_0^{a^2}}\|u(t_0)\|_{\dot{H}^k} \quad \textit{for all } k \in \mathbb{Z}.$$

Proof. Using the Fourier transform we write equation (14) as

$$0 = i\hat{u}_t(t,\xi) - \xi^2\hat{u}(t,\xi) \pm \frac{a^2}{t^{1\pm 2ia^2}}\hat{\bar{u}}(t,\xi) = i\hat{u}_t(t,\xi) - \xi^2\hat{u}(t,\xi) \pm \frac{a^2}{t^{1\pm 2ia^2}}\overline{\hat{u}(t,-\xi)}. \tag{20}$$

By multiplying by $\overline{\hat{u}(t,\xi)}$ and by taking the imaginary part,

$$\partial_t|\hat{u}(t,\xi)|^2 = \mp 2\Im \frac{a^2}{t^{1\pm 2ia^2}}\overline{\hat{u}(t,-\xi)}\,\overline{\hat{u}(t,\xi)}.$$

We obtain

$$\partial_t|\hat{u}(t,\xi)| \le \frac{a^2}{t}|\hat{u}(t,-\xi)|,$$

therefore

$$\partial_t(|\hat{u}(t,\xi)| + |\hat{u}(t,-\xi)|) \le \frac{a^2}{t}(|\hat{u}(t,\xi)| + |\hat{u}(t,-\xi)|),$$

so the lemma follows. □

Now we shall improve this control for some small frequencies.

Lemma 2.2. *Let $\delta > 0$. If u solves equation (14) then for all $\xi \neq 0$ and all $0 < t_0 \le t$,*

$$|\hat{u}(t,\xi)| \le \left(C(a) + \frac{C(a,\delta)}{(\xi^2\, t_0)^\delta}\right)(|\hat{u}(t_0,\xi)| + |\hat{u}(t_0,-\xi)|), \tag{21}$$

which is a better estimate than the one of Lemma 2.1 in the region $1/t^{a^2} \lesssim \xi^{2\delta}$.

Proof. We shall work with the solution $w(t) = u(t)e^{\pm ia^2 \log t}$ of (15):

$$i\partial_t w + w_{xx} \pm \frac{a^2}{t}(w + \overline{w}) = 0.$$

We have, by taking the Fourier modes of the real and imaginary part of w,

$$\partial_t \widehat{\Re w}(t,\xi) = \xi^2\, \widehat{\Im w}(t,\xi), \tag{22}$$

$$\partial_t\, \widehat{\Im w}(t,\xi) = -\xi^2\, \widehat{\Re w}(t,\xi) \pm \frac{2a^2}{t}\, \widehat{\Re w}(t,\xi). \tag{23}$$

We denote

$$Y_\xi(t) = \widehat{\Re w}(t/\xi^2,\xi), \qquad Z_\xi(t) = \widehat{\Im w}(t/\xi^2,\xi).$$

Equations (22) and (23) become

$$Y'_\xi(t) = Z_\xi(t), \qquad Z'_\xi(t) = \frac{1}{\xi^2}(-\xi^2 + 2a^2\xi^2/t)\, Y_\xi(t) = (-1 + 2a^2/t)\, Y_\xi(t). \tag{24}$$

For simplicity, we consider only the focusing case, which is slightly more complicated. For $0 < \epsilon \leq 1$ to be chosen later, the function

$$\sigma_\xi(t) = \frac{1}{\epsilon}|Y_\xi(t)|^2 + \epsilon |Z_\xi(t)|^2$$

satisfies

$$\sigma'_\xi = \left(\frac{1}{\epsilon} + \epsilon\left(-1 + \frac{2a^2}{t}\right)\right) 2\Re \overline{Y_\xi} Z_\xi \leq \left(\frac{1}{\epsilon} - \epsilon + \epsilon\frac{2a^2}{t}\right)\sigma_\xi.$$

Therefore

$$\left(\log \sigma_\xi - t\left(\frac{1}{\epsilon} - \epsilon\right) - 2a^2\epsilon \log t\right)' \leq 0,$$

and finally for all $0 < \tilde{t}_0 \leq t$,

$$\sigma_\xi(t) \leq e^{\Phi(t)}\sigma_\xi(\tilde{t}_0),$$

where

$$\Phi(t) = (t - \tilde{t}_0)\left(\frac{1}{\epsilon} - \epsilon\right) + 2a^2\epsilon(\log t - \log \tilde{t}_0).$$

Case 1: $0 < \tilde{t}_0 \leq t \leq \min\{a^2, 1/e\}$. In this region

$$\sigma_\xi(t) \leq e^{t/\epsilon - 2a^2\epsilon \log \tilde{t}_0}\sigma_\xi(\tilde{t}_0) \leq e^{a^2/\epsilon + 2a^2\epsilon|\log \tilde{t}_0|}\sigma_\xi(\tilde{t}_0).$$

By choosing $\epsilon = 1/\sqrt{|\log \tilde{t}_0|}$, we get

$$\sigma_\xi(t) \leq e^{3a^2\sqrt{|\log \tilde{t}_0|}}\sigma_\xi(\tilde{t}_0).$$

It follows that

$$|Y_\xi(t)|^2 \leq \left(|Y_\xi(\tilde{t}_0)|^2 + \frac{|Z_\xi(\tilde{t}_0)|^2}{|\log \tilde{t}_0|}\right)e^{3a^2\sqrt{|\log \tilde{t}_0|}},$$

$$|Z_\xi(t)|^2 \leq (|\log \tilde{t}_0|\, |Y_\xi(t_0)|^2 + |Z_\xi(\tilde{t}_0)|^2)e^{3a^2\sqrt{|\log \tilde{t}_0|}}.$$

Therefore, for all $\delta > 0$, there exists a constant $C(a, \delta)$ such that for all $0 < \tilde{t}_0 \le t \le \min\{a^2, 1/e\}$,

$$|Y_\xi(t)|^2 + |Z_\xi(t)|^2 \le \frac{C(a,\delta)}{\tilde{t}_0^{2\delta}}(|Y_\xi(\tilde{t}_0)|^2 + |Z_\xi(\tilde{t}_0)|^2).$$

Case 2: $\min\{a^2, 1/e\} \le \tilde{t}_0 \le t \le 4a^2$ (if such a situation exists). In this case, by taking $\epsilon = 1$, $\Phi(t)$ is bounded by a constant depending on a, and we get

$$|Y_\xi(t)|^2 + |Z_\xi(t)|^2 \le C(a)(|Y_\xi(\tilde{t}_0)|^2 + |Z_\xi(\tilde{t}_0)|^2).$$

Case 3: $4a^2 < \tilde{t}_0 \le t$. For this region we shall diagonalize the system

$$\partial_t \begin{pmatrix} Y_\xi \\ Z_\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(1-2a^2/t) & 0 \end{pmatrix} \begin{pmatrix} Y_\xi \\ Z_\xi \end{pmatrix}.$$

Let

$$\alpha(t) = \sqrt{1 - 2a^2/t}, \qquad P(t) = \begin{pmatrix} 1 & 1 \\ i\alpha(t) & -i\alpha(t) \end{pmatrix}.$$

In particular,

$$1/\sqrt{2} \le \alpha(t) \le 1, \qquad P^{-1}(t) = \begin{pmatrix} 1/2 & -i/2\alpha(t) \\ 1/2 & i/2\alpha(t) \end{pmatrix}.$$

Then

$$\begin{pmatrix} \tilde{Y}_\xi(t) \\ \tilde{Z}_\xi(t) \end{pmatrix} = P^{-1}(t) \begin{pmatrix} Y_\xi(t) \\ Z_\xi(t) \end{pmatrix}$$

satisfies

$$\partial_t \begin{pmatrix} \tilde{Y}_\xi \\ \tilde{Z}_\xi \end{pmatrix} = \partial_t(P^{-1})P \begin{pmatrix} \tilde{Y}_\xi \\ \tilde{Z}_\xi \end{pmatrix} + \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} \begin{pmatrix} \tilde{Y}_\xi \\ \tilde{Z}_\xi \end{pmatrix}.$$

Denote

$$\Phi(t) = t - a^2 \log t - \int_t^\infty \left(\alpha(s) - 1 + \frac{a^2}{s} \right) ds.$$

Finally,

$$\begin{pmatrix} \mathring{Y}_\xi(t) \\ \mathring{Z}_\xi(t) \end{pmatrix} = \begin{pmatrix} e^{-i\Phi(t)} & 0 \\ 0 & e^{i\Phi(t)} \end{pmatrix} \begin{pmatrix} \tilde{Y}_\xi(t) \\ \tilde{Z}_\xi(t) \end{pmatrix}$$

satisfies

$$\partial_t \begin{pmatrix} \mathring{Y}_\xi \\ \mathring{Z}_\xi \end{pmatrix} = M(t) \begin{pmatrix} \mathring{Y}_\xi \\ \mathring{Z}_\xi \end{pmatrix}, \tag{25}$$

where

$$M(t) = \begin{pmatrix} e^{-i\Phi(t)} & 0 \\ 0 & e^{i\Phi(t)} \end{pmatrix} \partial_t(P^{-1})P \begin{pmatrix} e^{i\Phi(t)} & 0 \\ 0 & e^{-i\Phi(t)} \end{pmatrix} = \frac{a^2}{2t^2\alpha^2} \begin{pmatrix} -1 & e^{-2i\Phi(t)} \\ e^{2i\Phi(t)} & -1 \end{pmatrix}$$

Since $1/\sqrt{2} \leq \alpha(t) \leq 1$, all the entries of $M(t)$ are upper-bounded by Ca^2/t^2. We infer that

$$\partial_t(|\mathring{Y}_\xi|^2 + |\mathring{Z}_\xi|^2) \leq \frac{Ca^2}{t^2}(|\mathring{Y}_\xi|^2 + |\mathring{Z}_\xi|^2),$$

so

$$\partial_t\left(\log(|\mathring{Y}_\xi|^2 + |\mathring{Z}_\xi|^2) + \frac{Ca^2}{t}\right) \leq 0.$$

We have $Ca^2/\tilde{t}_0 \leq C/4$, and we get

$$|\mathring{Y}_\xi(t)|^2 + |\mathring{Z}_\xi(t)|^2 \leq C(|\mathring{Y}_\xi(\tilde{t}_0)|^2 + |\mathring{Z}_\xi(\tilde{t}_0)|^2).$$

Finally, from the relation

$$|\mathring{Y}_\xi(t)|^2 + |\mathring{Z}_\xi(t)|^2 = \left|\frac{1}{2}Y_\xi - \frac{i}{2\alpha}Z_\xi\right|^2 + \left|\frac{1}{2}Y_\xi + \frac{i}{2\alpha}Z_\xi\right|^2 = \frac{1}{2}|Y_\xi|^2 + \frac{1}{2\alpha^2}|Z_\xi|^2$$

and from $1/\sqrt{2} \leq \alpha(t) \leq 1$ it follows that

$$|Y_\xi(t)|^2 + |Z_\xi(t)|^2 \leq C(|Y_\xi(\tilde{t}_0)|^2 + |Z_\xi(\tilde{t}_0)|^2). \tag{26}$$

Summarizing, we have found that for all $\delta > 0$, there exists a constant $C(a, \delta)$ such that for all $0 < \tilde{t}_0 \leq t$,

$$|Y_\xi(t)|^2 + |Z_\xi(t)|^2 \leq \left(C(a) + \frac{C(a,\delta)}{\tilde{t}_0^{2\delta}}\right)(|Y_\xi(\tilde{t}_0)|^2 + |Z_\xi(\tilde{t}_0)|^2). \tag{27}$$

By recovering the first unknowns, for all $0 < t_0 \leq t$,

$$|\widehat{\Re w}(t,\xi)|^2 + |\widehat{\Im w}(t,\xi)|^2 \leq \left(C(a) + \frac{C(a,\delta)}{(\xi^2 t_0)^{2\delta}}\right)(|\widehat{\Re w}(t_0,\xi)|^2 + |\widehat{\Im w}(t_0,\xi)|^2),$$

and by using the identity $2(|z_1|^2 + |z_2|^2) = |z_1 + iz_2|^2 + |z_1 - iz_2|^2$,

$$|\widehat{w}(t,\xi)|^2 + |\widehat{w}(t,-\xi)|^2 \leq \left(C(a) + \frac{C(a,\delta)}{(\xi^2 t_0)^{2\delta}}\right)(|\hat{w}(t_0,\xi)|^2 + |\hat{w}(t_0,-\xi)|^2).$$

Since $w(t) = u(t)e^{\pm ia^2 \log t}$ the lemma follows.

For further use we want to compute the asymptotic behavior of the solution u of (14). In view of (25) and (26) of Case 3, we can define, for $4a^2 \leq \tilde{t}_0$,

$$\begin{pmatrix} \mathring{Y}_\xi^+ \\ \mathring{Z}_\xi^+ \end{pmatrix} = \begin{pmatrix} \mathring{Y}_\xi(\tilde{t}_0) \\ \mathring{Z}_\xi(\tilde{t}_0) \end{pmatrix} + \int_{\tilde{t}_0}^\infty M(\tau)\begin{pmatrix} \mathring{Y}_\xi(\tau) \\ \mathring{Z}_\xi(\tau) \end{pmatrix} d\tau,$$

so that for $4a^2 \leq \tilde{t}_0 \leq t$,

$$\begin{pmatrix} \mathring{Y}_\xi^+ \\ \mathring{Z}_\xi^+ \end{pmatrix} = \begin{pmatrix} \mathring{Y}_\xi(t) \\ \mathring{Z}_\xi(t) \end{pmatrix} + \int_{t}^\infty M(\tau)\begin{pmatrix} \mathring{Y}_\xi(\tau) \\ \mathring{Z}_\xi(\tau) \end{pmatrix} d\tau, \tag{28}$$

and

$$|\mathring{Y}_\xi(t) - \mathring{Y}_\xi^+| + |\mathring{Z}_\xi(t) - \mathring{Z}_\xi^+| \leq \frac{C(a)}{t}(|Y_\xi(\tilde{t}_0)| + |Z_\xi(\tilde{t}_0)|). \tag{29}$$

We have

$$\begin{aligned}
\mathring{Y}_\xi^+ &= \mathring{Y}_\xi(t) + \int_t^\infty \frac{a^2}{2\tau^2\alpha^2}(-\mathring{Y}_\xi(\tau) + e^{-2i\Phi(\tau)}\mathring{Z}_\xi(\tau))\,d\tau \\
&= e^{-i\Phi(t)}\tilde{Y}_\xi(t) + \int_t^\infty \frac{a^2 e^{-i\Phi(\tau)}}{2\tau^2\alpha^2}(-\tilde{Y}_\xi(\tau) + \tilde{Z}_\xi(\tau))\,d\tau \\
&= e^{-i\Phi(t)}\left(\frac{1}{2}Y_\xi(t) - \frac{i}{2\alpha}Z_\xi(t)\right) + \int_t^\infty \frac{a^2 e^{-i\Phi(\tau)}}{2\tau^2\alpha^2}\,\frac{i}{\alpha}Z_\xi(\tau)\,d\tau,
\end{aligned}$$

and

$$\begin{aligned}
\mathring{Z}_\xi^+ &= \mathring{Z}_\xi(t) + \int_t^\infty \frac{a^2}{2\tau^2\alpha^2}(e^{2i\Phi(\tau)}\mathring{Y}_\xi(\tau) - \mathring{Z}_\xi(\tau))\,d\tau \\
&= e^{i\Phi(t)}\tilde{Z}_\xi(t) + \int_t^\infty \frac{a^2 e^{i\Phi(\tau)}}{2\tau^2\alpha^2}(\tilde{Y}_\xi(\tau) - \tilde{Z}_\xi(\tau))\,d\tau \\
&= e^{i\Phi(t)}\left(\frac{1}{2}Y_\xi(t) + \frac{i}{2\alpha}Z_\xi(t)\right) - \int_t^\infty \frac{a^2 e^{i\Phi(\tau)}}{2\tau^2\alpha^2}\,\frac{i}{\alpha}Z_\xi(\tau)\,d\tau,
\end{aligned}$$

therefore since $\overline{Y_\xi} = Y_{-\xi}$ and $\overline{Z_\xi} = Z_{-\xi}$ we get the relation

$$\overline{\mathring{Y}_\xi^+} = e^{i\Phi(t)}\left(\frac{1}{2}Y_{-\xi}(t) + \frac{i}{2\alpha}Z_{-\xi}(t)\right) - \int_t^\infty \frac{a^2 e^{i\Phi(\tau)}}{2\tau^2\alpha^2}\,\frac{i}{\alpha}Z_{-\xi}(\tau)\,d\tau = \mathring{Z}_{-\xi}^+. \tag{30}$$

As a conclusion, by (29) and (27) we get, for all $0 < \tilde{t}_0$ and all $t \geq \max\{\tilde{t}_0, 4a^2\}$,

$$\begin{aligned}
&\left|\left(\frac{1}{2}Y_\xi - \frac{i}{2\alpha}Z_\xi\right) - e^{i\Phi(t)}\overline{\mathring{Y}_\xi^+}\right| + \left|\left(\frac{1}{2}Y_\xi + \frac{i}{2\alpha}Z_\xi\right) - e^{-i\Phi(t)}\mathring{Z}_\xi^+\right| \\
&\quad = \left|\left(\frac{1}{2}Y_{-\xi} + \frac{i}{2\alpha}Z_{-\xi}\right) - e^{-i\Phi(t)}\mathring{Z}_{-\xi}^+\right| + \left|\left(\frac{1}{2}Y_\xi + \frac{i}{2\alpha}Z_\xi\right) - e^{-i\Phi(t)}\mathring{Z}_\xi^+\right| \\
&\quad \leq \frac{1}{t}\left(C(a) + \frac{C(a,\delta)}{\tilde{t}_0^{\,\delta}}\right)(|Y_\xi(\tilde{t}_0)| + |Z_\xi(\tilde{t}_0)|).
\end{aligned} \tag{31}$$

In particular, in view of the definition of $\alpha(t)$ and of estimate (26), we have

$$\left|\left(\frac{1}{2}Y_\xi + \frac{i}{2}Z_\xi\right) - e^{-i\Phi(t)}\mathring{Z}_\xi^+\right| \leq \frac{1}{t}\left(C(a) + \frac{C(a,\delta)}{\tilde{t}_0^\delta}\right)(|Y_\xi(\tilde{t}_0)| + |Z_\xi(\tilde{t}_0)|).$$

Hence noticing that $\Phi(t) = t - a^2\log t + \mathcal{O}(1/t)$ we see that u_+ defined by

$$2\mathring{Z}_\xi^+ = e^{-ia^2\log\xi^2}\hat{u}_+(\xi) \tag{32}$$

satisfies for all $t_0 > 0$ and all $t \geq \max\{t_0, 4a^2/\xi^2\}$ the estimate

$$|\hat{u}(t,\xi) - e^{-it\xi^2}\,\hat{u}_+(\xi)| \leq \frac{1}{\xi^2 t}\left(C(a) + \frac{C(a,\delta)}{(\xi^2 t_0)^\delta}\right)(|\hat{u}(t_0,\xi)| + |\hat{u}(t_0,-\xi)|). \tag{33}$$

By combining this estimate with (21) for $t = 4a^2/\xi^2$ and for $0 < t_0 \leq t \leq 4a^2/\xi^2$, we see that (33) is valid for all $0 < t_0 \leq t$. □

Remark 2.3. Let us notice that the logarithmic loss is generally unavoidable. Suppose $Y_\xi(\tilde{t_0}) = Z_\xi(\tilde{t_0}) = 1$ and $0 < \tilde{t_0} \leq t \leq \min\{a^2, 1/e\}$. Then in view of the system (24), we have $Y_\xi(t) > 1$ and $Z_\xi(t) > 1$, and so

$$Y_\xi(t) > Y_\xi(\tilde{t_0}), \quad Z'_\xi(t) > (-1 + 2a^2/t)Y_\xi(\tilde{t_0}) = -1 + 2a^2/t.$$

Then we get finally the logarithmic lower bound

$$Z_\xi(t) \geq Z_\xi(\tilde{t_0}) - 2a^2\log(t/\tilde{t_0}) - (t - \tilde{t_0}) \geq C(a)|\log\tilde{t_0}|.$$

Remark 2.4. In Appendix B.1 we shall see that if $\hat{u}(t_0, 0)$ is defined and if $\hat{u}(t_0, 0) \neq 0$, then also for $\xi = 0$ a logarithmic loss is unavoidable, independently of the size of $t_0 \leq t$:

$$\hat{u}(t,0) = e^{\pm ia^2\log(t_0/t)}\hat{u}(t_0,0) \pm 2ia^2 e^{\pm ia^2\log\frac{t_0}{t}}\Re\hat{u}(t_0,0)\log(t/t_0). \tag{34}$$

Moreover, under certain conditions on the initial data, a logarithmic loss will be shown in Appendix B.2 for the zero-modes of the solutions of the nonlinear equation (10).

We end this subsection with an estimate on the typical Duhamel term associated to (14).

Lemma 2.5. *Let $\delta > 0$. Let u be a solution of equation (14) and let*

$$A_{t_1,t_2}(\xi) = a^2\int_{t_1}^{t_2} e^{-i(t-\tau)\xi^2}\,\frac{\overline{\hat{u}(\tau,-\xi)}}{\tau^{1\pm 2ia^2}}\,d\tau$$

be the Fourier transform of the Duhamel term integrated between two arbitrary times $t_0 < t_1 \leq t_2$. Then for $\xi \neq 0$,

$$|A_{t_1,t_2}(\xi)| \leq \left(C(a) + \frac{C(a,\delta)}{(\xi^2 t_0)^\delta}\right)\frac{|\hat{u}(t_0,\xi)| + |\hat{u}(t_0,-\xi)|}{\xi^2 t_1}. \tag{35}$$

Proof. We perform an integration by parts

$$\begin{aligned} A_{t_1,t_2}(\xi) &= a^2 e^{-it\xi^2}\int_{t_1}^{t_2}\frac{\partial_\tau e^{i\tau\xi^2}}{i\xi^2}\,\frac{\overline{\hat{u}(\tau,-\xi)}}{\tau^{1\pm 2ia^2}}\,d\tau \\ &= \frac{a^2 e^{-i(t-\tau)\xi^2}}{i\xi^2\,\tau^{1\pm 2ia^2}}\overline{\hat{u}(\tau,-\xi)}\Bigg|_{t_1}^{t_2} - a^2\int_{t_1}^{t_2}\frac{e^{-i(t-\tau)\xi^2}}{i\xi^2}\,\frac{\partial_\tau\overline{\hat{u}(\tau,-\xi)}}{\tau^{1\pm 2ia^2}} \\ &\quad - \frac{(1\pm 2ia^2)e^{-i(t-\tau)\xi^2}}{i\xi^2\,\tau^{2\pm 2ia^2}}\overline{\hat{u}(\tau,-\xi)}\,d\tau. \end{aligned}$$

From (20) we get

$$i\hat{u}_t(t,-\xi)-\xi^2\hat{u}(t,-\xi)\pm\frac{a^2}{t^{1\pm 2ia^2}}\overline{\hat{u}(t,\xi)}=0,$$

and then

$$-i\overline{\hat{u}_t(t,-\xi)}-\xi^2\overline{\hat{u}(t,-\xi)}\pm\frac{a^2}{t^{1\mp 2ia^2}}\hat{u}(t,\xi)=0.$$

Therefore by replacing

$$\partial_\tau\overline{\hat{u}(\tau,-\xi)}=i\xi^2\overline{\hat{u}(\tau,-\xi)}\mp i\frac{a^2}{\tau^{1\mp 2ia^2}}\hat{u}(\tau,\xi)$$

we recover an $A_{t_1,t_2}(\xi)$ with minus sign, so that

$$A_{t_1,t_2}(\xi)=\frac{a^2e^{-i(t-\tau)\xi^2}}{2i\xi^2\,\tau^{1\pm 2ia^2}}\overline{\hat{u}(\tau,-\xi)}\Big|_{t_1}^{t_2}$$
$$-a^2\int_{t_1}^{t_2}\frac{e^{-i(t-\tau)\xi^2}}{2i\xi^2}\frac{\mp ia^2\hat{u}(\tau,\xi)}{\tau^2}-\frac{(1\pm 2ia^2)e^{-i(t-\tau)\xi^2}}{2i\xi^2\,\tau^{2\pm 2ia^2}}\overline{\hat{u}(\tau,-\xi)}\,d\tau.$$

Then we can upper-bound

$$|A_{t_1,t_2}(\xi)|\le\frac{a^2}{2\xi^2\,t_2}|\hat{u}(t_2,-\xi)|+\frac{a^2}{2\xi^2\,t_1}|\hat{u}(t_1,-\xi)|$$
$$+\frac{a^2}{2\xi^2}\int_{t_1}^{t_2}(a^2|\hat{u}(\tau,\xi)|+|1+2ia^2|\,|\hat{u}(\tau,-\xi)|)\,\frac{d\tau}{\tau^2}.$$

Now Lemma 2.2 allows us to conclude that

$$|A_{t_1,t_2}(\xi)|\le a^2(a^2+|1+2ia^2|)\left(C(a)+\frac{C(a,\delta)}{(\xi^2\,t_0)^\delta}\right)\frac{|\hat{u}(t_0,\xi)|+|\hat{u}(t_0,-\xi)|}{2\xi^2\,t_1},$$

as desired. □

2.2. *Global solutions*

For an initial data in H^s we see by Lemma 2.1 that the solution is globally in H^s, but with a growth of $\|u(t)\|_{H^s}$. To avoid this issue, we shall start with an initial data in a more restricted space. We recall the spaces defined in the Introduction by (12) and (13). Let $0\le\gamma<1/4$ throughout the rest of the paper. For a fixed t_0, we define a norm on functions depending only on the space variable,

$$\|f\|_{X^\gamma_{t_0}}=\frac{1}{t_0^{1/4}}\|f\|_{L^2}+\frac{t_0^\gamma}{\sqrt{t_0}}\left\||\xi|^{2\gamma}\hat{f}(\xi)\right\|_{L^\infty(\xi^2\le 1)},$$

and a norm on functions depending on both time and space,

$$\|g\|_{Y^\gamma_{t_0}}=\sup_{t\ge t_0}\left(\frac{1}{t_0^{1/4}}\|g(t)\|_{L^2}+\left(\frac{t_0}{t}\right)^{a^2}\frac{t_0^\gamma}{\sqrt{t_0}}\left\||\xi|^{2\gamma}\hat{g}(t,\xi)\right\|_{L^\infty(\xi^2\le 1)}\right),$$

and let $X^\gamma_{t_0}$ and $Y^\gamma_{t_0}$ be the corresponding spaces.

Proposition 2.6. *Let $t_0 \geq 1$. Let $u(t_0)$ be a function in $X^\gamma_{t_0}$. Then there exists a unique global solution $u \in Y^\gamma_{t_0}$ of equation (14) with initial data $u(t_0)$ at time t_0, and*

$$\|u\|_{Y^\gamma_{t_0}} \leq C(a)\|u(t_0)\|_{X^\gamma_{t_0}}.$$

More precisely,

$$\sup_{t\geq t_0} \frac{1}{t_0^{1/4}}\|u(t)\|_{L^2} \leq C(a)\|u(t_0)\|_{X^\gamma_{t_0}}, \tag{36}$$

$$\sup_{t\geq t_0} \left(\frac{t_0}{t}\right)^{a^2} \frac{t_0^\gamma}{\sqrt{t_0}} \left\||\xi|^{2\gamma}\hat{u}(t,\xi)\right\|_{L^\infty(\xi^2\leq 1)} \leq C\frac{t_0^\gamma}{\sqrt{t_0}}\left\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\right\|_{L^\infty(\xi^2\leq 1)}.$$

Proof. We first show the proposition with $t_0 = 1$ and then for an arbitrary t_0.

We start with $u(1) \in X^\gamma_1$, which means that $u(1) \in L^2$ with $|\xi|^{2\gamma}\hat{u}(1,\xi)$ bounded in the region $\xi^2 \leq 1$. We already know that a global solution $u(t) \in \mathcal{C}((1,\infty), L^2)$ exists, and we want to show that it belongs to Y^γ_1. By Lemma 2.1, for all $M > 0$,

$$\frac{1}{t^{a^2}}\left\||\xi|^{2\gamma}\hat{u}(t,\xi)\right\|_{L^\infty(\xi^2\leq M)} \leq 2\left\||\xi|^{2\gamma}\hat{u}(1,\xi)\right\|_{L^\infty(\xi^2\leq M)}, \tag{37}$$

so the second condition to be in Y^γ_1 is fulfilled by taking $M = 1$. To control the L^2 norm we split it into two parts,

$$\|u(t)\|_{L^2} = \|\hat{u}(t)\|_{L^2} = \|\hat{u}(t)\|_{L^2(\xi^2\leq 1)} + \|\hat{u}(t)\|_{L^2(1\leq\xi^2)} = I + J.$$

For both parts we use Lemma 2.2, with $\delta < 1/4 - \gamma$:

$$\begin{aligned} I &\leq C(a)\left\||\xi|^{-2\delta}|\hat{u}(1,\xi)|\right\|_{L^2(\xi^2\leq 1)} \\ &\leq C(a)\left\||\xi|^{-2(\gamma+\delta)}\right\|_{L^2(\xi^2\leq 1)}\left\||\xi|^{2\gamma}\hat{u}(1,\xi)\right\|_{L^\infty(\xi^2\leq 1)}, \\ J &\leq C(a)\|\hat{u}(1,\xi)\|_{L^2(1\leq\xi^2)} \leq C(a)\|\hat{u}(1)\|_{L^2}. \end{aligned}$$

Therefore we have the L^2 norm of $u(t)$ bounded in time,

$$\|u(t)\|_{L^2} \leq C(a)\|u(1)\|_{L^2} + C(a)\left\||\xi|^{2\gamma}\hat{u}(1,\xi)\right\|_{L^\infty(\xi^2\leq 1)} \leq C(a)\|u(1)\|_{X^\gamma_1},$$

and so u is in Y^γ_1.

Now we start with $u(t_0) \in X^\gamma_{t_0}$. We define $U(1)$ by

$$u(t_0,x) = U(1, x/\sqrt{t_0}).$$

We have

$$\|u(t_0)\|_{L^2} = t_0^{1/4}\|U(1)\|_{L^2}$$

and

$$\frac{t_0^\gamma}{\sqrt{t_0}}\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1)} = \frac{t_0^\gamma}{\sqrt{t_0}}\left\||\xi|^{2\gamma}\int e^{ix\xi}U(1,x/\sqrt{t_0})\,dx\right\|_{L^\infty(\xi^2\le 1)}$$
$$= t_0^\gamma\big\||\xi|^{2\gamma}\hat{U}(1,\xi\sqrt{t_0})\big\|_{L^\infty(\xi^2\le 1)}$$
$$= \big\||\xi|^{2\gamma}\hat{U}(1,\xi)\big\|_{L^\infty(\xi^2\le t_0)} \ge \big\||\xi|^{2\gamma}\hat{U}(1,\xi)\big\|_{L^\infty(\xi^2\le 1)}.$$

Hence

$$\|U(1)\|_{X_1^\gamma} \le \|u(t_0)\|_{X_{t_0}^\gamma},$$

and $U(1)$ is in X_1^γ. Therefore we can consider the global solution $U \in Y_1^\gamma$ of equation (14) with initial data $U(1)$ at time 1. The function u defined by

$$u(t,x) = U(t/t_0, x/\sqrt{t_0})$$

is the solution of equation (14) with initial data $u(t_0)$ at time t_0. We shall rewrite the L^2 estimate and (37) with $M = t_0$,

$$\sup_{t\ge 1}\|U(t)\|_{L^2} \le C(a)\|U(1)\|_{X_1^\gamma},$$

$$\sup_{t\ge 1}\frac{1}{t^{a^2}}\big\||\xi|^{2\gamma}\hat{U}(t,\xi)\big\|_{L^\infty(\xi^2\le t_0)} \le 2\big\||\xi|^{2\gamma}\hat{U}(1,\xi)\big\|_{L^\infty(\xi^2\le t_0)},$$

in terms of u. We have

$$\sup_{t\ge 1}\|U(t)\|_{L^2} = \sup_{t\ge 1}\|u(t\,t_0, x\sqrt{t_0})\|_{L^2} = \sup_{t\ge 1}\frac{1}{t_0^{1/4}}\|u(t\,t_0)\|_{L^2} = \sup_{t\ge t_0}\frac{1}{t_0^{1/4}}\|u(t)\|_{L^2},$$

and since we have already shown that $\|U(1)\|_{X_1^\gamma} \le \|u(t_0)\|_{X_{t_0}^\gamma}$, we get the first estimate of (36). We have also already computed

$$\big\||\xi|^{2\gamma}\hat{U}(1,\xi)\big\|_{L^\infty(\xi^2\le t_0)} = \frac{t_0^\gamma}{\sqrt{t_0}}\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1)},$$

and we get similarly

$$\sup_{t\ge 1}\frac{1}{t^{a^2}}\big\||\xi|^{2\gamma}\hat{U}(t,\xi)\big\|_{L^\infty(\xi^2\le t_0)} = \sup_{t\ge 1}\frac{1}{t^{a^2}}\left\||\xi|^{2\gamma}\int e^{ix\xi}u(t\,t_0, x\sqrt{t_0})\,dx\right\|_{L^\infty(\xi^2\le t_0)}$$
$$= \sup_{t\ge 1}\frac{1}{t^{a^2}}\frac{1}{\sqrt{t_0}}\big\||\xi|^{2\gamma}\hat{u}(t\,t_0,\xi/\sqrt{t_0})\big\|_{L^\infty(\xi^2\le t_0)}$$
$$= \sup_{t\ge t_0}\left(\frac{t_0}{t}\right)^{a^2}\frac{1}{\sqrt{t_0}}\big\||\xi|^{2\gamma}\hat{u}(t,\xi/\sqrt{t_0})\big\|_{L^\infty(\xi^2\le t_0)}$$
$$= \sup_{t\ge t_0}\left(\frac{t_0}{t}\right)^{a^2}\frac{t_0^\gamma}{\sqrt{t_0}}\big\||\xi|^{2\gamma}\hat{u}(t,\xi)\big\|_{L^\infty(\xi^2\le 1)},$$

so we also get the second estimate of (36) and the proof is complete. □

Since equation (14) is linear, we can apply Proposition 2.6 to higher order derivatives, and get the following statement.

Corollary 2.7. *Let $s \in \mathbb{N}$ and $t_0 \geq 1$. Let $u(t_0)$ be a function in $X^\gamma_{t_0}$ such that $\partial_x^k u(t_0) \in X^\gamma_{t_0}$ for all $0 \leq k \leq s$. Then there exists a unique global solution $u \in Y^\gamma_{t_0}$ of equation (14) with initial data $u(t_0)$ at time t_0, with $\partial_x^k u \in Y^\gamma_{t_0}$ for all $0 \leq k \leq s$, and*

$$\|\partial_x^k u\|_{Y^\gamma_{t_0}} \leq C(a)\|\partial_x^k u(t_0)\|_{X^\gamma_{t_0}}.$$

More precisely,

$$\sup_{t\geq t_0} \frac{1}{t_0^{1/4}}\|\partial_x^k u(t)\|_{L^2} \leq C(a)\|\partial_x^k u(t_0)\|_{X^\gamma_{t_0}},$$

$$\sup_{t\geq t_0}\left(\frac{t_0}{t}\right)^{a^2}\frac{t_0^\gamma}{\sqrt{t_0}}\left\||\xi|^{2\gamma}\widehat{\partial_x^k u}(t,\xi)\right\|_{L^\infty(\xi^2\leq 1)} \leq C(a)\frac{t_0^\gamma}{\sqrt{t_0}}\left\||\xi|^{2\gamma}\widehat{\partial_x^k u}(t_0,\xi)\right\|_{L^\infty(\xi^2\leq 1)}.$$

2.3. *Asymptotic completeness*

Proposition 2.8. *Let $t_0 \geq 1$ and let $u(t_0)$ be a function in $X^\gamma_{t_0}$. Then the unique global solution $u \in Y^\gamma_{t_0}$ of equation (14) with initial data $u(t_0)$ at time t_0 scatters in L^2. More precisely, there exists $u_+ \in L^2$ such that*

$$\|u(t) - e^{i(t-t_0)\partial_x^2}u_+\|_{L^2} \leq C(a,\delta)\frac{(1+\log t_0)t_0^{1/2-(\gamma+\delta)}}{t^{1/4-(\gamma+\delta)}}\|u(t_0)\|_{X^\gamma_{t_0}} \xrightarrow[t\to\infty]{} 0 \tag{38}$$

for any $0 < \delta < 1/4 - \gamma$.

Proof. First we shall show that $e^{-i(t-t_0)\partial_x^2}u(t,x)$ has a limit in L^2 as t goes to infinity. This is equivalent to

$$\|e^{-it_2\partial_x^2}u(t_2,x) - e^{-it_1\partial_x^2}u(t_1,x)\|_{L^2} \xrightarrow[t_1,t_2\to\infty]{} 0,$$

and to

$$\|e^{it_2\xi^2}\hat{u}(t_2,\xi) - e^{it_2\xi^2}\hat{u}(t_1,\xi)\|_{L^2} = \|A_{t_1,t_2}(\xi)\|_{L^2} \xrightarrow[t_1,t_2\to\infty]{} 0.$$

For $1/t_0 \leq \xi^2$, Lemma 2.5 gives

$$\|A_{t_1,t_2}(\xi)\|_{L^2(1/t_0\leq\xi^2)} \leq C(a)\frac{t_0}{t_1}\|u(t_0)\|_{L^2}.$$

In the region $\xi^2 \leq 1/t_2 \leq 1/t_0$ we use Lemma 2.2:

$$|A_{t_1,t_2}(\xi)| \leq a^2\int_{t_1}^{t_2}\frac{|\hat{u}(\tau,-\xi)|}{\tau}\,d\tau \leq C(a,\delta)\frac{|\hat{u}(t_0,-\xi)| + |\hat{u}(t_0,\xi)|}{(\xi^2\, t_0)^\delta}\log t_2,$$

so for $0 < \delta < 1/4 - \gamma$,

$$\|A_{t_1,t_2}\|_{L^2(\xi^2\le 1/t_2)} \le C(a,\delta)\frac{\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1)}}{t_0^\delta}\left\|\frac{\log\xi^2}{\xi^{2(\gamma+\delta)}}\right\|_{L^2(\xi^2\le 1/t_2)}$$
$$\le C(a,\delta)\frac{1+\log t_2}{t_0^\delta t_2^{1/4-(\gamma+\delta)}}\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1)}.$$

In the region $1/t_2 \le \xi^2 \le 1/t_1 \le 1/t_0$, we split

$$A_{t_1,t_2} = A_{t_1,1/\xi^2} + A_{1/\xi^2,t_2} = I + J.$$

For I we use again Lemma 2.2 to obtain

$$|I| \le a^2\int_{t_1}^{1/\xi^2}\frac{|\hat{u}(\tau,-\xi)|}{\tau}d\tau \le C(a,\delta)\frac{|\hat{u}(t_0,\xi)|+|\hat{u}(t_0,-\xi)|}{(\xi^2\,t_0)^\delta}|\log\xi^2|,$$

and for J we use Lemma 2.5:

$$|J| \le \frac{C(a,\delta)}{(\xi^2 t_0)^\delta}\,\frac{|\hat{u}(t_0,\xi)|+|\hat{u}(t_0,-\xi)|}{\xi^2\frac{1}{\xi^2}} = C(a,\delta)\frac{|\hat{u}(t_0,\xi)|+|\hat{u}(t_0,-\xi)|}{(\xi^2\,t_0)^\delta}.$$

Then for $0 < \delta < 1/4 - \gamma$,

$$\|A_{t_1,t_2}\|_{L^2(1/t_2\le\xi^2\le 1/t_1)} \le C(a,\delta)\frac{\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1/t_1)}}{t_0^\delta}\left\|\frac{\log\xi^2}{\xi^{2(\gamma+\delta)}}\right\|_{L^2(\xi^2\le 1)}$$
$$\le C(a,\delta)\frac{1+\log t_1}{t_0^\delta t_1^{1/4-(\gamma+\delta)}}\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1)}.$$

In the last region $1/t_1 \le \xi^2 \le 1/t_0$ we use Lemma 2.5:

$$\|A_{t_1,t_2}\|_{L^2(1/t_1\le\xi^2\le 1/t_0)} \le C(a,\delta)\frac{1}{t_1}\,\frac{\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1)}}{t_0^\delta}\left\|\frac{1}{\xi^{2+2(\gamma+\delta)}}\right\|_{L^2(1/t_1\le\xi^2\le 1)}$$
$$\le C(a,\delta)\frac{\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1)}}{t_1\,t_0^\delta}\,t_1^{3/4+(\gamma+\delta)}$$
$$= C(a,\delta)\frac{1}{t_0^\delta\,t_1^{1/4-(\gamma+\delta)}}\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1)}.$$

In conclusion, we have obtained

$$\|A_{t_1,t_2}\|_{L^2} \le C(a)\frac{t_0}{t_1}\|u(t_0)\|_{L^2} + C(a,\delta)\frac{1+\log t_1}{t_0^\delta\,t_1^{1/4-(\gamma+\delta)}}\big\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\big\|_{L^\infty(\xi^2\le 1)}$$
$$\le C(a,\delta)\left(t_0^{1/4}\frac{t_0}{t_1} + \frac{\sqrt{t_0}\,(1+\log t_1)}{t_0^{\gamma+\delta}\,t_1^{1/4-(\gamma+\delta)}}\right)\|u(t_0)\|_{X_{t_0}^\gamma}. \tag{39}$$

Therefore we have a limit $u_+ \in L^2$ of $e^{-i(t-t_0)\partial_x^2} u(t,x)$ as t goes to infinity. To get the decay rate (38) we fix $t_1 = t$ and $t_2 = \infty$, and obtain

$$\|u_+ - e^{-i(t-t_0)\partial_x^2} u(t,x)\|_{L^2} = \|A_{t,\infty}\|_{L^2} \leq C(a,\delta)\, t_0^{1/2-(\gamma+\delta)} \frac{1+\log t}{t^{1/4-(\gamma+\delta)}} \|u(t_0)\|_{X_{t_0}^\gamma},$$

for any $0 < \delta < 1/4 - \gamma$, and since $t_0 \geq 1$ the proposition follows. □

In this proof we have used Lemmas 2.1, 2.2 and 2.5, which are pointwise estimates in Fourier variables, so they apply to higher order derivatives. If $\partial_x^k u(t_0) \in X(t_0)^\gamma$ for $0 \leq k \leq s$, we then get similar estimates to (39),

$$\|\partial_x^k A_{t_1,t_2}\|_{L^2} \leq C(a,\delta) \frac{(1+\log t_0) t_0^{1/2-(\gamma+\delta)}}{t_1^{1/4-(\gamma+\delta)}} \|\partial_x^k u(t_0)\|_{X_{t_0}^\gamma}.$$

Therefore we get a limit $u_+ \in H^s$ of $e^{-i(t-t_0)\partial_x^2} u(t,x)$ as t goes to infinity and

$$\|u_+ - e^{-i(t-t_0)\partial_x^2} u(t,x)\|_{\dot H^k} = \|\partial_x^k A_{t,\infty}\|_{L^2} \leq C(a,\delta) \frac{(1+\log t_0) t_0^{1/2-(\gamma+\delta)}}{t^{1/4-(\gamma+\delta)}} \|\partial_x^k u(t_0)\|_{X_{t_0}^\gamma}.$$

Let us state this result.

Corollary 2.9. *Let $s \in \mathbb{N}$ and $t_0 \geq 1$. Let $u(t_0)$ be a function in $X_{t_0}^\gamma$ such that $\partial_x^k u(t_0) \in X_{t_0}^\gamma$ for all $0 \leq k \leq s$. Then the unique global solution $u \in Y_{t_0}^\gamma$ of equation (14) with initial data $u(t_0)$ at time t_0, with $\partial_x^k u \in Y_{t_0}^\gamma$ for all $0 \leq k \leq s$, scatters in H^s. More precisely, there exists $u_+ \in H^s$ such that*

$$\|u(t) - e^{i(t-t_0)\partial_x^2} u_+\|_{\dot H^k} \leq C(a,\delta) \frac{(1+\log t_0) t_0^{1/2-(\gamma+\delta)}}{t^{1/4-(\gamma+\delta)}} \|\partial_x^k u(t_0)\|_{X_{t_0}^\gamma} \underset{t\to\infty}{\longrightarrow} 0 \quad (40)$$

for any $0 < \delta < 1/4 - \gamma$.

2.4. *A posteriori estimates*

In this subsection we give some extra estimates first on the asymptotic state u_+, and then on $u(t)$, the solution of (14) with initial condition $u(t_0) \in X_{t_0}$ and $t_0 \geq 1$. By Proposition 2.8 we already know that $u_+ \in L^2$ with

$$\|u_+\|_{L^2} \leq \|u(t)\|_{L^2} + C(a,\delta), \frac{(1+\log t_0) t_0^{1/2-(\gamma+\delta)}}{t^{1/4-(\gamma+\delta)}} \|u(t_0)\|_{X_{t_0}^\gamma}$$

for all $t \geq t_0 \geq 1$, and by using (36) we obtain the bound

$$\|u_+\|_{L^2} \leq C(a) t_0^{1/4} \|u(t_0)\|_{X_{t_0}^\gamma}. \quad (41)$$

Next we shall derive a control of the asymptotic state u_+ in the spirit of the one in Lemma 2.2 on the solution $u(t)$.

Lemma 2.10. *Let $\delta > 0$. The function u_+ satisfies for all $\xi \neq 0$ the estimate*

$$|\hat{u}_+(\xi)| \leq \left(C(a) + C(a,\delta) \frac{1 + \big|\log|\xi|\big|}{(\xi^2 t_0)^\delta} \right) \Big(|\hat{u}(t_0,\xi)| + |\hat{u}(t_0,-\xi)| \Big). \tag{42}$$

Proof. We have in L^2 and pointwise in Fourier variables,

$$u_+(x) = u(t_0,x) + ia^2 \int_{t_0}^{\infty} e^{-i\tau\partial_x^2} \frac{\overline{u(\tau,x)}}{\tau^{1\pm 2ia^2}}\, d\tau, \tag{43}$$

so $\hat{u}_+(\xi) = \hat{u}(t_0,\xi) + e^{it\xi^2} A_{t_0,\infty}(\xi)$ and

$$|\hat{u}_+(\xi)| \leq |\hat{u}(t_0,\xi)| + |A_{t_0,\infty}(\xi)|.$$

For the region $1/t_0 \leq \xi^2$ the conclusion follows immediately from Lemma 2.5. For $\xi^2 \leq 1/t_0$ we have shown in the proof of Proposition 2.8 that

$$|A_{t_0,\infty}(\xi)| \leq C(a,\delta) \frac{|\hat{u}(t_0,-\xi)| + |\hat{u}(t_0,\xi)|}{(\xi^2 t_0)^\delta} |\log \xi^2| + C(a,\delta) \frac{|\hat{u}(t_0,\xi)| + |\hat{u}(t_0,-\xi)|}{(\xi^2 t_0)^\delta},$$

and the lemma follows. □

In particular, for all $\xi^2 \leq 1/t_0$ we have

$$\frac{|\xi|^{2(\gamma+\delta)}}{1 + \big|\log|\xi|\big|} |\hat{u}_+(\xi)| \leq C(a,\delta)\, t_0^{1/2-(\gamma+\delta)} \|u(t_0)\|_{X^\gamma_{t_0}}$$

for any $\delta > 0$. So, if $t_0 = 1$, we get, for all $\xi^2 \leq 1$ and for any $\delta > 0$,

$$|\xi|^{2(\gamma+\delta)} |\hat{u}_+(\xi)| \leq C(a,\delta)\, \|u(1)\|_{X^\gamma_1}. \tag{44}$$

We end this section with a regularity property of the solutions of (14).

Proposition 2.11. *Under the assumptions of Proposition 2.8, the solution $u(t)$ belongs to $L^4((t_0,\infty), L^\infty)$ with the bound*

$$\|u\|_{L^4((t_0,\infty),L^\infty)} \leq C(a) t_0^{1/4} (1 + \log^2 t_0) \|u(t_0)\|_{X^\gamma_{t_0}},$$

and so does also $u(t) - e^{i(t-t_0)\partial_x^2} u_+$.

Proof. We use the Duhamel formulae

$$\begin{aligned} u(t) &= e^{i(t-t_0)\partial_x^2} u(t_0) + ia^2 \int_{t_0}^{t} e^{i(t-\tau)\partial_x^2} \frac{\overline{u(\tau)}}{\tau^{1\pm 2ia^2}}\, d\tau \\ &= e^{i(t-t_0)\partial_x^2} u(t_0) + ia^2 \int_{t_0}^{t} e^{i(t-\tau)\partial_x^2} \frac{\overline{u(\tau) - e^{i(\tau-t_0)\partial_x^2} u_+}}{\tau^{1\pm 2ia^2}}\, d\tau \\ &\quad + ia^2 \int_{t_0}^{t} e^{i(t-2\tau)\partial_x^2} \frac{e^{it_0\partial_x^2}\overline{u_+}}{\tau^{1\pm 2ia^2}}\, d\tau. \end{aligned}$$

Since $(4,\infty)$ is a Strichartz 1-d admissible couple, we can upper-bound the $L^4((t_0,\infty),L^\infty)$ norm of the first and of the second term by

$$M = C\|u(t_0)\|_{L^2} + a^2\int_{t_0}^\infty \frac{\|u(\tau)-e^{i(\tau-t_0)\partial_x^2}u_+\|_{L^2}}{\tau}\,d\tau,$$

and by using the rate of decay of Proposition 2.8, for some $0<\delta<1/4-\gamma$,

$$\begin{aligned} M &\leq C\|u(t_0)\|_{L^2} + C(a)(1+\log t_0)t_0^{1/2-(\gamma+\delta)}\|u(t_0)\|_{X_{t_0}}\int_{t_0}^\infty \frac{d\tau}{\tau^{5/4-(\gamma+\delta)}} \\ &\leq C(a)(1+\log t_0)t_0^{1/4}\,\|u(t_0)\|_{X^\gamma_{t_0}}. \end{aligned}$$

Therefore we only need to estimate the last term in $L^4((t_0,\infty),L^\infty)$. Let $\theta(x)$ be a cut-off function with $\theta(x)=0$ for $|x|<1/2$ and $\theta(x)=1$ for $|x|>1$. We decompose as usual the domain of the Fourier variable into three regions, $\xi^2\lesssim 1/t$, $1/t\leq\xi^2\leq 1/t_0$ and $1/t_0\leq\xi^2$,

$$\begin{aligned} \int_{t_0}^t e^{i(t-2\tau)\partial_x^2}\frac{e^{it_0\partial_x^2}\overline{u_+}}{\tau^{1\pm 2ia^2}}\,d\tau &= \int e^{ix\xi}e^{-it\xi^2}e^{-it_0\xi^2}\overline{\hat{u}_+(-\xi)}\int_{t_0}^t \frac{e^{i2\tau\xi^2}}{\tau^{1\pm 2ia^2}}\,d\tau\,d\xi \\ &= \int(1-\theta)(t\xi^2) + \int\theta(t\xi^2)(1-\theta)(t_0\xi^2) + \int\theta(t\xi^2)\,\theta(t_0\xi^2) = I+J+K. \end{aligned}$$

For I we integrate directly in τ,

$$|I(t)| \leq \int_{\xi^2\leq 1/t}|\hat{u}_+(-\xi)|\log t\,d\xi,$$

and we apply Lemma 2.10, for some $0<\delta<1/4-\gamma$, to obtain

$$\begin{aligned} |I(t)| &\leq C(a)\frac{\log t}{t_0^\delta}\int_{\xi^2\leq 1/t}\frac{1+\left|\log|\xi|\right|}{\xi^{2\delta}}(|\hat{u}(t_0,\xi)|+|\hat{u}(t_0,-\xi)|)\,d\xi \\ &\leq C(a)\frac{\left\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\right\|_{L^\infty(\xi^2\leq 1)}}{t_0^\delta}\,\frac{\log^2 t}{t^{1/2-(\gamma+\delta)}}. \end{aligned}$$

Then

$$\begin{aligned} \|I\|_{L^4((t_0,\infty),L^\infty)} &\leq C(a)\frac{\left\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\right\|_{L^\infty(\xi^2\leq 1)}}{t_0^\delta}\,\frac{1+\log^2 t_0}{t_0^{1/4-(\gamma+\delta)}} \\ &\leq C(a)t_0^{1/4}(1+\log^2 t_0)\|u(t_0)\|_{X^\gamma_{t_0}}. \end{aligned}$$

To treat J we first split the integral in τ into two parts:

$$\begin{aligned} J &= \int e^{ix\xi}e^{-it\xi^2}\theta(t\xi^2)\,(1-\theta)(t_0\xi^2)e^{-it_0\xi^2}\overline{\hat{u}_+(-\xi)}\int_{t_0}^{1/\xi^2}\frac{e^{i2\tau\xi^2}}{\tau}\,d\tau\,d\xi \\ &+ \int e^{ix\xi}e^{-it\xi^2}\theta(t\xi^2)(1-\theta)(t_0\xi^2)e^{-it_0\xi^2}\overline{\hat{u}_+(-\xi)}\int_{1/\xi^2}^t\frac{e^{i2\tau\xi^2}}{\tau}\,d\tau\,d\xi = J_1+J_2. \end{aligned}$$

We need the following lemma.

Lemma 2.12. *Define $U_t f$ as $\widehat{U_t f}(\xi) = \phi(\sqrt{|t|}\xi)e^{-it\xi^2}\hat{f}(\xi)$, with $\|\phi\|_{L^\infty} + \|\phi'\|_{L^1} \le C$. Then*

$$\|U_t f\|_{L^4_t L^\infty_x} \le C\|f\|_{L^2}.$$

Proof. The lemma follows from the usual TT^* argument and the elementary inequality

$$\int e^{-it\xi^2+ix\xi}\phi(\sqrt{|t|}\xi)\,d\xi \le \frac{C}{\sqrt{|t|}}(\|\phi\|_{L^\infty} + \|\phi'\|_{L^1}). \qquad \square$$

Therefore we get the following estimate for J_1:

$$\begin{aligned}\|J_1\|_{L^4((t_0,\infty),L^\infty)} &\le C\left\|(1-\theta)(t_0\,\xi^2)\,\overline{\hat{u}_+(-\xi)}\int_{t_0}^{1/\xi^2}\frac{e^{i2\tau\xi^2}}{\tau}\,d\tau\right\|_{L^2}\\ &\le C\left\|\hat{u}_+(-\xi)\log|\xi|\right\|_{L^2(\xi^2\le 1/t_0)}.\end{aligned}$$

Now we use Lemma 2.10 to get

$$\begin{aligned}\|J_1\|_{L^4((t_0,\infty),L^\infty)} &\le C(a)\frac{\left\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\right\|_{L^\infty(\xi^2\le 1)}}{t_0^\delta}\left\|\frac{1+\log^2|\xi|}{\xi^{2(\gamma+\delta)}}\right\|_{L^2(\xi^2\le 1/t_0)}\\ &\le C(a)\frac{1+\log^2 t_0}{t_0^{1/4-\gamma}}\left\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\right\|_{L^\infty(\xi^2\le 1)} \le C(a)t_0^{1/4}\,(1+\log^2 t_0)\,\|u(t_0)\|_{X^\gamma_{t_0}}.\end{aligned}$$

For J_2 we perform first the integration by parts

$$\begin{aligned}\int_{1/\xi^2}^t \frac{e^{i2\tau\xi^2}}{\tau}\,d\tau &= \frac{e^{i2\tau\xi^2}}{2i\xi^2\tau}\bigg|_{1/\xi^2}^t + \int_{1/\xi^2}^t \frac{e^{i2\tau\xi^2}}{2i\xi^2\tau^2}\,d\tau = \frac{e^{i2t\xi^2}}{2i\xi^2 t} - \frac{e^{i2}}{2i} + \int_1^{t\xi^2}\frac{e^{i2\tau}}{2i\tau^2}\,d\tau\\ &= \frac{e^{i2t\xi^2}}{2i\xi^2 t} - \int_{t\xi^2}^\infty \frac{e^{i2\tau}}{2i\tau^2}\,d\tau - \frac{e^{i2}}{2i} + \int_1^\infty \frac{e^{i2\tau}}{2i\tau^2}\,d\tau.\end{aligned}$$

Therefore

$$\begin{aligned}|J_2(t)| \le{}& \frac{C}{t}\int_{1/2t\le\xi^2\le 1/t_0}\frac{|\hat{u}_+(-\xi)|}{\xi^2}\,d\xi\\ &+ C\left|\int e^{ix\xi}e^{-it\xi^2}\theta(t\xi^2)(1-\theta)(t_0\xi^2)e^{-it_0\xi^2}\overline{\hat{u}_+(-\xi)}\,d\xi\right|.\end{aligned}$$

For the first term we use again Lemma 2.10 to get

$$\begin{aligned}&\frac{C}{t}\int_{1/2t\le\xi^2\le 1/t_0}\frac{|\hat{u}_+(-\xi)|}{\xi^2}\,d\xi\\ &\qquad\le C(a)\frac{\left\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\right\|_{L^\infty(\xi^2\le 1)}}{t t_0^\delta}\left\|\frac{1+\left|\log|\xi|\right|}{\xi^{2+2(\gamma+\delta)}}\right\|_{L^1(1/2t\le\xi^2\le 1/t_0)}\\ &\qquad\le C(a)\frac{1+\log t}{t^{1/2-(\gamma+\delta)}}\,\frac{\left\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\right\|_{L^\infty(\xi^2\le 1)}}{t_0^\delta}.\end{aligned}$$

The second term of J_2 is similar to a linear evolution as J_1. We obtain

$$\|J_2\|_{L^4((t_0,\infty),L^\infty)} \leq C(a)\frac{1+\log^2 t_0}{t_0^{1/4-\gamma}}\left\||\xi|^{2\gamma}\hat{u}(t_0,\xi)\right\|_{L^\infty(\xi^2\leq 1)},$$

so

$$\|J\|_{L^4((t_0,\infty),L^\infty)} \leq C(a)t_0^{1/4}(1+\log^2 t_0)\|u(t_0)\|_{X_{t_0}^\gamma}.$$

For K we use again the integration by parts

$$\int_{t_0}^t \frac{e^{i2\tau\xi^2}}{\tau}\,d\tau = \frac{e^{i2t\xi^2}}{2i\xi^2 t} - \int_{t\xi^2}^\infty \frac{e^{i2\tau}}{2i\tau^2}\,d\tau - \frac{e^{i2t_0\xi^2}}{2i\xi^2 t_0} + \int_{t_0}^\infty \frac{e^{i2\tau\xi^2}}{2i\xi^2\tau^2}\,d\tau,$$

hence

$$|K(t)| \leq \frac{C}{t}\int_{1/2t_0\leq\xi^2}\frac{|\hat{u}_+(-\xi)|}{\xi^2}\,d\xi$$
$$+\left|\int e^{ix\xi}e^{-it\xi^2}\theta(t\xi^2)\theta(t_0\xi^2)e^{-it_0\xi^2}\overline{\hat{u}_+(-\xi)}\left(-\frac{e^{i2t_0\xi^2}}{2i\xi^2 t_0}+\int_{t_0}^\infty\frac{e^{i2\tau\xi^2}}{2i\xi^2\tau^2}\,d\tau\right)d\xi\right|.$$

By Cauchy–Schwarz's inequality, the first term is upper-bounded by $C\frac{t_0^{3/4}}{t}\|u_+\|_{L^2}$. By (41) this in turn is smaller than $C(a)\frac{t_0}{t}\|u(t_0)\|_{X_0^\gamma}$. We get again, as for J_2,

$$\|K\|_{L^4((t_0,\infty),L^\infty)} \leq C(a)t_0^{1/4}(1+\log^2 t_0)\|u(t_0)\|_{X_{t_0}^\gamma}.$$

Summarizing, we have obtained the desired estimate

$$\|u\|_{L^4((t_0,\infty),L^\infty)} \leq C(a)t_0^{1/4}(1+\log^2 t_0)\|u(t_0)\|_{X_{t_0}^\gamma}.$$

The Strichartz inequalities for a free evolution together with (41) give

$$\|e^{i(t-t_0)\partial_x^2}u_+\|_{L^4((t_0,\infty),L^\infty)} \leq C\|u_+\|_{L^2} \leq C(a)t_0^{1/4}\|u(t_0)\|_{X_{t_0}^\gamma},$$

so we also have

$$\|u(t)-e^{i(t-t_0)\partial_x^2}u_+\|_{L^4((t_0,\infty),L^\infty)} \leq C(a)t_0^{1/4}(1+\log^2 t_0)\|u(t_0)\|_{X_{t_0}^\gamma}. \qquad \square$$

Lemma 2.10 is a pointwise estimate for Fourier transforms, so it fits for higher order derivatives. Again by linearity we have the results of Proposition 2.11 at higher Sobolev order, if $\partial_x^k u(t_0) \in X(t_0)$: $\partial_x^k u(t)$ belongs to $L^4((t_0,\infty),L^\infty)$ with the bound

$$\|\partial_x^k u\|_{L^4((t_0,\infty),L^\infty)} \leq C(a)t_0^{1/4}(1+\log^2 t_0)\,\|\partial_x^k u(t_0)\|_{X_{t_0}^\gamma}.$$

3. Scattering for the nonlinear equation

In this section we prove Theorem 1.1. By using the results on the linear equation (14) obtained in the previous section, we first infer in §3.1 a global existence result for the nonlinear equation (10). Then we prove in §3.2 asymptotic completeness for these solutions. In the last subsection we give new information about the regularity of the asymptotic state, which completes the proof of Theorem 1.1.

We start by writing the nonlinear solutions of (10) in terms of solutions of the linear equation (14). Let us notice that the estimates obtained in the previous section are independent of the sign in (14), so in what follows we shall consider only one of the signs—the other case can be treated the same way. We denote by $S(t, t_0)f$ the solution of (14) with a plus sign,

$$iu_t + u_{xx} + \frac{a^2}{t^{1+2ia^2}}\overline{u} = 0,$$

with initial data f at time $t_0 \geq 1$. With this notation, for $t_0 \leq t$ we have the estimates (36) of Proposition 2.6,

$$\|S(t, t_0)f\|_{L^2} \leq C(a)t_0^{1/4}\|f\|_{X^\gamma_{t_0}}, \tag{45}$$

and

$$\left\||\xi|^{2\gamma}\widehat{S(t, t_0)f}(\xi)\right\|_{L^\infty(\xi^2\leq 1)} \leq C(t/t_0)^{a^2}\left\||\xi|^{2\gamma}\hat{f}(\xi)\right\|_{L^\infty(\xi^2\leq 1)}, \tag{46}$$

and the one of Proposition 2.11,

$$\|S(\cdot, t_0)f\|_{L^4((t_0,\infty),L^\infty)} \leq C(a)t_0^{1/4}(1 + \log^2 t_0)\|f\|_{X^\gamma_{t_0}}, \tag{47}$$

as well as all their equivalents for higher order derivatives, if $\partial_x^k f \in X^\gamma_{t_0}$.

Now the solution of

$$u_t = i\left(u_{xx} + \frac{a^2}{t^{1+2ia^2}}\overline{u} + \frac{F}{t}\right)$$

with initial data $u(1)$ at time $t = 1$ reads

$$u(t, x) = S(t, 1)u(1) + \int_1^t S(t, \tau)\frac{iF(\tau)}{\tau}\, d\tau. \tag{48}$$

It is enough to verify this formula for $u(1) = 0$. Indeed,

$$\begin{aligned}
\partial_t u &= \partial_t \int_1^t S(t, \tau)\frac{iF(\tau)}{\tau}\, d\tau \\
&= i\frac{F}{t} + \int_1^t i\left(\partial_{xx}S(t, \tau)\frac{iF(\tau)}{\tau} + \frac{a^2}{t^{1+2ia^2}}\overline{S(t, \tau)\frac{iF(\tau)}{\tau}}\right) d\tau \\
&= i\left(\frac{F}{t} + u_{xx} + \frac{a^2}{t^{1+2ia^2}}\overline{u}\right).
\end{aligned}$$

In our case of (10), F is composed of cubic and quadratic powers of u.

3.1. Global existence

Let us recall again the definitions of the norms of X_1^γ and Y_1^γ, for $0 \leq \gamma < 1/4$:

$$\|f\|_{X_1^\gamma} = \|f\|_{L^2} + \left\||\xi|^{2\gamma}\hat{f}(\xi)\right\|_{L^\infty(\xi^2\leq 1)},$$

$$\|g\|_{Y_1^\gamma} = \sup_{t\geq 1}\left(\|g(t)\|_{L^2} + \frac{1}{t^{a^2}}\left\||\xi|^{2\gamma}\hat{g}(t,\xi)\right\|_{L^\infty(\xi^2\leq 1)}\right).$$

We have the following global existence result on the nonlinear equation (10).

Proposition 3.1. *Let $u(1)$ be a function in X_1^γ small with respect to a. Then there exists a unique global solution $u \in Z^\gamma = Y_1^\gamma \cap L^4((1,\infty), L^\infty)$ of equation (10) with initial data $u(1)$ at time $t = 1$, and*

$$\|u\|_{Z^\gamma} \leq C(a)\|u(1)\|_{X_1^\gamma}.$$

Proof. In view of (48) we shall prove the proposition by a fixed point argument in Z^γ for the operator

$$\Phi(u)(t) = S(t,1)u(1) + \int_1^t S(t,\tau)\frac{iF(u(\tau))}{\tau}\,d\tau.$$

The estimates (45)–(47) ensure that

$$\|S(t,1)u(1)\|_{Z^\gamma} \leq C(a)\|u(1)\|_{X_1^\gamma}.$$

We start with a property that we shall frequently use in the following.

Lemma 3.2. *Let $u \in Z^\gamma$ and $\alpha < 1/2 - \gamma$. Then for $1 \leq t_1 \leq t_2$,*

$$\int_{t_1}^{t_2} \tau^\alpha \frac{\|F(u(\tau))\|_{X_\tau^\gamma}}{\tau}\,d\tau \leq C\frac{\sum_{j\in\{1,2\}}(a\|u\|^2_{L^{p_j}((t_1,t_2),L^{q_j})} + \|u\|^3_{L^{p_j}((t_1,t_2),L^{q_j})})}{t_1^{1/2-\alpha-\gamma}}$$

$$\leq C\frac{a\|u\|^2_{Z^\gamma} + \|u\|^3_{Z^\gamma}}{t_1^{1/2-\alpha-\gamma}},$$

where $(p_1,q_1) = (\infty,2)$ and $(p_2,q_2) = (4,\infty)$.

Proof. By definition (12) of X_τ^γ, and since $|\hat{f}(\xi)| \leq \|f\|_{L^1}$, we get

$$\int_{t_1}^{t_2} \tau^\alpha \frac{\|F(u(\tau))\|_{X_\tau^\gamma}}{\tau}\,d\tau$$

$$= \int_{t_1}^{t_2}\left(\frac{1}{\tau^{1/4}}\|F(u(\tau))\|_{L^2} + \frac{\tau^\gamma}{\sqrt{\tau}}\left\||\xi|^{2\gamma}\widehat{F(u(\tau))}\right\|_{L^\infty(\xi^2\leq 1)}\right)\frac{d\tau}{\tau^{1-\alpha}}$$

$$\leq ca\int_{t_1}^{t_2}\|u(\tau)\|^2_{L^4}\frac{d\tau}{\tau^{5/4-\alpha}} + ca\int_{t_1}^{t_2}\|u(\tau)\|^2_{L^2}\frac{d\tau}{\tau^{3/2-\alpha-\gamma}}$$

$$+ c\int_{t_1}^{t_2}\|u(\tau)\|^3_{L^6}\frac{d\tau}{\tau^{5/4-\alpha}} + c\int_{t_1}^{t_2}\|u(\tau)\|^3_{L^3}\frac{d\tau}{\tau^{3/2-\alpha-\gamma}}.$$

We apply Hölder's inequality L^4-$L^{4/3}$ in the first and the last integral, and Cauchy–Schwarz's inequality for the third one, to obtain

$$\int_{t_1}^{t_2} \tau^{\alpha} \frac{\|F(u(\tau))\|_{X_\tau^\gamma}}{\tau}\, d\tau$$
$$\leq ca\|u\|^2_{L^8((t_1,t_2),L^4)} \left\| \frac{1}{\tau^{5/4-\alpha}} \right\|_{L^{4/3}(t_1,t_2)} + ca\|u\|^2_{L^\infty((1,\infty),L^2)} \int_{t_1}^{t_2} \frac{d\tau}{\tau^{3/2-\alpha-\gamma}}$$
$$+ c\|u\|^3_{L^6((t_1,t_2),L^6)} \left\| \frac{1}{\tau^{5/4-\alpha}} \right\|_{L^2(t_1,t_2)} + c\|u\|^3_{L^{12}((t_1,t_2),L^3)} \left\| \frac{1}{\tau^{3/2-\alpha-\gamma}} \right\|_{L^{4/3}(t_1,t_2)}.$$

The spaces L^8L^4, L^6L^6 and $L^{12}L^3$ are interpolation spaces between $L^\infty L^2$ and L^4L^∞, therefore the lemma follows. □

Let $u \in Z^\gamma$. The L^4L^∞ norm of the integral in $\Phi(u)$ can be bounded by

$$\left\| a \int_1^t S(t,\tau) \frac{iF(u(\tau))}{\tau}\, d\tau \right\|_{L^4((1,\infty),L^\infty)} \leq a \int_1^\infty \left\| S(t,\tau) \frac{iF(u(\tau))}{\tau} \right\|_{L^4((\tau,\infty),L^\infty)} d\tau.$$

By using (47),

$$\left\| a \int_1^t S(t,\tau) \frac{iF(u(\tau))}{\tau}\, d\tau \right\|_{L^4((1,\infty),L^\infty)} \leq C(a) \int_1^\infty \tau^{1/4}(1+\log^2\tau) \frac{\|F(u(\tau))\|_{X_\tau^\gamma}}{\tau}\, d\tau,$$

so Lemma 3.2 with $\alpha = (1/4)^+$ gives us

$$\left\| a \int_1^t S(t,\tau) \frac{iF(u(\tau))}{\tau}\, d\tau \right\|_{L^4((1,\infty),L^\infty)} \leq C(a)(\|u\|^2_{Z^\gamma} + \|u\|^3_{Z^\gamma}).$$

Next we upper-bound the $L^\infty L^2$ norm:

$$\left\| a \int_1^t S(t,\tau) \frac{iF(u(\tau))}{\tau}\, d\tau \right\|_{L^2} \leq a \int_1^\infty \left\| S(t,\tau) \frac{iF(u(\tau))}{\tau} \right\|_{L^2} d\tau,$$

and by using (45),

$$\left\| a \int_1^t S(t,\tau) \frac{iF(u(\tau))}{\tau}\, d\tau \right\|_{L^2} \leq C(a) \int_1^t \tau^{1/4} \frac{\|F(u(\tau))\|_{X_\tau^\gamma}}{\tau}\, d\tau.$$

Again, Lemma 3.2 with $\alpha = 1/4$ gives us

$$\left\| a \int_1^t S(t,\tau) \frac{iF(u(\tau))}{\tau}\, d\tau \right\|_{L^2} \leq C(a)(\|u\|^2_{Z^\gamma} + c\,\|u\|^3_{Z^\gamma}).$$

Finally, we compute (the contribution of the other quadratic term $|u|^2$ can be treated the same)

$$\frac{1}{t^{a^2}}\left\||\xi|^{2\gamma}\mathcal{F}\left(a\int_1^t S(t,\tau)\frac{iu^2(\tau)}{\tau^{1-ia^2}}\,d\tau\right)\right\|_{L^\infty(\xi^2\le 1)} \le \frac{a}{t^{a^2}}\int_1^t\left\||\xi|^{2\gamma}\mathcal{F}\left(S(t,\tau)\frac{iu^2(\tau)}{\tau^{1-ia^2}}\right)\right\|_{L^\infty(\xi^2\le 1)}d\tau,$$

and by (46)

$$\frac{1}{t^{a^2}}\left\||\xi|^{2\gamma}\mathcal{F}\left(a\int_1^t S(t,\tau)\frac{iu^2(\tau)}{\tau^{1-ia^2}}\,d\tau\right)\right\|_{L^\infty(\xi^2\le 1)} \le \frac{Ca}{t^{a^2}}\int_1^t\left(\frac{t}{\tau}\right)^{a^2}\left\||\xi|^{2\gamma}\widehat{u^2}(\tau,\xi)\right\|_{L^\infty(\xi^2\le 1)}\frac{d\tau}{\tau} \le Ca\|u\|^2_{L^\infty((1,\infty),L^2)}\int_1^\infty\frac{d\tau}{\tau^{1+a^2}} \le \frac{C}{a}\|u\|^2_{Z^\gamma}.$$

Also, by (46) and by Hölder's inequality,

$$\frac{1}{t^{a^2}}\left\||\xi|^{2\gamma}\mathcal{F}\left(\int_1^t S(t,\tau)\frac{i|u|^2u(\tau)}{\tau}\,d\tau\right)\right\|_{L^\infty(\xi^2\le 1)} \le \frac{C}{t^{a^2}}\int_1^t\left\||\xi|^{2\gamma}\mathcal{F}\left(S(t,\tau)\frac{i|u|^2u(\tau)}{\tau}\right)\right\|_{L^\infty(\xi^2\le 1)}d\tau$$
$$\le \frac{C}{t^{a^2}}\int_1^t\left(\frac{t}{\tau}\right)^{a^2}\left\||\xi|^{2\gamma}\widehat{|u|^2u}(\tau,\xi)\right\|_{L^\infty(\xi^2\le 1)}\frac{d\tau}{\tau} \le C\int_1^\infty\|u(\tau)\|^3_{L^3}\frac{d\tau}{\tau^{1+a^2}} \le C\|u\|^3_{L^{12}((1,\infty),L^3)}.$$

So we have shown that the contribution of the quadratic and cubic term is in Z^γ,

$$\left\|\int_1^t S(t,\tau)\frac{iF(u(\tau))}{\tau}\,d\tau\right\|_{Z^\gamma} \le C(a)(\|u\|^2_{Z^\gamma}+\|u\|^3_{Z^\gamma}).$$

Summarizing, we have

$$\|\Phi(u)\|_{Z^\gamma} \le C(a)(\|u(1)\|_{X_1^\gamma}+\|u\|^2_{Z^\gamma}+\|u\|^3_{Z^\gamma}),$$

so for $u(1)\in X_1^\gamma$ small with respect to a, by the fixed point argument we get a global solution $u\in Z^\gamma$ of (10) with norm bounded by

$$\|u\|_{Z^\gamma} \le C(a)\|u(1)\|_{X_1^\gamma}.$$ □

We now state the result in Sobolev spaces. This is a direct corollary of Proposition 3.1, by using the Leibniz rule and the fact that estimating the Fourier norm in Z^γ on derivative terms creates powers of ξ which are bounded by 1.

Corollary 3.3. *Let $s \in \mathbb{N}$. Let $\partial_x^k u(1)$ be a function in X_1^γ small with respect to a, for all $0 \le k \le s$. Then there exists a unique global solution $u \in Z^\gamma = Y_1^\gamma \cap L^4((1,\infty), L^\infty)$, with $\partial_x^k u \in Z^\gamma$, of equation (10) with initial data $u(1)$ at time $t = 1$, and*

$$\sum_{0\le k\le s} \|\partial_x^k u\|_{Z^\gamma} \le C(a) \sum_{0\le k\le s} \|\partial_x^k u(1)\|_{X_1^\gamma}.$$

3.2. Asymptotic completeness

Now we prove the second part of Theorem 1.1, namely the asymptotic completeness of the global solutions obtained by Proposition 3.1.

Proposition 3.4. *Let $u(1)$ be a function in X_1^γ small with respect to a. Then the unique global solution $u \in Z^\gamma = Y_1^\gamma \cap L^4((1,\infty), L^\infty)$ of equation (10) with initial data $u(1)$ at time $t = 1$ scatters in L^2. More precisely, there exists $f_+ \in L^2$ for which*

$$\|u(t) - e^{i(t-1)\partial_x^2} f_+\|_{L^2} \le \frac{C(a,\delta)}{t^{1/4-(\gamma+\delta)}} \|u(1)\|_{X_1^\gamma} \xrightarrow[t\to\infty]{} 0 \tag{49}$$

for any $0 < \delta < 1/4 - \gamma$.

Proof. The nonlinear solution reads

$$u(t) = S(t,1)u(1) + \int_1^t S(t,\tau)\frac{iF(u(\tau))}{\tau}\, d\tau.$$

The scattering result of Proposition 2.8 guarantees the existence of $u_+ \in L^2$ such that

$$\|S(t,1)\, u(1) - e^{i(t-1)\partial_x^2} u_+\|_{L^2} \le \frac{C(a,\tilde\delta)}{t^{1/4-(\gamma+\tilde\delta)}} \|u(1)\|_{X_1^\gamma},$$

for some $\tilde\delta$ to be chosen later. Since $u \in Z^\gamma$ we have $F(u(\tau)) \in X_\tau^\gamma$ a.e. and we can again apply Proposition 2.8. There exists $u_+(\tau) \in L^2$ such that

$$\|S(t,\tau) i F(u(\tau)) - e^{i(t-\tau)\partial_x^2} i u_+(\tau)\|_{L^2} \le C(a,\tilde\delta) \frac{(1+\log\tau)\tau^{1/2-(\gamma+\tilde\delta)}}{t^{1/4-(\gamma+\tilde\delta)}} \|F(u(\tau))\|_{X_\tau^\gamma}.$$

In view of (43) the expression of $u_+(\tau)$ is

$$u_+(\tau) = F(u(\tau)) + a^2 \int_\tau^\infty e^{-is\partial_x^2} \frac{\overline{S(s,\tau) i F(u(\tau))}}{s^{1+2ia^2}}\, ds. \tag{50}$$

We define

$$f_+ = u_+ + i\int_1^\infty e^{-i(\tau-1)\partial_x^2} u_+(\tau)\, \frac{d\tau}{\tau} \tag{51}$$

and we have

$$
\begin{aligned}
&u(t) - e^{i(t-1)\partial_x^2} f_+ \\
&\quad = S(t,1)u(1) - e^{i(t-1)\partial_x^2} u_+ + \int_1^t S(t,\tau) i F(u(\tau)) \frac{d\tau}{\tau} - i \int_1^\infty e^{i(t-\tau)\partial_x^2} u_+(\tau) \frac{d\tau}{\tau} \\
&\quad = S(t,1)\, u(1) - e^{i(t-1)\partial_x^2} u_+ + \int_1^t \big(S(t,\tau) i F(u(\tau)) - e^{i(t-\tau)\partial_x^2} i u_+(\tau)\big) \frac{d\tau}{\tau} \\
&\qquad - i \int_t^\infty e^{i(t-\tau)\partial_x^2} u_+(\tau) \frac{d\tau}{\tau}.
\end{aligned}
$$

The first term has the right decay in L^2, and the second is upper-bounded by

$$
\begin{aligned}
&\left\| \int_1^t \big(S(t,\tau) i F(u(\tau)) - e^{i(t-\tau)\partial_x^2} i u_+(\tau)\big) \frac{d\tau}{\tau} \right\|_{L^2} \\
&\qquad \le C(a,\tilde\delta) \int_1^t \frac{(1+\log\tau)\tau^{1/2-(\gamma+\tilde\delta)}}{t^{1/4-(\gamma+\tilde\delta)}} \|F(u(\tau))\|_{X_\tau^\gamma} \frac{d\tau}{\tau},
\end{aligned}
$$

so we can use Lemma 3.2 with $\alpha = 1/2 - (\gamma + \tilde\delta)$,

$$
\left\| \int_1^t \big(S(t,\tau) i F(u(\tau)) - e^{i(t-\tau)\partial_x^2} i u_+(\tau)\big) \frac{d\tau}{\tau} \right\|_{L^2} \le C(a,\tilde\delta) \frac{1+\log t}{t^{1/4-(\gamma+\tilde\delta)}} (\|u\|_{Z^\gamma}^2 + \|u\|_{Z^\gamma}^3).
$$

For the last term we use (41),

$$
\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} u_+(\tau) \frac{d\tau}{\tau} \right\|_{L^2} \le \int_t^\infty \|u_+(\tau)\|_{L^2} \frac{d\tau}{\tau} \le C(a) \int_t^\infty \tau^{1/4} \|F(u(\tau))\|_{X_\tau^\gamma} \frac{d\tau}{\tau},
$$

and again Lemma 3.2 with $\alpha = 1/4$,

$$
\left\| \int_t^\infty e^{i(t-\tau)\partial_x^2} u_+(\tau) \frac{d\tau}{\tau} \right\|_{L^2} \le C(a) \frac{\|u\|_{Z^\gamma}^2 + \|u\|_{Z^\gamma}^3}{t^{1/4-\gamma}}.
$$

In conclusion we have

$$
\begin{aligned}
\|u(t) - e^{i(t-1)\partial_x^2} f_+\|_{L^2} &\le C(a,\tilde\delta) \frac{1+\log t}{t^{1/4-(\gamma+\tilde\delta)}} (\|u(1)\|_{X_1^\gamma} + \|u\|_{Z^\gamma}^2 + \|u\|_{Z^\gamma}^3) \\
&\le C(a,\tilde\delta) \frac{1+\log t}{t^{1/4-(\gamma+\tilde\delta)}} (\|u(1)\|_{X_1^\gamma} + \|u(1)\|_{X_1^\gamma}^2 + \|u(1)\|_{X_1^\gamma}^3) \\
&\le C(a,\tilde\delta) \frac{1+\log t}{t^{1/4-(\gamma+\tilde\delta)}} \|u(1)\|_{X_1^\gamma},
\end{aligned}
$$

and the proposition follows by choosing $0 < \tilde\delta < \delta < 1/4 - \gamma$. □

Similarly we also get the statement for Sobolev spaces.

Corollary 3.5. *Let $s \in \mathbb{N}$. Let $u(1)$ be a function in X_1^γ such that $\partial_x^k u(1) \in X_1^\gamma$ for all $0 \le k \le s$, small with respect to a. Then the unique global solution $u \in Y_1^\gamma$ of equation (14) with initial data $u(1)$ at time $t = 1$, with $\partial_x^k u \in Y_1^\gamma$ for all $0 \le k \le s$, scatters in H^s. More precisely, there exists $f_+ \in H^s$ such that*

$$\|u(t) - e^{i(t-1)\partial_x^2} f_+\|_{H^s} \le \frac{C(a,\delta)}{t^{1/4-(\gamma+\delta)}} \sum_{0 \le k \le s} \|\partial_x^k u(1)\|_{X_1^\gamma} \xrightarrow[t\to\infty]{} 0 \tag{52}$$

for any $0 < \delta < 1/4 - \gamma$.

3.3. Regularity of the asymptotic state

As an extra information on f_+, we have the following result, in the spirit of (44). It completes the proof of Theorem 1.1.

Proposition 3.6. *If $\|u(1)\|_{X_1^\gamma}$ is small enough with respect to a, the function f_+ satisfies, for all $\xi^2 \le 1$ and $0 < \delta < 1/4 - \gamma$,*

$$|\xi|^{2(\gamma+\delta)} |\hat{f}_+(\xi)| \le C(a,\delta) \|u(1)\|_{X_1^\gamma}.$$

Proof. By the definition (51) of f_+, we have

$$\begin{aligned}
&|\xi|^{2(\gamma+\delta)} \hat{f}_+(\xi)\\
&\quad = |\xi|^{2(\gamma+\delta)} \left(\hat{u}_+(\xi) + i \int_1^{1/\xi^2} e^{i(\tau-1)\xi^2} \hat{u}_+(\tau,\xi) \frac{d\tau}{\tau} + i \int_{1/\xi^2}^{\infty} e^{i(\tau-1)\xi^2} \hat{u}_+(\tau,\xi) \frac{d\tau}{\tau} \right),
\end{aligned}$$

so on $\xi^2 \le 1$ the estimate (44) ensures that the first term is upper-bounded by $C(a,\delta)\|u(1)\|_{X_1^\gamma}$.

By Lemma 2.10 and (50) we can treat the first integral:

$$\begin{aligned}
&\int_1^{1/\xi^2} |\xi|^{2(\gamma+\delta)} |\hat{u}_+(\tau,\xi)| \frac{d\tau}{\tau}\\
&\quad \le \int_1^{1/\xi^2} \frac{C(a,\delta)}{\tau^\delta} |\xi|^{2\gamma} \left(1 + \big|\log|\xi|\big|\right) \left(|\widehat{F(u(\tau))}(\xi)| + |\widehat{F(u(\tau))}(-\xi)| \right) \frac{d\tau}{\tau}\\
&\quad \le C(a,\delta) \int_1^{1/\xi^2} (\|u(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^3}^3) \frac{1+\log\tau}{\tau^{1+\gamma+\delta}} \, d\tau.
\end{aligned}$$

As usual, we use Hölder's inequality for the second term to get

$$\begin{aligned}
\int_1^{1/\xi^2} |\xi|^{2(\gamma+\delta)} |\hat{u}_+(\tau,\xi)| \frac{d\tau}{\tau} &\le C(a,\delta) \|u\|_{L^\infty((1,\infty),L^2)}^2 \int_1^{1/\xi^2} \frac{1+\log\tau}{\tau^{1+\gamma+\delta}} \, d\tau\\
&\quad + C(a,\delta) \|u\|_{L^{12}((1,\infty),L^3)}^3 \left\| \frac{1+\log\tau}{\tau^{1+\gamma+\delta}} \right\|_{L^{4/3}(1,1/\xi^2)}\\
&\le C(a,\delta)(\|u\|_{Z^\gamma}^2 + \|u\|_{Z^\gamma}^3) \le C(a,\delta)\|u(1)\|_{X_1^\gamma}.
\end{aligned}$$

It remains to estimate the last integral which, in view of (50), is

$$\begin{aligned}
&|\xi|^{2(\gamma+\delta)}\int_{1/\xi^2}^{\infty} e^{i(\tau-1)\xi^2}\hat{u}_+(\tau,\xi)\,\frac{d\tau}{\tau}\\
&\quad= |\xi|^{2(\gamma+\delta)}e^{-i\xi^2}\int_{1/\xi^2}^{\infty} e^{i\tau\xi^2}\left(\widehat{F(u(\tau))}(\xi)+a^2\int_\tau^\infty e^{is\xi^2}\frac{\overline{\widehat{S(s,\tau)iF(u(\tau))}(\xi)}}{s^{1+2ia^2}}\,ds\right)\frac{d\tau}{\tau}\\
&\quad= I_1(\xi)+I_2(\xi)+I_3(\xi), \qquad (53)
\end{aligned}$$

where we denote by I_1 the cubic contributions of $\widehat{F(u(\tau))}$, by I_2 the quadratic ones, and by I_3 the double integral. By Lemma 2.5 applied for some $\tilde{\delta}>0$, since $\xi^2\leq 1$,

$$\begin{aligned}
|I_3(\xi)| &\leq |\xi|^{2(\gamma+\delta)}\int_{1/\xi^2}^{\infty} C(a)\frac{|\widehat{F(u(\tau))}(\xi)|+|\widehat{F(u(\tau))}(-\xi)|}{\xi^2\tau}\,\frac{d\tau}{\tau}\\
&\leq C(a)\frac{|\xi|^{2(\gamma+\delta)}}{\xi^2}\int_{1/\xi^2}^{\infty}(\|u(\tau)\|_{L^2}^2+\|u(\tau)\|_{L^3}^3)\,\frac{d\tau}{\tau^2}\\
&\leq C(a)\frac{|\xi|^{2(\gamma+\delta)}}{\xi^2}\left(\|u\|^2_{L^\infty((1,\infty),L^2)}\int_{1/\xi^2}^{\infty}\frac{d\tau}{\tau^2}+\|u\|^3_{L^{12}((1,\infty),L^3)}\left\|\frac{1}{\tau^2}\right\|_{L^{4/3}(1/\xi^2,\infty)}\right)\\
&\leq C(a)\|u(1)\|_{X_1^\gamma}.
\end{aligned}$$

On $\xi^2\leq 1$ we have

$$|I_1(\xi)|\leq\int_1^\infty\|u(\tau)\|_{L^3}^3\,\frac{d\tau}{\tau}\leq C\|u\|^3_{L^{12}((1,\infty),L^3)}\leq C(a)\|u\|^3_{Z^\gamma}\leq C(a)\|u(1)\|_{X_1^\gamma}.$$

For the quadratic terms I_2 we first notice that quadratic powers of $u(\tau)$ can be replaced by the quadratic powers of $e^{i(\tau-1)\partial_x^2}f_+$, because, in view of Proposition 3.4,

$$\begin{aligned}
&a\left|\int_{t_1}^{t_2} e^{i\tau\xi^2}\left(\mathcal{F}u^2(\tau,\xi)-\mathcal{F}(e^{i(\tau-1)\partial_x^2}f_+)^2(\xi)\right)\frac{d\tau}{\tau^{1-ia^2}}\right|\\
&\qquad\leq a\int_{t_1}^{t_2}\|u^2(\tau)-(e^{i(\tau-1)\partial_x^2}f_+)^2\|_{L^1}\,\frac{d\tau}{\tau}\leq\frac{C(a,\delta)}{t_1^{1/4-(\gamma+\delta)}}\|u(1)\|_{X_1^\gamma}.
\end{aligned}$$

Therefore we have obtained, for $\xi^2\leq 1$,

$$\begin{aligned}
|\xi|^{2(\gamma+\delta)}|\hat{f}_+(\xi)| &\leq C(a,\delta)\|u(1)\|_{X_1^\gamma}\\
&\quad+a|\xi|^{2(\gamma+\delta)}\left|\int_{1/\xi^2}^{\infty} e^{i\tau\xi^2}\,\frac{\mathcal{F}(e^{i(\tau-1)\partial_x^2}f_+)^2(\xi)+2\mathcal{F}|e^{i(\tau-1)\partial_x^2}f_+|^2(\xi)}{\tau^{1-ia^2}}\,d\tau\right|.
\end{aligned}$$

By writing explicitly the Fourier transforms, we get

$$\begin{aligned}
|\xi|^{2(\gamma+\delta)}|\hat{f}_+(\xi)| &\leq C(a,\delta)\|u(1)\|_{X_1^\gamma}\\
&\quad+\sum_{j\in\{1,2\}}a|\xi|^{2(\gamma+\delta)}\int|\hat{f}_+(\eta)|\,|\hat{f}_+(\xi-\eta)|\left|\int_{1/\xi^2}^{\infty} e^{i\tau h_j(\xi,\eta)}\,\frac{d\tau}{\tau^{1-ia^2}}\right|d\eta, \qquad (54)
\end{aligned}$$

where

$$h_1(\xi,\eta)=2\eta(\xi-\eta),\qquad h_2(\xi,\eta)=2\xi(\xi-\eta).$$

By integrating by parts, for $\eta\neq\xi$, $\eta\neq 0$,

$$\int_{1/\xi^2}^{\infty} e^{i\tau h_j(\xi,\eta)}\,\frac{d\tau}{\tau^{1-ia^2}}=\frac{e^{i\tau h_j(\xi,\eta)}}{ih_j(\xi,\eta)\tau^{1-ia^2}}\bigg|_{1/\xi^2}^{\infty}+(1-ia^2)\int_{1/\xi^2}^{\infty}\frac{e^{i\tau h_j(\xi,\eta)}}{ih_j(\xi,\eta)}\,\frac{d\tau}{\tau^{2-ia^2}}.$$

On one hand, if $|h_j(\xi,\eta)|\geq c\xi^2$ for some positive constant c, we get the uniform estimate

$$\left|\int_{1/\xi^2}^{\infty} e^{i\tau h_j(\xi,\eta)}\,\frac{d\tau}{\tau^{1-ia^2}}\right|\leq C(a).$$

On the other hand, in the region $|h_j(\xi,\eta)|\leq c\xi^2$, the integral from $1/(\xi^2|h_j(\xi,\eta)|)$ to infinity can be treated the same way. Finally, since

$$\left|\int_{1/\xi^2}^{1/(\xi^2|h_j(\xi,\eta)|)} e^{i\tau h_j(\xi,\eta)}\,\frac{d\tau}{\tau^{1-ia^2}}\right|\leq\big|\log|h_j(\xi,\eta)|\big|,$$

we get

$$\left|\int_{1/\xi^2}^{\infty} e^{i\tau h_j(\xi,\eta)}\,\frac{d\tau}{\tau^{1-ia^2}}\right|\leq C(a)+\big|\log|h_j(\xi,\eta)|\big|\mathbb{I}_{|h_j(\xi,\eta)|\leq c|\xi|^2}.$$

Summarizing, we have obtained

$$\begin{aligned}|\xi|^{2(\gamma+\delta)}|\hat{f}_+(\xi)|&\leq C(a,\delta)\|u(1)\|_{X_1^\gamma}\\&\quad+\big(C(a)+\big|\log|\xi|\big|\big)|\xi|^{2(\gamma+\delta)}\int|\hat{f}_+(\eta)|\,|\hat{f}_+(\xi-\eta)|\,d\eta\\&\quad+2a|\xi|^{2(\gamma+\delta)}\int_{|\eta|<C}|\hat{f}_+(\eta)|\,|\hat{f}_+(\xi-\eta)|\,\big|\log|\eta|\big|\,d\eta.\end{aligned}\tag{55}$$

We have also used here the fact that since $|\xi|<1$, both $|\eta|$ and $|\eta-\xi|$ are bounded in the regions $|h_j(\xi,\eta)|\leq c|\xi|^2$.

The function f_+ is in L^2 with norm bounded by $\|u(1)\|_{X_1^\gamma}$, so Cauchy–Schwarz's inequality yields

$$\begin{aligned}|\xi|^{2(\gamma+\delta)}|\hat{f}_+(\xi)|&\leq C(a,\delta)\|u(1)\|_{X_1^\gamma}\\&\quad+C(a)|\xi|^{2(\gamma+\delta)}\int_{|\eta|<C}|\hat{f}_+(\eta)|\,|\hat{f}_+(\xi-\eta)|\,\big|\log|\eta|\big|\,d\eta.\end{aligned}\tag{56}$$

On the region $\{|\eta|<C\}\cap\{|\xi|/2<|\eta|\}$ we can upper-bound $\big|\log|\eta|\big|\leq|\log(|\xi|/2)|$ and treat this term as before, to finally get

$$\begin{aligned}|\xi|^{2(\gamma+\delta)}|\hat{f}_+(\xi)|&\leq C(a,\delta)\|u(1)\|_{X_1^\gamma}\\&\quad+C(a)|\xi|^{2(\gamma+\delta)}\int_{|\eta|\leq|\xi|/2}|\hat{f}_+(\eta)|\,|\hat{f}_+(\xi-\eta)|\,\big|\log|\eta|\big|\,d\eta.\end{aligned}\tag{57}$$

By Cauchy–Schwarz's inequality we obtain

$$|\hat{f}_+(\xi)|^2 \leq C(a,\delta)\frac{1}{|\xi|^{4(\gamma+\delta)}}\|u(1)\|^2_{X_1^\gamma} + C(a)\|u(1)\|_{X_1^\gamma}\int_{|\eta|\leq|\xi|/2}|\hat{f}_+(\xi-\eta)|^2\log^2|\eta|\,d\eta.$$

For $0<r<|\xi|$ we then get

$$\left(|\hat{f}_+|^2\star\frac{1}{2r}\mathbb{I}_{[-r,r]}\right)(\xi) \leq \|u(1)\|_{X_1^\gamma}\left(\frac{C(a,\delta)}{2r}\int_{|\xi'|\leq r}\frac{d\xi'}{|\xi-\xi'|^{4(\gamma+\delta)}}+C(a)I_r(\xi)\right),$$

where

$$I_r(\xi)=\frac{1}{2r}\int_{|\xi'|\leq r}\int_{|\eta|\leq|\xi-\xi'|/2}|\hat{f}_+(\xi-\xi'-\eta)|^2\,\log^2|\eta|\,d\eta\,d\xi'.$$

Since $\delta<1/4-\gamma$, we have

$$\frac{1}{2r}\int_{|\xi'|\leq r}\frac{d\xi'}{|\xi-\xi'|^{4(\gamma+\delta)}}\leq C\,\frac{|\xi+r|^{1-4(\gamma+\delta)}-|\xi-r|^{1-4(\gamma+\delta)}}{2r}.$$

For $|\xi|/2<r<|\xi|$ we immediately get the upper bound $C/|\xi|^{4\delta}$, while for $0<r<|\xi|/2$ we get the same upper bound by noticing that $|\xi|/2<|\xi-\xi'|<3|\xi|/2$. As a consequence, for $0<r<|\xi|$ and $\xi^2\leq 1$,

$$\left(|\hat{f}_+|^2\star\frac{1}{2r}\mathbb{I}_{[-r,r]}\right)(\xi)\leq\|u(1)\|_{X_1^\gamma}\left(\frac{C(a,\delta)}{|\xi|^{4(\gamma+\delta)}}+C(a)I_r(\xi)\right). \tag{58}$$

We define, for $\xi\neq 0$ and $h\in L^1$,

$$Mh(\xi)=\sup_{0<r<|\xi|}\left(h\star\frac{1}{2r}\mathbb{I}_{[-r,r]}\right)(\xi)=\sup_{0<r<|\xi|}\frac{1}{2r}\int_{|\xi'|\leq r}h(\xi-\xi')\,d\xi'.$$

We find that $Mh(\xi)$ is well defined almost everywhere in ξ: for r large we use $h\in L^1$, and for $r\to 0$ we get $h(\xi)<\infty$ a.e. in ξ. As a property of this operator we have, for $h\geq 0$ and for ϕ even and decreasing,

$$\begin{aligned}\int_{|\eta|\leq|\xi|}h(\xi-\eta)\phi(\eta)\,d\eta &= \sum_{j=0}^{+\infty}\int_{|\xi|/2^{j+1}\leq|\eta|\leq|\xi|/2^j}h(\xi-\eta)\phi(\eta)\,d\eta\\ &\leq\sum_{j=0}^{+\infty}\frac{|\xi|}{2^{j-1}}\phi\left(\frac{|\xi|}{2^{j+1}}\right)\frac{2^{j-1}}{|\xi|}\int_{|\xi|/2^{j+1}\leq|\eta|\leq|\xi|/2^j}h(\xi-\eta)\,d\eta\\ &\leq 2Mh(\xi)\int_{|\eta|\leq|\xi|}\phi(\eta)\,d\eta.\end{aligned} \tag{59}$$

We have the following lemma.

Lemma 3.7. *For* $0 < r < |\xi| \le 1$,

$$I_r(\xi) \le C(a)(1+\log^2|\xi|)\|u(1)\|^2_{X_1^\gamma} + C|\xi|(1+\log^2|\xi|)M|\hat{f}_+|^2(\xi).$$

Proof. First, for $|\xi|/4 < r < |\xi|$, we make the change of variable $\eta = \eta' - \xi'$, so

$$I_r(\xi) = \frac{1}{2r}\int_{|\xi'|\le r}\int_{|\eta'-\xi'|\le|\xi-\xi'|/2} |\hat{f}_+(\xi-\eta')|^2 \log^2|\eta'-\xi'|\,d\eta'\,d\xi'.$$

In particular, $|\eta'| \le |\xi|/2 + 3r/2 \le 2|\xi|$, and

$$\begin{aligned} I_r(\xi) &\le \int_{|\eta'|\le 2|\xi|} |\hat{f}_+(\xi-\eta')|^2 \frac{1}{2r}\int_{|\xi'|\le r} \log^2|\eta'-\xi'|\,d\xi'\,d\eta' \\ &\le C\int_{|\eta'|\le 2|\xi|} |\hat{f}_+(\xi-\eta')|^2 \frac{(|\eta'|+r)\log^2(|\eta|'+r)}{2r}\,d\eta' \\ &\le C\frac{|\xi|(1+\log^2|\xi|)}{r}\|f_+\|^2_{L^2}, \end{aligned}$$

so for $|\xi|/4 < r < |\xi|$ we have the upper bound $C(1+\log^2|\xi|)\|f_+\|^2_{L^2}$.

For $0 < r < |\xi|/4$ we perform the same change of variable, and get $|\eta'| \le |\xi|/2 + 3r/2 \le |\xi|$, so

$$I_r(\xi) \le \int_{|\eta'|\le|\xi|} |\hat{f}_+(\xi-\eta')|^2 \frac{1}{2r}\int_{|\xi'|\le r} \log^2|\eta'-\xi'|\,d\xi'\,d\eta'.$$

In the region $|\eta'| \ge 2r$ we have $|\xi'| \le |\eta'|/2$, so $|\eta'-\xi'| \ge \eta'/2$, and by using (59), we get the desired upper bound

$$\begin{aligned} \int_{|\eta'|\le|\xi|} |\hat{f}_+(\xi-\eta')|^2 \log^2\frac{|\eta'|}{2}\,d\eta' &\le 2M|\hat{f}_+|^2(\xi)\int_{|\eta'|\le|\xi|} \log^2\frac{|\eta'|}{2}\,d\eta' \\ &\le C|\xi|(1+\log^2|\xi|)M|\hat{f}_+|^2(\xi). \end{aligned}$$

In the remaining region $|\eta'| \le 2r$ we decompose the integral in η' into three parts:

$$\begin{aligned} \int_{|\eta'|\le\min\{|\xi|,2r\}} |\hat{f}_+(\xi-\eta')|^2 \frac{1}{2r}\left(\int_{|\xi'|\le|\eta'|/2} + \int_{|\eta'|/2\le|\xi'|\le 3|\eta'|/2} + \int_{\frac{3}{2}|\eta'|\le|\xi'|\le r}\right) d\eta' \\ = I_r^1(\xi) + I_r^2(\xi) + I_r^3(\xi). \end{aligned}$$

In the first one, $|\eta'-\xi'| \ge |\eta'|/2$, so

$$I_r^1(\xi) \le C\int_{|\eta'|\le\min\{|\xi|,2r\}} |\hat{f}_+(\xi-\eta')|^2 \log^2\frac{|\eta'|}{2}\,d\eta',$$

so as before we recover the upper bound $C|\xi|(1+\log^2|\xi|)M|\hat{f}_+|^2(\xi)$. In the second region we integrate in ξ', and since ξ' is of the size of η', we end up as before:

$$I_r^2(\xi) \le C\int_{|\eta'|\le\min\{|\xi|,2r\}} |\hat{f}_+(\xi-\eta')|^2\,\frac{|\eta'|\log^2|\eta'|}{2r}\,d\eta'$$
$$\le C\int_{|\eta'|\le\min\{|\xi|,2r\}} |\hat{f}_+(\xi-\eta')|^2\log^2|\eta'|\,d\eta'.$$

In the last region $|\eta'-\xi'|\ge|\eta'|/2$, so we get again

$$I_r^3(\xi) \le C\int_{|\eta'|\le\min\{|\xi|,2r\}} |\hat{f}_+(\xi-\eta')|^2\,\log^2\frac{|\eta'|}{2}\,d\eta'.$$

In conclusion for $|\xi|/4 < r < |\xi|$ we get the upper bound $C(1+\log^2|\xi|)\|f_+\|_{L^2}^2$ and for $0 < r < |\xi|/4$ we get the upper bound $C|\xi|(1+\log^2|\xi|)M|\hat{f}_+|^2(\xi)$, so the lemma follows. □

By using this lemma, estimate (58) gives us, for $0 < r < |\xi|$,

$$\left(|\hat{f}_+|^2\star\frac{1}{2r}\mathbb{I}_{[-r,r]}\right)(\xi)$$
$$\le \|u(1)\|_{X_1^\gamma}\left(\frac{C(a,\delta)}{|\xi|^{4(\gamma+\delta)}}+C(a)(1+\log^2|\xi|)\|u(1)\|_{X_1^\gamma}^2+C(a)|\xi|(1+\log^2|\xi|)M|\hat{f}_+|^2(\xi)\right).$$

The constant is independent of r, so by taking the supremum in $0 < r < |\xi|$ we obtain, for $\xi^2 \le 1$,

$$M|\hat{f}_+|^2(\xi) \le$$
$$\|u(1)\|_{X_1^\gamma}\left(\frac{C(a,\delta)}{|\xi|^{4(\gamma+\delta)}}+C(a)(1+\log^2|\xi|)\|u(1)\|_{X_1^\gamma}^2+C(a)|\xi|(1+\log^2|\xi|)M|\hat{f}_+|^2(\xi)\right). \tag{60}$$

Since $M|\hat{f}_+|^2(\xi) < \infty$ almost everywhere in ξ, for $\|u(1)\|_{X_1^\gamma}C(a)|\xi|(1+\log^2|\xi|) < 1/2$, so for $C(a)\|u(1)\|_{X_1^\gamma} < 1/2$, we get the estimate

$$M|\hat{f}_+|^2(\xi) \le \frac{C(a,\delta)}{|\xi|^{4(\gamma+\delta)}}\|u(1)\|_{X_1^\gamma}+C(a)(1+\log^2|\xi|)\|u(1)\|_{X_1^\gamma}^3 \le \frac{C(a,\delta)}{|\xi|^{4(\gamma+\delta)}}\|u(1)\|_{X_1^\gamma}.$$

Then

$$|\hat{f}_+|^2(\xi) = \lim_{r\to 0}\left(|\hat{f}_+|^2\star\frac{1}{2r}\mathbb{I}_{[-r,r]}\right)(\xi) \le M|\hat{f}_+|^2(\xi) \le \frac{C(a,\delta)}{|\xi|^{4(\gamma+\delta)}}\|u(1)\|_{X_1^\gamma},$$

and the proposition follows. □

Appendix A. Wave operators

In this section we prove the existence of wave operators for the nonlinear equation (10). The difference with respect to the wave operators constructed in [3] is that here we shall weaken the conditions on the final data by working in spaces that fit the conditions of Theorem 1.1.

We first study the existence of the wave operators for the linearized equation (14).

Proposition A.1. *Let $0 \le \gamma < 1/4$, $\nu > 0$, and let $u_+ \in X_1^{\gamma-\nu}$. Then the equation (14) has a unique solution $u \in Z^\gamma$ satisfying, as t goes to infinity,*

$$\|u(t) - e^{it\partial_x^2}u_+\|_{L^2} \le \frac{C(a,\nu,\delta)}{t^{1/4-(\gamma+\delta)}}\|u_+\|_{X_1^{\gamma-\nu}}$$

for any $0 < \delta < 1/4-\gamma$. In particular, $u(1) \in X_1^\gamma$, with norm bounded by $\|u_+\|_{X_1^{\gamma-\mu}}$.

Proof. We are going to use similar arguments to those in Lemma 2.2. We define, as in (32),

$$2\mathring{Z}_\xi^+ = e^{-ia^2\log\xi^2}\hat{u}_+(\xi), \qquad \mathring{Y}_\xi^+ = \overline{\mathring{Z}_{-\xi}^+}.$$

We define for $0 \le \tilde{t} \le 1/4a^2$ the solutions $(\mathring{y}_\xi, \mathring{z}_\xi)(\tilde{t})$ of

$$\begin{pmatrix}\mathring{y}_\xi(\tilde{t})\\ \mathring{z}_\xi(\tilde{t})\end{pmatrix} = \begin{pmatrix}\mathring{Y}_\xi^+\\ \mathring{Z}_\xi^+\end{pmatrix} + \int_0^{\tilde{t}} M\left(\frac{1}{\tau}\right)\begin{pmatrix}\mathring{y}_\xi(\tau)\\ \mathring{z}_\xi(\tau)\end{pmatrix}\left(-\frac{1}{\tau^2}\right)d\tau.$$

Then

$$\sup_{0\le\tilde{t}\le 1/4a^2}(|\mathring{y}_\xi(\tilde{t})| + |\mathring{z}_\xi(\tilde{t})|) \le (|\mathring{Y}_\xi^+| + |\mathring{Z}_\xi^+|) + \int_0^{1/4a^2}\frac{a^2}{\alpha^2(1/\tau)}\,d\tau \sup_{0\le\tilde{t}\le 1/4a^2}(|\mathring{y}_\xi(\tilde{t})| + |\mathring{z}_\xi(\tilde{t})|),$$

and as $\alpha(1/\tau) = \sqrt{1-2a^2\tau}$, we get

$$\sup_{0\le\tilde{t}\le 1/4a^2}(|\mathring{y}_\xi(\tilde{t})| + |\mathring{z}_\xi(\tilde{t})|) \le 2(|\mathring{Y}_\xi^+| + |\mathring{Z}_\xi^+|).$$

Now, for $4a^2 \le t < \infty$, the functions $(\mathring{Y}_\xi(t), \mathring{Z}_\xi(t)) = (\mathring{y}_\xi(1/t), \mathring{z}_\xi(1/t))$ solve (28) and

$$|\mathring{Y}_\xi(t)|^2 + |\mathring{Z}_\xi(t)|^2 \le C(|\mathring{Y}_\xi^+|^2 + |\mathring{Z}_\xi^+|^2) = C(|\hat{u}_+(\xi)|^2 + |\hat{u}_+(-\xi)|^2).$$

In particular,

$$\partial_t\begin{pmatrix}\mathring{Y}_\xi\\ \mathring{Z}_\xi\end{pmatrix} = M(t)\begin{pmatrix}\mathring{Y}_\xi\\ \mathring{Z}_\xi\end{pmatrix} = \frac{a^2}{2t^2\alpha^2(t)}\begin{pmatrix}-1 & e^{-2i\Phi(t)}\\ e^{2i\Phi(t)} & -1\end{pmatrix}\begin{pmatrix}\mathring{Y}_\xi\\ \mathring{Z}_\xi\end{pmatrix}.$$

Since $\mathring{Y}_\xi^+ = \overline{\mathring{Z}_{-\xi}^+}$ and

$$\partial_t\begin{pmatrix}\mathring{Y}_\xi - \overline{\mathring{Z}_{-\xi}}\\ \mathring{Z}_\xi - \overline{\mathring{Y}_{-\xi}}\end{pmatrix} = M(t)\begin{pmatrix}\mathring{Y}_\xi - \overline{\mathring{Z}_{-\xi}}\\ \mathring{Z}_\xi - \overline{\mathring{Y}_{-\xi}}\end{pmatrix},$$

we obtain $\mathring{Y}_\xi(t) = \overline{\mathring{Z}_{-\xi}(t)}$. Therefore, with the notation of Lemma 2.2, we can define, for $4a^2 \le t < \infty$,

$$\begin{pmatrix} Y_\xi(t) \\ Z_\xi(t) \end{pmatrix} = P(t) \begin{pmatrix} e^{i\Phi(t)} & 0 \\ 0 & e^{-i\Phi(t)} \end{pmatrix} \begin{pmatrix} \mathring{Y}_\xi(t) \\ \mathring{Z}_\xi(t) \end{pmatrix} = \begin{pmatrix} e^{i\Phi(t)} \mathring{Y}_\xi(t) + e^{-i\Phi(t)} \mathring{Z}_\xi(t) \\ i\alpha(t) e^{i\Phi(t)} \mathring{Y}_\xi(t) - i\alpha(t) e^{-i\Phi(t)} \mathring{Z}_\xi(t) \end{pmatrix},$$

a solution of (24):

$$\partial_t \begin{pmatrix} Y_\xi \\ Z_\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(1 - 2a^2/t) & 0 \end{pmatrix} \begin{pmatrix} Y_\xi \\ Z_\xi \end{pmatrix},$$

which satisfies $Y_\xi(t) = \overline{Y_{-\xi}(t)}$ and $Z_\xi(t) = \overline{Z_{-\xi}(t)}$. Moreover, since

$$\begin{aligned} |\mathring{Y}_\xi(t)|^2 + |\mathring{Z}_\xi(t)|^2 &= \left| \frac{1}{2} Y_\xi(t) - \frac{i}{2\alpha(t)} Z_\xi(t) \right|^2 + \left| \frac{1}{2} Y_\xi(t) + \frac{i}{2\alpha(t)} Z_\xi(t) \right|^2 \\ &= \frac{|Y_\xi(t)|^2}{2} + \frac{|Z_\xi(t)|^2}{2\alpha^2(t)}, \end{aligned}$$

and $1/\sqrt{2} \le \alpha(t) \le 1$, it follows that

$$|Y_\xi(t)|^2 + |Z_\xi(t)|^2 \le C(|\hat{u}_+(\xi)|^2 + |\hat{u}_+(-\xi)|^2). \tag{61}$$

We continue the definition of $(Y_\xi(t), Z_\xi(t))$ for the remaining $0 < t < \infty$ as the solution of (24). It follows that $u(t,x)$ defined by $u(t,x) = w(t,x)e^{-ia^2 \log t}$, where

$$Y_\xi(t) = \widehat{\Re w}(t/\xi^2, \xi), \qquad Z_\xi(t) = \widehat{\Im w}(t/\xi^2, \xi),$$

is a solution of (14). In particular, (61) is satisfied for all $1 \le t < \infty$, so for large frequencies $\xi^2 \ge 1/t$ we get

$$|\hat{u}(t,\xi)|^2 + |\hat{u}(t,-\xi)|^2 \le C(a)(|\hat{u}_+(\xi)|^2 + |\hat{u}_+(-\xi)|^2). \tag{62}$$

We next define

$$y_\xi(t) = Y_\xi(1/t), \qquad z_\xi(t) = Z_\xi(1/t),$$

a solution for $1 \le t < \infty$ of

$$y_\xi' = -\frac{1}{t^2} z_\xi, \qquad z_\xi' = \left(\frac{1}{t^2} - \frac{2a^2}{t} \right) y_\xi,$$

with initial data $(y_\xi(1), z_\xi(1)) = (Y_\xi(1), Z_\xi(1))$. We take $\sigma_\epsilon = y_\xi^2/\epsilon + \epsilon z_\xi^2$ and proceeding as in Lemma 2.2, for all $t > 1$, we obtain

$$\sigma_\epsilon(t) \le e^{1/\epsilon + \epsilon + 2a^2 \epsilon \log t} \sigma_\epsilon(1).$$

By choosing $\epsilon = 1/\sqrt{\log t}$ for $t > 3/2$ and $\epsilon = 1$ for $1 \le t \le 3/2$, we get for all $t \ge 1$ the estimate

$$\begin{aligned} |y_\xi(t)|^2 + |z_\xi(t)|^2 &\le C(1 + \log t) e^{2 + 2a^2 \sqrt{\log t}} (|y_\xi(1)|^2 + |z_\xi(1)|^2) \\ &\le C(a)(1 + \log t) e^{2 + 2a^2 \sqrt{\log t}} (|\hat{u}_+(\xi)|^2 + |\hat{u}_+(-\xi)|^2). \end{aligned}$$

So we finally get the estimates for low frequencies $\xi^2 \leq 1/t$ of the solution u of (14),

$$|\hat{u}(t,\xi)|^2 + |\hat{u}(t,-\xi)|^2 \leq C(a)(1+|\log t\xi^2|)e^{2+2a^2\sqrt{|\log t\xi^2|}}(|\hat{u}_+(\xi)|^2 + |\hat{u}_+(-\xi)|^2). \tag{63}$$

Therefore setting $f_+(x) = u(1,x)$ we see by (62) and (63) that $f_+ \in X_1^\gamma$ with norm bounded by $\|u_+\|_{X_1^{\gamma-\mu}}$. From Propositions 2.6, 2.8, and (32), (33) it follows that $u = S(t,1)f_+ \in Z^\gamma$ and for all $0 < \delta < 1/4 - \gamma$,

$$\|S(t,1)f_+ - e^{i(t-1)\partial_x^2}u_+\|_{L^2} \leq \frac{C(a,\nu,\delta)}{t^{1/4-(\gamma+\delta)}}\big(\|u_+\|_{L^2} + \big\||\xi|^{2(\gamma-\nu)}\hat{u}_+(\xi)\big\|_{L^\infty(|\xi|\leq 1)}\big). \qquad \square$$

Remark A.2. Let us notice that (62) together with (63) imply that for $t_1, \delta_1 > 0$,

$$|\hat{u}(t_1,\xi)| \leq \left(C(a) + \frac{C(a,\delta_1)}{(t_1\xi^2)^{\delta_1}}\right)(|\hat{u}_+(\xi)| + |\hat{u}_+(-\xi)|).$$

This, combined with Lemma 2.10, yields, for any times $t_1, t_2 \geq 1$ and for any positive δ_1, δ_2,

$$|\hat{u}(t_1,\xi)| \leq \left(C(a) + \frac{C(a,\delta_1)}{(t_1\xi^2)^{\delta_1}}\right)\left(C(a) + \frac{C(a,\delta_2)}{(t_2\xi^2)^{\delta_2}}\right)(|\hat{u}(t_2,\xi)| + |\hat{u}(t_2,-\xi)|). \tag{64}$$

It follows that for $1 \leq t_1 \leq t_2$ and $\gamma + \delta_1 + \delta_2 < 1/4$,

$$\begin{aligned}\|u(t_1)\|_{L^2} &\leq \|u(t_2)\|_{L^2}\left(C(a) + C(a,\delta_1)\frac{t_2^{\delta_1}}{t_1^{\delta_1}}\right)\\ &\quad + C(a,\delta_2)\frac{\big\||\xi|^{2\gamma}\hat{u}(t_2,\xi)\big\|_{L^\infty(\xi^2\leq 1/t_2)}}{t_1^{\delta_1}t_2^{\delta_2}}\left\|\frac{1}{\xi^{2\gamma+2(\delta_1+\delta_2)}}\right\|_{L^2(\xi^2\leq 1/t_2)}\\ &\leq C(a,\delta_1,\delta_2)\frac{t_2^{1/4+\delta_1}}{t_1^{\delta_1}}\|u(t_2)\|_{X_{t_2}^\gamma}.\end{aligned} \tag{65}$$

Now we show the existence of wave operators for the nonlinear equation (10) with respect to the linear solutions of (14).

Proposition A.3. *Let* $0 \leq \gamma < 1/4$. *For all* $f_+ \in X_1^\gamma$ *small with respect to a, equation* (10) *has a unique solution* $u \in L^\infty((1,\infty), L^2(\mathbb{R})) \cap L^4((1,\infty), L^\infty(\mathbb{R}))$ *satisfying, as t goes to infinity,*

$$\|u(t) - S(t,1)f_+\|_{L^2} + \|u(\tau) - S(\tau,1)f_+\|_{L^4((t,\infty),L^\infty)} \leq \frac{C(a,\delta)}{t^{1/4-(\gamma+\delta)}}\|f_+\|_{X_1^\gamma}$$

for any $0 < \delta < 1/4 - \gamma$.

Proof. We shall perform a fixed point argument for the operator

$$Bu = S(t,1)f_+ + \int_t^\infty S(t,\tau)\frac{iF(u(\tau))}{\tau}\,d\tau$$

in the closed ball $A_R = \{u \mid \|u\|_A \le R\}$, where

$$\|u\|_A = \sup_{t\in[1,\infty)} t^{1/4-(\gamma+\delta)}(\|u(t) - S(t,1)f_+\|_{L^2} + \|u(\cdot) - S(\cdot,1)f_+\|_{L^4((t,\infty),L^\infty)}),$$

with R to be specified later.

Let $u \in A_R$. In particular for all admissible couples (p,q), interpolated between $(\infty,2)$ and $(4,\infty)$, we have

$$\sup_{t\in[1,\infty)} t^{1/4-(\gamma+\delta)}\,\|u(\cdot) - S(\cdot,1)f_+\|_{L^p((t,\infty),L^q)} \le CR,$$

and therefore, by the estimates (45) and (47),

$$\|u\|_{L^p((t,\infty),L^q)} \le C\|S(\cdot,1)f_+\|_{L^p((1,\infty),L^q)} + C\|u\|_A \le C\|f_+\|_{X_1^\gamma} + C\|u\|_A. \quad (66)$$

We want to estimate

$$Bu - S(t,1)f_+ = \int_t^\infty S(t,\tau)\frac{iF(u(\tau))}{\tau}\,d\tau = J(t).$$

We have

$$\begin{aligned}\|J(t)\|_{L^2} + \|J\|_{L^4((t,\infty),L^\infty)} \le &\int_t^\infty \|S(t,\tau)iF(u(\tau))\|_{L^2}\frac{d\tau}{\tau}\\ &+\int_t^\infty \|S(\cdot,\tau)iF(u(\tau))\|_{L^4((\tau,\infty),L^\infty)}\frac{d\tau}{\tau}\\ &+\int_t^\infty \|S(\cdot,\tau)iF(u(\tau))\|_{L^4((t,\tau),L^\infty)}\frac{d\tau}{\tau}.\end{aligned}$$

We upper-bound the first term by using the backwards estimates (65) with $(t_1,t_2) = (t,\tau)$ and $\delta_1 = \delta \in (0, 1/4-\gamma)$,

$$\int_t^\infty \|S(t,\tau)iF(u(\tau))\|_{L^2}\frac{d\tau}{\tau} \le C(a,\delta)\int_t^\infty \frac{\tau^{1/4+\delta}}{t^\delta}\|F(u(\tau))\|_{X_\tau^\gamma}\frac{d\tau}{\tau}.$$

For the second we use the forward estimate (47),

$$\int_t^\infty \|S(\cdot,\tau)iF(u(\tau))\|_{L^4((\tau,\infty),L^\infty)}\frac{d\tau}{\tau} \le C(a)\int_t^\infty \tau^{1/4}(1+\log^2\tau)\|F(u(\tau))\|_{X_\tau^\gamma}\frac{d\tau}{\tau}.$$

We write the third term as

$$\begin{aligned}&\int_t^\infty \|S(\cdot,\tau)iF(u(\tau))\|_{L^4((t,\tau),L^\infty)}\frac{d\tau}{\tau}\\ &\quad= \int_t^\infty \left\| e^{i(s-\tau)\partial_x^2}F(u(\tau)) - ia^2\int_s^\tau e^{i(\tau-r)\partial_x^2}\frac{\overline{S(r,\tau)iF(u(\tau))}}{r^{1+2ia^2}}dr\right\|_{L^4((t,\tau),L^\infty)}\frac{d\tau}{\tau}.\end{aligned}$$

We use the Strichartz estimates and the backwards estimates (65) to get

$$\begin{aligned}\int_t^\infty \|S(\cdot,\tau)iF(u(\tau))\|_{L^4((t,\tau),L^\infty)}\frac{d\tau}{\tau} &\le C(a)\int_t^\infty\left(\|F(u(\tau))\|_{L^2}+\left\|\frac{S(r,\tau)iF(u(\tau))}{r}\right\|_{L^1((t,\tau),L^2)}\right)\frac{d\tau}{\tau}\\ &\le \int_t^\infty \|F(u(\tau))\|_{X_\tau^\gamma}\left(C\tau^{1/4}+C(a)\left\|\frac{\tau^{1/4+\delta}}{r^{1+\delta}}\right\|_{L^1(t,\tau)}\right)\frac{d\tau}{\tau}\\ &\le C(a)\int_t^\infty \frac{\tau^{1/4+\delta}}{t^\delta}\|F(u(\tau))\|_{X_\tau^\gamma}\frac{d\tau}{\tau}.\end{aligned}$$

Summarizing, we have obtained

$$\|J(t)\|_{L^2}+\|J\|_{L^4((t,\infty),L^\infty)}\le C(a,\delta)\int_t^\infty \tau^{1/4+\delta}\|F(u(\tau))\|_{X_\tau^\gamma}\frac{d\tau}{\tau}.$$

Now Lemma 3.2 with $(t_1,t_2)=(t,\infty)$ and $\alpha=1/4+\delta$ gives

$$\|J(t)\|_{L^2}+\|J\|_{L^4((t,\infty),L^\infty)}\le\frac{C(a,\delta)}{t^{1/4-(\gamma+\delta)}}\sum_{j\in\{1,2\}}(a\|u\|^2_{L^{p_j}((t,\infty),L^{q_j})}+\|u\|^3_{L^{p_j}((t,\infty),L^{q_j})}),$$

where $(p_1,q_1)=(\infty,2)$ and $(p_2,q_2)=(4,\infty)$. Therefore, in view of (66),

$$\|J\|_A\le Ca\|f_+\|^2_{X_1^\gamma}+Ca\|u\|_A^2+C\|f_+\|^3_{X_1^\gamma}+C\|u\|_A^3.$$

For all $f_+\in X_1^\gamma$ small with respect to a, there exists R small with respect to a such that the operator B is a contraction on A_R, and the proposition follows. □

The last two propositions imply the following result.

Theorem A.4. *Let* $0\le\gamma<1/4$, $\nu>0$ *and* $u_+\in X_1^{\gamma-\nu}$ *with norm small with respect to* a. *Then the equation* (10) *has a unique solution* $u\in L^\infty((1,\infty),L^2(\mathbb{R}))\cap L^4((1,\infty),L^\infty(\mathbb{R}))$ *satisfying, as* t *goes to infinity,*

$$\|u(t)-e^{i(t-1)\partial_x^2}u_+\|_{L^2}\le\frac{C(a,\nu,\delta)}{t^{1/4-(\gamma+\delta)}}\|u_+\|_{X_1^{\gamma-\nu}}$$

for any $0<\delta<1/4-\gamma$.

Appendix B. Remarks on the growth of the zero-Fourier modes

B.1. Growth of the zero-Fourier modes for the linear equation

Let u be the global H^2 solution of (14) obtained as a consequence of Lemma 2.1. We shall get here some extra information on $u(t)$, via estimates done directly on $w(t)=u(t)e^{\pm ia^2\log t}$, the solution of (15):

$$i\partial_t w+w_{xx}\pm\frac{a^2}{t}(w+\overline{w})=0.$$

We shall use the fact that $w \in H^2$ to get proper integration by parts at the level of the Laplacian.

Let us notice that since u is a solution of the linear equation (14), if $\hat{u}(t_0)$ is continuous, so will be $\hat{u}(t)$. In this case, by integrating in space, we get the law of evolution of the zero-Fourier modes,

$$i\partial_t \int w = \mp \frac{a^2}{t} \int \Re w,$$

so

$$\partial_t \int \Re w = 0, \qquad \partial_t \int \Im w = \pm \frac{a^2}{t} \int \Re w = \pm \frac{a^2}{t} \int \Re w(t_0).$$

Therefore

$$\int \Im w(t) = \int \Im w(t_0) \pm 2a^2 \int \Re w(t_0) \log \frac{t}{t_0}.$$

In conclusion, if the zero-mode $\int w(t_0)$ is null, then it will be the same for all times, $\int w(t) = 0$. Furthermore, if the real part of the zero-modes $\Re \int w(t_0)$ is not null, then we have a logarithmic growth of the zero-modes $\int w(t)$, independently of the size of t_0, which cannot be avoided:

$$\int w(t) = \int w(t_0) \pm 2ia^2 \int \Re w(t_0) \log \frac{t}{t_0}. \tag{67}$$

Recovering the expression of u, we obtain (34).

B.2. Growth of the Fourier modes for the nonlinear equation

Let u be the global H^1 solution of (10) obtained by Corollary 3.3. In particular,

$$\sum_{0\leq k\leq 1} \|\partial_x^k u\|_Z \leq C(a) \sum_{0\leq k\leq 1} \|\partial_x^k u(1)\|_{X_1} \leq C(a, u(1)).$$

For the computations on Fourier modes in this subsection, the existence of $\hat{u}(t, 0)$ has to be justified. We have the following control.

Lemma B.1. *If $xu(1) \in L^2$, then*

$$\|xu(t)\|_{L^2} \leq C(a, u(1)) t^{\tilde{C}(a,u(1))}.$$

Proof. Let φ be a positive radial cut-off function, equal to x^2 on $B(0, 1)$, such that $(\partial_x\varphi)^2 \leq C\varphi$. For $R > 0$ we define

$$\varphi_R(x) = R^2\varphi(x/R).$$

We multiply equation (10) by $\varphi_R\overline{u}$ and integrate the imaginary part to get

$$\begin{aligned}\partial_t\int\varphi_R|u(t)|^2 &= -\Im\int u_{xx}\varphi_R\overline{u}\mp\Im\int\frac{a^2}{t^{1\pm 2ia^2}}\varphi_R\overline{u}\,\overline{u}-\Im\int\frac{F(u)}{t}\varphi_R\overline{u}\\ &=\Im\int u_x\partial_x\varphi_R\overline{u}\mp\Im\int\frac{a^2}{t^{1\pm 2ia^2}}\varphi_R\overline{u}\,\overline{u}-\Im\int\frac{F(u)}{t}\varphi_R\overline{u}\\ &\le\|\partial_x u\|_{L^2}\left(\int(\partial_x\varphi_R)^2|u(t)|^2\right)^{1/2}\\ &\quad+\frac{a^2}{t}\int\varphi_R|u(t)|^2+\frac{\|u\|_{L^\infty}+\|u\|^2_{L^\infty}}{t}\int\varphi_R|u(t)|^2.\end{aligned}$$

Therefore, by using $(\partial_x\varphi)^2\le C\varphi$ and Sobolev embeddings,

$$\partial_t\left(\int\varphi_R|u(t)|^2\right)^{1/2}\le C(a,u(1))+\frac{C(a,u(1))}{t}\left(\int\varphi_R|u(t)|^2\right)^{1/2},$$

so

$$\left(\int\varphi_R|u(t)|^2\right)^{1/2}\le C(a,u(1))t^{\tilde{C}(a,u(1))}.$$

The estimate is uniform in R, and the lemma follows by letting R go to infinity. □

In particular, the lemma ensures that $\hat{u}(t)\in H^1$, so in particular $\hat{u}(t)$ is continuous and the existence of $\hat{u}(t,0)$ is justified. Now we shall get information on the zero-mode of $u(t)$, via estimates on the solution w of (8):

$$iw_t+w_{xx}=\mp\frac{1}{t}(|a+w|^2-a^2)(a+w).$$

We shall use the following conservation law:

$$\partial_t\int(|w+a|^2-a^2)=0, \tag{68}$$

obtained by multiplying (8) by $\overline{w}+a$ and by taking the imaginary part. We integrate (8) in space to get

$$i\partial_t\int w\pm\int\frac{1}{t}(|w+a|^2-a^2)(w+a)=0.$$

By using (68) we get the evolution of the zero-modes:

$$\begin{aligned}\int w(t)-\int w(t_0)&=\pm i\int_{t_0}^{t}\int(|w(\tau)+a|^2-a^2)(w(\tau)+a)\,dx\,\frac{d\tau}{\tau}\\ &=\pm ia\int(|w(t_0)+a|^2-a^2)\,dx\log\frac{t}{t_0}\pm i\int_{t_0}^{t}\int(|w(\tau)|^2+2a\Re w(\tau))w(\tau)\,dx\,\frac{d\tau}{\tau}.\end{aligned}$$

The Strichartz estimates imply that the part coming from the cubic power of w is bounded in time, so we can bound the second term:

$$\left|\int_{t_0}^{t}\int (|w(\tau)|^2+2a\Re w(\tau))w(\tau)\,dx\,\frac{d\tau}{\tau}\right| \leq C(a)\|u(t_0)\|_{X_{t_0}}+2a\|w\|^2_{L^\infty((t_0,t),L^2)}\log\frac{t}{t_0}$$
$$\leq C(a)\|u(t_0)\|_{X_{t_0}}+C(a)\|u(t_0)\|^2_{X_{t_0}}\log\frac{t}{t_0}.$$

Therefore we get a logarithmic upper bound for $\int w(t)$, and implicitly for $\hat{u}(t,0)$. This growth is sharp provided that

$$C(a)\|w(t_0)\|^2_{X_{t_0}} = C(a,t_0)(\|w(t_0)\|^2_{L^2}+\|\hat{w}(t_0)\|^2_{L^\infty(\xi^2\leq 1/t_0)})$$
$$< a\left|\int (|w(t_0)+a|^2-a^2)\,dx\right|,$$

for which a sufficient condition is

$$C(a,t_0)(\|w(t_0)\|^2_{L^2}+\|\hat{w}(t_0)\|^2_{L^\infty(\xi^2\leq 1/t_0)}) < \left|\int \Re w(t_0)\,dx\right|.$$

We also get a logarithmic growth for $\Im\int w(t)$, provided that $\int(|w(t_0)+a|^2-a^2)\,dx > 0$.

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References

[1] Alekseenko, S. V., Kuibin, P. A., Okulov, V. L.: Theory of Concentrated Vortices. An Introduction. Springer, Berlin (2007) Zbl 1132.76001 MR 2398034

[2] Banica, V., Vega, L.: On the Dirac delta as initial condition for nonlinear Schrödinger equations. Ann. Inst. H. Poincaré Anal. Non Linéaire **25**, 97–711 (2008) Zbl 1147.35092 MR 2436789

[3] Banica, V., Vega, L.: On the stability of a singular vortex dynamics. Comm. Math. Phys. **286**, 593–627 (2009) Zbl 1183.35029 MR 2472037

[4] Batchelor, G. K.: An Introduction to Fluid Dynamics. Cambridge Univ. Press, Cambridge (1967) Zbl 0152.44402 MR 1744638

[5] Carles, R.: Geometric optics and long range scattering for one-dimensional nonlinear Schrödinger equations. Comm. Math. Phys. **220**, 41–67 (2001) Zbl 1029.35211 MR 1882399

[6] Christ, M.: Power series solution of a nonlinear Schrödinger equation. In: Mathematical Aspects of Nonlinear Dispersive Equations, Ann. of Math. Stud. 163, Princeton Univ. Press, Princeton, NJ, 131–155 (2007) Zbl 1142.35084 MR 2333210

[7] Da Rios, L. S.: On the motion of an unbounded fluid with a vortex filament of any shape. Rend. Circ. Mat. Palermo **22**, 117–135 (1906) (in Italian) JFM 37.0764.01

[8] de la Hoz, F.: Self-similar solutions for the 1-D Schrödinger map on the hyperbolic plane. Math. Z. **257**, 61–80 (2007) Zbl 1128.35099 MR 2318570

[9] de la Hoz, F., García-Cervera, C., Vega, L.: A numerical study of the self-similar solutions of the Schrödinger map. SIAM J. Appl. Math. **70**, 1047–1077 (2009) Zbl 1219.65139 MR 2546352

[10] Grünrock, A.: Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS. Int. Math. Res. Notices **2005**, no. 41, 2525–2558 Zbl 1088.35063 MR 2181058

[11] Gustafson, S., Nakanishi, K., Tsai, T.-P.: Global dispersive solutions for the Gross–Pitaevskii equation in two and three dimensions. Ann. Henri Poincaré **8**, 1303–1331 (2007) Zbl pre05218113 MR 2360438

[12] Gustafson, S., Nakanishi, K., Tsai, T.-P.: Scattering theory for the Gross–Pitaevskii equation in three dimensions. Comm. Contemp. Math. **11**, 657–707 (2009) Zbl 1180.35481 MR 2559713

[13] Gutiérrez, S., Rivas, J., Vega, L.: Formation of singularities and self-similar vortex motion under the localized induction approximation. Comm. Partial Differential Equations **28**, 927–968 (2003) Zbl 1044.35089 MR 1986056

[14] Hasimoto, H.: A soliton on a vortex filament. J. Fluid Mech. **51**, 477–485 (1972) Zbl 0237.76010

[15] Hayashi, N., Naumkin, P.: Domain and range of the modified wave operator for Schrödinger equations with critical nonlinearity. Comm. Math. Phys. **267**, 477–492 (2006) Zbl 1113.81121 MR 2249776

[16] Nahmod, A., Shatah, J., Vega, L., Zeng, C.: Schrödinger maps and their associated frame systems. Int. Math. Res. Notices **2007**, no. 21, art. ID rnm088, 29 pp. Zbl 1142.35087 MR 2352219

[17] Ozawa, T.: Long range scattering for nonlinear Schrödinger equations in one space dimension. Comm. Math. Phys. **139**, 479–493 (1991) Zbl 0742.35043 MR 1121130

[18] Ricca, R. L.: The contributions of Da Rios and Levi-Civita to asymptotic potential theory and vortex filament dynamics. Fluid Dynam. Res. **18**, 245–268 (1996) Zbl 1006.01505 MR 1408546

[19] Saffman, P. G.: Vortex Dynamics. Cambridge Monogr. Mech. Appl. Math., Cambridge Univ. Press, New York (1992) Zbl 0777.76004 MR 1217252

[20] Vargas, A., Vega, L.: Global wellposedness of 1D non-linear Schrödinger equation for data with an infinite L^2 norm. J. Math. Pures Appl. **80**, 1029–1044 (2001) Zbl 1027.35134 MR 1876762