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Asymptotic behaviour of numerical invariants of algebraic varieties

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Abstract. We show that if the degree of a nonsingular projective variety is high enough, maximization of any of the most important numerical invariants, such as class, Betti number, and any of the Chern or middle Hodge numbers, leads to the same class of extremal varieties. Moreover, asymptotically (say, for varieties whose total Betti number is large enough) the ratio of any two of these invariants tends to a well-defined constant.

Introduction

Let $\mathfrak{C}$ denote the set of all nonsingular complex projective varieties. An integral-valued function $\nu\colon \mathfrak{C} \to \mathbb{Z}$ is called a *numerical invariant* (resp. *projective numerical invariant*) if $\nu(X') = \nu(X)$ for X' isomorphic to X (resp. X' projectively isomorphic to X). Of course, good numerical invariants should satisfy additional conditions, such as functoriality. Most interesting numerical invariants involve the tangent bundle or tensors associated to it. Among these, a particularly important role is played by the *Hodge numbers* $h^{p,q}(X)$ and *Chern numbers* $c_I(X)$, where I is a multiindex of weight $n = \dim X$. Examples of projective numerical invariants are given by the *classes* $\mu_i(X)$, $0 \le i \le n$, and the *degrees* $\deg c_I(X)$ *of Chern classes of weight* $0 < |I| < n = \dim X$. A classical approach to studying algebraic varieties consists in choosing a numerical invariant ν, finding the range in which it varies, and then examining the moduli for a given value of ν.

Since the class of X in the cohomology algebra of the ambient linear space $\mathbb{P}^N$ is determined by the *dimension* $n = \dim X$, *codimension* $a = N - n$, and *degree* $d = \deg X$, it is natural to look for bounds for a numerical invariant ν in terms of d, n and a; a function $\mathcal{C}_\nu(d, n, a)$ is called an *upper bound* for ν if $\nu(X) \le \mathcal{C}_\nu(d, n, a)$ for all $X \in \mathfrak{C}$. A bound $\mathcal{C}_\nu$ is called *sharp* if $\nu(X) = \mathcal{C}_\nu(d, n, a)$ for some variety X. A bound $\mathcal{C}_\nu$ is called *asymptotically sharp* if, for given n and a, there exists a sequence of varieties $X_i \in \mathfrak{C}$ with $\dim X_i = n$, $\operatorname{codim} X_i = a$, and $\deg X_i = d_i \to \infty$ such that $\lim_{i\to\infty} \nu(X_i)/\mathcal{C}_\nu(d_i, n, a) = 1$.

Lower bounds for numerical invariants can be defined in a similar way, but they only make sense for invariants ν assuming large negative values. Replacing such an invariant

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ν by $-\nu$, one is reduced to studying the upper bounds. The most important example when $\nu(X) = c_I(X)$ will be discussed in more detail below.

The only important numerical invariant of a curve is its *genus* g. Well over a century ago, Guido Castelnuovo gave a sharp upper bound for the genus in terms of the degree d and codimension a and classified the curves for which the maximum is attained (cf. e.g. [GH, pp. 251–253, 527–533]). Castelnuovo's work implies that $g < \mathcal{C}_g(d, 1, a) = d^2/2a$ is an asymptotically sharp bound and that the curves whose genus is sufficiently close to this bound are contained in surfaces of minimal degree. Joe Harris [Ha] extended Castelnuovo's bound to the geometric genus $p_g = h^{0,n}$ of varieties of an arbitrary dimension n and described the varieties of maximal genus which he called *Castelnuovo varieties*. Under certain additional assumptions (such as numerical effectivity of the canonical line bundle K_X), Castelnuovo type bounds for K_X^n were obtained in [DG]. In [Za] the author obtained similar asymptotically sharp bounds for the middle Betti number $b_n(X) = \sum_{p+q=n} h^{p,q}(X)$ and the class $\mu = \mu_n(X)$. It turns out that all the above bounds have the form $\mathcal{C}_\nu(d, n, a) \sim \alpha_\nu d^{n+1}/a^n$, where α_ν is a constant depending on ν. Thus, $\alpha_{p_g} = 1/(n+1)!$ and $\alpha_{b_n} = \alpha_\mu = 1$ (more generally, in [Za] it is shown that for $i < n$ the Betti number $b_i(X)$ and the class $\mu_i(X)$ are bounded by a polynomial in d whose leading term is d^{i+1}/a^i, and so, asymptotically, these invariants are subject to more severe constraints). Also, if ν is one of the above invariants (i.e. $\nu = p_g$, $\nu = b_n$ or $\nu = \mu$) and X is a variety of sufficiently large degree for which $\nu(X)$ is large enough (i.e. close to the asymptotic bound), then X is a hypersurface in a variety of minimal degree in the ambient linear space, viz. $X \subset V \subset \mathbb{P}^N$, where $\dim V = n + 1$, $\deg V = a$.

However, the geometric genus p_g and the middle Betti number b_n (resp. the class μ) are only special cases of numerical (resp. projective numerical) invariants playing important role in algebraic geometry. Already for surfaces, the list of standard invariants includes the Chern numbers c_1^2 and c_2 and the Hodge numbers $h^{2,0} = h^{0,2} = p_g$ and $h^{1,1}$. Little seems to be known about the relationship between p_g and $h^{1,1}$, but the results of [Per] on the 'geography' of surfaces suggest that, subject to certain constraints, the Chern numbers can be almost arbitrary. As $n = \dim X$ grows, the number of different and apparently independent numerical invariants, such as Hodge and Chern numbers, grows, and for each of these invariants it is natural to consider the following problems:

- Find (sharp) upper bounds for the invariant in terms of dimension, codimension and degree.
- Describe the varieties for which the invariant attains maximal value.
- Study the relations between various numerical invariants.

As we already saw, it is hard to give a satisfactory solution of the above problems, even in the case of surfaces. However, these problems tend to be easier if one assumes an 'asymptotic' point of view, i.e. considers the varieties of sufficiently high degree for which the invariants are large enough. In the present paper we solve the 'asymptotic' versions of the three problems. In particular, we deal with the Chern and Hodge numbers. Unlike the Hodge numbers, the Chern numbers are not necessarily positive. If I is a multiindex of weight $n = \dim X$, T_X is the tangent bundle of X, and $T_X^* = \Omega_X^1$ is the cotangent bundle, then $c_I(X) = c_I(T_X) = (-1)^n c_I(T_X^*)$, and we show that the Chern number $c_I(T_X^*)$

is bounded from below by a polynomial of degree n in d (cf. Theorems 1.5 and 1.14). In particular, if $|c_I(T_X^*)|$ is large with respect to d^n, then $c_I(T_X^*)$ is necessarily positive. We find asymptotically sharp upper bounds for the Chern numbers $c_I(T_X^*)$ (or, which is the same, for the absolute values $|c_I(X)|$) and middle Hodge numbers and study the relationship between these invariants provided that at least one of them is large enough (cf. Theorems 1.14 and 2.2 and Corollaries 1.15, 1.16, and 2.3). Our bounds are of the form $\mathcal{C}_\nu \sim \alpha_\nu d^{n+1}/a^n$, and we compute the values α_ν for the invariants in question. Thus, while in general there are few restrictions on the 'geography' of Chern and Hodge numbers, the 'asymptotic geography' turns out to be perfectly rigid.

We also describe the extremal varieties provided that the degree is large enough (cf. Theorems 1.17 and 2.5). It turns out that the extremal varieties for the above numerical invariants are all of the same type, viz. they are hypersurfaces in varieties of minimal degree (although the concrete type of hypersurface may depend on the invariant). For example, maximization of any of the Hodge numbers $h^{p,q}(X)$, $p+q=n=\dim X$ (or their weighted sum, such as the Betti number $b_n(X)$) or, say, $(K_X^n)=(-1)^n c_1^n(X)$ or the absolute value of the Euler–Poincaré characteristic $|e(X)|=|c_n(X)|$ leads to extremal varieties of the same type provided that the degree is large enough, and the ratios of these invariants tend to well defined constants.

Our methods are mainly geometric. Basing on the results of [Za], we reduce the problem for Chern numbers to inequalities between (intersections of) polar classes and ramification loci of generic projections. Hodge numbers are dealt with using the Riemann–Roch–Todd–Hirzebruch theorem.

1. Bounds for the degrees of Chern classes

1.1. Definition. Let $L \subset \mathbb{P}^N$, $\dim L = a+i-2$, $0 \le i \le n$ be a general linear subspace. The subset $P_i = P_i(L) = \{x \in X \mid \dim T_{X,x} \cap L \ge i-1\}$ is called the *i-th polar locus* or *polar subvariety* of X with respect to L. It is clear that $\operatorname{codim}_X P_i = i$ and that $x \in P_i$ if and only if there exists a hyperplane passing through L and tangent to X at x. The degree $\mu_i = \deg P_i$ is called the *i-th class* of X (cf. [Pi]). In particular, $P_0 = X$, $P_1 = R_L$ is the ramification divisor of the projection $\pi_L\colon X \to \mathbb{P}^n$ with center at L, and P_n consists of μ_n distinct points.

Let X_j, $j=0,\dots,n$ denote the section of X by a general linear subspace $M_j \subset \mathbb{P}^N$, $\operatorname{codim} M_j = n-j$, so that $X_j \subset \mathbb{P}^{N+j-n}$ is a nondegenerate smooth projective variety of dimension j. From the above definition it is clear that $\mu_i(X)=\mu_i(X_j)$ for $j \ge i$. Furthermore,

$$\mu_i = \begin{cases} d_i^*, & \operatorname{def} X_i = 0, \\ 0, & \operatorname{def} X_i > 0, \end{cases} \qquad d_i^* = \deg X_i^*, \tag{1.1.1}$$

where X_i^* is the dual variety of X_i and $\operatorname{def} X_i = N-n+i-1-\dim X_i^* = \max\{0, \operatorname{def} X - n + i\}$ is the (dual) defect of X_i (cf. [Za, Definitions 1.8 and 1.11]).

Let $\mathcal{P}_X^1$ be the *sheaf of principal parts* or the *jet bundle* of $\mathcal{O}_X(1)$. By definition, the fibre $\mathcal{P}_x^1$ of $\mathcal{P}_X^1$ at a point $x \in X$ is the quotient of the space of sections of $\mathcal{O}_X(1)$ in a neighbourhood of x by the sections vanishing at x together with their first derivatives.

In other words, the dual bundle $\mathcal{P}_X^{1*}$ coincides with the inverse image of the universal subbundle over the Grassmann variety $G(N,n)$ of linear subspaces of dimension n in $\mathbb{P}^N$ under the Gauss map $\gamma\colon X \to G(N,n)$, so that the fibre of $\mathcal{P}_X^{1*}$ over x is naturally isomorphic to the tangent space to the affine cone over X along the generatrix corresponding to x. Thus $\mathcal{P}_X^1$ fits into an Euler type exact sequence

$$0 \to \Omega_X^1(1) \to \mathcal{P}_X^1 \to \mathcal{O}_X(1) \to 0 \qquad (1.1.2)$$

(cf. [Pi, §2] or [Tev, Chapter V]). It is clear that the bundle $\mathcal{P}_X^1$ is globally generated. Furthermore, if $p_i = [P_i]$ denotes the i-th *polar class*, that is, the equivalence class of the i-th polar subvariety in the group of algebraic cycles modulo rational equivalence, then from the geometric definition of Chern classes (cf. e.g. [Fu, Example 14.4.3]) it immediately follows that

$$p_i = c_i(\mathcal{P}_X^1), \quad i = 1, \dots, n. \qquad (1.1.3)$$

For $I = \{i_1, \dots, i_k\}$, $1 \le i_1, \dots, i_k \le n$, we denote by $|I| = i_1 + \dots + i_k$ the weight of the multiindex I and put

$$p_I = p_{i_1} \cdots p_{i_k} = c_I(\mathcal{P}_X^1) = c_{i_1}(\mathcal{P}_X^1) \cdots c_{i_k}(\mathcal{P}_X^1). \qquad (1.1.4)$$

Let $\mathfrak{C}_n$ denote the class[1] of all nondegenerate nonsingular projective varieties of dimension n. For $X \in \mathfrak{C}_n$ we denote by $A_{\mathbb{Q}}^i(X)$ the group of algebraic cycles of codimension i with rational coefficients modulo rational equivalence.

1.2. Definition. A map E associating to each variety $X \in \mathfrak{C}_n$ a vector bundle E of rank e on X is called a *bundle assignment of rank e* on $\mathfrak{C}_n$.

A map κ associating to each $X \in \mathfrak{C}_n$ a cycle in $A_{\mathbb{Q}}^i(X)$ is called a *cycle assignment of* (pure) *weight i* on $\mathfrak{C}_n$. The scalar function $\nabla\kappa\colon \mathfrak{C}_n \to \mathbb{Q}$, $\nabla\kappa(X) = \int_X h^{n-i}\kappa(X)$, where $h = h_X$ is the class of hyperplane section, is called the *degree function* of the assignment κ.

1.3. Example. (i) Let $\kappa(X) = (\alpha_X h_X)^i$, where α is a rational-valued scalar function on $\mathfrak{C}_n$. Then $\nabla\kappa(X) = d\alpha^i$, where $d = \deg X$. In particular, for $\alpha = d/a$ (where, as usual, $a = \operatorname{codim} X$) one gets a cycle assignment $\hbar_i = ((d/a)h)^i$ with $\nabla\hbar_i = d^{i+1}/a^i$.

(ii) Let $\kappa(X) = p_1^i$. Then $\nabla\kappa(X) = (R_L^i \cdot h_X^{n-i}) = r_i(X)$ is the i-th ramification volume of X (cf. [Za, 1.4]).

(iii) Let $\kappa(X) = p_i(X)$. Then $\nabla\kappa(X) = (p_i \cdot h_X^{n-i}) = \mu_i(X)$ is the i-th class of X (cf. Definition 1.1).

(iv) More generally, for any multiindex $I = \{i_1, \dots, i_k\}$ one can consider the degree function $\nabla p_I(X)$ of weight $|I| = i_1 + \dots + i_k$ (cf. (1.1.4)).

(v) Let $\kappa(X) = c_1^i(\Omega_X^1)$. Then $\nabla\kappa(X) = \deg K_X^i$, where $K = K_X$ is the canonical class of X (in particular, for $i = n$ one has $\nabla\kappa(X) = (K_X^n)$).

(vi) More generally, for any $1 \le k \le n$ and any multiindex $I = \{i_1, \dots, i_k\}$ we can consider the characteristic function $\nabla c_I(\Omega_X^k)$ of weight $|I| = i_1 + \dots + i_k$. The function

[1] Here (and on several occasions below) the word *class* is used in its conventional sense, not the one introduced in Definition 1.1. We hope that this will not lead to ambiguity.

$\nabla c_I(\Omega^k_X)$ can be expressed in terms of the degree functions of the bundle Ω^1_X (cf. [Fu, Remark 3.2.3c)]).

(vii) Let $T_X = \Omega^{1\,*}_X$ be the tangent bundle, and let $I = \{i_1, \dots, i_k\}$ be a multiindex. Then $\nabla c_I = \nabla c_I(T_X) = (-1)^{|I|} c_I(\Omega^1_X)$ (cf. [Fu, Remark 3.2.3a)]).

(viii) The next example subsumes most of the preceding ones. Let E be a bundle assignment on $\mathfrak{C}_n$. Then E gives rise to cycle assignments $\kappa_i : \mathfrak{C}_n \to \mathbb{Q}$, $1 \le i \le \min\{n, e\}$, by putting $\kappa_i(X) = c_i(E_X)$. The assignment κ_i will be denoted simply by $c_i(E)$ and the corresponding degree function by $\nabla c_i(E)$. In a similar way, for any multiindex $I = \{i_1, \dots, i_k\}$ one can define a cycle assignment $c_I(E)$ of weight $|I|$ and the corresponding degree function $\nabla c_I(E)$.

1.4. Definition. We say that a cycle assignment κ of weight i is *asymptotically small* and write $\kappa \sim 0$ if there exists a polynomial $Q_i \in \mathbb{Q}[T]$ such that $\deg Q_i \le i$ and $|\nabla\kappa| < Q_i(d)$ everywhere on $\mathfrak{C}_n$ (here, as usual, $d = \deg X$).

We say that two cycle assignments κ_1, κ_2 of weight i on $\mathfrak{C}_n$ are *asymptotically equivalent* and write $\kappa_1 \sim \kappa_2$ if $\kappa_1 - \kappa_2 \sim 0$; clearly this is an equivalence relation on $\mathfrak{C}_n$.

We say that a cycle assignment κ of weight i is *asymptotically bounded* by an assignment κ' of the same weight and write $\kappa \precsim \kappa'$ if there exists a polynomial $Q_i \in \mathbb{Q}[T]$ with $\deg Q_i \le i$ such that $\nabla\kappa < \nabla\kappa' + Q_i(d)$ everywhere on $\mathfrak{C}_n$; in this case we also write $\kappa' \succsim \kappa$. It is clear that $\kappa \sim \kappa'$ if and only if $\kappa \precsim \kappa'$ and $\kappa' \precsim \kappa$. We say that κ is *strictly asymptotically bounded* by κ' and write $\kappa \prec \kappa'$ or $\kappa' \succ \kappa$ if $\kappa \precsim \kappa'$, but $\kappa \nsim \kappa'$.

The results of Section 1 in [Za] can now be interpreted as follows.

1.5. Theorem. *Let $X \in \mathfrak{C}_n$. Then*

$$0 \precsim \left\{ \begin{matrix} [K^i] \\ [R^i] \\ p_i \end{matrix} \right\} \precsim \hbar_i,$$

where $[K^i]$ (resp. $[R^i]$) is the class of the i-th selfintersection of the canonical class, cf. Example 1.3(v) *(resp. ramification divisor, cf. Example* 1.3(ii)*), p_i is the i-th polar class (cf. formula* (1.1.3), *and Example* 1.3(iii)*), and $\hbar_i$ is defined in Example* 1.3(i).

Proof. The upper bound is an immediate consequence of Definition 1.4 and Corollaries 1.5, 1.6(i) and 1.13 in [Za].

As for the lower bound, we observe that $r_i = \nabla[R^i] \ge 0$ and $\mu_i = \nabla p_i \ge 0$ (cf. Example 1.3(ii), (iii)). Thus it remains to show that $\nabla[K^i] \succsim 0$. In fact,

$$\begin{aligned} \nabla[K^i] &= \nabla[(R - (n+1)H)^i] \\ &= \sum_{j=0}^{[i/2]} \binom{i}{2j} (n+1)^{2j}\, \nabla[R^{i-2j}] - \sum_{j=0}^{[(i-1)/2]} \binom{i}{2j+1} (n+1)^{2j+1} \nabla[R^{i-2j-1}] \\ &\ge - \sum_{j=0}^{[(i-1)/2]} \binom{i}{2j+1} (n+1)^{2j+1} \nabla\hbar_{i-2j-1} = - \sum_{j=0}^{[(i-1)/2]} \binom{i}{2j+1} \frac{(n+1)^{2j+1}}{a^{i-2j-1}}\, d^{\,i-2j}, \end{aligned} \tag{1.5.1}$$

where the last term is a polynomial of degree i (here we used the upper bound proved in the preceding paragraph and [Za, (1.4.2) and the proof of Corollary 1.6(i)]). □

1.6. Lemma. *Let E be a bundle assignment of rank e on $\mathfrak{C}_n$ (cf. Definition 1.2), and let $c_i(E)$, $1 \le i \le \min\{n, e\}$ be the corresponding cycle assignment (cf. Example 1.3(viii)). Suppose that $|c_i(E)| \prec \beta_i \hbar_i$, where $\beta_i > 0$ is a constant function on $\mathfrak{C}_n$ and $\hbar$ is defined in Example 1.3(i). Then $c_i(E(1)) \sim c_i(E)$. More generally, $c_I(E(m)) \sim c_I(E)$ for arbitrary $m \in \mathbb{Z}$ and multiindex $I = \{i_1, \dots, i_k\}$.*

Proof. By a well known formula for Chern classes (cf. e.g. [Fu, Example 3.2.2]), we have

$$\begin{aligned} c_i(E(1)) &= \sum_{j=0}^{i} \binom{e-j}{i-j} h^{i-j} c_j(E) \\ &= c_i(E) + (e-i+1)hc_{i-1}(E) + \cdots + \binom{e-1}{i-1} h^{i-1} c_1(E) + \binom{e}{i} h^i \end{aligned} \tag{1.6.1}$$

and

$$\begin{aligned} c_i(E(-1)) = \sum_{j=0}^{i} (-1)^{i-j} \binom{e-j}{i-j} h^{i-j} c_j(E) = c_i(E) - (e-i+1)hc_{i-1}(E) + \cdots \\ + (-1)^{i-1} \binom{e-1}{i-1} h^{i-1} c_1(E) + (-1)^i \binom{e}{i} h^i. \end{aligned} \tag{1.6.2}$$

Thus

$$|c_i(E(\pm 1)) - c_i(E)| \prec \sum_{j=0}^{i-1} \beta_j \binom{e-j}{i-j} \hbar_j, \tag{1.6.3}$$

and, iterating, we get

$$c_i(E(m)) \sim c_i(E) \tag{1.6.4}$$

for an arbitrary $m \in \mathbb{Z}$.

To prove Lemma 1.6 for a multiindex $I = \{i_1, \dots, i_k\}$ one should consider a product of expansions (1.6.1) or (1.6.2) for $i = i_1, \dots, i_k$. □

1.7. Corollary. *For each multiindex $I = \{i_1, \dots, i_k\}$ one has $c_I(\Omega^1) \sim p_I$. In particular, $c_i(\Omega^1) \precsim \hbar_i$, $1 \le i \le n$.*

Proof. In view of (1.1.2), one has

$$c_i(\Omega^1) = c_i(\mathcal{P}^1(-1)), \quad i = 0, \dots, n, \tag{1.7.1}$$

and therefore

$$c_I(\Omega^1) = c_I(\mathcal{P}^1(-1)), \quad I = \{i_1, \dots, i_k\}. \tag{1.7.2}$$

The corollary now follows from Theorem 1.5 and Lemma 1.6 (for $E = \mathcal{P}^1$, $e = n + 1$, $\beta_i = 1$, and $m = -1$). □

1.8. Corollary. *Let $e\colon \mathfrak{C}_n \to \mathbb{Z}$ be the Euler–Poincaré characteristic, and let $1 \le i \le n$. For $X \in \mathfrak{C}_n$, one has*:

$$\begin{aligned}(-1)^i c_i(X_i) = c_i(\Omega^1_{X_i}) &= \mu_i - 2\mu_{i-1} + \cdots + (-1)^{i-1} i\mu_1 + (-1)^i (i+1)\mu_0\\ &< \frac{d^{i+1}}{a^i} + Q_i(d),\\ \mu_i &= c_i(\Omega^1_{X_i}) + 2c_{i-1}(\Omega^1_{X_i})h + \cdots + ic_1(\Omega^1_{X_i})h^{i-1} + (i+1)d\\ &= (-1)^i c_i(X_i) + (-1)^{i-1} 2c_{i-1}(X_i)h + \cdots - c_1(X_i)h^{i-1} + (i+1)d\\ &< \frac{d^{i+1}}{a^i} + Q_i(d).\end{aligned}$$

Furthermore, $c_i(\Omega^1_{X_i}) > Q'_i(d)$ and $|e(X_i)| = |c_i(X_i)| < d^{i+1}/a^i + Q_i(d)$, where Q_i and Q'_i are polynomials of degree i in d.

Proof. The equality in the first (resp. second) formula follows from (1.6.2) and (1.7.1) (resp. (1.6.1) and (1.7.1)). The inequalities follow from Theorem 1.5 or Corollary 1.13 in [Za]. The lower bound for $c_i(\Omega^1_{X_i})$ and the bound for absolute values follow from the first inequality and Theorem 1.5. □

1.9. Remarks. (i) The polynomials Q_i (and Q'_i) of degree i in d occurring in the preceding results and realizing asymptotic smallness have the form $Q_i(d) = dq_{i-1}(d/a)$, where q_{i-1} is a polynomial of degree $i-1$. The same applies to many other polynomials arising in this note.

(ii) The first formula in Corollary 1.8 gives another proof of the formula for the Euler–Poincaré characteristic from Corollary 2.9(i) in [Za]. Moreover, arguing by induction and using Lefschetz's theorem, we conclude that $b_i(X) < d^{i+1}/a^i + Q_i(d)$, where Q_i is a polynomial of degree i, $i = 1, \dots, n$; hence $b(X) = \sum_{j=0}^{2n} b_j(X) < d^{n+1}/a^n + Q(d)$, where Q is a polynomial of degree n. Thus Corollary 1.8 allows one to obtain bounds similar to those given in Section 2 of [Za] using the language of Chern classes (of course one can write out an explicit expression for the polynomial Q_i in each particular case to get better bounds with the same highest terms). This method is well known in the theory of characteristic classes; it was used in [LV] to obtain a bound for the total Betti number, but the bound in [LV] is far from being asymptotically sharp.

(iii) The second formula expressing projective classes in terms of the degrees of Chern classes is known; cf. e.g. [Tev, Theorem 5.18(a)].

1.10. Proposition. *Let $X \in \mathfrak{C}_n$, and let $I = \{i_1, \dots, i_k\}$, $1 \le i_1, \dots, i_k \le n$, be a multiindex of weight $|I| = i_1 + \cdots + i_k$. Then $0 < \nabla p_{|I|} \le \nabla p_I \le \nabla p_1^{|I|}$, where the first* (*resp. second*) *inequality is strict unless $I = \{|I|\}$* (*resp. $I = \{1, \dots, 1\}$*).

Proof. As already observed, for each $X \in \mathfrak{C}_n$ the bundle $\mathcal{P}^1_X$ is globally generated. The claim now follows from the geometric definition of Chern classes [Fu, Example 14.4.3]. Alternatively, one can argue by induction using basic facts about Schur polynomials (cf. [Laz, 8.3], [DPS, p. 317] or [LV, §4]). □

1.11. Remark. Theorem 1.12 in [Za] is a special case of Proposition 1.10. In fact, for $I = \{i\}$ Proposition 1.10 yields

$$\mu_i = \nabla p_i \le \nabla p_1^i = r_i \tag{1.11.1}$$

(cf. Example 1.3(ii), (iii)).

1.12. Proposition. *In the notation of Example* 1.3(ii), (iii) *one has* $p_i \sim p_1^i$ *for* $i = 1, \dots, n$. *More precisely,*

$$\mu_i < r_i < \mu_i + \sum_{j=1}^{[i/2]} \binom{i}{j} (d-1)^j \mu_{i-2j} < \mu_i + Q_i(d), \qquad 2 \le i \le n,\ a = \operatorname{codim} X \ge 2,$$

where Q_i *is a polynomial of degree* i *in* d.

Proof. Let $L_1, \dots, L_n \subset \mathbb{P}^N$, $\dim L_i = a - 1$, $a = N - n = \operatorname{codim} X$, $i = 1, \dots, n$, be a general sequence of linear subspaces such that $\dim \boldsymbol{L}_i = i + a - 2$, where $\boldsymbol{L}_i = \langle L_1, \dots, L_i \rangle$ is the linear span of $L_1, \dots, L_i$, and let $P_i = P_i(\boldsymbol{L}_i) = \{x \in X \mid \dim T_{X,x} \cap \boldsymbol{L}_i \ge i - 1\}$, $\dim P_i = n - i$, $i = 0, \dots, n$ be the corresponding polar subvarieties. Denote by $R_i = R(L_i)$ the ramification locus of the projection $\pi_i \colon X \to \mathbb{P}^n$ with center at L_i, $R_i = \{x \in X \mid T_{X,x} \cap L_i \ne \emptyset\}$. It is clear that $P_i \subset R_1 \cap \dots \cap R_i$, $X = P_0 \supset R_1 = P_1 \supset \dots \supset P_n$ and $\mu_i = \deg P_i$, where μ_i is the i-th class of X (cf. Definition 1.1). Put $\mathcal{L}_i = L_i \cap \boldsymbol{L}_{i-1}$, $\dim \mathcal{L}_i = a - 2$, $i = 2, \dots, n$, and denote by $\mathcal{R}_i$ the ramification locus of the projection $\varpi_i \colon X \to \mathbb{P}^{n+1}$ with center at $\mathcal{L}_i$, $\mathcal{R}_i = \{x \in X \mid T_{X,x} \cap \mathcal{L}_i \ne \emptyset\}$. It is clear that $\dim R_i = n - 1$, $\dim \mathcal{R}_i = n - 2$ and $\mathcal{R}_i \subset R_i$. Furthermore, it is easy to see that $\mathcal{R}_i \ne \emptyset$ for $a > 1$ and

$$P_{i-1} \cap R_i = P_i \cup (P_{i-1} \cap \mathcal{R}_i), \quad P_{i-1} \cap \mathcal{R}_i = P_{i-2} \cap \mathcal{R}_i, \quad 2 \le i \le n. \tag{1.12.1}$$

Let $\mathfrak{r}_k \in A^k_{\mathbb{Q}}(X)$ denote the class of the ramification locus of a general projection of X to $\mathbb{P}^{n+k-1}$; in particular, for $i = 1, \dots, n$ one has $[R_i] = \mathfrak{r}_1 = p_1$, $[\mathcal{R}_i] = \mathfrak{r}_2$, and from (1.12.1) it follows that

$$p_1 p_{i-1} = p_i + p_{i-2}\mathfrak{r}_2 \tag{1.12.2}$$

(for $i = 2$ this formula is classical, cf. [SR, Ch. IX, §2]).

We denote by $\mathfrak{d}_k \in A^k_{\mathbb{Q}}(X)$ the class of the double locus of a general projection of X to $\mathbb{P}^{n+k}$. It is easy to see that

$$\mathfrak{r}_1 + \mathfrak{d}_1 = (d-1)h \tag{1.12.3}$$

(this classical relation was found by Plücker in the case $n = 1$ and by Clebsch in the case $n = 2$; cf. [Za, Remark 1.17(ii)]). By (1.12.3) and K. Johnson's formula (cf. e.g. [Fu, Example 9.3.13]),

$$\mathfrak{r}_2 = h\mathfrak{d}_1 - \mathfrak{d}_2 = (d-1)h^2 - hp_1 - \mathfrak{d}_2 \tag{1.12.4}$$

(in the case $n = 2$ this formula is due to Zeuthen; cf. [Za, Remark 1.17(iv)]). Since the codimension two cycles hp_1 and $\mathfrak{d}_2$ are 'positive' (and, in particular, 'movable'),

$p_{i-2} \cdot (hp_1 + \mathfrak{d}_2) > 0$ and from (1.12.4), (1.12.2) and Proposition 1.10 (applied to $I = \{1, i-1\}$) it follows that

$$\mu_i \le \nabla p_1 p_{i-1} = \nabla p_i + \nabla p_{i-2}\mathfrak{r}_2 < \mu_i + (d-1)\mu_{i-2}, \quad 2 \le i \le n. \tag{1.12.5}$$

Furthermore, since $\mathfrak{r}_2 \neq 0$ for $a \ge 2$, from (1.12.2) it follows that the first inequality in (1.12.5) is strict provided that $a > 1$; cf. [Za, Remark 1.17(iv)].

Next we show that for each $k \ge 1$ and each $i \ge \max\{2, k\}$, $i \le n$, one has

$$p_1^k p_{i-k} = \sum_{j=0}^{\min\{k,[i/2]\}} \binom{k}{j} p_{i-2j}\mathfrak{r}_2^j. \tag{1.12.6}$$

For $k = 1$, (1.12.6) reduces to (1.12.2). Suppose that (1.12.6) holds for $k' = k-1$. By (1.12.2) one has

$$p_1^k p_{i-k} = p_1^{k-1}(p_{i-k+1} + p_{i-k-1}\mathfrak{r}_2). \tag{1.12.7}$$

Substituting (1.12.6) for $k' = k-1$ and $i' = i$ and $i-2$ in (1.12.7), one gets

$$p_1^k p_{i-k} = \sum_{j=0}^{\min\{k-1,[i/2]\}} \binom{k-1}{j} p_{i-2j}\mathfrak{r}_2^j + \sum_{j=1}^{\min\{k,[i/2]\}} \binom{k-1}{j-1} p_{i-2j}\mathfrak{r}_2^j, \tag{1.12.8}$$

which yields (1.12.6). Substituting (1.12.4) in (1.12.6) and taking the degrees, one arrives at the inequality

$$\mu_i \le \nabla p_1^k p_{i-k} < \sum_{j=0}^{\min\{k,[i/2]\}} \binom{k}{j} (d-1)^j \mu_{i-2j}, \quad k \le i,\ 2 \le i \le n; \tag{1.12.9}$$

as above, for $a > 1$ the first inequality in (1.12.9) is strict. In particular, for $k = i$, (1.12.9) yields

$$\mu_i = \nabla p_i < \nabla p_1^i = r_i < \mu_i + \sum_{j=1}^{[i/2]} \binom{i}{j} (d-1)^j \mu_{i-2j}, \quad 2 \le i \le n. \tag{1.12.10}$$

Proposition 1.12 now follows from Theorem 1.5 (for details, cf. [Za, Corollary 1.13 or Theorem 3.16]). □

1.13. Remarks. (i) We recall that $\mu_1 = r_1$ and $\mu_i = r_i$ for all i if $a = 1$ (cf. Examples 1.14(i), (ii) in [Za]).

(ii) Proposition 1.12 should be compared with Remark 1.17(iv) in [Za] giving a *lower* bound for the difference $r_i - \mu_i$.

1.14. Theorem. *Let I, I' be two multiindices with $|I| = |I'| = i$, $1 \le i \le n$. Then*

$$0 \precsim c_I(\Omega^1) \sim p_I \sim p_{I'} \sim c_{I'}(\Omega^1) \precsim \hbar_i.$$

Proof. From Proposition 1.10 it follows that $p_i \precsim p_I \precsim p_1^i$ while by Proposition 1.12, $p_i \sim p_1^i$, $i = 1, \dots, n$. This proves that $p_I \sim p_i \sim p_1^i$ and similarly $p_{I'} \sim p_i \sim p_1^i$;

hence $p_I \sim p_{I'}$. By Corollary 1.7, $c_I(\Omega^1) \sim p_I$, $c_{I'}(\Omega^1) \sim p_{I'}$, and by transitivity $c_I \sim c_{I'}$. The proof is completed by applying Theorem 1.5. □

1.15. Corollary. *For any two multiindices I, I' with $|I| = |I'| = i$ one has $c_I \sim c_{I'}$, where $c_I = c_I(T) = (-1)^i c_I(\Omega^1)$ and $|\nabla c_I| < d^{i+1}/a^i + Q_I(d)$, where Q_I is a polynomial of degree i in d.*

1.16. Corollary. (i) *Let $1 \le i \le n$ be an integer. For each $\epsilon > 0$ there exists a number $M = M(\epsilon)$ such that*

$$\left|1 - \frac{\nabla p_I(X)}{\nabla p_J(X)}\right| < \epsilon \quad \textit{and} \quad \left|1 - \frac{\nabla c_I(X)}{\nabla c_J(X)}\right| < \epsilon$$

for all multiindices I, J of weight i and all varieties $X \in \mathfrak{C}_n$ for which $\nabla p_K(X) > Md^i$ or $|\nabla c_K(X)| > Md^i$ for some multiindex K with $|K| = i$.

(ii) *Let $\mathrm{P}(c_1, \dots, c_n)$ be a weighted homogeneous polynomial of weight n, and let $\mathrm{P} = \mathrm{P}(X) = \nabla \mathrm{P}(c_1(T_X), \dots, c_n(T_X))$ be the corresponding Chern number (cf. e.g. [Fu, Example 15.2.13]). Then $\mathrm{P}(c_1, \dots, c_n) \sim \mathrm{P}(1, \dots, 1)c_n$ and for each $\epsilon > 0$ there exists a number $M = M(\epsilon)$ such that*

$$\left|\mathrm{P}(1, \dots, 1) - \frac{\mathrm{P}(X)}{c_n(X)}\right| < \epsilon$$

for all varieties $X \in \mathfrak{C}_n$ for which one of the following conditions holds:

$$c_1^n(X) > Md^n; \quad c_n(X) > Md^n; \quad b(X) > Md^n; \quad b_n(X) > Md^n.$$

1.17. Theorem. *Let I be a multiindex of weight $|I| = i$, and let $X_0 \in \mathfrak{C}_n$, $X_0 = X_0(a, d, I)$, be a variety for which $|\nabla c_I|$ attains its maximum (for given a and d). Then, for $d \gg 0$, $X_0 \subset V$, where $V^{n+1} \subset \mathbb{P}^{n+a}$, $\deg V = a$, i.e. X is a codimension one subvariety in a variety of minimal degree. The same is true if we replace $|\nabla c_I|$ by ∇p_I or by $|\mathrm{P}(c_1, \dots, c_n)|$ or by $\mathrm{P}(p_1, \dots, p_n)$, where $\mathrm{P}(c_1, \dots, c_n)$ is an arbitrary weighted homogeneous polynomial of weight i such that $\mathrm{P}(1, \dots, 1) \ne 0$.*

Proof. This follows from Theorem 1.14 and from Theorems 3.1 and 3.15 of [Za]. □

1.18. Remarks. (i) Basing on formulas (1.6.2) and (1.7.2), the inequalities proved in Proposition 1.10 and the results of [Za, §3] (cf. particularly [Za, Theorem 3.1]), one can find explicit upper bounds for the numerical invariants considered in Theorem 1.17 and a number d_0 (depending on a and I or P) such that the claim of this theorem holds for all $d > d_0$. However in the present paper we lay emphasis on qualitative asymptotic results.

(ii) Just as in Section 3 of [Za], one can obtain better bounds for subvarieties of codimension one in varieties of minimal degree and a description of maximal elements in the set of such varieties. Then, applying [Za, Theorem 3.1], one can get good bounds for the above numerical invariants of arbitrary varieties from $\mathfrak{C}_n$ provided that $d > d_0$ (with d_0 depending on a and I or P; compare with Theorem 3.16 in [Za]).

(iii) Theorem 1.17 holds not only for varieties with maximal numerical invariants, but also for varieties whose numerical invariants are (asymptotically) large enough (e.g. $c_I \gtrsim ((d/a')h)^i$, where $a < a' < a + 1$; cf. [Za, Theorem 3.1]).

2. Bounds for Hodge numbers

Let $X \in \mathfrak{C}_n$ be a smooth projective n-fold. We denote by $h^{p,q} = h_X^{p,q} = \dim H^q(X, \Omega_X^p)$ the Hodge numbers of X, so that $\sum_{p=0}^{i} h^{p,i-p} = b_i$, $i = 0, \ldots, 2n$, and put $\chi_X^p = \chi(\Omega_X^p) = \sum_{q=0}^{n} (-1)^q h^{p,q}$. Let $\alpha^p = h^{p,n-p}/b_n$, $p = 0, \ldots, n$ denote the 'weight' of the corresponding Hodge number; then $\alpha^p = \alpha^{n-p}$ and $\sum_{p=0}^{n} \alpha^p = 1$.

2.1. Example. Let $\boldsymbol{X} \in \mathfrak{C}_n$ be a hypersurface. Then

$$b_n(\boldsymbol{X}) = d^{n+1} + Q_n(d), \tag{2.1.1}$$

where Q_n is a polynomial of degree n (cf. Remark 1.9(ii) or [KK, Ch. 4, 5.10] or [Za, (2.6.1) and (2.6.4)]). By Lefschetz's theorem,

$$h^{p,q}(\boldsymbol{X}) = \begin{cases} 0, & p+q \neq n,\ p \neq q, \\ 1, & p+q \neq n,\ p = q, \end{cases} \tag{2.1.2}$$

and so the *middle Hodge numbers* $h^{p,n-p}$, $p = 0, \ldots, n$ are the only (possibly) nontrivial Hodge numbers. There exist formulas for computing $h^{p,n-p}(\boldsymbol{X})$ (cf. [Hi, §22]), but they are rather complicated. These formulas have the form

$$h^{p,n-p} = \alpha_0^p d^{n+1} + Q_n^p(d), \qquad p = 0, \ldots, n, \tag{2.1.3}$$

where $\alpha_0^p = \alpha_0^p(n) \in \mathbb{Q}$ and $Q_n^p(d)$ are polynomials of degree n. Thus

$$\alpha_0^p = \alpha_0^{n-p}, \qquad \sum_{p=0}^{n} \alpha_0^p = 1, \qquad \lim_{d \to \infty} \alpha^p(\boldsymbol{X}) = \alpha_0^p, \qquad p = 0, \ldots, n. \tag{2.1.4}$$

One can give a nice description of the numbers $\alpha^p(n)$ for $\boldsymbol{X} \in \mathfrak{C}_n$ in terms of the number of integer points in the slabs of an $(n+1)$-dimensional cube with edge length $d-2$ obtained by cutting the cube by hyperplanes perpendicular to a main diagonal (cf. [KK, Ch. 4, 5.10], [ChL]). From this it easily follows that the limit weights α_0^p are given by the volumes of slabs of the unit cube $\mathbf{K} \subset \mathbb{R}^{n+1}$, viz.

$$\alpha_0^p = \operatorname{vol}\{(x_0, \ldots, x_n) \in \mathbf{K} \mid p \leq x_0 + \cdots + x_n \leq p+1\}. \tag{2.1.5}$$

Let $A_{n+1,p+1}$ denote the *Eulerian number* equal to the number of permutations of $n+1$ elements having exactly p descents (we recall that the number of descents of a permutation $a_0 \ldots a_n$ is equal to card $\{i \mid a_i > a_{i+1}\}$). By (2.1.5) and the Laplace–Pólya–Stanley theorem (cf. [St] and the award-winning expository article [ChL]),

$$\alpha_0^p = \frac{A_{n+1,p+1}}{(n+1)!}, \qquad p = 0, \ldots, n \tag{2.1.6}$$

(I am grateful to F. Sottile and J. De Loera for pointing out the above references to me). In particular,

$$\alpha_0^0(n) = \alpha_0^n(n) = \frac{1}{(n+1)!} \tag{2.1.7}$$

and by (2.1.1),

$$\lim_{d\to\infty} \frac{p_g(\boldsymbol{X})}{b_n(\boldsymbol{X})} = \frac{1}{(n+1)!}, \tag{2.1.8}$$

where $d = \deg \boldsymbol{X}$ and $p_g = h^{0,n}$ denotes the geometric genus. The computations in [Ha] and [Za, §3] show that (2.1.8) still holds if the hypersurface $\boldsymbol{X} \in \mathfrak{C}_n$ is replaced by an arbitrary *Castelnuovo variety* in $\mathfrak{C}_n$, i.e. any variety having maximal geometric genus (for given a and d). Our next result shows that this is not a pure coincidence.

2.2. Theorem. *For each $\epsilon > 0$ there exists a number $M = M(\epsilon)$ such that $|\alpha^p(X) - \alpha_0^p| < \epsilon$ for all $p = 0, \dots, n$ and all $X \in \mathfrak{C}_n$ for which $b_n(X) > Md^n$.*

Proof. From Remark 1.9(ii) (or Corollary 2.14(i) in [Za]) it follows that

$$h^{pq} \le b_{p+q} < \begin{cases} \dfrac{d^{p+q+1}}{a^{p+q}} + Q_{p+q}(d), & p+q \le n, \\ \dfrac{d^{2n-p-q+1}}{a^{2n-p-q}} + Q_{2n-p-q}(d), & p+q \ge n, \end{cases} \tag{2.2.1}$$

where Q_i is a polynomial of degree i. Thus

$$|h_X^{p,n-p} - (-1)^{n-p}\chi_X^p| < Q_n^p(d), \tag{2.2.2}$$

where Q_n^p is a polynomial of degree n.

By the Riemann–Roch–Todd–Hirzebruch theorem (cf. [Hi, Ch. IV, §21] or [Fu, Corollary 15.2.1]),

$$\chi_X^p = \int_X (\mathrm{ch}(\Omega_X^p) \cdot \mathrm{td}(T_X))_n = \sum_{i=0}^{n} \nabla(\mathrm{ch}(\Omega_X^p)_i\, \mathrm{td}(T_X)_{n-i}), \tag{2.2.3}$$

where $(\ )_i$ denotes the component of codimension i. In particular, for $p = 0$ we get the well known formula

$$\chi_X^0 = \mathrm{Td}(X), \tag{2.2.4}$$

where $\mathrm{Td}(X) = \nabla(\mathrm{td}(T_X))_n$ is the Todd genus of X (cf. [Hi], [Fu, Examples 15.2.12 and 15.2.13]).

From (2.2.2) and (2.2.3) it follows that *there exists a weighted homogeneous polynomial $\mathrm{P}^p(c_1, \dots, c_n)$ of weight n in the Chern classes $c_i = c_i(T)$ such that*

$$|h_X^{p,n-p} - \mathrm{P}^p(X)| < Q_n^p(d), \quad 0 \le p \le n. \tag{2.2.5}$$

By (2.2.5), Corollary 1.16(ii), and Remark 1.9(ii) (or Corollary 2.14(i) from [Za]), for each $\epsilon > 0$ there exists a number $M = M(\epsilon)$ such that

$$|\alpha^p(X) - \mathrm{P}^p(1, \dots, 1)| < \epsilon \tag{2.2.6}$$

for all varieties $X \in \mathfrak{C}_n$ with $b_n(X) > Md^n$. In view of (2.1.1), (2.2.6) also holds for any hypersurface $X \in \mathfrak{C}_n$ of sufficiently high degree d, and applying (2.1.4) we conclude that

$$\alpha_0^p = \mathrm{P}^p(1, \dots, 1). \tag{2.2.7}$$

Theorem 2.2 now follows from (2.2.6). □

2.3. Corollary. *For any p with $0 \le p \le n$ and any $X \in \mathfrak{C}_n$ one has $h^{p,n-p} < \alpha_0^p d^{n+1}/a^n + Q_n^p(d)$, where Q_n^p is a polynomial of degree n.*

Next we give an example where the hypotheses of Corollary 1.16 and Theorem 2.2 are satisfied.

2.4. Example. Let $V^{n+1} \subset \mathbb{P}^N$ be a nondegenerate nonsingular variety, $\deg V = s$, let $W \subset \mathbb{P}^N$ be a general hypersurface of degree $m > 1$, and let $X = X_m = V \cap W \subset \mathbb{P}^N$ be a nonsingular nondegenerate variety of dimension n, codimension $a = N - n$ and degree $d = ms$. Arguing as in [Za, Example 3.5], one can show that

$$\mu_n(m) \sim d^{n+1}/s^n. \tag{2.4.1}$$

Alternatively, using the adjunction formula, one concludes that

$$K_X^n = X_m \cdot (X_m + K_V)^n \sim sm^{n+1} = \frac{d^{n+1}}{s^n}. \tag{2.4.2}$$

Since $b_n(X_m) \sim \mu_n(X_m) \sim K_X^n$, (2.4.1) (or (2.4.2)) shows that, for large m, X_m satisfies the hypotheses of Corollary 1.16 and Theorem 2.2.

For example, if V is a complete intersection of $a - 1$ hypersurfaces of fixed degrees $d_1, \dots, d_{a-1}$, $s = d_1 \cdots d_{a-1}$, then X_m is a complete intersection and there are classical formulas for $\mu_n(m)$ due to Severi (cf. [Sev, Cap. II, §6] and [SR, Ch. IX, §5]) and for $h^{p,n-p}$ due to Hirzebruch (cf. [Hi, §22]). Using these formulas, it is easy to verify Corollary 1.16 and Theorem 2.2 in this special case.

2.5. Theorem. *Let $X_0 \in \mathfrak{C}_n$, $X_0 = X_0(a, d, p)$, be a variety for which the Hodge number $h^{p,n-p}$ attains its maximum (for given a, d and $0 \le p \le n$). Then, for $d \gg 0$, $X_0 \subset V$, where $V^{n+1} \subset \mathbb{P}^{n+a}$, $\deg V = a$, i.e. X is a codimension one subvariety in a variety of minimal degree.*

Proof. This is a consequence of Theorem 2.2 and of Theorems 3.1 and 3.16 in [Za]. □

2.6. Remarks. (i) Writing out the polynomials $Q_n^p(d)$ in (2.1.3) and $\mathrm{P}^p(c_1, \dots, c_n)$ in (2.2.5) explicitly, one can get an explicit form of the function $M(\epsilon)$ in the statement of Theorem 2.2. We do not need it here.

(ii) It might be more convenient to apply the Riemann–Roch–Hirzebruch theorem for the generalized Euler–Poincaré characteristic and the generalized Todd genus. According to this theorem, $\chi_y(X) = T_y(X)$, where $\chi_y(X) = \sum_{p=0}^n \chi_X^p y^p$, $T_y(X) = \sum_{p=0}^n T^p(c_1, \dots, c_n) y^p$ (cf. [Hi, 1.8, 15.5, and 21.3]).

(iii) Theorem 2.2 shows that, asymptotically, if one of the middle Hodge numbers $h^{p,n-p}$ is large (close to maximum for a given codimension and (large) degree), then *all* the middle Hodge numbers $h^{p,q}$, $p+q = n$, and the Betti number b_n are also large (close to their respective maxima).

Furthermore, as in Corollary 1.16 and Theorem 1.17, we get statements similar to Theorems 2.2 and 2.5 for linear combinations $\sum_p \lambda_p h^{p,n-p}$ provided that $\sum_p \lambda_p \alpha_0^p \neq 0$. A useful example is that of cohomology of *Hodge level* not exceeding j, viz. $\sum_{|n-2p|\le j} h^{p,n-p}$.

(iv) If X is a codimension one subvariety in a variety of minimal degree, then, by Lefschetz's theorem, $h^{p,n-p}$ are the only (possibly) nontrivial Hodge numbers (cf. e.g. [Za, (3.5.12)]; compare with (2.1.2)). As in Remark 1.18(ii), one can obtain good and even sharp bounds for the (middle) Hodge numbers of codimension one subvarieties of varieties of minimal degree as well as a description of subvarieties with maximal Hodge numbers whose existence is granted by Theorem 2.5. An interesting question here is whether these maximal subvarieties are the same for different values of p. For example, let $V^{n+1} \subset \mathbb{P}^{n+a}$, $\deg V = a$, be a rational normal scroll, let $X^n \subset V$ be a nonsingular subvariety of degree d, and let m denote the degree of the hypersurface cut out by X on a general linear subspace $\mathbb{P}^n \subset V$. In [Za, Proposition 3.11(iii)] it was shown that, for given d, the maximal value of $b_n(X)$ corresponds to either $m = [d/a]$ or $m = [d/a] + 1$. A priori, for $p = 0, \dots, n$ one can get several different values of m corresponding to the varieties $X^p = X_0(a, d, p)$ furnished by Theorem 2.5 for $d \gg 0$. However, it turns out that in many special cases the variety X^p does not actually depend on p. It would be interesting to check whether this is always the case. The same question makes sense in the setup of Theorem 1.17. Answering these questions involves combinatorial computations and should not be very difficult, but it does not fit into the scope of the present paper.

(v) Theorem 2.5 holds not only for varieties with maximal Hodge numbers, but also for varieties whose Hodge numbers are (asymptotically) large enough (e.g. $h^{p,n-p} \ge \alpha_0^p d^{n+1}/a'^n$, where $a < a' < a + 1$; cf. Remark 1.18(iii)).

(vi) For $p = 0$, Corollary 2.3 yields

$$p_g \le \frac{d^{n+1}}{(n+1)!a^n} + Q_n^g(d), \tag{2.6.1}$$

where Q_n^g is a polynomial of degree n (which, by (i), can be written out explicitly).

Alternatively, from (2.2.2) and (2.2.4) it follows that (in the obvious notation)

$$p_g \sim p_a \sim (-1)^n \operatorname{Td}(X), \tag{2.6.2}$$

so that, in a more traditional, but also more cumbersome way, (2.6.1) can be obtained by computing the coefficients in the expansion of the Todd genus. For example, in the case when $X = S$ is a surface, (2.6.2) and Noether's formula yield

$$p_g(S) \sim p_a(S) = \frac{1}{2}\nabla(c_1^2 + c_2), \tag{2.6.3}$$

and combining (2.6.3) with Corollary 1.15 one gets

$$p_g(S) \le \frac{d^3}{6a^2} + Q_2^g(d), \qquad \deg Q_2^g = 2, \tag{2.6.4}$$

in accordance with (2.6.1).

As far as I am aware, the case $p = 0$ is the only one for which a sharp upper bound for $h^{p,n-p}$ and the structure of maximal varieties were known previously; cf. [Ha].

3. Further problems, examples, and speculations

3.1. (i) By Theorem 2.2, asymptotically, varieties with large (middle) Betti number tend to have Eulerian distribution of Hodge weights α^p, that is, the distribution of normalized Eulerian numbers α_0^p from (2.1.6) which is known to converge to the normal distribution as $n \to \infty$ (cf. [WX] and references therein). It might be tempting to introduce 'infinite-dimensional' varieties, study their invariants, and interpret the limit distribution.

(ii) Corollary 1.16 and Theorem 2.2 describe the asymptotic behaviour of numerical invariants of varieties whose Betti number is large. It would be interesting to study the asymptotic behaviour of numerical invariants for classes (cf. the footnote preceding Definition 1.2) of varieties with smaller Betti numbers. For example, denote by ${}_c\mathfrak{C}_n^\rho$ the class of nondegenerate nonsingular n-dimensional projective varieties X such that $\mu_1(X) < cd^{\rho+1}$, where $0 < \rho \leq 1$ and $c > 0$. Since $\mu_1(X) = 2\pi_1(X) + 2d - 2$, where $\pi_1(X) = g(X_1)$ is the sectional genus of X (cf. Definition 1.1), bounding μ_1 is equivalent to bounding the sectional genus π_1. Castelnuovo's bound shows that ${}_c\mathfrak{C}_n^1 = \mathfrak{C}_n$ provided that c is large enough, e.g. $c = 1$ (asymptotically it suffices to take $c = 1/a$, but a larger c should be chosen to kill a linear term in Castelnuovo's bound). One can introduce an equivalence relation $\overset{\rho}{\sim}$ and precedence relation $\overset{\rho}{\precsim}$ on cycle assignments on ${}_c\mathfrak{C}_n^\rho$ generalizing those introduced in Definition 1.4. For example, if κ is a cycle assignment of weight i, then we say that κ is *asymptotically ρ-small* and write $\kappa \overset{\rho}{\sim} 0$ if $|\nabla\kappa| = O(d^\alpha)$, where $\alpha < i\rho + 1$. The notions of *ρ-equivalence* and *ρ-boundedness* are defined in a similar way.

Analyzing the arguments used in the proof of Theorems 1.14 and 2.2 and Corollaries 1.15 and 1.16, one sees that the only point that might fail for their analogues in our setting (that is, on ${}_c\mathfrak{C}_n^\rho$) is Proposition 1.12. Since

$$\mu_k \leq r_k \leq r_1^k/d^{k-1} < c^k d^{k\rho+1}, \quad k = 1, \dots, n, \tag{3.1.1}$$

(cf. [Za, Corollary 1.5]), (1.12.10) shows that

$$r_i < \mu_i + \sum_{j=1}^{[i/2]} \binom{i}{j} (d-1)^j \mu_{i-2j} < \mu_i + \sum_{j=1}^{[i/2]} \binom{i}{j} c^i (d-1)^j d^{(i-2j)\rho+1}, \tag{3.1.2}$$

and so

$$\rho > 1/2, \quad \mu_i \overset{\rho}{\sim} r_i, \quad i = 1, \dots, n \tag{3.1.3}$$

(with $\alpha = (i-2)\rho + 2$). Thus the analogue of Proposition 1.12 is still true provided that $\rho > 1/2$. Therefore, analogues of Theorems 1.14 and 2.2 and Corollaries 1.15 and 1.16 hold on ${}_c\mathfrak{C}_n^\rho$ for arbitrary $c > 0$ and $\rho > 1/2$ (and actually the results cited above are special cases of these analogues for $\rho = 1$).

For $\rho \leq 1/2$ the situation is different, and one needs to get more information on the ramification cycle used in the proof of Proposition 1.12. Hopefully, this will yield properties of the limit distribution of the Hodge weights α^p. To illustrate this point, consider the following example.

3.2. Example. Let $X \subset \mathbb{P}^N$, $n = 2$, $N = a + 2$ be a nonsingular complete intersection of a_j hypersurfaces of degree d^{λ_j}, $j = 1, \dots, k$, $a_1 + \dots + a_k = a$, $\lambda_1 > \dots > \lambda_k$, $a_1\lambda_1 + \dots + a_k\lambda_k = 1$. It is easy to see that, for such a surface X, $\mu_1(X) \sim a_1 d^{\lambda_1+1}$, and so $X \in {}_{a_1}\mathfrak{C}_2^{\lambda_1}$. Furthermore, using Severi's computations for classes (cf. [Sev, Cap. II, §6] or [SR, Ch. IX, §5] and also [Za, Theorems 1.21 and 2.20]) and Hirzebruch's computations for Hodge numbers (cf. [Hi, §22]), it is easy to check that

$$\begin{aligned} &\mu_2(X) \sim c_2(X) = e(X) \sim b_2(X) \sim \binom{a_1+1}{2} d^{2\lambda_1+1}, \\ &c_1^2(X) = (K_X^2) \sim \frac{\mu_1^2}{d} \sim a_1^2 d^{2\lambda_1+1}, \qquad c_1^2 \sim \frac{2a_1}{a_1+1} c_2, \end{aligned} \tag{3.2.1}$$

and so, by Noether's formula,

$$\lim_{d\to\infty} \alpha^0 = \lim_{d\to\infty} \alpha^2 = \frac{1}{4} - \frac{1}{6(a_1+1)}, \qquad \lim_{d\to\infty} \alpha^1 = \frac{1}{2} + \frac{1}{3(a_1+1)}. \tag{3.2.2}$$

If $\lambda_1 > 1/2$, then clearly $a_1 = 1$, and so $c_1^2 \sim c_2$ and $\alpha^p = \alpha_0^p$, $0 \le p \le 2$, in accordance with our preceding remark. The maximal values of c_1^2/c_2 and α^0 and the minimal value of α^1 are attained when $a_1 = a$, i.e. $k = 1$; as a_1 and a grow to infinity, c_1^2/c_2 and $\alpha^0 = \alpha^2$ grow from 1 to 2 and from $1/6$ to $1/4$ respectively, and α^1 decreases from $2/3$ to $1/2$. It is instructive to observe that the limit values of c_1^2/c_2 and α^p do not actually depend on $\rho = \lambda_1$ and that for general complete intersections the limit distribution of Hodge weights coincides with the Eulerian distribution α_0^p while for special complete intersections (when there are several equations of maximal degree) the limit distribution becomes more flat.

(iii) It is desirable to give a nice general definition of a class of projective varieties, to prove analogues of Corollary 1.16 and Theorem 2.2 for varieties in such a class having large Betti numbers, to study the limit distribution of Hodge weights α^p, and to understand the role played by examples, such as complete intersections. It would also be nice to give a combinatorial interpretation of the limit distribution of Hodge weights of complete intersections generalizing the one given for hypersurfaces in terms of volumes of slabs of the unit cube and Eulerian numbers.

(iv) An analogue of Theorem 2.2 does not hold for the Hodge weights corresponding to Hodge numbers $h^{p,q}$ with $p + q \neq n$. This is connected with the fact that the classes $\mu_i(X)$ are related to the Betti numbers $b_i(X_i)$ of linear sections of X rather than to the Betti numbers $b_i(X)$ of X itself, and, by Lefschetz's theory, these last numbers are usually smaller than the first ones (cf. [Za, Section 4, Remark 10]). It would be interesting to classify varieties of large degree for which $h^{p,q}$ attains its maximum (i.e. to prove an analogue of Theorem 2.5) and to study the asymptotic behaviour of the Hodge weights $\alpha_i^p = h^{p,q}/b_{p+q}$ for $p + q = i < n$ (by Lefschetz's theory, it suffices to solve this problem for $n = i + 1$).

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