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Idempotent semigroups and tropical algebraic sets

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Abstract. The tropical semifield, i.e., the real numbers enhanced by the operations of addition and maximum, serves as a base of tropical mathematics. Addition is an abelian group operation, whereas the maximum defines an idempotent semigroup structure. We address the question of the geometry of idempotent semigroups, in particular, tropical algebraic sets carrying the structure of a commutative idempotent semigroup. We show that commutative idempotent semigroups are contractible, that systems of tropical polynomials, formed from univariate monomials, define subsemigroups with respect to coordinatewise tropical addition (maximum); and, finally, we prove that the subsemigroups in $\mathbb{R}^n$ which are either tropical hypersurfaces, or tropical curves in the plane or in the three-space have the above polynomial description.

Keywords. Tropical geometry, polyhedral complexes, tropical polynomials, idempotent semigroups, simple polynomials

1. Introduction

Tropical geometry is a geometry over the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with the operations of tropical addition and multiplication

$$a \oplus b = \max\{a, b\}, \quad a \odot b = a + b$$

(cf. [IMS, M1, M4, RST]). We equip $\mathbb{T}^* = \mathbb{R}$ with Euclidean topology, assuming that $\mathbb{T}$ is homeomorphic to $[0, \infty)$. In this setting, tropical varieties appear to be certain finite rational polyhedral complexes. The simplest examples of tropical varieties, $\mathbb{R} = \mathbb{T}^*$ and $\mathbb{T}$, carry algebraic structures: for instance, $\mathbb{R}$ is an abelian group with respect to the tropical multiplication and is a commutative idempotent semigroup with respect to tropical addition, whereas $\mathbb{T}$ is a semigroup with respect to both the operations. Thus, it is natural to inquire about algebraic and geometric properties of tropical varieties, equipped with one of these structures.

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Tropical varieties with a group structure have been studied in [G, MZ]. In particular, tropical abelian varieties, i.e., tropical varieties which are abelian groups whose operations are regular tropical functions, say, tropical Jacobians, are just real tori (products of circles and lines).

On the other hand, tropical varieties (and more generally, tropical algebraic sets) enhanced with a structure of an idempotent semigroup, have not been considered yet. In this paper, we focus on the geometric and algebraic properties of such tropical varieties. After general considerations, resulting in the claim that connected topological idempotent semigroups with a nontrivial center are contractible (Theorem 2.3), we turn to an interesting particular case of subsemigroups in $\mathbb{R}^n$ equipped with the coordinatewise tropical addition $\oplus$. Observing that the tropical power induces an endomorphism of $(\mathbb{R}, \oplus)$, we conclude that tropical polynomials consisting of only univariate monomials (termed *simple polynomials*) define subsemigroups of $(\mathbb{R}^n, \oplus)$. Yet, not every polyhedral complex which is a subsemigroup of $(\mathbb{R}^n, \oplus)$ can be defined by only simple polynomials. However, we conjecture that such subsemigroups which are also tropical varieties (called *additive tropical varieties*) can be defined by simple polynomials, and we prove this conjecture for additive tropical hypersurfaces of arbitrary dimension (Theorem 5.1) and for additive tropical curves in the plane and in the three-space (Theorem 6.4). As a consequence, we show that, for any additive tropical variety, its skeletons support additive tropical subvarieties (Theorem 7.1), and thus the connected components of all skeletons are contractible.

2. Topology of idempotent semigroups

A *topological semigroup* is a pair (U, ψ), where U is a topological space, and $\psi : U^2 \to U$ is continuous and associative, i.e.,

$$\psi(u, \psi(v, w)) = \psi(\psi(u, v), w), \qquad u, v, w \in U.$$

In what follows, we consider only topological semigroups and therefore we will omit the word "topological" and write semigroups, for short. Moreover, when the operation is clear from the context, we write U for (U, ψ). Also, we will often write uv instead of $\psi(u, v)$; no confusion will arise.

The *center* of a semigroup (U, ψ) is defined to be the set

$$C(U, \psi) := \{u \in U : \psi(u, v) = \psi(v, u) \text{ for all } v \in U\}.$$

A semigroup (U, ψ) is called *idempotent* if $\psi(u, u) = u$ for all $u \in U$.

We start with two simple observations.

Lemma 2.1. *Any connected component of an idempotent semigroup is a subsemigroup.*

Proof. Let (U, ψ) be an idempotent semigroup with $U_0 \subset U$ a connected component. Then $\psi(U_0 \times U_0)$ is connected, and since the diagonal remains in U_0, it is contained in U_0. □

Lemma 2.2. *Any commutative idempotent semigroup U is a directed poset with respect to the relation*

$$v \succ u \;\Leftrightarrow\; v = au \text{ for some } a \in U, \tag{2.1}$$

which in its turn is compatible with the semigroup operation in the following sense:

$$v \succ u \;\Rightarrow\; vw \succ uw \text{ for all } w \in U.$$

Proof. Reflexivity and transitivity of relation (2.1) are immediate. Next, if $u \succ v$ and $v \succ u$, then $u = va$, $v = ub$, and we obtain

$$u = va = uab = uabb = ub = v.$$

Hence, relation (2.1) defines a partial order. Since $u \prec uv$ and $v \prec uv$ for any $u, v \in U$, we obtain a directed set. Finally,

$$v \succ u \;\Rightarrow\; v = au \;\Rightarrow\; vw = a(uw) \;\Rightarrow\; vw \succ uw. \qquad \square$$

Our main observation is the following:

Theorem 2.3. *Let (U, ψ) be an idempotent semigroup with a nonempty center, and let U be a connected topological space homotopy equivalent to a CW-complex. Then U is contractible.*

Proof. By assumption, there exists $u_0 \in C(U, \psi)$. We shall show that $\pi_k(U, u_0) = 0$ for all $k \geq 1$. This will yield the contractibility by the classical Whitehead theorem.

Represent the elements of $\pi_k(U, u_0)$ by maps $\gamma : I^k \to U$, where $I = [0, 1]$, $\gamma(\partial I^k) = u_0$, taken up to homotopy relative to ∂I^k. The operation in $\pi_k(U, u_0)$ is then induced by the composition of $\gamma_1, \gamma_2 : I^k \to U$ with $\gamma_1(\partial I^k) = \gamma_2(\partial I^k) = u_0$, defined as

$$\gamma_1 * \gamma_2 : I^k \to U, \qquad \gamma_1 * \gamma_2(t_1, \ldots, t_k) = \begin{cases} \gamma_1(2t_1, t_2, \ldots, t_k), & 0 \leq t_1 \leq 1/2, \\ \gamma_2(2t_1 - 1, t_2, \ldots, t_k), & 1/2 \leq t_1 \leq 1. \end{cases}$$

For each map $\gamma : I^k \to U$ with $\gamma(\partial I^k) = u_0$, we have

$$\gamma = \psi(\gamma, \gamma) \overset{h}{\sim} \psi(u_0 * \gamma, \gamma * u_0) = \psi(u_0, \gamma) * \psi(\gamma, u_0) = \psi(u_0, \gamma) * \psi(u_0, \gamma),$$

where $\overset{h}{\sim}$ stands for homotopy relative to ∂I^k, and u_0 means the constant map. Furthermore,

$$\begin{aligned} \psi(u_0, \gamma) &\overset{h}{\sim} \psi(u_0, \psi(u_0, \gamma) * \psi(u_0, \gamma)) \\ &= \psi(u_0, \psi(u_0, \gamma)) * \psi(u_0, \psi(u_0, \gamma)) = \psi(u_0, \gamma) * \psi(u_0, \gamma); \end{aligned}$$

which altogether means that $\gamma \overset{h}{\sim} u_0$, and we are done. $\square$

Remark 2.4. The hypothesis on the nonemptiness of the center cannot be discarded from Theorem 2.3, since, for example, in any topological space U one can define the structure of an idempotent semigroup by letting $\psi(u, v) = u$ for any $u, v \in U$.

A natural question arising from the preceding discussion is:

Question 2.5. *Does any contractible topological space admit the structure of an idempotent semigroup with a nonempty center?*

This is indeed so in the following particular situation.

Proposition 2.6. *Any one-dimensional contractible CW-complex admits the structure of a commutative idempotent semigroup.*

Proof. Let U be a 1-dimensional contractible CW-complex and pick a point $u_0 \in U$. For any point $u \in U$ there is a unique path $\gamma_u \subset U$ joining u to u_0 that is homeomorphic either to $I = [0, 1]$, or to a point, according as $u \neq u_0$ or $u = u_0$. The intersection of two paths γ_u and γ_v, $u, v \in U$, is a path γ_w, for some $w \in U$, and thus, we define

$$uv = w \quad \text{whenever} \quad \gamma_u \cap \gamma_v = \gamma_w.$$

It is easy to check that this operation is associative, commutative, and idempotent. □

Contractible tropical curves are called *rational* [M3]. Accordingly, our results show that a tropical curve carries the structure of a commutative idempotent semigroup iff it is rational. This contrasts with the case of compact tropical curves having the structure of an abelian group: these are elliptic (or more precisely, homeomorphic to a circle [MZ]).

3. Basic tropical algebraic geometry

For the reader's convenience we first recall some necessary definitions and facts in tropical geometry; these can be found in [AR, EKL, G, GKM, IMS, M2, RST]. We also introduce some new notions to be used throughout the text.

3.1. Tropical polynomials and tropical algebraic sets

A *tropical polynomial* is an expression of the form

$$f = \bigoplus_{\omega \in \Omega} A_\omega \odot \lambda_1^{\omega_1} \odot \cdots \odot \lambda_n^{\omega_n}, \tag{3.1}$$

where $\Omega \subset \mathbb{Z}^n$ is a finite nonempty set of points $\omega = (\omega_1, \dots, \omega_n)$ with nonnegative coordinates, and $A_\omega \in \mathbb{R}$ for all $\omega \in \Omega$; here and throughout, the power a^m means a repeated m times, i.e., $a^m = \underbrace{a \odot \cdots \odot a}_{m} = ma$. We write a polynomial as $f = \bigoplus_{\omega \in \Omega} A_\omega \odot \Lambda^\omega$, where Λ^ω stands for $\lambda_1^{\omega_1} \odot \cdots \odot \lambda_n^{\omega_n}$, and we denote the semiring of tropical polynomials by $\mathbb{T}[\Lambda]$. Abusing notation, we will sometimes write $f(\lambda_{i_1}, \dots, \lambda_{i_m})$ for $f \in \mathbb{T}[\Lambda]$, indicating that f involves only the variables $\lambda_{i_1}, \dots, \lambda_{i_m}$.

Any tropical polynomial $f \in \mathbb{T}[\Lambda] \setminus \{-\infty\}$ determines a piecewise linear convex function $f : \mathbb{R}^n \to \mathbb{R}$:

$$f = \bigoplus_{\omega \in \Omega} A_\omega \odot \Lambda^\omega \mapsto f(\boldsymbol{u}) = \max_{\omega \in \Omega}(\langle \boldsymbol{u}, \omega \rangle + A_\omega) , \tag{3.2}$$

where $\boldsymbol{u}$ stands for the n-tuple $(u_1, \dots, u_n) \in \mathbb{R}^n$, and $\langle *, * \rangle$ is the standard scalar product. Unlike the classical polynomials over an infinite field, here the map of $\mathbb{T}[\Lambda]$ to the space of functions is not injective. Some of the linear functions on the right-hand side of (3.2) can be omitted without changing the function; we call the corresponding monomials of f *inessential*, while the other monomials are called *essential*.

Remark 3.1. We denote a tropical polynomial and the corresponding function by the same symbol; no confusion will arise. Whenever we write an expression with formal variables λ_i, we assume a polynomial, otherwise we mean a function. The value of the function corresponding to a polynomial $f \in \mathbb{T}[\Lambda]$ at a point $\boldsymbol{u} \in \mathbb{T}^n$ is denoted by $f(\boldsymbol{u})$ or $f(\Lambda)|_{\boldsymbol{u}}$.

Given a tropical polynomial f, $Z_{\mathbb{T}}(f)$ is defined to be the set of points $\boldsymbol{u} \in \mathbb{T}^n$ on which the value $f(\boldsymbol{u})$ is either equal to $-\infty$, or attained by at least two of the monomials on the left-hand side of (3.2). When $f \in \mathbb{T}[\Lambda] \setminus \{-\infty\}$ is nonconstant, the set $Z_{\mathbb{T}}(f)$ is a proper nonempty subset of $\mathbb{T}^n$, and is called an *affine tropical hypersurface*. Note that for the constant polynomial $f = -\infty$ we have $Z_{\mathbb{T}}(-\infty) = \mathbb{T}^n$.

Letting $I = \langle f_1, \dots, f_s \rangle \subset \mathbb{T}[\Lambda]$ be a finitely generated ideal, the set

$$Z_{\mathbb{T}}(I) := \bigcap_{f \in I} Z_{\mathbb{T}}(f) \subset \mathbb{T}^n$$

is called an *affine tropical* (*algebraic*) *set*. Clearly, $Z_{\mathbb{T}}(I) = Z_{\mathbb{T}}(f_1) \cap \dots \cap Z_{\mathbb{T}}(f_s)$. Indeed, taking a polynomial $f = g_1 \odot f_1 \oplus \dots \oplus g_s \odot f_s \in I$ and a point $\boldsymbol{u} \in Z_{\mathbb{T}}(f_1) \cap \dots \cap Z_{\mathbb{T}}(f_s)$, we have $f(\boldsymbol{u}) = g_i(\boldsymbol{u}) + f_i(\boldsymbol{u})$ for some $i = 1, \dots, s$. Suppose $f(\boldsymbol{u}) \neq -\infty$. Then the value of $g_i(\boldsymbol{u})$ is attained by a monomial $B_\mu \odot \Lambda^\mu$ of g_i and the value of $f_i(\boldsymbol{u})$ is attained by some pair of monomials of f_i, say $A_{\omega'} \odot \Lambda^{\omega'}$ and $A_{\omega''} \odot \Lambda^{\omega''}$. Thus, the value of $f(\boldsymbol{u})$ is attained by the two monomials $A_{\omega'} \odot B_\mu \odot \Lambda^{\omega' + \mu}$ and $A_{\omega''} \odot B_\mu \odot \Lambda^{\omega'' + \mu}$ of f; that is, $\boldsymbol{u} \in Z_{\mathbb{T}}(f)$.

It is more convenient (and customary) to consider tropical algebraic sets in $\mathbb{R}^n \subset \mathbb{T}^n$ (a tropical torus, cf. [M4]). So, for a tropical polynomial $f \in \mathbb{T}[\Lambda] \setminus \{-\infty\}$, we let

$$Z(f) := Z_{\mathbb{T}}(f) \cap \mathbb{R}^n .$$

This set can be viewed as the corner locus of the function f, i.e., the set of points $\boldsymbol{u} \in \mathbb{R}^n$ on which f is not differentiable, or equivalently, the set of points $\boldsymbol{u} \in \mathbb{R}^n$ where the value $f(\boldsymbol{u})$ is attained by at least two of the linear functions on the right-hand side of (3.2). For example, $Z(f)$ is nonempty as long as f contains at least two monomials. Given a finitely generated ideal $I = \langle f_1, \dots, f_s \rangle \subset \mathbb{T}[x_1, \dots, x_n]$, the set $Z(I) := Z_{\mathbb{T}}(I) \cap \mathbb{R}^n$ is called a *tropical* (*algebraic*) *set in* $\mathbb{R}^n$.

3.2. Tropical varieties

A *finite polyhedral complex* (briefly, *FPC*) in $\mathbb{R}^n$ is a pair $(P, \mathcal{P})$, where $P \subset \mathbb{R}^n$ and $\mathcal{P}$ is a finite set of distinct convex closed polyhedra in $\mathbb{R}^n$, such that:

- $P = \bigcup_{\sigma \in \mathcal{P}} \sigma$;
- if $\sigma \in \mathcal{P}$, then any proper face of σ also belongs to $\mathcal{P}$;
- if $\delta, \sigma \in \mathcal{P}$, then $\delta \cap \sigma$ is either empty, or is a common face (not necessarily proper) of δ and σ.

Let $\dim(P, \mathcal{P}) = \max\{\dim(\sigma) : \sigma \in \mathcal{P}\}$. An FPC $(P, \mathcal{P})$ is said to be *pure-dimensional* if any $\delta \in \mathcal{P}$ is a face of some $\sigma \in \mathcal{P}$ with $\dim(\sigma) = \dim(P, \mathcal{P})$. An FPC $(P, \mathcal{P})$ is called *rational* if all the linear spaces

$$\mathbb{R}\sigma := \{\boldsymbol{u} - \boldsymbol{u}' : \boldsymbol{u}, \boldsymbol{u}' \in \sigma\}, \qquad \sigma \in \mathcal{P},$$

are defined over $\mathbb{Q}$.

It is not difficult to see that tropical sets are rational FPC and vice versa.

An m-dimensional *tropical variety* in $\mathbb{R}^n$, $n > m$, is a rational FPC $(P, \mathcal{P})$ of pure dimension m equipped with the weight function w which is defined on the set of top-dimensional cells of $(P, \mathcal{P})$, has positive integral values, and satisfies the balancing condition at any cell $\tau \in \mathcal{P}$ of dimension $m - 1$:

$$\sum w(\sigma)\boldsymbol{v}_\tau(\sigma) = 0 \in \mathbb{Z}^n/\mathbb{Z}\tau \ , \tag{3.3}$$

where the sum is taken over all m-dimensional $\sigma \in \mathcal{P}$ containing τ as a face, $\mathbb{Z}\tau = \mathbb{R}\tau \cap \mathbb{Z}^n$, and $\boldsymbol{v}_\tau(\sigma)$ is a generator of the lattice $\mathbb{Z}\sigma/\mathbb{Z}\tau$ oriented inside the cone based at τ and directed by σ.

In this paper we deal mainly with a weaker notion of tropical variety which we call a *tropical set-variety*:

Definition 3.2. Let $(P, \mathcal{P}, w)$ be a tropical variety in $\mathbb{R}^n$. We call the set P a *tropical set-variety*.

Namely, when working with a tropical set-variety, we get rid of the weight function and the FPC structure. However, a tropical set-variety can be canonically represented as a union of convex polyhedra. Given an m-dimensional tropical set-variety P, we denote by $\mathrm{Reg}(P)$ the set of points of P where P is locally homeomorphic to $\mathbb{R}^m$.

Lemma 3.3. *Let P be an m-dimensional tropical set-variety. Then:*

- *the closures of the connected components of* $\mathrm{Reg}(P)$ *are rational m-dimensional convex polyhedra;*
- *if K_1, K_2 are two connected components of* $\mathrm{Reg}(P)$*, and* $\dim(\overline{K}_1 \cap \overline{K}_2) = m - 1$*, then $\sigma = \overline{K}_1 \cap \overline{K}_2$ is a common face of $\overline{K}_1$ and $\overline{K}_2$.*

Proof. The case of $m = 1$ is evident, and we assume that $m \geq 2$.

Suppose that the closure $\overline{K}$ of a connected component K of $\mathrm{Reg}(X)$ is not convex, that is, there are closed convex polyhedra $\sigma, \tau, \xi \subset \partial\overline{K}$, $\dim(\sigma) = \dim(\tau) = m - 1$, $\dim(\xi) = m - 2$, $\xi = \sigma \cap \tau$, such that $\overline{K}$ is not convex in a neighborhood of a point $\boldsymbol{x} \in \mathrm{Int}(\xi)$.[1] Without loss of generality, we may assume that P is a cone with vertex $\boldsymbol{x}$

[1] By the interior of a convex polyhedron we always mean its relative interior (i.e., in its closure).

(the weight function and the balancing condition will be naturally inherited by the cone from any structure of tropical variety on P).

Take an $(n-m+2)$-dimensional subspace V of $\mathbb{R}^n$ defined over $\mathbb{Q}$, passing through $\boldsymbol{x}$ and transverse to ξ. It supports a tropical variety with one cell of weight 1 whose intersection with P is a two-dimensional tropical set-variety (see [AR, GKM] for details) which possesses a connected component $K \cap V$ of its regular part with a nonconvex closure $\overline{K} \cap V$; more precisely, this component is the complement of the convex sector S spanned by the rays $\sigma \cap V$ and $\tau \cap V$ in the two-plane $\Pi = \boldsymbol{x} + \mathbb{R}K \cap V$.

Let $W \subset \mathbb{R}^n$ be a hyperplane defined over $\mathbb{Q}$ (again a tropical set-variety) containing the plane Π and transverse to each one-dimensional cell of $P \cap V$ which is not parallel to Π. Then $P \cap V \cap (a + W)$, for a small generic vector $\boldsymbol{a} \in \mathbb{R}^n$, is a tropical set-curve, whose projection to Π (being a plane tropical set-curve, push-forward in the terminology of [AR, GKM]) is contained in a neighborhood of the sector S, which contradicts the balancing condition.

Now suppose that, for some two connected components K_1, K_2 of $\mathrm{Reg}(P)$, their intersection $\sigma = \overline{K}_1 \cap \overline{K}_2$ is not a common face and has dimension $m-1$. Two situations are possible: either $\sigma \subset \partial\overline{K}_1 \cap \partial\overline{K}_2$, or $\sigma \cap \mathrm{Int}(\overline{K}_1) \neq \emptyset$. They both can be viewed as a limit case of the preceding considerations when either σ, τ lie in the same $(m-1)$-face of $\overline{K}_1$, or $\sigma = \tau$. Then the above argument literally leads either to a plane tropical set-curve different from a straight line and lying in a half-plane, or to a plane set-curve in a neighborhood of a ray. Both the cases contradict the balancing condition. □

Definition 3.4. For an m-dimensional tropical set-variety X (in $\mathbb{R}^n$), denote by $X^{(m-1)}$ the union of the $(m-1)$-dimensional faces of the closures of the connected components of $\mathrm{Reg}(X)$.

Question 3.5. *Is $X^{(m-1)}$ a tropical set-variety?*

The answer is yes for tropical set-curves (evident), tropical set-hypersurfaces (commented in the next section), and for additive tropical set-varieties (shown in Section 7).

3.3. Tropical hypersurfaces

An important example of a tropical variety is a *tropical hypersurface*, i.e., a tropical variety in $\mathbb{R}^n$ of dimension $n-1$. By [M2, Proposition 2.4, Corollary 2.5], for any tropical hypersurface $(P, \mathcal{P}, w)$ in $\mathbb{R}^n$, there exists a tropical polynomial $f(\lambda_1, \dots, \lambda_n)$ which satisfies $P = Z(f)$ and possesses a number of properties listed below.

If f is given by (3.1), then the function ν_f Legendre dual to f is defined on the Newton polytope Δ_f of f (the convex hull of the set Ω of (3.1)), is convex and piecewise linear. The graph of ν_f can be viewed as the lower part of the convex hull of the set $\{(\omega, -A_\omega) \in \mathbb{R}^{n+1} : \omega \in \Delta \cap \mathbb{Z}^n\}$, with A_ω being the coefficients from formula (3.1). The maximal linearity domains of ν_f and their faces (which are all convex lattice polytopes) define an FPC structure $S(f)$ on Δ_f. This structure is dual to the FPC structure Σ_P on $\mathbb{R}^n$, given by $\mathcal{P}$ and the closures of the connected components of $\mathbb{R}^n \setminus P$. Namely, there is a one-to-one correspondence between the cells of $S(f)$ and the cells of Σ_P which inverts the incidence relation and is such that:

- the vertices of $S(f)$ on $\partial\Delta_f$ correspond to the closures of the unbounded components of $\mathbb{R}^n \setminus P$, and the vertices of $S(f)$ in $\mathrm{Int}(\Delta_f)$ correspond to the closures of the bounded components of $\mathbb{R}^n \setminus P$,
- cells of dimension $m > 0$ in $S(f)$ correspond to cells of dimension $n - m$ in $\mathcal{P}$, and the corresponding cells are orthogonal,
- the weight of an $(n-1)$-dimensional cell of $\mathcal{P}$ equals the lattice length of the dual segment of $S(f)$.

It follows immediately that the vertices of the subdivision $S(f)$, or equivalently, the components of $\mathbb{R}^n \setminus P$, correspond bijectively to the essential monomials of f; in particular, the vertices of Δ always correspond to the essential monomials of f. Another immediate consequence is that the Newton polytopes of tropical polynomials g such that $Z(g) = P$ and their FPC structure $S(g)$ have the same combinatorial type so that the corresponding cells are parallel.

In connection to Question 3.5, we recall the following well-known fact, supplying it with a simple proof.

Lemma 3.6. *The proper faces of the closures of the connected components of the complement in $\mathbb{R}^n$ to a tropical set-hypersurface P define a FPC structure $\mathcal{P}$ on P, and, for any $k = 0, \dots, n-2$, the set $P^{(k)} = \bigcup_{\sigma\in\mathcal{P},\,\dim(\sigma)\le k}\sigma$ is a k-dimensional tropical set-variety.*

Proof. If $k = 0$, then $P^{(0)}$ is a finite set which is always a zero-dimensional tropical set-variety. So, fix $0 < k < n-1$ and let $P = Z(f)$ for some tropical polynomial f. Let w_k be the weight function $w_k(\sigma) = \mathrm{Vol}_{\mathbb{Z}}(\sigma^*)$, $\sigma \in \mathcal{P}$, $\dim(\sigma) = k$, where σ^* is the dual polytope in the subdivision $S(f)$ of the Newton polytope Δ of f, and $\mathrm{Vol}_{\mathbb{Z}}(\sigma^*)$ is the lattice volume of σ^* (i.e., the ratio of the Euclidean k-dimensional volume $\mathrm{Vol}_{\mathbb{R}}(\sigma^*)$ and $\mathrm{Vol}_{\mathbb{R}}(\Delta_\sigma)$, with Δ_σ being the minimal lattice simplex in $\mathbb{R}\sigma$).

We will show that $P^{(k)}$ with the FPC structure $\mathcal{P}^{(k)} = \{\sigma \in \mathcal{P} : \dim(\sigma) \le k\}$ and the weight function $w_k(\sigma)$ is a k-dimensional topical variety. We pick $\tau \in \mathcal{P}$ with $\dim(\tau) = k-1$, and prove that

$$\sum_{\substack{\sigma\in\mathcal{P},\,\dim(\sigma)=k\\ \tau\subset\sigma}} \mathrm{Vol}_{\mathbb{Z}}(\sigma^*)\cdot \boldsymbol{v}_\tau(\sigma) = 0 \in \mathbb{Z}^n/\mathbb{Z}\tau,$$

or equivalently,

$$\sum_{\substack{\sigma\in\mathcal{P},\,\dim(\sigma)=k\\ \tau\subset\sigma}} \mathrm{Vol}_{\mathbb{Z}}(\sigma^*)\cdot \boldsymbol{v}_\tau^{\perp}(\sigma) = 0, \tag{3.4}$$

where

$$\boldsymbol{v}_\tau(\sigma) = \boldsymbol{v}_\tau^{\perp}(\sigma) + \boldsymbol{v}_\tau^{\parallel}(\sigma), \qquad \boldsymbol{v}_\tau^{\parallel}(\sigma) \in \mathbb{R}\tau,\ \boldsymbol{v}_\tau^{\perp}(\sigma) \perp \mathbb{R}\tau.$$

Notice that

$$\boldsymbol{v}_\tau^{\perp}(\sigma) = \frac{\mathrm{Vol}_{\mathbb{R}}(\Delta_\sigma)}{k\mathrm{Vol}_{\mathbb{R}}(\Delta_\tau)}\boldsymbol{n}_\tau(\sigma)\ ,$$

where $\boldsymbol{n}_\tau(\sigma)$ is the unit vector in $\mathbb{R}\sigma$ orthogonal to $\mathbb{R}\tau$ and directed inside σ. Observing that $\mathrm{Vol}_{\mathbb{R}}(\Delta_\sigma) = \mathrm{Vol}_{\mathbb{R}}(\Delta_{\sigma^*})$, we rewrite (3.4) as

$$\sum_{\substack{\tau \subset \sigma \in \mathcal{P} \\ \dim(\sigma)=k}} \mathrm{Vol}_{\mathbb{R}}(\sigma^*) \cdot \boldsymbol{n}_\tau(\sigma) = 0. \tag{3.5}$$

Finally, observe that $\boldsymbol{n}_\tau(\sigma)$ is the outer normal in $\mathbb{R}\tau^*$ to the facet σ^* of the polytope τ^*, and thus (3.5) turns into the polytopal Stokes formula. ◻

4. Simple additive tropical sets

Subsemigroups of $(\mathbb{R}^n, \oplus)$ which are tropical algebraic sets are called *additive tropical sets*.

Lemma 4.1. *Let $u_1, \dots, u_n \in \mathbb{R}$. Then $\left(\bigoplus_{i=1}^n u_i\right)^s = \bigoplus_{i=1}^n u_i^s$ for all $n, s \in \mathbb{N}$.*

Proof. We apply double induction on s and n. Fixing $n = 2$, the case $s = 1$ is evident. Then the induction step from $s-1$ to s (where $s \geq 2$) goes as follows:

$$\begin{aligned}(u_1 \oplus u_2)^s &= (u_1 \oplus u_2) \odot (u_1 \oplus u_2)^{s-1} = (u_1 \oplus u_2) \odot (u_1^{s-1} \oplus u_2^{s-1}) \\ &= u_1^s \oplus u_1^{s-1} \odot u_2 \oplus u_1 \odot u_2^{s-1} \oplus u_2^s.\end{aligned}$$

When $u_1 = u_2$ the required equality is clear: $(u_1 \oplus u_1)^s = u_1^s = u_1^s \oplus u_1^s$. If $u_1 > u_2$, then $u_1^s > u_1^{s-1} \odot u_2 \oplus u_1 \odot u_2^{s-1} \oplus u_2^s$, and hence $(u_1 \oplus u_2)^s = u_1^s = u_1^s \oplus u_2^s$. The case of $u_1 < u_2$ is treated similarly. The proof is then completed by induction on n. ◻

A tropical polynomial $f \in \mathbb{T}[\Lambda]$ is called *simple* if each of its monomials is univariate, or a constant.

Corollary 4.2. *Any tropical algebraic set $Z(I) \subset \mathbb{R}^n$, where the ideal $I \subset \mathbb{T}[\Lambda]$ is finitely generated by simple polynomials, is additive.*

Proof. It is sufficient to prove that, for any simple polynomial $f \in \mathbb{T}[\Lambda]$, the set $Z(f) \subset \mathbb{R}^n$ is closed under the operation $\oplus$.

Given $\boldsymbol{u}, \boldsymbol{v} \in Z(f)$, by Lemma 4.1, we have

$$f(\boldsymbol{u} \oplus \boldsymbol{v}) = f(\boldsymbol{u}) \oplus f(\boldsymbol{v}) = \max\{f(\boldsymbol{u}), f(\boldsymbol{v})\}.$$

Suppose that $f(\boldsymbol{u} \oplus \boldsymbol{v}) = f(\boldsymbol{u})$. Since $\boldsymbol{u} \in Z(f)$, we have $f(\boldsymbol{u}) = M_1(\boldsymbol{u}) = M_2(\boldsymbol{u})$ for some two distinct monomials M_1 and M_2 of f. By our assumption and by Lemma 4.1,

$$f(\boldsymbol{u} \oplus \boldsymbol{v}) = f(\boldsymbol{u}) = M_1(\boldsymbol{u}) = M_2(\boldsymbol{u}) \geq \max\{M_1(\boldsymbol{v}), M_2(\boldsymbol{v})\},$$

and hence $M_i(\boldsymbol{u} \oplus \boldsymbol{v}) = M_i(\boldsymbol{u}) \oplus M_i(\boldsymbol{v}) = M_i(\boldsymbol{u}) = f(\boldsymbol{u} \oplus \boldsymbol{v})$, $i = 1, 2$; that is, $\boldsymbol{u} \oplus \boldsymbol{v} \in Z(f)$. ◻

An additive tropical set of the form $Z(I)$ with an ideal $I \subset \mathbb{T}[\Lambda]$ finitely generated by simple polynomials is called a *simple additive tropical set*.

Not all additive tropical sets are simple; for example, the horizontal ray

$$R = \{(t, 0) : t \geq 0\} \subset \mathbb{R}^2$$

is a tropical algebraic set defined by the ideal

$$I = \langle \lambda_1 \odot \lambda_2 \oplus \lambda_1 \oplus \lambda_2,\ \lambda_1 \odot \lambda_2 \oplus \lambda_1 \oplus (-1) \odot \lambda_2 \rangle \subset \mathbb{T}[\lambda_1, \lambda_2],$$

and it is additive. On the other hand, R is not simple. Indeed, due to the duality described in Section 3.3, the Newton polygon of a tropical polynomial $f \in \mathbb{T}[\lambda_1, \lambda_2]$ such that $Z(f) \supset R$ must have a (vertical) side with the outer normal $(1, 0)$. For a simple polynomial f with two variables, which may contain only monomials of the form A_0, $A_i \odot \lambda_1^i$, or $B_j \odot \lambda_2^j$, this is possible only when $f = f(\lambda_2)$, i.e., λ_1 is not involved in f. But then $Z(f)$ must contain the whole straight line through the ray R, and so does $Z(I)$ for I generated by such simple polynomials.

However, we propose the following converse to Corollary 4.2.

Conjecture 4.3. *Any additive tropical set-variety in $\mathbb{R}^n$ is simple.*

We prove this conjecture for the three particular cases: tropical set-hypersurfaces, affine subspaces of $\mathbb{R}^n$, and tropical set-curves in $\mathbb{R}^2$ and $\mathbb{R}^3$.

5. Additive tropical set-hypersurfaces and affine subspaces

Theorem 5.1. *A tropical set-hypersurface $P \subset \mathbb{R}^n$ is additive if and only if $P = Z(f)$ for some simple tropical polynomial $f \in \mathbb{T}[\Lambda]$.*

Proof. It is sufficient to prove the "only if" implication.

Step 1. Let $P = Z(f)$ be additive. Without loss of generality, since multiplication by a monomial and removal of inessential monomials does not affect $Z(f)$ (see details in Section 3.3), we may assume that f is not divisible by any monomial and it contains only essential monomials. Then all the monomials of f are encoded by points lying on the boundary of the Newton polytope Δ of f. Indeed, otherwise we would have an essential monomial corresponding to a vertex of the subdivision $S(f)$ in $\mathrm{Int}(\Delta)$, and thus this vertex would be dual to a bounded component of $\mathbb{R}^n \setminus Z(f)$. But, in view of Theorem 2.3, the latter is impossible, since $Z(f)$ is contractible while the boundary of a bounded component of $\mathbb{R}^n \setminus P$ would give a nontrivial $(n-1)$-cycle in $Z(f)$.

Step 2. Since $P \neq \emptyset$, f has at least two monomials. Let $A_\omega \odot \Lambda^\omega$ and $A_\tau \odot \Lambda^\tau$ be two monomials of f such that $\omega \neq \tau$, where $\omega, \tau \in \mathbb{Z}^n$, and the corresponding hyperplane (cf. (3.2))

$$\langle \boldsymbol{u}, \omega \rangle + A_\omega = \langle \boldsymbol{u}, \tau \rangle + A_\tau \tag{5.1}$$

contains an $(n-1)$-dimensional cell D of P. We claim that the n-tuple $\omega - \tau$ has at most two nonzero coordinates, and the product of any pair of coordinates of $\omega - \tau$ is nonpositive. Indeed, otherwise, one could write (5.1) as $a_1\lambda_1 + \cdots + a_n\lambda_n = b$ with

$a_i, a_j > 0$ for some $i \neq j$, and then one could choose two sufficiently close points $\boldsymbol{u}' = (u'_1, \dots, u'_n)$, $\boldsymbol{u}'' = (u''_1, \dots, u''_m)$ in the interior of D for which

$$u'_i > u''_i, \quad u'_j < u''_j, \quad u'_k = u''_k \quad \text{for all } k \neq i, j.$$

But then $\boldsymbol{u}' \oplus \boldsymbol{u}'' \notin D$, since this sum does not satisfy (5.1).

Step 3. Suppose that $n = 2$, and $P = Z(f)$ for $f \in \mathbb{T}[\lambda_1, \lambda_2]$. Let f contain the monomials $A_i \odot \lambda_1^i$ and $B_j \odot \lambda_2^j$ with $i, j > 0$. Assuming that f has an (essential) monomial $A_{k\ell} \odot \lambda_1^k \odot \lambda_2^\ell$ with some $k, \ell > 0$, and taking into account the conclusions of Step 1, we obtain the three vertices $(i, 0)$, $(0, j)$, and (k, ℓ) of the subdivision $S(f)$ lying on the boundary of the Newton polygon Δ. According to the conclusion of Step 2, the sides of Δ cannot be directed by vectors with positive coordinates, and hence the tropical curve U necessarily has either

- a pair of rays directed by vectors with negative coordinates (see Figure 1(a,b)), or
- a pair of rays directed by vectors with positive coordinates (see Figure 1(c,d)), or
- a pair of non-parallel rays directed by vectors with nonnegative coordinates (see Figure 1(e,f)).

(The labels e_1 and e_2 in Figure 1 denote the edges of $S(f)$ adjacent to the point (k, ℓ), the symbol Δ designates the part of the plane adjacent to the depicted fragment of the boundary in which the Newton polygon lies; we also note that in cases (a), (c), (e), and (f), the rays drawn in bold may merge to the same vertex, this does not affect our argument.)

In all the situations described above, the tropical sums of points lying on such pairs of rays sweep a two-dimensional domain in $\mathbb{R}^2$, contradicting the one-dimensionality of P.

Assume f does not contain a monomial $A_i \odot \lambda_1^i$ with $i > 0$. Since f is not divisible by any monomial, it should also contain a constant term $A_0 \in \mathbb{R}$. This implies that f has no mixed monomials $A_{k\ell} \odot \lambda_1^k \odot \lambda_2^\ell$ with $k, \ell > 0$, since otherwise the Newton polygon Δ would have a side with an outer normal whose coordinates are nonzero and have distinct signs—a contradiction to the conclusion of Step 2. Therefore f is simple.

Step 4. Suppose that $n \geq 3$, and $P = Z(f)$ with $f \in \mathbb{T}[\Lambda]$. Write f as the sum of essential monomials: $f = \bigoplus_{\omega \in \Omega} M_\omega$, where $\Omega \subset \mathbb{Z}^n$ is finite. Assume that f has an essential monomial M_τ, $\tau \in \Omega$, depending on at least two variables, say λ_1, λ_2. By definition, there are $c_1, \dots, c_n \in \mathbb{R}$ for which

$$M_\tau(c_1, \dots, c_n) > M_\omega(c_1, \dots, c_n) \quad \text{for each } \omega \in \Omega \setminus \{\tau\}. \tag{5.2}$$

Since a small variation of $c_1, \dots, c_n$ does not violate (5.2), we can take these numbers to be generic. The precise requirement is as follows: denoting by $\mathrm{pr}_{12} : \mathbb{Z}^n \to \mathbb{Z}^2$ the projection of $\mathbb{Z}^n$ to the first two coordinates of $\mathbb{Z}^n$, we rewrite f as

$$f(\lambda_1, \dots, \lambda_n) = \bigoplus_{(k_1,k_2) \in \mathrm{pr}_{12}(\Omega)} \lambda_1^{k_1} \odot \lambda_2^{k_2} \odot f_{k_1 k_2}(\lambda_3, \dots, \lambda_n),$$

where

$$f_{k_1 k_2}(\lambda_3, \dots, \lambda_n) = \bigoplus_{(k_1,k_2,k_3,\dots,k_n) \in \Omega} A_{k_1 \cdots k_n} \odot \lambda_3^{k_3} \odot \cdots \odot \lambda_n^{k_n} \tag{5.3}$$

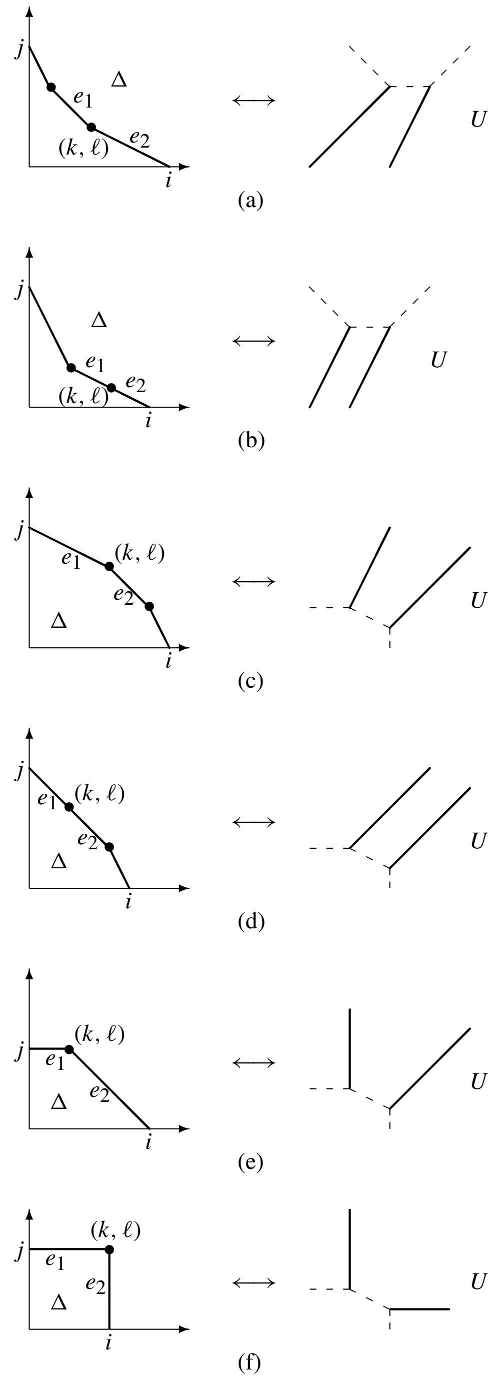

Fig. 1. Illustration to the proof of Theorem 5.1.

for $(k_1, k_2) \in \mathrm{pr}_{12}(\Omega)$. Our demand is that, for each polynomial (5.3), the values of the monomials at $(c_3, \dots, c_n)$ be distinct. Geometrically, this means that $(c_3, \dots, c_n)$ lies outside $\bigcup_{(k_1,k_2)\in \mathrm{pr}_{12}(\Omega)} Z(f_{k_1k_2}) \subset \mathbb{R}^{n-2}$, and that such a generic choice is always possible, since the latter set is a finite polyhedral complex of dimension $n-3$ in $\mathbb{R}^{n-2}$.

Let $\Pi = \{\lambda_3 = c_3, \dots, \lambda_n = c_n\} \subset \mathbb{R}^n$ be a plane in $\mathbb{R}^n$, and let $g \in \mathbb{T}[\lambda_1, \lambda_2]$ be the polynomial obtained from f by substituting $c_3, \dots, c_n$ for $\lambda_3, \dots, \lambda_n$, respectively. We claim that $P_2 := P \cap \Pi$ is the tropical set-curve in Π given by g. Indeed, if $(u_1, u_2) \in Z(g)$, then

$$u_1^{k_1} \odot u_2^{k_2} \odot f_{k_1k_2}(c_3, \dots, c_n) = u_1^{\ell_1} \odot u_2^{\ell_2} \odot f_{\ell_1\ell_2}(c_3, \dots, c_n) \tag{5.4}$$

for some $(k_1, k_2) \neq (\ell_1, \ell_2) \in \mathrm{pr}_{12}(\Omega)$, and thus f has a pair of monomials reaching the value $f(u_1, u_2, c_3, \dots, c_n)$, which, in particular, means that $(u_1, u_2, c_3, \dots, c_n) \in Z(f)$. On the other hand, if $(u_1, u_2, c_3, \dots, c_n) \in Z(f)$, then the value $f(u_1, u_2, c_3, \dots, c_n)$ is attained by a pair of monomials $M_{\omega'}, M_{\omega''}$ of f, which in addition must satisfy

$$\mathrm{pr}_{12}(\omega') = (k_1, k_2) \neq \mathrm{pr}_{12}(\omega'') = (\ell_1, \ell_2)$$

due to the choice of $(c_3, \dots, c_n)$. Hence, equality (5.4) is satisfied and thus $(u_1, u_2) \in Z(g)$.

The 2-plane $\Pi = \{\lambda_3 = c_3, \dots, \lambda_n = c_n\}$ is a subsemigroup of $(\mathbb{R}^n, \oplus)$ isomorphic to $(\mathbb{R}^2, \oplus)$, and therefore P_2 is an additive tropical set-curve in $\mathbb{R}^2$.

To summarize, we have $\mathrm{pr}_{12}(\tau) = (k, \ell)$ with $k, \ell > 0$ for the monomial M_τ initially chosen to be essential. Then, the monomial $N_{k\ell} = \lambda_1^k \odot \lambda_2^\ell \odot f_{k\ell}(c_3, \dots, c_n)$ of g is essential as well, since, due to (5.2), its value at (c_1, c_2) is greater than that for all the other monomials of g. As shown in Step 3, the sum $\widehat{g}$ of the essential monomials of g must be a simple polynomial, possibly multiplied by a monomial. Since $N_{k\ell}$ is essential and depends both on λ_1 and λ_2, the polynomial $\widehat{g}$ is divisible either by λ_1 or by λ_2. If $\widehat{g}$ is divisible by λ_1, then so is g. Indeed, otherwise at least one of the monomials of g depends only on λ_2, and would correspond to a vertex of the Newton polygon. Hence it must be essential (see Section 3.3). Finally, notice that if g is divisible by λ_1, then so is f, contradicting the assumption of Step 1.

The proof of Theorem 5.1 is complete. □

Another example of simple additive tropical sets is provided by additive affine subspaces of $\mathbb{R}^n$. Note that a hyperplane in $\mathbb{R}^n$ is a tropical set-hypersurface, since it can be defined by a tropical binomial. Moreover, any rational affine subspace of $\mathbb{R}^n$ is a tropical set-variety defined by a number of tropical binomials.

Theorem 5.2. *An affine subspace $P \subset \mathbb{R}^n$, parallel to a linear subspace defined over $\mathbb{Q}$, is additive if and only if P is simple.*

Proof. As before, given an additive affine subspace $P \subset \mathbb{R}^n$, the task is to find simple tropical binomials that define P. In view of Theorem 5.1, we may assume that $k = \dim(P) \leq n-2$. Choose a base $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ of the linear space parallel to P and, without loss of generality, assume that the first $k \times k$ minor of the coordinate matrix of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$

is nonsingular. Then P projects onto a hyperplane P_i in the coordinate $(k+1)$-plane $\Pi_i = \{\lambda_j = 0,\ k < j \le n,\ j \ne i\}$, $i = k+1, \dots, n$. Using Theorem 5.1 again, we have $P_i = Z(f_i) \cap \Pi_i$, where f_i is a simple binomial for each $i = k+1, \dots, n$, and hence $P = \bigcap_{i=k+1}^{n} Z(f_i) = Z(f_{k+1}, \dots, f_n)$. □

6. Additive tropical set-curves

The treatment of additive tropical set-curves appears to be more involved and delicate than one may expect. Our exposition appeals to the natural idea of considering the projections of a given curve to the coordinate planes and taking the intersection of the cylinders built over all these projections. This intersection can be greater than the original curve, and the central problem is then to remove unnecessary pieces; this is what we are doing below. So, we proceed as follows: first, we clarify several geometric properties of additive tropical set-curves, later we construct some auxiliary additive tropical sets, and finally we prove that additive tropical set-curves are simple.

6.1. Geometry of additive tropical set-curves

Let $U \subset \mathbb{R}^n$ $(n \ge 2)$ be an additive tropical set-curve. Without loss of generality, we may assume that U does not lie entirely in any hyperplane $\lambda_j = \text{const}$, $1 \le j \le n$.

We denote by U^0 and U^1 the sets of vertices and of edges of U respectively. Let us outline some useful geometric properties of additive tropical set-curves.

(i) The directing vectors of the edges of U cannot have a pair of coordinates having distinct signs, since otherwise, by the arguments of Steps 2 and 3 in the proof of Theorem 5.1, the sums of points on such an edge would fill a two-dimensional domain.

We shall equip all the edges e of U with an orientation, taking their (integral primitive) directing vectors $\boldsymbol{a}(e)$ to have only nonnegative coordinates. Note that this orientation agrees with the order given by (2.1). In addition, this orientation defines a partial order in U^1 by letting $e \succ e'$ when e and e' have a common vertex $\boldsymbol{u}$, e' coming to $\boldsymbol{u}$, and e emanating from $\boldsymbol{u}$. The poset U^1 has a unique maximal element, which is a ray pointing to $\mathbb{R}^n_{\ge 0} := \{x_1 \ge 0, \dots, x_n \ge 0\}$. Indeed, if one had two rays pointing to $\mathbb{R}^n_{\ge 0}$, then, as was shown in Step 3 of the proof of Theorem 5.1, the sums of points on such two edges would sweep a two-dimensional domain.

(ii) Let $\boldsymbol{u} \in U^0$ and $C^1_{\boldsymbol{u}} = \{e \in U^1 : \boldsymbol{u} \in e\}$. As pointed out above, in the notation of Section 3.2, we have

$$\boldsymbol{a}_{\boldsymbol{u}}(e) \in \mathbb{R}^n_{\ge 0} \cup \mathbb{R}^n_{\le 0} \quad \text{for all } e \in U^1_{\boldsymbol{u}},$$

where $\mathbb{R}^n_{\le 0} := \{x_1 \le 0, \dots, x_n \le 0\}$. Furthermore, due to (3.3), $U^1_{\boldsymbol{u}}$ must contain at least one edge e with $\boldsymbol{a}_{\boldsymbol{u}}(e) \in \mathbb{R}^n_{\ge 0}$ and at least one edge e' with $\boldsymbol{a}_{\boldsymbol{u}}(e') \in \mathbb{R}^n_{\le 0}$. We also claim that $U^1_{\boldsymbol{u}}$ contains precisely one edge e with $\boldsymbol{a}_{\boldsymbol{u}}(e) \in \mathbb{R}^n_{\ge 0}$. Indeed, otherwise, the sums of points on such two edges would sweep a two-dimensional domain. We denote this edge by $e_{\boldsymbol{u}}$. A similar reasoning shows that there is at most one edge e' with $\boldsymbol{a}_{\boldsymbol{u}}(e') \in \mathbb{R}^n_{<0} := \{x_1 < 0, \dots, x_n < 0\}$.

(iii) Next, we notice that if $\boldsymbol{a}_{\boldsymbol{u}}(e_{\boldsymbol{u}}) \in \{x_i = 0\}$, then $\boldsymbol{a}_{\boldsymbol{u}}(e') \in \{x_i = 0\}$ for all $e' \in U^1_{\boldsymbol{u}}$. Indeed, if $\boldsymbol{a}_{\boldsymbol{u}}(e')$ has a nonzero i-th coordinate for some $e' \in U^1_{\boldsymbol{u}}$, then, due to (3.3), there should be some $\boldsymbol{a}_{\boldsymbol{u}}(e'')$, $e'' \in U^1_{\boldsymbol{u}}$, with a positive i-th coordinate, contrary to $\boldsymbol{a}_{\boldsymbol{u}}(e'') \in \mathbb{R}^n_{\le 0}$ for all $e'' \in C^1_{\boldsymbol{u}} \setminus \{e_{\boldsymbol{u}}\}$.

(iv) Let U_+ denote the union of those edges $e \in U^1$ whose directing vectors satisfy

$$\boldsymbol{a}(e) \in \mathbb{R}^n_{>0} := \{x_1 > 0, \dots, x_n > 0\}.$$

We point out that $U_+ \neq \emptyset$, since it contains the maximal edge-ray $e \in U^1$. Indeed, otherwise, by (iii) the whole tropical set-curve U would lie in a hyperplane $x_i = \text{const}$, contrary to the initial assumption. Furthermore, due to (ii), U_+ must be connected and homeomorphic either to $[0, \infty)$ or to $\mathbb{R}$. We call U_+ the *spine* of the additive tropical set-curve U.

(v) Let $U^0_+ = U_+ \cap U^0 := \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\}$ be the set of vertices of U that lie on U_+. Pick $i \in \{1, \dots, m\}$, and to each edge $e \in U^1_{\boldsymbol{u}_i}$ such that $\boldsymbol{a}_{\boldsymbol{u}_i}(e) \in \mathbb{R}^n_{\le 0}$ associate the set $J(e) \subset \{1, \dots, n\}$ consisting of the indices for which the coordinates of $\boldsymbol{a}_{\boldsymbol{u}_i}(e)$ are negative. The additivity condition implies that

- the map $e \mapsto J(e)$ restricted to $U^1_{\boldsymbol{u}_i}$ is injective,
- if $e_1, e_2 \in U^1_{\boldsymbol{u}_i}$ emanate from $\boldsymbol{u}_i$ in nonpositive directions, then either $J(e_1) \cap J(e_2) = \emptyset$, or there is an edge $e \in U^1_{\boldsymbol{u}_i}$ with $\boldsymbol{a}_{\boldsymbol{u}_i}(e) \in \mathbb{R}^n_{\le 0}$ such that $J(e) = J(e_1) \cap J(e_2)$.

(vi) Let $U_i = \{\boldsymbol{u} \in U : \boldsymbol{u} \prec \boldsymbol{u}_i\}$. This is the part of the curve U that lies in the shifted orthant $\boldsymbol{u}_i + \mathbb{R}^n_{\le 0}$. Since this orthant is a subsemigroup of $\mathbb{R}^n$, U_i is an additive tropical set.

6.2. *Auxiliary additive tropical sets*

Introducing the cone

$$\Sigma_0 := \mathbb{R}^n_{\le 0} \setminus \mathbb{R}^n_{<0} = \{(u_1, \dots, u_n) \in \mathbb{R}^n_{\le 0} : u_1, \dots, u_n = 0\},$$

and denoting by $\Sigma_{\boldsymbol{u}}$ the shift of Σ_0 to the cone with vertex at $\boldsymbol{u} \in \mathbb{R}^n$, and by the results of Section 6.1, we have

$$U \subset \widetilde{U} := U_+ \cup \bigcup_{\boldsymbol{u} \in U^0_+} \Sigma_{\boldsymbol{u}}.$$

The cone $\Sigma_{\boldsymbol{u}}$ divides $\mathbb{R}^n$ into two components which we denote by

$$\operatorname{Int}(\Sigma_{\boldsymbol{u}}) = \boldsymbol{u} + \mathbb{R}^n_{<0} \quad \text{and} \quad \operatorname{Ext}(\Sigma_{\boldsymbol{u}}) = \mathbb{R}^n \setminus (\Sigma_{\boldsymbol{u}} \cup \operatorname{Int}(\Sigma_{\boldsymbol{u}})).$$

The cone Σ_0 (and each $\Sigma_{\boldsymbol{u}_i}$, $i = 1, \dots, m$) splits naturally into the disjoint union of open cells, labeled by subsets $J \subsetneq \{1, \dots, n\}$, and defined by

$$\Sigma_0(J) := \{(u_1, \dots, u_n) \in \Sigma_0 : u_j < 0 \text{ for } j \in J,\ u_j = 0 \text{ for } j \notin J\}.$$

Observing that

$$\overline{\Sigma_0(J)} = \bigcup_{K\subset J} \Sigma_0(K),$$

we let

$$\mathcal{J}_i(U) = \{J \subsetneq \{1,\dots,n\} : \Sigma_{\boldsymbol{u}_i}(J)\cap U \neq \emptyset\} \quad \text{and} \quad \Sigma^U_{\boldsymbol{u}_i} = \bigcup_{J\in\mathcal{J}_i(U)} \overline{\Sigma_{\boldsymbol{u}_i}(J)},$$

for each $i = 1,\dots,m$, and define

$$\widetilde{U}_{\mathrm{red}} := U_+ \cup \bigcup_{\boldsymbol{u}\in U^0_+} \Sigma^U_{\boldsymbol{u}}.$$

Note that $U \subset \widetilde{U}_{\mathrm{red}} \subset \widetilde{U}$, and $U = \widetilde{U}_{\mathrm{red}}$ for $n = 2$.

Lemma 6.1. *$\widetilde{U}$ and $\widetilde{U}_{\mathrm{red}}$ are simple additive tropical sets.*

Proof. We shall define $\widetilde{U}$ and $\widetilde{U}_{\mathrm{red}}$ by simple tropical polynomials.

(1) We first consider $\widetilde{U}$, and organize our argument in a few steps.

Step 1. Assume that U_+ is homeomorphic to $\mathbb{R}$. We intend to determine a (finite) set Φ consisting of simple tropical polynomials in n variables such that $\bigcap_{f\in\Phi} Z(f) = \widetilde{U}$. In this step we show that $\bigcap_{f\in\Phi} Z(f) \supset \widetilde{U}$.

Let

$$U^0_+ = \{\boldsymbol{u}_1,\dots,\boldsymbol{u}_m\}, \qquad \boldsymbol{u}_i = (u_{i1},\dots,u_{in}),\ i = 1,\dots,m,$$

with $u_{ij} < u_{kj}$ for all $1 \le i < k \le m$, $j = 1,\dots,n$. The set U_+ contains $m+1$ edges, in order $e_0 \prec e_1 = [\boldsymbol{u}_1,\boldsymbol{u}_2] \prec \dots \prec e_{m-1} = [\boldsymbol{u}_{m-1},\boldsymbol{u}_m] \prec e_m$, where e_0 and e_m are rays, whose primitive integral directing vectors are

$$\boldsymbol{a}(e_i) = (a_{i1},\dots,a_{in}) \in \mathbb{R}^n_{>0}, \qquad i = 0,\dots,m.$$

In particular, $u_{i+1,s} - u_{is} = a_{is}\mu_i$ for some $\mu_i > 0$ for each $1 \le i < m$ and $s = 1,\dots,n$.

Let $p_0,\dots,p_m$ and $b(i,j)$, $i = 0,\dots,m$, $j = 1,\dots,n$, be positive integers such that

(P1) p_i is divisible by $2a_{i1}\cdots a_{in}$, $i = 0,\dots,m$;
(P2) $b(0,j) \gg n$ and $b(i,j) - b(i-1,j) \gg n$ for all $i = 1,\dots,m$, $j = 1,\dots,n$, where $b(i,j) := p_i/a_{ij}$.

By definition

$$a_{ik}b(i,k) = a_{i\ell}b(i,\ell) \quad \text{for all } k,\ell = 1,\dots,n,\ i = 1,\dots,m-1. \tag{6.1}$$

Now we introduce the set $\Phi \subset \mathbb{T}[\Lambda]$ of $n(n-1)/2$ simple tropical polynomials $f_{k\ell}$, $1 \le k < \ell \le n$, given by

$$f_{k\ell} = \Big(\bigoplus_{i=0}^{m}\big(A_{ik}^{k\ell} \odot \lambda_k^{b(i,k)} \oplus A_{i\ell}^{k\ell} \odot \lambda_\ell^{b(i,\ell)}\big)\Big) \oplus \Big(\bigoplus_{i=0}^{m-1} \bigoplus_{\substack{1\le j\le n \\ j\ne k,\ell}} \big(B_{i+1,j}^{k\ell} \odot \lambda_j^{b(i+1,j)-j} \oplus C_{ij}^{k\ell} \odot \lambda_j^{b(i,j)+j}\big)\Big), \tag{6.2}$$

whose coefficients A_*^* are as specified below. The monomials of $f_{k\ell}$ correspond to the following integral points:

1. $m+1$ points $P_{ki} = b(i,k)\varepsilon_k$, $i = 0, \dots, m$, on the k-th axis;
2. $m+1$ points $P_{\ell i} = b(i,\ell)\varepsilon_\ell$, $i = 0, \dots, m$, on the ℓ-th axis;
3. $2m$ points $P_{ji}^+ = (b(i,j)+j)\varepsilon_j$, $i = 0, \dots, m-1$, and $P_{ji}^- = (b(i,j)-j)\varepsilon_j$, $i = 1, \dots, m$, on the j-th axis for all $1 \le j \le n$, $j \ne k, \ell$ (here ε_j denotes the unit vector of the j-th coordinate axis).

The Newton polytope of $f_{k\ell}$ naturally splits into the subpolytopes

$$\Pi_{k\ell}^i = \mathrm{conv}\{P_{k,i-1}, P_{ki}, P_{\ell,i-1}, P_{\ell i}, P_{j,i-1}^+, P_{ji}^-,\ j \ne k,\ell\}, \qquad i = 1, \dots, m. \tag{6.3}$$

Now we impose conditions on the coefficients of $f_{k\ell}$:

$$\begin{aligned}\big(A_{i-1,k}^{k\ell} \odot \lambda_k^{b(i-1,k)}\big)\big|_{\boldsymbol{u}_i} &= \big(A_{i-1,\ell}^{k\ell} \odot \lambda_\ell^{b(i-1,\ell)}\big)\big|_{\boldsymbol{u}_i} = \big(A_{ik}^{k\ell} \odot \lambda_k^{b(i,k)}\big)\big|_{\boldsymbol{u}_i} \\ &= \big(A_{i\ell}^{k\ell} \odot \lambda_\ell^{b(i,\ell)}\big)\big|_{\boldsymbol{u}_i} \qquad = \big(B_{ij}^{k\ell} \odot \lambda_j^{b(i,j)-j}\big)\big|_{\boldsymbol{u}_i} \\ &= \big(C_{i-1,j}^{k\ell} \odot \lambda_j^{b(i-1,j)+j}\big)\big|_{\boldsymbol{u}_i},\end{aligned} \tag{6.4}$$

for all $i = 1, \dots, m$, $1 \le k < \ell \le n$, $1 \le j \le n$, $j \ne k, \ell$. We should check the consistency of system (6.4), since each of the coefficients $A_{ik}^{k\ell}$, $A_{i\ell}^{k\ell}$, $i = 1, \dots, m-1$, enters two equations in this system. The verification goes as follows: the restriction of (6.4) to the variables A_*^* reads

$$\begin{aligned}A_{i-1,k}^{k\ell} + u_{ik}b(i-1,k) = A_{i-1,\ell}^{k\ell} + u_{i\ell}b(i-1,\ell) &= A_{ik}^{k\ell} + u_{ik}b(i,k) \\ &= A_{i\ell}^{k\ell} + u_{i\ell}b(i,\ell)\end{aligned} \tag{6.5}$$

for $1 \le i \le m$; hence we have to show that

$$A_{ik}^{k\ell} + u_{ik}b(i,k) = A_{i\ell}^{k\ell} + u_{i\ell}b(i,\ell) \ \Rightarrow\ A_{ik}^{k\ell} + u_{i+1,k}b(i,k) = A_{i\ell}^{k\ell} + u_{i+1,\ell}b(i,\ell)$$

for all $1 \le i < m$, or equivalently, that

$$(u_{i+1,k} - u_{ik})b(i,k) = (u_{i+1,\ell} - u_{i\ell})b(i,\ell), \qquad 1 \le i < m,$$

which finally reduces to assumption (6.1). The solutions to system (6.4) form a one-parameter family, and we pick one of these solutions.

Consider now the truncation of $f_{k\ell}$ to one variable (i.e., the sum of monomials only containing the chosen variable). From (6.5) and property (P2) above, we derive

$$A^{k\ell}_{i+1,k} + \alpha_{i+1,k} b(i+1,k) = A^{k\ell}_{ik} + \alpha_{i+1,k} b(i,k)$$
$$\Rightarrow\ A^{k\ell}_{i+1,k} + \alpha_{ik} b(i+1,k) < A^{k\ell}_{ik} + \alpha_{ik} b(i,k) = A^{k\ell}_{i-1,k} + \alpha_{ik} b(i-1,k),$$

which immediately generalizes to

$$\begin{cases} \big(A^{k\ell}_{ik} \odot \lambda_k^{b(i,k)}\big)\big|_{\boldsymbol{u}_i} > \big(A^{k\ell}_{sk} \odot \lambda_k^{b(s,k)}\big)\big|_{\boldsymbol{u}_i} \\ \big(A^{k\ell}_{i\ell} \odot \lambda_\ell^{b(i,\ell)}\big)\big|_{\boldsymbol{u}_i} > \big(A^{k\ell}_{s\ell} \odot \lambda_\ell^{b(s,\ell)}\big)\big|_{\boldsymbol{u}_i} \end{cases} \quad \text{when } |i-s| \geq 2. \tag{6.6}$$

Similarly, we have

$$\begin{cases} \big(B^{k\ell}_{i+1,j} \odot \lambda_j^{b(i+1,j)-j}\big)\big|_{\boldsymbol{u}_i} > \big(B^{k\ell}_{s+1,j} \odot \lambda_j^{b(s+1,j)-j}\big)\big|_{\boldsymbol{u}_i} \\ \big(C^{k\ell}_{ij} \odot \lambda_j^{b(i,j)+j}\big)\big|_{\boldsymbol{u}_i} > \big(C^{k\ell}_{sj} \odot \lambda_j^{b(s,j)+j}\big)\big|_{\boldsymbol{u}_i} \end{cases} \quad \text{when } i \neq s. \tag{6.7}$$

Altogether, this means that all the monomials of the truncation are essential.

The latter property and condition (6.4) imply that the subdivision $S(f_{k\ell})$ of the Newton polytope $\Delta(f_{k\ell})$ contains the polytopes $\Pi^i_{k\ell}$, $i = 1, \ldots, m$, defined by (6.3). Furthermore, each polytope $\Pi^i_{k\ell}$ is dual to the vertex $\boldsymbol{u}_i$ (in the sense of Section 3.3), and the polytope's edges lying on the coordinate axes are dual to the facets of the cone Σ_{u_i}.

Next, the edge $[P_{ki}, P_{\ell i}]$ of the subdivision $S(f_{k\ell})$ is dual to a convex $(n-1)$-dimensional polyhedron in $Z(f_{k\ell})$ that contains either the point $\boldsymbol{u}_1$ if $i = 0$, or the points $\boldsymbol{u}_{i-1}$ and $\boldsymbol{u}_i$ if $1 \leq i < m$, or the point $\boldsymbol{u}_m$ if $i = m$. In the case of $1 \leq i < m$, due to convexity, the relevant polyhedron contains the whole edge e_i of U_+. In the case when $i = 0$ or m, due to the orthogonality of $[P_{ki}, P_{\ell i}]$ to this polyhedron and to the edge e_i (the latter orthogonality comes from (6.1)), the hyperplane spanned by the polyhedron contains the edge e_i. Moreover, the following comparison of monomials says that the polyhedron itself contains e_i: due to (6.1) and (6.4), for $\boldsymbol{u} = \boldsymbol{u}_1 - t\boldsymbol{a}(e_0) \in e_0$, $t > 0$, and any $j \neq k, \ell$, $1 \leq j \leq n$, one has

$$\begin{aligned} \big(C^{k\ell}_{0j} \odot \lambda_j^{b(0,j)+j}\big)\big|_{\boldsymbol{u}} &= \big(C^{k\ell}_{0j} \odot \lambda_j^{b(0,j)+j}\big)\big|_{\boldsymbol{u}_1} - tb(0,j)a_{0j} - tja_{0j} \\ &= \big(A^{k\ell}_{0k} \odot \lambda_j^{b(0,k)}\big)\big|_{\boldsymbol{u}_1} - tb(0,k)a_{0k} - tja_{0j} \\ &= \big(A^{k\ell}_{0k} \odot \lambda_j^{b(0,k)}\big)\big|_{\boldsymbol{u}} - tja_{0j} < \big(A^{k\ell}_{0k} \odot \lambda_j^{b(0,k)}\big)\big|_{\boldsymbol{u}}, \end{aligned}$$

and similarly, for $\boldsymbol{u} = \boldsymbol{u}_m + t\boldsymbol{a}(e_m) \in e_m$, $t > 0$,

$$\big(B^{k\ell}_{mj} \odot \lambda_j^{b(m,j)-j}\big)\big|_{\boldsymbol{u}} < \big(A^{k\ell}_{mk} \odot \lambda_k^{b(m,k)}\big)\big|_{\boldsymbol{u}}.$$

Thus, $\bigcap_{k,\ell} Z(f_{k\ell}) \supset \widetilde{U}$.

Step 2. Let us prove the inverse relation $\bigcap_{k,\ell} Z(f_{k\ell}) \subset \widetilde{U}$. More precisely, we have to show that outside the cones $\Sigma_{\boldsymbol{u}}$, $\boldsymbol{u} \in U^0_+$, the ideal generated by the polynomials $f_{k\ell}$, $1 \leq k < \ell \leq n$, defines a subset of U_+.

First, we introduce extra notation referring to the splitting of each polynomial $f_{k\ell}$, $1 \le k < \ell \le n$, into the following (tropical) sum:

$$\begin{aligned} f_{k\ell} &= f_{k\ell}^{(0)} \oplus f_{k\ell}^{(1)} \oplus \cdots \oplus f_{k\ell}^{(m)}, \\ f_{k\ell}^{(0)} &= A_{k\ell}^{0k} \odot \lambda_k^{b(0,k)} \oplus A_{k\ell}^{0\ell} \odot \lambda_\ell^{b(0,\ell)} \oplus \bigoplus_{j \ne k,\ell} C_{k\ell}^{0j} \odot \lambda_j^{b(0,j)+j}, \\ f_{k\ell}^{(m)} &= A_{k\ell}^{mk} \odot \lambda_k^{b(m,k)} \oplus A_{k\ell}^{m\ell} \odot \lambda_\ell^{b(m,\ell)} \oplus \bigoplus_{j \ne k,\ell} B_{k\ell}^{mj} \odot \lambda_j^{b(m,j)-j}, \\ f_{k\ell}^{(i)} &= A_{k\ell}^{ik} \odot \lambda_k^{b(i,k)} \oplus A_{k\ell}^{i\ell} \odot \lambda_\ell^{b(i,\ell)} \oplus \bigoplus_{j \ne k,\ell} \big(B_{k\ell}^{ij} \odot \lambda_j^{b(i,j)-j} \oplus C_{k\ell}^{ij} \odot \lambda_j^{b(i,j)+j}\big), \\ &\qquad\qquad 1 \le i < m. \end{aligned}$$

Let $\boldsymbol{u}' = (u'_1, \dots, u'_n) \in \bigcap_{k,\ell} Z(f_{k\ell}) \cap \mathrm{Ext}(\Sigma_{\boldsymbol{u}_m})$. Without loss of generality, we may assume that $\boldsymbol{u}_m = 0$ and $f_{k\ell}(\boldsymbol{u}_m) = 0$ for all $1 \le k < \ell \le n$. Then, in particular,

$$f_{k\ell}^{(m)} = \lambda_k^{b(m,k)} \oplus \lambda_\ell^{b(m,\ell)} \oplus \bigoplus_{j \ne k,\ell} \lambda_j^{b(m,j)-j},$$

and the point $\boldsymbol{u}$ is such that $u'_i > 0$ if i belongs to a nonempty subset $J \subset \{1, \dots, n\}$, and $u'_j \le 0$ if $j \notin J$. It follows immediately from (6.4), (6.6), and (6.7) that, for any fixed $1 \le k < \ell \le n$, the top degree monomials of $f_{k\ell}$ in the variables λ_i, $i \in J$, take positive values at $\boldsymbol{u}'$. These values are greater than the values taken by the other monomials in λ_i, $i \in J$, at $\boldsymbol{u}'$. Similarly, the monomials in λ_j, $j \notin J$, take negative values at $\boldsymbol{u}'$. This means that the geometry of $Z(f_{k\ell})$ in $\mathrm{Ext}(\Sigma_{\boldsymbol{u}_m})$ is determined by the top degree monomials of $f_{k\ell}$, i.e., by $f_{k\ell}^{(m)}$.

Next, we have

$$\max_{r=1,\dots,n} \lambda_r^{b(m,r)}\big|_{u'_r} = \lambda_i^{b(m,i)}\big|_{u'_i} > \lambda_j^{b(m,j)}\big|_{u'_j}, \qquad i \in J',\ j \notin J',$$

for some set $J' \subset J$. Assuming that $J' \subsetneq \{1, \dots, n\}$, we pick $k \in J'$ and $\ell \in \{1, \dots, n\} \setminus J'$, and obtain the following:

$$\begin{aligned} \lambda_k^{b(m,k)}\big|_{u'_k} = \lambda_i^{b(m,i)}\big|_{u'_i} > \lambda_i^{b(m,i)-i}\big|_{u'_i} &\quad \text{for all } i \in J' \setminus \{k\}, \\ \lambda_k^{b(m,k)}\big|_{u'_k} > \lambda_i^{b(m,i)}\big|_{u'_i} &\quad \text{for all } i \in J \setminus J', \text{ and} \\ \lambda_k^{b(m,k)}\big|_{u'_k} > 0 \ge \lambda_i^{b(m,i)-i}\big|_{u'_i} &\quad \text{for all } i \notin J; \end{aligned}$$

this means that the value $f_{k\ell}^{(m)}(\boldsymbol{u}')$ is attained by a unique monomial, i.e., $\boldsymbol{u}' \notin Z(f_{k\ell})$. Hence, $J' = \{1, \dots, n\}$, which, due to (6.1), implies that $\boldsymbol{u}' = \mu \boldsymbol{a}(e_m)$ with $\mu > 0$; that is, $\boldsymbol{u}' \in U_+$.

Let $\boldsymbol{u}' \in \bigcap_{k,\ell} Z(f_{k\ell}) \cap \mathrm{Ext}(\Sigma_{\boldsymbol{u}_{m-1}}) \cap \mathrm{Int}(\Sigma_{\boldsymbol{u}_m})$, that is, $\boldsymbol{u}' = (u'_1, \dots, u'_n)$ with $0 > u'_i > u_{m-1,i}$ for $i \in J$ and $0 > u_{m-1,j} \ge u'_j$ for $j \notin J$, where $J \subset \{1, \dots, n\}$ is some nonempty set. These relations, together with equalities (6.4) and inequalities (6.6), (6.7), yield

$$f_{k\ell}^{(m-1)}(\boldsymbol{u}') > \max\{f_{k\ell}^{(0)}(\boldsymbol{u}'), \dots, f_{k\ell}^{(m-2)}(\boldsymbol{u}'), f_{k\ell}^{(m)}(\boldsymbol{u}')\}, \qquad 1 \le k < \ell \le n;$$

nevertheless the value $f_{k\ell}^{(m-1)}(\boldsymbol{u}')$ can be attained only by monomials which depend on λ_j, $j \in J$.

In view of $\boldsymbol{u}_m = 0$, $f_{k\ell}(\boldsymbol{u}_m) = 0$, and equalities (6.4) for $i = m$, we then get

$$f_{k\ell}^{(m-1)} = \lambda_k^{b(m-1,k)} \oplus \lambda_\ell^{b(m-1,\ell)} \oplus \bigoplus_{j \ne k,\ell} \big(B_{k\ell}^{m-1,j} \odot \lambda_j^{b(m-1,j)-j} \oplus \lambda_j^{b(m-1,j)+j}\big).$$

Furthermore, due to equalities (6.4) for $i = m-1$ and inequalities (6.6), (6.7), we have

$$\begin{aligned} \lambda_i^{b(m-1,i)}\big|_{u_i'} > \lambda_i^{b(m-1,i)}\big|_{u_{m-1,i}} &= f_{ij}(\boldsymbol{u}_{m-1}) \qquad \text{for all } i \in J,\ j \ne i,\\ \lambda_j^{b(m-1,j)}\big|_{u_j'} \le \lambda_j^{b(m-1,j)}\big|_{u_{m-1,j}} &= f_{ij}(\boldsymbol{u}_{m-1}) \qquad \text{for all } j \notin J,\ i \ne j. \end{aligned}$$

Hence,

$$\max_{r=1,\dots,n} \lambda_r^{b(m-1,r)}\big|_{u_r'} = \lambda_i^{b(m-1,i)}\big|_{u_i'} > \lambda_j^{b(m-1,j)}\big|_{u_j'} \qquad \text{if } i \in J',\ j \notin J' \tag{6.8}$$

for some nonempty set $J' \subset J$. Suppose that $J' \subsetneq \{1, \dots, n\}$, and pick $k \in J'$, $\ell \notin J'$. Then

$$(6.8) \ \Rightarrow\ \lambda_k^{b(m-1,k)}\big|_{u_k'} > \lambda_\ell^{b(m-1,\ell)}\big|_{u_\ell'}, \tag{6.9}$$

$$(6.8)\ \&\ u_i' < 0 \ \Rightarrow\ \lambda_k^{b(m-1,k)}\big|_{u_k'} \ge \lambda_i^{b(m-1,i)}\big|_{u_i'} > \lambda_i^{b(m-1,i)+i}\big|_{u_i'},\ i \ne k, \ell, \tag{6.10}$$

(6.4) & $i \in J \setminus \{k\}$ $\Rightarrow$

$$\begin{aligned} &\lambda_k^{b(m-1,k)}\big|_{u_k'} - \big(B_{k\ell}^{m-1,i} \odot \lambda_i^{b(m-1,i)-i}\big)\big|_{u_i'}\\ &= u_k' b(m-1,k) - (B_{k\ell}^{m-1,i} + u_i'(b(m-1,i) - i))\\ &= (u_k' - u_{m-1,k}) b(m-1,k) - (u_i' - u_{m-1,i})(b(m-1,i) - i)\\ &\quad + \big(u_{m-1,k} b(m-1,k) - (B_{k\ell}^{m-1,i} + u_{m-1,i}(b(m-1,i) - i))\big)\\ &= (u_k' - u_{m-1,k}) b(m-1,k) - (u_i' - u_{m-1,i})(b(m-1,i) - i)\\ &\quad + \big(\lambda_k^{b(m-1,k)}\big|_{u_{m-1,k}} - \big(B_{k\ell}^{m-1,i} \odot \lambda_i^{b(m-1,i)-i}\big)\big|_{u_{m-1,i}}\big)\\ &\overset{(6.4)}{=} (u_k' - u_{m-1,k}) b(m-1,k) - (u_i' - u_{m-1,i})(b(m-1,i) - i)\\ &> (u_k' - u_{m-1,k}) b(m-1,k) - (u_i' - u_{m-1,i}) b(m-1,i)\\ &= \big(\lambda_k^{b(m-1,k)}\big|_{u_k'} - \lambda_i^{b(m-1,i)}\big|_{u_i'}\big) + \big(\lambda_k^{b(m-1,k)}\big|_{u_{m-1,k}} - \lambda_i^{b(m-1,i)}\big|_{u_{m-1,i}}\big). \end{aligned}$$

In view of $k \in J'$ and (6.8), the former expression on the last line is nonnegative. In its turn, the latter expression vanishes; this follows from (6.4) and (6.5) (cf. the verification of the consistency of system (6.4) performed in Step 1). Hence,

$$\lambda_k^{b(m-1,k)}\big|_{u'_k} > \big(B_{k\ell}^{m-1,i} \odot \lambda_i^{b(m-1,i)-i)}\big)\big|_{u'_i}, \quad i \in J \setminus \{k\}. \tag{6.11}$$

Finally, for $i \notin J$ and $i \neq \ell$, one has

$$\begin{aligned}\lambda_k^{b(m-1,k)}\big|_{u'_k} > \lambda_k^{b(m-1,k)}\big|_{u_{m-1,k}} = f_{k\ell}(\boldsymbol{u}_{m-1}) = \big(B_{k\ell}^{m-1,i} \odot \lambda_i^{b(m-1,i)-i)}\big)\big|_{u_{m-1,k}} \\ \geq \big(B_{k\ell}^{m-1,i} \odot \lambda_i^{b(m-1,i)-i)}\big)\big|_{u'_i}.\end{aligned} \tag{6.12}$$

Thus, the assumption $J' \subsetneq \{1, \dots, n\}$ together with (6.9)–(6.12) has led to the fact that the value $f_{k\ell}(\boldsymbol{u}')$ is attained by the unique monomial $\lambda_k^{b(m-1,k)}$, namely $\boldsymbol{u}' \notin Z(f_{k\ell})$—a contradiction. Hence, $J' = \{1, \dots, n\}$, which, due to (6.1), implies $\boldsymbol{u}' - \boldsymbol{u}_{m-1} = \lambda \boldsymbol{a}(e_{m-1})$, that is, $\boldsymbol{u}' \in e_{m-1} \subset U_+ \subset \widetilde{U}$.

In the same manner we proceed further showing that if $\boldsymbol{u}' \in \bigcap_{k,\ell} Z(f_{k\ell}) \cap \mathrm{Ext}(\Sigma_{\boldsymbol{u}_{r-1}}) \cap \mathrm{Int}(\Sigma_{\boldsymbol{u}_r})$, then $\boldsymbol{u}' \in e_{r-1} \subset \widetilde{U}$, $r < m$, which allows us to derive the required relation $\bigcap_{k,\ell} Z(f_{k\ell}) \subset \widetilde{U}$.

Step 3. In the remaining situation, when U_+ is homeomorphic to $[0, \infty)$, we modify the preceding construction in order to exclude any ray e_0 attached to the vertex u_1 and directed to the negative infinity. Namely, in formula (6.2) for $f_{k\ell}$, we replace all the terms having exponents $b(0, j)$, $j = 1, \dots, n$, by a constant $A_0^{k\ell}$ which satisfies condition (6.4) for $i = 0$. The equality $\bigcap_{k,\ell} Z(f_{k\ell}) = \widetilde{U}$ is then obtained in the same way as in Steps 1 and 2 for the case $U_+ \simeq \mathbb{R}$.

(2) To prove that $\widetilde{U}_{\mathrm{red}}$ is simple, we extend the ideal $I := \langle f_{k\ell} : 1 \leq k < \ell \leq n \rangle$, defining $\widetilde{U}$, with the extra simple tropical polynomials constructed below.

We assume that U_+ is homeomorphic to $\mathbb{R}$. As in the preceding situation, in order to cover the case of U_+ homeomorphic to $[0, \infty)$, the forthcoming construction should be slightly modified; however, we skip this case.

Fix some $i = 1, \dots, m$, and choose a set $K \subsetneq \{1, \dots, n\}$ such that $\Sigma_{\boldsymbol{u}_i}(K) \not\subset \widetilde{U}_{\mathrm{red}}$ (or equivalently, $\Sigma_{\boldsymbol{u}_i}(K) \cap \widetilde{U}_{\mathrm{red}} = \emptyset$). Then, we shall construct a simple polynomial $f_{i,K}$ such that the set $Z(f_{i,K})$ contains the following:

- the spine U_+,
- all the cones $\Sigma_{\boldsymbol{u}_k}$ for $1 \leq k \leq m$, $k \neq i$, and
- all the orthants $\Sigma_{\boldsymbol{u}_i}(J)$ such that $\Sigma_{\boldsymbol{u}_i}(J) \subset \widetilde{U}_{\mathrm{red}}$, but $Z(f_{i,K}) \cap \Sigma_{\boldsymbol{u}_i}(K) = \emptyset$.

Taking an appropriate $K \subsetneq \{1, \dots, n\}$ and adding such simple polynomials for all $i = 1, \dots, m$, we obtain the required ideal.

Further, the required polynomial $f_{i,K}$ will be defined by an explicit formula. Aiming to obtain uniform expressions, we (formally) pick two extra vertices in U_+: a point $\boldsymbol{u}_0 \in e_0 \setminus \{\boldsymbol{u}_1\}$ and a point $\boldsymbol{u}_{m+1} \in e_m \setminus \{\boldsymbol{u}_m\}$. Accordingly, we add two more elements

$$b(m+1, j) := 2b(m, j), \quad b(-1, j) := b(0, j)/2, \quad j = 1, \dots, n,$$

to the sequence $b(k, j)$, $0 \leq k \leq m$, $1 \leq j \leq n$, defined in (P2).

To make clearer the construction and properties of $f_{i,K}$, we start with an auxiliary polynomial which can be viewed as a simplified version of the polynomial $f_{k\ell}$, as introduced in the preceding step,

$$f = \sum_{\substack{-1 \le k \le m+1 \\ 1 \le j \le n}} A_{k,j} \odot \lambda_j^{b(k,j)}, \tag{6.13}$$

where, for all $k = 0, \dots, m+1$,

$$\big(A_{k-1,j} \odot \lambda_j^{(k-1,j)}\big)\big|_{\boldsymbol{u}_k} = \big(A_{k,l} \odot \lambda_j^{(k,\ell)}\big)\big|_{\boldsymbol{u}_k}, \qquad j, \ell = 1, \dots, n.$$

The argument of Step 1 implies immediately that: the conditions imposed on the coefficients A_{kj} are consistent; all the monomials of f are essential; and the subdivision $S(f)$ of $\Delta(f)$ consists of the polytopes (cf. (6.3))

$$\Pi^k = \operatorname{conv}\{P_{jk},\ P_{\ell,k-1},\ j, \ell = 1, \dots, n\}, \qquad k = 0, \dots, m+1\ , \tag{6.14}$$

which are dual to the vertices $\boldsymbol{u}_0, \dots, \boldsymbol{u}_{m+1}$ (here $P_{kj} = b(k,j)\varepsilon_k$ are the vertices of the polytopes (6.3)). In addition, we obtain $Z(f) \supset \widetilde{U}$.

Next we modify formula (6.13). Pick an element $j_0 \in \{1, \dots, n\} \setminus K$ and define the desired polynomial $f_{i,K}$ to be

$$f_{i,K} = \sum_{\substack{-1 \le k \le m+1 \\ 1 \le j \le n}} \widehat{A}_{k,j} \odot \lambda_j^{\widehat{b}(k,j)},$$

where $\widehat{b}(k,j) = b(k,j)$ for all $k = -1, \dots, m+1$, $j = 1, \dots, n$, except for the cases

$$\widehat{b}(i,j) = b(i,j) + 1,\ \ j \notin K, \qquad \widehat{b}(i-1,j) = b(i-1,j) - 1,\ \ j \notin K \cup \{j_0\},$$

while, for all $k = 0, \dots, m+1$, $k \ne i$, the coefficients $\widehat{A}_{kj}$ satisfy the condition

$$\big(A_{k-1,j} \odot \lambda_j^{(k-1,j)}\big)\big|_{\boldsymbol{u}_k} = \big(A_{k,l} \odot \lambda_j^{(k,\ell)}\big)\big|_{\boldsymbol{u}_k}, \qquad j, \ell = 1, \dots, n\ , \tag{6.15}$$

and the new condition

$$\big(A_{i-1,j} \odot \lambda_j^{(i-1,j)}\big)\big|_{\boldsymbol{u}_i} = \big(A_{i,\ell} \odot \lambda_j^{(i,\ell)}\big)\big|_{\boldsymbol{u}_i}, \qquad j \in K \cup \{j_0\},\ \ell \in K. \tag{6.16}$$

Again, as in Step 1, from (6.1), we derive the consistence of conditions (6.15) and (6.16), as well as the inequalities

$$\big(\widehat{A}_{is} \odot \lambda_s^{b(i,s)+1}\big)\big|_{\boldsymbol{u}_i} < \big(\widehat{A}_{ij} \odot \lambda_j^{b(i,j)}\big)\big|_{\boldsymbol{u}_i}, \qquad j \in K,\ s \notin K,$$
$$\big(\widehat{A}_{i-1,\ell} \odot \lambda_\ell^{b(i-1,\ell)-1}\big)\big|_{\boldsymbol{u}_i} < \big(\widehat{A}_{i-1,s} \odot \lambda_s^{b(i-1,s)}\big)\big|_{\boldsymbol{u}_i}, \qquad s \in K \cup \{j_0\},\ \ell \notin K \cup \{j_0\}.$$

These inequalities imply that all the monomials of $f_{i,K}$ are essential, and that the subdivision $S(f_{i,K})$ of the Newton polytope $\Delta(f_{i,K})$ contains the n-dimensional polytopes

$$\widehat{\Pi}^k = \operatorname{conv}\{\widehat{P}_{jk}, \widehat{P}_{\ell,k-1},\ j, \ell = 1, \dots, n\}, \quad k = 0, \dots, m+1,\ k \ne i, \quad \widehat{P}_{jk} = b(k,j)\varepsilon_j$$

($P_{jk} = b(k, j)\varepsilon_j$ being the vertices of the polytopes (6.3)), and the polytope

$$\widehat{\Pi}_i = \operatorname{conv}\{\widehat{P}_{j,i-1}, \widehat{P}_{\ell i},\ j \in K \cup \{j_0\},\ \ell \in K\}.$$

Further, the above polytopes $\widehat{\Pi}^k$, $k = 0, \dots i-1, i+1, \dots, m+1$, are dual to the vertices $\boldsymbol{u}_0, \dots, \boldsymbol{u}_{i-1}, \boldsymbol{u}_{i+1}, \dots, \boldsymbol{u}_{m+1}$ of U_+, and the polytope $\widehat{\Pi}_i$ is dual to a face of $Z(f_{i,K})$ which passes through $\boldsymbol{u}_i$.

So, we immediately deduce that $Z(f_{i,K})$ contains: the part of U_+ preceding the vertex $\boldsymbol{u}_{i-1}$, the part of U_+ following the vertex $\boldsymbol{u}_{i+1}$, and all the cones $\Sigma_{\boldsymbol{u}_k}$, $k = 0, \dots, m+1$, $k \neq i$. Now, we observe that K contains at least two elements. Indeed, otherwise, if $K = \{j_1\}$, then the vectors $\boldsymbol{a}_{\boldsymbol{u}_i}(e)$, with $e \in U^1_{u_i}$, oriented to $\mathbb{R}^n_{\le 0}$, would lie in the same hyperplane $\{\lambda_{j_1} = u_{ij_1}\}$, and thus could not be balanced by a vector $\boldsymbol{a}_{\boldsymbol{u}_i}(e_{\boldsymbol{u}_i}) \in \mathbb{R}^n_{>0}$, which contradicts (3.3). So, if $j_1, j_2 \in K$, then the $(n-1)$-face of $Z(f_{i,K})$, dual to the edge $[P_{j_1,i-1}, P_{j_2,i-1}]$, contains the points $\boldsymbol{u}_{i-1}$ and $\boldsymbol{u}_i$, and hence also contains the edge e_i of U_+. Similarly, the $(n-1)$-face of $Z(f_{i,K})$, dual to the edge $[P_{j_1 i}, P_{j_2 i}]$, contains the points $\boldsymbol{u}_i$ and $\boldsymbol{u}_{i+1}$, and hence also contains the edge e_{i+1} of U_+. That is, $U_+ \subset Z(f_{i,K})$.

Next we verify that $\Sigma_{\boldsymbol{u}_i}(J) \subset \widetilde{U}_{\mathrm{red}}$ implies $\Sigma_{\boldsymbol{u}_i}(J) \subset Z(f_{i,K})$. Indeed, if $\Sigma_{u_i}(J) \subset \Sigma^U_{\boldsymbol{u}_i}$, then by construction $J \not\supset K$. Hence, there exists $s \in K \setminus J$, and thus the value $f_{i,K}(\boldsymbol{u}_i)$ is attained (among others) by the two monomials $\widehat{A}_{is} \odot \lambda_s^{b(i,s)}$ and $\widehat{A}_{i-1,s} \odot \lambda_s^{b(i-1,s)}$. Then, the $(n-1)$-dimensional orthant

$$\{\lambda_s = u_{is},\ \lambda_j \le u_{ij},\ 1 \le j \le n,\ j \ne s\},$$

having the vertex $\boldsymbol{u}_i$, is contained in $Z(f_{i,K})$, and in its turn contains $\Sigma_{\boldsymbol{u}_i}(J)$.

The last task is to check that $\Sigma_{\boldsymbol{u}_i}(K) \cap Z(f_{i,K}) = \emptyset$. To show this, we note that the polynomial $f_{i,K}$ is constant along

$$\Sigma_{\boldsymbol{u}_i}(K) = \{\lambda_j = \alpha_{ij},\ j \notin K,\ \lambda_l < u_{i\ell},\ \ell \in K\}$$

and its value is attained only by the monomial $\widehat{A}_{ij_0} \odot \lambda_{j_0}^{b(i,j_0)}$.

This completes the proof of Lemma 6.1. □

The construction of the ideals defining $\widetilde{U}$ and $\widetilde{U}_{\mathrm{red}}$ depends on the choice of the parameters $p_0, \dots, p_m$. Next we define these parameters so that Proposition 6.2 below holds true.

Given two strictly increasing sequences $\overline{\xi} = \{\xi_1, \dots, \xi_r\}$ and $\overline{\eta} = \{\eta_1, \dots, \eta_r\}$ of real numbers, we say that $\overline{\eta}$ is $\overline{\xi}$*-convex* if

$$\frac{\eta_k - \eta_{k-1}}{\xi_k - \xi_{k-1}} < \frac{\eta_{k+1} - \eta_k}{\xi_{k+1} - \xi_k} \quad \text{for all } k = 2, \dots, r-1. \tag{6.17}$$

Proposition 6.2. *In the above notation, let* $\overline{\xi}^{(k)} = \{\xi_1^{(k)}, \dots, \xi_m^{(k)}\}$, $k = 1, \dots, s$, *be an arbitrary strictly increasing sequence of real numbers. Then there are integers* $p_0, \dots, p_m$, *satisfying conditions* (P1), (P2) *from the first part of the proof of Lemma* 6.1, *such that, for each generator* f *of the defining ideal of* $\widetilde{U}_{\mathrm{red}}$ *and for every* $k = 1, \dots, s$, *the* (*strictly increasing*) *sequence* $f(\boldsymbol{u}_1), \dots, f(\boldsymbol{u}_m)$ *is* $\overline{\xi}^{(k)}$*-convex.*

We leave the proof of this elementary statement to the reader, remarking only that one should choose the sequence $p_0, \dots, p_m$ which grows sufficiently quickly.

6.3. Remark on plane additive tropical set-curves

The above geometric treatment, as well as the algebraic one, becomes quite transparent in the case of additive tropical plane set-curves.

Geometrically, one obtains an additive tropical set-curve $U \subset \mathbb{R}^2$ from its spine U_+ by attaching to each vertex $\boldsymbol{u}_i$, $1 \le i \le m$, one or two negatively directed horizontal and vertical rays. Furthermore, if $\boldsymbol{u}_1$ is the minimal point of the spine U_+ (i.e., $U_+ \simeq [0, \infty)$), then we call $\boldsymbol{u}_1$ a *terminal vertex* of U. In particular, if $\boldsymbol{u}_1$ is terminal, then it is a common vertex of a horizontal and a vertical negatively directed rays of U.

By Theorem 5.1, such a set-curve U can be defined by one simple tropical polynomial. Furthermore, Lemma 6.1 provides a family of such polynomials with parameters $p_0, \dots, p_m$ subject to conditions (P1), (P2). Keeping the property declared in Proposition 6.2, we claim that one can vary these parameters and get the following additional property:

Proposition 6.3. *In the above notation, assume that the point $\boldsymbol{u}_m \in U_+^0$ is a common vertex of a horizontal and a vertical negatively directed rays. Then, for any polynomial $f(\lambda_1, \lambda_2)$ constructed for U as in the proof of Lemma 6.1, keeping the values $f(\boldsymbol{u}_1), \dots, f(\boldsymbol{u}_m)$ and the set $Z(f)$ unchanged, one can make*

$$p_m \gg p_k \quad \textit{and} \quad p_0 \ll p_k \quad \textit{for all } k = 1, \dots, m-1. \tag{6.18}$$

Again, the proof is an easy exercise left to the reader. We only observe that if $\boldsymbol{u}_1$ is not terminal then $p_0 = 0$ satisfies the requirements of the proposition. Also, in such a variation of p_0 and p_m, the convexity property required in Proposition 6.2 persists, since it depends only on $f(u_1), \dots, f(u_m)$.

6.4. Simplicity of spatial additive tropical set-curves

Theorem 6.4. *A tropical set-curve $U \subset \mathbb{R}^n$, where $n = 2$ or 3, is additive if and only if it is simple.*

Proof. In view of Corollary 4.2 and Theorem 5.1, it remains to prove that an additive tropical set-curve $U \subset \mathbb{R}^3$ is simple.

Pick $i = 1, \dots, m$ and $J \in \mathcal{J}_i(U)$, and consider the set $U_{i,J} = U \cap \overline{\Sigma_{\boldsymbol{u}_i}(J)}$. We intend to construct a pair of simple polynomials, denoted F and F', for which

$$Z(F) \cap Z(F') \supset U \quad \text{and} \quad Z(F) \cap Z(F') \cap \overline{\Sigma_{\boldsymbol{u}_i}(J)} = U_{i,J}. \tag{6.19}$$

Then, varying $i = 1, \dots, m$ and $J \in \mathcal{J}_i(U)$, and adding all the newly acquired polynomials to the ideal of $\widetilde{U}_{\text{red}}$, we obtain the desired simple ideal defining U.

The case of $\#J = 1$ is easy. Indeed, $U_{i,J}$ is just the ray parallel to one of the coordinate axes, say the λ_1-axis, emanating from $\boldsymbol{u}_i$, and pointing to $-\infty$. We project U to the (λ_1, λ_2)-plane and obtain an additive tropical plane curve V, which is defined by a simple polynomial $F(\lambda_1, \lambda_2)$. It is then clear that $Z(F) \cap \overline{\Sigma_{\boldsymbol{u}_i}(J)} = U_{i,J}$.

So, it remains to consider the case when $\#J = 2$; thus, from now on we assume $J = \{1, 2\}$. We identify $\mathbb{R}^2$ with the plane $\{\lambda_3 = 0\} \subset \mathbb{R}^3$ and introduce the natural projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$. Our strategy is to construct the polynomials F and F' to be of the form $f(\lambda_1, \lambda_2) \oplus g(\lambda_3)$, where $f(\lambda_1, \lambda_2)$ is a simple polynomial defining a certain modification of the projection $\pi(U)$ in $\mathbb{R}^2$, and $g(\lambda_3)$ provides a correction of the intersections of $Z(f)$ with the quadrants $\Sigma_{\boldsymbol{u}_k}(J)$, $k = 1, \dots, m$.

We proceed further in several steps.

Step 1. In general, $\pi(\overline{\Sigma_{\boldsymbol{u}_i}(J)}) \cap \pi(U)$ is greater than $\pi(\overline{\Sigma_{\boldsymbol{u}_i}(J)}) \cap \pi(U_{i,J})$. In this step, we shall decide which parts of U contribute to $\pi(\overline{\Sigma_{\boldsymbol{u}_i}(J)}) \cap \pi(U)$ beyond $\pi(U_{i,J})$, and which do not.

The set $V_{i,J} = \pi(U_{i,J})$ is an additive tropical plane set, which can be extended to an additive tropical set-curve $\widehat{V}_{i,J}$ by attaching a ray with vertex $\pi(\boldsymbol{u}_i)$ directed to $\mathbb{R}^2_{>0}$. If $\widehat{V}_{i,J}$ has a terminal vertex $\boldsymbol{v} = (v_1, v_2)$ (which must then differ from $\boldsymbol{u}_i$, since $\Sigma_{\boldsymbol{u}_i}(J) \cap U \neq \emptyset$), we let

$$Q_{i,J} := \{(x_1, x_2) \in \mathbb{R}^2 : \max_{j=1,2}(x_j - u_{ij}) \leq 0 \leq \max_{j=1,2}(x_j - v_j)\},$$

and otherwise we set

$$Q_{i,J} := \{(x_1, x_2) \in \mathbb{R}^2 : \max_{j=1,2}(x_j - u_{ij}) \leq 0\}.$$

Geometrically, in the latter case, $Q_{i,J}$ is just a shifted negative quadrant, while in the former case, $Q_{i,J}$ is the closed difference of two such quadrants, one lying in the interior of the other. Moreover, $V_{i,J} \subset Q_{i,J}$, and $Q_{i,J}$ is the minimal figure of the given shape, containing $V_{i,J}$.

We claim that:

- for each $k > i$ and $K \subset \{1, 2, 3\}$ such that $K \not\supset J$, one has $\pi(U_{k,K}) \cap Q_{i,J} = \emptyset$,
- $\pi(U_i) \cap Q_{i,J} = V_{i,J}$, where $U_i = \{\boldsymbol{u} \in U : \boldsymbol{u} \prec \boldsymbol{u}_i\}$.

The first relation is easy: if $\ell \in J \setminus K$, then any point of $U_{k,K}$ has the ℓ-th coordinate $u_{k\ell} > u_{i\ell}$, and hence its π-projection lies outside $Q_{i,J}$. To prove the second relation, we note that for any point $\boldsymbol{u} = (u_1, u_2, u_3) \in U_i$ for which $\pi(\boldsymbol{u}) \in Q_{i,J}$, one has $u_3 \leq u_{i3}$, and there always exists a point $\boldsymbol{v} = (u_1', u_2', u_{i3}) \in V_{i,J}$ such that $u_1' \leq u_1$ and $u_2' \leq u_2$ (the latter property is evident if $\widehat{V}_{i,J}$ has no terminal vertex, and it follows from (vii) in Section 6.1 if $\widehat{V}_{i,J}$ has a terminal vertex). Then,

$$\boldsymbol{u} \oplus \boldsymbol{v} = (u_1, u_2, u_{i3}) \in V_{i,J},$$

which, in particular, yields $\pi(\boldsymbol{u}) = \pi(\boldsymbol{u} \oplus \boldsymbol{v})$.

Thus, the only parts of U whose π-projections may contribute to $\pi(U) \cap Q_{i,J}$ beyond $\pi(U_{i,J})$ are $U_{k,J}$ with $k > i$.

Step 2. Assume that $\widehat{V}_{i,J}$ has a terminal vertex $\boldsymbol{v}_1$ as in Step 1. In this situation we shall construct just one required polynomial $F = F'$; we start by constructing the part $f(\lambda_1, \lambda_2)$ of F.

Pick a point $\boldsymbol{u}_{m+1} = (u_{m+1,1}, u_{m+1,2}, u_{m+1,3})$ on the ray $e_m \subset U$, and attach to it the three rays parallel to the coordinate axes and pointing to $-\infty$. The newly obtained set $\widehat{U}$ is again a tropical additive curve. Now, let

$$W_{i,J} = \widehat{U} \cup \bigcup_{k>i} \overline{\Sigma_{\boldsymbol{u}_k}(J)} \setminus \bigcup_{k>i} \Sigma_{\boldsymbol{u}_k}(J).$$

Geometrically, $W_{i,J}$ is obtained from $\widehat{U}$ as follows: for each vertex $\boldsymbol{u}_k$ with $k > i$, we delete the part of U attached to $\boldsymbol{u}_k$ and contained in the quadrant $Q_{k,J}$ and, instead, we add two negatively directed rays emanating from $\boldsymbol{u}_k$ and parallel to the λ_1-axis and to the λ_2-axis, respectively. It is easy to see that $W_{i,J}$ is an additive tropical curve. Hence, $\pi(W_{i,J}) \subset \mathbb{R}^2$ is an additive tropical plane curve. The results of Step 1 imply that

$$\pi(W_{i,J}) \cap Q_{i,J} = U_{i,J}. \tag{6.20}$$

Note that the points $\pi(\boldsymbol{v}_1)$ and $\pi(\boldsymbol{u}_1), \dots, \pi(\boldsymbol{u}_m), \pi(\boldsymbol{u}_{m+1})$ belong to the spine of $\pi(W_{i,J})$. Let them be ordered (cf. (2.1)) as follows:

$$\pi(\boldsymbol{u}_1) \prec \cdots \prec \pi(\boldsymbol{u}_s) \prec \pi(\boldsymbol{v}_1) \prec \pi(\boldsymbol{u}_{i+1}) \prec \cdots \prec \pi(\boldsymbol{u}_{m+1}),$$

where $0 \le s < i$ and s is the maximal possible index satisfying this ordering with $\pi(\boldsymbol{u}_s) \ne \pi(\boldsymbol{v}_1)$. Due to Propositions 6.2 and 6.3, we may assume that $\pi(W_{i,J})$ is defined by a simple polynomial $f(\lambda_1, \lambda_2)$ satisfying the following condition: the sequence

$$f(\boldsymbol{u}_1) < \cdots < f(\boldsymbol{u}_s) < f(\boldsymbol{v}_1) < f(\boldsymbol{u}_{i+1}) < \cdots < f(\boldsymbol{u}_{m+1})$$

is convex with respect to the sequence $u_{13} < \cdots < u_{s3} < u_{i3} < \cdots < u_{m3} < u_{m+1,3}$, and relation (6.18) holds true as well.

Step 3. Now, we define the polynomial

$$F(\lambda_1, \lambda_2, \lambda_3) = f(\lambda_1, \lambda_2) \oplus g(\lambda_3) \quad \text{with} \quad g(\lambda_3) = \bigoplus_{k=0}^{m+1} A_k \odot \lambda_3^{c_k}, \tag{6.21}$$

whose parameters A_k and c_k, $k = 0, \dots, m+1$, satisfy the following conditions:

(a) $c_{m+1} = c_m + 1$;
(b) for $i < k \le m$,

$$c_k = \frac{f(\boldsymbol{u}_{k+1}) - f(\boldsymbol{u}_k)}{u_{k+1,3} - u_{k3}}, \qquad (A_k \odot \lambda_3^{c_k})|_{\boldsymbol{u}_{k+1}} = (A_{k+1} \odot \lambda_3^{c_{k+1}})|_{\boldsymbol{u}_{k+1}} = f(\boldsymbol{u}_{k+1});$$

(c) for the i-th monomial,

$$c_i = \frac{f(\boldsymbol{u}_{i+1}) - f(\boldsymbol{v}_1)}{u_{i+1,3} - u_{i3}}, \qquad (A_i \odot \lambda_3^{c_i})|_{\boldsymbol{u}_{i+1}} = (A_{i+1} \odot \lambda_3^{c_{i+1}})|_{\boldsymbol{u}_{i+1}} = f(\boldsymbol{u}_{i+1}),$$
$$(A_i \odot \lambda_3^{c_i})|_{\boldsymbol{u}_i} = f(\boldsymbol{v}_1);$$

(d) $c_{i-1} = c_i - 1$, and $(A_{i-1} \odot \lambda_3^{c_{i-1}})_{\lambda_3 = u_{i3} - \varepsilon} = (A_i \odot \lambda_3^{c_i})_{\lambda_3 = u_{i3} - \varepsilon}$, where $\varepsilon > 0$ is small (we specify this later);

(e) for $s < k < i - 1$,

$$c_k = c_{k+1} - 1, \quad (A_k \odot \lambda_3^{c_k})|_{\boldsymbol{u}_{k+1}} = (A_{k+1} \odot \lambda_3^{c_{k+1}})|_{\boldsymbol{u}_{k+1}};$$

(f) $(A_s \odot \lambda_3^{c_s})|_{\boldsymbol{u}_{s+1}} = (A_{s+1} \odot \lambda_3^{c_{s+1}})|_{\boldsymbol{u}_{s+1}}$;

(g) for $0 \le k < s$,

$$(A_k \odot \lambda_3^{c_k})|_{\boldsymbol{u}_{k+1}} = (A_{k+1} \odot \lambda_3^{c_{k+1}})|_{\boldsymbol{u}_{k+1}} = f(\boldsymbol{u}_{k+1});$$

(h) $c_0 = c_1 - 1$.

We observe that these relations uniquely determine the values of the parameters A_k and $c_k, k = 0, \dots, m+1$, out of the values $f(\boldsymbol{u}_1), \dots, f(\boldsymbol{u}_{m+1}), f(\boldsymbol{v}_1)$. Multiplying the parameters $p_0, \dots, p_m$ in the construction of the polynomial f by a suitable natural number, we multiply the values of f by that number, and thus we can achieve the integrality of the exponents c_k in the above definition.

Due to the assumed convexity property of the values of f, each monomial of g is essential (here we specify the value of ε, taking into account that for $\varepsilon = 0$ all the monomials of g are essential).

Let us verify that $Z(F) \supset U$. Observing that

$$Z(F) = \{f = g\} \cup (Z(f) \cap \{f \ge g\}) \cup (Z(g) \cap \{f \le g\}),$$

we note that, for any point $\boldsymbol{u} \in \{f = g\}$, the set $\{f \ge g\}$ contains the negative ray with vertex $\boldsymbol{u}$, parallel to the λ_3-axis, and that the set $\{f \le g\}$ contains the negative quadrant with vertex $\boldsymbol{u}$, parallel to the (λ_1, λ_2)-plane. Notice also that $W_{i,J} \subset Z(f)$. Then:

1. For any $k > i$, due to relations (b), (c), and the construction in the proof of Lemma 6.1, the value $F(\boldsymbol{u}_k)$ is attained by the pair of monomials $A_{k-1} \odot \lambda_3^{c_{k-1}}$ and $A_k \odot \lambda_3^{c_k}$ of $g(\lambda_3)$; the same value $F(\boldsymbol{u}_k)$ is also attained by some four monomials of $f(\lambda_1, \lambda_2)$, since by construction the plane tropical curve $\pi(W_{i,J})$ has four edges incident to its vertex $\pi(\boldsymbol{u}_k)$, two of them with positive slopes, and the other two being negatively directed vertical and horizontal rays. It is then easy to derive that $Z(F) \supset \Sigma_{\boldsymbol{u}_k}$.
2. Since $f(\boldsymbol{u}_{m+1}) = g(\boldsymbol{u}_{m+1})$ and $p_{m+1} \gg \max_\ell f(\boldsymbol{u}_\ell)$ (cf. Proposition 6.3 and relation (a) above), and since the polynomial g is linear in the half-space $\{\lambda_3 \ge u_{m+1,3}\}$, we derive that $f(\boldsymbol{u}) \ge g(\boldsymbol{u})$ along the ray $e_{\boldsymbol{u}_{m+1}}$, and hence $e_{\boldsymbol{u}_{m+1}} \subset Z(F)$. Similarly, $f(\boldsymbol{u}_1) = g(\boldsymbol{u}_1)$ and $p_0 \ll \min_\ell f(\boldsymbol{u}_\ell)$ (cf. Proposition 6.3 and relation (h) above), and hence the ray of $\widehat{U}$, emanating from $\boldsymbol{u}_1$ and proceeding to $\mathbb{R}^n_{<0}$, is contained in $Z(F)$.
3. By construction, for $k > i$, the values $f(\boldsymbol{u}_k)$ and $f(\boldsymbol{u}_{k+1})$ are attained by the same monomial, and the same also holds for g. Hence, due to the linearity of f and g along the segment $[\boldsymbol{u}_k, \boldsymbol{u}_{k+1}]$, this segment is contained in $Z(F)$. In the same way, when $1 \le k < s$ we have $[\boldsymbol{u}_k, \boldsymbol{u}_{k+1}] \subset Z(F)$.
4. Since $g(\boldsymbol{u}_{i+1}) = f(\boldsymbol{u}_{i+1})$ and $g(\boldsymbol{u}_i) = f(\boldsymbol{v}_1) < f(\boldsymbol{u}_i)$, we derive that the segment $[\boldsymbol{u}_i, \boldsymbol{u}_{i+1}]$ lies in the domain $\{f \ge g\}$, and hence is contained in $Z(F)$.

5. We have $g(\boldsymbol{u}_s) = f(\boldsymbol{u}_s)$ and $g(\boldsymbol{u}_k) < g(\boldsymbol{u}_i) = f(\boldsymbol{v}_1) \le f(\boldsymbol{u}_k)$ for all $s < k < i$, since $\boldsymbol{v}_1 \prec \pi(\boldsymbol{u}_k)$ on $\pi(W_{i,J})$. Hence, the segments $[\boldsymbol{u}_l, \boldsymbol{u}_{l+1}]$, $s \le l < i$, lie in the domain $\{f \ge g\}$, and thus are contained in $Z(F)$.
6. If $U \cap \Sigma_{\boldsymbol{u}_k}(K) \ne \emptyset$ for some $k = 1, \dots, i$ and $K = \{1, 3\}$, then $\pi(W_{i,J})$ contains the negatively directed ray starting at $\pi(\boldsymbol{u}_k)$ and parallel to the λ_1-axis. Hence, the value $f(\boldsymbol{u}_k)$ is attained by at least two monomials involving λ_2, which keep their value along the negatively directed ray starting at $\boldsymbol{u}_k$ and parallel to the λ_1-axis. As we have seen earlier, $f(\boldsymbol{u}_k) \ge g(\boldsymbol{u}_k)$, and thus the latter ray lies entirely in the domain $\{f \ge g\}$. Hence, $\overline{\Sigma_{\boldsymbol{u}_k}(K)} \subset Z(F)$. The case of $K = \{2, 3\}$ is treated in the same way.
7. For each $k = 1, \dots, i-1$, the value of g along $\overline{\Sigma_{\boldsymbol{u}_k}(J)}$ is attained by two monomials of g, and thus
$$\overline{\Sigma_{\boldsymbol{u}_k}(J)} \cap Z(F) = (\overline{\Sigma_{\boldsymbol{u}_k}(J)} \cap \{g \ge f\}) \cup (\overline{\Sigma_{\boldsymbol{u}_k}(J)} \cap \{f \ge g\} \cap Z(f)) \supset \overline{\Sigma_{\boldsymbol{u}_k}(J)} \cap U.$$
8. Finally, the value of g along $\overline{\Sigma_{\boldsymbol{u}_i}(J)}$ is attained exactly by one monomial of g, and hence
$$\overline{\Sigma_{\boldsymbol{u}_k}(J)} \cap Z(F) = (\overline{\Sigma_{\boldsymbol{u}_k}(J)} \setminus \{g > f\}) \cap \{f \ge g\} \cap Z(f).$$

Recall that U contains two negatively directed rays, starting at $\boldsymbol{v}_1$ and parallel to the λ_1-axis and to the λ_2-axis, respectively, as in Section 6.3. Now, since the value $g(\boldsymbol{u}_i) = f(\boldsymbol{v}_1)$ is attained by four monomials of f, two in λ_1 and two in λ_2, we conclude that
$$\pi(\overline{\Sigma_{\boldsymbol{u}_k}(J)} \setminus \{g > f\}) = Q_{i,J}$$
(see the definition at the beginning of Step 2).

Summarizing, we have shown that $Z(F) \supset U$ for all suitable generators f of the simple ideal of $\pi(W_{i,J})$, and thus, due to (6.20), that
$$\bigcap_f Z(f) \cap \overline{\Sigma_{\boldsymbol{u}_i}(J)} = U_{i,J}.$$

Step 4. In the case when $\widehat{V}_{i,J}$ has no terminal vertex we shall suitably modify the preceding construction of the polynomials $f(\lambda_1, \lambda_2)$ and $g(\lambda_3)$, constituting F, and at the end we shall append an additional simple polynomial F' meeting requirements (6.19).

Consider the additive tropical plane curve $\pi(U) \subset \mathbb{R}^2$ and denote the minimal vertex of $(\pi(U))_+$ by $\boldsymbol{w} = (w_1, w_2)$. Note that $\boldsymbol{w} \prec \pi(\boldsymbol{u}_k)$ for all $k = 1, \dots, m+1$, and
$$\pi(U) \cap \{\lambda_1 < w_1,\ \lambda_2 < w_2\} = V_{i,J} \cap \{\lambda_1 < w_1,\ \lambda_2 < w_2\} \tag{6.22}$$
(this is just an open ray).

Again, using Propositions 6.2 and 6.3, we can choose a simple polynomial $f(\lambda_1, \lambda_2)$ defining $\pi(W_{i,J})$ and satisfying the following conditions: the sequence
$$f(\boldsymbol{w}) < f(\boldsymbol{u}_{i+1}) < \cdots < f(\boldsymbol{u}_{m+1})$$
is convex with respect to the sequence $u_{i3} < u_{i+1,3} < \cdots < u_{m3} < u_{m+1,3}$, and relation (6.18) holds true as well. Then, we define the polynomial F as in formula (6.21), where the parameters A_k and c_k, $0 \le k \le m+1$, are determined by conditions (a) and (b) in Step 3 and by the following requirements:

(c′) for the i-th monomial,

$$c_i = \frac{f(\boldsymbol{u}_{i+1}) - f(\boldsymbol{w}_1)}{u_{i+1,3} - u_{i3}}, \qquad (A_i \odot \lambda_3^{c_i})|_{\boldsymbol{u}_{i+1}} = (A_{i+1} \odot \lambda_3^{c_{i+1}})\big|_{\boldsymbol{u}_{i+1}} = f(\boldsymbol{u}_{i+1}),$$
$$(A_i \odot \lambda_3^{c_i})|_{\boldsymbol{u}_i} = f(\boldsymbol{w});$$

(d′) for $0 \le k < i$,

$$c_k = c_{k+1} - 1, \qquad (A_k \odot \lambda_3^{c_k})|_{\boldsymbol{u}_{k+1}} = (A_{k+1} \odot \lambda_3^{c_{k+1}})|_{\boldsymbol{u}_{k+1}}.$$

Conditions (a), (b), (c′), and (d′) uniquely determine the values of the parameters A_k and c_k, $0 \le k \le m+1$, out of the values $f(\boldsymbol{w}), f(\boldsymbol{u}_1), \dots, f(\boldsymbol{u}_{m+1})$.

Using the argument of Step 3, where $\boldsymbol{w}$ plays the role of $\boldsymbol{v}_1$, we show that $Z(F) \supset U$. However, in the quadrant $\overline{\Sigma_{\boldsymbol{u}_i}(J)}$ we obtain

$$Z(F) \cap \overline{\Sigma_{\boldsymbol{u}_i}(J)} = (\overline{\Sigma_{\boldsymbol{u}_i}(J)} \cap \{g \ge f\}) \cup (\overline{\Sigma_{\boldsymbol{u}_i}(J)} \cap Z(f)),$$

since the value $g(\boldsymbol{u}_i) = f(\boldsymbol{w})$ is attained by two monomials of g. Here, $\{g \ge f\}$ cuts off the quadrant $Q = \{\lambda_1 \le w_1,\ \lambda_2 \le w_2,\ \lambda_3 = u_{i3}\}$ from the set $\overline{\Sigma_{\boldsymbol{u}_i}(J)}$, and $Z(f)$ cuts off the set $(\overline{\Sigma_{\boldsymbol{u}_i}(J)} \setminus Q) \cap U_{i,J}$ from the set $\overline{\Sigma_{\boldsymbol{u}_i}(J)} \setminus Q$. Hence,

$$\overline{\Sigma_{\boldsymbol{u}_i}(J)} \cap \bigcap_f Z(f) \cap Z(F') = U_{i,J}$$

as required, where $F'(\lambda_1, \lambda_2)$ is a simple polynomial defining the tropical set-curve $\pi(U)$.

The proof of Theorem 6.4 is complete. □

7. Additive tropical subvarieties of additive tropical varieties

Theorem 7.1. *Let $P \subset \mathbb{R}^n$ be an m-dimensional additive tropical set-variety. Then*

- *the faces of the closures of the connected components of* Reg(P) *define an FPC structure $\mathcal{P}$ on P;*
- *for any $k = 0, \dots, m-1$, the k-skeleton $P^{(k)} = \bigcup_{\sigma \in \mathcal{P},\, \dim(\sigma) \le k} \sigma$ and each of its connected components are additive tropical set-varieties.*

Proof. Let $K_1, \dots, K_N$ be all the connected components of $\mathrm{Reg}(P)$. We extend the result of Lemma 3.3 by showing that

$$\overline{K}_i \cap \overline{K}_j \subset \partial \overline{K}_i \cap \partial \overline{K}_j \qquad \text{for every } 1 \le i < j \le N.$$

Arguing towards a contradiction, in view of Lemma 3.3, we assume that

$$\tau = \mathrm{Int}(\overline{K}_i) \cap \bigcup_{j \ne i} \overline{K}_j \ne \emptyset, \qquad \dim(\tau) = k < m-1,$$

for some $i \ne j$. Let $\boldsymbol{x} \in \tau$ be a generic point. Then there is an $(n-k)$-plane

$$\Pi = \{x_{i_1} = \dots = x_{i_k} = 0\} \subset \mathbb{R}^n$$

which is transverse to $\mathbb{R}\tau$. This $(n-k)$-plane is an additive tropical set-variety, and hence so is $P' = P \cap (\boldsymbol{x} + \Pi)$.

Let K_i' be the germ of $\mathrm{Int}(\overline{K}_i) \cap \Pi$ at $\boldsymbol{x}$, which is the germ of an affine space of dimension $m-k \geq 2$, and let K_i'' be the germ of $\Pi \cap \bigcup_{j \neq i} \overline{K}_j$ at $\boldsymbol{x}$ such that $\dim(K_i'') = m-k \geq 2$. Furthermore, we may assume that K_i' is the whole affine $(m-k)$-space, and K_i'' is a cone with vertex $\boldsymbol{x}$.

Given a neighborhood of $\boldsymbol{x}$, P' is the union of the germ K_i' at $\boldsymbol{x}$ and of the germ K_i'' at $\boldsymbol{x}$ such that $K_i' \cap K_i'' = \boldsymbol{x}$. Clearly there is $\boldsymbol{u} \in K_i' \setminus \{\boldsymbol{x}\}$ such that $\boldsymbol{x} \oplus \boldsymbol{u} \neq \boldsymbol{x}$. Then $\boldsymbol{x} \notin \boldsymbol{u} \oplus (K_i' \cup K_i'')$, and, due to $\boldsymbol{u} \oplus \boldsymbol{u} = \boldsymbol{u} \in K_i'$, we conclude that $\boldsymbol{u} \oplus (K_i' \cup K_i'') \subset K_i' \setminus \{\boldsymbol{x}\}$. By the same token, we see that $\boldsymbol{x} \oplus K_i'' = \boldsymbol{x}$, or equivalently, $K_i'' \subset \boldsymbol{x} + \mathbb{R}^n_{\leq 0}$, which in turn contradicts the fact that K_i'' is a tropical set-variety. (Clearly, K_i' does not affect the balancing condition for cells of K_i''.)

Now we show that $P^{(m-1)}$ (see Definition 3.4) is an additive tropical set-variety. Pick a set $I \subset \{1, \dots, n\}$, $|I| = m+1$, and introduce the projection

$$\pi_I : \mathbb{R}^n \to \mathbb{R}^I = \{x_i = 0,\ i \notin I\}.$$

If, for some $j = 1, \dots, N$, $\dim(\pi_I(\overline{K}_j)) = m$, then $\pi_I(P)$ is an m-dimensional tropical set-hypersurface in $\mathbb{R}^I$ (i.e., push-forward, as defined in [AR]), which is additive, since π_I is a semigroup homomorphism. Furthermore, $Q_I = \pi_I^{-1}(\pi_I(P))$ is an additive tropical set-hypersurface in $\mathbb{R}^n$ (i.e., pull-back, as defined in [AR]) which can be viewed as the union of the $(n-1)$-dimensional polyhedra $\overline{K}_i + V_I$, where $\dim(\pi_I(\overline{K}_i)) = m$ and $V_I = \{x_j = 0,\ j \in I\}$.

Observe that the two $(n-1)$-dimensional polyhedra $\overline{K}_i + V_I$ and $\overline{K}_j + V_I$ cannot intersect so that

$$\dim(\mathrm{Int}(\overline{K}_i + V_I) \cap \mathrm{Int}(\overline{K}_j + V_I)) = n-2.$$

Indeed, otherwise the $(n-2)$-cell $(\overline{K}_i + V_I) \cap (\overline{K}_j + V_I)$ of Q_I would be dual to a parallelogram in the subdivision $S(f)$ of the Newton polytope of a simple tropical polynomial f with $Z(f) = Q_I$ (see Theorem 5.1), and thus we reach a contradiction, since no four distinct points on coordinate axes span a parallelogram.

Consider now the intersection $R_{I,t} = P \cap (at + Q_I)$ for a generic vector $a \in \mathbb{R}^n$ and a small positive parameter t. This intersection is transversal and makes $R_{I,t}$ an additive tropical set-variety of dimension $m-1$. The top dimensional cells of $R_{I,t}$ appear as the intersections $\overline{K}_i \cap (at + \overline{K}_j + V_I)$, $i \neq j$, where $\overline{K}_i$ and $\overline{K}_j + V_I$ are transverse and intersect along their interiors. As $t \to 0$, such an intersection either contracts or converges to an $(m-1)$-dimensional polyhedron in P. Two situations may occur:

- If $\overline{K}_i \cap \overline{K}_j$ is a common $(m-1)$-dimensional face σ, then $\overline{K}_i \cap (at + \overline{K}_j + V_I)$ converges to σ as $t \to 0$.
- If $\dim(\overline{K}_i \cap \overline{K}_j) < m-1$, but $\dim(\overline{K}_i \cap (at + \overline{K}_j + V_I)) = m-1$, we necessarily obtain that $\overline{K}_i + V_I$ and $\overline{K}_j + V_I$ intersect transversally along their interior, which is not possible as observed in the preceding paragraph.

Thus, we see that the limit R_I of $R_{I,t}$ as $t \to 0$ is an additive tropical set-variety which is the union of some $(m-1)$-faces of $\overline{K}_i$, $i = 1, \dots, N$. Noticing that $R = \bigcup_I R_I =$

$P^{(m-1)}$, where I runs over all $(m+1)$-subsets of $\{1,\dots,n\}$, we prove that $P^{(m-1)}$ is an additive tropical set-variety.

The rest of the proof goes by induction on descending dimension. □

Corollary 7.2. *Given an additive tropical set-variety, the connected components of its skeletons are contractible.*

We finish by describing additive tropical set-varieties having a disconnected skeleton of a positive dimension:

Lemma 7.3. *If a connected additive m-dimensional tropical set-variety P in $\mathbb{R}^n$ has a disconnected skeleton $P^{(d)}$, $0 < d < m$, then there are additive transversal linear subspaces $U, V \subset \mathbb{R}^n$ of dimension d and $n-d$ respectively such that $P = U + Q$, with $Q = P \cap V$ being an additive $(m-d)$-dimensional tropical set-variety.*

Proof. We use the following fact which we leave to the reader as an elementary exercise: if the union of the d-dimensional faces of an m-dimensional convex polyhedron σ in $\mathbb{R}^n$ $(n > m > d)$ is not connected, then there are transversal linear subspaces $U', V' \subset \mathbb{R}^n$ of dimension d and $n-d$ respectively, such that $\sigma = U' + \tau$, with $\tau = \sigma \cap V'$ being an $(m-d)$-dimensional convex polyhedron.

We can assume that $P^{(d+1)}$ is connected. Let σ be a $(d+1)$-cell of $P^{(d+1)}$ joining two connected components of $X^{(d)}$. As noticed above, $\sigma = U + \tau$ with τ a segment. Then $\partial\sigma$ is the union of two affine spaces $a+U, b+U, a, b \in \mathbb{R}^n$. By Theorem 7.1, $P^{(d)}$ is a tropical set-variety, and hence, due to the balancing condition, $a+U$, $b+U$ must be separate connected components of $P^{(d)}$. This immediately implies that $P^{(d)}$ is the union of several affine spaces $a+U$. Since each of them is additive (Theorem 7.1), U is additive. Again the above observation on polyhedra with a disconnected d-skeleton shows that each cell σ of P of dimension $> d$ is represented as $U + \tau$, and hence $P = U + Q$, where Q can be obtained as intersection of P with an additive $(n-d)$-dimensional linear subspace $V \subset \mathbb{R}^n$ transversal to U. □

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