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Varieties with generically nef tangent bundles

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Abstract. We study various "generic" nefness and ampleness notions for holomorphic vector bundles on a projective manifold. We apply this in particular to the tangent bundle and investigate the relation to the geometry of the manifold.

Contents

1. Introduction

The notion of generic nefness of a vector bundle is a concept which appeared first in Miyaoka's important paper [Mi87], in which he studied the cotangent bundles of non-uniruled manifolds:

The cotangent bundle of a non-uniruled manifold is generically nef.

A vector bundle E on a projective n-dimensional manifold X is *generically nef* if the following holds. Given any ample line bundles H_j, $1 \le j \le n-1$, let C be a curve cut out by general elements in $|m_j H_j|$ for $m_j \gg 0$. Then the restriction $E|C$ is nef.

In the same way *generic ampleness* is defined. If we fix the H_j, we speak of generic nefness with respect to $(H_1, \dots, H_{n-1})$.

The complete intersection curves C in the above definition play an important role in the theory of (slope-)semi-stability: the rank r vector bundle E is semi-stable with respect

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to $h = (H_1, \dots, H_{n-1})$ if and only if the $\mathbb{Q}$-bundle $E \otimes \frac{\det E^*}{r} | C$ is nef. So if E is h-semi-stable with $c_1(E) \cdot h \geq 0$, then E is h-generically nef. Therefore generic nefness can be viewed as a weak version of semi-stability.

In this paper we are mainly interested in projective manifolds X with generically nef and generically ample *tangent* bundles T_X. Our motivation is twofold. First, these classes seem to be appropriate algebraic notions for expressing that the manifold X has (semi-)positive bisectional curvature in the general direction. These manifolds have interesting geometric properties and could be good models for Campana's special varieties (see e.g. [Ca10]). Second, the aim is to better understand certain significant classes of uniruled varieties, in particular those with nef anti-canonical bundles. Both issues will be explained in more detail below.

A projective manifold X whose tangent bundle T_X is generically nef with respect to some polarization is uniruled, unless $K_X \equiv 0$; this is a special case of Miyaoka's theorem. Actually much more holds: if $C \subset X$ is an irreducible curve with $K_X \cdot C < 0$ and $T_X | C$ is nef, then X is uniruled. If $T_X | C$ is even ample, then X is rationally connected. Uniruledness however is by far not enough to have a generically nef tangent bundle. E.g., one can take the product of $\mathbb{P}_1$ with a manifold of general type. More generally this is demonstrated by the following mapping property.

1.1. Theorem. *If T_X is generically nef with respect to some polarization and if $f : X \to Y$ is a surjective holomorphic map to a normal projective variety Y, then either Y is uniruled or a smooth model of Y has Kodaira dimension* 0.

In particular, the Albanese map of X is surjective.

On the other hand, there is of course no restriction on images of uniruled varieties. This picture, which might be rephrased by saying that varieties with generically nef tangent bundle have a *pure* geometry, is—as already mentioned—very similar to Campana's class of special varieties, and one might even speculate that there is a deep connection between both classes.

Generic ampleness should be seen as a birational version of ampleness. Recall in this context Mori's theorem that $\mathbb{P}_n$ is the only manifold with ample tangent bundle. In the birational context we want to characterize rationally connected manifolds. As said before, if the tangent bundle T_X is ample on one single curve, then X is already rationally connected. Conversely, we expect

1.2. Conjecture. Let X be a rationally connected projective manifold (with $-K_X$ pseudo-effective, possibly). Then T_X is generically ample (at least for some polarization).

The assumption that $-K_X$ is pseudo-effective might be necessary in order to avoid "too many blow-ups". We support the conjecture by proving

1.3. Theorem. *Let X be a projective manifold whose anti-canonical bundle $-K_X$ is big and nef. Then T_X is generically ample.*

Notice that manifolds with $-K_X$ big and nef are rationally connected.

Particularly interesting are Fano manifolds X, i.e., $-K_X$ is ample. Suppose also that $b_2(X) = 1$. A long-standing conjecture says that then T_X is stable. Although known to be valid in many cases, this conjecture is wide open in general. Since, as already explained, the stability of T_X in this case implies generic ampleness, Theorem 1.3 can be seen as a weak substitute for the stability of the tangent bundle. It should also be noted that the proofs of the generic ampleness and nefness theorem 1.3 and 1.4 use the theory of stable bundles, namely the Mehta–Ramanathan theorem [MR82].

The proof of Theorem 1.3 also uses a cone theorem for the movable cone proved by Birkar–Cascini–Hacon–McKernan [BCHM10] and further results from that paper.

Another step towards Conjecture 1.2 is done in Section 7: if X is rationally connected, we prove in all dimensions that there is an irreducible C whose numerical equivalence class is in the interior of the movable cone and $T_X|C$ is ample.

Towards generic nefness, we show

1.4. Theorem. *Let X be a projective manifold.*

- *If $-K_X$ is semi-ample, i.e., some multiple $|-mK_X|$ is spanned, then T_X is generically nef.*
- *If X is a rational surface with $-K_X$ pseudo-effective, then T_X is H-generically ample for a suitable ample divisor H.*
- *If X is a surface with $-K_X$ pseudo-effective and irregularity $q(X) = 1$, then T_X is generically nef for some H.*

Observe that there are ruled surfaces X over curves of genus $g \geq 2$ with $-K_X$ big. Therefore the class of varieties with $-K_X$ generically nef is much larger than the class of varieties with generically nef tangent bundles. Theorem 1.4 motivates

1.5. Conjecture. Let X be a projective manifold with $-K_X$ nef. Then T_X is generically nef.

For applications which are of biregular nature, generic nefness is sometimes not sufficient, simply because one would like to have nefness of $T_X|C$ for some curves passing through a fixed, not necessarily general point. For that purpose we introduce the notion of *sufficient nefness:*

A vector bundle E is *sufficiently nef* if through any point of X there is a family of (generically irreducible) curves covering the whole manifold X such that the bundle E is nef on the general member of this family. In the same way we define *sufficiently ample* bundles.

There is a relation between both ampleness notions:

1.6. Theorem. *If E is generically ample with respect to some polarization, then E is sufficiently ample.*

This is proved in Section 2 (Theorem 2.10). Unfortunately the analogous statement in the nef case fails in general, as shown by an example in Remark 2.11 (with E a vector bundle of rank 2 on $\mathbb{P}_3$).

So the logical implications of the four nefness/ampleness notions are as follows. E generically ample implies E generically nef and sufficiently ample; E sufficiently ample implies E sufficiently nef. E generically ample does not imply E sufficiently nef. The remaining possible implications: E sufficiently nef (ample) implies E generically nef (ample) for some polarization are likely to be false, even for line bundles, since the cone of movable curves is in general much larger than the cone of complete intersection curves.

The importance of the notion of sufficient nefness lies in particular in

1.7. Theorem. *If T_X is sufficiently nef, then the Albanese map is a (surjective) submersion.*

It is conjectured in [DPS96] that the Albanese map of a projective manifold X with $-K_X$ nef is a (surjective) submersion, the surjectivity being known by Zhang [Zh05]. If $-K_X$ is hermitian semi-positive, i.e., X has a metric with semi-positive Ricci curvature, then the Albanese map is indeed a surjective submersion [DPS96]. Since manifolds with nef anti-canonical bundles also have the mapping property of Theorem 1.1, we are led to

1.8. Conjecture. Let X be a projective manifold. If $-K_X$ is nef, then T_X is sufficiently nef.

Theorem 1.7 says that Conjecture 1.8 implies the smoothness of the Albanese map of a manifold with nef anti-canonical class. In this paper we prove Conjecture 1.8 in the following cases.

1.9. Theorem. *Let X be a projective manifold with $-K_X$ semi-ample. Then T_X is sufficiently nef.*

As a consequence of the above results we conclude that the Albanese map for a manifold with semi-ample anti-canonical bundle is a surjective submersion, that sections in a tensor power of the cotangent bundle allow zeroes and that the rational quotient is a submersion at all points where it is defined. These properties also follow from [DPS96], since the anti-canonical bundle is hermitian semi-positive (using Kähler geometry), but the method presented here has the advantage of immediately generalizing also to singular situations. For simplicity we will however formulate all results only in the smooth case.

2. Generically nef vector bundles

Unless otherwise stated, X denotes a projective manifold of dimension n.

2.1. Definition. (1) Let $H_1, \dots, H_{n-1}$ be ample divisors. A vector bundle E is said to be $(H_1, \dots, H_{n-1})$-*generically nef* (resp. *ample*) if $E|C$ is nef (resp. ample) for a general curve $C = D_1 \cap \cdots \cap D_{n-1}$ with $D_i \in |m_i H_i|$ general and $m_i \gg 0$. Such a curve is called *MR-general*, which is to say "general in the sense of Mehta–Ramanathan".

(2) The vector bundle E is called *generically nef* (resp. *ample*) if E is $(H_1, \dots, H_{n-1})$-generically nef (resp. ample) for all H_i.

(3) E is *almost nef* [DPS01] if there is a countable union S of algebraic subvarieties such that $E|C$ is nef for all curves $C \not\subset S$.

If Y is a normal projective variety and L a Weil divisor on Y, then L is locally free on the regular part of Y. Let C be MR-general with respect to $(H_1, \dots, H_{n-1})$. Then C does not meet the singular locus of Y and thus $L \cdot C$ makes perfect sense. *Hence the notion of generic nefness* (*and ampleness*) *is well-defined also for Weil divisors.*

2.2. Notation. Let X be a normal variety with smooth locus X_0 and inclusion $i : X_0 \to X$. Then we define the tangent sheaf

$$\mathcal{T} = i_*(\mathcal{T}_{X_0})$$

and the sheaf of p-forms

$$\Omega_X^{[p]} = i_*(\Omega_{X_0}^p).$$

2.3. Notation. Fix ample line bundles H_i on a projective variety X and let E be a vector bundle on X. Then we define the slope

$$\mu_{(H_1,\dots,H_{n-1})}(E) = \det E \cdot H_1 \cdot \ldots \cdot H_{n-1}$$

and obtain the usual notion of (semi-)stability with respect to $(H_1, \dots, H_{n-1})$. This makes also sense if the H_i are big and nef, or, more generally, if $H_1 \cdot \ldots \cdot H_{n-1}$ is substituted by a class α.

We shall use the following result, proved in [CP11, Lemma 5.6].

2.4. Proposition. *Let X be a projective manifold of dimension n and $\mathcal{E}$ a reflexive sheaf on X. Let $\alpha \in \overline{ME}(X) \cap H^{2n-2}(X, \mathbb{Q})$ be a non-zero class. Let ω_i be a rational Kähler classes and set*

$$\alpha_t = \alpha + t(\omega_1 \wedge \cdots \wedge \omega_{n-1})$$

for $t \geq 0$. Assume that $\mathcal{E}$ is α-stable. Then $\mathcal{E}$ is α_t-semi-stable for sufficiently small t.

The relation between semi-stability and generic nefness is provided by

2.5. Proposition. *Let X be a projective manifold of dimension n and E a rank r-vector bundle on X. Suppose that E is $(H_1, \dots, H_{n-1})$-semi-stable and*

$$\det E \cdot H_1 \cdot \ldots \cdot H_{n-1} \geq 0.$$

Then E is $(H_1, \dots, H_{n-1})$-generically nef. Moreover E is even generically ample with respect to $(H_1, \dots, H_{n-1})$ unless $\det E \cdot H_1 \cdot \ldots \cdot H_{n-1} = 0$. In the latter case $E|C$ is flat for C MR-general.

Proof. Let C be MR-general with respect to $(H_1, \dots, H_{n-1})$. Then $E|C$ is semi-stable. By Miyaoka [Mi87] this is equivalent to saying that the $\mathbb{Q}$-bundle

$$\left(E \otimes \frac{\det E^*}{r} \right) \Big| C$$

is nef (i.e. flat since the first Chern class of this $\mathbb{Q}$-bundle vanishes). Since $\det E \cdot C \geq 0$, we conclude that $E|C$ is nef. ◻

2.6. Remark. Suppose that $\det E \cdot H_1 \cdot \ldots \cdot H_{n-1} = 0$. Then semi-stability and generic nefness with respect to $(H_1, \ldots, H_{n-1})$ are the same. In fact, if C is MR-general with respect to $(H_1, \ldots, H_{n-1})$, so that $E|C$ is nef, hence numerically flat (see 2.13, possibly normalize) and consequently $E|C$ is semi-stable (2.5). Thus by [MR82], E is semi-stable with respect to $(H_1, \ldots, H_{n-1})$. The remaining claim is clear.

For reasons which become clear later (and are already mentioned in the introduction), we will also need nefness conditions for curves through any point. The relevant notion is provided by

2.7. Definition. We say that a vector bundle E is *sufficiently nef* (resp. *ample*) if the following holds. Given any point $x \in X$, there exists a covering family $(C_t)_{t \in T}$ of (generically irreducible) curves through x such that $E|C_t$ is nef (resp. ample) for general $t \in T$.

2.8. Theorem. *Let E be a vector bundle on the projective manifold X_n and $x \in X$. Assume that E is $(H_1, \ldots, H_{n-1})$-stable. Then there exists a covering family (C_t) through x such that $E|C_t$ is stable for general t, i.e. $f_t^*(E)$ is stable, where f_t is the normalization of C_t.*

Proof. Let $\pi : \tilde{X} \to X$ be the blow-up of x with exceptional divisor D and fix an ample divisor A on $\tilde{X}$. Then for any $\epsilon > 0$, the divisor

$$\tilde{H}_i = \pi^*(H_i) + \epsilon A$$

is ample for all i and $\pi^*(E)$ is stable with respect to $(\pi^*(H_1), \ldots, \pi^*(H_{n-1}))$. Hence by Proposition 2.4 (with a trivial modification) $\pi^*(E)$ is $(\tilde{H}_1, \ldots, \tilde{H}_{n-1})$-stable for suitable sufficientl small ϵ. Now we apply Mehta–Ramanathan: Let $\tilde{C}$ be MR-general with respect to $(\tilde{H}_1, \ldots, \tilde{H}_{n-1})$, so that $\pi^*(E)|\tilde{C}$ is stable. Since $\pi|\tilde{C}$ is the normalization of $\pi(\tilde{C})$ and since $D \cap \tilde{C} \neq \emptyset$, so that $x \in \pi(\tilde{C})$, the proof is complete. □

2.9. Corollary. *Suppose the vector bundle E is $(H_1, \ldots, H_{n-1})$-stable and* $\det E \cdot H_1 \cdot \ldots \cdot H_{n-1} \geq 0$. *Then E is sufficiently nef. If* $\det E \cdot H_1 \cdot \ldots \cdot H_{n-1} > 0$, *then E is sufficiently ample.*

Proof. Fix $x \in X$. By 2.7 there exists a covering family C_t through x such that $E|C_t$ is stable for general t. Since $\det E \cdot H_1 \cdot \ldots \cdot H_{n-1} \geq 0$, we conclude as in the proof of 2.4 that $E|C_t$ is nef, and analogously for ampleness. □

2.10. Theorem. *Assume that the vector bundle E is generically ample with respect to $(H_1, \ldots, H_{n-1})$. Let $x \in X$. Then there exists a covering family (C_t) through x such that $E|C_t$ is ample for general t, i.e., E is sufficiently ample.*

Proof. Let $\pi : \tilde{X} \to X$ be the blow-up of x. Fix an ample divisor A on $\tilde{X}$ and write

$$\tilde{H}_{i,\epsilon} = \pi^*(H_i) + \epsilon A.$$

For $\epsilon > 0$ rational and sufficiently small, $\tilde{H}_{i,\epsilon}$ is an ample $\mathbb{Q}$-divisor. Assuming our claim false, we conclude that for $\tilde{C} \subset \tilde{X}$ MR-general with respect to $\tilde{H}_{1,\epsilon}, \ldots, \tilde{H}_{n-1,\epsilon}$ the bundle $\pi^*(E)|\tilde{C}$ is not ample. Let

$$\mathcal{S}_\epsilon \subset \pi^*(E)|\tilde{C}$$

be maximally ample. Then as in 5.4, $\mathcal{S}_\epsilon$ extends to a global subsheaf $\tilde{\mathcal{S}}_\epsilon \subset \pi^*(E)$. Notice that $\tilde{\mathcal{S}}_\epsilon$ is a part of the Harder–Narasimhan filtration with respect to $\tilde{H}_{1,\epsilon}, \dots, \tilde{H}_{n-1,\epsilon}$, moreover by maximal ampleness we have

$$c_1(\tilde{\mathcal{S}}_\epsilon) \cdot \tilde{H}_{1,\epsilon} \cdot \ldots \cdot \tilde{H}_{n-1,\epsilon} \geq \pi^*(\det E) \cdot \tilde{H}_{1,\epsilon} \cdot \ldots \cdot \tilde{H}_{n-1,\epsilon}. \qquad (*)$$

By (the proof of) [CP11, 5.6] (Proposition 2.4), the sheaf $\tilde{\mathcal{S}}_\epsilon$ does not depend on ϵ for ϵ suitable and sufficiently small; we therefore drop the index (the proof of [CP11, 5.6] actually gives that for the maximal destabilizing subsheaf; otherwise consider the quotient by the maximal destabilizing subsheaf and proceed by induction). Now we pass to the limit $\epsilon \to 0$ in $(*)$ and obtain

$$c_1(\tilde{\mathcal{S}}) \cdot \pi^*(H_1) \cdot \ldots \cdot \pi^*(H_{n-1}) \geq \pi^*(\det E) \cdot \pi^*(H_1) \cdot \ldots \cdot \pi^*(H_{n-1}).$$

Let $\mathcal{S} = \pi_*(\tilde{\mathcal{S}}) \subset E$. Then

$$(\det E^* + \det \mathcal{S}) \cdot H_1 \cdot \ldots \cdot H_{n-1} \geq 0.$$

On the other hand, $\det E^* + \det \mathcal{S} \subset \bigwedge^p E^*$ for a suitable p and thus we obtain a contradiction to the assumption that E is generically ample with respect to $(H_1, \dots, H_{n-1})$. □

2.11. Remark. The analogous statement: *E generically nef implies E sufficiently nef*, is unfortunately false. E.g. let $X = \mathbb{P}_3$ and consider a rank 2-vector bundle $\mathcal{E}$ on $\mathbb{P}_3$ given as an extension

$$0 \to \mathcal{O}(a) \to \mathcal{E} \to \mathcal{I}_Z \to 0$$

with a suitable $a > 0$ and a suitable locally complete intersection curve $Z \subset \mathbb{P}_3$. Then obviously $\mathcal{E}$ is generically nef, since the general curve does not meet T, but neither generically ample nor sufficiently nef ($\mathcal{E}|C$ is never nef for a curve C which meets Z in a finite set).

The difficulty with going to the limit in 2.10 is the following. Fix $x \in X$ and an ample line bundle $\mathcal{L}$. We now apply 2.10 to the h-generically ample bundle $S^m E \otimes \mathcal{L}$. So if $x \in X$ we find a covering family (C_t) through x such that $S^m \otimes \mathcal{L}|C_t$ is ample for general t. The family (C_t) however depends on m and therefore one cannot conclude.

Although Theorem 2.10 fails in the generically nef case, the methods of the proof still give something.

2.12. Theorem. *Let E be generically nef with respect to $(H_1, \dots, H_{n-1})$, and suppose the same is true for small pertubations of the H_j. Suppose furthermore that sufficient nefness fails at x, i.e., there is no covering family (C_t) through x such that $E|C_t$ is nef for the general t. Then*

(1) *there exists a torsion free quotient $E \to Q$ such that* $\det Q \equiv 0$;
(2) *$Q|C_t$ is negative modulo torsion for t general.*

Proof. Following the lines of argument in 2.10 (with maximal ampleness replaced by maximal nefness), we obtain a subsheaf $\tilde{\mathcal{S}} \subset \pi^*(E)$ such that

$$c_1(\tilde{\mathcal{S}}) \cdot \pi^*(H_1) \cdot \ldots \cdot \pi^*(H_{n-1}) \geq \pi^*(\det E) \cdot \pi^*(H_1) \cdot \ldots \cdot \pi^*(H_{n-1}).$$

We again consider $\mathcal{S} = \pi_*(\tilde{\mathcal{S}}) \subset E$ so that

$$(\det E^* + \det \mathcal{S}) \cdot H_1 \cdot \ldots \cdot H_{n-1} \geq 0. \tag{$*$}$$

If we have strict inequality in $(*)$, then we can conclude as in 2.10 to get a contradiction. Thus equality in $(*)$ must happen. Then for small pertubations of H_j, we also cannot have a strict inequality $(*)$; consequently,

$$\det E^* + \det \mathcal{S} \equiv 0.$$

Introducing the torsion free quotient $Q = E/\mathcal{S}$, we obtain $\det Q \equiv 0$, proving (1).

Moreover we find a covering family (C_t) through x such that $Q|C$ is negative modulo torsion. To see this, consider $\tilde{Q} = \pi^*(E)/\tilde{\mathcal{S}}$ and observe that by construction $\tilde{Q}|\tilde{C}$ is negative, where $\tilde{C}$ is MR-general for $\tilde{H}_{1,\epsilon}, \ldots, \tilde{H}_{n-1,\epsilon}$ and suitable small ϵ. We obtain an exact sequence

$$0 \to \pi_*(\tilde{\mathcal{S}}) \to E \to \pi_*(\tilde{Q}) \to R^1\pi_*(\tilde{\mathcal{S}}) \to 0,$$

so that Q is a subsheaf of $\pi_*(\tilde{Q})$ which coincides with $\pi_*(\tilde{Q})$ outside x. Since $\pi^*\pi_*(\tilde{Q})$ is a subsheaf of $\tilde{Q}$ modulo torsion, it follows that $\pi_*(Q)|\pi(\tilde{C})$ is negative and so does $Q|\pi(\tilde{C})$, proving (2). □

In the following we collect some material used in this paper.

2.13. Definition. A vector bundle E on a compact Kähler manifold is *numerically flat* [DPS94] iff E is nef with $c_1(E) = 0$.

Numerically flat bundles are filtered by hermitian flat bundles (which by definition are given by unitary representations of the fundamental group):

2.14. Theorem. *A vector bundle E is numerically flat iff there is a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_p = E$$

of subbundles such that the graded pieces E_{j+1}/E_j are hermitian flat. In particular E is semi-stable with respect to any polarization.

For the proof we refer to [DPS94].

2.15. Remark. Let C be a smooth curve and E a nef vector bundle on C. Then E contains a maximal ample subsheaf F. Thus F is ample and any ample subsheaf of E is contained in F. We refer to [PS00] and [KST07], the latter also showing that F is part of the Harder–Narasimhan filtration of E.

2.16. Notation. As usual, $\kappa(L)$ is the Kodaira dimension of the line bundle L, and $\nu(L)$ denotes the numerical dimension of a nef line bundle L, i.e., the largest number m so that $L \not\equiv 0$. The Kodaira dimension of a normal projective variety will be the Kodaira dimension of a desingularization.

Given a projective manifold X, the movable cone is denoted by $\overline{ME}(X)$ (see [BDPP04]). Also the subcone $\overline{ME}(X)^-$ consisting of those $\gamma \in \overline{ME}(X)$ for which $K_X \cdot \gamma \leq 0$ will be of importance. The closed cone generated by the MR-general curves will be denoted by $\mathcal{CI}(X)$ ("complete intersection").

2.17. Remark. We recall for later use the nef reduction of a nef line bundle L on a normal projective variety X [B–W02]. Namely, there is an almost holomorphic map $f : X \dashrightarrow Q$ (i.e., f is meromorphic, but the general fiber is compact) such that L is numerically trivial on the general fiber of f and if $x \in X$ is a general point and C an irreducible curve through x with $\dim f(C) > 0$, then $L \cdot C > 0$.

3. Generically nef anti-canonical divisors

In this section we study varieties with generically nef anti-canonical divisors. Recall that the Kodaira dimension $\kappa(Y)$ of a normal projective variety is by definition the Kodaira dimension of a desingularization.

3.1. Theorem. *Let Y be a normal projective variety. Assume that $-K_Y$ is generically nef with respect to $(H_1, \ldots, H_{n-1})$. Then either Y is uniruled or $\kappa(Y) = 0$.*

Proof. Let $\pi : \hat{Y} \to Y$ be a desingularization. Let $C \subset Y$ be MR-general with respect to $(H_1, \ldots, H_{n-1})$ and let $\hat{C}$ denote its strict transform in $\hat{Y}$. Since C does not meet the center of π, we have

$$K_{\hat{Y}} \cdot \hat{C} = K_Y \cdot C \leq 0.$$

If $K_{\hat{Y}} \cdot \hat{C} < 0$, then $\hat{Y}$ (hence Y) is uniruled by [MM86]. Therefore we may assume that

$$K_{\hat{Y}} \cdot \pi^*(H_1) \cdot \ldots \cdot \pi^*(H_{n-1}) = 0. \tag{$*$}$$

We assume that $\hat{Y}$ is not uniruled and must show $\kappa(\hat{Y}) = 0$. So by [BDPP04], $K_{\hat{Y}}$ is pseudo-effective. Hence $c_1(K_{\hat{Y}})$ is represented by a positive closed current T, e.g. via a singular metric on $K_{\hat{Y}}$. From $(*)$ we deduce that

$$[\pi_*(T)] \cdot H_1 \cdot \ldots \cdot H_{n-1} = 0$$

and therefore $\pi_*(T) = 0$. Therefore the support of T is contained in the exceptional locus of π, hence by Siu's theorem [Si74] (see also [Sk82, Corollaire 1 and the following remark]), we obtain a decomposition $T = \sum a_i T_{E_i}$, where $a_i > 0$, the E_i are π-exceptional and T_{E_i} denotes the current "integration over E_i". By [BDPP04, 3.7, 3.10], the a_i are rational, so that some multiple $mK_{\hat{Y}}$ is numerically equivalent to an effective divisor. From [CP11] we finally obtain $\kappa(\hat{Y}) \geq 0$, hence $\kappa(\hat{Y}) = 0$. □

An application of Theorem 3.1 is

3.2. Corollary. *Let X be a projective manifold, and $f : X \to Y$ a surjective holomorphic map to a normal projective variety. Assume one of the following.*

(1) $-K_X$ *is nef;*
(2) $-mK_X = \mathcal{O}_X(D)$ *for some positive m and (X, D) is log canonical;*
(3) *there exists a positive number m such that $h^0(-mK_X) > 0$ and the base locus of $|-mK_X|$ does not project onto Y.*

Then either Y is uniruled or $\kappa(Y) = 0$. In particular the Albanese map of X is surjective.

Proof. By [Zh05], $-K_Y$ is generically nef. This is not explicitly stated in [Zh05], but this is what the proofs of Theorems 2–4 there actually give. Now apply 3.1. □

In [Zh05], even more is shown in case $-K_X$ nef: if $f : X \dashrightarrow Y$ is a dominant rational map to the projective manifold Y, then Y is uniruled or $\kappa(Y) = 0$.

It is natural to ask for a common generalization for the second and third assumptions in Corollary 3.2; if f is a submersion, this is contained in [DPS01, Theorem 2.11]; if Y is smooth and f is flat, it follows from Höring [Hoe06]. We generalize this as follows.

3.3. Theorem. *Let $f : X \to Y$ be a surjective holomorphic map from the projective manifold to the normal projective variety Y. Assume that $-K_X$ is pseudo-effective and let h be a singular metric on $-K_X$ with positive curvature current. Assume that the support of the multiplier ideal $\mathcal{I}(h)$ does not project on Y. Then $-K_Y$ is generically nef.*

Proof. Let $\tau : \hat{Y} \to Y$ be a desingularization; choose a birational map $\pi : \hat{X} \to X$ with $\hat{X}$ such that the induced map $\hat{f} : \hat{X} \dashrightarrow \hat{Y}$ is holomorphic. Then the main result of Berndtsson–Paun [BP08] implies that

$$K_{\hat{X}/\hat{Y}} + \pi^*(-K_X)$$

is pseudo-effective. Write $K_{\hat{X}} = \pi^*(K_X) + \sum a_i E_i$ with $a_i > 0$ and E_i being π-exceptional. Then we conclude that

$$\hat{f}^*(-K_{\hat{Y}}) + \sum a_i E_i$$

is pseudo-effective and consequently $-K_{\hat{Y}}$ is pseudo-effective. Now take an MR-general curve $C \subset Y$ and let $\hat{C}$ be its strict transform. Since C does not meet the exceptional locus of τ, we conclude that

$$-K_Y \cdot C = -K_{\hat{Y}} \cdot \hat{C} \geq 0,$$

the last inequality being justified by the pseudo-effectivity of $-K_{\hat{Y}}$. □

Notice that the support condition is really necessary; in fact, Zhang observed that there are many ruled surfaces over curves of genus ≥ 2 with big (but not nef) anti-canonical bundle.

3.4. Question. *Let $f : X \to Y$ be a surjective map of projective manifolds. Assume $-K_X$ nef. Is $-K_Y$ pseudo-effective?*

In general, a generically nef line bundle is not pseudo-effective. The reason is simply that the movable cone $\overline{ME}(X)$ in general is larger than the closed cone $\mathcal{CI}(X)$ generated by MR-general curves. See [Pe10] for an example (joint with J. P. Demailly). However we might ask:

3.5. Question. *Suppose X is a projective manifold with K_X generically nef. Is K_X pseudo-effective? In other words, if K_X is not pseudo-effective, are there ample divisors H_i such that*

$$K_X \cdot H_1 \cdot \ldots \cdot H_{n-1} < 0?$$

See [CP98] for some positive results.

4. Nefness properties of the tangent bundle

Miyaoka–Mori [MM86] proved that if $K_X \cdot C < 0$ for some MR-general curve (actually a curve moving in a covering family suffices), then X is uniruled. This can be seen as a special case of Miyaoka's theorem [Mi87] stating that if $T_X^*|C$ is not nef on the MR-general curve $C = m_1H_1 \cap \cdots \cap m_{n-1}H_{n-1}$, then X is uniruled. It is natural to ask whether one really needs to consider MR-general curves; in fact movable curves are often easier to handle. This should be seen in the context of the difference between the movable and the complete intersection cone: the movable cone behaves well, the complete intersection cone does not. We refer to [Pe10].

4.1. Question. *Let $T \subset \mathrm{Chow}(X)$ be an irreducible component and assume that $T_X^*|C_t$ is not nef for general $t \in T$. Is X uniruled?*

4.2. Remark. It is not sufficient to require $T_X^*|C_t$ to be non-nef for the general member C_t of some covering family of curves. In fact, in [BDPP04] it is shown that any K3 surface and any Calabi–Yau threefold admit a covering family (C_t) of curves such that $T_X^*|C_t$ is not nef for general t. In other words T_X^* (as well as T_X) is not almost nef. Consequently, we also find projective n-folds X with $\kappa(X) = n - 2$ such that T_X^* is not almost nef.

4.3. Question. *Let X be a projective manifold with $\kappa(X) \geq \dim X - 1$. Is T_X^* almost nef?*

Since Question 4.1 seems to be too hard at the moment, we consider the stronger condition that $T_X|C$ be nef. In [Pe06] the following is proved; for the ampleness statement see also [KST07], [BM01].

4.4. Theorem. *Let X be a projective manifold, and $C \subset X$ an irreducible curve. If $T_X|C$ is nef, then $\kappa(X) < \dim X$. If additionally $K_X \cdot C < 0$, then X is uniruled. If $T_X|C$ is ample, then X is rationally connected.*

4.5. Corollary. *Suppose X is uniruled with rational quotient $f : X \dashrightarrow W$, where W is smooth. Let $C \subset X$ be an irreducible curve such that f is holomorphic near C and assume $\dim f(C) > 0$. Suppose that $T_X|C$ is nef and let $\mathcal{S} \subset T_X|C$ be the maximal ample subsheaf. If $\mathrm{rk}\,\mathcal{S} = m > 0$, then $\dim W \leq n - m$.*

Proof. We simply observe that the natural morphism $\mathcal{S} \to f^*T_W|C$ must vanish, because otherwise $T_W|f(C)$ would contain an ample subsheaf. Since $T_W|f(C)$ is nef (the map $T_X|C \to f^*T_W|C$ being generically surjective), we conclude by 4.4 that W is uniruled, a contradiction. □

The next theorem shows that manifolds with generically nef tangent bundles have the same mapping properties as manifolds with nef anti-canonical bundles.

4.6. Theorem. *Assume that T_X is generically nef with respect to $(H_1, \dots, H_{n-1})$. Let $f : X \to Y$ be a surjective holomorphic map to the normal projective variety Y. Then either Y is uniruled or $\kappa(Y) = 0$.*

Proof. Let $C \subset X$ be MR-general. Fix a desingularization $\tau : \hat{Y} \to Y$ and choose a smooth birational model $\pi : \hat{X} \to X$ such that the induced map $\hat{f} : \hat{X} \to \hat{Y}$ is holomorphic. Then C can be considered as a curve on $\hat{X}$ (but it is not an MR-curve on $\hat{X}$), and $T_{\hat{X}}|C$ is nef. We claim that

$$K_{\hat{Y}} \cdot \hat{f}(C) \leq 0. \tag{$*$}$$

In fact, otherwise we find a positive integer N and a non-zero section

$$\omega \in H^0(NK_{\hat{Y}}|\hat{f}(C)) = H^0((\Omega^m_{\hat{Y}})^{\otimes N}|\hat{f}(C))$$

having a zero ($m = \dim \hat{Y}$). Hence we obtain a section

$$\hat{f}^*(\omega) \in H^0((\Omega^m_{\hat{X}})^{\otimes N}|C)$$

with a zero. This contradicts the nefness of $(\bigwedge^m T_{\hat{X}})^{\otimes N}|C$.

Now we proceed similarly to 3.1. If we have strict inequality in ($*$), then Y is uniruled. So suppose $K_{\hat{Y}} \cdot \hat{f}(C) = 0$. Assuming Y is not uniruled, we must show that $\kappa(\hat{Y}) = 0$. By [BDPP04], $K_{\hat{Y}}$ is pseudo-effective. Hence $L = \hat{f}^*(K_{\hat{Y}})$ is pseudo-effective with $L \cdot C = 0$. Then we just follow the arguments of 3.1 to conclude that $\kappa(\hat{Y}) = 0$. □

In particular the Albanese map of a manifold X whose tangent bundle T_X is generically $(H_1, \dots, H_{n-1})$-nef is surjective. Now we ask for conditions under which the Albanese map is a submersion. Here we naturally need conditions concerning curves through every point.

4.7. Definition. We say that T_X is *sufficiently nef* (resp. *ample*) if the following holds. Given any point $x \in X$, there exists a covering family (C_t) of (generically irreducible) curves through x such that $T_X|C_t$ is nef (resp. ample) for general t.

With this notion we have

4.8. Proposition. *If T_X is sufficiently nef, then the Albanese map of X is a surjective submersion.*

Proof. If the Albanese map is not surjective or not a submersion, then we find a nonzero holomorphic 1-form ω on X admitting a zero x. Now consider a curve C through x such that $T_X|C$ is nef and such that $\omega|C \in H^0(\Omega^1_X|C)$ does not vanish identically. Such a curve exists by our assumption. Since $\omega(x) = 0$, we obtain a contradiction. □

4.9. Corollary. *If T_X is almost nef, then the Albanese map is a surjective submersion.*

4.10. Remark. The arguments of 4.8 show more generally the following. If

$$0 \neq \omega \in H^0(X, (\Omega^1_X)^{\otimes N}),$$

then ω has no zeroes. As an application, if $f : X \to Y$ is a surjective holomorphic map to a projective manifold Y with $\kappa(Y) \geq 0$ (where T_X is sufficiently nef) then $mK_Y = \mathcal{O}_Y$ for some $m > 0$ (and any non-zero $\eta \in H^0(Y, (\Omega^1_Y)^{\otimes N})$ is without zeroes). It seems however impossible in general to conclude with these methods that f is a submersion.

Coming back to manifolds with nef anti-canonical bundles, we recall the following

4.11. Conjecture. Let X be a projective manifold with $-K_X$ nef. Then the Albanese map is a (surjective) submersion.

This is known in dimension 3 [PS98] and in all dimensions if $-K_X$ is hermitian semi-positive [DPS96] and, more generally, if $-K_X$ has a singular metric h with trivial multiplier ideal $\mathcal{I}(h) = \mathcal{O}_X$ [DPS01]—even in the Kähler case. As already mentioned, the surjectivity is due to Zhang [Zh05]. However in the Kähler case, surjectivity in general is still unknown.

Conjecture 4.11 is potentially a consequence of Proposition 4.8 via the first part of the following

4.12. Conjecture. Let X be a projective manifold.

(1) If $-K_X$ nef, then T_X is sufficiently nef and also generically nef.
(2) If X is rationally connected and $-K_X$ is pseudo-effective, e.g. Fano, then T_X is generically ample.

4.13. Example. Let X be a projective surface with $-K_X$ nef. Then T_X is sufficiently nef, as shown by the following considerations, using classification.

If $q(X) \geq 2$, then X is a torus, and there is nothing to prove. If $q(X) = 1$, then either X is hyperelliptic and we are done, or $X = \mathbb{P}(E)$ with a rank 2-vector bundle E over an elliptic curve A. Then automatically T_X is nef [CP91].

If $q(X) = 0$ and X is not uniruled, then X is K3 (or Enriques), this being settled by 7.2 below. So it remains to treat the rational case. Here we may assume that X is the plane blown up in at most nine points $p_1, \dots, p_r$. Fix $x \in X$. If x is disjoint from the exceptional locus of $\pi : X \to \mathbb{P}_2$, then the claim is clear (take lines). Otherwise we have say $\pi(x) = p_1$. Since $-K_X$ is nef, at most three points can be infinitely near and it follows easily that there is a 1-dimensional family of strict transforms of conics through p_1 which all contain x.

5. Semi-stability

In this section we shall bring methods from the theory of stable vector bundles into the game.

5.1. Proposition. (1) *Assume T_X is semi-stable with respect to $(H_1, \dots, H_{n-1})$. If $-K_X \cdot H_1 \cdot \dots \cdot H_{n-1} > 0$, then T_X is generically ample with respect to $(H_1, \dots, H_{n-1})$ and X is rationally connected.*

(2) *Assume T_X is stable with respect to $(H_1, \dots, H_{n-1})$. If $-K_X \cdot H_1 \cdot \dots \cdot H_{n-1} \geq 0$ and $K_X \not\equiv 0$, then again T_X is generically ample with respect to $(H_1, \dots, H_{n-1})$ and X is rationally connected.*

Proof. (1) If $-K_X \cdot H_1 \cdot \dots \cdot H_{n-1} > 0$, the stability follows from 2.5. Moreover $T_X|C$ is ample for C MR-general, hence we may apply 4.4.

(2) The stability is again 2.5. For the rational connectedness, we may assume by (1) that $-K_X \cdot H_1 \cdot \dots \cdot H_{n-1} = 0$. Since $K_X \not\equiv 0$, we find $H'_1, \dots, H'_{n-1}$ ample such that for all positive ϵ,

$$K_X \cdot (H_1 + \epsilon H'_1) \cdot \dots \cdot (H_{n-1} + \epsilon H'_{n-1}) < 0.$$

Then we find a sequence (t_ν) of positive numbers converging to 0 such that T_X is stable with respect to $(H_1 + t_\nu H'_1, \dots, H_{n-1} + t_\nu H'_{n-1})$ (adapt [CP11, 5.6] = Proposition 2.4, where the case of all H_i equal is treated; actually by a convexity argument we have stability for all small t). Now we apply (1). $\square$

5.2. Theorem. *Let X be a projective manifold.*

(1) *Assume that T_X is not generically ample with respect to $(H_1, \dots, H_{n-1})$ and that $-K_X \cdot H_1 \cdot \dots \cdot H_{n-1} > 0$. Then there exists a reflexive subsheaf $\mathcal{E} \subset T_X$ such that*

$$c_1(\mathcal{E}) \cdot H_1 \cdot \dots \cdot H_{n-1} \geq -K_X \cdot H_1 \cdot \dots \cdot H_{n-1}$$

and such that $\mathcal{E}|C$ is the maximal ample subsheaf, where C is MR-general with respect to $(H_1, \dots, H_{n-1})$.

(2) *Assume that T_X is not generically nef with respect to $(H_1, \dots, H_{n-1})$ and that $-K_X \cdot H_1 \cdot \dots \cdot H_{n-1} > 0$. Then there exists a reflexive subsheaf $\mathcal{E} \subset T_X$ such that*

$$c_1(\mathcal{E}) \cdot H_1 \cdot \dots \cdot H_{n-1} > -K_X \cdot H_1 \cdot \dots \cdot H_{n-1}$$

and such that $\mathcal{E}|C$ is the maximal nef subsheaf, where C is MR-general with respect to $(H_1, \dots, H_{n-1})$.

Proof. (1) Let $C \subset X$ be MR-general with respect to $(H_1, \dots, H_{n-1})$. Since $T_X|C$ is not ample, $T_X|C$ has a quotient of non-positive degree. Since $-K_X \cdot C > 0$, we conclude that $T_X|C$ is not semi-stable. Consider the Harder–Narasimhan filtration of $T_X|C$,

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = T_X|C.$$

By [KST07], there exists a positive s such that $\mathcal{E}_s$ is ample, and all ample subsheaves $\mathcal{F}_C \subset T_X|C$ are contained in $\mathcal{E}_s$. In other words, $\mathcal{E}_s$ is *maximally ample*. Consider the exact sequence

$$0 \to \mathcal{E}_s \to T_X|C \to Q \to 0.$$

We claim that $c_1(Q) \leq 0$. In fact, if $c_1(Q) > 0$, then, being non-ample, Q must contain an ample subsheaf, which contradicts the maximality of $\mathcal{E}_k$. Hence

$$c_1(\mathcal{E}_k) \geq -K_X \cdot C. \tag{$*$}$$

Now by [MR82], $\mathcal{E}_s$ extends to a torsion free sheaf $\mathcal{E} \subset T_X$, and by $(*)$ we conclude that

$$c_1(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-1} \geq -K_X \cdot H_1 \cdot \ldots \cdot H_{n-1}.$$

For (2), we argue in the same way with nef instead of ample bundles, using the obvious nef version of [KST07, Prop. 29]. □

As a corollary we obtain (compare also [RC00]):

5.3. Corollary. *Let X be a Fano manifold with $b_2(X) = 1$. Then T_X is generically ample (with respect to $-K_X$).*

Proof. Let $C \subset T_X$ be MR-general (necessarily with respect to $-K_X$) and assume $T_X|C$ not ample. Then by Theorem 5.2 we obtain a subsheaf $\mathcal{E} \subset T_X$ of some rank k such that

$$c_1(\mathcal{E}) \geq -K_X \cdot C.$$

Hence

$$\det \mathcal{E} = -K_X \otimes \mathcal{O}_X(a) \subset \bigwedge^k T_X$$

with an integer $a \geq 0$ (here $\mathcal{O}_X(1)$ is the ample generator of $\mathrm{Pic}(X)$). Thus

$$\mathcal{O}_X(a) \subset \bigwedge^k T_X \otimes K_X = \Omega_X^{n-k},$$

which is clearly impossible, e.g. by restricting to curves on which T_X is ample (these exist by the rational connectedness of X). □

5.4. Remark. The same argument as in 5.3 also works for a variety X which is $\mathbb{Q}$-Fano with $\rho(X) = 1$, i.e. X is a normal projective $\mathbb{Q}$-factorial variety with at most canonical singularities. In fact, most things go over verbatim (of course one needs to deal with Weil $\mathbb{Q}$-Cartier divisors), only at the end the following additions need to be made. The variety X is rationally connected by [Zh06]; we have a Weil divisor $\mathcal{L} \subset \tilde{\Omega}_X^{n-k}$ which is ample or trivial, and the sheaf $\tilde{\Omega}_X^{n-k}$ is the extension of the sheaf of $(n-k)$-forms on the regular part of X. Then we pass to a desingularization and argue there as before.

In the following we investigate the existence of *some* polarization H such that T_X is H-semi-stable resp. generically nef (ample).

5.5. Proposition. *Let X and Y be projective manifolds, $\pi : X \to Y$ birational. Assume that T_Y is $(H_1, \ldots, H_{n-1})$-(semi-)stable. Then T_X is (semi-)stable with respect to $(\pi^*(H_1), \ldots, \pi^*(H_{n-1}))$.*

Proof. We argue for stability only. Write $H'_j = \pi^*(H_j)$ for short and $H' = (H'_1, \dots, H'_{n-1})$; similarly introduce H. Assume that T_X is H'-unstable and let $\mathcal{S}'$ be maximally destabilizing. Then

$$\mu_{H'}(\mathcal{S}') \geq \mu_{H'}(T_X).$$

Let $\mathcal{S} := \pi_*(\mathcal{S}') \subset \pi_*(T_X) \subset T_Y$. Now trivial calculations show

$$\mu_{H'}(T_X) = \mu_H(T_Y) \quad \text{and} \quad \mu_{H'}(\mathcal{S}') = \mu_H(\mathcal{S}).$$

Thus we obtain $\mu_H(\mathcal{S}) \geq \mu_H(T_Y)$, contradicting the stability of T_Y. □

5.6. Corollary. *Let $\pi : X \to Y$ be a birational map of projective manifolds. Assume that T_Y is $(H_1, \dots, H_{n-1})$-stable. Then there are ample divisors H'_j on X such that T_X is $(H'_1, \dots, H'_{n-1})$-stable.*

Proof. Combine the last proposition and Proposition 2.4. □

5.7. Corollary. *Let Y be a projective manifold with $(H_1, \dots, H_{n-1})$-stable tangent bundle. Assume that $-K_Y \cdot H_1 \cdot \ldots \cdot H_{n-1} > 0$ (e.g. Y is a Fano manifold of dimension 3 with $b_2 = 1$ or a rational-homogenenous manifold with $b_2 = 1$). Let $\pi : X \to Y$ be birational. Then there exist $H'_1, \dots, H'_{n-1}$ ample on X such that T_X is $(H'_1, \dots, H'_{n-1})$-stable and generically nef with respect to $(H'_1, \dots, H'_{n-1})$.*

Proof. By Proposition 2.4 we can take $H'_j = \pi^*(H_j) + \epsilon A$ where A is ample and $\epsilon > 0$ small. In order to prove generic nefness, we need to verify that

$$-K_X \cdot H'_1 \cdot \ldots \cdot H'_{n-1} > 0. \qquad (*)$$

From $-K_Y \cdot H_1 \cdot \ldots \cdot H_{n-1} > 0$, we deduce

$$-K_X \cdot \pi^*(H_1) \cdot \ldots \cdot \pi^*(H_{n-1}) > 0,$$

hence $(*)$ for ϵ small enough. □

5.8. Corollary. *Let X be a rational surface. Then there exists H ample such that T_X is H-generically ample.*

Proof. Choose a birational morphism $\pi : \hat{X} \to X$ such that $\hat{X}$ is a blow-up of $\mathbb{P}_2$. By the last corollary we find $\hat{H}$ ample on $\hat{X}$ such that $T_{\hat{X}}$ is $\hat{H}$-generically ample. Arguing by induction, we may assume that π is just one blow-up; let E be the exceptional curve. Write

$$\hat{H} = \pi^*(H) + aE$$

with $a < 0$ and H ample on X. Let $\hat{C} \subset \hat{X}$ be MR-general with respect to $\hat{H}$. Then $T_{\hat{X}}|\hat{C}$ is ample. Since $T_{\hat{X}} \subset \pi^*(T_X)$, also $\pi^*(T_X)|\hat{C}$ is ample. Therefore T_X is ample on the image curve $C = \pi(\hat{C})$. Since ampleness is a Zariski-open property, $T_X|C'$ is ample for C' MR-general with respect to H. □

Notice that for a Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))$ with $e \geq 3$, the tangent bundle is never stable for any polarization H; however it is generically ample for all H.

Corollary 5.8 can be strengthened as follows.

5.9. Theorem. *Let X be a del Pezzo surface or a rational surface with $-K_X$ pseudo-effective. Let H be ample. Then T_X is H-generically ample.*

Proof. Suppose T_X is not H-generically ample. By 5.2 there exists a line bundle $\mathcal{L} \subset T_X$ such that

$$\mathcal{L} \cdot H \geq -K_X \cdot H$$

and $\mathcal{L}|C \subset T_X|C$ is maximally ample for C MR-general with respect to H. Moreover $\mathcal{L}$ is a subbundle outside a finite set. Notice that our assumptions guarantee $-K_X \cdot H > 0$. By [KST07], [BM01], the general leaf of the foliation $\mathcal{L}$ is algebraic and its closure is rational. So the closures of the (general) leaves form an algebraic family and we find a (smooth) blow-up $p : \mathcal{C} \to X$ which is a generic $\mathbb{P}_1$-bundle $q : \mathcal{C} \to B \simeq \mathbb{P}_1$ such that

$$\mathcal{L} = p_*(T_{\mathcal{C}/B})^{**} = p_*(-K_{\mathcal{C}/B})^{**} \subset p_*(T_{\mathcal{C}}) = T_X.$$

Now

$$\begin{aligned}\mathcal{L} \cdot H &= (-K_{\mathcal{C}/B}) \cdot p^*(H) = -K_{\mathcal{C}} \cdot p^*(H) + q^*(K_B) \cdot p^*(H)\\ &= p^*(-K_X) \cdot p^*(H) + q^*(K_B) \cdot p^*(H).\end{aligned}$$

Since $B \simeq \mathbb{P}_1$, we have $q^*(K_B) \cdot p^*(H) < 0$, which leads to

$$\mathcal{L} \cdot H < -K_X \cdot H,$$

a contradiction. □

5.10. Remark. The same scheme of proof stills works for birationally ruled surfaces over an elliptic curve with $-K_X$ pseudo-effective:

The tangent bundle of a surface with pseudo-effective anti-canonical bundle, which is birationally equivalent to a ruled surface over an elliptic curve, is generically nef.

In the proof of 5.9 we take a priori $\mathcal{L} \subset T_X$ such that $\mathcal{L}_C$ is maximally nef. Since $-K_X \cdot C > 0$, it is clear that $\mathcal{L}_C$ is also maximally ample. Then we proceed as before, having in mind that $\mathcal{L} \cdot H > -K_X \cdot H$.

Combining Theorem 5.9 and Remark 5.10 we obtain

5.11. Corollary. *Let X be a smooth projective surface with $-K_X$ nef. Then T_X is generically nef for some H.*

6. The movable cone

6.1. Theorem. *Let X be a $\mathbb{Q}$-Fano variety, i.e. a normal projective $\mathbb{Q}$-Gorenstein variety with at most terminal singularities such that $-K_X$ ample. Let $\mathcal{L} \subset \Omega_X^{[r]}$ be a reflexive rank 1 subsheaf which is $\mathbb{Q}$-Cartier. Then $\kappa(\mathcal{L}) = -\infty$.*

Proof. Let $\pi : \hat{X} \to X$ be a desingularization. By the Extension Theorem in [GKKP11], there is a generically injective map

$$\pi^*(\Omega_X^{[r]}) \to \Omega^r_{\hat{X}}.$$

This induces an injective map

$$\pi^*(\mathcal{L})/\mathrm{tor} \to \Omega^r_{\hat{X}}.$$

If $\kappa(\mathcal{L}) \geq 0$, then there is an integer N such that

$$H^0(\hat{X}, (\Omega^r_{\hat{X}})^{\otimes N}) = 0.$$

But $\hat{X}$ is rationally connected by [Zh06], which is absurd (e.g. by [Ko95, IV.3.8]). □

We next study the behaviour of the movable cone under surjective morphisms; the results will be used in Theorem 6.3.

6.2. Theorem. *Let $f : X \to W$ be a surjective holomorphic map with connected fibers from the projective manifold X to the normal projective variety W. Let F be a general fiber of f. Then*

(1) *The inclusion map $i_* : N_1(F) \to N_1(X)$ maps $\overline{ME}(F)$ to $\overline{ME}(X)$.*
(2) *$\overline{ME}(X/W)$ is the closed cone generated by covering families (C_t) in X such that $f_*(C_t) = 0$.*
(3) *$i_* : \overline{ME}(F) \to \overline{ME}(X/W)$ is surjective.*

Proof. (1) The following argument, simplifying my original proof, was provided by the referee. The cone $\overline{ME}(F)$ is generated by the classes of F-covering families. So let a be the class of an F-covering family. Since F is general, the countability of the number of components of the Hilbert scheme shows that there is an X-covering family (C_t) with $f_*(C_t) = 0$ inducing the original family in F. Hence $a \in \overline{ME}(X)$.

(2) Let L be a line bundle such that

$$L \cdot C_t \geq 0$$

for all covering families (C_t) in X with $f_*(C_t) = 0$. We need to show that

$$L \cdot \gamma \geq 0 \tag{$*$}$$

for $\gamma \in \overline{ME}(X/W)$. By (1), L_F is pseudo-effective for the general fiber F. Choose an ample line bundle A on X. Then $L + \epsilon A$ is f-big for all $\epsilon > 0$. Hence $(L + \epsilon A) \cdot \gamma > 0$, and $(*)$ follows by letting ϵ converge to 0.

(3) is a consequence of (2). □

The following theorem will be important for the next section; we state it in a more general form than necessary for future use.

6.3. Theorem. *Let X be a projective manifold, and L a nef line bundle on X with (almost holomorphic) nef reduction $f : X \dashrightarrow W$. Let F be a general fiber of f and $i : F \to X$ the inclusion map. Assume that*

(1) *$L|F = \mathcal{O}_F$;*
(2) $\dim W = \nu(L)$.

Let $\gamma \in \overline{ME}(X)$ with $L \cdot \gamma = 0$. Then there exists $\delta \in \overline{ME}(F)$ such that $\gamma = i_*(\delta)$.

Proof. In a first step we show that we may assume f is holomorphic. In fact, take a birational map $\pi : \hat{X} \to X$ from a projective manifold $\hat{X}$ such that the induced map $\hat{f} : \hat{X} \to W$ is holomorphic. Of course, we blow up only inside the set of indeterminacies of f. Then $\hat{f}$ is a nef reduction for $\hat{L} = \pi^*(L)$ and obviously $\nu(\hat{L}) = \nu(L)$. Next observe that

$$\hat{\gamma} := \pi^*(\gamma) \in \overline{ME}(\hat{X}).$$

To see that, take a pseudo-effective divisor $\hat{D}$. Then the divisor $\pi_*(\hat{D})$ is pseudo-effective too, and therefore

$$\hat{D} \cdot \hat{\gamma} = \pi_*(D) \cdot \gamma \geq 0.$$

Now apply [BDPP04] to conclude.

Since $\hat{L} \cdot \hat{\gamma} = 0$ and since we assume in this step that the assertion of the proposition is true in the case of a holomorphic nef reduction, we find an element $\hat{\delta} \in \overline{ME}(\hat{F})$ (where $\hat{F}$ is a general fiber of $\hat{f}$) such that

$$i_*(\hat{\delta}) = \hat{\gamma}.$$

But $F = \hat{F}$, since f is almost holomorphic, and thus our claim follows.

From now on we assume f holomorphic. By a birational base change and the above arguments, we may also assume W is smooth and f is flat, in particular equidimensional. However X is now only normal.

Consider the line bundle

$$A = f_*(L)^{**};$$

here we use the assumption $L|F = \mathcal{O}_F$ to see that A has indeed rank 1. The canonical map $f^* f_*(L) \to L$ gives rise to a map $\psi : f^*(A) \to L$ which is defined outside of the preimage of the singular locus S of the torsion free sheaf $f_*(L)$. But S has codimension at least 2, and so does $f^{-1}(S)$, since f is equidimensional. Therefore ψ is defined everywhere, and we can write

$$L = f^*(A) + E$$

with an effective divisor E. Since E is f-nef with $f_*(\mathcal{O}_X(E)) = \mathcal{O}_W$, it is easy to show [Fu86, 1.5] that

$$mE = f^*(B)$$

for some multiple m. In total we can write

$$mL = f^*(A') \tag{$*$}$$

with some nef line bundle A'. Notice that A' also big by our assumption $\dim W = \nu(L)$. By passing to a desingularization of X, we may again assume that X is smooth; the new map $X \to W$ might no longer be equidimensional, but this is not important for the rest of the considerations.

We show that $f_*(\gamma) = 0$. By $(*)$,

$$0 = L \cdot \gamma = A' \cdot f_*(\gamma).$$

Since $f_*(\gamma) \in \overline{ME}(W)$ (again test by intersecting with a pseudo-effective divisor on W), we conclude from the bigness of A' that $f_*(\gamma) = 0$.

Now that we know $f_*(\gamma) = 0$, we apply Theorem 6.2 to conclude. □

6.4. Remark. Suppose $-K_X \cdot C > 0$ for all movable curves C. One might expect that then $q(X) = 0$ or even that X is rationally connected. One reason for such an expectation is Serrano's theorem that if $-K_X \cdot C > 0$ for *all* curves in a threefold X, then X is Fano (the same being expected in all dimensions). This expectation however is completely false, even if $-K_X$ is nef. Take an elliptic curve C and a rank 2-vector bundle E given by a non-split extension

$$0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0.$$

Let $X = \mathbb{P}(E)$. Then $-K_X \cdot B = 0$ only for one curve, the section defined by the epimorphism $E \to \mathcal{O}_C \to 0$.

We next prove a weak substitute for generic nefness for rationally connected manifolds. To do that we first observe

6.5. Theorem. *Let X be a rationally connected projective manifold. Then there are covering families (C_t^j) of rational curves, $1 \leq j \leq m$, such that $\sum_j [C_t^j]$ is in the interior $ME^0(X)$.*

Proof. Applying [Ko95, IV.3.9] we find for all $x_1, x_2 \in X$ an irreducible rational curve C_{x_1,x_2} joining x_1 and x_2. By the usual Chow scheme argument, there are finitely many connecting families (C_t^j) of rational curves, with $T_X|C_t^j$ ample for general t and all j, $1 \leq j \leq m$, such that any two x_1, x_2 can be joined by an irreducible member of some of the families. Moreover at least one of the families, say (C_t^1), has the property that two general points of X can be joined by an irreducible member of this family. We may also assume that for any j the general C_t^j is smooth (resp. has only nodes when X is a surface). In order to prove that $\alpha := \sum_j [C_t^j]$ is in the interior $ME^0(X)$, we check that

$$L \cdot \alpha > 0$$

for all pseudo-effective line bundles L on X which are not numerically trivial. Suppose to the contrary that $L \cdot \alpha = 0$ for some L, so that

$$L \cdot C_t^j = 0 \tag{$*$}$$

for all j, in particular $L \cdot C_t^1 = 0$. Let $L = P + N$ be the divisorial Zariski decomposition [Bo04] of L; see also [BDPP04] for a discussion. So P is an effective $\mathbb{R}$-divisor and N is nef in codimension 1. Thus

$$P \cdot C_t^1 = N \cdot C_t^1 = 0$$

and by [BDPP04, 8.7], we first obtain $N = 0$ and consequently [ibid., 3.11] P is numerically equivalent to a multiple of an effective divisor, hence (without loss of generality) effective. Let P_k be the irreducible components of the support of P. Since $P \cdot C_t^j = 0$ and $P_k \cdot C_t^j \geq 0$ for all j, we have $P_k \cdot C_t^j = 0$ for all j. Now pick x_1 general and $x_2 \in P_k$ and choose some irreducible C_t^j joining x_1 and x_2. Thus $0 < P \cdot C_t^j = L \cdot C_t^j$, contradicting $(*)$. □

It seems likely that we can find a single "connecting" family (C_t) of rational curves (with C_t irreducible!) such that $[C_t]$ is in the interior of the movable cone, but we cannot prove that at the moment.

As a consequence of Theorem 6.5 we obtain

6.6. Theorem. *Let X be a rationally connected manifold. Then there exists a smooth curve $C \subset X$ such that*

(1) *the deformations of C cover X;*
(2) *$T_X|C$ is ample;*
(3) $[C] \in ME^0(X)$.

Proof. Choose families (C_t^j) on X as in Theorem 6.5. We form a reducible connected curve

$$C' = \sum C_t^j,$$

where the C_t^j are chosen general so that the intersections are all transversal and that through any point of X there are at most two components. Then C' is smoothable ([Ko95, II.7.9]); let C be a smoothing so that $[C] \in ME^0(X)$. Since $T_X|C_t^j$ is ample for all j and t general, so does $T_X|C$, proving (2). Claim (1) is also clear since $[C']$ belongs to a component of the Chow scheme which covers X. □

The fact that $[C] \in ME^0(X)$ can be rephrased as follows. There are birational morphisms $\pi_j : \tilde{X}_j \to X$ from projective manifolds $\tilde{X}_j$ and $\tilde{h}_j = (\tilde{H}_1^j, \ldots, \tilde{H}_{n-1}^j)$ ample such that $[C] = \sum_j a_j \pi_{j*}(\tilde{h}_j)$ with positive numbers a_j. It is not clear whether one can simply achieve $[C] = \pi_*(\tilde{H} \cdot \ldots \cdot \tilde{H}_{n-1})$.

7. Almost Fano manifolds and manifolds with semi-positive curvature

A projective manifold X is *almost Fano* if $-K_X$ is big and nef. Our first purpose here is to generalize Theorem 5.4 to the case of almost Fano manifolds. We will use the following cone theorem [BCHM10].

7.1. Theorem. *Let X be a projective manifold with $-K_X$ big and nef.*

(1) *$\overline{ME}^-(X) = \overline{ME}(X)$ is rationally polyhedral: $\overline{ME}^-(X) = \sum_{j=1}^r R_j$, where the R_j are the extremal rays in $\overline{ME}(X)$.*
(2) *if $R \subset \overline{ME}^-(X)$ is an extremal ray, then there is a sequence of contractions and flips $X \dashrightarrow X'$ and a Mori fiber space $\phi : X' \to Y$ such that the pull-back of a sufficiently general curve contracted by ϕ is in R.*

Proof. In case $-K_X$ is ample, this is [BCHM10, 1.3.5]. If $-K_X$ is merely big and nef, write

$$-K_X = A + D$$

where A is an ample $\mathbb{Q}$-divisor and D is an effective $\mathbb{Q}$-divisor. Consider the birational morphism $\psi : X \to Y$ defined by $|-mK_X|$ for large m. Then $-D$ is ψ-ample; consequently, $-(K_X + \epsilon D)$ is ample for small positive ϵ. Moreover the pair $(X, \epsilon D)$ is klt, so that we can apply [BCHM10, 1.3.5] to $K_X + \epsilon D$. □

One should expect that R contains the class of an irreducible rational curve whose deformations cover X. These curves should come from suitable rational curves in a general fiber F' of ϕ. The technical difficulty lies in the fact that F' is singular (Fano) and one needs to establish rational curves which avoid a set of codimension 2 in this singular variety.

Of course one could ask for more:

7.2. Question. *Let X be a projective manifold.*

(1) *Under which conditions is $\overline{ME}^-(X)$ locally rationally polyhedral?*
(2) *If $-K_X$ is nef, is there a substitute of* 7.1(2) *with log-contractions and log-flips?*
(3) *If X is rationally connected, is an extremal ray of $\overline{ME}^-(X)$ represented by a covering family of rational curves?*

Observe however that 7.1(2) without nefness assumption will in general not be true, even if $-K_X$ is big. Indeed, consider a ruled surface $X = \mathbb{P}(E)$ over a curve C of genus $g \geq 2$. Assume X has a negative section C_0 with $C_0^2 = -e < 0$ and $e = 2g(C) - 1$ so that $-K_X$ is big. Let F be a ruling line. Then the boundary components of $\overline{ME}(X)$ are represented by F and $C_0 + eF$. Then 7.1(2) obviously does not hold, since the contraction of C_0 is not a Mori contraction; moreover the ray determined by $C_0 + eF$ does not contain a covering family of (generically irreducible) curves.

Lehmann [Le08] has shown that the cone $\overline{NE}(X) + \overline{ME}(X)$ has a good decomposition in the sense of the cone theorem of Mori theory.

If X is an irrational surface with $-K_X$ nef, then obviously 7.2(1) & (2) hold; here we have only one extremal ray.

7.3. Theorem. *Let X be a projective manifold with $-K_X$ big and nef. Then T_X is generically ample and therefore also sufficiently ample.*

Proof. Generic ampleness implies sufficient ampleness by Theorem 2.8, therefore only the first claim needs to be proved. If the claim were false, we would find by 5.5 some ample $(H_1, \ldots, H_{n-1})$, and a subsheaf $\mathcal{E} \subset T_X$ such that

$$(K_X + \det \mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-1} \geq 0.$$

Notice that $\det \mathcal{E} \subset \bigwedge^k T_X$ with k the rank of $\mathcal{E}$, and therefore, as in 5.4,

$$K_X + \det \mathcal{E} \subset \bigwedge^k T_X \otimes K_X = \Omega_X^{n-k}. \qquad (*)$$

We want to show that by a change of the polarizations H_j, we may achieve

$$(K_X + \det \mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-1} > 0. \qquad (**)$$

So assume equality in $(**)$. Since the interior of $\mathcal{CI}(X)$ is open in $N_1(X)$, we find ample $\mathbb{Q}$-divisors H_j' sufficiently near to H_j such that

$$(K_X + \det \mathcal{E}) \cdot H_1' \cdot \ldots \cdot H_{n-1}' > 0$$

unless $K_X + \det \mathcal{E} \equiv 0$. But this last alternative cannot happen: since $\pi_1(X) = 0$, we would have $K_X + \det \mathcal{E} \simeq \mathcal{O}_X$ and therefore by $(*)$, $h^0(\Omega_X^{n-k}) \neq 0$, contradicting $H^{n-k}(\mathcal{O}_X) = 0$ (Kodaira vanishing). So $(**)$ can always be achieved.

By 6.1 there is an extremal ray $R \subset \overline{ME}^-(X)$ such that

$$(K_X + \det \mathcal{E}) \cdot R > 0.$$

We apply Theorem 7.1 and run a suitable MMP $X \dashrightarrow X'$ to obtain a Mori fiber space $\phi : X' \to Y$. Moreover, this fiber space is given by an extremal ray R' which is spanned by the images of the curves in R. Let $\mathcal{E}' \subset T_{X'}$ be the induced reflexive subsheaf, so that

$$K_{X'} + \det \mathcal{E}' \subset \tilde{\Omega}_{X'}^{n-k}$$

and

$$(K_{X'} + \det \mathcal{E}') \cdot R' > 0,$$

which is an easy calculation of intersection numbers (for birational contractions and flips separately). Therefore $(K_{X'} + \det \mathcal{E}')$ is a ϕ-ample $\mathbb{Q}$-Cartier divisor. Now we restrict to a general fiber F' of ϕ and, by the same techniques as fully explained in part (I) of the proof of the next theorem and by applying Lemma 6.6, we obtain a contradiction. □

One might ask whether the above method also works if $-K_X$ is merely nef, granted that Questions 7.2(1) & (2) have positive answers. Of course we only ask for generic nefness. We try to argue by induction as before and this time obtain $\mathcal{E} \subset T_X$ such that

$$(K_X + \det \mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-1} > 0.$$

So $(**)$ automatically holds. The problem is now that there is no way to deduce the existence of an extremal ray R in the movable cone such that

$$(K_X + \det \mathcal{E}) \cdot R > 0.$$

If we had R, then things go through smoothly. It is clear that we arrived here at a subtle point: up to now all arguments also work if $-K_X$ is merely pseudo-effective, but we know that under this weaker assumption T_X is in general not generically nef.

What we actually can conclude when $-K_X$ is nef is the existence of an element $\gamma \in \overline{ME}(X)$ such that

$$K_X \cdot \gamma = 0 \quad \text{and} \quad (K_X + \det \mathcal{E}) \cdot \gamma > 0.$$

We continue to discuss the case where $-K_X$ is not big (but still nef). Here $-K_X$ might no longer be semi-ample. Geometrically speaking, we consider the nef reduction

$$\phi : X \dashrightarrow W$$

of $-K_X$ and assume that $0 < \dim W < \dim X = n$ (that is, $K_X \not\equiv 0$ and there is a covering family (C_t) of curves such that $K_X \cdot C_t = 0$), and that $\nu(-K_X) = \dim W$. Then $\kappa(X) = \nu(X)$ and by Kawamata [Ka92] a suitable multiple $-mK_X$ is spanned and thus ϕ is holomorphic and given by $|-mK_X|$. Then $-K_X$ is automatically hermitian semi-positive, but we do not use this information. The proof of Theorem 7.4 can also be achieved without using Kawamata's result, just using $\nu(-K_X) = \dim W$. The reason is that ϕ being almost holomorphic suffices to make our arguments work. We shall not use the finer structure of manifold with $-K_X$ hermitian semi-positive [DPS96] (see also 8.5) because we are aiming also for the general nef case; moreover the arguments in the proof of 7.4 also work in a singular setting (canonical singularities).

7.4. Theorem. *Let X be a projective manifold with $-K_X$ semi-ample. Then T_X is generically nef.*

Proof. **(I)** If $K_X \equiv 0$, then Miyaoka's theorem [Mi87] already gives the result; hence we shall assume $K_X \not\equiv 0$.

Let $\phi : X \to W$ be the fibration given by a suitable multiple $-mK_X$. In this first step of the proof, we connect this morphism to the generic nefness of the tangent bundle.

By 7.3 we may assume that $-K_X$ is not big, hence $\dim W < \dim X$. Let F be a general fiber of ϕ and set $d = n - k = \dim W$. Then we have an exact sequence

$$0 \to \mathcal{O}_F^d \to \Omega_X^1|F \to \Omega_F^1 \to 0. \tag{1}$$

Assuming that T_X is not generically nef, we apply 5.2(2) and obtain a subsheaf $\mathcal{E} \subset T_X$ of rank k such that

$$(K_X + \det \mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-1} > 0$$

for suitable ample divisors H_j.

We restrict the inclusion $K_X + \det \mathcal{E} \subset \Omega_X^{n-k}$ to F and obtain an injective map

$$K_F + \det \mathcal{E}_F \subset \Omega_X^{n-k}|F. \tag{2}$$

We apply $\bigwedge^{n-k}$ to (1) and obtain sequences

$$0 \to \mathcal{F}_j \to \mathcal{F}_{j-1} \to \mathcal{F}_{j-1}/\mathcal{F}_j \to 0$$

with

$$\mathcal{F}_{j-1}/\mathcal{F}_j \simeq \Omega_F^{n-k-j+1} \otimes \bigwedge^{j-1} \mathcal{O}_F^d.$$

Now we chase the inclusion (2) through these sequences, using the fact that any morphism

$$\psi : K_F + \det \mathcal{E}_F \to (\Omega_F^r)^{\oplus N}$$

must vanish for $r > 0$. This will be proved below in Steps (II)–(IV). The outcome is a non-vanishing morphism

$$\psi : K_F + \det \mathcal{E}_F \to \bigwedge^{n-k} \mathcal{O}_F^d = \mathcal{O}_F^N.$$

Such a morphism however cannot exist, since

$$(K_X + \det \mathcal{E}) \cdot B > 0$$

for a suitable movable curve $B \subset F$, the existence of which will also be proved below.

(II) It remains to prove the vanishing of ψ and the existence of B. By projection to a suitable direct summand we obtain a non-zero morphism

$$\tau : K_F + \det \mathcal{E}_F \to \Omega^j_F, \tag{3}$$

for some $j \geq 0$. Since F is a fiber of ϕ, the morphism associated with $|-mK_X|$, we have $K_F \equiv 0$.

(III) This step is the core of the proof. We want to reduce ourselves to the case where

$$(K_X + \det \mathcal{E})_{>0} \cap \overline{ME}(X) \cap K_X^{\perp} \neq \{0\}. \tag{4}$$

This means that we can find an element $\gamma \in \overline{ME}(X) \setminus \{0\}$ with $K_X \cdot \gamma = 0$ such that

$$(K_X + \det \mathcal{E}) \cdot \gamma > 0.$$

This will lead to a contradiction in step (IV). Suppose that (4) is false and let $H = (K_X + \det \mathcal{E})^{\perp}$, so that

$$\overline{ME}(X) \cap K_X^{\perp} \subset H_{\leq 0}. \tag{4$'$}$$

(III.1) We choose a pseudo-effective $\mathbb{R}$-divisor P which is not big such that

$$P^{\perp} \cap \partial \overline{ME}(X) \subset H_{>0} \tag{5}$$

and such that P is ϕ-ample, i.e. positive on $K_X^{\perp}$. The divisor P is constructed as a linear form

$$\lambda : N_1(X) \to \mathbb{R}$$

which is non-negative on $\overline{ME}(X)$ (using [BDPP04] to conclude that P is pseudo-effective). More specifically, we apply Lemma 7.5 below with $V = N_1(X)$; $K = \overline{ME}(X) + \overline{NE}(X/W)$, the closed cone generated by $\overline{ME}(X)$; and $\overline{NE}(X/W)$, and L_2 the vector space generated by $\overline{NE}(X/W)$. Since we do not know that $L_2 \cap K = \overline{NE}(X/W) \subset H_{\leq 0}$, we choose $\tilde{H}$ such that $K \cap H_{\leq 0} \subset K \cap \tilde{H}_{\leq 0}$ and $L_2 \cap K \subset \tilde{H}_{\leq 0}$. Clearly we can still achieve $\overline{ME}(X) \not\subset \tilde{H}_{\leq 0}$, since $\overline{ME}(X) \not\subset H_{\leq 0}$. Therefore we may choose a linear subspace $L_1 \subset N_1(X)$ such that $L_1 \cap K \subset \partial K \cap \tilde{H}_{>0}$.

Now we apply, working with $\tilde{H}$, Lemma 7.5 to obtain a linear form $\lambda : N_1(X) \to \mathbb{R}$ with the properties stated in 7.5. The form λ is given by intersection with an $\mathbb{R}$-divisor P which is pseudo-effective by 7.5(1) and [BDPP04]. Moreover P is ϕ-ample by 7.5(2) and is not big by 7.5(3). Since $K \cap \tilde{H}_{>0} \subset K \cap H_{>0}$, claim (5) follows from 7.5(2).

(III.2) Since $-K_X$ is the pull-back of an ample $\mathbb{Q}$-divisor on W, and since P is ϕ-ample, we conclude that $A_0 = \lambda P - K_X$ is an ample $\mathbb{R}$-divisor for $0 < \lambda \ll 1$. Fix such a λ so that $K_X + A_0 = \lambda P$. Choose ample $\mathbb{Q}$-divisors A_k whose classes converge to the class of λP. Let t_k be the effective threshold or pseudo-effective value, i.e., the smallest positive number such that $K_X + t_k A_k$ is pseudo-effective. Since P is pseudo-effective but not big, the sequence (t_k) converges to 1 and therefore $K_X + t_k A_k$ converges to λP (numerically). From (5) we deduce

$$(K_X + t_0 A)^{\perp} \cap \partial \overline{ME}(X) \subset H_{>0} \tag{6}$$

for $k \gg 0$. We fix such a large k, choose m such that $A := mA_k$ is Cartier and set $t_0 = mt_k$. By [BCHM10, 1.1.7], t_k is in $\mathbb{Q}_+$ and so is t_0.

For $\epsilon > 0$ rational and sufficiently small the $\mathbb{Q}$-divisor $K_X + (t_0 - \epsilon)A$ is not pseudo-effective, so by [BCHM10] we can run the MMP and obtain a birational rational map

$$\sigma : X \dashrightarrow X'$$

(composed of birational contractions and flips) together with a Mori contraction

$$f : X' \to Y$$

such that $\dim Y < \dim X$. Notice that there is an algebraic set $B \subset X'$ of codimension at least 2 such that σ is an isomorphism over $X' \setminus B$. Let A' be the divisor on X' induced by A. Then by construction $K_{X'} + t_0 A'$ is the pull-back of an ample $\mathbb{Q}$-divisor on Y. Let M' be the divisor on X' induced by $\det \mathcal{E}$ and let l' be a general curve in F', a general fiber of f. Then $l' \cap B = \emptyset$ and we can consider its isomorphic preimage $l \subset X$. We claim that

$$(K_{X'} + M') \cdot l' > 0. \tag{7}$$

To prove (7), we need to verify $(K_X + \det \mathcal{E}) \cdot l > 0$, i.e. $[l] \in H_{>0}$. By (6) this holds as soon as we know that

$$(K_X + t_0 A) \cdot l = 0,$$

which however is obvious from

$$(K_X + t_0 A) \cdot l = (K_{X'} + t_0 A') \cdot l' = 0.$$

So (7) holds and thus $K_{X'} + M' | F'$ is ample (since $\rho(X'/Y) = 1$). Now, taking $\bigwedge^{n-k}$ of the exact sequence of Kähler differentials on the regular locus of the normal variety F',

$$N^*_{F'|X'} \to \Omega^1_{X'} | F' \to \Omega^1_{F'} \to 0,$$

we conclude via Theorem 6.1 that $K_{X'} + M' | F'$ is a subsheaf of some $\tilde{\Omega}^j_{F'}$. This is impossible, again by Theorem 6.1.

(IV) So (4) holds and consequently there exists $\gamma \in \overline{ME}(X) \setminus \{0\}$ with $K_X \cdot \gamma = 0$ such that

$$(K_X + \det \mathcal{E}) \cdot \gamma > 0.$$

By 6.3 there exists $\delta \in \overline{ME}(F)$ such that

$$\gamma = i_*(\delta).$$

By perturbation, we may assume δ is in the interior of $ME(F)$. Moreover we have a movable curve

$$B = \mu_*(H'_1 \cdot \ldots \cdot H'_{n-k})$$

(for some modification $\mu : \tilde{F} \to F$) such that

$$(K_F + \det \mathcal{E}_F) \cdot B > 0.$$

Since $K_F \equiv 0$ and since $K_F + \det \mathcal{E}_F \subset \Omega^j_F$ by (3), this contradicts [CP11] (and is actually trivial for $j = 0$). □

We still need to establish the following lemma.

7.5. Lemma. *Let V be a finite-dimensional real vector space, $K \subset V$ a closed cone with $K \cap -K = \{0\}$, and $L_1, L_2 \subset V$ linear subspaces such that $L_1 \cap K \subset \partial K$. Let $H \in V^* \setminus \{0\}$ be such that $L_2 \cap K \subset H_{\leq 0}$ and $L_1 \cap K \subset H_{>0}$. Then there exists a non-trivial linear form $\lambda : V \to \mathbb{R}$ such that*

(1) $\lambda|K \geq 0$,
(2) $\lambda|K \cap H_{\leq 0} > 0$,
(3) $\lambda|L_2 \cap K > 0$, *and*
(4) $\lambda|L_1 = 0$.

Proof. We approximate H by some $\tilde{H}$ such that $K \cap H_{\leq 0}$ is in the interior of $K \cap \tilde{H}_{\leq 0}$ and still $L_1 \cap K \subset \tilde{H}_{>0}$. We choose a sequence (K_i) of closed cones $K_i \subset K$ with the following properties:

- $K_i \subset K_{i+1}$,
- $K_i \cap L_1 = \{0\}$ for all i,
- $K = \bigcup K_i$, and
- $K_i \cap \tilde{H}_{\leq 0} = K \cap \tilde{H}_{\leq 0}$ for all i.

By Hahn–Banach we may choose $\lambda_i : V \to \mathbb{R}$ linear such that $\lambda_i|K_i > 0$, $\lambda_i|L_1 = 0$ and $\|\lambda_i\| = 1$ (for some norm in V^*). Then we extract a limit $\lambda : V \to \mathbb{R}$, and automatically (1), (2) and (4) hold. (3) is finally a consequence of (2). □

8. Base points

Generic nefness is sometimes not good enough for applications, e.g. for the smoothness of the Albanese map. Therefore we need to consider nefness concepts which deal with curves through any point, introduced in Definition 2.7. Here we apply the results of Sect. 2 to the tangent bundle.

8.1. Theorem. *Let X be a projective manifold with $K_X \equiv 0$. Then T_X is sufficiently nef.*

Proof. Fix ample divisors H_i. A priori T_X is only semi-stable with respect to any $(H_1, \dots, H_{n-1})$. The Beauville–Bogomolov decomposition gives a splitting (possibly after a finite étale cover which we may of course ignore)

$$X \simeq \prod X_j \times A$$

such that T_{X_j} is stable for all j and A is abelian. Hence T_{X_j} is sufficiently nef for all j and so is $p_j(T_{X_j})$. Since

$$T_X = \bigoplus p_j^*(T_{X_j}) \oplus p^*(T_A)$$

where $p_j : X \to X_j$ and $p : X \to A$ are the projections, we conclude easily that T_X is sufficiently nef. □

8.2. Remark. (1) The arguments of 8.1 actually show the following. If $X \simeq \prod X_j$ with all T_{X_j} sufficiently nef, then T_X is sufficiently nef.

(2) Using [Pe94], Theorem 8.1 also generalizes to the case where X has canonical singularities.

Theorem 2.12 also gives

8.3. Proposition. *Let X be a rationally connected manifold. If T_X is generically nef, it is sufficiently nef.*

Proof. We assume that T_X is not sufficiently nef at x. By 2.12 we obtain an epimorphism $T_X \to Q \to 0$ with $\det Q \equiv 0$. Since $-K_X$ is big and nef, $\det Q = \mathcal{O}_X$, hence $H^0(X, \Omega^j_X) \neq 0$ for some $j \geq 1$, a contradiction. □

The same proof actually shows the following (using the almost holomorphy of the rational quotient):

8.4. Proposition. *Suppose that T_X is generically nef, but not sufficiently nef. Let $f : X \dashrightarrow W$ be the rational quotient and Q the quotient sheaf of T_X constructed in 2.12. Then the composite rational map $T_{X/W} \to Q$ vanishes; in particular* $\operatorname{rk} Q \leq \dim W$.

We next discuss the case where X has semi-positive curvature.

8.5. Theorem. *Suppose that $-K_X$ is hermitian semi-positive.*

(1) *The rational quotient has a model which is a holomorphic fiber bundle $f : X \to Z$. Possibly after finite étale cover, $K_Z = \mathcal{O}_Z$ and $Z \simeq A \times B$ with A the Albanese torus of X and B a product of hyper-Kähler and Calabi–Yau manifolds.*
(2) *The tangent bundle T_X is sufficiently nef.*

Proof. (1) By [DPS96], and [Zh05] for the rational connectedness statement, the Albanese map $\alpha : X \to A$ is a fiber bundle over A and possibly after a finite étale cover of X, the fiber F of α is a product $F = \prod F_j$ such that the F_j are hyper-Kähler, Calabi–Yau or rationally connected. From that it is immediately clear that the rational quotient has a model which is a holomorphic fiber bundle $X \to Z$. Moreover Z is a holomorphic fiber bundle over its Albanese such that the fibers F have $K_F \equiv 0$. Therefore $K_Z \equiv 0$ and after a finite étale cover Z is a product as stated.

(2) The tangent bundles T_{F_j} are sufficiently nef by 8.1, resp. by 7.4 and 8.4. Therefore T_F is sufficiently nef by 8.2. This already solves the problem when $\pi_1(X)$ is finite.

Now assume that T_X is not sufficiently nef at x and apply 2.12. In the following we will freely pass to a finite étale cover if necessary. Consider the canonical morphism

$$\lambda : T_{X/A} \to T_X \to Q$$

and set

$$Q' = \operatorname{Im} \lambda, \qquad \mathcal{S}' = \operatorname{Ker} \lambda.$$

Notice that the map $\tilde{\mathcal{S}} \to \pi^*\alpha^*(T_A) = \mathcal{O}^d_{\tilde{X}}$ has non-maximal rank at most in codimension 2 (restrict to MR-general curves and use nefness of $\tilde{\mathcal{S}}$ on those curves). Now an easy diagram chase shows that Q' has the same negativity property as Q in 2.12 since there is an exact sequence

$$0 \to Q' \to Q \to \mathcal{T} \oplus \mathcal{O}^e_X \to 0.$$

Here $\mathcal{T}$ is a torsion sheaf, supported in codimension 2, and e is a non-negative integer. In particular

$$\det Q' \equiv 0.$$

Now consider the relative tangent bundle sequence

$$0 \to T_{X/Z} \to T_{X/A} \to f^*(T_{Z/A}) = p_B^*(T_B) \to 0.$$

By restricting to a fiber of f (which is rationally connected) and by recalling $\det Q' \equiv 0$ the induced map

$$\lambda : T_{X/Z} \to Q'$$

vanishes. In fact we consider the exact sequence

$$0 \to \operatorname{Im} \lambda_F \to Q' \to \operatorname{Coker} \lambda_F \to 0.$$

Since $\operatorname{Coker} \lambda_F$ is the quotient of a trivial bundle, namely $f^*(T_{Z/A})|F$, and since F is rationally connected, this clearly contradicts $\det Q' = 0$. So $\lambda_F = 0$ and hence $\lambda = 0$. Therefore we obtain an epimorphism

$$f^*(T_{Z/A}) = p_B^*(T_B) \to Q'.$$

But T_B is generically nef, and so is $p_B^*(T_B)$. This contradicts the negativity of Q' on certain curves through x. $\square$

9. Threefolds

We end by considering Conjecture 1.5 in dimension 3.

9.1. Theorem. *Let X be a smooth projective threefold with $-K_X$ nef. Then T_X is generically nef for some (H_1, H_2) and even generically nef for all (H_1, H_2) unless (possibly) we are in the following case: X is rationally connected, $-K_X$ is ample on all movable curves; moreover there is a fibration $f : X \to B \simeq \mathbb{P}_1$, whose general fiber has non-semi-ample anti-canonical bundle.*

Proof. By 7.4 we may assume that $-K_X$ is not semi-ample. Thus by [BP04] we are in one of the following cases:

(1) There is a finite étale cover $\tilde{X} \to X$ such that $\tilde{X} \simeq B \times S$ with B an elliptic curve and $-K_S$ is not semi-ample.
(2) X is rationally connected, $-K_X$ is ample on all movable curves; moreover there is a fibration $f : X \to B \simeq \mathbb{P}_1$ whose general fiber has non-semi-ample anti-canonical bundle.
(3) $-K_X$ is ample on all movable curves and X is a $\mathbb{P}_1$-bundle over an abelian surface.

In case (3) T_X is even nef and there is nothing to prove. In case (1) clearly $q(\tilde{X}) \leq 2$. If $q(\tilde{X}) = 2$, so that $q(S) = 1$, then the Albanese of $\tilde{X}$, being smooth by [PS98], realizes $\tilde{X}$ as a $\mathbb{P}_1$-bundle over an abelian surface. Hence $T_{\tilde{X}}$ is nef and so is T_X. So we may assume $q(S) = 0$ and thus S is $\mathbb{P}_2$ blown up in nine points in general position. By 5.9, T_S is generically ample, hence $T_{\tilde{X}}$ is easily seen to be generically nef.

We finally treat the most difficult case (2) and need to prove the existence of some $h = (H_1, H_2)$ for which T_X is generically nef. We fix an ample divisor H on X and set $A = \mathcal{O}_B(1)$. Introduce

$$h_0 = p^*(A) \cdot H$$

and notice that

$$K_X \cdot h_0 < 0.$$

We claim that there is an open neighborhood U of h_0 in $\overline{ME}(X)$ such that for all line bundles $L \subset T_X$ and all h with $[h] \in U$,

$$(K_X + L) \cdot h < 0. \tag{3.a}$$

For the proof of (3.a) we first observe that

$$(K_X + L) \cdot h_0 < 0. \tag{3.b}$$

In fact, by the definition of h_0 this comes down to

$$(K_F + L_F) \cdot H_F < 0. \tag{$*$}$$

If the compos map $K_F \otimes L_F \to K_F \otimes T_X|F \to K_F \otimes N_F = K_F$ is non-zero, then $(*)$ and (3.b) are already clear. If this map vanishes, then we have an inclusion

$$K_F \otimes L_F \subset K_F \otimes T_F = \Omega^1_F.$$

Since $-K_F$ is not semi-ample, F is either the plane blown up in nine points in sufficiently general position, or the projectivization of a suitable semi-stable rank 2-bundle on an elliptic curve. In the first case T_X is generically ample, hence $(*)$ follows. In the second case T_F is nef and $(*)$ is also easily checked. This establishes (3.b).

From (3.b) the claim (3.a) is an immediate consequence since the set

$$\{c_1(L) \mid L \subset T_X,\ L \cdot h_0 > 0\}$$

is finite.

We claim that T_X is h-generically nef for $[h] \in U(h_0)$. Supposing the contrary for a fixed h, we find by 5.4 a subsheaf $\mathcal{E} \subset T_X$ such that

$$(K_X + \det \mathcal{E}) \cdot h > 0 \qquad (**)$$

and $\mathcal{E}$ is h_ϵ-maximally nef. But then (3.a) conflicts with $(**)$, so we obtain a contradiction and conclude. □

9.2. Theorem. *Let X be a smooth projective threefold with $-K_X$ nef. Then T_X is sufficiently nef.*

Proof. If $-K_X$ is hermitian semi-positive, we simply apply 8.5. Otherwise we are in of the three "exceptional" cases listed in the proof of 9.1. In cases (1) and (3) it is clear that T_X is sufficiently nef; in case (2) we can argue as in 9.1 using 7.8, and everything remains the same except that the positivity $(**)$ is replaced by

$$K_X + \det \mathcal{E} \equiv 0. \qquad □$$

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Added in proof. Question 3.4 has now been positively answered by Meng Chen and Qi Zhang ("On a question of Demailly–Peternell–Schneider", arXiv:1110.1824). They also provide an example where the anti-canonical bundle of the target variety is not nef.

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