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Universal lifting theorem and quasi-Poisson groupoids

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Abstract. We prove the universal lifting theorem: for an α-simply connected and α-connected Lie groupoid Γ with Lie algebroid A, the graded Lie algebra of multi-differentials on A is isomorphic to that of multiplicative multi-vector fields on Γ. As a consequence, we obtain the integration theorem for a quasi-Lie bialgebroid, which generalizes various integration theorems in the literature in special cases.

The second goal of the paper is the study of basic properties of quasi-Poisson groupoids. In particular, we prove that a group pair (D, G) associated to a quasi-Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ induces a quasi-Poisson groupoid on the transformation groupoid $G \times D/G \rightrightarrows D/G$. Its momentum map corresponds exactly with the D/G-momentum map of Alekseev and Kosmann-Schwarzbach.

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1. Introduction

According to the classical Lie's third theorem, there is a bijection between (finite-dimensional) Lie algebras and (finite-dimensional) connected and simply connected Lie groups. Indeed it is a fundamental principle in Lie theory that given a connected and simply connected Lie group G, there should exist a one-one correspondence between various notions concerning the Lie group G and their infinitesimal counterparts in its Lie algebra $\mathfrak{g}$. The latter is easier to deal with, and the passage from a Lie algebra notion to its Lie group counterpart is normally referred to as "integration". Poisson groups are the classical limit of quantum groups and have been extensively studied in the past two decades. It is a theorem of Drinfel'd [13, 14] that there is a bijection between connected and simply connected Poisson groups and their infinitesimal invariants: Lie bialgebras. A Poisson group is a Lie group equipped with a compatible Poisson structure. By a compatible Poisson structure π on a Lie group G, we mean that the Poisson tensor π is multiplicative, i.e., the group multiplication $G \times G \to G$ is a Poisson map. In [23], motivated by the theory of Poisson groups, Lu studied multiplicative multi-vector fields on a Lie group and proved that they are closed under the Schouten bracket. Therefore, the space of multiplicative multi-vector fields on a Lie group constitutes a Lie subalgebra of the Lie algebra of multi-vector fields. Moreover, Lu proved that this Lie subalgebra is isomorphic, under the assumption that the Lie group is connected and simply connected, to the Lie algebra of derivations of the algebra $(\bigoplus \bigwedge^\bullet \mathfrak{g}, \wedge)$. The latter implies Drinfel'd's theorem above regarding integration of Lie bialgebras.

On the other hand, symplectic groupoids were introduced by Karasev [18], Weinstein [36] and Zakrzewski [43] independently in their study of Poisson geometry. Symplectic groupoids are Lie groupoids equipped with compatible symplectic structures. They have played an increasingly important role in quantization theory [7, 36]. In order to explore the intrinsic relation between symplectic groupoid theory and Drinfel'd theory, Weinstein introduced the notion of Poisson groupoids [37], i.e., Lie groupoids equipped with compatible Poisson structures. In [27], Mackenzie and one of the authors introduced the notion of Lie bialgebroids: pairs of Lie algebroids (A, A^*) in duality which satisfy a certain compatibility condition. Furthermore, they proved that Lie bialgebroids are in bijection with Poisson groupoids under a suitable simply-connectedness assumption [29]. This result extends the well-known result of Drinfel'd that a Lie bialgebra is the Lie bialgebra of a Poisson group [13, 14]. At the other extreme, they obtained, as a consequence, a new proof of the existence of local symplectic groupoids for any Poisson manifolds, a remarkable theorem of Karasev and Weinstein [18, 36].

In their study of the moduli space of flat connections on surfaces by using finite-dimensional techniques, Alekseev, Malkin and Meinrenken introduced the notion of quasi-Hamiltonian manifolds [3]. In [1, 2], the more general notion of quasi-Poisson manifold was introduced by Alekseev and Kosmann-Schwarzbach. These are quasi-Poisson spaces with the so-called D/G-momentum maps. Here $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a quasi-Manin triple, and D and G are connected and simply connected Lie groups with Lie algebras $\mathfrak{d}$ and $\mathfrak{g}$ respectively. They also developed the theory of reduction for these quasi-Poisson spaces. According to the Weinstein guiding principle [39], such a quasi-Poisson space can be considered as a "symmetric space" by a Lie groupoid equipped with a certain Poisson type structure. It is thus natural to explore the underlying structure on a groupoid, which gives rise to such a D/G-momentum map. For this purpose, it turns out that one must enlarge the notion of Poisson groupoids to consider groupoids equipped with multiplicative bivector fields which are "Poisson up to homotopy", i.e., quasi-Poisson groupoids. The related notion of quasi-Lie bialgebroids was introduced by Roytenberg [33]. And for a quasi-Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, there is a naturally associated quasi-Lie bialgebroid. However, it remains an open question whether every quasi-Lie bialgebroid integrates to a quasi-Poisson groupoid.

The main goal of this paper is to study various integration problems from a general perspective and prove an analogue of Lu's theorem for Lie groupoids. More precisely, we study multiplicative multi-vector fields on a Lie groupoid. Since left and right translations are not well defined on a Lie groupoid, Lu's definition does not have a straightforward generalization to the groupoid setting. This difficulty can however be overcome by using a generalized version of Weinstein's coisotropic calculus [37]. The space of multiplicative k-vector fields on Γ is denoted by $\mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$. We prove that $\bigoplus_i \mathfrak{X}^i_{\mathrm{mult}}(\Gamma)$ is closed under the Schouten bracket, and therefore is a Gerstenhaber subalgebra of $\bigoplus_i \mathfrak{X}^i(\Gamma)$.

To study the infinitesimal version of multiplicative multi-vector fields, we introduce the notion of multi-differentials on a Lie algebroid. Recall that for a given Lie algebroid A, the anchor map together with the Lie bracket on $\Gamma(A)$ extends to a graded Lie bracket on $\bigoplus_i \Gamma(\bigwedge^i A)$, which makes it into a Gerstenhaber algebra $(\bigoplus_i \Gamma(\bigwedge^i A), [\![\cdot,\cdot]\!], \wedge)$ [42]. By a *k-differential* on a Lie algebroid A, we mean a linear operator $\delta : \Gamma(\bigwedge^\bullet A) \to \Gamma(\bigwedge^{\bullet+k-1} A)$ satisfying

$$\begin{aligned} \delta(P \wedge Q) &= (\delta P) \wedge Q + (-1)^{p(k-1)} P \wedge \delta Q, \\ \delta[\![P, Q]\!] &= [\![\delta P, Q]\!] + (-1)^{(p-1)(k-1)} [\![P, \delta Q]\!], \end{aligned} \tag{1.1}$$

for all $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge^q A)$ (see Definition 2.23 for an equivalent definition). In other words, δ is a differential (of degree $k-1$) of the Gerstenhaber algebra $(\oplus_i \Gamma(\bigwedge^i A), [\![\cdot,\cdot]\!], \wedge)$. The space of all multi-differentials, $\mathcal{A} = \bigoplus_k \mathcal{A}_k$, becomes a graded Lie algebra under the graded commutator. The main theorem is the following

Universal lifting theorem. Assume that $\Gamma \rightrightarrows M$ is an α-simply connected and α-connected Lie groupoid with Lie algebroid A. Then $\bigoplus_i \mathcal{A}_i$ is isomorphic to $\bigoplus_i \mathfrak{X}^i_{\mathrm{mult}}(\Gamma)$ as graded Lie algebras.

Here, α-connected and α-simply connected Lie groupoid means that the fibers of the fibration $\alpha : \Gamma \to M$ are connected and simply connected, respectively (see the end of the introduction for more details on the notation). To prove this theorem, one direction is straightforward. Given a multiplicative k-vector field $\Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$, for any

$P \in \Gamma(\bigwedge^p A)$, one proves that $[\Pi, \overrightarrow{P}]$ is again right-invariant, where $\overrightarrow{P}$ denotes the right-invariant p-vector field on Γ corresponding to $P \in \Gamma(\bigwedge^p A)$. Therefore, there exists $\delta_\Pi P \in \Gamma(\bigwedge^{k+p-1} A)$ such that for any $P \in \Gamma(\bigwedge^p A)$,

$$\overrightarrow{\delta_\Pi P} = [\Pi, \overrightarrow{P}].$$

Thus δ_Π is indeed a k-differential. When Γ is a Lie group, the above construction is called the *inner derivative* [23]. To prove the other direction, we realize the groupoid Γ as the moduli space of the space $P(A)$ of all A-paths modulo gauge transformations. Such a characterization was first obtained by Cattaneo–Felder motivated by the Poisson sigma model when the Lie algebroid is the cotangent Lie algebroid associated to a Poisson manifold [7]. The general case was due to Crainic–Fernandes [11] (see [6, 12] for applications). Heuristically, our idea can be described as follows. Let δ be a k-differential. Then δ naturally induces a k-vector field π_δ on A, which is linear along the fibers. It in turn gives rise to a k-vector field $\widetilde{\pi}_\delta$ on the path space $\widetilde{P}(A)$. We then prove that $\widetilde{\pi}_\delta$ induces a k-vector field on the moduli space of the space $P(A)$ of all A-paths.

As an important application, in the second part of the paper, we study quasi-Poisson groupoids and their infinitesimals: quasi-Lie bialgebroids. A *quasi-Poisson groupoid* is a triple (Γ, Π, Ω), where Γ is a Lie groupoid, Π is a multiplicative bivector field on Γ and $\Omega \in \Gamma(\bigwedge^3 A)$ such that the following compatibility conditions hold:

$$\tfrac{1}{2}[\Pi, \Pi] = \overrightarrow{\Omega} - \overleftarrow{\Omega}, \tag{1.2}$$

$$[\Pi, \overrightarrow{\Omega}] = 0. \tag{1.3}$$

Infinitesimally, a quasi-Poisson groupoid corresponds to a quasi-Lie bialgebroid, i.e. to a 2-differential whose square is a coboundary: $\delta \in \mathcal{A}_2$ such that $\delta^2 = [\![\Omega, \cdot]\!]$ for some $\Omega \in \Gamma(\bigwedge^3 A)$ satisfying $\delta\Omega = 0$. The notion of quasi-Lie bialgebroids, first introduced by Roytenberg [33], is a natural generalization of Lie bialgebroids [27]. It also generalizes Drinfeld's quasi-Lie bialgebras [15], the classical limit of quasi-Hopf algebras.

As an immediate consequence of the universal lifting theorem, one concludes that there is a bijection between quasi-Lie bialgebroids (A, δ, Ω) and quasi-Poisson groupoids (Γ, Π, Ω), where Γ is an α-simply connected and α-connected Lie groupoid integrating the Lie algebroid A. In particular, when $\Omega = 0$, one obtains a simpler proof of the Lie bialgebroid integration theorem of Mackenzie–Xu [29]. On the other hand, when (A, δ, Ω) is the quasi-Lie bialgebroid corresponding to a twisted Poisson manifold [34], one recovers the integration theorem of Cattaneo–Xu [8].

A fundamental example, which is also a driving force for our study, is the quasi-Poisson groupoid induced by a quasi-Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. Given such a quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, there is an associated quasi-Lie bialgebra $(\mathfrak{g}, \delta, \phi)$, where $\delta : \bigwedge^\bullet \mathfrak{g} \to \bigwedge^{\bullet+1} \mathfrak{g}$ is a derivation of the Gerstenhaber algebra $\bigwedge \mathfrak{g}$ such that $\delta^2 = [\phi, \cdot]$. It is easy to see that δ extends to a 2-differential of the transformation Lie algebroid $\mathfrak{g} \times D/G \to D/G$, where D and G are connected and simply connected Lie groups with Lie algebras $\mathfrak{d}$ and $\mathfrak{g}$ respectively, and $\mathfrak{g}$ acts on D/G as the infinitesimal action of the left G-multiplication on D/G. We explicitly describe the corresponding quasi-Poisson groupoid structure on $G \times D/G \rightrightarrows D/G$.

Similar to the case of Poisson groupoids, a quasi-Poisson groupoid Γ also defines a momentum theory via the so-called Hamiltonian Γ-spaces. Important properties of such spaces are also studied in this paper. For the quasi-Poisson groupoid $G \times D/G \rightrightarrows D/G$ above, we prove that the corresponding Hamiltonian Γ-spaces are equivalent to the quasi-Poisson spaces with D/G-momentum maps in the sense of Alekseev and Kosmann–Schwarzbach [1]. However, our approach does not require $\mathfrak{h}$ to be admissible. A particularly interesting case is the quasi-Manin triple $(\mathfrak{g} \oplus \mathfrak{g}, \Delta(\mathfrak{g}), \frac{1}{2}\Delta_-(\mathfrak{g}))$ corresponding to a Lie algebra $\mathfrak{g}$ equipped with an ad-invariant nondegenerate symmetric pairing. In this case, one obtains a quasi-Poisson groupoid structure on $G \times G \rightrightarrows G$, where G acts on G by conjugation. The discussion on this topic occupies Section 4.

When $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple and G is a complete Poisson group, we recover the Poisson groupoid $G \times G^* \rightrightarrows G^*$ [25] (which is symplectic in this case), G^* being the dual Poisson group. When G is not necessarily complete, the Poisson groupoid $G \times D/G \rightrightarrows D/G$, which is in fact a symplectic groupoid integrating the Poisson structure on D/G, can be considered as a replacement of the Lu–Weinstein symplectic groupoid $G \times G^* \rightrightarrows G^*$.

We note that multiplicative multi-vector fields on a Lie groupoid are also related to super-groupoids studied by Mehta [30]. We refer the interested reader to [30] for more details. After the paper has been submitted, an alternative approach to the universal lifting theorem has been proposed by Bursztyn–Cabrera [5]. Moreover, a number of important applications of the universal lifting theorem have recently appeared, including, for instance, in the study of Poisson quasi-Nijenhuis structures [35] and holomorphic Lie algebroids [21].

Notation. Some remarks about notation are in order. For a Lie groupoid $\Gamma \rightrightarrows M$, we denote by $\alpha, \beta \colon \Gamma \to M$ the source and target maps. Two elements $g, h \in \Gamma$ are composable if $\beta(g) = \alpha(h)$. We denote by $\Gamma^{(2)} \subset \Gamma \times \Gamma$ the subset of composable pairs in $\Gamma \times \Gamma$. By $i : \Gamma \to \Gamma$, $g \mapsto i(g) = g^{-1}$, we denote the inversion, and $\epsilon : M \to \Gamma$, $x \mapsto \epsilon(x) = \widetilde{x}$, is the unit map. We denote by $A \to M$ the Lie algebroid of Γ (for any $m \in M$, $A_m := \operatorname{Ker} \beta_{*|\epsilon(m)}$). Moreover, given a section X of A, $\overrightarrow{X}$ denotes the right-invariant vector field on Γ corresponding to X. The cotangent Lie groupoid of Γ is denoted by $T^*\Gamma \rightrightarrows A^*$, A^* being the dual of the Lie algebroid A, where the source and target maps are denoted by $\widetilde{\alpha}$ and $\widetilde{\beta}$, respectively, the inversion by $\widetilde{\iota}$, and the unit map by $\widetilde{\epsilon}$.

2. Multiplicative k-vector fields on Lie groupoids

2.1. Coisotropic submanifolds

In this section, we generalize Weinstein's coisotropic calculus [37] to multi-vector fields.

Definition 2.1. Let V be a vector space and $\Pi \in \bigwedge^k V$. We say that a subspace W of V is *coisotropic* with respect to Π if

$$\Pi(\xi^1, \ldots, \xi^k) = 0$$

for all $\xi^1, \ldots, \xi^k \in W^\circ$, where W° is the annihilator of W, that is, $W^\circ = \{\xi \in V^* \mid \xi_{|W} = 0\}$.

A slight modification of coisotropic calculus for bivector fields developed in [37] will allow us to show several properties. We now give the following important lemma.

Lemma 2.2. *Let V_1, V_2 be vector spaces and $\Pi_i \in \bigwedge^k V_i$ for $i = 1, 2$. If $R \subseteq V_1 \times V_2$ is coisotropic with respect to $\Pi_1 \oplus \Pi_2$ and $C \subseteq V_2$ is coisotropic with respect to Π_2 then*

$$R(C) = \{u \in V_1 \mid \exists\, v \in C \text{ with } (u, v) \in R\}$$

is coisotropic with respect to Π_1.

Proof. Let $R^* \subseteq V_1^* \times V_2^*$ be the subspace given by

$$R^* = \{(\xi^1, \xi^2) \in V_1^* \times V_2^* \mid \langle \xi^1, v_1\rangle = \langle \xi^2, v_2\rangle,\ \forall\, (v_1, v_2) \in R\}.$$

Then we have $R(C)^\circ = R^*(C^\circ)$.

Now, let $\xi^1, \dots, \xi^k \in R(C)^\circ = R^*(C^\circ)$. Then there exist $\varphi^1, \dots, \varphi^k \in C^\circ$ such that $(\xi^i, \varphi^i) \in R^*$, i.e., $(\xi^i, -\varphi^i) \in R^\circ$. Thus,

$$\begin{aligned} 0 &= (\Pi_1 \oplus \Pi_2)((\xi^1, -\varphi^1), \dots, (\xi^k, -\varphi^k)) = \Pi_1(\xi^1, \dots, \xi^k) + (-1)^k \Pi_2(\varphi^1, \dots, \varphi^k) \\ &= \Pi_1(\xi^1, \dots, \xi^k). \end{aligned}$$

That is, $R(C)$ is coisotropic. □

A generalization of the notion of coisotropy to manifolds is the following.

Definition 2.3. Let M be an arbitrary manifold and $\Pi \in \mathfrak{X}^k(M)$. A submanifold S of M is said to be *coisotropic* with respect to Π if $T_x S$ is coisotropic with respect to $\Pi(x)$ for all $x \in S$.

Remark 2.4. It is easy to see that S is coisotropic with respect to the multi-vector field Π if and only if $\Pi(\mathrm{d}f^1, \dots, \mathrm{d}f^k)_{|S} = 0$ for any $f^1, \dots, f^k \in C^\infty(M)$ such that $f^i_{|S} = 0$ (see [37] for the case of bivector fields).

On any manifold M, the usual Lie bracket of vector fields can be extended to multi-vector fields, yielding the so-called Schouten bracket, denoted by $[\cdot, \cdot]$, in such a way that $(\bigoplus_k \mathfrak{X}^k(M), [\cdot, \cdot])$ is endowed with a structure of Gerstenhaber algebra. More precisely, this bracket can be defined as follows. If A is any subset of $\{1, \dots, k+k'-1\}$, let A' denote its complement and $|A|$ the number of elements in A. If $|A| = l$ and the elements in A are $\{i_1, \dots, i_l\}$ in increasing order, let us write f_A for the ordered k-tuple $(f^{i_1}, \dots, f^{i_l})$. Furthermore, we write ε_A for the sign of the permutation which rearranges the elements of the ordered $(k + k' - 1)$-tuple (A', A) in the original order. Then the *Schouten bracket* of $\Pi \in \mathfrak{X}^k(M)$ and $\Pi' \in \mathfrak{X}^{k'}(M)$ is given [4] by

$$\begin{aligned} [\Pi, \Pi'](\mathrm{d}f^1, \dots, \mathrm{d}f^{k+k'-1}) = (-1)^{k+1}\Big(&\sum_{|A|=k'} \varepsilon_A \Pi(\mathrm{d}(\Pi'(f_A)), \mathrm{d}f_{A'}) \\ &+ (-1)^{kk'} \sum_{|B|=k} \varepsilon_B \Pi'(\mathrm{d}(\Pi(f_B)), \mathrm{d}f_{B'})\Big). \end{aligned} \tag{2.4}$$

Using (2.4) and the characterization of coisotropic submanifolds in Remark 2.4, one can deduce the following result.

Proposition 2.5. *If S is coisotropic with respect to the multi-vector fields Π and Π', then S is coisotropic with respect to the multi-vector field $[\Pi, \Pi']$, defined by the Schouten bracket of Π and Π'.*

2.2. *Definition and examples*

In this section, we will introduce the notion of multiplicative multi-vector fields and show that it generalizes several concepts which have previously appeared in the literature.

Throughout this section, we fix a Lie groupoid $\Gamma \rightrightarrows M$ and denote its Lie algebroid by A. Moreover, we denote by Λ the graph of the groupoid multiplication, that is,

$$\Lambda = \{(g, h, gh) \mid \beta(g) = \alpha(h)\}.$$

Definition 2.6. Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\Pi \in \mathfrak{X}^k(\Gamma)$ a k-vector field on Γ. We say that Π is *multiplicative* if Λ is coisotropic with respect to $\Pi \oplus \Pi \oplus (-1)^{k+1}\Pi$. The space of multiplicative vector fields is denoted by $\mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$.

An interesting characterization of multiplicative multi-vector fields is the following:

Proposition 2.7. *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\Pi \in \mathfrak{X}^k(\Gamma)$ a k-vector field on Γ. Then the following are equivalent:*

(i) *Π is multiplicative;*
(ii) *for any $\mu^i_g, \nu^i_h \in T^*\Gamma$, such that $\widetilde{\beta}(\mu^i_g) = \widetilde{\alpha}(\nu^i_h)$,*

$$\Pi(gh)(\mu^1_g \cdot \nu^1_h, \ldots, \mu^k_g \cdot \nu^k_h) = \Pi(g)(\mu^1_g, \ldots, \mu^k_g) + \Pi(h)(\nu^1_h, \ldots, \nu^k_h); \tag{2.5}$$

(iii) *the linear skew-symmetric function F_Π on $T^*\Gamma \times_\Gamma \overset{(k)}{\ldots} \times_\Gamma T^*\Gamma$ induced by Π,*

$$F_\Pi(\mu^1, \ldots, \mu^k) = \Pi(\mu^1, \ldots, \mu^k),$$

is a 1-cocycle with respect to the Lie groupoid

$$T^*\Gamma \times_\Gamma \overset{(k)}{\ldots} \times_\Gamma T^*\Gamma \rightrightarrows A^* \times_M \overset{(k)}{\ldots} \times_M A^*. \tag{2.6}$$

Here $T^*\Gamma \times_\Gamma \overset{(k)}{\ldots} \times_\Gamma T^*\Gamma = \{(\mu^1, \ldots, \mu^k) \in T^*\Gamma \times \ldots \times T^*\Gamma \mid \tau(\mu^1) = \cdots = \tau(\mu^k)\}$, $\tau : T^*\Gamma \to \Gamma$ is the bundle projection onto its base, $A^* \times_M \overset{(k)}{\ldots} \times_M A^* = \{(\eta^1, \ldots, \eta^k) \in A^* \times \ldots \times A^* \mid p_*(\eta^1) = \ldots = p_*(\eta^k)\}$, with $p_* : A^* \to M$ the projection onto the base, and $T^*\Gamma \times_\Gamma \overset{(k)}{\ldots} \times_\Gamma T^*\Gamma \rightrightarrows A^* \times_M \overset{(k)}{\ldots} \times_M A^*$ is considered as a Lie subgroupoid of the direct product groupoid $(T^*\Gamma)^k \rightrightarrows (A^*)^k$.

Proof. It is well-known [9] that if Λ is the graph of the groupoid multiplication for a groupoid $\Gamma \rightrightarrows M$, then the graph of the groupoid multiplication for the cotangent groupoid $T^*\Gamma \rightrightarrows A^*$ is related to the conormal bundle of Λ, $N^*\Lambda$, in the following way:

$$(\mu_g, \nu_h, \gamma_{gh}) \in N^*_{(g,h,gh)}\Lambda \quad \text{if and only if} \quad \gamma_{gh} = -\mu_g \cdot \nu_h \text{ for } \mu_g, \nu_h \in T^*\Gamma. \tag{2.7}$$

Thus, for Π a k-vector field, it follows from Definition 2.6 and (2.7) that Π is multiplicative if and only if for all $(\mu_g^1, \nu_h^1, \gamma_{gh}^1), \dots, (\mu_g^k, \nu_h^k, \gamma_{gh}^k) \in N^*_{(g,h,gh)}\Lambda$, we have

$$\begin{aligned}0 &= (\Pi \oplus \Pi \oplus (-1)^{k+1}\Pi)((\mu_g^1, \nu_h^1, \gamma_{gh}^1), \dots, (\mu_g^k, \nu_h^k, \gamma_{gh}^k)) \\ &= \Pi(g)(\mu_g^1, \dots, \mu_g^k) + \Pi(h)(\nu_h^1, \dots, \nu_h^k) - \Pi(gh)(\mu_g^1 \cdot \nu_h^1 \dots, \mu_g^k \cdot \nu_h^k).\end{aligned}$$

From this relation, the equivalences trivially follow. □

Remark 2.8. From Proposition 2.7, one can deduce the well-known fact that a bivector field Π on a Lie groupoid Γ is multiplicative if and only if $\Pi^\sharp : T^*\Gamma \to T\Gamma$ is a Lie groupoid morphism, i.e.,

$$\Pi^\sharp(\mu_g \cdot \nu_h) = \Pi^\sharp(\mu_g) \cdot \Pi^\sharp(\nu_h)$$

for all $\mu_g, \nu_h \in T^*\Gamma$ such that $\widetilde{\beta}(\mu_g) = \widetilde{\alpha}(\nu_h)$.

A direct consequence of Proposition 2.5 and Definition 2.6 is the following.

Proposition 2.9. *The Schouten bracket of multiplicative multi-vector fields is still multiplicative. That is,* $(\bigoplus_k \mathfrak{X}^k_{\mathrm{mult}}(\Gamma), [\cdot,\cdot])$ *is a graded Lie subalgebra of* $(\bigoplus_k \mathfrak{X}^k(\Gamma), [\cdot,\cdot])$.

Below we list some well-known examples.

Example 2.10. Let G be a Lie group and $\Pi \in \mathfrak{X}^k(G)$ a k-vector field. From $\widetilde{\beta}(\mu_g^i) = \widetilde{\alpha}(\nu_h^i)$ we deduce that

$$\mu_g^i = (L_{g^{-1}})^*(R_h)^*\nu_h^i.$$

Moreover, we have

$$\mu_g^i \cdot \nu_h^i = (L_{g^{-1}})^*\nu_h^i.$$

Therefore, we see that (2.5) is equivalent to

$$((L_{g^{-1}})_*\Pi(gh) - (L_{g^{-1}})_*(R_h)_*\Pi(g) - \Pi(h))(\nu_h^1, \dots, \nu_h^k) = 0.$$

That is, $\Pi(gh) = (R_h)_*\Pi(g) + (L_g)_*\Pi(h)$. The converse is obvious. Therefore, our definition of multiplicative k-vector fields is indeed a generalization of the usual notion for Lie groups (see [24]).

Example 2.11. We say that an $\mathbb{R}$-valued function σ on Γ is *multiplicative* if $\sigma(gh) = \sigma(g) + \sigma(h)$ for $(g,h) \in \Gamma^{(2)}$. Therefore, multiplicative functions are multiplicative 0-vector fields.

Example 2.12. A vector field $X \in \mathfrak{X}(\Gamma)$ is said to be *multiplicative* if it is a Lie groupoid morphism $X : \Gamma \to T\Gamma$ from a Lie groupoid $\Gamma \rightrightarrows M$ to the corresponding tangent Lie groupoid $T\Gamma \rightrightarrows TM$ (see [28]), so we have $X(gh) = X(g) \cdot X(h)$ for $\beta(g) = \alpha(h)$. Using this fact and that

$$(\mu_g \cdot \nu_h)(u_g \cdot v_h) = \mu_g(u_g) + \nu_h(v_h) \qquad \text{for all } (u_g, v_h) \in T\Gamma^{(2)},$$

we deduce that X is a multiplicative 1-vector field in the sense of Definition 2.6.

Example 2.13. From Definition 2.6, we see that the Poisson tensor on a Poisson groupoid is a multiplicative bivector field [37].

Example 2.14. Given a Lie groupoid Γ with Lie algebroid A, if $P \in \Gamma(\bigwedge^k A)$, then $\overrightarrow{P} - \overleftarrow{P}$ is a multiplicative k-vector field on $\Gamma \rightrightarrows M$.

2.3. *Multiplicative and affine multi-vector fields*

An *affinoid structure* on a space X is defined to be a subset of X^4 whose elements are designated as parallelograms, with axioms modeled on the properties of the quaternary relation $\{(x, y, z, w) \mid yx^{-1} = wz^{-1}\}$ on a group or groupoid. In the group case, this concept boils down to the standard one of an affine space as developed originally by R. Baer. An axiomatic approach to affinoid structures on a groupoid is given by Weinstein [38] in connection with the study of Poisson geometry. For a Lie groupoid Γ, let Ω be the submanifold of $\Gamma \times \Gamma \times \Gamma \times \Gamma$ consisting of elements (l, h, g, w) such that $w = hl^{-1}g$. Then Ω is called the *affinoid diagram* corresponding to the groupoid Γ by Weinstein [38]. A characterization of multiplicative k-vector fields in terms of the affinoid diagram, analogous to the classification of multiplicative Poisson structures in [38, Thm. 4.5], is the following:

Proposition 2.15. *Let Π be a k-vector field on a Lie groupoid. Then Π is multiplicative if and only if Ω is coisotropic with respect to $\Pi \oplus (-1)^{k+1}\Pi \oplus (-1)^{k+1}\Pi \oplus \Pi$ and M is coisotropic with respect to Π.*

Proof. Suppose that Π is a multiplicative k-vector field on Γ. Using (2.5) and the identity $\widetilde{\epsilon}(\phi_m) \cdot \widetilde{\epsilon}(\phi_m) = \widetilde{\epsilon}(\phi_m)$ for all $\phi_m \in A^*_m$, A^*_m being the fiber at $m \in M$ of the vector bundle A^*, we have

$$\Pi(\epsilon(m))(\widetilde{\epsilon}(\phi^1_m), \ldots, \widetilde{\epsilon}(\phi^k_m)) = 2\Pi(\epsilon(m))(\widetilde{\epsilon}(\phi^1_m), \ldots, \widetilde{\epsilon}(\phi^k_m))$$

for all $\phi^1_m, \ldots, \phi^k_m \in A^*_m, m \in M$. Since $\widetilde{\epsilon}(A^*) = N^*M$, we deduce that M is coisotropic with respect to Π.

Now we mimic the proof of Theorem 4.5 in [38]. In the product $\Gamma \times \bar{\Gamma} \times \bar{\Gamma} \times \Gamma \times \Gamma \times \bar{\Gamma}$ (where $\bar{\Gamma}$ denotes Γ endowed with the k-vector field $(-1)^{k+1}\Pi$), we consider the coisotropic submanifold $R = \{(g, h, l, x, y, z) \mid gy = l \text{ and } hz = x\}$. On the other hand, it is easy to see that the diagonal $\Delta \subset \Gamma \times \bar{\Gamma}$ is a coisotropic submanifold. Therefore, using Lemma 2.2, we find that $R(\Delta)$ is coisotropic submanifold of $\Gamma \times \bar{\Gamma} \times \bar{\Gamma} \times \Gamma$. It is easy to see that $R(\Delta) = \Omega$. Our result thus follows.

Conversely, let Π be a k-vector field on a Lie groupoid such that Ω is coisotropic with respect to $\Pi \oplus (-1)^{k+1}\Pi \oplus (-1)^{k+1}\Pi \oplus \Pi$ and M is coisotropic with respect to Π. Applying Lemma 2.2 to $R = \Omega$ and $C = M$, we conclude that Λ is coisotropic with respect to $\Pi \oplus \Pi \oplus (-1)^{k+1}\Pi$. That is, Π is multiplicative. ◻

Next we recall the definition of an affine multi-vector field on a Lie groupoid. We will also show its relation to multiplicative multi-vector fields.

Definition 2.16. A multi-vector field Π on Γ is *affine* if for any $g, h \in \Gamma$ such that $\beta(g) = \alpha(h) = m$ and any bisections $\mathcal{X}, \mathcal{Y}$ through the points g, h, we have

$$\Pi(gh) = (R_{\mathcal{Y}})_*\Pi(g) + (L_{\mathcal{X}})_*\Pi(h) - (R_{\mathcal{Y}} \circ L_{\mathcal{X}})_*\Pi(\epsilon(m)). \tag{2.8}$$

A useful characterization of affine multi-vector fields is the following:

Proposition 2.17 ([29]). *Let Π be a k-vector field on a Lie groupoid $\Gamma \rightrightarrows M$. Then Π is affine if and only if $[\overrightarrow{X}, \Pi]$ is right-invariant for all $X \in \Gamma(A)$.*

Multiplicative multi-vector fields are a particular case of affine multi-vector fields, as shown in the following proposition.

Proposition 2.18. *If Π is a multiplicative k-vector field on a Lie groupoid Γ, then Π is affine.*

Proof. If Π is multiplicative, then according to Proposition 2.15, Ω is coisotropic with respect to $\Pi \oplus (-1)^{k+1}\Pi \oplus (-1)^{k+1}\Pi \oplus \Pi$, and M is coisotropic with respect to Π. For any $\mu \in T_{gh}\Gamma$, it follows from Lemma 2.6 in [41] that $(-\mu, L^*_{\mathcal{X}}\mu, R^*_{\mathcal{Y}}\mu, -L^*_{\mathcal{X}}R^*_{\mathcal{Y}}\mu)$ is conormal to Ω. Therefore, for any $\mu^1, \ldots, \mu^k \in T_{gh}\Gamma$, we have

$$\begin{aligned} -\Pi(gh)(\mu^1, \ldots, \mu^k) + \Pi(h)(L^*_{\mathcal{X}}\mu^1, \ldots, L^*_{\mathcal{X}}\mu^k) + \Pi(g)(R^*_{\mathcal{Y}}\mu^1, \ldots, R^*_{\mathcal{Y}}\mu^k) \\ - \Pi(\epsilon(m))(L^*_{\mathcal{X}}R^*_{\mathcal{Y}}\mu^1, \ldots, L^*_{\mathcal{X}}R^*_{\mathcal{Y}}\mu^k) = 0. \end{aligned}$$

Thus (2.8) follows immediately. □

Similar to the case of multiplicative bivector fields, we can give another useful characterization of multiplicative k-vector fields.

Theorem 2.19. *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\Pi \in \mathfrak{X}^k(\Gamma)$ a k-vector field on Γ. Then Π is multiplicative if and only if the following conditions hold:*

(i) *Π is affine, i.e., (2.8) holds;*
(ii) *M is a coisotropic submanifold of Γ;*
(iii) *$\alpha_*\Pi(g)$ and $\beta_*\Pi(g)$ only depend on $\alpha(g)$ and $\beta(g)$, respectively;*
(iv) *for all $\eta^1, \eta^2 \in \Omega^1(M)$, we have $(\alpha^*\eta^1 \wedge \beta^*\eta^2) \lrcorner \Pi = 0$;*
(v) *for all $\theta \in \Omega^p(M)$, $1 \leq p < k$, then $(\beta^*\theta) \lrcorner \Pi$ is a left-invariant $(k-p)$-vector field on Γ.*

Proof. Let Π be a multiplicative k-vector field on Γ. From Propositions 2.15 and 2.18, we obtain (i) and (ii).

Next, since

$$0_g \cdot (\beta^*\eta)_h = (\beta^*\eta)_{gh} \tag{2.9}$$

for any $\eta \in \Omega^1(M)$, from (2.5) it follows that

$$\begin{aligned} \Pi((\beta^*\eta^1)_{gh}, \ldots, (\beta^*\eta^k)_{gh}) &= \Pi(0_g \cdot (\beta^*\eta^1)_h, \ldots, 0_g \cdot (\beta^*\eta^k)_h) \\ &= \Pi((\beta^*\eta^1)_h, \ldots, (\beta^*\eta^k)_h) \end{aligned}$$

for all $\eta^1, \ldots, \eta^k \in \Omega^1(M)$. Hence $\beta_*\Pi_g$ only depends on $\beta(g)$. Similarly, from the equality

$$(\alpha^*\eta)_g \cdot 0_h = (\alpha^*\eta)_{gh} \tag{2.10}$$

for all $\eta \in \Omega^1(M)$, we also deduce that $\alpha_*\Pi_g$ only depends on $\alpha(g)$. Hence, (iii) holds.

Let $\theta \in \Omega^p(M)$ be a p-form on M, $1 \le p < k$. Then, using (2.5), (2.9) and (2.10), we see that $(\beta^*\theta) \lrcorner \Pi$ is tangent to α-fibers. Hence we have (iv). To prove (v), it suffices to prove that

$$(L_{\mathcal{X}})_*((\beta^*\theta) \lrcorner \Pi)_{(h)}) = ((\beta^*\theta) \lrcorner \Pi)_{(gh)}, \quad \forall (g, h, gh) \in \Lambda,$$

where $\mathcal{X}$ is an arbitrary bisection through g. According to [41, Eq. (5)], for any $\mu \in T^*_{gh}\Gamma$, there exists $\nu \in T^*_g\Gamma$ which is characterized by the equation

$$\langle \nu, v_g\rangle = \langle \mu, (R_h)_*(v_g - (L_{\mathcal{X}})_*\beta_* v_g)\rangle, \quad \forall v_g \in T_g\Gamma,$$

and $\nu \cdot (L_{\mathcal{X}})^*\mu = \mu$. Using this fact and (2.10), it follows that

$$((\beta^*\theta) \lrcorner \Pi)_{(h)}((L_{\mathcal{X}})^*\mu^1, \dots, (L_{\mathcal{X}})^*\mu^{p-k}) = ((\beta^*\theta) \lrcorner \Pi)_{(gh)}(\mu^1, \dots, \mu^{p-k})$$

for all $\mu^1, \dots, \mu^{p-k} \in \Omega^1(\Gamma)$. Thus, (v) follows.

To prove the converse, we first note that the following three types of vectors span the whole conormal space of Ω at a point (g, h, l, w): $(-\mu, L^*_{\mathcal{X}}\mu, R^*_{\mathcal{Y}}\mu, -L^*_{\mathcal{X}}R^*_{\mathcal{Y}}\mu)$ for any $\mu \in T^*_l\Gamma$, $(-\beta^*\eta, \beta^*\eta, 0, 0)$ for any $\eta \in T^*_{\alpha(k)}\Gamma$, and $(-\alpha^*\zeta, 0, \alpha^*\zeta, 0)$ for any $\zeta \in T^*_{\beta(z)}\Gamma$, where $\mathcal{X}$ and $\mathcal{Y}$ are any bisections through the points g and h, respectively (see [41, Thm. 2.8]). From (i), (iii), (iv) and (v) we deduce that Ω is coisotropic with respect to $\Pi \oplus (-1)^{k+1}\Pi \oplus (-1)^{k+1}\Pi \oplus \Pi$ (see Theorem 2.8 in [41]). From this fact, (ii) and Proposition 2.15, the conclusion follows. □

The following proposition can be proved in a similar fashion to item (v) of Theorem 2.19.

Proposition 2.20. *If Π is a multiplicative k-vector field on Γ, then for all $\theta \in \Omega^p(M)$, $1 \le p < k$, $(\alpha^*\theta) \lrcorner \Pi$ is a right-invariant $(k-p)$-vector field on Γ.*

Finally, let us show an interesting property that generalizes the one obtained for multiplicative bivector fields in [37, Thm. 4.2.3].

Proposition 2.21. *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\Pi \in \mathfrak{X}^k(\Gamma)$ be a multiplicative k-vector field on Γ. Then there exists a unique k-vector field π on M such that*

$$\alpha_*\Pi = \pi, \qquad \beta_*\Pi = (-1)^{k+1}\pi.$$

Proof. Since Π is multiplicative, using property (iii) in Theorem 2.19, we can define a k-vector field π on M by setting $\pi = \alpha_*\Pi$. Now, let us investigate the relation between Π, π and the map β.

First, we show that if $i : \Gamma \to \Gamma$ is the groupoid inversion, then

$$i_*\Pi = (-1)^{k+1}\Pi. \tag{2.11}$$

This is an immediate consequence of property (ii) in Theorem 2.19 and the fact that the inverse $(\mu_g)^{-1}$ of $\mu_g \in T^*_g\Gamma$ is given by $(\mu_g)^{-1} = -i^*(\mu_g)$. In fact,

$$\begin{aligned} 0 &= \Pi(\widetilde{\epsilon}(\widetilde{\beta}(\mu^1_g)), \dots, \widetilde{\epsilon}(\widetilde{\beta}(\mu^k_g))) = \Pi(\mu^1_g \cdot (\mu^1_g)^{-1}, \dots, \mu^k_g \cdot (\mu^k_g)^{-1}) \\ &= \Pi(\mu^1_g, \dots, \mu^k_g) + (-1)^k (i_*\Pi)(\mu^1_g, \dots, \mu^k_g) \end{aligned}$$

for any $\mu^1_g, \dots, \mu^k_g \in T^*_g\Gamma$. Finally, using (2.11) and the relation $\alpha \circ i = \beta$, we conclude that $\beta_*\Pi = (-1)^{k+1}\pi$. □

Example 2.22. From Proposition 2.21 and property (iv) in Theorem 2.19, we deduce that the map $(\alpha, \beta) : \Gamma \to M \times M$ satisfies

$$(\alpha, \beta)_* \Pi = \pi \oplus (-1)^{k+1} \pi.$$

In particular, when Γ is a pair groupoid $M \times M \rightrightarrows M$, since $(\alpha, \beta) : M \times M \to M \times M$ is a diffeomorphism, the only multiplicative k-vector fields are of the form $\pi \oplus (-1)^{k+1}\pi$, $\pi \in \mathfrak{X}^k(M)$ (see [28, p. 67] and [37, Cor. 4.2.8] for the case $k = 1$ and $k = 2$, respectively).

2.4. k-differentials on Lie algebroids

We now turn to the study of the Lie algebroid counterpart of multiplicative k-vector fields, namely, k-differentials.

Definition 2.23. Let $(A, [\![\cdot, \cdot]\!], \rho)$ be a Lie algebroid over M. An *almost k-differential* is a pair of linear maps $\delta : C^\infty(M) \to \Gamma(\bigwedge^{k-1} A)$ and $\delta : \Gamma(A) \to \Gamma(\bigwedge^k A)$ satisfying

(i) $\delta(fg) = g(\delta f) + f(\delta g)$ for all $f, g \in C^\infty(M)$;
(ii) $\delta(fX) = (\delta f) \wedge X + f \delta X$ for all $f \in C^\infty(M)$ and $X \in \Gamma(A)$.

An almost k-differential is said to be a *k-differential* if it satisfies the compatibility condition

$$\delta[\![X, Y]\!] = [\![\delta X, Y]\!] + [\![X, \delta Y]\!] \tag{2.12}$$

for all $X, Y \in \Gamma(A)$.

For a given Lie algebroid A, it is known that the anchor map together with the bracket on $\Gamma(A)$ extends to a graded Lie bracket on $\bigoplus_k \Gamma(\bigwedge^k A)$, which makes it into a Gerstenhaber algebra $(\bigoplus_k \Gamma(\bigwedge^k A), [\![\cdot, \cdot]\!], \wedge)$ [42].

A k-differential δ extends naturally to sections of $\bigwedge A$ as follows:

$$\delta(X_1 \wedge \cdots \wedge X_s) = \sum_{i=1}^{s} (-1)^{(i+1)(k+1)} X_1 \wedge \cdots \wedge (\delta X_i) \wedge \cdots \wedge X_s \tag{2.13}$$

for $X_1, \ldots, X_s \in \Gamma(A)$. In this way, we obtain a linear operator $\delta : \Gamma(\bigwedge^\bullet A) \to \Gamma(\bigwedge^{\bullet+k-1} A)$. The following proposition can be directly verified.

Proposition 2.24. *A k-differential on a given Lie algebroid A is equivalent to a derivation of degree of $k-1$ of the associated Gerstenhaber algebra $(\bigoplus \Gamma(\bigwedge^\bullet A), [\![\cdot, \cdot]\!], \wedge)$, i.e., a linear operator $\delta : \Gamma(\bigwedge^\bullet A) \to \Gamma(\bigwedge^{\bullet+k-1} A)$ satisfying*

$$\begin{aligned} \delta(P \wedge Q) &= (\delta P) \wedge Q + (-1)^{p(k+1)} P \wedge \delta Q, \\ \delta[\![P, Q]\!] &= [\![\delta P, Q]\!] + (-1)^{(p+1)(k+1)} [\![P, \delta Q]\!], \end{aligned} \tag{2.14}$$

for all $P \in \Gamma(\bigwedge^p A)$ and $Q \in \Gamma(\bigwedge^q A)$.

As we see below, k-differentials reduce to various well-known notions in special cases.

Example 2.25. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. A k-differential on $\mathfrak{g}$ is just a linear map $\delta : \mathfrak{g} \to \bigwedge^k \mathfrak{g}$ such that

$$\delta[X, Y]_{\mathfrak{g}} = \mathrm{ad}_X(\delta(Y)) - \mathrm{ad}_Y(\delta(X)) \quad \text{for } X, Y \in \mathfrak{g},$$

that is, δ is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation ad of $\mathfrak{g}$ on $\bigwedge^k \mathfrak{g}$ [23, Prop. 2.14].

Example 2.26. Let δ be a 0-differential, that is, $\delta f = 0$ for $f \in C^\infty(M)$, and $\delta X \in C^\infty(M)$ for $X \in \Gamma(A)$. From (ii) in Definition 2.23, we deduce that $\delta(fX) = f\delta X$. Therefore, there exists $\phi \in \Gamma(A^*)$ such that

$$\delta X = \phi(X) \quad \text{for } X \in \Gamma(A). \tag{2.15}$$

Moreover, using (2.12), we find that ϕ is a 1-cocycle in the Lie algebroid cohomology of A. Thus, 0-differentials are just Lie algebroid 1-cocycles with trivial coefficients.

Example 2.27. Let δ be an almost 1-differential, that is, $\delta f \in C^\infty(M)$ for $f \in C^\infty(M)$, and $\delta X \in \Gamma(A)$ for $X \in \Gamma(A)$. From (i) in Definition 2.23, we deduce that there exists $X_0 \in \mathfrak{X}(M)$ such that

$$\delta f = X_0(f) \quad \text{for } f \in C^\infty(M). \tag{2.16}$$

Moreover, using (ii), we obtain

$$\delta(fX) = f\delta(X) + X_0(f)X,$$

that is, δ is a covariant differential operator on A, with anchor X_0 (see [26, 28]). If, moreover, δ is a 1-differential, from (2.12) we see that the covariant differential operator δ is a derivation of the bracket on $\Gamma(A)$.

Example 2.28. Assume that $A \to M$ is a Lie algebroid whose dual vector bundle $A^* \to M$ is also furnished with a Lie algebroid structure. The pair (A, A^*) is said to be a *Lie bialgebroid* if

$$\mathrm{d}_{A^*}[\![X, Y]\!] = [\![X, \mathrm{d}_{A^*}Y]\!] - [\![Y, \mathrm{d}_{A^*}X]\!] \quad \text{for } X, Y \in \Gamma(A),$$

where d_{A^*} is the Lie algebroid differential associated to A^* (see [27]). It is well-known that a Lie algebroid structure on a vector bundle $V \to M$ is equivalent to an almost 2-differential $\delta : \Gamma(\bigwedge^\bullet V^*) \to \Gamma(\bigwedge^{\bullet+1} V^*)$ of square 0 (see, for instance, [20, 42]). Thus, we see that a Lie bialgebroid corresponds to a 2-differential of square 0 on a Lie algebroid A.

Example 2.29. If $P \in \Gamma(\bigwedge^k A)$, then $\mathrm{ad}(P) = [\![P, \cdot]\!]$ is clearly a k-differential, which is called the *coboundary* k-differential associated to P.

An easy but long computation shows that the space of almost differentials can be endowed with a graded Lie algebra structure.

Proposition 2.30. *Let $\hat{\mathcal{A}}_k$ denote the space of almost k-differentials and $\hat{\mathcal{A}} = \bigoplus_k \hat{\mathcal{A}}_k$. Define*

$$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - (-1)^{(k+1)(l+1)} \delta_2 \circ \delta_1 \tag{2.17}$$

for $\delta_1 \in \hat{\mathcal{A}}_k$ and $\delta_2 \in \hat{\mathcal{A}}_l$. Then

(i) $[\delta_1, \delta_2] \in \hat{\mathcal{A}}_{(k+l-1)}$;

(ii) $(\hat{\mathcal{A}}, [\cdot, \cdot])$ *is a graded Lie algebra.*

Moreover, the subspace $\mathcal{A} = \bigoplus_k \mathcal{A}_k$ of all differentials is a graded Lie subalgebra.

Remark 2.31. From (2.14) and (2.17), we deduce that

$$[\delta, \mathrm{ad}(P)] = \mathrm{ad}(\delta P) \tag{2.18}$$

for any k-differential $\delta \in \mathcal{A}$ and any multisection $P \in \Gamma(\bigwedge^\bullet A)$.

To end this section, we note that one can introduce a graded Lie algebra structure on $\hat{\mathcal{A}} \oplus \Gamma(\bigwedge A)$, where $\Gamma(\bigwedge A) = \bigoplus_k \Gamma(\bigwedge^k A)$. This is defined by

$$[(\delta_1, P_1), (\delta_2, P_2)] = ([\delta_1, \delta_2], \delta_1(P_2) - \delta_2(P_1))$$

for $(\delta_1, P_1), (\delta_2, P_2) \in \hat{\mathcal{A}} \oplus \Gamma(\bigwedge A)$.

Note that this is the semi-direct product Lie bracket when we consider the natural representation of $\hat{\mathcal{A}}$ on $\Gamma(\bigwedge A)$.

Lemma 2.32. *If δ is a k-differential on a Lie algebroid A, then there exists a k-vector field π_M on M given by*

$$\pi_M(\mathrm{d}f_1, \ldots, \mathrm{d}f_k) = (-1)^{k+1} \langle \rho(\delta f_1), \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_k \rangle \quad \textit{for } f_1, \ldots, f_k \in C^\infty(M),$$

where $\rho : \Gamma(\bigwedge^k A) \to \mathfrak{X}^k(M)$ is the natural extension of the anchor $\rho : \Gamma(A) \to \mathfrak{X}(M)$.

Proof. Since $\delta f \in \Gamma(\bigwedge^{k-1} A)$, we have, for any $i \geq 2$,

$$\begin{aligned}\{f_1, f_2, \ldots, f_i, f_{i+1}, \ldots, f_k\} &= (-1)^{k+1} \langle \rho(\delta f_1), \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_i \wedge \mathrm{d}f_{i+1} \wedge \cdots \wedge \mathrm{d}f_k \rangle \\ &= -(-1)^{k+1} \langle \rho(\delta f_1), \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_{i+1} \wedge \mathrm{d}f_i \wedge \cdots \wedge \mathrm{d}f_k \rangle \\ &= -\{f_1, f_2, \ldots, f_{i+1}, f_i, \ldots, f_k\}.\end{aligned}$$

On the other hand, since δ is a k-differential, using (2.14) we find that, for all $f, g \in C^\infty(M)$,

$$0 = \delta[\![f, g]\!] = [\![\delta f, g]\!] + (-1)^{k+1} [\![f, \delta g]\!] = (-1)^k \mathrm{d}_A g \lrcorner \delta f + (-1)^k \mathrm{d}_A f \lrcorner \delta g,$$

where d_A is the differential of the Lie algebroid A. Thus for any $f_1, \ldots, f_k \in C^\infty(M)$,

$$\begin{aligned}0 &= (-1)^{k+1} \langle \mathrm{d}f_2 \lrcorner \rho(\delta f_1), \mathrm{d}f_3 \wedge \cdots \wedge \mathrm{d}f_k \rangle + (-1)^{k+1} \langle \mathrm{d}f_1 \lrcorner \rho(\delta f_2), \mathrm{d}f_3 \wedge \cdots \wedge \mathrm{d}f_k \rangle \\ &= \{f_1, f_2, \ldots, f_k\} + \{f_2, f_1, \ldots, f_k\}.\end{aligned}$$

Therefore $\{\cdot, \ldots, \cdot\}$ is indeed skew-symmetric. Moreover, from the fact that both δ and d are derivations, we can deduce that $\{\cdot, \ldots, \cdot\}$ is a derivation with respect to each argument. That is, $\{\cdot, \ldots, \cdot\}$ induces a k-vector field $\pi_M \in \mathfrak{X}^k(M)$. □

Example 2.33. If $P \in \Gamma(\bigwedge^k A)$ and $\mathrm{ad}(P)$ is the coboundary k-differential associated to P, then the corresponding k-vector field on M is just $\rho(P)$.

2.5. From multiplicative k-vector fields to k-differentials

Assume that Π is a multiplicative k-vector field on Γ. For any $f \in C^\infty(M)$ and $X \in \Gamma(A)$, it is known from Propositions 2.17, 2.18 and 2.20 that $[\Pi, \alpha^* f]$ and $[\Pi, \overrightarrow{X}]$ are right-invariant, where $\overrightarrow{X}$ denotes the right-invariant vector field on Γ corresponding to $X \in \Gamma(A)$. Therefore there exist $\delta_\Pi f \in \Gamma(\bigwedge^{k-1} A)$ and $\delta_\Pi X \in \Gamma(\bigwedge^k A)$ such that

$$\overrightarrow{\delta_\Pi f} = [\Pi, \alpha^* f], \quad \forall f \in C^\infty(M), \qquad \overrightarrow{\delta_\Pi X} = [\Pi, \overrightarrow{X}], \quad \forall X \in \Gamma(A). \tag{2.19}$$

It is easy to see that δ_Π is indeed a k-differential. We are now ready to state the main theorem of the paper.

Theorem 2.34. *Assume that $\Gamma \rightrightarrows M$ is an α-simply connected and α-connected Lie groupoid with Lie algebroid A. Then the map*

$$\delta : \bigoplus_k \mathfrak{X}^k_{\mathrm{mult}}(\Gamma) \to \bigoplus_k \mathcal{A}_k, \qquad \Pi \mapsto \delta_\Pi,$$

is a graded Lie algebra isomorphism.

We divide the proof into several steps. The proof of the surjectivity of δ will be postponed to Section 3. Here we prove the following result.

Proposition 2.35. *Under the hypothesis of Theorem 2.34, δ is an injective graded Lie algebra homomorphism.*

Proof. Using the graded Jacobi identity for the Schouten brackets and (2.17), we deduce that if $\Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\Gamma)$ and $\Pi' \in \mathfrak{X}^l_{\mathrm{mult}}(\Gamma)$ then $\delta_{[\Pi,\Pi']} = [\delta_\Pi, \delta_{\Pi'}]$. Therefore, δ is a graded Lie algebra homomorphism.

Next, let us prove that δ is injective. We will use the following lemma (see Theorem 2.6 in [29]):

Lemma 2.36. *If Π is an affine multi-vector field on an α-connected Lie groupoid $\Gamma \rightrightarrows M$, then $\Pi = 0$ if and only if $\delta_\Pi X = 0$ for all $X \in \Gamma(A)$, and Π vanishes on the unit space M.*

Suppose that Π is a multiplicative k-vector field on Γ such that $\delta_\Pi = 0$. It remains to show that $\Pi_{|M} = 0$. We know that $T^*_{\epsilon(m)}\Gamma$ is spanned by the differentials of functions of the type $\alpha^* f$ with $f \in C^\infty(M)$, and the differentials of functions F which are constant along M, i.e., such that $\mathrm{d}f \in N^*M$. Since M is coisotropic, we get

$$\Pi(\epsilon(m))(\mathrm{d}F_1, \dots, \mathrm{d}F_k) = 0.$$

Moreover,

$$i_{\alpha^* f}\Pi = (-1)^{k+1}[\Pi, \alpha^* f] = (-1)^{k+1}\delta_\Pi(f) = 0,$$

which implies that

$$\Pi(\epsilon(m))(\alpha^* f_1, \dots, \alpha^* f_j, \mathrm{d}F_1, \dots, \mathrm{d}F_l) = 0$$

for $j + l = k$ and $j \geq 1$. Therefore, $\Pi_{|M} = 0$.

From Lemma 2.36, it follows that $\Pi = 0$. Thus δ is injective. □

Following [24], we also call δ_Π the *inner derivative* of Π.

Theorem 2.34 has many interesting corollaries. Below is a list of well-known results which are in the literature.

Example 2.37. Let G be a simply connected and connected Lie group with Lie algebra $\mathfrak{g}$ and $\Pi \in \mathfrak{X}^k_{\text{mult}}(G)$. Then $\delta_\Pi : \mathfrak{g} \to \bigwedge^k \mathfrak{g}$ is the 1-cocycle obtained by taking the inner derivative of Π [24]. Thus we have a one-to-one correspondence between 1-cocycles $\mathfrak{g} \to \bigwedge^* \mathfrak{g}$ and multiplicative multi-vector fields on G (see [23, Prop. 2.14]).

Example 2.38. If σ is a multiplicative function on Γ, i.e., $\sigma : \Gamma \to \mathbb{R}$ is a groupoid 1-cocycle, then $\delta_\sigma \in \Gamma(A^*)$ is exactly the corresponding Lie algebroid 1-cocycle. Thus for an α-connected and α-simply connected Lie groupoid, there is a one-to-one correspondence between groupoid 1-cocycles $\sigma : \Gamma \to \mathbb{R}$ and Lie algebroid 1-cocycles $\delta \in \Gamma(A^*)$. This result was first proved in [40].

Example 2.39. Multiplicative vector fields on a Lie groupoid are exactly infinitesimals of Lie groupoid automorphisms. 1-differentials on a Lie algebroid, on the other hand, are covariant differential operators on A which are derivations with respect to the bracket. These are exactly infinitesimals of the Lie algebroid automorphisms. Thus for an α-connected and α-simply connected Lie groupoid, we have a one-to-one correspondence between infinitesimals of the Lie groupoid automorphisms and infinitesimals of the corresponding Lie algebroid automorphisms [28, Prop. 3.8 and Thm. 4.9].

Example 2.40. Let $P \in \Gamma(\bigwedge^k A)$ be a k-section of A, and $\Pi = \overrightarrow{P} - \overleftarrow{P}$ the corresponding multiplicative k-vector field on Γ. From the definition of δ_Π, we see that it is just the coboundary k-differential $\mathrm{ad}(P) = [\![P, \cdot]\!]$. Thus for an α-connected and α-simply connected Lie groupoid, there is a one-to-one correspondence between coboundary multiplicative multi-vector fields on the Lie groupoid and coboundary k-differentials on its Lie algebroid.

Example 2.41. Let $(\Gamma \rightrightarrows M, \Pi)$ be a Poisson groupoid, i.e., $\Pi \in \mathfrak{X}^2_{\text{mult}}(\Gamma)$ such that $[\Pi, \Pi] = 0$. From Theorem 2.34, there exists a 2-differential δ_Π on A. Moreover,

$$\delta_\Pi^2 = \delta_\Pi \circ \delta_\Pi = \tfrac{1}{2}[\delta_\Pi, \delta_\Pi] = \tfrac{1}{2}\delta_{[\Pi,\Pi]} = 0.$$

Thus, δ_Π defines a Lie algebroid structure on A^*. Moreover, since $\delta_\Pi[\![X, Y]\!] = [\![\delta_\Pi X, Y]\!] + [\![X, \delta_\Pi Y]\!]$ for all $X, Y \in \Gamma(A)$, we deduce that (A, A^*) is a Lie bialgebroid. As a consequence, we obtain the integration theorem of Mackenzie–Xu [29, Thm. 4.1]: there is a one-to-one correspondence between α-connected and α-simply connected Poisson groupoids and Lie bialgebroids.

3. Lifting of k-differentials

This section is devoted to the proof of the surjectivity of δ in Theorem 2.34.

3.1. A-paths

From now on, we use the notation $I = [0, 1]$. Let $(A, [\![\cdot, \cdot]\!], \rho)$ be the Lie algebroid of an α-connected and α-simply connected Lie groupoid Γ. Following [11], we denote by

$\widetilde{P}(A)$ the Banach manifold of all C^1-paths in A. A C^1-path $a : I \to A$ is said to be an *A-path* if

$$\rho(a(t)) = \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}, \tag{3.20}$$

where $\gamma(t) = (p \circ a)(t)$ is the base path ($p : A \to M$ is the bundle projection). The set of A-paths, denoted by $P(A)$, is a Banach submanifold of $\widetilde{P}(A)$.

It is well-known that integrating along A-paths yields a Γ-path [11, Prop 1.1]. We recall this construction in order to be self-contained. Following [11], a Γ*-path* is a C^2-path $r(t)$ on the groupoid Γ such that $r(0) \in M$ and $\alpha(r(t)) = r(0)$ for all $t \in I$.

There is a diffeomorphism $\mathfrak{I}$ from the space of A-paths to the space of Γ-paths. For any $a \in P(A)$, we define $\mathfrak{I}(a)$ to be the solution $r(t)$ of the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}r(t)}{\mathrm{d}t} = \overrightarrow{a(t)}_{r(t)}, \\ r(0) = \gamma(0), \end{cases} \tag{3.21}$$

where $\gamma = p \circ a$ is the base path. Conversely, for any Γ-path $r(t)$, the corresponding A-path is given by $a(t) = L_{r^{-1}(t)*} \frac{\mathrm{d}r(t)}{\mathrm{d}t}$.

The purpose of this section is to study properties of the smooth map τ from $P(A)$ to Γ given by

$$\tau(a) = \mathfrak{I}(a)(1). \tag{3.22}$$

First, we study the covariance of τ. Recall that local bisections of $\Gamma \rightrightarrows M$ act on Γ by

$$r \mapsto \widetilde{g}^{-1} \cdot r \cdot \widetilde{g}$$

for any bisection $\widetilde{g}$ and $r \in \Gamma$ (this last expression makes sense provided that the local bisection is chosen so that the above products are defined). Differentiating with respect to r, one obtains an automorphism of the Lie algebroid $A \to M$, denoted by $\mathrm{Ad}_{\widetilde{g}^{-1}} : A_m \to A_{\widetilde{g}^{-1}\cdot m}$. Since the Lie algebra of the group of bisections (over a contractible open subset $U \subset M$) is the Lie algebra of sections of A (over U), by differentiating furthermore with respect to $\widetilde{g}$, one constructs, for all $\xi \in \Gamma(A)$ and $b \in A$, an element in T_bA, which we denote by $\mathrm{ad}_\xi\, b \in T_bA$. The reader should not confuse $b \mapsto \mathrm{ad}_\xi\, b$, which is a tangent vector on A, with the adjoint action of the Lie algebra $\Gamma(A)$ on itself.

For any Γ-path $r(t)$ and any C^2-path $g(t)$ in Γ, such that $r(t)$ and $g(t)$ are composable for all $t \in I$ (i.e. $\beta(r(t)) = \alpha(g(t))$), the path $g(0)^{-1} \cdot r(t) \cdot g(t)$ is a Γ-path again. It is easy to check that $\mathfrak{I}^{-1}(g(0)^{-1} \cdot r(t) \cdot g(t))$ is equal to $\mathrm{Ad}_{\widetilde{g}_t}\, \mathfrak{I}^{-1}(r(t)) + \big(R_{\widetilde{g}_t^{-1}*} \frac{\mathrm{d}\widetilde{g}_t}{\mathrm{d}t}\big)_{|\widetilde{g}_t \gamma(t)}$, where $\widetilde{g}_t$ is, for all $t \in I$, a (local) bisection of $\Gamma \rightrightarrows M$ through $g(t)$ (defined in a neighborhood of $g(t)$). In short, for any $a \in P(A)$, and any time-dependent (local) bisection $\widetilde{g}_t$ through $g(t)$, with a C^2-dependence on t,

$$\tau\left(\mathrm{Ad}_{\widetilde{g}_t}\, a(t) + \left(R_{\widetilde{g}_t^{-1}*} \frac{\mathrm{d}\widetilde{g}_t}{\mathrm{d}t}\right)_{|\widetilde{g}_t \cdot \gamma(t)}\right) = g(0)^{-1} \cdot \tau(a) \cdot g(1). \tag{3.23}$$

For any A-paths $a_1(t)$ and $a_2(t)$, if $\tau(a_1) = \tau(a_2)$, then $\Im(a_1) = \Im(a_2) \cdot g(t)$ for some path $g(t)$ on Γ such that $g(0), g(1) \in M$. Hence, it follows from (3.23) that $\tau(a_1) = \tau(a_2)$ if and only if there exists a time-dependent bisection $\widetilde{g_t}$ such that

$$a_2(t) = \mathrm{Ad}_{\widetilde{g_t}}\, a_1(t) + R_{\widetilde{g_t}^{-1}(t)*} \frac{\mathrm{d}\widetilde{g_t}}{\mathrm{d}t}_{|\widetilde{g_t}\cdot\gamma(t)}$$

and $\widetilde{g_0}$ and $\widetilde{g_1}$ are unital elements of the pseudo-group of local bisections.

Since Γ is α-simply connected, the pseudo-group of time-dependent local bisections $\widetilde{g_t}$ is connected. Therefore, $\tau(a_1) = \tau(a_2)$ if and only if a_1 and a_2 can be linked by a differentiable path in $P(A)$ which is tangent to the differential of the action described by (3.23). But vectors in $T_a P(A)$ tangent to this action are precisely the vectors of the form, at a given A-path $a(t)$ with base path $\gamma(t)$,

$$(G_\xi)_{|a} : t \mapsto \mathrm{ad}_{\xi(t)}\, a(t) + \frac{\mathrm{d}\xi(t)}{\mathrm{d}t}_{|\gamma(t)}, \tag{3.24}$$

where $\xi(t)$ is a C^2-time-dependent section of $A \to M$ with $\xi(0) = \xi(1) = 0$, and $\frac{\mathrm{d}\xi(t)}{\mathrm{d}t}_{|\gamma(t)}$, an element of $A_{\gamma(t)}$, is considered as an element of $T_{a(t)}A$.

For any C^2-time-dependent section $\xi(t)$ of $A \to M$ with $\xi(0) = \xi(1) = 0$, the vector field G_ξ on $P(A)$ given by (3.24) is called a *gauge vector field*. A smooth function $f : P(A) \to \mathbb{R}$ is said to be *invariant under the gauge transformation* if $G_\xi(f) = 0$ for any G_ξ of the form described by (3.24) with $\xi(0) = \xi(1) = 0$. The following proposition summarizes the above discussion.

Proposition 3.1. *Assume that Γ is an α-connected and α-simply connected Lie groupoid. Then the map $\tau : P(A) \to \Gamma$ induces an isomorphism between $\mathcal{C}^\infty(\Gamma)$ and the algebra of smooth functions on $P(A)$ invariant under the gauge transformation.*

Note that it follows from (3.23)–(3.24) that for any time-dependent section $\xi(t)$ of $\Gamma(A)$, we have

$$\tau_*\left(\mathrm{ad}_{\xi(t)}\, a(t) + \frac{\mathrm{d}\xi(t)}{\mathrm{d}t}_{|\gamma(t)}\right) = \overrightarrow{\xi(1)} - \overleftarrow{\xi(0)}. \tag{3.25}$$

This relation will be useful later on.

Next we need to introduce regular extensions to $\widetilde{P}(A)$ of the 1-form $\mathrm{d}\tau^* f$ for a smooth function f on Γ. First, we give some definitions related to the cotangent spaces of $P(A)$ and $\widetilde{P}(A)$.

For any $a \in \widetilde{P}(A)$ (resp. $P(A)$), the cotangent space of $\widetilde{P}(A)$ at a is denoted by $T_a^*\widetilde{P}(A)$ (resp. $T_a^* P(A)$). For any real-valued function f on $\widetilde{P}(A)$ (resp. $P(A)$), its differential at the point $a \in \widetilde{P}(A)$ (resp. $a \in P(A)$), if it exists, is an element of $T_a^*\widetilde{P}(A)$ (resp. $T_a^* P(A)$), which is denoted by $\mathrm{d}f_{|a}$.

For any $a \in \widetilde{P}(A)$, denote by $P_a(T^*A)$ the space of C^1-maps $\eta : I \to T^*A$ such that for all $t \in I$ the identity $\pi \circ \eta(t) = a(t)$ holds, where π stands for the projection

$T^*A \to A$. The space $P_a(T^*A)$ can be considered as a subspace of $T_a^*\widetilde{P}(A)$ in a natural manner: to every $\eta(t) \in P_a(T^*A)$, one can associate a linear form on $T_a\widetilde{P}(A)$ by

$$T_a\widetilde{P}(A) \to \mathbb{R}, \qquad X(t) \mapsto \int_0^1 \langle \eta(t), X(t)\rangle \,\mathsf{d}t, \tag{3.26}$$

where $\langle\ ,\ \rangle$ denotes the pairing between the cotangent and tangent vectors. With this identification, throughout this section, we will always consider $P_a(T^*A)$ as a subspace of $T_a^*\widetilde{P}(A)$, and therefore, the vector bundle $P(T^*A) \to \widetilde{P}(A)$ as a vector subbundle of $T^*\widetilde{P}(A) \to \widetilde{P}(A)$. We denote by $P(T^*A)_{|P(A)} \to P(A)$ and $T^*\widetilde{P}(A)_{|P(A)} \to P(A)$ the restrictions of these vector bundles to $P(A)$.

Definition 3.2. Given a 1-form ω on the Banach submanifold $P(A)$ of A-paths, by an *extension* (resp. *regular extension*), we mean a smooth section Φ_ω of $T^*\widetilde{P}(A)_{|P(A)} \to P(A)$ (resp. $PT^*A_{|P(A)} \to P(A)$) such that $\Phi_{\omega_{|a}}$ is, for any $a \in P(A)$, an extension of $\omega_{|a}$ (i.e., the restriction of $\Phi_{\omega_{|a}}$ to $T_aP(A)$ is $\omega_{|a}$ for any $a \in P(A)$).

By a *regular extension of a smooth function* $g \in C^\infty(P(A))$, we mean a regular extension of its differential. A regular extension of the zero function on $P(A)$ will be called a *regular extension of zero*. Also, we use the following notation: for any regular extension Φ_ω, we denote by $\Phi_\omega(t)$ the corresponding path in P_aT^*A.

Given a vector field X on $\widetilde{P}(A)$ tangent to $P(A)$, and a 1-form ω on $P(A)$, the Lie derivative of a regular extension Φ_ω of ω is defined by

$$(\mathcal{L}_X\Phi_\omega)(Y) = \Phi_\omega([Y, X]) + X(\Phi_\omega(Y)), \tag{3.27}$$

where Y is the restriction to $P(A)$ of a vector field on $\widetilde{P}(A)$. This definition needs to be justified. It is clear that the right hand side of (3.27) is C^∞-linear with respect to Y and depends only on its restriction to $P(A)$. It therefore defines a section of $T^*\widetilde{P}(A)_{|P(A)} \to P(A)$. It follows from (3.27) that if Y itself is tangent to $P(A)$, then $(\mathcal{L}_X\Phi_\omega)(Y) = (\mathcal{L}_X\omega)(Y)$. Therefore, $\mathcal{L}_X\Phi_\omega$ is an extension of $\mathcal{L}_X\omega$.

In the following subsections, we need to investigate regular extensions $\Phi_{\mathsf{d}\tau^*f}$ of $\mathsf{d}\tau^*f$ for a smooth function $f \in C^\infty(\Gamma)$. The following technical lemma will be very useful.

Lemma 3.3. (i) *For any smooth function $f : \Gamma \to \mathbb{R}$, the pull-back function $\tau^*f : P(A) \to \mathbb{R}$ admits a regular extension $\Phi_{\mathsf{d}\tau^*f}$.*

(ii) *For any smooth function $f : \widetilde{P}(A) \to \mathbb{R}$ whose restriction to $P(A)$ vanishes, there exists $g_t : I \to C^\infty(M)$ with $g_0 = g_1 = 0$ such that $\mathsf{d}f_{|a} = \mathsf{d}\mathcal{F}_{\mathsf{d}g_{|a}}$. Here for any time dependent 1-form $\omega : t \mapsto \omega_t$ on M and any $a \in \widetilde{P}(A)$ with base path $\gamma(t)$,*

$$\mathcal{F}_\omega(a) = \int_I \left\langle \omega_{t_{|\gamma(t)}}, \frac{\mathsf{d}\gamma(t)}{\mathsf{d}t} - \rho(a)\right\rangle \mathsf{d}t.$$

Proof. (i) We divide the proof of (i) into four steps. In what follows, γ always denotes the base path of $a \in P(A)$. We say that a path $e(t)$ in a vector bundle $p : E \to M$ is *over a base path* γ if $p \circ e = \gamma$.

Step 1. We first describe the differential τ_* of the map τ defined in (3.22).

The map $(r, a) \mapsto \overrightarrow{a}_{|r}$ from the fibered product $\Gamma \times_{\alpha, M, p} A$ to $T\Gamma$ maps a pair (r, a) (with $r \in \Gamma$, $a \in A$) to an element of $T_r\Gamma$. Considering the fibered product $\Gamma \times_{\alpha, M, p} A$ as a submanifold of $\Gamma \times A$, one can extend this map to a smooth map $F(\alpha, a)$ from $\Gamma \times A$ to $T\Gamma$ with $F(r, a) \in T_r\Gamma$ for all $r \in \Gamma$. This allows us to rewrite τ more conveniently. Consider the map $\widetilde{\tau} : \widetilde{P}(A) \times M \to \Gamma$ given by

$$\widetilde{\tau}(a, m) = r(1), \tag{3.28}$$

where $r : I \to \Gamma$ is the solution of the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}r(t)}{\mathrm{d}t} = F(r(t), a(t)), \\ r(0) = m. \end{cases}$$

By definition, we have $\tau(a) = \widetilde{\tau}(a, (p \circ a)(0))$.

Linearizing the previous equation shows that the differential of the map $\widetilde{\tau}$ can be written as

$$\widetilde{\tau}_*(\delta a, \delta m) = L(\delta m(0)) + \int_I L_t(\delta a(t))\,\mathrm{d}t, \tag{3.29}$$

where $\delta a, \delta m$ are elements of $T_a\widetilde{P}(A)$ and $T_m M$ respectively, and L_t, L are linear maps from $T_{r(t)}\Gamma$ and $T_m M$ to $T_{\widetilde{\tau}(a,m)}\Gamma$ respectively. Thus the differential of $\tau : P(A) \to \Gamma$ can be expressed as

$$\tau_*(\delta a) = \int_I L_t(\delta a(t))\,\mathrm{d}t + L(p_*(\delta a(0))) \tag{3.30}$$

where $\delta a \in T_a P(A)$.

At this point, we need to explain why some technical difficulties arise in the construction of a regular extension of $\tau^* f$ and what remains to be done in order to avoid it.

It follows from (3.30) that the differential $\mathrm{d}\tau^* f = \mathrm{d}f \circ \tau_*$ is the sum of two 1-forms, namely $\omega_1 : \delta a \mapsto \int_I L_t(\delta a(t))\,\mathrm{d}t$ and $\omega_2 : \delta a \mapsto L(p_*(\delta a(0)))$. It is easy to find a regular extension of ω_1: we can just choose, for any $\delta a \in T_a\widetilde{P}(A)$,

$$\Phi_{\omega_1}(\delta a) = \int_I L_t(\delta a(t))\,\mathrm{d}t.$$

Unfortunately, it is not so easy to find a regular extension of ω_2, since this 1-form is "concentrated" at 0. The remaining steps describe such an extension.

Step 2. We describe explicitly the tangent space $T_a P(A)$ of $P(A)$.

Let us choose a connection ∇^A on the vector bundle $A \to M$. This allows us to decompose the tangent space of A as a direct sum $T_b A = T_{p(b)}M \oplus A_{p(b)}$ for any $b \in A$. With this convention, for any $a \in \widetilde{P}(A)$, an element δa of $T_a\widetilde{P}(A)$ becomes a pair $\delta a = (\epsilon, \beta)$ where $\epsilon : I \to TM$ and $\beta : I \to A$ are C^1-maps over the base path $\gamma = p \circ a$.

We now choose a connection ∇^M on the tangent bundle $TM \to M$. By differentiating the relation $\mathrm{d}\gamma/\mathrm{d}t = \rho(a)$, we see that $\delta a = (\epsilon, \beta) \in T_a\widetilde{P}(A)$ is an element of the tangent space of $T_a P(A)$ if and only if

$$\nabla^M_{\mathrm{d}\gamma/\mathrm{d}t}\epsilon = (\nabla_\epsilon \rho)(a) + \rho(\beta), \tag{3.31}$$

where $(\nabla_\epsilon\rho)(a) = \nabla^M_\epsilon \rho(a) - \rho(\nabla^A_\epsilon a)$ is a path in TM (over γ again).

Step 3. For any A-path a, we want to construct a linear map Π_a from $T_a\widetilde{P}(A)$ to $T_{\gamma(0)}M$, where $\gamma = p \circ a$ is the base path, depending smoothly on $a \in P(A)$, whose restriction to $T_a P(A)$ is simply the differential of the map $a \mapsto \gamma(0)$, and which is "given by an integral". That is, the differential is of the form

$$\Pi_a(\delta a) = \int_I M_t(\delta a(t))\,\mathrm{d}t, \tag{3.32}$$

where M_t is, for all $t \in I$, a linear map from $T_{a(t)}A$ to $T_{\gamma(0)}M$. We proceed as follows. Set $\delta a = (\epsilon, \beta) \in T\widetilde{P}(A)$ as in Step 2.

First, for any $s \in I$, we define $\eta_s(t) : I \to TM$ to be the unique solution of the initial value problem

$$\begin{cases} \nabla^M_{\mathrm{d}\gamma(t)/\mathrm{d}t}\eta_s(t) = (\nabla_{\eta_s(t)}\rho)(a) + \rho(\beta(t)), \\ \eta_s(s) = \epsilon(s) \end{cases} \tag{3.33}$$

(this is a linear equation of order 1, which guarantees the existence and uniqueness of solution). Then we define $\Pi_a(\epsilon, \beta)$ by

$$\Pi_a(\epsilon, \beta) = \int_I \eta_s(0)\,\mathrm{d}s.$$

It follows from a classical result of ordinary linear differential equations (see, for instance, [32]) that

$$\eta_s(0) = L(s)(\epsilon(s)) + \int_s^0 M(s, u)(\beta(u))\,\mathrm{d}u$$

for some smooth function $M(s, u)$ from $A_{\gamma(u)}$ to $T_{\gamma(0)}M$. It is easy to check that for any $\delta a = (\epsilon, \beta)$,

$$\Pi_a(\delta a) = \int_I L(s)(\epsilon(s))\,\mathrm{d}s - \int_I \int_0^s M(s, u)(\beta(u))\,\mathrm{d}u\,\mathrm{d}s.$$

The right hand side of this equation is of the form given by (3.32). Now it remains to check that the restriction of Π_a to $TP(A)$ is equal to the map $\delta a \mapsto p_*(\delta a(0))$. For any $\delta = (\epsilon, \beta)$ tangent to $P(A)$, by the uniqueness of solution of the initial value problem, we have $\eta_s = \epsilon$ for all $s \in I$. Therefore, the restriction of Π_a to $T_a P(A)$ is equal to

$$\Pi_a(\delta a) = \int_I \epsilon(0)\,\mathrm{d}t = \epsilon(0) = p_*(\delta a(0)).$$

Step 4. We can now define, for any $f \in C^\infty(M)$,

$$\Phi_{\mathrm{d}\tau^* f}(\delta a) := \widetilde{\tau}_*(\delta a, \Pi_a(\delta a)).$$

It follows from (3.28)–(3.32) that $\Phi_{\mathrm{d}\tau^* f}$ is a regular extension of $\tau^* f$. This completes the proof of (i).

(ii) We identify $\delta a \in T_a\widetilde{P}(A)$ with a pair of paths $\epsilon(t)$ in TM and $\beta(t)$ in A, over the same base path $\gamma(t)$ as previously.

Since $\Phi_{\mathrm{d}\tau^* f}$ is a regular extension, we have

$$\Phi_{\mathrm{d}\tau^* f}(\epsilon, \beta) = \int_I \langle M(t), \epsilon(t)\rangle \, \mathrm{d}t + \int_I \langle A(t), \beta(t)\rangle \, \mathrm{d}t$$

for some C^1-maps $M(t)$, $A(t)$ from I to T^*M and A^* respectively, which are over the base path $\gamma(t)$.

Let $\omega : t \mapsto T^*_{\gamma(t)}M$ be a path over γ which is a solution of the initial value problem

$$\begin{cases} \nabla^M_{\mathrm{d}\gamma(t)/\mathrm{d}t}\omega(t) + ((\nabla\rho)a(t))^*(\omega(t)) = M(t), \\ \omega(1) = 0, \end{cases}$$

where, for any fixed $t \in I$, $((\nabla\rho)a(t))^* \in \mathrm{End}(T^*_{\gamma(t)}M)$ is the dual of the endomorphism $v \mapsto (\nabla_v\rho)(a(t))$ of $T_{\gamma(t)}M$.

Using integration by parts, we obtain

$$\begin{aligned} \int_I \langle M(t), \epsilon(t)\rangle \, \mathrm{d}t = &- \int_I \langle \omega(t), \nabla^M_{\mathrm{d}\gamma(t)/\mathrm{d}t}\epsilon(t)\rangle \, \mathrm{d}t \\ &+ \int_I \langle \omega(t), (\nabla_{\epsilon(t)}\rho)(a(t))\rangle \, \mathrm{d}t + \langle \omega(0), \epsilon(0)\rangle. \end{aligned}$$

Since $\Phi_{\mathrm{d}\tau^* f}(\epsilon, \beta)$ vanishes as long as the conditions

$$\begin{cases} \beta(t) = 0, \\ \nabla^M_{\mathrm{d}\gamma(t)/\mathrm{d}t}\epsilon(t) - (\nabla_{\epsilon(t)}\rho)a(t) = 0 \end{cases}$$

are satisfied, we must have $\omega(0) = 0$.

Now, since

$$\Phi_{\mathrm{d}\tau^* f}(\epsilon, \beta) = \int_I \langle \omega(t), \nabla^M_{\mathrm{d}\gamma(t)/\mathrm{d}t}\epsilon(t) + (\nabla_{\epsilon(t)}\rho)a(t)\rangle \, \mathrm{d}t + \int_I \langle A(t), \beta(t)\rangle \, \mathrm{d}t$$

must vanish whenever ϵ, β satisfy (3.31), we must have $A(t) = \rho^*\omega(t)$. As a consequence,

$$\Phi_{\mathrm{d}f}(\delta a) = \int_I \langle \omega(t), \nabla^M_{\mathrm{d}\gamma(t)/\mathrm{d}t}\epsilon(t) - (\nabla_{\epsilon(t)}\rho)a(t) + \rho(\beta(t))\rangle \, \mathrm{d}t.$$

According to Lemma 3.4(i), there exists a family of time-dependent functions g_t from I to $C^\infty(M)$ vanishing at $t = 0, 1$ such that $\mathrm{d}g_{t_{|\gamma(t)}} = \omega(t)$. The result now follows from (ii) of the lemma below. □

Lemma 3.4. (i) *For any C^1-path $\omega : I \to T^*M$, there exists a time-dependent function $g : t \mapsto g_t$ vanishing at $t = 0, 1$ such that $\mathrm{d}g_{t_{|\gamma(t)}} = \omega(t)$ for any $t \in]0, 1[$.*

(ii) *The function* $a \mapsto \mathrm{d}\mathcal{F}_{\mathrm{d}g_{|a}}$ *is a regular extension of zero whose differential is of the form*

$$\mathrm{d}\mathcal{F}_{\mathrm{d}g_{|a}}(\epsilon, \beta) = \int_I \langle \mathrm{d}g_t, \nabla^M_{\mathrm{d}\gamma(t)/\mathrm{d}t}\epsilon(t) - (\nabla_{\epsilon(t)}\rho)a(t) + \rho(\beta(t))\rangle\, \mathrm{d}t, \tag{3.34}$$

where $\epsilon, \beta, \nabla^M, \nabla$ *are as in* (3.31).

Proof. (i) We denote by $\exp_m : T_m M \to M$ the exponential map associated to the connection ∇^M. Recall that $\exp_m$ is a local diffeomorphism from a neighborhood of $0 \in T_m M$ to a neighborhood of m. Let $f(t)$ be a real-valued smooth function that vanishes at $t = 0$ and $t = 1$.

Define $g_t(m) = f(t)\psi(t, m)\langle\omega(t), \exp^{-1}_{\gamma(t)}(m)\rangle$, where $\psi(t, m)$ is a smooth function on $I \times M$ satisfying two conditions: (1) $\psi(t, m)$ is identically equal to 1 in a neighborhood of the curve $(t, \gamma(t))$ for all $t \in I$, and (2) $\psi(t, m)$ vanishes outside an open set on which $\exp^{-1}_{\gamma(t)}$ is well-defined. We leave it to the reader to check that these conditions can be satisfied and that the function g_t has the reqired properties.

(ii) The function $\mathcal{F}_{\mathrm{d}g}$ is identically zero on $P(A)$ by definition. We check that its differential is of the form (3.34). Let $a(t), \gamma(t), \epsilon(t), \beta(t)$ be as in Lemma 3.4. Recall that, by the definition of an A-path, we have $\mathrm{d}\gamma(t)/\mathrm{d}t + \rho(a(t)) = 0$. For any $t \in [0, 1]$, consider the functional

$$b \mapsto \left\langle \mathrm{d}g_t(t), \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} + \rho(b(t))\right\rangle.$$

Taking its derivative at the A-path a in the direction of the tangent vector $(\epsilon(t), \beta(t)) \in T_a\widetilde{P}(A)$, we obtain, using (3.31),

$$\begin{aligned}(\epsilon, \beta) \mapsto \langle \mathrm{d}g_t(t), \nabla^M_{\mathrm{d}\gamma(t)/\mathrm{d}t}\epsilon(t) - (\nabla_{\epsilon(t)}\rho)a(t) + \rho(\beta(t))\rangle \\ = \langle \mathrm{d}g_t(t), \nabla^M_{\mathrm{d}\gamma(t)/\mathrm{d}t}\epsilon(t) - (\nabla_{\epsilon(t)}\rho)a(t) + \rho(\beta(t)))\rangle.\end{aligned}$$

Integrating with respect to t, we obtain (3.34). □

3.2. *Almost differentials and linear multi-vector fields*

In this subsection, we study a particular case of multi-vector fields on a vector bundle.

Let $p : A \to M$ be a vector bundle. It is clear that there exists a bijection between the space $\Gamma(A^*)$ of sections of the dual bundle $p_* : A^* \to M$ and the set $C^\infty_{\mathrm{lin}}(A)$ of functions which are linear on each fiber. In fact, for any section $\phi \in \Gamma(A^*)$ the corresponding linear function ℓ_ϕ is given by $\ell_\phi(X_m) = \langle\phi(m), X_m\rangle$ for $X_m \in A_m$. On the other hand, by basic functions, we mean functions on A which are the pull-back functions from M.

Definition 3.5. Let $p : A \to M$ be a vector bundle and $\pi \in \mathfrak{X}^k(A)$ a k-vector field on A. We say that π is *linear* if $\pi(\mathrm{d}f_1, \ldots, \mathrm{d}f_k)$ is a linear function whenever all $f_1, \ldots, f_k \in C^\infty(A)$ are linear functions.

Linear multi-vector fields were called homogeneous in [16]. They have the following properties.

Proposition 3.6. *Let $p : A \to M$ be a vector bundle and $\pi \in \mathfrak{X}^k(A)$ a linear k-vector field on A with $k \geq 2$. Then $\pi(\mathrm{d}\ell_{\phi_1}, \dots, \mathrm{d}\ell_{\phi_{k-1}}, \mathrm{d}(p^* f_1))$ is a basic function on A and*

$$(\mathrm{d}(p^* f_1) \wedge \mathrm{d}(p^* f_2)) \lrcorner \pi = 0 \quad \forall \phi_1, \dots, \phi_{k-1} \in \Gamma(A^*) \text{ and } f_1, f_2 \in C^\infty(M).$$

Proof. Proceed as in [10, pp. 643–644], using the Leibniz rule and the fact that $(p^* f)\ell_\phi = \ell_{f\phi}$ for $f \in C^\infty(M)$ and $\phi \in \Gamma(A^*)$. □

Let $\pi \in \mathfrak{X}^k_{\mathrm{lin}}(A)$ be a linear k-vector field on A. Proposition 3.6 enables us to introduce a pair of operations

$$\rho_* : \Gamma(A^*) \times \overset{(k-1)}{\dots} \times \Gamma(A^*) \to \mathfrak{X}(M) \quad \text{and} \quad [\cdot, \dots, \cdot]_* : \Gamma(A^*) \times \overset{(k)}{\dots} \times \Gamma(A^*) \to \Gamma(A^*)$$

by

$$\begin{gathered} \rho_*(\phi_1, \dots, \phi_{k-1})(f) \circ p = \pi(\mathrm{d}\ell_{\phi_1}, \dots, \mathrm{d}\ell_{\phi_{k-1}}, \mathrm{d}(f \circ p)), \\ \ell_{[\phi_1, \dots, \phi_k]_*} = \pi(\mathrm{d}\ell_{\phi_1}, \dots, \mathrm{d}\ell_{\phi_k}), \end{gathered} \tag{3.35}$$

for $\phi_1, \dots, \phi_k \in \Gamma(A^*)$ and $f \in C^\infty(M)$.

It is easy to see that the following identities are satisfied:

$$\begin{aligned} \rho_*(\phi_1, \dots, f\phi_i, \dots, \phi_{k-1}) &= f\rho_*(\phi_1, \dots, \phi_i, \dots, \phi_{k-1}), \\ [\phi_1, \dots, f\phi_i, \dots, \phi_k]_* &= (-1)^{i+1}\rho_*(\phi_1, \dots, \hat{\phi}_i, \dots, \phi_{k-1})(f)\phi_i \\ &\quad + f[\phi_1, \dots, \phi_i, \dots, \phi_k]_* \end{aligned} \tag{3.36}$$

for any $\phi_1, \dots, \phi_k \in \Gamma(A^*)$ and $f \in C^\infty(M)$. In particular, ρ_* induces a bundle map $\bigwedge^{k-1} A^* \to TM$.

Conversely, if we have a pair $(\rho_*, [\cdot, \dots, \cdot]_*)$ satisfying (3.36), we can define a linear k-vector field on A using (3.35).

Now, assume that δ is an almost k-differential. We construct a pair $([\cdot, \dots, \cdot]_*, \rho_*)$ satisfying (3.36) as follows. Let

$$\begin{aligned} \rho_*(\phi_1, \dots, \phi_{k-1})(f) &= \langle \delta f, \phi_1 \wedge \dots \wedge \phi_{k-1} \rangle, \\ \langle [\phi_1, \dots, \phi_k]_*, X \rangle &= \sum_{i=1}^{k} (-1)^{i+k} \rho_*(\phi_1, \dots, \hat{\phi}_i, \dots, \phi_k)(\phi_i(X)) \\ &\quad - \langle \delta X, \phi_1 \wedge \dots \wedge \phi_k \rangle, \end{aligned}$$

for all $\phi_1, \dots, \phi_k \in \Gamma(A^*)$, $X \in \Gamma(A)$ and $f \in C^\infty(M)$. Conversely, using these equations, one can construct an almost k-differential from $([\cdot, \dots, \cdot]_*, \rho_*)$.

A combination of the above discussion leads to the following

Proposition 3.7. *For a given Lie algebroid A, there is a one-to-one correspondence between linear multi-vector fields and almost differentials on A.*

For an almost differential δ on A, we denote the corresponding linear multi-vector field by π_δ. In local coordinates, the correspondence between almost k-differentials and k-vector fields π_δ on A can be described as follows. Let $(x_1, \dots, x_n)$ be local coordinates on M and $\{e_1, \dots, e_s\}$ a local basis of $\Gamma(A)$. Assume that

$$\delta x_i = \sum a_i^{i_1 \dots i_{k-1}}(x) e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}, \qquad \delta e_i = \sum c_i^{i_1 \dots i_k}(x) e_{i_1} \wedge \cdots \wedge e_{i_k},$$

Then

$$\pi_\delta = \sum \left(a_i^{i_1 \dots i_{k-1}}(x) \frac{\partial}{\partial v_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial v_{i_{k-1}}} \wedge \frac{\partial}{\partial x_i} - c_i^{i_1 \dots i_k}(x) v^i \frac{\partial}{\partial v_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial v_{i_k}} \right),$$

where $(v_1, \dots, v_s)$ are the corresponding linear coordinates on the fibers.

The correspondence between almost differentials and linear multi-vector fields preserves the graded Lie algebra structure, as shown in the following

Proposition 3.8. *Given a Lie algebroid A, the map $\delta \mapsto \pi_\delta$ is a graded Lie algebra isomorphism from almost differentials on A to linear multi-vector fields on A. That is, for any almost differentials δ_1 and δ_2,*

$$[\pi_{\delta_1}, \pi_{\delta_2}] = \pi_{[\delta_1, \delta_2]}.$$

Proof. Note that if $P \in \Gamma(\bigwedge^p A)$ is given locally by $\sum_{1 \le j_1 < \cdots < j_p \le n} P^{j_1 \dots j_p}(x) e_{j_1} \wedge \cdots \wedge e_{j_p}$ for a given local basis $\{e_1, \dots, e_s\}$ of $\Gamma(A)$, then

$$\begin{aligned} \delta P = & \sum_{1 \le j_1 < \cdots < j_p \le n} \sum_i \frac{\partial P^{j_1 \dots j_p}(x)}{\partial x_i} a_i^{i_1 \cdots i_{k-1}}(x) e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{j_1} \wedge \cdots \wedge e_{j_p} \\ & + \sum_{1 \le j_1 < \cdots < j_p \le n} \sum_i (-1)^{(i+1)(k+1)} P^{j_1 \dots j_p}(x) e_{j_1} \wedge \cdots \wedge \delta e_{j_i} \wedge \cdots \wedge e_{j_p}. \end{aligned}$$

The assertion follows from a tedious computation and is left to the reader. □

Example 3.9. If δ is an almost 0-differential, then δ is just the contraction operator by an element $\phi \in \Gamma(A^*)$. In this case, one shows that $\pi_\delta \in \mathfrak{X}^0_{\mathrm{lin}}(A) = C^\infty_{\mathrm{lin}}(A)$ is just $-\ell_\phi$.

Example 3.10. When $k = 1$, as a consequence of Proposition 3.8, one obtains a Lie algebra isomorphism between $\Gamma(\mathrm{CDO}(A))$, the space of covariant differential operators on A, and $\Gamma(T^{\mathrm{LIN}}(A))$, the space of linear vector fields on A (see [28]).

Example 3.11. Let δ be a 2-differential of square zero. We know that δ induces a Lie algebroid structure on A^* (see [20, 42]). On the other hand, from Proposition 3.8, it follows that $[\pi_\delta, \pi_\delta] = 0$, and therefore π_δ defines a Poisson structure on A. Such a correspondence between Lie algebroid structures on A^* and linear Poisson structures on the dual bundle is standard (see [10] for instance). A generalization to arbitrary linear 2-vector fields and pre-Lie algebroids was considered in [17].

We end this subsection by recalling two kinds of liftings from $\Gamma(\bigwedge^\bullet A)$ to multi-vector fields on A, the complete and vertical lifts. Let $P \in \Gamma(\bigwedge^p A)$, and $\mathrm{ad}(P)$ its corresponding coboundary differential. Then the corresponding linear p-vector field $\pi_{\mathrm{ad}(P)}$ is the so-called *complete lift* of P, which is denoted by P^c (see [16]).

On the other hand, for any multisection $P \in \Gamma(\bigwedge^p A)$, there exists another kind of lift, called the *vertical lift*, denoted by P^v. It is obtained via the natural identification $A_m \simeq T_a^{\mathrm{Vert}}(A)$ for $a \in A$. In local coordinates, if $P = \sum_{1\le j_1<\cdots<j_p\le n} P^{j_1\cdots j_p}(x) e_{j_1} \wedge \cdots \wedge e_{j_p}$ for a given local basis $\{e_1, \ldots, e_s\}$, then

$$P^v = \sum_{1\le j_1<\cdots<j_p\le n} P^{j_1\cdots j_p}(x) \frac{\partial}{\partial v_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial v_{j_p}}.$$

Complete and vertical lifts have the following properties [16]:

$$f^v = p^* f, \tag{3.37}$$
$$f^c = \ell_{\mathrm{d}_A f}, \tag{3.38}$$
$$(P \wedge Q)^c = P^c \wedge Q^v + P^v \wedge Q^c, \tag{3.39}$$
$$(P \wedge Q)^v = P^v \wedge Q^v, \tag{3.40}$$
$$[P^c, Q^c] = [\![P, Q]\!]^c, \tag{3.41}$$
$$[P^c, Q^v] = [\![P, Q]\!]^v, \tag{3.42}$$
$$[P^v, Q^v] = 0, \tag{3.43}$$

for all $f \in C^\infty(M)$ and $P, Q \in \Gamma(\bigwedge^\bullet A)$.

Proposition 3.12. *Let δ be a k-differential and $P \in \Gamma(\bigwedge^\bullet A)$. Then*

$$[\pi_\delta, P^c] = (\delta P)^c, \qquad [\pi_\delta, P^v] = (\delta P)^v.$$

Proof. Using (2.18) and Proposition 3.8, we have

$$[\pi_\delta, P^c] = [\pi_\delta, \pi_{\mathrm{ad}(P)}] = \pi_{[\delta, \mathrm{ad}(P)]} = \pi_{\mathrm{ad}(\delta P)} = (\delta P)^c.$$

The second identity can be checked directly using local coordinates. □

Let

$$P\Gamma(\textstyle\bigwedge^\bullet A) = \{I \ni t \mapsto P(t) \in \Gamma(\bigwedge^\bullet A) : P(t) \text{ is of class } C^2 \text{ in } t\},$$
$$P_0\Gamma(\textstyle\bigwedge^\bullet A) = \{I \ni t \mapsto P(t) \in \Gamma(\bigwedge^\bullet A) : P(0) = P(1) = 0,\ P(t) \text{ is of class } C^2 \text{ in } t\}.$$

If $P \in P_0\Gamma(\bigwedge^\bullet A)$, we define $G(P)$ as the time-dependent multi-vector field on A given by

$$G(P) = P^c + \left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)^v. \tag{3.44}$$

The multi-vector fields $G(P)$ have the following properties.

Proposition 3.13. *If $P, Q \in P\Gamma(\bigwedge^\bullet A)$ and δ is a multi-differential on A, then*

$$[G(P), G(Q)] = G([P, Q]), \tag{3.45}$$

$$G(P \wedge Q) = G(P) \wedge Q^v + P^v \wedge G(Q), \tag{3.46}$$

$$[\pi_\delta, G(P)] = G(\delta(P)). \tag{3.47}$$

In particular, (3.46) *implies that*

$$G(f(t)P) = f(t)G(P) + \frac{\mathrm{d}f(t)}{\mathrm{d}t} P^v. \tag{3.48}$$

Proof. Using (3.41), (3.42) and (3.43) and the fact that $\mathrm{d}/\mathrm{d}t$ is a derivation with respect to the Schouten bracket, we deduce that (3.45) holds.

(3.47) is a direct consequence of Proposition 3.12 and the fact that δ commutes with $\mathrm{d}/\mathrm{d}t$. □

3.3. *From k-vector fields on A to k-vector fields on $\widetilde{P}(A)$*

We start this subsection with a construction that should be thought of, at least heuristically, as a lifting of a time-dependent multi-vector field on A to a multi-vector field on $\widetilde{P}(A)$.

Let $\pi(t)$ be a time-dependent k-vector field on A with $k \geq 1$. For given k 1-forms $\eta_1, \ldots, \eta_k \in \Omega^1(P(A))$ and their regular extensions $\Phi_{\eta_1}, \ldots, \Phi_{\eta_k}$, define a smooth function $\widetilde{\pi}(\Phi_{\eta_1}, \ldots, \Phi_{\eta_k})$ on $P(A)$ as follows. For any $a \in P(A)$,

$$\widetilde{\pi}(\Phi_{\eta_1}, \ldots, \Phi_{\eta_k})(a) := \int_I \pi(t)_{|a(t)}(\Phi_{\eta_1}(a(t)), \ldots, \Phi_{\eta_k}(a(t)))\, \mathrm{d}t. \tag{3.49}$$

Remark 3.14. (3.49) still makes sense when one of the extensions Φ_{η_i} is not necessarily regular, with the following modification of its definition. Assume that Φ_{η_1} is not necessarily regular, and that $(\Phi_{\eta_2}, \ldots, \Phi_{\eta_k})$ are. Then $\pi_t(\Phi_{\eta_2}(a(t)), \ldots, \Phi_{\eta_k}(a(t)))$ is an element of $T\widetilde{P}(A)$ and we can therefore define

$$\widetilde{\pi}(\Phi_{\eta_1}, \ldots, \Phi_{\eta_k})(a) = \Phi_{\eta_1}\big(\pi_t(\Phi_{\eta_2}(a(t)), \ldots, \Phi_{\eta_k}(a(t)))\big).$$

We recover of course the previous definition of $\widetilde{\pi}$ if Φ_{η_1} is regular.

For instance, consider the case $k = 0$. If f_t is a time-dependent function on A, i.e., $f \in C^\infty(I \times A)$, then (3.49) gives

$$\widetilde{f}(a) = \int_I f(t, a(t))\, \mathrm{d}t. \tag{3.50}$$

This still makes sense for any $a \in \widetilde{P}(A)$ and hence defines a function on $\widetilde{P}(A)$ that we denote by $\widetilde{f}$ again. Note also that for any time-dependent vector field $X : t \mapsto X_t$ on A, $\widetilde{X}$ is a vector field on $\widetilde{P}(A)$, which, at any $a(t) \in \widetilde{P}(A)$, is given by $t \mapsto X_{t_{|a(t)}}$.

For any time-dependent vector field $X : t \mapsto X_t$ on $\widetilde{P}(A)$, tangent to $P(A)$, we define the Lie derivative $\mathcal{L}_{\widetilde{X}}\widetilde{\pi}$ of $\widetilde{\pi}$ as follows:

$$(\mathcal{L}_{\widetilde{X}}\widetilde{\pi})(\Phi_{\eta_1}, \dots, \Phi_{\eta_k}) = \widetilde{X}(\widetilde{\pi}(\Phi_{\eta_1}, \dots, \Phi_{\eta_k})) - \sum_{i=1}^{k} \widetilde{\pi}(\Phi_{\eta_1}, \dots, \mathcal{L}_{\widetilde{X}}\Phi_{\eta_i}, \dots, \Phi_{\eta_k}). \tag{3.51}$$

This definition has to be justified. Of course, the Lie derivative of a regular extension is not necessarily a regular extension, but it is an extension and by Remark 3.14, the left hand side of (3.51) makes sense.

Lemma 3.15. (i) *For any time-dependent function $g : t \mapsto g_t$ from I to $C^\infty(M)$ with $g_0 = g_1 = 0$ (i.e., $g \in P\Gamma(\bigwedge^0 A)$), we have $\widetilde{G(g)} = \mathcal{F}_{\mathrm{d}g}$.*

(ii) *If $\xi \in P_0\Gamma(A)$, then $\widetilde{G(\xi)}$ is the gauge vector field given by* (3.24).

(iii) *If $\xi \in P\Gamma(A)$, then $\tau_*(\widetilde{G(\xi)}) = \overrightarrow{\xi(0)} - \overleftarrow{\xi(1)}$, where $\tau : P(A) \to \Gamma$ is the map defined by* (3.22).

(iv) *For any multi-differential δ on A,*

$$\mathcal{L}_{\widetilde{G(\xi)}}\widetilde{\pi}_\delta = \widetilde{G(\delta(\xi))}.$$

(v) *For any multi-differential δ on A and time-dependent function $g : t \mapsto g_t$ on M,*

$$\widetilde{\pi}_\delta(\widetilde{G(g)}, \Phi_{\eta_2}, \dots, \Phi_{\eta_k}) = \widetilde{G(\delta(g))}(\Phi_{\eta_2}, \dots, \Phi_{\eta_k}).$$

Proof. (i) For any time-dependent function $g : t \mapsto g_t$ from I to $C^\infty(M)$, it follows from (3.38) and (3.50) that

$$\widetilde{g^c} = \int_0^1 \langle \mathrm{d}g_t, \rho(a(t))\rangle \, \mathrm{d}t. \tag{3.52}$$

Now using (3.37) and (3.50) we have

$$\widetilde{\left(\frac{\mathrm{d}g}{\mathrm{d}t}\right)^v} = \int_0^1 \frac{\mathrm{d}g_t}{\mathrm{d}t}(p \circ a(t))\, \mathrm{d}t = -\int_0^1 \left\langle \mathrm{d}g_t, \frac{\mathrm{d}p \circ a(t)}{\mathrm{d}t}\right\rangle \mathrm{d}t, \tag{3.53}$$

where the last equality is obtained by integration by parts using the boundary condition $g_0 = g_1 = 0$. Thus (i) follows.

(ii) follows from the fact that for any $\xi \in \Gamma(A)$, ad_ξ is equal to ξ^c, a fact that can be easily checked in local coordinates.

(iii) is easily deduced from (ii) and (3.25).

(iv) From the definition of Lie derivatives given by (3.27) and (3.51), it follows that

$$\mathcal{L}_{\widetilde{X_t}}\widetilde{\pi_t} = \widetilde{[X_t, \pi_t]} \tag{3.54}$$

for any time-dependent vector field $X : t \mapsto X_t$ on A such that $\widetilde{X}$ is tangent to $P(A)$. Therefore $\mathcal{L}_{\widetilde{G(\xi)}}\widetilde{\pi}_\delta = [\widetilde{G(\xi), \pi_\delta}] = G(\widetilde{\delta(\xi)})$, where the last identity follows from (3.47).

(v) For any time-dependent smooth function f_t, $t \in I$, on A and any A-path $a(t)$, the differential at a of $\widetilde{f_t}$ is a regular extension of the restriction of $\widetilde{f_t}$ to $P(A)$ given by

$$\mathsf{d}\widetilde{f_t}_{|a}(X) = \int_I \langle \mathsf{d} f_{t|a(t)}, X(t)\rangle\, \mathsf{d}t$$

for any $X \in T_a \widetilde{P}(A)$, where $\mathsf{d} f_{t|a(t)}$ is the differential of the smooth function f_t at the point $a(t)$.

By the definition of $\widetilde{\pi_\delta}$, we have

$$\widetilde{\pi_\delta}(\mathsf{d}\widetilde{G(g)}, \Phi_{\eta_2}, \ldots, \Phi_{\eta_k})_{|a} = \int_I \pi_\delta(\mathsf{d}G(g)_{|a(t)}, \Phi_{\eta_2}(t), \ldots, \Phi_{\eta_k}(t))\, \mathsf{d}t$$

for any A-path $a(t)$. By (3.47), we have

$$\begin{aligned}\widetilde{\pi_\delta}(\mathsf{d}\widetilde{G(g)}, \Phi_{\eta_2}, \ldots, \Phi_{\eta_k})_{|a} &= \int_I G(\delta(g))_{|a(t)}(\Phi_{\eta_2}(t), \ldots, \Phi_{\eta_k}(t))\, \mathsf{d}t\\ &= \widetilde{G(\delta(g))}(\Phi_{\eta_2}, \ldots, \Phi_{\eta_k}).\end{aligned}$$

This proves (v). □

To a given $P \in P\Gamma(\bigwedge^k A)$, we have so far associated $\widetilde{G(P)}$, which should be interpreted as a "multi-vector field on $\widetilde{P}(A)$" (defined at points of $P(A)$). It turns out that, if one applies $\widetilde{G(P)}$ to the functions obtained by pulling back those on Γ, the resulting functions depend only on $P(0)$ and $P(1)$, which are, heuristically, obtained by integrating some total differentials on [0, 1]. However, several technical difficulties arise which need to be addressed in order to justify the computation.

Proposition 3.16. *For any $P \in P\Gamma(\bigwedge^k A)$, any functions $f_1, \ldots, f_k \in C^\infty(\Gamma)$, and any regular extensions $\Phi_{\mathsf{d}\tau^* f_1}, \ldots, \Phi_{\mathsf{d}\tau^* f_k}$ of $\tau^* f_1, \ldots, \tau^* f_k$, we have*

$$\widetilde{G(P)}(\Phi_{\mathsf{d}\tau^* f_1}, \ldots, \Phi_{\mathsf{d}\tau^* f_k}) = \overrightarrow{P(1)}(\mathsf{d}f_1, \ldots, \mathsf{d}f_k) - \overleftarrow{P(0)}(\mathsf{d}f_1, \ldots, \mathsf{d}f_k). \tag{3.55}$$

First we need the following

Lemma 3.17. *For any $\xi \in P\Gamma(A)$, $a \in P(A)$, any covector $\eta \in T^*_a P(A)$ conormal to the gauge orbit, and any regular extension Φ_η of η, the following identity holds:*

$$\frac{\mathsf{d}\langle \Phi_\eta(a(t)), \xi^v(t)\rangle}{\mathsf{d}t} = \langle \Phi_\eta(t), G(\xi)(t)\rangle. \tag{3.56}$$

In particular, for any smooth function $f : \Gamma \to \mathbb{R}$ and any regular extension $\Phi_{\mathsf{d}\tau^* f}$ of $\tau^* f$,

$$\frac{\mathsf{d}\langle \Phi_{\mathsf{d}\tau^* f}(t), \xi^v(t)\rangle}{\mathsf{d}t} = \langle \Phi_{\mathsf{d}\tau^* f}(t), G(\xi)(t)\rangle. \tag{3.57}$$

Proof. Since η is conormal to the gauge orbits, for any smooth map $\chi : I \to \Gamma(A)$ with $\chi(0) = \chi(1) = 0$ we have

$$\langle \eta, \widetilde{G(\chi)}_{|a} \rangle = \langle \Phi_\eta, \widetilde{G(\chi)}_{|a} \rangle = \int_I \langle \Phi_\eta(t), G(\chi(t))_{|a(t)} \rangle \, \mathrm{d}t = 0.$$

Applying this equality to $\chi(t) := \psi(t)\xi(t)$, where $\psi(t)$ is any smooth function with $\psi(0) = \psi(1) = 0$, and using (3.48), we obtain

$$0 = \int_0^1 \psi(t) \langle \Phi_\eta(t), G(\xi(t)) \rangle \, \mathrm{d}t + \int_0^1 \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} \langle \Phi_\eta(t), \xi^v(t) \rangle \, \mathrm{d}t. \tag{3.58}$$

The result follows by integration by parts. $\square$

Now we are ready to prove Proposition 3.16.

Proof of Proposition 3.16. By Lemma 3.15, we have $\widetilde{G(\xi)}(\tau^* f) = \overrightarrow{\xi(1)}(f) - \overleftarrow{\xi(0)}(f)$ for all $f \in C^\infty(\Gamma)$. On the other hand, by definition, $\widetilde{G(\xi)}(\tau^* f) = \int_0^1 \langle \Phi_{\mathrm{d}\tau^* f}(t), G(\xi)(t) \rangle \, \mathrm{d}t$. Therefore,

$$\int_0^1 \langle \Phi_{\mathrm{d}\tau^* f}(t), G(\xi)(t) \rangle \, \mathrm{d}t = \overrightarrow{\xi(1)}(f) - \overleftarrow{\xi(0)}(f).$$

By Lemma 3.17, it follows that

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} (\langle \Phi_{\mathrm{d}\tau^* f}(t), \xi^v(t) \rangle) \, \mathrm{d}t = \overrightarrow{\xi(1)}(f) - \overleftarrow{\xi(0)}(f).$$

Hence

$$\langle \Phi_{\mathrm{d}\tau^* f}(1), \xi^v(1) \rangle - \langle \Phi_{\mathrm{d}\tau^* f}(0), \xi^v(0) \rangle = \overrightarrow{\xi(1)}(f) - \overleftarrow{\xi(0)}(f).$$

Since this identity holds for any time-dependent section ξ, we have

$$\langle \Phi_{\mathrm{d}\tau^* f}(1), \xi^v(1) \rangle = \overrightarrow{\xi(1)}(f) \quad \text{and} \quad \langle \Phi_{\mathrm{d}\tau^* f}(0), \xi^v(0) \rangle = \overleftarrow{\xi(0)}(f).$$

This implies that for any $P \in P\Gamma(\bigwedge^k A)$,

$$\begin{cases} P^v(1)(\Phi_{\mathrm{d}\tau^* f_1}(1), \ldots, \Phi_{\mathrm{d}\tau^* f_k}(1)) = \overrightarrow{P(1)}(f_1, \ldots, f_k), \\ P^v(0)(\Phi_{\mathrm{d}\tau^* f_1}(0), \ldots, \Phi_{\mathrm{d}\tau^* f_k}(0)) = \overleftarrow{P(0)}(f_1, \ldots, f_k). \end{cases} \tag{3.59}$$

Now assume that $P(t) = \xi_1(t) \wedge \cdots \wedge \xi_k(t)$ for some time-dependent sections $\xi_1, \ldots, \xi_k$ of $\Gamma(A)$. Then, according to (3.46),

$$G(P)(t) = \sum_{i=1}^k \xi_1^v(t) \wedge \cdots \wedge G(\xi_i(t)) \wedge \cdots \wedge \xi_k^v(t).$$

Therefore, for any $a(t) \in P(A)$,

$$\widetilde{G(P)}(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k})(a(t)) = \int_0^1 G(P)(t)(\Phi_{\mathrm{d}\tau^* f_1}(t), \dots, \Phi_{\mathrm{d}\tau^* f_k}(t))\, \mathrm{d}t$$

$$= \sum_{i=1}^{k} \int_0^1 (\xi_1^v(t) \wedge \cdots \wedge G(\xi_i(t))_{|a(t)} \wedge \cdots \wedge \xi_k^v(t))(\Phi_{\mathrm{d}\tau^* f_1}(t), \dots, \Phi_{\mathrm{d}\tau^* f_k}(t))\, \mathrm{d}t$$

$$= \sum_{\sigma \in S_k} \sum_{i=1}^{k} (-1)^{|\sigma|}$$

$$\times \int_0^1 \langle \Phi_{\mathrm{d}\tau^* f_1}(t), \xi^v_{\sigma(1)}(t)\rangle \dots \langle \Phi_{\mathrm{d}\tau^* f_1}(t), G(\xi_{\sigma(i)}(t))\rangle \dots \langle \Phi_{\mathrm{d}\tau^* f_k}(t), \xi^v_{\sigma(k)}(t)\rangle\, \mathrm{d}t$$

(by Lemma 3.17)

$$= \sum_{\sigma \in S_k} \sum_{i=1}^{k} (-1)^{|\sigma|}$$

$$\times \int_0^1 \langle \Phi_{\mathrm{d}\tau^* f_1}(t), \xi^v_{\sigma(1)}(t)\rangle \dots \frac{\mathrm{d}\langle \Phi_{\mathrm{d}\tau^* f_i}(t), \xi_{\sigma(i)}(t)\rangle}{\mathrm{d}t} \dots \langle \Phi_{\mathrm{d}\tau^* f_k}(t), \xi^v_{\sigma(k)}(t)\rangle\, \mathrm{d}t$$

$$= \sum_{\sigma \in S_k} (-1)^{|\sigma|} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(\langle \Phi_{\mathrm{d}\tau^* f_1}(t), \xi^v_{\sigma(1)}(t)\rangle \dots \langle \Phi_{\mathrm{d}\tau^* f_k}(t), \xi^v_{\sigma(k)}(t)\rangle\big)\, \mathrm{d}t$$

$$= P^v(t)(\Phi_{\mathrm{d}\tau^* f_1}(t), \dots, \Phi_{\mathrm{d}\tau^* f_k}(t))|_{t=0}^1.$$

The result now follows from (3.59). □

3.4. *From k-vector fields on $\widetilde{P}(A)$ to k-vector fields on Γ*

Assume that Γ is an α-simply connected and α-connected Lie groupoid with Lie algebroid A. Let δ be a k-differential on A and $\widetilde{\pi}_\delta$ the corresponding k-vector field on $\widetilde{P}(A)$. The goal of this section is to construct a k-vector field Π_δ on Γ from $\widetilde{\pi}_\delta$.

Proposition 3.18. *Let $f_1, \dots, f_k \in C^\infty(\Gamma)$ be a family of smooth functions on Γ and $\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k}$ regular extensions of $\tau^* f_1, \dots, \tau^* f_k$. Then $\widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k})$ is a smooth function on $P(A)$, which is*

- (i) *independent of the choice of the regular extensions $\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k}$ of $\mathrm{d}\tau^* f_1, \dots, \mathrm{d}\tau^* f_k$, and*
- (ii) *invariant under gauge transformations.*

Proof. (i) It suffices to prove that if $\Phi_{\mathrm{d}\tau^* f_1}$ is a regular extension of zero, the function $\widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k})$ vanishes. By Lemma 3.3(ii), for any $a \in P(A)$, there exists $g : I \to C^\infty(M)$ with $g_0 = g_1 = 0$ such that $\Phi_{\mathrm{d}\tau^* f_1}{}_{|a} = \mathrm{d}\mathcal{F}_{\mathrm{d}g_{|a}}$. Then

$$\widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k})_{|a} = \widetilde{\pi}_\delta(\mathrm{d}\mathcal{F}_{\mathrm{d}g}, \Phi_{\mathrm{d}\tau^* f_2}, \dots, \Phi_{\mathrm{d}\tau^* f_k}).$$

According to Lemma 3.15 (i), we have $\mathcal{F}_{\mathrm{d}g} = \widetilde{G(g)}$. Lemma 3.15 (v) implies that

$$\widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k})_{|a} = \widetilde{G(\delta(g))}(\Phi_{\mathrm{d}\tau^* f_2}, \dots, \Phi_{\mathrm{d}\tau^* f_k})_{|a}.$$

The latter vanishes according to Proposition 3.16, since by assumption $\xi(0) = \xi(1) = 0$. This proves (i).

Before starting the proof of (ii), we would like to add a comment. We have proven in (i) that for any regular extension Φ of 0, and any regular extensions $\Phi_{\mathrm{d}\tau^* f_2}, \dots, \Phi_{\mathrm{d}\tau^* f_k}$ of $\tau^* f_2, \dots, \tau^* f_n$, we have

$$\widetilde{\pi}_\delta(\Phi, \Phi_{\mathrm{d}\tau^* f_2}, \dots, \Phi_{\mathrm{d}\tau^* f_k}) = 0.$$

Regular extensions are dense with respect to the induced topology of $T_a^* \widetilde{P}(A)$. Thus,

$$\widetilde{\pi}_\delta(\Phi, \Phi_{\mathrm{d}\tau^* f_2}, \dots, \Phi_{\mathrm{d}\tau^* f_k}) = 0 \tag{3.60}$$

for any extension Φ of 0.

(ii) Since the functions $\tau^* f_1, \dots, \tau^* f_k$ are invariant under the gauge transformation, for each $\xi \in P_0\Gamma(A)$ the Lie derivative $\mathcal{L}_{\widetilde{G(\xi)}} \Phi_{\mathrm{d}\tau^* f_i}$ is, for all $i \in \{1, \dots, k\}$, an extension of zero. Therefore, by (3.60), $\widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \mathcal{L}_{\widetilde{G(\xi)}} \Phi_{\mathrm{d}\tau^* f_i}, \dots \Phi_{\mathrm{d}\tau^* f_k}) = 0$ for all $i \in \{1, \dots, k\}$. By (3.51),

$$\widetilde{G(\xi)}(\widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k})) = (\mathcal{L}_{\widetilde{G(\xi)}} \widetilde{\pi}_\delta)(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k}).$$

By Lemma 3.15(v),

$$\widetilde{G(\xi)}(\widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k})) = \widetilde{G(\delta(\xi))}(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k}),$$

which is identically 0 according to Proposition 3.16. This proves (ii). $\square$

By Propositions 3.1 and 3.18(ii), the function $\widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k})$ descends to a smooth function on Γ, which will be denoted by $\{f_1, \dots, f_k\}$. That is,

$$\widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k}) = \tau^*\{f_1, \dots, f_k\}. \tag{3.61}$$

It is straightforward to check that the map $f_1, \dots, f_k \mapsto \{f_1, \dots, f_k\}$ indeed defines a k-vector field Π_δ on Γ, i.e.,

$$\{f_1, \dots, f_k\} = \Pi_\delta(\mathrm{d}f_1, \dots, \mathrm{d}f_k). \tag{3.62}$$

Proposition 3.19. *Π_δ is a multiplicative k-vector field on Γ.*

Proof. By definition, Π_δ is given, for any $\eta_1, \dots, \eta_k \in T_g^*\Gamma$, by

$$\Pi_\delta(\eta_1, \dots, \eta_k) = \int_I (\pi_\delta)_{|a(t)}(\Phi_{\tau^*\eta_1}(t), \dots, \Phi_{\tau^*\eta_k}(t))\, \mathrm{d}t \tag{3.63}$$

where $\Phi_{\tau^*\eta_1}(t), \dots, \Phi_{\tau^*\eta_k}(t)$ are any regular extensions of $\tau^*\eta_1, \dots, \tau^*\eta_k$, with a smooth dependence on the variable t, and $a(t)$ is any A-path with $\tau(a) = g$. Since smooth functions are dense in the space of piecewise continuous functions with finitely many discontinuities, (3.63) remains valid when the extensions $\Phi_{\tau^*\eta_1}(t), \dots, \Phi_{\tau^*\eta_k}(t)$ are just assumed to be piecewise continuous in t (with finitely many discontinuities).

It is straightforward to check that for any $g \in \Gamma$ with $\alpha(g) = m$ and $\beta(g) = n$, there exists an A-path $a(t)$ with $\tau(a) = g$ such that $a(t)$ is constantly equal to m in a neighborhood of $t = 0$ and constantly equal to n in a neighborhood of $t = 1$. Consider two composable elements $g_1, g_2 \in \Gamma$ and two composable elements $\eta_1 \in T^*_{g_1}\Gamma$ and $\eta_2 \in T^*_{g_2}\Gamma$. Choose now A-paths $a_1(t)$ and $a_2(t)$ satisfying the previous condition. Define $a(t)$ by $a(t) = 2a_1(2t)$ for $t \in [0, 1/2]$ and by $a(t) = 2a_2(2t-1)$ for $t \in [1/2, 1]$. By construction, $a(t)$ is an A-path and $\tau(a) = g_1 g_2$.

Let $\Phi_{\tau^*\eta_1}(t)$ and $\Phi_{\tau^*\eta_2}(t)$ be two regular extensions of $\tau^*\eta_1$ and $\tau^*\eta_2$ respectively. Then the map defined by $\Phi(t) = \Phi_{\tau^*\eta_1}(2t)$ for $t \in [0, 1/2]$ and $\Phi(t) = \Phi_{\tau^*\eta_2}(2t-1)$ for $t \in [1/2, 1]$ is a regular extension of $\Phi_{\eta_1\cdot\eta_2}$ for $\eta_1 \cdot \eta_2 \in T^*_{g_1g_2}\Gamma$, and it may have a point of discontinuity at $t = 1/2$.

We now choose k compatible pairs η_1^i, η_2^i for $i = 1, \dots, k$ of elements of $T^*_{g_1}\Gamma$ and $T^*_{g_2}\Gamma$. For all $i = 1, \dots, k$ we consider two regular extensions $\Phi_{\tau^*\eta_1^i}$ and $\Phi_{\tau^*\eta_2^i}$ of $\tau^*\eta_1^i$ and $\tau^*\eta_2^i$ respectively. And, for all $i = 1, \dots, k$ again, we form $\Phi_{\eta_1^i\cdot\eta_2^i}(t)$ as above.

(3.63) being valid even for piecewise regular extensions, the first of the identities below is valid; the other ones are routine.

$$\begin{aligned}
\Pi_\delta(\eta_1^1\cdot\eta_2^1, \dots, \eta_1^k\cdot\eta_2^k) &= \int_I (\pi_\delta)_{|a(t)}(\Phi_{\eta_1^1\cdot\eta_2^1}(t), \dots, \Phi_{\eta_1^k\cdot\eta_2^k}(t))\,\mathrm{d}t \\
&= \int_0^1 (\pi_\delta)_{|a_1(t)}\left(\Phi_{\eta_1^1\cdot\eta_2^1}\left(\frac{t}{2}\right), \dots, \Phi_{\eta_1^k\cdot\eta_2^k}\left(\frac{t}{2}\right)\right)\mathrm{d}t \\
&\quad + \int_0^1 (\pi_\delta)_{|a_2(t)}\left(\Phi_{\eta_1^1\cdot\eta_2^1}\left(\frac{1+t}{2}\right), \dots, \Phi_{\eta_1^k\cdot\eta_2^k}\left(\frac{1+t}{2}\right)\right)\mathrm{d}t \\
&= \int_I (\pi_\delta)_{|a_1(t)}(\Phi_{\tau^*\eta_1^1}(t), \dots, \Phi_{\tau^*\eta_1^k}(t))\,\mathrm{d}t \\
&\quad + \int_I (\pi_\delta)_{|a_2(t)}(\Phi_{\tau^*\eta_2^1}(t), \dots, \Phi_{\tau^*\eta_2^k}(t))\,\mathrm{d}t \\
&= \Pi_\delta(\eta_1^1, \dots, \eta_1^k) + \Pi_\delta(\eta_2^1, \dots, \eta_2^k).
\end{aligned} \tag{3.64}$$

By Proposition 3.6(ii), Π_δ is multiplicative. □

Finally to complete the proof of Theorem 2.34, we need to show that the map $\delta \mapsto \Pi_\delta$ constructed above is indeed the inverse of $\Pi \mapsto \delta_\Pi$. This is due to the following

Proposition 3.20. *For any k-differential δ,*

$$[\Pi_\delta, \overrightarrow{X}] = \overrightarrow{\delta X}, \qquad [\Pi_\delta, \alpha^* f] = \overrightarrow{\delta f}.$$

Proof. For any $f_1, \dots, f_k \in C^\infty(\Gamma)$ and any $X \in \Gamma(A)$, one has

$$[\Pi_\delta, \overrightarrow{X}](f_1, \dots, f_k) = \overrightarrow{X}(\Pi_\delta(f_1, \dots, f_k)) - \sum_{i=1}^k \Pi_\delta(f_1, \dots, \overrightarrow{X}(f_i), \dots, f_k). \tag{3.65}$$

Let $\xi \in P\Gamma(A)$ be such that $\xi(1) = X$ and $\xi(0) = 0$. According to Lemma 3.15, we have $\tau_*(\widetilde{G(\xi)}) = \overrightarrow{X}$. Therefore, for any regular extension $\Phi_{\mathrm{d}\tau^* f}$ of $\tau^* f$, where $f \in C^\infty(\Gamma)$, $\mathcal{L}_{\widetilde{G(\xi)}}\Phi_{\mathrm{d}\tau^* f}$ is a regular extension of $\tau^*(\overrightarrow{X}(f))$.

Let us choose some regular extensions $\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k}$ of $\tau^* f_1, \dots, \tau^* f_k$. Applying τ^* to both sides of (3.65) and using (3.62), we obtain

$$\begin{aligned}\tau^*([\Pi_\delta, \overrightarrow{X}](f_1, \dots, f_k)) = \widetilde{G(\xi)} \cdot \widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k}) \\ - \sum_{i=1}^{k} \widetilde{\pi}_\delta(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \mathcal{L}_{\widetilde{G(\xi)}}\Phi_{\mathrm{d}\tau^* f_i}, \dots, \Phi_{\mathrm{d}\tau^* f_k}).\end{aligned}$$

By (3.51), the right hand side above is

$$(\mathcal{L}_{\widetilde{G(\xi)}}\widetilde{\pi}_\xi)(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k}).$$

which is again, by Lemma 3.15(iv), equal to $\widetilde{G(\delta(\xi))}(\Phi_{\mathrm{d}\tau^* f_1}, \dots, \Phi_{\mathrm{d}\tau^* f_k})$. By Proposition 3.16, the latter is equal to $\tau^*\overrightarrow{\delta(X)}(f_1, \dots, f_k)$. This proves the first equality. The proof of the other one is similar. □

4. Quasi-Poisson groupoids

As an application of the general theory developed in the previous sections, we now introduce the notion of quasi-Poisson groupoids and study their properties. In particular, we study the relation with their infinitesimal invariants, namely quasi-Lie bialgebroids, and prove the integration theorem. Application to momentum map theory will also be discussed.

4.1. Definition and properties

Definition 4.1. A *quasi-Poisson groupoid* is a triple $(\Gamma \rightrightarrows M, \Pi, \Omega)$, where $\Gamma \rightrightarrows M$ is a Lie groupoid, Π is a multiplicative bivector field on Γ and $\Omega \in \Gamma(\bigwedge^3 A)$, such that the following compatibility conditions hold:

$$\tfrac{1}{2}[\Pi, \Pi] = \overrightarrow{\Omega} - \overleftarrow{\Omega}, \tag{4.66}$$

$$[\Pi, \overrightarrow{\Omega}] = 0. \tag{4.67}$$

Using Proposition 2.21 and (4.66) and (4.67), we obtain the following

Proposition 4.2. *Given a quasi-Poisson groupoid* $(\Gamma \rightrightarrows M, \Pi, \Omega)$ *there exists a bivector field* Π_M *on* M *such that*

$$\Pi_M = \alpha_*\Pi = -\beta_*\Pi.$$

Π_M satisfies the relations

$$\tfrac{1}{2}[\Pi_M, \Pi_M] = \Omega_M, \tag{4.68}$$
$$[\Pi_M, \Omega_M] = 0, \tag{4.69}$$

where Ω_M is the 3-vector field $\Omega_M = \rho(\Omega)$, and $\rho : \Gamma(\bigwedge^3 A) \to \mathfrak{X}^3(M)$ is the extension of the anchor map.

Some interesting examples of quasi-Poisson groupoids are listed below.

Example 4.3. If $\Omega = 0$ then we just have a multiplicative Poisson structure Π on a Lie groupoid $\Gamma \rightrightarrows M$. That is, $(\Gamma \rightrightarrows M, \Pi)$ is a Poisson groupoid.

Example 4.4. If G is a Lie group, then we recover the notion of quasi-Poisson structures on a Lie group of Kosmann-Schwarzbach [19], that is, a multiplicative bivector field Π on G and an element $\Omega \in \bigwedge^3 \mathfrak{g}$ such that $\frac{1}{2}[\Pi, \Pi] = \overrightarrow{\Omega} - \overleftarrow{\Omega}$ and $[\Pi, \overrightarrow{\Omega}] = 0$.

Proposition 4.5. *Let $(\Gamma \rightrightarrows M, \Pi, \Omega)$ be a quasi-Poisson groupoid such that $\Pi \in \mathfrak{X}^2(\Gamma)$ is nondegenerate. Let $\omega \in \Omega^2(\Gamma)$ be the corresponding nondegenerate 2-form and $\phi \in \Omega^3(M)$ be the 3-form on M defined by $(\bigwedge^3 \omega^\flat)(\overrightarrow{\Omega}) = \alpha^*\phi$. Then $(\Gamma \rightrightarrows M, \omega, \phi)$ is a nondegenerate twisted symplectic groupoid in the sense of* [8, *Def.* 2.1]. *That is,*

1. $\mathrm{d}\phi = 0$*;*
2. $\mathrm{d}\omega = \alpha^*\phi - \beta^*\phi$*;*
3. *ω is multiplicative, i.e., the 2-form $(\omega, \omega, -\omega)$ vanishes when restricted to the graph of the groupoid multiplication $\Lambda \subset \Gamma \times \Gamma \times \Gamma$.*

Proof. Since Π is multiplicative, it follows from Remark 2.8 that $m^*\omega = \mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\omega$. That is, ω is multiplicative.

Since ω is multiplicative, we know (see [8]) that the Lie algebroid A of Γ is isomorphic to T^*M as a vector bundle, and the isomorphism $\lambda : A \to T^*M$ is characterized by $\omega^\flat(\overrightarrow{X}) = \alpha^*\eta$ for $X \in \Gamma(A)$ and $\eta \in \Omega^1(M)$. In general, we have

$$(\textstyle\bigwedge^k \omega^\flat)(\overrightarrow{P}) = \alpha^*\varphi, \qquad (\textstyle\bigwedge^k \omega^\flat)(\overleftarrow{P}) = \beta^*\varphi,$$

for all $P \in \Gamma(\bigwedge^k A)$ and $\varphi \in \Omega^k(M)$.

Define $\phi \in \Omega^3(M)$ as the 3-form on M such that $(\bigwedge^3 \omega^\flat)(\overrightarrow{\Omega}) = \alpha^*\phi$ or, equivalently, $\overrightarrow{\Omega} = (\bigwedge^3 \Pi^\sharp)(\alpha^*\phi)$. Using the fact that $\Pi^\sharp$ is the inverse of $\omega^\flat$, $(\bigwedge^3 \Pi^\sharp)(\mathrm{d}\omega) = \frac{1}{2}[\Pi, \Pi]$, and (4.66), we deduce that

$$(\textstyle\bigwedge^3 \Pi^\sharp)(\mathrm{d}\omega) = \frac{1}{2}[\Pi, \Pi] = \overrightarrow{\Omega} - \overleftarrow{\Omega} = (\textstyle\bigwedge^3 \Pi^\sharp)(\alpha^*\phi - \beta^*\phi).$$

As a consequence, $\mathrm{d}\omega = \alpha^*\phi - \beta^*\phi$.

Let $\Pi_M = \alpha_*\Pi$. We will prove that

$$\lambda(-\delta_\Pi P) = \delta(\lambda(P)) \quad \text{for } P \in \Gamma(\textstyle\bigwedge^\bullet A), \tag{4.70}$$

where δ_Π is the 2-differential corresponding to Π (see Theorem 2.34), and $\delta : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is the map characterized by

$$\delta f = \mathrm{d} f, \quad \forall f \in C^\infty(M), \qquad \delta\eta = \mathrm{d}\eta - \Pi_M^\sharp(\eta) \lrcorner \phi, \quad \forall \eta \in \Omega^1(M).$$

From Lemma 2.4 in [8], we know that $\delta f = \mathrm{d} f = -\lambda(\delta_\Pi f)$. If $\eta \in \Omega^1(M)$ and $X \in \Gamma(A)$ are such that $\lambda(X) = \eta$ (that is, $\overrightarrow{X} = \Pi^\sharp(\alpha^*\eta)$), then using the relation

$$(\textstyle\bigwedge^2 \Pi^\sharp)\mathrm{d}\gamma = [\Pi^\sharp(\gamma), \Pi] - \frac{1}{2}(\gamma \lrcorner [\Pi, \Pi]), \qquad \forall \gamma \in \Omega^1(M),$$

and (2.19) and (4.66), we obtain

$$\begin{aligned}(\textstyle\bigwedge^2 \Pi^\sharp)\mathrm{d}\alpha^*\eta &= [\Pi^\sharp(\alpha^*\eta), \Pi] - (\alpha^*\eta \lrcorner (\overrightarrow{\Omega} - \overleftarrow{\Omega}))\\ &= -\overrightarrow{\delta_\Pi X} - (\alpha^*\eta \lrcorner ((\textstyle\bigwedge^3 \Pi^\sharp)(\alpha^*\phi - \beta^*\phi))).\end{aligned}$$

Applying $\bigwedge^2 \omega^\flat$ to both sides of this equation and using

$$(\textstyle\bigwedge^2 \omega^\flat)(\alpha^*\eta \lrcorner ((\textstyle\bigwedge^3 \Pi^\sharp)(\alpha^*\phi - \beta^*\phi))) = -\alpha^*(\Pi_M^\sharp(\eta) \lrcorner \phi),$$

we conclude that

$$-(\textstyle\bigwedge^2 \omega^\flat)(\overrightarrow{\delta_\Pi X}) = \alpha^*(\mathrm{d}\eta - \Pi_M^\sharp(\eta) \lrcorner \phi).$$

That is, $-\lambda(\delta_\Pi X) = \delta\eta$. As a consequence, by (4.67), (4.70) and since $\lambda(\Omega) = \phi$, we have $\delta\phi = 0$, and $\mathrm{d}\phi = 0$. Thus, we conclude that $(\Gamma \rightrightarrows M, \omega, \phi)$ is a nondegenerate twisted symplectic groupoid. □

4.2. Quasi-Poisson groupoids and quasi-Lie bialgebroids

In this subsection, we will describe the infinitesimal invariants of quasi-Poisson groupoids, namely quasi-Lie bialgebroids. The notion of quasi-Lie bialgebroid was first introduced by Roytenberg [33]. Here, we give an alternative definition using 2-differentials.

Definition 4.6. A *quasi-Lie bialgebroid* corresponds to a 2-differential whose square is a coboundary, i.e., $\delta : \Gamma(\bigwedge^\bullet A) \to \Gamma(\bigwedge^{\bullet+1} A)$ such that $\delta \circ \delta = [\![\Omega, \cdot]\!]$ for some $\Omega \in \Gamma(\bigwedge^3 A)$ satisfying $\delta\Omega = 0$.

An interesting example of a quasi-Lie bialgebroid is the following:

Example 4.7. Recall that [34, (1)] a *twisted Poisson structure* (M, π, ϕ) is a bivector field $\pi \in \mathfrak{X}^2(M)$ and a closed 3-form $\phi \in \Omega^3(M)$ such that

$$\frac{1}{2}[\pi, \pi] = (\textstyle\bigwedge^3 \pi^\sharp)(\phi), \tag{4.71}$$

where $\pi^\sharp$ is the bundle map $T^*M \to TM$ induced by π (i.e. $\pi^\sharp(x)(\sigma) := \pi(x)(\sigma, \cdot)$ for $x \in M$ and $\sigma \in T_x^*M$). As explained in [34], a twisted Poisson structure induces a

Lie algebroid structure on T^*M with anchor map $\pi^\sharp$ and Lie bracket of sections σ and τ defined by

$$[\sigma, \tau] := \mathcal{L}_{\pi^\sharp(\sigma)}\tau - \mathcal{L}_{\pi^\sharp(\tau)}\sigma - \mathrm{d}\pi(\sigma, \tau) + \phi(\pi^\sharp(\sigma), \pi^\sharp(\tau), \cdot). \tag{4.72}$$

We will denote this Lie algebroid by $(T^*M)_{(\pi,\phi)}$. Sections of its exterior algebra are ordinary differential forms. There is a derivation $\delta_{\pi,\phi} : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ deforming the de Rham differential d by ϕ, which is defined as follows. For $f \in C^\infty(M)$, $\delta_{\pi,\phi} f = \mathrm{d}f$, and $\delta_{\pi,\phi}\sigma = \mathrm{d}\sigma - \pi^\sharp(\sigma) \lrcorner \phi$ if $\sigma \in \Omega^1(M)$. It turns out that $\delta_{\pi,\phi}[\sigma, \tau] = [\delta_{\pi,\phi}\sigma, \tau] + [\sigma, \delta_{\pi,\phi}\tau]$ for all $\sigma, \tau \in \Omega^1(M)$, and $\delta^2_{\pi,\phi} = [\phi, \cdot]$. Thus, one obtains a quasi-Lie bialgebroid $((T^*M)_{(\pi,\phi)}, \delta_{\pi,\phi})$, which was first described in [8, p. 188].

A direct consequence of Lemma 2.32 is the following

Proposition 4.8. *Let (A, δ, Ω) be a quasi-Lie bialgebroid. Then δ induces a bivector field π_M on M such that*

$$\tfrac{1}{2}[\pi_M, \pi_M] = \Omega_M, \qquad [\pi_M, \Omega_M] = 0,$$

where Ω_M is the 3-vector field $\Omega_M = \rho(\Omega)$, and $\rho : \Gamma(\bigwedge^3 A) \to \mathfrak{X}^3(M)$ is the extension of the anchor map.

Now we are ready to state the main theorem of this section: quasi-Lie bialgebroids are indeed the infinitesimal invariants of quasi-Poisson groupoids.

Theorem 4.9. *If $(\Gamma \rightrightarrows M, \Pi, \Omega)$ is a quasi-Poisson groupoid, then there exists a natural quasi-Lie bialgebroid structure (δ, Ω) on the Lie algebroid A of Γ.*

Conversely, if (A, δ, Ω) is a quasi-Lie bialgebroid, where A is the Lie algebroid of an α-connected and α-simply connected Lie groupoid Γ, then there exists a quasi-Poisson groupoid structure (Π, Ω) on Γ such that the corresponding quasi-Lie bialgebroid is (δ, Ω).

Proof. Let $(\Gamma \rightrightarrows M, \Pi, \Omega)$ be a quasi-Poisson groupoid. Since Π is a multiplicative bivector field, it induces a 2-differential δ_Π on A according to Theorem 2.34. In addition, since $\Pi \mapsto \delta_\Pi$ preserves the graded Lie algebra structures, (2.17) and (4.66) imply that

$$\delta_\Pi \circ \delta_\Pi = \tfrac{1}{2}[\delta_\Pi, \delta_\Pi] = \mathrm{ad}(\Omega).$$

Moreover (4.67) yields $\delta_\Pi(\Omega) = 0$. As a consequence, (A, δ_Π, Ω) is a quasi-Lie bialgebroid.

Conversely, let (A, δ, Ω) be a quasi-Lie bialgebroid. Since δ is a 2-differential, according to Theorem 2.34, there exists a multiplicative bivector field Π on Γ such that $\delta_\Pi = \delta$. On the other hand, the 3-differential $\mathrm{ad}(\Omega)$ can be integrated to the multiplicative 3-vector field $\overrightarrow{\Omega} - \overleftarrow{\Omega}$. In addition, using the identity $\delta \circ \delta = [\![\Omega, \cdot]\!]$, we have

$$\delta_{[\Pi,\Pi]} = [\delta, \delta] = 2\delta \circ \delta = 2\,\mathrm{ad}(\Omega) = 2\delta_{\overrightarrow{\Omega} - \overleftarrow{\Omega}}.$$

Thus, by Theorem 2.34 again, we have

$$\tfrac{1}{2}[\Pi, \Pi] = \overrightarrow{\Omega} - \overleftarrow{\Omega}.$$

Moreover, $[\Pi, \overrightarrow{\Omega}] = \overrightarrow{\delta_\Pi \Omega} = \overrightarrow{\delta\Omega} = 0$. Thus, we conclude that $(\Gamma \rightrightarrows M, \Pi, \Omega)$ is a quasi-Poisson groupoid integrating the quasi-Lie bialgebroid (A, δ, Ω). □

Remark. Theorem 4.9 generalizes a result of Kosmann-Schwarzbach regarding quasi-Poisson Lie groups and quasi-Lie bialgebras [19]. On the other hand, when $\Omega = 0$, that is, (A, δ) is a Lie bialgebroid, we recover the classical results in [27, 29]: there exists a one-to-one correspondence between Lie bialgebroids and Poisson groupoids.

As another consequence of Theorem 4.9, we recover the construction of the nondegenerate twisted symplectic groupoid associated with a twisted Poisson manifold [8].

Corollary 4.10. *Let (M, π, ϕ) be a twisted Poisson structure. If the Lie algebroid $(T^*M)_{(\pi,\phi)}$ can be integrated to an α-simply connected and α-connected Lie groupoid Γ, then Γ is a twisted symplectic groupoid.*

Proof. Let $\lambda : T^*M \to A$ be the Lie algebroid isomorphism between the Lie algebroid A of Γ and T^*M. Denote by δ and Ω the almost 2-differential and the 3-section of A respectively such that

$$\Omega = \lambda(\phi), \qquad -\lambda(\delta_{\pi,\phi}\varphi) = \delta(\lambda(\varphi)), \qquad \forall \varphi \in \Omega^\bullet(M), \tag{4.73}$$

where $\delta_{\pi,\phi}$ is defined as in Example 4.7. Then (A, δ, Ω) is clearly a quasi-Lie bialgebroid induced by the quasi-Lie bialgebroid structure $(T^*M, \delta_{\pi,\phi}, \phi)$. Therefore, according to Theorem 4.9, there exists a bivector field Π such that $(\Gamma \rightrightarrows M, \Pi, \Omega)$ is a quasi-Poisson groupoid satisfying $\delta_\Pi = \delta$. Using (2.19) and (4.73), we have

$$\lambda(\mathrm{d}f) = \lambda(\delta_{\pi,\phi} f) = -\delta f = \Pi^\sharp(\alpha^*\mathrm{d}f), \qquad \forall f \in C^\infty(M).$$

Thus, $\lambda(\eta) = \Pi^\sharp(\alpha^*\eta)$ for all $\eta \in T^*M$. As a consequence, $\lambda^* : A^* \to TM$ is given by $\lambda^*(\xi) = -\alpha_*\Pi^\sharp(\xi)$ for $\xi \in A^*$, where A^* is identified with the conormal bundle of M. Hence, $\lambda^*(\xi) = -\Pi^\sharp(\xi)$, because $\Pi^\sharp(\xi)$ is tangent to M. Therefore,

$$\Pi^\sharp(\xi + \alpha^*\mathrm{d}f) = -\lambda^*(\xi) + \lambda(\mathrm{d}f), \qquad \forall \xi \in A^* \text{ and } f \in C^\infty(M).$$

Since any element of the cotangent space $T^*_{\epsilon(m)}\Gamma$ $(m \in M)$ can be written as $\xi + \alpha^*\mathrm{d}f$ with $\xi \in A^*$ and $f \in C^\infty(M)$, and λ and λ^* are injective (λ is a vector bundle isomorphism), it follows that Π is nondegenerate along M. By Theorem 5.3 in [29], one can extend this nondegeneracy to all points in Γ. From Proposition 4.5, we conclude that Γ is the twisted symplectic groupoid integrating (M, π, ϕ). □

We end this section with the following proposition, which reveals the relationship between the bivector fields obtained by Propositions 4.2 and 4.8.

Proposition 4.11. *Let $(\Gamma \rightrightarrows M, \Pi, \Omega)$ be a quasi-Poisson groupoid, and (A, δ_Π, Ω) the corresponding quasi-Lie bialgebroid. Denote by Π_M and π_M the bivector fields on M induced from the quasi-Poisson groupoid structure on Γ and the quasi-Lie bialgebroid structure on A as in Propositions 4.2 and 4.8 respectively. Then*

$$\Pi_M = \pi_M.$$

Proof. Let $f, g \in C^\infty(M)$. Then, using (2.19),

$$-\langle \rho(\delta f), \mathrm{d}g\rangle = -\langle \alpha_*([\Pi, \alpha^* f]), \mathrm{d}g\rangle = \Pi(\alpha^*\mathrm{d}f, \alpha^*\mathrm{d}g) = (\alpha_*\Pi)(\mathrm{d}f, \mathrm{d}g),$$

and the conclusion follows. □

4.3. *Hamiltonian Γ-spaces of quasi-Poisson groupoids*

Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Recall that a Γ*-space* is a smooth manifold X with a map $J : X \to M$, called the *momentum map*, and an action

$$\Gamma \times_M X = \{(g, x) \in \Gamma \times X \mid \beta(g) = J(x)\} \to X, \qquad (g, x) \mapsto g \cdot x,$$

satisfying

1. $J(g \cdot x) = \alpha(g)$ for $(g, x) \in \Gamma \times_M X$;
2. $(gh) \cdot x = g \cdot (h \cdot x)$ for $g, h \in \Gamma$ and $x \in X$ such that $\beta(g) = \alpha(h)$ and $J(x) = \beta(h)$;
3. $\epsilon(J(x)) \cdot x = x$ for $x \in X$.

For a Poisson groupoid Γ, Hamiltonian Γ-spaces were introduced in [22]; they generalize the notion of Poisson actions of Poisson groups. In particular, Poisson reduction theory works for Hamiltonian Γ-spaces, which gives rise to new Poisson manifolds. For quasi-Poisson groupoids, one can introduce Hamiltonian Γ-spaces in a similar fashion.

Definition 4.12. Let $(\Gamma \rightrightarrows M, \Pi, \Omega)$ be a quasi-Poisson groupoid. A *Hamiltonian* Γ-*space* is a Γ-space X with momentum map $J : X \to M$ and a bivector field $\Pi_X \in \mathfrak{X}^2(X)$ such that:

1. the graph of the action $\{(g, x, g \cdot x) \mid J(x) = \beta(g)\}$ is a coisotropic submanifold of $(\Gamma \times X \times X, \Pi \oplus \Pi_X \oplus -\Pi_X)$;
2. $\frac{1}{2}[\Pi_X, \Pi_X] = \hat{\Omega}$, where the hat denotes the map $\Gamma(\bigwedge^3 A) \to \mathfrak{X}^3(X)$, induced by the infinitesimal action of the Lie algebroid on X: $\Gamma(A) \to \mathfrak{X}(X)$, $Y \mapsto \hat{Y}$.

Proposition 4.13. *Let $(\Gamma \rightrightarrows M, \Pi, \Omega)$ be a quasi-Poisson groupoid. If (X, Π_X) is a Hamiltonian Γ-space with momentum map J, then J maps Π_X to Π_M, where Π_M is given by Proposition 4.2.*

Proof. The result follows using the coisotropy condition and the fact that $(-\beta^*\eta, J^*\eta, 0)$, with $\eta \in T^*M$, is conormal to the graph of the groupoid action. Indeed,

$$0 = \Pi(-\beta^*\eta_1, -\beta^*\eta_2) + \Pi(J^*\eta_1, J^*\eta_2) = (\beta_*\Pi + J_*\Pi_X)(\eta_1, \eta_2)$$

for any $\eta_1, \eta_2 \in T^*_m M$. Thus, $J_*\Pi_X = -\beta_*\Pi = \Pi_M$. □

The following theorem gives an equivalent description of Hamiltonian Γ-spaces of quasi-Poisson groupoids in terms of their infinitesimal objects.

Theorem 4.14. *Let $(\Gamma \rightrightarrows M, \Pi, \Omega)$ be a quasi-Poisson groupoid with the corresponding quasi-Lie bialgebroid (A, δ_Π, Ω). Then (X, Π_X) is a Hamiltonian Γ-space with momentum map $J : X \to M$ if and only if $\frac{1}{2}[\Pi_X, \Pi_X] = \hat{\Omega}$ and*

$$[\Pi_X, J^* f] = \widehat{\delta_\Pi f}, \quad \forall f \in C^\infty(M), \qquad [\Pi_X, \hat{Y}] = \widehat{\delta_\Pi(Y)}, \quad \forall Y \in \Gamma(A). \tag{4.74}$$

Proof. Using Theorem 7.1 in [22] we know that the coisotropy condition is equivalent to the following conditions:

1. For any $f \in C^\infty(M)$, $X_{J^*f}(x) = (r_x)_* X_{\alpha^* f}(u)$, where $x \in X$, $u = J(x)$ and r_x denotes the map $g \mapsto g \cdot x$ from $\beta^{-1}(u)$ to X.
2. For any compatible $(g, x) \in \Gamma \times_M X$,

$$\Pi_X(g \cdot x) = (L_{\mathcal{X}})_* \Pi_X(x) + (R_{\mathcal{Y}})_* \Pi(g) - (R_{\mathcal{Y}})_* (L_{\mathcal{X}})_* \Pi(u),$$

   where $u = \beta(g) = J(x)$, $\mathcal{X}$ is any local bisection through g, and $\mathcal{Y}$ is any local section of J through the point x.

From the first condition and the equality $\overrightarrow{\delta_\Pi f} = [\Pi, \alpha^* f]$, it follows that $[\Pi_X, J^* f] = \widehat{\delta_\Pi f}$.

On the other hand, let $x \in X$ and $u = J(x)$. Applying the second condition to the family of (local) bisections $\mathcal{X}_t = \exp tY$ associated to $Y \in \Gamma(A)$ and $g_t = (\exp tY)(u)$, one obtains

$$(L_{\mathcal{X}_t}^{-1})_* \Pi_X(g_t \cdot x) = \Pi_X(x) + (R_{\mathcal{Y}})_* (L_{\mathcal{X}_t}^{-1})_* \Pi(g_t) - (R_{\mathcal{Y}})_* \Pi(u)$$

for any section $\mathcal{Y}$ of J. Then, taking derivatives on both sides, we have

$$[\Pi_X, \hat{Y}] = \widehat{\delta_\Pi(Y)}.$$

The other direction follows just by going backwards. $\square$

Remark. Note that (4.74) is equivalent to $[\Pi_X, \hat{P}] = \widehat{\delta_\Pi P}$ for any $P \in \Gamma(\bigwedge^k A)$, $k \geq 0$. That is, the following diagram is commutative:

$$\begin{tikzcd}
\Gamma(\bigwedge^k A) \arrow[r, "\widehat{\ }"] \arrow[d, "\delta_\Pi"'] & \mathfrak{X}^k(X) \arrow[d, "{[\Pi_X, \cdot]}"] \\
\Gamma(\bigwedge^{k+1} A) \arrow[r, "\widehat{\ }"] & \mathfrak{X}^{k+1}(X)
\end{tikzcd} \qquad \forall k \geq 0.$$

4.4. Twists of quasi-Lie bialgebroids

As for quasi-Lie bialgebras [15], one can talk about twists of a quasi-Lie bialgebroid. Given a quasi-Lie bialgebroid (A, δ, Ω) and a section $t \in \Gamma(\bigwedge^2 A)$, let $\delta^t = \delta + [\![t, \cdot]\!]$, and $\Omega^t = \Omega + \delta t + \frac{1}{2}[\![t, t]\!]$. Using (2.14) and the properties of the Schouten bracket, it is easy to see that (A, δ^t, Ω^t) is also a quasi-Lie bialgebroid, which will be called the *twist of* (A, δ, Ω) *by* $t \in \Gamma(\bigwedge^2 A)$.

The following proposition describes how a twist affects integration (see [1, (3.3.8)] for the case of quasi-Lie bialgebras).

Proposition 4.15. *Assume that* $(\Gamma \rightrightarrows M, \Pi, \Omega)$ *is a quasi-Poisson groupoid with corresponding quasi-Lie bialgebroid* (A, δ, Ω)*, and* $t \in \Gamma(\bigwedge^2 A)$*. Then* $(\Gamma \rightrightarrows M, \Pi^t, \Omega^t)$ *is a quasi-Poisson groupoid with quasi-Lie bialgebroid* (A, δ^t, Ω^t)*, where* $\Pi^t = \Pi + \overrightarrow{t} - \overleftarrow{t}$.

Proof. This is a direct consequence of Theorem 2.34. Note that if Π and Π' are multiplicative bivector fields then $\delta_{\Pi+\Pi'} = \delta_\Pi + \delta_{\Pi'}$. Also $\delta_{\overrightarrow{t} - \overleftarrow{t}} = \mathrm{ad}(t)$ (see Example 2.40). □

Next, we will show how a Hamiltonian Γ-space of a quasi-Poisson groupoid $(\Gamma \rightrightarrows M, \Pi, \Omega)$ must be modified in order to obtain a Hamiltonian Γ-space for the twisted quasi-Poisson structure (Π^t, Ω^t).

Proposition 4.16. *Assume that* $(\Gamma \rightrightarrows M, \Pi, \Omega)$ *is a quasi-Poisson groupoid and* $t \in \Gamma(\bigwedge^2 A)$*. There is a bijection between Hamiltonian* Γ*-spaces of* $(\Gamma \rightrightarrows M, \Pi, \Omega)$ *and those of* $(\Gamma \rightrightarrows M, \Pi^t, \Omega^t)$.

More precisely, if $(X \to M, \Pi_X)$ *is a Hamiltonian* Γ*-space of* $(\Gamma \rightrightarrows M, \Pi, \Omega)$*, then* $(X \to M, \Pi_X + \hat{t})$ *is a Hamiltonian* Γ*-space of* $(\Gamma \rightrightarrows M, \Pi^t, \Omega^t)$.

Proof. Let $(X \to M, \Pi_X)$ be a Hamiltonian Γ-space of the quasi-Poisson groupoid $(\Gamma \rightrightarrows M, \Pi, \Omega)$. Using Theorem 4.14, we deduce that

$$[\Pi_X, \hat{P}] = \widehat{\delta_\Pi P} \quad \text{for any } P \in \Gamma(\textstyle\bigwedge^k A),\ k \geq 0.$$

Therefore, from the fact that $\bigoplus_k \Gamma(\bigwedge^k A) \to \bigoplus_k \mathfrak{X}^k(X)$, $Y \mapsto \hat{Y}$, is a graded Lie algebra morphism, one gets

$$[\Pi_X + \hat{t}, \hat{P}] = \widehat{\delta_\Pi P} + \widehat{[\![t, P]\!]} = \widehat{\delta_{\Pi'} P}.$$

On the other hand, it is trivial to see that

$$\tfrac{1}{2}[\Pi_X + \hat{t}, \Pi_X + \hat{t}] = \widehat{\Omega^t}.$$

As a consequence of Theorem 4.14, $(X \to M, \Pi_X + \hat{t})$ is a Hamiltonian Γ-space of the quasi-Poisson groupoid $(\Gamma \rightrightarrows M, \Pi^t, \Omega^t)$. □

4.5. Quasi-Poisson groupoids associated to Manin pairs

In this subsection we will describe an example of a quasi-Poisson groupoid associated to a quasi-Manin triple.

Let $(\mathfrak{d}, \mathfrak{g})$ be a *Manin pair*, that is, $\mathfrak{d}$ is an even-dimensional Lie algebra with an invariant, nondegenerate symmetric bilinear form, and $\mathfrak{g}$ is a maximal isotropic subalgebra of $\mathfrak{d}$. In this case, one can integrate the Manin pair $(\mathfrak{d}, \mathfrak{g})$ to the so-called *group pair* (D, G), where D and G are connected and simply connected Lie groups with Lie algebras $\mathfrak{d}$ and $\mathfrak{g}$ respectively. Furthermore, the action of the Lie group D on itself by left multiplication induces an action of D on $S = D/G$, and in particular a G-action on S, which is called the *dressing action*. As in [1], the infinitesimal dressing action is denoted by $v \mapsto v_S$ for any $v \in \mathfrak{d}$.

If $\mathfrak{h}$ is an isotropic complement of $\mathfrak{g}$ in $\mathfrak{d}$, by identifying $\mathfrak{h}$ with $\mathfrak{g}^*$ we obtain a quasi-Lie bialgebra structure on $\mathfrak{g}$, with cobracket $F : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ and 3-vector $\Omega \in \bigwedge^3 \mathfrak{g}$. If $\{e_i\}$ is a basis of $\mathfrak{g}$ and $\{\epsilon^i\}$ the dual basis of $\mathfrak{g}^* \cong \mathfrak{h}$, then $F(e_i) = \frac{1}{2} \sum_{j,k} F_i^{jk} e_j \wedge e_k$ and $\Omega = \frac{1}{6} \sum_{i,j,k} \Omega^{ijk} e_i \wedge e_j \wedge e_k$. Moreover, the bracket on $\mathfrak{d} \cong \mathfrak{g} \oplus \mathfrak{h}$ can be written as

$$\begin{gathered}
[e_i, e_j]_{\mathfrak{d}} = \sum_{k=1}^{n} c_{ij}^k e_k, \qquad [e_i, \epsilon^j]_{\mathfrak{d}} = \sum_{k=1}^{n} (-c_{ik}^j \epsilon^k + F_i^{jk} e_k), \\
[\epsilon^i, \epsilon^j]_{\mathfrak{d}} = \sum_{k=1}^{n} (F_k^{ij} \epsilon^k + \Omega^{ijk} e_k),
\end{gathered} \tag{4.75}$$

where c_{ij}^k are the structure constants of the Lie algebra $\mathfrak{g}$ with respect to the basis $\{e_i\}$.

Example 4.17. Let $\mathfrak{g}$ be a Lie algebra endowed with a nondegenerate symmetric bilinear form K. On the direct sum $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ one can construct a scalar product $(\cdot|\cdot)$ by

$$((u_1, u_2)|(v_1, v_2)) = K(u_1, v_1) - K(u_2, v_2)$$

for $(u_1, u_2), (v_1, v_2) \in \mathfrak{d}$. Then, $(\mathfrak{d}, \Delta(\mathfrak{g}), \frac{1}{2}\Delta_-(\mathfrak{g}))$ is a quasi-Manin triple, where $\Delta(v) = (v, v)$ and $\Delta_-(v) = (v, -v)$ for $v \in \mathfrak{g}$ (see [1]). In this case, as far as the corresponding quasi-Lie bialgebra is concerned, the cobracket F vanishes and Ω can be identified with the trilinear form on $\mathfrak{g}$ given by $(u, v, w) \mapsto \frac{1}{4} K(w, [u, v]_{\mathfrak{g}})$.

Let $\lambda : T_s^* S \to \mathfrak{g}$ be the dual map of the infinitesimal dressing action $\mathfrak{g}^* \cong \mathfrak{h} \to T_s S$. That is,

$$\langle \lambda(\theta_s), \eta \rangle = \langle \theta_s, \eta_S(s) \rangle, \qquad \forall \theta_s \in T_s^* S \text{ and } \eta \in \mathfrak{h}.$$

A direct consequence is that

$$\lambda(\mathrm{d} f) = \sum_{i=1}^{n} (\epsilon^i)_S(f) e_i \qquad \text{for } f \in C^\infty(S). \tag{4.76}$$

Remark 4.18. Recall that an isotropic complement $\mathfrak{h}$ is said to be *admissible* at a point $s \in S = D/G$ if the infinitesimal dressing action restricted to $\mathfrak{h}$ defines an isomorphism from $\mathfrak{h}$ onto $T_s S$ [1]. In this case, we can define an isomorphism from $\mathfrak{g}$ to $T_s^* S$, $\xi \mapsto \xi_{\mathfrak{h}}(s)$, by

$$\langle \xi_{\mathfrak{h}}(s), \eta_S(s) \rangle = -(\xi \mid \eta), \qquad \forall \eta \in \mathfrak{h},$$

where on the right hand side, $(\cdot|\cdot)$ is the bilinear form on $\mathfrak{d}$. $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is said to be an *admissible* quasi-Manin triple if $\mathfrak{h}$ is admissible at every point of S. In this case, if $\lambda(\theta_s) = \xi$ then $\xi_{\mathfrak{h}}(s) = -\theta_s$.

Next, we will show that on the transformation Lie algebroid $\mathfrak{g} \times S \to S$ there exists a natural quasi-Lie bialgebroid structure.

Proposition 4.19. *Assume that* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ *is a quasi-Manin triple with associated quasi-Lie bialgebra* $(\mathfrak{g}, F, \Omega)$*. On the transformation Lie algebroid* $\mathfrak{g} \times S \to S$ (*where* $\mathfrak{g}$ *acts on S by the infinitesimal dressing action*) *define an almost* 2*-differential*

$$\delta(f) = \lambda(\mathrm{d}f) \quad \text{for } f \in C^\infty(S), \qquad \delta\xi = -F(\xi) \quad \text{for } \xi \in \mathfrak{g}, \tag{4.77}$$

where λ *is defined by* (4.76) *and* $\xi \in \mathfrak{g}$ *is considered as a constant section of the Lie algebroid* $\mathfrak{g} \times S \to S$*, extending this operation to all sections using the derivation law. Then* $(\mathfrak{g} \times S, \delta, \Omega)$ *is a quasi-Lie bialgebroid.*

Proof. We remark that it suffices to check the axioms of a quasi-Lie bialgebroid for functions on S and constant sections of $\mathfrak{g} \times S$, since the general case follows from the derivation law.

If $f \in C^\infty(S)$, then by (4.75),

$$\begin{aligned}
\delta^2 f &= \sum_{i=1}^n \delta((\epsilon^i)_S(f) e_i) = \sum_{i,j=1}^n (\epsilon^j)_S(\epsilon^i)_S(f) e_j \wedge e_i + \sum_{i=1}^n (\epsilon^i)_S(f)\delta(e_i) \\
&= \sum_{i<j} \big((\epsilon^i)_S(\epsilon^j)_S(f) - (\epsilon^j)_S(\epsilon^i)_S(f)\big) e_i \wedge e_j + \sum_{i=1}^n (\epsilon^i)_S(f)\delta(e_i) \\
&= \sum_{i<j} ([\epsilon^i, \epsilon^j]_{\mathfrak{d}})_S(f) e_i \wedge e_j + \sum_{k=1}^n (\epsilon^k)_S(f)\delta(e_k) \\
&= \sum_{i<j}\sum_{k=1}^n \big(F_k^{ij}(\epsilon^k)_S(f) + \Omega^{ijk}(e_k)_S(f)\big) e_i \wedge e_j - \sum_{k=1}^n (\epsilon^k)_S(f) \sum_{i<j} F_k^{ij} e_i \wedge e_j \\
&= \sum_{i<j}\sum_{k=1}^n \Omega^{ijk}(e_k)_S(f) e_i \wedge e_j = \frac{1}{2}\sum_{i,j,k=1}^n \Omega^{ijk}(e_k)_S(f) e_i \wedge e_j = [\![\Omega, f]\!].
\end{aligned}$$

Moreover, since $(\mathfrak{g}, F, \Omega)$ is a quasi-Lie bialgebra, we have

$$\delta^2 \xi = [\![\Omega, \xi]\!], \qquad \forall \xi \in \mathfrak{g}.$$

Next, let us show that $\delta[\![\xi, f]\!] = [\![\delta\xi, f]\!] + [\![\xi, \delta f]\!]$ for $\xi \in \mathfrak{g}$ and $f \in C^\infty(S)$. Taking $\xi = e_i$ and using (4.75), we obtain

$$
\begin{aligned}
[\![\delta e_i, f]\!]+[\![e_i, \delta f]\!]-\delta[\![e_i, f]\!] &= \sum_{j,k=1}^{n} (e_j)_S(f) F_i^{jk} e_k + \sum_{j=1}^{n} (e_i)_S(\epsilon^j)_S(f) e_j \\
&\quad + \sum_{j,k=1}^{n} (\epsilon^j)_S(f) c_{ij}^k e_k - \sum_{j=1}^{n} (\epsilon^j)_S(e_i)_S(f) e_j \\
&= \sum_{j=1}^{n} \big((e_i)_S(\epsilon^j)_S(f) e_j - (\epsilon^j)_S(e_i)_S(f) e_j\big) \\
&\quad + \sum_{j,k=1}^{n} \big((e_j)_S(f) F_i^{jk} + (\epsilon^j)_S(f) c_{ij}^k\big) e_k \\
&= \sum_{j=1}^{n} [(e_i)_S, (\epsilon^j)_S](f) e_j - \sum_{j,k=1}^{n} (F_i^{jk} e_k - c_{ik}^j \epsilon^k)_S(f) e_j \\
&= \sum_{j=1}^{n} [(e_i)_S, (\epsilon^j)_S](f) e_j - \sum_{j=1}^{n} ([e_i, \epsilon^j]_{\mathfrak{d}})_S(f) e_j = 0.
\end{aligned}
$$

Since also $\delta[\xi_1, \xi_2] = [\delta\xi_1, \xi_2] + [\xi_1, \delta\xi_2]$ for any $\xi_1, \xi_2 \in \mathfrak{g}$, the conclusion follows. □

Now, consider the transformation groupoid $\Gamma : G \times S \rightrightarrows S$ associated to the dressing action. Theorem 4.9 implies that Γ is a quasi-Poisson groupoid. In what follows, we will explicitly describe the multiplicative bivector field Π on Γ.

The fact that $(\mathfrak{g}, F, \Omega)$ is a quasi-Lie bialgebra implies that G is a quasi-Poisson Lie group with multiplicative bivector field denoted by Π_G. Moreover, there exists a bivector field Π_S on S given by $\Pi_S = -\sum_{i=1}^n (e_i)_S \otimes (\epsilon^i)_S$, i.e.,

$$
\Pi_S(\mathrm{d}f, \mathrm{d}g) = -\sum_{i=1}^{n} (\epsilon^i)_S(f)(e_i)_S(g), \qquad \forall f, g \in C^\infty(S). \tag{4.78}
$$

Using Proposition 4.8 and (4.77) and (4.78), we directly deduce the following:

Proposition 4.20. *Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ be a quasi-Manin triple and $(\mathfrak{g} \times S \to S, \delta, \Omega)$ the corresponding quasi-Lie bialgebroid. Then the bivector field on S induced by δ as in Proposition 4.8 coincides with $\Pi_S = -\sum_{i=1}^n (e_i)_S \otimes (\epsilon^i)_S$.*

Proof. For all $f, g \in C^\infty(S)$,

$$
-(\delta f)_S(g) = -\sum_{i=1}^{n} (\epsilon^i)_S(f)(e_i)_S(g) = \Pi_S(\mathrm{d}f, \mathrm{d}g).
$$
□

Example 4.21. Let $\mathfrak{g}$ be a Lie algebra endowed with a nondegenerate symmetric bilinear form K and consider the corresponding quasi-Manin triple $(\mathfrak{d}, \Delta(\mathfrak{g}), \frac{1}{2}\Delta_-(\mathfrak{g}))$ (see Example 4.17). Then $\mathfrak{g}$ acts on $S = G$ by the adjoint action and the 2-differential is given by

$$\delta(f) = \frac{1}{2}\sum_{i=1}^{n}(\overrightarrow{e_i} + \overleftarrow{e_i})(f)e_i, \quad \forall f \in C^\infty(G), \qquad \delta\xi = 0, \quad \forall \xi \in \mathfrak{g},$$

where $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}$. Moreover, the bivector field on G induced by δ is

$$\Pi_G = \frac{1}{2}\sum_{i=1}^{n} \overleftarrow{e_i} \wedge \overrightarrow{e_i},$$

which was first obtained in [1] (see also [2]).

Now we are ready to describe the quasi-Poisson groupoid structure on the transformation groupoid $G \times S \rightrightarrows S$.

Theorem 4.22. *Assume that $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a quasi-Manin triple. Define a bivector field Π on $G \times S$ by*

$$\begin{aligned}\Pi((\theta_g, \theta_s), (\theta'_g, \theta'_s)) = {} & \Pi_G(\theta_g, \theta'_g) - \Pi_S(\theta_s, \theta'_s) \\ & + \langle \theta'_s, (L^*_g\theta_g)_S\rangle - \langle \theta_s, (L^*_g\theta'_g)_S\rangle \end{aligned} \tag{4.79}$$

for any $(g, s) \in G \times S$, $\theta_g, \theta'_g \in T^*_g G$, and $\theta_s, \theta'_s \in T^*_s S$, where $(L^*_g\theta_g)_S$ denotes the vector field on S corresponding to the dressing action of $L^*_g\theta_g \in \mathfrak{g}^* \cong \mathfrak{h} \subset \mathfrak{d}$, and similarly for $(L^*_g\theta'_g)_S$. Then $(G \times S \rightrightarrows S, \Pi, \Omega)$ is a quasi-Poisson groupoid integrating the 2-differential δ given by (4.77).

Proof. Let Π be the multiplicative bivector field on $G \times S$ integrating δ. Then, by Propositions 4.11 and 4.20 and the fact that $\beta(g, s) = s$ for any $(g, s) \in G \times S$,

$$\Pi((0, \mathrm{d}f), (0, \mathrm{d}g)) = \Pi(\beta^*(\mathrm{d}f), \beta^*(\mathrm{d}g)) = (\beta_*\Pi)(\mathrm{d}f, \mathrm{d}g) = -\Pi_S(\mathrm{d}f, \mathrm{d}g).$$

Therefore,

$$\Pi((0, \theta_s), (0, \theta'_s)) = -\Pi_S(\theta_s, \theta'_s), \qquad \forall \theta_s, \theta'_s \in T^*_s S.$$

Next, if $f \in C^\infty(S)$ and $\theta_g \in T^*_g G$, then from (2.19), (4.76) and (4.77), it follows that

$$\begin{aligned}\Pi((0, \mathrm{d}f), (\theta_g, 0)) &= \langle \Pi^\sharp(\beta^*(\mathrm{d}f)), (\theta_g, 0)\rangle = -\langle L_{(g,s)*}(\delta f), (\theta_g, 0)\rangle \\ &= -\langle (L_g)_*(\delta f), \theta_g\rangle = -\langle \delta f, L^*_g\theta_g\rangle = -\langle \mathrm{d}f, (L^*_g\theta_g)_S\rangle.\end{aligned}$$

Thus,

$$\Pi((0, \theta_s), (\theta_g, 0)) = -\langle \theta_s, (L^*_g\theta_g)_S\rangle \quad \text{ for } \theta_g \in T^*_g G,\ \theta_s \in T^*_s S.$$

Finally, fixing $s \in S$, we write Π_s for the bivector field on G defined by $\Pi_s(g)(\theta_g, \theta'_g) = \Pi(g, s)((\theta_g, 0), (\theta'_g, 0))$. Then it is clear that Π_s is a multiplicative bivector field on G. Moreover, if $\xi \in \mathfrak{g}$ is considered as a constant section of $\mathfrak{g} \times S \to S$, then $\overrightarrow{\xi} = (\overrightarrow{\xi}_G, 0)$, where $\overrightarrow{\xi}_G$ is the unique right invariant vector field on G through ξ. Using (2.19) and (4.77), we get

$$([\Pi_s, \overrightarrow{\xi}_G], 0) = [\Pi, \overrightarrow{\xi}\,] = \overrightarrow{\delta\xi} = (-\overrightarrow{F(\xi)}_G, 0) = (-[\overrightarrow{\xi}_G, \Pi_G], 0) = ([\Pi_G, \overrightarrow{\xi}\,]_G, 0).$$

As a consequence, Π_s coincides with Π_G, that is,

$$\Pi((\theta_g, 0), (\theta'_g, 0)) = \Pi_G(\theta_g, \theta'_g), \qquad \forall \theta_g, \theta'_g \in T^*_g G.$$

Summing up, the multiplicative bivector field Π integrating δ is the one given by (4.79), and $(\Gamma \rightrightarrows S, \Pi, \Omega)$ is a quasi-Poisson groupoid. □

Example 4.23. In the particular case when $\mathfrak{h}$ is also a Lie subalgebra of $\mathfrak{d}$, that is, $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple, then $\Omega = 0$ and (G, Π_G) is a Poisson group. Moreover, if the dressing action is complete, we have $D/G \cong G^*$, the dual Poisson group. Thus, we recover a Poisson groupoid structure on $G \times G^* \rightrightarrows G^*$, whose Poisson bivector field is described in [23].

On the other hand, when the dressing action is not complete, we can still have a Poisson groupoid $G \times (D/G) \rightrightarrows D/G$, while the groupoid $G \times G^* \rightrightarrows G^*$ does not exist any more. It would be interesting to study the relation between this Poisson groupoid and the symplectic groupoid of Lu–Weinstein [25].

From Examples 4.17 and 4.21 and Theorem 4.22, one can deduce the following:

Corollary 4.24. *Assume that $\mathfrak{g}$ is a Lie algebra endowed with a nondegenerate symmetric bilinear form K, and G is the corresponding connected and simply connected Lie group. Then the transformation groupoid $G \times G \rightrightarrows G$, where G acts on G by conjugation, together with the multiplicative bivector field Π on $G \times G$:*

$$\Pi(g, s) = \frac{1}{2}\sum_{i=1}^{n}\Big(\overleftarrow{e_i^2} \wedge \overrightarrow{e_i^2} - \overleftarrow{e_i^2} \wedge \overleftarrow{e_i^1} - \overrightarrow{(\mathrm{Ad}_{g^{-1}}\, e_i)^2} \wedge \overrightarrow{e_i^1}\Big),$$

and the bi-invariant 3-form $\Omega := \frac{1}{4}K(\cdot, [\cdot, \cdot]_{\mathfrak{g}}) \in \bigwedge^3 \mathfrak{g}^* \cong \Omega^3(G)^G$ on G, is a quasi-Poisson groupoid. Here $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}$ and the superscript refers to the respective G-component.

Remark. We remark that, under the change of coordinates $(g, s) \mapsto (a, b) = (s^{-1}g^{-1}, g)$, Π becomes the bivector field on $G \times G$ obtained in Example 5.3 of [2], i.e.,

$$\Pi = \frac{1}{2}\sum_{i=1}^{n}\Big(\overleftarrow{e_i^1} \wedge \overrightarrow{e_i^2} + \overrightarrow{e_i^1} \wedge \overleftarrow{e_i^2}\Big).$$

4.6. D/G momentum maps

Next, we investigate the relation between quasi-Hamiltonian spaces with D/G-momentum map in the sense of [1] and Hamiltonian Γ-spaces, where Γ is the quasi-Poisson groupoid $G \times S \rightrightarrows S$ associated to a quasi-Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ as described in Section 4.5.

First, we recall the notion of a quasi-Hamiltonian space with D/G-momentum map.

Definition 4.25 ([1, Def. 4.1.5]). Let (G, Π_G, Ω) be a connected quasi-Poisson Lie group acting on a manifold X with a bivector field Π_X. The action $\Phi : G \times X \to X$ of G on X is said to be a *quasi-Poisson action* if

$$\Phi_*(\Pi_G \oplus \Pi_X) = \Pi_X, \tag{4.80}$$

$$\tfrac{1}{2}[\Pi_X, \Pi_X] = \Omega_X, \tag{4.81}$$

where $\Omega_X \in \mathfrak{X}^3(X)$ is defined using the map $\bigwedge^3 \mathfrak{g} \to \mathfrak{X}^3(X)$ induced by the infinitesimal action.

Now, we recall the definition of a quasi-Hamiltonian action with a momentum map J.

Definition 4.26. Let Φ be a quasi-Poisson action of a quasi-Poisson Lie group (G, Π_G, Ω) on (X, Π_X). A G-equivariant map $J : X \to S$ is called a *momentum map* if

$$\Pi_X^\sharp(J^*\theta_s) = -(\lambda(\theta_s))_X \quad \text{for } \theta_s \in T_s^*S, \tag{4.82}$$

where G acts on S by dressing action. The action is called *quasi-Hamiltonian* if it admits a momentum map and X is then called a *quasi-Hamiltonian space*.

Remark. Our definition does not require any assumption on $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. In [1, Def. 5.5.1], the isotropic complement $\mathfrak{h}$ to $\mathfrak{g}$ in $\mathfrak{d}$ is required to be admissible on $J(U)$, U being any open subset of X (see Remark 4.18 for the notion of admissibility). Therefore, our definition is more general than that in [1].

Proposition 4.27. *Let (G, Π_G, Ω) be a quasi-Poisson Lie group. If X is a quasi-Hamiltonian space then the momentum map is a bivector map from (X, Π_X) to (S, Π_S), i.e.,*

$$J_*\Pi_X = \Pi_S. \tag{4.83}$$

Proof. Let $f, g \in C^\infty(S)$. Using (4.76), (4.82) and G-equivariance, we have

$$\begin{aligned}\Pi_X(J^*\mathrm{d}f, J^*\mathrm{d}g) &= \langle \Pi_X^\sharp(J^*\mathrm{d}f), J^*\mathrm{d}g\rangle = -J_*(\lambda(\mathrm{d}f)_X)(g)\\ &= -(\lambda(\mathrm{d}f))_S(g) = \Pi_S(\mathrm{d}f, \mathrm{d}g).\end{aligned}$$ □

Theorem 4.28. *Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ be a quasi-Manin triple. If (X, Π_X) is a quasi-Hamiltonian space with momentum map $J : X \to S$, then X is a Hamiltonian Γ-space, where Γ is the quasi-Poisson groupoid $(G \times S \rightrightarrows S, \Pi, \Omega)$. Here the Γ-action is given by*

$$(g, s) \cdot x = \Phi(g, x) \quad \text{for } g \in G, s \in S \text{ and } x \in X \text{ such that } J(x) = s, \tag{4.84}$$

and $\Phi(g, x)$ denotes the G-action on X.

Conversely, if (X, Π_X) *is a Hamiltonian* Γ*-space with momentum map* $J : X \to S$*, then* (X, Π_X) *is a quasi-Hamiltonian space with momentum map* $J : X \to S$*, where the* G*-action on* X *is given by*

$$\Phi(g, x) = (g, J(x)) \cdot x \quad \textit{for } g \in G \textit{ and } x \in X.$$

Proof. First of all, using the fact that $J : X \to S$ is G-equivariant, we know that Φ can be extended to an action of Γ on X by (4.84). In addition, the conormal space of the graph of the Γ-action at a point $((g, x), s, \Phi(g, x))$ is spanned by vectors of the form $(-(\Phi_x)^*\theta, 0, -(\Phi_g)^*\theta, \theta)$ for $\theta \in T^*_{\Phi(g,x)}X$ and $(0, \gamma, -J^*\gamma, 0)$ for $\gamma \in T^*_s S$. We accordingly divide our proof into three cases:

Case 1. The two covectors are of the first type. From the definition of Π (see (4.79)) we see that the coisotropy condition is equivalent to

$$\Phi_*(\Pi_G \oplus \Pi_X) = \Pi_X.$$

Case 2. The two covectors are of the second type. From (4.79), we deduce that $\Pi \oplus \Pi_X \oplus -\Pi_X$ being coisotropic is equivalent to

$$J_*\Pi_X = \Pi_S.$$

Case 3. One covector is of the first type and the other of the second. In this case, the coisotropy condition is just the momentum map condition, i.e.,

$$\Pi_X^\sharp(J^*\gamma) + (\lambda(\gamma))_X = 0.$$

Finally, we note that (4.81) is equivalent to $\frac{1}{2}[\Pi_X, \Pi_X] = \hat{\Omega}$. □

In [2], the authors study quasi-Hamiltonian spaces for the particular case when the Manin pair is the one associated with a Lie algebra $\mathfrak{g}$ endowed with a nondegenerate symmetric bilinear form K (see Example 4.17). We now recall their definitions and discuss the relation with Hamiltonian Γ-spaces of quasi-Poisson groupoids.

Definition 4.29 ([1, 2]). A *quasi-Poisson manifold* is a G-manifold X equipped with an invariant bivector field Π_X such that

$$\tfrac{1}{2}[\Pi_X, \Pi_X] = \Omega_X.$$

Moreover, one can also introduce the following specific definition of momentum maps.

Definition 4.30 ([1, 2]). An Ad-equivariant map $J : X \to G$ is called a *momentum map* for the quasi-Poisson manifold (X, Π_X) if

$$\Pi_X^\sharp(\mathrm{d}(J^*f)) = (J^*(\mathcal{D}f))_X, \quad \forall f \in C^\infty(G),$$

where $\mathcal{D} : C^\infty(G) \to C^\infty(G, \mathfrak{g})$ is defined by $\mathcal{D}f = \frac{1}{2}\sum_{i=1}^n(\overrightarrow{e_i} + \overleftarrow{e_i})(f)e_i$. The triple (X, Π_X, J) is then called a *Hamiltonian quasi-Poisson manifold.*

Remark. Note that the operator $\mathcal{D}$ is exactly the 2-differential δ on the Lie algebroid $\mathfrak{g} \times S \to S$ applying to functions (see Example 4.21).

As a consequence of Theorem 4.28 we deduce the following

Corollary 4.31. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ endowed with a nondegenerate symmetric bilinear form K. If (X, Π_X) is a Hamiltonian quasi-Poisson manifold with momentum map $J : X \to G$, where the action is denoted by $\Phi(g, x)$, then X is a Hamiltonian Γ-space, where Γ is the quasi-Poisson groupoid $(G \times G \rightrightarrows G, \Pi, \Omega)$ obtained in Corollary 4.24. Here the Γ-action is given by*

$$(g, s) \cdot x = \Phi(g, x) \quad \text{for } g \in G, s \in S \text{ and } x \in X \text{ such that } J(x) = s.$$

Conversely, if (X, Π_X) is a Hamiltonian Γ-space with momentum map $J : X \to G$, then (X, Π_X) is a quasi-Poisson manifold with momentum map $J : X \to G$, where the G-action on X is given by

$$\Phi(g, x) = (g, J(x)) \cdot x \quad \text{for } g \in G \text{ and } x \in X.$$

It is thus natural to expect that, for any Hamiltonian Γ-space, there is a general Poisson reduction theory, which generalizes the reduction theory of Alekseev, Kosmann-Schwarzbach and Meinrenken [2] for quasi-Poisson spaces. This will be investigated elsewhere.

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