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Counterexamples to the Strichartz inequalities for the wave equation in general domains with boundary

Abstract. We consider a smooth and bounded domain $\Omega \subset \mathbb{R}^d$ of dimension $d \geq 2$ with boundary and we construct sequences of solutions to the wave equation with Dirichlet boundary condition which fail the Strichartz estimates of the free space, providing losses of derivatives at least for a subset of the usual range of indices. This is due to microlocal phenomena such as caustics generated in arbitrarily small time near the boundary. Moreover, the result holds for microlocally strictly convex domains in $\mathbb{R}^d$.

Keywords. Microlocal analysis, wave equation, Dirichlet boundary condition, Strichartz estimates, propagation and reflection of singularities, conormal waves with cusps, caustics

1. Introduction

Let Ω be a smooth manifold of dimension $d \geq 2$ with C^∞ boundary $\partial\Omega$, equipped with a Riemannian metric g. Let Δ_g be the Laplace–Beltrami operator associated to g on Ω, acting on $L^2(\Omega)$ with Dirichlet boundary condition. Let $0 < T < \infty$ and consider the wave equation with Dirichlet boundary conditions:

$$\begin{cases} (\partial_t^2 - \Delta_g)u = 0 & \text{on } \Omega \times [0, T], \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

Strichartz estimates are a family of dispersive estimates on solutions $u : \Omega \times [0, T] \to \mathbb{C}$ to the wave equation (1.1). In their most general form, local Strichartz estimates state that

$$\|u\|_{L^q([0,T],L^r(\Omega))} \leq C(\|u_0\|_{\dot{H}^\gamma(\Omega)} + \|u_1\|_{\dot{H}^{\gamma-1}}), \tag{1.2}$$

where $\dot{H}^\gamma(\Omega)$ denotes the homogeneous Sobolev space over Ω and where the pair (q, r) is *wave admissible in dimension d*, i.e. $2 \leq q \leq \infty$, $2 \leq r < \infty$ and

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma, \quad \frac{2}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}. \tag{1.3}$$

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When equality holds in (1.3) the pair (q, r) is called *sharp wave admissible in dimension d*. Estimates involving $r = \infty$ hold when $(q, r, d) \neq (2, \infty, 3)$, but typically require the use of Besov spaces.

Our main result is work in the opposite direction. Roughly speaking, we show that if $\Omega \subset \mathbb{R}^d$ is a smooth and bounded domain of $\mathbb{R}^d$ and (q, r) is a sharp wave admissible pair in dimension $d \geq 2$ with $r > 4$, then there exists $T = T(\Omega)$ such that the quotient

$$\frac{\|u\|_{L^q([0,T], L^r(\Omega))}}{\|u_0\|_{\dot{H}^{\gamma + \frac{1}{6}(\frac{1}{4} - \frac{1}{r}) - \epsilon}(\Omega)} + \|u_1\|_{\dot{H}^{\gamma + \frac{1}{6}(\frac{1}{4} - \frac{1}{r}) - 1 - \epsilon}(\Omega)}}$$

takes arbitrarily large values for suitable initial data (u_0, u_1), which means that a loss of at least $\frac{1}{6}(\frac{1}{4} - \frac{1}{r}) - \epsilon$ derivatives is unavoidable in the Strichartz estimates for the flow.

The main motivation for the above types of Strichartz estimates comes from applications to harmonic analysis and the study of nonlinear dispersive equations. Estimates like (1.2) can be used to prove existence theorems for nonlinear wave equations.

In $\mathbb{R}^d$ and for $g_{ij} = \delta_{ij}$, Strichartz estimates in the context of the wave and Schrödinger equations have a long history, beginning with Strichartz's pioneering work [25], where he proved the particular case $q = r$ for the wave and (classical) Schrödinger equation. This was later generalized to mixed $L^q((-T, T), L^r(\Omega))$ norms by Ginibre and Velo [7] for the Schrödinger equation, where (q, r) is sharp admissible and $q > 2$; the wave estimates were obtained independently by Ginibre–Velo [8] and Lindblad–Sogge [17], following earlier work by Kapitanski [13]. The remaining endpoints for both equations were finally settled by Keel and Tao [15]. In that case $\gamma = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})$ and one can obtain a global estimate with $T = \infty$ (see also Kato [14] and Cazenave–Weissler [5]).

Let us recall the result for the flat space: if Δ is the Euclidian Laplace operator, then the Strichartz estimates for the wave equation on $\mathbb{R}^d$ read as follows (see [15]):

Proposition 1.1. *Let (q, r) be a wave admissible pair in dimension $d \geq 2$. If u satisfies*

$$(\partial_t^2 - \Delta)u = 0 \quad \text{on } [0, T] \times \mathbb{R}^d, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1$$

for some $0 < T < \infty$, $u_0, u_1 \in C^\infty(\mathbb{R}^d)$, then there is a constant $C = C_T$ such that

$$\|u\|_{L^q([0,T], L^r(\mathbb{R}^d))} \leq C(\|u_0\|_{\dot{H}^{\frac{d+1}{2}(\frac{1}{2} - \frac{1}{r})}(\mathbb{R}^d)} + \|u_1\|_{\dot{H}^{\frac{d+1}{2}(\frac{1}{2} - \frac{1}{r}) - 1}(\mathbb{R}^d)}).$$

In the variable coefficients case, even without boundary, the situation is much more complicated: we simply recall here the pioneering work of Staffilani and Tataru [24], dealing with compact, nontrapping perturbations of the flat metric and recent work of Bouclet and Tzvetkov [2] in the context of the Schrödinger equation, which considerably weakens the decay of the perturbation (retaining the nontrapping character at spatial infinity). On compact manifolds without boundary, due to the finite speed of propagation, it is enough to work in coordinate charts and to establish local Strichartz estimates for variable coefficients wave operators in $\mathbb{R}^d$: we recall here the works by Kapitanski [12] and Mockenhaupt, Seeger and Sogge [19] in the case of smooth coefficients when one can use the Lax parametrix construction to obtain the appropriate dispersive estimates. In the case of $C^{1,1}$ coefficients, Strichartz estimates were shown in the works by Smith [21] and by

Tataru [26], the latter work establishing the full range of local estimates; here the lack of smoothness prevents the use of Fourier integral operators and instead wave packets and coherent state methods are used to construct parametrices for the wave operator. In these situations, if the metric is sufficiently smooth (if it has at least two derivatives bounded), the Strichartz estimates hold as in the Euclidian case.

Even though the case without boundary has been well understood for some time, obtaining results for manifolds with boundary has been surprisingly elusive.

For a manifold with smooth, strictly geodesically concave boundary (i.e. for which the second fundamental form is strictly negative definite), the Melrose and Taylor parametrix yields the Strichartz estimates for the wave equation with Dirichlet boundary condition for the range of exponents in (1.3) (not including the endpoints), as shown in the paper of Smith and Sogge [22]. If the concavity assumption is removed, however, the presence of multiply reflecting geodesic and their limits, the gliding rays, prevents the construction of a similar parametrix.

Note that on an exterior domain a source point does not generate caustics and that the presence of caustics generated in small time near a source point is what makes things difficult inside a strictly convex set.

Recently, Burq, Lebeau and Planchon [3], [4] established Strichartz type inequalities on a manifold with boundary using the $L^r(\Omega)$ estimates for the spectral projectors obtained by Smith and Sogge [23]. The range of triples (q, r, γ) that can be obtained in this manner, however, is restricted by the allowed range of r in the square function estimate for the wave equation, which controls the norm of u in the space $L^r(\Omega, L^2(-T, T))$ (see [23]). In dimension 3, for example, this restricts the indices to $q, r \geq 5$. The work of Blair, Smith and Sogge [1] expands the range of indices q and r obtained in [3]: specifically, they show that if Ω is a compact manifold with boundary and (q, r, γ) is a triple satisfying the first condition in (1.3) together with the restriction

$$\begin{cases} \dfrac{3}{q} + \dfrac{d-1}{r} \leq \dfrac{d-1}{2}, & d \leq 4, \\ \dfrac{1}{q} + \dfrac{1}{r} \leq \dfrac{1}{2}, & d \geq 4, \end{cases} \tag{1.4}$$

then the Strichartz estimates (1.2) hold true for solutions u to (1.1) satisfying Dirichlet or Neumann homogeneous boundary conditions, with a constant C depending on Ω and T.

In this paper we prove that Strichartz estimates for the wave equation inside the domain Ω suffer losses when compared to the usual case of $\mathbb{R}^d$, at least for a subset of the usual range of indices, under our assumption that there exists a point in $T^*\partial\Omega$ where the second fundamental form on the boundary of the manifold has a strictly positive eigenfunction. Precisely, our assumption reads as follows:

Assumption 1.2. Let Ω be a smooth manifold of dimension $d \geq 2$ with C^∞ boundary $\partial\Omega$. We assume that there exists a bicharacteristic that intersects $\partial\Omega \times \mathbb{R}$ tangentially at some point $(\rho_0, \vartheta_0) \in T^*(\partial\Omega \times \mathbb{R})$ having exactly second order contact with the boundary at (ρ_0, ϑ_0) and which remains in the complement of $\bar{\Omega} \times \mathbb{R}$. We call the point (ρ_0, ϑ_0) a *gliding point*.

Remark 1.3. In particular, any smooth and bounded domain of $\mathbb{R}^d$, $d \geq 2$, satisfies the condition in Assumption 1.2.

Our main result reads as follows:

Main Theorem 1.4. *Let* (Ω, g) *satisfy the conditions in Assumption* 1.2 *with* $d \in \{2, 3, 4\}$. *Then there exists* $T = T(\Omega) \in (0, \infty)$ *and for every small* $\epsilon > 0$ *there exist sequences* $V_{h,j,\epsilon} \in C^{\infty}(\bar{\Omega})$, $j \in \{0, 1\}$, *such that the solution* $V_{h,\epsilon}$ *to the wave equation with Dirichlet boundary conditions*

$$\begin{cases} (\partial_t^2 - \Delta_g) V_{h,\epsilon} = 0, \\ V_{h,\epsilon}|_{t=0} = V_{h,0,\epsilon}, \quad \partial_t V_{h,\epsilon}|_{t=0} = V_{h,1,\epsilon}, \\ V_{h,\epsilon}|_{\partial\Omega \times [0,T]} = 0, \end{cases} \tag{1.5}$$

satisfies

$$\sup_{\epsilon > 0, \, h \in (0,1], \, j} h^{-\frac{d+1}{2}(\frac{1}{2}-\frac{1}{r}) - \frac{1}{6}(\frac{1}{4}-\frac{1}{r}) + 2\epsilon + j} \|V_{h,j,\epsilon}\|_{L^2(\Omega)} \leq 1 \tag{1.6}$$

and

$$\lim_{h \to 0} \|V_{h,\epsilon}\|_{L_t^q([0,T], L^r(\Omega))} = \infty, \tag{1.7}$$

for every sharp wave admissible pair (q, r) *in dimension* d *with* $r > 4$. *Moreover* $V_{h,\epsilon}$ *has compact support for the normal variable in a neighborhood of the boundary of size* $h^{(1-\epsilon)/2}$ *and is well localized at spatial frequency* $1/h$ *in the tangential variable.*

Remark 1.5. Notice that Theorem 1.4 shows an explicit loss of at least $\frac{1}{6}(\frac{1}{4} - \frac{1}{r})$ derivatives in the Strichartz estimates for domains Ω satisfying Assumption 1.2 compared to the Euclidian case (see Proposition 1.1).

Remark 1.6. The proof of Theorem 1.4 will show that the restriction on the dimension comes only from the fact that for $d \geq 5$ all admissible pairs (q, r) satisfy $r \leq 4$.

Remark 1.7. From Remarks 1.3 and 1.6 it follows that Theorem 1.4 holds true if Ω is any smooth, bounded domain in $\mathbb{R}^d$ with $d \in \{2, 3, 4\}$.

Remark 1.8. In [10] we proved Theorem 1.4 in the particular case of the two-dimensional half-space $\{(x, y) \mid x > 0, y \in \mathbb{R}\}$ with Laplace operator given by $\partial_x^2 + (1 + x)\partial_y^2$. We notice that the half-space together with the metric inherited from the above Laplace operator becomes a strictly convex domain (called *Friedlander's model domain*).

In this paper we generalize the result of [10] to any smooth domain satisfying Assumption 1.2 using Melrose's theorem on glancing surfaces.

Remark 1.9. Notice that Theorem 1.4 states for instance that the scale-invariant Strichartz estimates fail for $3/q + 1/r > 15/24$, whereas the result of Blair, Smith and Sogge states that such estimates hold if $3/q + 1/r \leq 1/2$. Of course, the counterexample places a lower bound on the loss for such indices (q, r), and the work [1] would place some

upper bounds, but this concise statement shows one explicit gap in our knowledge that remains to be filled.

A very interesting and natural question would be to determine the sharp range of exponents for (1.2) in any dimension $d \geq 2$.

A classical way to prove Strichartz inequalities is to use dispersive estimates: the fact that weakened dispersive estimates can *still* imply optimal (and scale invariant) Strichartz estimates for the solution of the wave equation was first noticed by Lebeau who announced in [16] that a loss of derivatives is unavoidable for the wave equation inside a strictly convex domain, and this appears because of swallowtail type caustics in the wave front set of u:

$$|\chi (hD_t)u(t, x)| \lesssim h^{-d} \min(1, (h/t)^{\frac{d-2}{2}+\frac{1}{4}}). \tag{1.8}$$

As shown in [11] (where a detailed proof of (1.8) is given), the estimates (1.8), although optimal for the dispersion, imply Strichartz type inequalities without losses, but with indices (q, r, d) satisfying

$$\frac{1}{q} \leq \left(\frac{d-2}{2} + \frac{1}{4}\right)\left(\frac{1}{2} - \frac{1}{r}\right).$$

In dimension $d \geq 3$ and in the case of strictly convex domains, this range of indices generalizes the one in (1.4) obtained in [1].

In the proof of Theorem 1.4 we use conormal waves with multiply reflected cusps at the boundary, together with Melrose's theorem on glancing rays, to reduce the study of the iterated boundary operators to Friedlander's model, and hence to [10]. The construction of a parametrix for (1.1) is in the same spirit as in the classical paper by J. Ralston [20], which however deals with a very different geometric situation.

The organization of the paper is as follows: In Section 2 we show that in order to prove Theorem 1.4 it is enough to consider the two-dimensional case. In Section 3 we recall the construction in the model case of a strictly convex domain of dimension two we dealt with in [10] and use it to determine an approximate solution of (1.5) which satisfies the conclusion of Theorem 1.4.

2. Reduction to the two-dimensional case

Let Ω satisfy the assumptions of Theorem 1.4. Write local coordinates on Ω as $(x, y_1, \ldots, y_{d-1})$ with $x > 0$ on Ω, $\partial\Omega = \{(0, y) \mid y = (y_1, \ldots, y_{d-1}) \in \mathbb{R}^{d-1}\}$, and local coordinates induced by the product $X = \Omega \times \mathbb{R}_t$ as (x, y, t).

Local coordinates on the base induce local coordinates on the cotangent bundle, namely $(\rho, \vartheta) = (x, y, t, \xi, \eta, \tau)$ on T^*X near $\pi^{-1}(q)$, $q \in T^*\partial X$, where $\pi : T^*X \to {}^bT^*X$ is the canonical inclusion from the cotangent bundle into the b-cotangent bundle defined by ${}^bT^*X = T^*\mathring{X} \cup T^*\partial X$. The corresponding local coordinates on the

boundary are denoted (y, t, η, τ) (on a neighborhood of a point q in $T^*\partial X$). The metric function in $T^*\Omega$ has the form

$$g(x, y, \xi, \eta) = A(x, y)\xi^2 + 2\sum_{j=1}^{d-1} C_j(x, y)\xi\eta_j + \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\eta_j\eta_k,$$

with A, $B_{j,k}$, C_j smooth. Moreover, these coordinates can be chosen so that $A(x, y) = 1$ and $C_j(x, y) = 0$ (see [9, Appendix C]). Thus, in this coordinate chart the metric on the boundary reads

$$g(0, y, \xi, \eta) = \xi^2 + \sum_{j,k=1}^{d-1} B_{j,k}(0, y)\eta_j\eta_k.$$

On $T^*\partial\Omega$ the metric g takes an even simpler form, since introducing geodesic coordinates we can assume moreover that, locally,

$$B_{1,1}(0, y) = 1, \qquad B_{1,j}(0, y) = 0 \quad \forall j \in \{2, \ldots, d-1\}.$$

Hence, if we write $R(x, y, \eta) := \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\eta_j\eta_k$, then for small x we have

$$R(x, y, \eta) = (1 + x\partial_x B_{1,1}(0, y_1, y'))\eta_1^2$$
$$+ \sum_{j=1}^{d-1}(x\partial_x B_{1,j}(0, y) + O(x^2))\eta_1\eta_j + \sum_{j,k=2}^{d-1} B_{j,k}(x, y)\eta_j\eta_k. \qquad (2.1)$$

Assumption 1.2 on the domain Ω is equivalent to saying that there exists a point $(0, y_0, \xi_0, \eta_0)$ on $T^*\Omega$ where the boundary is microlocally strictly convex, i.e. there exists a bicharacteristic passing through this point that intersects $\partial\Omega$ tangentially having exactly second order contact with the boundary and remaining in the complement of $\partial\bar{\Omega}$. If $p \in C^\infty(T^*X \setminus o)$ (where we write o for the zero section) denotes the principal symbol of the wave operator $\partial_t^2 - \Delta_g$, this last condition translates into

$$\tau^2 = R(0, y_0, \eta_0), \qquad \{p, x\} = \frac{\partial p}{\partial \xi} = 2\xi_0 = 0, \qquad (2.2)$$

$$\{\{p, x\}, p\} = \left\{\frac{\partial p}{\partial \xi}, p\right\} = 2\partial_x R(0, y_0, \eta_0) > 0, \qquad (2.3)$$

where $\{f_1, f_2\}$ denotes the Poisson bracket

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial \vartheta}\frac{\partial f_2}{\partial \rho} - \frac{\partial f_1}{\partial \rho}\frac{\partial f_2}{\partial \vartheta}.$$

Denote the gliding point (in $T^*\Omega \times \mathbb{R}$) provided by Assumption 1.2 by

$$(\rho_0, \vartheta_0) = \left(0, y_0, 0, 0, \eta_0, \tau_0 = -\sqrt{R(0, y_0, \eta_0)}\right).$$

We start the proof of Theorem 1.4 by reducing the problem to the two-dimensional case. Consider the following assumption:

Assumption 2.1. Let $(\tilde{\Omega}, \tilde{g})$ be a smooth manifold of dimension 2 with C^∞ boundary and Riemannian metric $\tilde{g}$. Assume that in local coordinates $\tilde{\Omega} = \{(x, \tilde{y}) \mid x > 0, \ \tilde{y} \in \mathbb{R}\}$ and that the Laplace–Beltrami operator associated to $\tilde{g}$ is given by

$$\partial_x^2 + (1 + xb(\tilde{y}))\partial_{\tilde{y}}^2,$$

where $b(\tilde{y})$ is a smooth function. Suppose in addition that there exists a bicharacteristic which intersects the boundary tangentially at $(0, \tilde{y}_0, \tilde{\xi}_0, \tilde{\eta}_0) \in T^*\tilde{\Omega}$ having exactly second order contact with the boundary and which remains in the complement of $\overline{\tilde{\Omega}}$. This is equivalent to saying that at $(0, \tilde{y}_0, \tilde{\xi}_0, \tilde{\eta}_0)$,

$$\tilde{\xi}_0 = 0, \quad 2b(\tilde{y}_0) > 0.$$

We suppose in addition (without loss of generality, since we can always rescale the normal variable x) that $b(\tilde{y}_0) = 1$.

Theorem 2.2. *Let $(\tilde{\Omega}, \tilde{g})$ satisfy Assumption 2.1. Then there exists $T = T(\tilde{\Omega}) \in (0, \infty)$ and for every $\epsilon > 0$ small enough there exist sequences $\tilde{V}_{h,j,\epsilon}$, $j \in \{0, 1\}$, and approximate solutions $\tilde{V}_{h,\epsilon}$ to the wave equation on $\tilde{\Omega}$ with Dirichlet boundary conditions*

$$\begin{cases} \partial_t^2 V - \partial_x^2 V - (1 + xb(\tilde{y}))\partial_{\tilde{y}}^2 V = 0 & \text{on } \tilde{\Omega} \times \mathbb{R}, \\ V|_{t=0} = \tilde{V}_{h,0,\epsilon}, \quad \partial_t V|_{t=0} = \tilde{V}_{h,1,\epsilon}, \\ V|_{\partial\tilde{\Omega} \times [0,T]} = 0, \end{cases} \tag{2.4}$$

which satisfy the following conditions:

(i) *First, $\tilde{V}_{h,\epsilon}$ is an approximate solution to (2.4) in the sense that*

$$\partial_t^2 \tilde{V}_{h,\epsilon} - \partial_x^2 \tilde{V}_{h,\epsilon} - (1 + xb(\tilde{y}))\partial_{\tilde{y}}^2 \tilde{V}_{h,\epsilon} = O_{L^2(\tilde{\Omega})}(1/h), \quad \|\tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \leq 1.$$

(ii) *Secondly, $\tilde{V}_{h,\epsilon}$ can be written as a sum*

$$\tilde{V}_{h,\epsilon}(x, \tilde{y}, t) = \sum_{n=0}^{N} v_{h,\epsilon}^n(x, \tilde{y}, t), \tag{2.5}$$

where $1 \leq N \simeq h^{-(1-\epsilon)/4}$ and where the functions $v_{h,\epsilon}^n(x, \tilde{y}, t)$ satisfy the following conditions:

- *for $4 < r < \infty$:*

$$\begin{cases} \|v_{h,\epsilon}^n(\cdot, t)\|_{L^r(\tilde{\Omega})} \geq Ch^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{6}(\frac{1}{4}-\frac{1}{r})+2\epsilon}, \\ \sup_{\epsilon>0}\left(\|v_{h,\epsilon}^n(\cdot, t)\|_{L^2(\tilde{\Omega})} + h\|\partial_t v_{h,\epsilon}^n(\cdot, t)\|_{L^2(\tilde{\Omega})}\right) \leq 1, \end{cases} \tag{2.6}$$

  where the constant $C = C(T) > 0$ is independent of h, ϵ and n;
- *$v_{h,\epsilon}^n(x, \tilde{y}, t)$ are essentially supported for the time variable t in almost disjoint intervals of time I_n satisfying $|I_0| \simeq |I_n| \simeq h^{(1-\epsilon)/4}$ for all $n \in \{0, \ldots, N\}$, and also supported for the tangential variable $\tilde{y}$ in almost disjoint intervals;*

- $\tilde{V}_{h,\epsilon}$ are supported for the normal variable $0 \le x \lesssim h^{(1-\epsilon)/2}$ (where the implied constant depends only on $\tilde{\Omega}$) and localized at spatial frequency $1/h$ in the tangential variable $\tilde{y}$. Moreover, uniformly in $\epsilon > 0$,

$$\sup_{\epsilon>0} \|\tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \lesssim 1, \quad \sup_{\epsilon>0} \|\partial_{\tilde{y}}\tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h}, \quad \sup_{\epsilon>0} \|\partial_{\tilde{y}}^2\tilde{V}_{h,\epsilon}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h^2}. \tag{2.7}$$

In the rest of this section we show how Theorem 2.2 implies Theorem 1.4. Assume we have proved Theorem 2.2. Let (Ω, g) be a Riemannian manifold of dimension $d > 2$ satisfying our assumptions in Theorem 1.4 and let $(0, y_0, \xi_0, \eta_0) \in T^*\Omega$ be a point satisfying (2.2), (2.3). From (2.1) it follows that local coordinates can be chosen such that $y_0 = 0 \in \mathbb{R}^{d-1}$, $\eta_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{d-1}$ and such that the Laplace–Beltrami operator Δ_g is given by

$$\Delta_g = \partial_x^2 + \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\partial_j\partial_k, \tag{2.8}$$

where for x small enough

$$B_{1,1}(x, y) = 1 + x\partial_x B_{1,1}(0, y) + O(x^2), \quad \partial_x B_{1,1}(0, y) > 0$$

and for $j \in \{2, \ldots, d-1\}$ we have $B_{1,j}(0, y) = 0$. By rescaling the normal variable x, we can assume without loss of generality that $\partial_x B_{1,1}(0, 0) = 1$.

We can now define $\tilde{\Omega}$, locally in a neighborhood of $(x = 0, y_1 = 0, \xi = 0, \eta_1 = 0)$, to be the two-dimensional half-space $\tilde{\Omega} := \{(x, y_1) \mid x > 0, \ y_1 \in \mathbb{R}\}$ equipped with the metric

$$\tilde{g}(x, y_1, \xi, \eta_1) := \xi^2 + (1 + xb(y_1))\eta_1^2, \quad b(y_1) := \partial_x B_{1,1}(0, y_1, 0).$$

Recall that we have assumed $b(0) = 1$. Applying Theorem 2.2 near $(0, y_1 = 0, 0, \eta_1 = 1) \in T^*\tilde{\Omega}$ we obtain, for $\epsilon > 0$ small enough, sequences $\tilde{V}_{h,\epsilon,j}$, $j \in \{0, 1\}$, such that the solution $\tilde{V}_{h,\epsilon}$ to (2.4) satisfies (2.5)–(2.7). Let $\chi \in C_0^\infty(\mathbb{R}^{d-2})$ be a cut-off function supported in the coordinate chart such that $\chi = 1$ in a neighborhood of $0 \in \mathbb{R}^{d-2}$, and for $j \in \{0, 1\}$ set

$$V_{h,\epsilon,j}(x, y_1, y') := h^{-(d-2)/4}\tilde{V}_{h,\epsilon/3,j}(x, y_1)e^{-\frac{|y'|^2}{2h}}\chi(y'). \tag{2.9}$$

Proposition 2.3. *Let Δ_g be given by* (2.8). *Then the solution $V_{h,\epsilon}$ to the wave equation* (1.5) *with Dirichlet boundary conditions and with initial data $(V_{h,\epsilon,0}, V_{h,\epsilon,1})$ defined in* (2.9) *satisfies* (1.6), (1.7).

Remark 2.4. Notice that Proposition 2.3 immediately implies Theorem 1.4.

Proof. We proceed by contradiction. Let (q, r) be a sharp wave admissible pair in dimension $d \in \{2, 3, 4\}$ with $r > 4$ and set

$$\beta(r, d) = \frac{d + 1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{1}{6}\left(\frac{1}{4} - \frac{1}{r}\right).$$

We suppose to the contrary that the operator

$$\sin(t\sqrt{-\Delta_g}) : L^2(\Omega) \to L^q([0, T], L^r(\Omega))$$

is bounded by $h^{-\beta(r,d)+2\epsilon}$, where $T = T(\tilde{\Omega})$ is given by Theorem 2.2. Let $\tilde{V}_{h,\epsilon/3}$ be the approximate solution to (2.4) with initial data $(\tilde{V}_{h,\epsilon/3,j})_{j=0,1}$ satisfying all the conditions in Theorem 2.2. For $t \in [0, T]$ we define

$$W_{h,\epsilon}(x, y, t) := h^{-(d-2)/4}\tilde{V}_{h,\epsilon/3}(x, y_1, t)e^{-\frac{|y'|^2}{2h}}\chi(y').$$

Lemma 2.5. *There exists a constant $c(T) > 0$ independent of h such that*

$$\|W_{h,\epsilon}\|_{L^q([0,T],L^r(\Omega))} \geq c(T)h^{-\beta(r,d)+2\epsilon/3}, \tag{2.10}$$

$$\|W_{h,\epsilon}|_{t=0}\|_{L^2(\Omega)} + h\|\partial_t W_{h,\epsilon}|_{t=0}\|_{L^2(\Omega)} \lesssim 1. \tag{2.11}$$

Proof. Indeed, using the special form of $\tilde{V}_{h,\epsilon}$ provided by Theorem 2.2 we can estimate

$$\|W_{h,\epsilon}\|^q_{L^q([0,T],L^r(\Omega))} = \int_0^T \|W_{h,\epsilon}\|^q_{L^r(\Omega)}\,dt$$

$$= \left(\int_0^T \left\|\sum_{n=0}^N v^n_{h,\epsilon/3}\right\|^q_{L^r(\tilde{\Omega})}\,dt\right) \times \|h^{-(d-2)/4}e^{-\frac{|y'|^2}{2h}}\chi(y')\|^q_{L^r(\mathbb{R}^{d-2})}$$

$$\geq ch^{-\frac{q(d-2)}{2}(\frac{1}{2}-\frac{1}{r})}\sum_{k\leq N}\int_{I_k}\left\|\sum_{n=0}^N v^n_{h,\epsilon/3}\right\|^q_{L^r(\tilde{\Omega})}\,dt + O(h^\infty)$$

$$\simeq ch^{-\frac{q(d-2)}{2}(\frac{1}{2}-\frac{1}{r})}\sum_{k\leq N}|I_k|\,\|v^0_{h,\epsilon/3}\|^q_{L^r(\tilde{\Omega})} + O(h^\infty)$$

$$\simeq cTh^{-\frac{q(d-2)}{2}(\frac{1}{2}-\frac{1}{r})}\|v^0_{h,\epsilon/3}\|^q_{L^r(\tilde{\Omega})} + O(h^\infty)$$

$$\geq cTh^{(-\beta(r,d)+2\epsilon/3)q}.$$

We used here the fact that each $v^n_{h,\epsilon/3}$ provided by Theorem 2.2 is essentially supported in time in an interval I_n of size $1/N$ and that $(I_n)_{n\in\{0,\dots,N\}}$ are almost disjoint. Take $c(T) = (cT)^{1/q}$, where c is the bound from below of the integral in the $d - 2$ tangential variables.

To estimate the $L^2(\Omega)$ norm we use again the fact that $v^n_{h,\epsilon}$ and its time derivative have disjoint essential supports in the tangential variable y_1. For $W_{h,\epsilon}(\cdot, 0)$ we have, for instance,

$$\|W_{h,\epsilon}|_{t=0}\|_{L^2(\Omega)} = \|\tilde{V}_{h,\epsilon/3,0}\|_{L^2(x,y_1)}\|h^{-(d-2)/4}e^{-\frac{|y'|^2}{2h}}\chi(y')\|_{L^2(\mathbb{R}^{d-2})} \lesssim 1. \qquad \square$$

Let $V_{h,\epsilon}$ be the solution to (1.5) with data $(V_{h,\epsilon,j})_{j=0,1}$ defined in (2.9) and write

$$V_{h,\epsilon} = W_{h,\epsilon} + w_{h,\epsilon,\text{err}}.$$

If we denote $\Delta_{\tilde{g}} = \partial_x^2 + (1 + xb(y_1))\partial_{y_1}^2$, $\Box_{\tilde{g}} = \partial_t^2 - \Delta_{\tilde{g}}$, then $W_{h,\epsilon}$ solves

$$\begin{cases} \Box_{\tilde{g}} W_{h,\epsilon} = \Box_{\tilde{g}} \tilde{V}_{h,\epsilon/3} h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y'), \\ W_{h,\epsilon}|_{t=0} = V_{h,\epsilon,0}, \quad \partial_t W_{h,\epsilon}|_{t=0} = V_{h,\epsilon,1}, \\ W_{h,\epsilon}|_{\partial\Omega \times [0,T]} = 0. \end{cases}$$

Since $V_{h,\epsilon}$ is a solution to (1.5), $w_{h,\epsilon,\text{err}}$ must satisfy

$$\begin{cases} \Box_g w_{h,\epsilon,\text{err}} = -\Box_{\tilde{g}} \tilde{V}_{h,\epsilon/3} h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y') - (1 + xb(y_1))\partial_{y_1}^2 W_{h,\epsilon} \\ \qquad\qquad + \sum_{j,k=1}^{d-1} B_{j,k}(x,y)\partial_{y_j,y_k}^2 W_{h,\epsilon}, \\ w_{h,\epsilon,\text{err}}|_{t=0} = 0, \quad \partial_t w_{h,\epsilon,\text{err}}|_{t=0} = 0, \\ w_{h,\epsilon,\text{err}}|_{\partial\Omega \times [0,T]} = 0, \end{cases} \qquad (2.12)$$

where we set $\Box_g := \partial_t^2 - \Delta_g$ and we used that

$$\Delta_g - \Delta_{\tilde{g}} = -(1 + xb(y_1))\partial_{y_1}^2 + \sum_{j,k=1}^{d-1} B_{j,k}(x,y)\partial_{y_j,y_k}^2.$$

Lemma 2.6. *For $t \in [0, T]$ the solution $w_{h,\epsilon,\text{err}}$ to the wave equation* (2.12) *satisfies*

$$\|(\partial_t^2 - \Delta_g)w_{h,\epsilon,\text{err}}(\cdot,t)\|_{L^2(\Omega)} \lesssim h^{-2(1-(1-\epsilon/3)/2)}\|w_{h,\epsilon,\text{err}}\|_{L^2(\Omega)} \simeq h^{-1-\epsilon/3}, \quad (2.13)$$

$$\|(\partial_t^2 - \Delta_g)w_{h,\epsilon,\text{err}}(\cdot,t)\|_{\dot{H}^{-1}(\Omega)} \lesssim h^{-\epsilon/3}\|w_{h,\epsilon,\text{err}}\|_{L^2(\Omega)} \simeq h^{-\epsilon/3}, \qquad\qquad (2.14)$$

where the estimates hold uniformly in $t \in [0, T]$ with constants independent of ϵ and of h. Moreover,

$$\|w_{h,\epsilon,\text{err}}\|_{L^q([0,T],L^r(\Omega))} \leq C h^{-\beta(r,d)+2\epsilon-\epsilon/3}, \qquad (2.15)$$

where $C = C(T) > 0$ is independent of ϵ.

Proof. We start with (2.15). Assume we have already proved (2.14). The Duhamel formula for $w_{h,\epsilon,\text{err}}$ reads

$$w_{h,\epsilon,\text{err}}(x,y,t) = \int_0^t \frac{\sin((t-s)\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\left((\partial_t^2 - \Delta_g)w_{h,\epsilon,\text{err}}(x,y,s)\right) ds.$$

Using the Minkowski inequality together with (2.13) we find

$$
\begin{aligned}
\|w_{h,\epsilon,\mathrm{err}}(\cdot,t)\|_{L^r(\Omega)} &= \left\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}} \big((\partial_t^2 - \Delta_g)w_{h,\epsilon,\mathrm{err}}(\cdot,s)\big)\, ds \right\|_{L^r(\Omega)} \\
&\lesssim \int_0^t \left\| \frac{\sin((t-s)\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}} \big((\partial_t^2 - \Delta_g)w_{h,\epsilon,\mathrm{err}}(\cdot,s)\big) \right\|_{L^r(\Omega)} ds \\
&\lesssim h^{-\beta(r,d)+2\epsilon} \|(\sqrt{-\Delta_g})^{-1}(\partial_t^2 - \Delta_g)w_{h,\epsilon,\mathrm{err}}\|_{L^1([0,T],L^2(\Omega))} \\
&\simeq h^{-\beta(r,d)+2\epsilon} \|(\partial_t^2 - \Delta_g)w_{h,\epsilon,\mathrm{err}}\|_{L^1([0,T],\dot{H}^{-1}(\Omega))} \\
&\lesssim h^{-\beta(r,d)+2\epsilon-\epsilon/3},
\end{aligned}
$$

where in the third line we used that the wave operator $\sin(t\sqrt{-\Delta_g})$ was supposed to be bounded by $h^{-\beta(r,d)+2\epsilon}$ and where in the last line we used (2.13). The implied constant in the last inequality depends only on T and the estimates hold uniformly with respect to $t \in [0,T]$.

To prove (2.13) and (2.14), we use the special form of Δ_g and the fact that $\tilde{V}_{h,\epsilon/3}(x, y_1, t)$ (and therefore $V_{h,\epsilon}$) is supported in $0 \le x \lesssim h^{(1-\epsilon/3)/2}$. The inhomogeneous part of the equation in (2.12) reads

$$
\Box \tilde{V}_{h,\epsilon/3} h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y') + (1+xb(y_1))\partial_{y_1}^2 W_{h,\epsilon} - \sum_{j,k=1}^{d-1} B_{j,k}(x, y)\partial_{y_j, y_k}^2 W_{h,\epsilon}. \quad (2.16)
$$

The $L^2(\Omega)$ norm of $\Box \tilde{V}_{h,\epsilon/3} h^{-(d-2)/4} e^{-|y'|^2/(2h)} \chi(y')$ is estimated using the last condition in Theorem 2.2 and its contribution to the norm of the nonlinear term of (2.12) is $O_{L^2(\Omega)}(1/h)$. We estimate the second term in (2.16) as follows:

$$
\begin{aligned}
-(1+xb(y_1))\partial_{y_1}^2 W_{h,\epsilon} &+ B_{1,1}(x, y)\partial_{y_1, y_1}^2 W_{h,\epsilon} \\
&= h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \chi(y')\partial_{y_1}^2 \tilde{V}_{h,\epsilon/3}\big(B_{1,1}(x, y) - 1 - b(y_1)\big).
\end{aligned}
$$

The last term in (2.16) splits into two sums, corresponding to $k = 1$,

$$
\begin{aligned}
\sum_{j=1}^{d-1} B_{1,j}(x, y)\partial_{y_1, y_j}^2 W_{h,\epsilon} \\
= -h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \frac{1}{h}\partial_{y_1} \tilde{V}_{h,\epsilon/3} \sum_{j=2}^{d-1} B_{1,j}(x, y)\big(y_j \chi(y') + h\partial_{y_j}\chi(y')\big), \quad (2.17)
\end{aligned}
$$

and $k \in \{2, \ldots, d-1\}$, respectively,

$$
\begin{aligned}
\sum_{j,k=2}^{d-1} B_{j,k}(x, y)\partial_{y_j, y_k}^2 W_{h,\epsilon} &= h^{-(d-2)/4} e^{-\frac{|y'|^2}{2h}} \frac{1}{h^2} B_{j,k}(x, y)\tilde{V}_{h,\epsilon/3} \\
&\times \sum_{j,k=2}^{d-1} \big(y_j y_k \chi(y') - h(y_j \partial_{y_k}\chi(y') + y_k \partial_{y_j}\chi(y') + \delta_{j=k}) + h^2\partial_{y_j, y_k}^2 \chi(y')\big). \quad (2.18)
\end{aligned}
$$

If $|y'| \geq h^{(1-\epsilon')/2}$ for some $\epsilon' > 0$, then $e^{-|y'|^2/(2h)} \leq C_M h^M$ for all $M \geq 0$, thus taking $\epsilon' = \epsilon/3$ we can estimate the $L^2(\Omega)$ norm of (2.17) and (2.18) as follows:

$$\left\| -(1 + xb(y_1))\partial_{y_1}^2 W_{h,\epsilon} + \sum_{j,k=1}^{d-1} B_{j,k}(x,y)\partial_{y_j}\partial_{y_k} W_{h,\epsilon} \right\|_{L^2(\Omega)}$$
$$\lesssim h^{-2+(1-\epsilon/3)} \|\tilde{V}_{h,\epsilon/3}\|_{L^2(\tilde{\Omega})} \lesssim h^{-1-\epsilon/3},$$

where we used that

$$\sup_{\epsilon>0} \|\tilde{V}_{h,\epsilon/3}\|_{L^2(\tilde{\Omega})} \lesssim 1, \quad \sup_{\epsilon>0} \|\partial_{y_1}\tilde{V}_{h,\epsilon/3}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h}, \quad \sup_{\epsilon>0} \|\partial_{y_1}^2 \tilde{V}_{h,\epsilon/3}\|_{L^2(\tilde{\Omega})} \lesssim \frac{1}{h^2}.$$

In the same way we obtain the bounds

$$\left\| -(1 + xb(y_1))\partial_{y_1}^2 W_{h,\epsilon} + \sum_{j,k=1}^{d-1} B_{j,k}(x,y)\partial_{y_j,y_k}^2 W_{h,\epsilon} \right\|_{\dot{H}^{-1}(\Omega)}$$
$$\lesssim h\left\| -(1 + xb(y_1))\partial_{y_1}^2 W_{h,\epsilon} + \sum_{j,k=1}^{d-1} B_{j,k}(x,y)\partial_{y_j,y_k}^2 W_{h,\epsilon} \right\|_{L^2(\Omega)} \lesssim h^{-\epsilon/3}.$$

For the last inequality we used the following lemma (see [10, Prop. 5.4] for the proof):

Lemma 2.7. *Let $f(x,y) : \Omega \to \mathbb{R}$ be localized at frequency $1/h$ in the $y \in \mathbb{R}^{d-1}$ variable, i.e. such that there exists $\psi \in C_0^\infty(\mathbb{R}^{d-1} \setminus \{0\})$ with $\psi(hD_y)f = f$. Then there exists a constant $C > 0$ independent of h such that*

$$\|f\|_{\dot{H}^{-1}(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}. \qquad \square$$

End of proof of Proposition 2.3. Recall that we have assumed that the operator

$$\sin(t\sqrt{-\Delta_g}) : L^2(\Omega) \to L^q([0,T], L^r(\Omega))$$

is bounded by $h^{-\beta(r,d)+2\epsilon}$. This assumption implies

$$\|V_{h,\epsilon}\|_{L^q([0,T],L^r(\Omega))} \leq Ch^{-\beta(r,d)+2\epsilon}(\|V_{h,\epsilon,0}\|_{L^2(\Omega)} + \|V_{h,\epsilon,1}\|_{\dot{H}^{-1}(\Omega)})$$
$$\leq \tilde{C}h^{-\beta(r,d)+2\epsilon}, \qquad (2.19)$$

where $C, \tilde{C} > 0$ are independent of h. Now (2.19) together with (2.10) gives

$$h^{-\beta(r,d)+2\epsilon/3} \lesssim \|W_{h,\epsilon}\|_{L^q([0,T],L^r(\Omega))}$$
$$\lesssim (\|V_{h,\epsilon}\|_{L^q([0,T],L^r(\Omega))} + \|w_{h,\epsilon,\text{err}}\|_{L^q([0,T],L^r(\Omega))}).$$

The last estimate together with (2.15) and (2.19) gives a contradiction, since it would imply

$$h^{-\beta(r,d)+2\epsilon/3} \lesssim h^{-\beta(r,d)+2\epsilon} + h^{-\beta(r,d)+2\epsilon-\epsilon/3},$$

which obviously cannot be true. The proof is complete. $\square$

3. Proof of Theorem 1.4

In order to finish the proof of Theorem 1.4 it remains to prove Theorem 2.2. Let (Ω, g) be a Riemannian manifold of dimension $d = 2$ with C^∞ boundary $\partial\Omega$. Suppose that local coordinates can be chosen near $(x = 0, y = 0)$ such that in a neighborhood of this point Ω is given by

$$\Omega = \{(x, y) \mid x > 0, \ y \in \mathbb{R}\},$$

and the Laplace–Beltrami operator Δ_g associated to the metric g is given by

$$\Delta_g = \partial_x^2 + (1 + xb(y))\partial_y^2, \tag{3.1}$$

where b is a smooth function with $b(0) = 0$. Let $Y > 0$ be such that for $y \in [0, Y]$ we have $|b^{1/3}(y) - 1| \leq \frac{1}{10}$. We proceed with the proof of Theorem 2.2, where we write (Ω, g) instead of $(\tilde\Omega, \tilde g)$ and where Δ_g is given by (3.1).

Remark 3.1. In what follows we fix $\epsilon > 0$ small enough and we do not mention anymore the dependence on ϵ of the solution to the wave equation (2.4) we are going to construct.

Before starting the proof of Theorem 2.2 we briefly recall some definitions we shall use in the rest of the paper (for details see [9] or [27], for example).

Set $X = \Omega \times \mathbb{R}_t$, let $\square_g = \partial_t^2 - \Delta_g$ denote the wave operator on X and let $p \in C^\infty(T^*X \setminus o)$ be the principal symbol of $\square_g$, which is homogeneous of degree 2 in $T^*X \setminus o$,

$$p(x, y, t, \xi, \eta, \tau) = \xi^2 + (1 + xb(y))\eta^2 - \tau^2. \tag{3.2}$$

The characteristic set $P := \mathrm{Char}(p) \subset T^*X \setminus o$ of $\square_g$ is defined to be $p^{-1}(\{0\})$.

Let us consider the Dirichlet problem for $\square_g$:

$$\square_g u = 0, \quad u|_{\partial X} = 0. \tag{3.3}$$

The statement of the propagation of singularities of solutions to (3.3) has two main ingredients: locating singularities of a distribution, as captured by the wave front set, and describing the curves along which they propagate, namely the bicharacteristics. Both of these are closely related to an appropriate notion of "phase space", in which both the wave front set and the bicharateristics are located. On manifolds without boundary, this phase space is the standard cotangent bundle T^*X. In the presence of boundaries it is the b-cotangent bundle, $^bT^*X$. There is a natural noninjective "inclusion" $\pi : T^*X \to {}^bT^*X$. We define the elliptic, glancing and hyperbolic sets in $T^*\partial X$ as follows:

$$\mathcal{E} = \{q \in \pi(T^*X) \setminus o \mid \pi^{-1}(q) \cap \mathrm{Char}(p) = \emptyset\},$$
$$\mathcal{G} = \{q \in \pi(T^*X) \setminus o \mid \mathrm{Card}(\pi^{-1}(q) \cap \mathrm{Char}(p)) = 1\},$$
$$\mathcal{H} = \{q \in \pi(T^*X) \setminus o \mid \mathrm{Card}(\pi^{-1}(q) \cap \mathrm{Char}(p)) \geq 2\},$$

with Card denoting the cardinality of a set; each of these is a conic subset of $\pi(T^*X) \setminus o$. Note that in $T^*\mathring{X}$, π is the identity map, so every point $q \in T^*\mathring{X}$ is either elliptic or glancing, depending on whether $q \notin \mathrm{Char}(p)$ or $q \in \mathrm{Char}(p)$.

The canonical local coordinates on T^*X will be denoted $(x, y, t, \xi, \eta, \tau)$, so 1-forms are $\alpha = \xi \, dx + \eta \, dy + \tau \, dt$. Let $(\rho, \vartheta) = (x, y, t, \xi, \eta, \tau)$ on T^*X near $\pi^{-1}(q)$, $q \in T^*\partial X$, with corresponding coordinates (y, t, η, τ) on a neighborhood $\mathcal{U}$ of q in $T^*\partial X$. Consequently,

$$\mathcal{E} \cap \mathcal{U} = \{(y, t, \eta, \tau) \mid \tau^2 < \eta^2\},$$

$$\mathcal{G} \cap \mathcal{U} = \{(y, t, \eta, \tau) \mid \tau^2 = \eta^2\},$$

$$\mathcal{H} \cap \mathcal{U} = \{(y, t, \eta, \tau) \mid \tau^2 > \eta^2\}.$$

Let $\rho = \rho(s) = (x, y, t)(s)$, $\vartheta = \vartheta(s) = (\xi, \eta, \tau)(s)$ be a bicharacteristic of $p(\rho, \vartheta)$, i.e. such that (ρ, ϑ) satisfies

$$\frac{d\rho}{ds} = \frac{\partial p}{\partial \vartheta}, \quad \frac{d\vartheta}{ds} = -\frac{\partial p}{\partial \rho}, \quad p(\rho(0), \vartheta(0)) = 0.$$

We say that $(\rho(s), \vartheta(s))|_{s=0}$ on the boundary ∂X is a *gliding point* if

$$x(\rho(0)) = 0, \quad \frac{d}{ds} x(\rho(0)) = 0, \quad \frac{d^2}{ds^2} x(\rho(0)) < 0.$$

This is equivalent to saying that $(\rho, \vartheta) \in T^*X \setminus o$ is a gliding point if

$$p(\rho, \vartheta) = 0, \quad \{p, x\}|_{(\rho, \vartheta)} = 0, \quad \{\{p, x\}, p\}|_{(\rho, \vartheta)} > 0. \tag{3.4}$$

Our assumption on the domain Ω in Theorem 2.2 near the point $(x = 0, y = 0)$ is equivalent to saying that there exists a bicharacteristic intersecting tangentially the boundary at $(0, 0)$ and having second order contact with the boundary at this point. From (3.4) this last condition translates into the following: there exists (ξ_0, η_0, τ_0) such that

$$\tau_0^2 = (1 + xb(y))\eta^2_{|(0,0,\xi_0,\eta_0)}, \quad \{p, x\}|_{(0,0,\xi_0,\eta_0)} = \frac{\partial p}{\partial \xi}\bigg|_{(0,0,\xi_0,\eta_0)} = 2\xi_0 = 0,$$

$$\{\{p, x\}, p\}|_{(0,0,\xi_0,\eta_0)} = \left\{\frac{\partial p}{\partial \xi}, p\right\}\bigg|_{(0,0,\xi_0,\eta_0)} = 2b(0)\eta_0^2 > 0.$$

Since $b(0) = 1$ we must have $\xi_0 = 0$ and $\eta_0 \neq 0$. Suppose without loss of generality that $\eta_0 = 1$. Let $(\rho_0, \vartheta_0) = (x = 0, y = 0, t = 0, \xi_0 = 0, \eta_0 = 1, \tau_0 = -1)$ and denote the gliding point (in $T^*\partial X$) by

$$\pi(\rho_0, \vartheta_0) = (y = 0, t = 0, \eta_0, \tau_0 = -\eta_0) = (0, 0, 1, -1) \in \mathcal{G}. \tag{3.5}$$

We define the semi-classical wave front set $\mathrm{WF}_h(u)$ of a distribution u on $\mathbb{R}^3$ to be the complement of the set of points $(\rho = (x, y, t), \zeta = (\xi, \eta, \tau)) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus 0)$ for which there exists a symbol $a(\rho, \zeta) \in \mathcal{S}(\mathbb{R}^6)$ such that $a(\rho, \zeta) \neq 0$ and for all integers $m \geq 0$,

$$\|a(\rho, hD_\rho)u\|_{L^2} \leq c_m h^m.$$

3.1. Choice of an approximate solution

We look for an approximate solution to the equation (2.4) of the form

$$u_h(x, y, t) = \int e^{\frac{i}{h}(\theta + \zeta\xi + \xi^3/3)} g_h \, d\xi \, d\eta \, d\tau, \tag{3.6}$$

where the phase functions $\theta(x, y, t, \eta, \tau)$, $\zeta(x, y, \eta, \tau)$ are real valued and homogeneous in (η, τ) of degree 1 and $2/3$, respectively, and where g_h is a symbol to be determined in the next sections. In order for u_h to solve (2.4), the functions θ, ζ must solve an eikonal equation that we derive in what follows. We denote by $\langle \cdot, \cdot \rangle$ the symmetric bilinear form obtained by polarization of the second order homogeneous principal symbol p of the wave operator $\Box_g$,

$$\langle da, db \rangle = \partial_x a \partial_x b + (1 + xb(y))\partial_y a \partial_y b - \partial_t a \partial_t b.$$

Applying the wave operator $h^2 \Box_g$ to u_h, the main contribution becomes

$$(\partial_x \theta + \xi \partial_x \zeta)^2 + (1 + xb(y))(\partial_y \theta + \xi \partial_y \zeta)^2 - (\partial_t \theta + \xi \partial_t \zeta)^2$$
$$= \langle d\theta, d\theta \rangle - 2\xi \langle d\theta, d\zeta \rangle + \xi^2 \langle d\zeta, d\zeta \rangle. \tag{3.7}$$

In order to eliminate this term after integrations by parts in ξ we require the right hand side of (3.7) to be a nontrivial multiple of $\partial_\xi \Phi$, where we set

$$\Phi = \theta + \zeta\xi + \xi^3/3.$$

This is equivalent to determining θ, ζ solutions to

$$\begin{cases} \langle d\theta, d\theta \rangle - \zeta \langle d\zeta, d\zeta \rangle = 0, \\ \langle d\theta, d\zeta \rangle = 0. \end{cases} \tag{3.8}$$

The system (3.8) is a nonlinear system of partial differential equations, which is elliptic where $\zeta > 0$ (shadow region), hyperbolic where $\zeta < 0$ (illuminated region) and parabolic where $\zeta = 0$ (caustic curve or surface). It is crucial that there is a solution of the form

$$\phi^{\pm} = \theta \mp \frac{2}{3}(-\zeta)^{3/2}$$

with θ, ζ smooth. Solutions with such singularities arise from solving the initial value problem for (3.8) off an initial surface which does not have the usual transversality condition, corresponding to the fact that there are bicharacteristics tangent to the boundary.

3.1.1. Geometric reduction.
Let $X = \Omega \times \mathbb{R}$ as before. Let p and q be functions on T^*X with independent differentials at a point $(\rho, \vartheta) \in T^*X \setminus o$. We denote by P and Q the hypersurfaces defined by p and q, respectively.

Definition 3.2. We say that the hypersurfaces P, Q in the symplectic manifold T^*X are *glancing surfaces* at (ρ, ϑ) if

1. $\{p, q\}((\rho, \vartheta)) = 0$,
2. $\{p, \{p, q\}\}((\rho, \vartheta)) \neq 0$ and $\{q, \{q, p\}\}((\rho, \vartheta)) \neq 0$.

In our case we take q to be the defining function of the boundary $\partial\Omega$, therefore $q = x$, and p the symbol of the wave operator $\Box_g$ defined in (3.2). Precisely,

$$Q = \{q(x, y, t, \xi, \eta, \tau) = x = 0\}, \quad P = \{p = \xi^2 + (1 + xb(y))\eta^2 - \tau^2 = 0\}, \quad (3.9)$$

which are glancing at (ρ_0, ϑ_0) defined in (3.5). The nondegeneracy conditions in Definition 3.2 hold at a point (ρ, ϑ) with $\{p, q\} = 0$ if and only if $\partial\Omega$ is strictly convex at (ρ, ϑ).

Remark 3.3. A model case of a pair of glancing surfaces is given by

$$Q_F = \{q_F(x, y, \xi, \eta, \tau) = x = 0\}, \quad P_F = \{p_F = \xi^2 + (1 + x)\eta^2 - \tau^2 = 0\}, \quad (3.10)$$

which have a second order intersection at the point

$$(\bar{\rho}_0, \bar{\vartheta}_0) := (0, y_0 = 0, t_0 = 0, 0, \eta_0 = 1, \tau_0 = -1) \in T^*X_F \setminus o.$$

This model case was studied in [10]. There is a deep geometrical reason underlying the similarity of the general gliding ray parametrix for (3.9) and the one for the model example (3.10), which will facilitate solution to the eikonal equation.

Theorem 3.4. *Let P and Q be two hypersurfaces in $T^*X \setminus o$ satisfying the glancing conditions in Definition 3.2 at $(\rho_0, \vartheta_0) \in P \cap Q \subset T^*X \setminus o$. Then there exist real functions θ and ζ which are C^∞ in a conic neighborhood $\mathcal{U}$ of $(\rho_0, 1, -1) \in X \times \mathbb{R}^2$, are homogeneous of degree one and two-thirds, respectively, and have the following properties:*

(i) *$\zeta_0 := \zeta|_{x=0} = -(\tau^2 - \eta^2)\eta^{-4/3}$ and $\partial_x\zeta|_{\partial X} > 0$ on $\mathcal{U} \cap \partial X \times \mathbb{R}^2$,*
(ii) *$d_{y,t}(\partial_\eta\theta, \partial_\tau\theta)$ are linearly independent on $\mathcal{U}$,*
(iii) *the system (3.8) holds in $\zeta \leq 0$; moreover, the phases θ and ζ also satisfy (3.8) to infinite order at $x = 0$.*

Moreover, ζ is a defining function for the fold set denoted Σ. By translation invariance in time, ζ is independent of t while the phase function θ is linear in the time variable.

Remark 3.5. Theorem 3.4 has been proved independently by Melrose in [18] and by Eskin in [6, Thm. 1] for the canonical glancing surfaces

$$Q_{\text{can}} = \{x = 0\}, \quad P_{\text{can}} = \{\xi^2 + x\eta^2 - \tau\eta = 0\},$$

near the glancing point $(x = 0, y, t, \xi = 0, \eta = 1, \tau = 0)$. It is not difficult to see that the phase functions θ and ζ can be chosen to satisfy the conditions stated above and therefore we leave the details of the proof of Theorem 3.4 to the reader.

Remark 3.6. Notice that if P and Q are the hypersurfaces in the symplectic space $T^*X \setminus o$ defined in (3.9) and glancing at $(\rho_0, \vartheta_0) \in T^*X \setminus o$, then there exists a canonical transformation

$$\chi : \Gamma \subset T^*X_F \setminus o \to T^*X \setminus o,$$

defined in a conic neighborhood Γ of $(\bar{\rho}_0, \bar{\vartheta}_0)$ and taking $(\bar{\rho}_0, \bar{\vartheta}_0)$ to (ρ_0, ϑ_0) and the model pair P_F and Q_F to P and Q. The fact that χ, which is symplectic, maps Q_F onto Q means that it defines a local canonical transformation from the quotient space of Q_F, modulo its Hamilton fibration, to the corresponding quotient space of Q, which is naturally identified as the cotangent space of the hypersurface

$$Q/\mathbb{R}H_q \simeq T^*\partial X.$$

Now, as we just said, on Q (and similarly on Q_F) the symplectic form gives a Hamilton foliation. Let it determine an equivalence relation $\sim$. Then $Q \cap P/\sim$ has the structure of a symplectic manifold with boundary and it is naturally isomorphic to the closure of the "hyperbolic" set in $T^*\partial X$, the region over which real rays pass, and similarly for $Q_F \cap P_F/\sim$. Therefore, the restriction of χ to $T^*\partial X_F$, denoted χ_∂, is also a canonical transformation from a neighborhood $\gamma \subset T^*\partial X_F \setminus o$ of $\pi(\bar{\rho}_0, \bar{\vartheta}_0)$ to a neighborhood of $\pi(\rho_0, \vartheta_0) \in T^*\partial X \setminus o$,

$$\chi_\partial : \gamma \to T^*\partial X \setminus o, \quad \gamma \subset T^*\partial X_F \setminus o,$$
$$\gamma = \{(y, t, \eta, \tau) \in T^*\partial X_F \mid \exists \xi, (0, y, t, \xi, \eta, \tau) \in \Gamma\}$$

defined in the hyperbolic region by

$$\chi_\partial^{-1} : (y, t, d_y\theta_0, d_t\theta_0) \mapsto (d_\eta\theta_0, d_\tau\theta_0, \eta, \tau), \quad \chi_\partial^{-1}(\pi(\rho_0, \vartheta_0)) = \pi(\bar{\rho}_0, \bar{\vartheta}_0),$$

where $\theta_0 := \theta|_{\partial X}$ is the restriction to ∂X of the phase θ introduced in Theorem 3.4.

The map χ_∂ has the property that near $\pi(\bar{\rho}_0, \bar{\vartheta}_0)$ it conjugates the billiard ball map $\delta^\pm \subset (T^*\partial X \setminus o) \times (T^*\partial X \setminus o)$ to the normal form $\delta_F^\pm$ introduced in (3.14) below. An interpolating Hamiltonian for the billiard ball maps $\delta^\pm$ is ζ_0 and $\delta^\pm(y, t, \eta, \tau) = \exp\left(\pm\frac{4}{3}H_{(-\zeta_0)^{3/2}}\right)$.

Remark 3.7. In Friedlander's model domain where $\Omega_F := \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}\}$ and $\Delta_F := \partial_x^2 + (1 + x)\partial_y^2$, which we have dealt with in [10], the equation (3.8) has a solution of the form

$$\phi_F^\pm = \theta_F \mp \frac{2}{3}(-\zeta_F)^{3/2},$$

where

$$\theta_F(x, y, t, \eta, \tau) = y\eta + t\tau, \quad \zeta_F(x, y, \eta, \tau) = \left(x - \frac{\tau^2 - \eta^2}{\eta^2}\right)\eta^{2/3},$$

as can be seen by direct computation. This solution serves very much as a guide to the general construction as we shall see in the next sections.

3.2. A model operator

Since the heart of the matter is well illustrated by Friedlander's example, we start by recalling the main steps of the construction in [10] where we proved Theorem 1.4 in the case of a two-dimensional, strictly convex domain $\Omega_F = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}\}$ with Laplace operator given by

$$\Delta_F = \partial_x^2 + (1 + x)\partial_y^2. \tag{3.11}$$

Let $X_F = \Omega_F \times \mathbb{R}$ and let $p_F \in C^\infty(T^* X_F \setminus o)$ denote the homogeneous symbol of the model wave operator $\Box_F$, $p_F(x, y, t, \xi, \eta, \tau) = \xi^2 + (1 + x)\eta^2 - \tau^2$. Consider the wave equation

$$\begin{cases} \partial_t^2 v - \partial_x^2 v - (1 + x)\partial_y^2 v = 0, \\ v|_{\partial\Omega_F \times [0,T]} = 0. \end{cases} \tag{3.12}$$

In [10] we have constructed a parametrix for (3.12) of the form

$$U_{F,h}(x, y, t) = \sum_{n=0}^{N} u_{F,h}^n(x, y, t),$$

where $u_{F,h}^n$ are approximate solutions to (3.12) of the form

$$u_{F,h}^n(x, y, t) = \int e^{\frac{i}{h} \eta(y - t(1+a)^{1/2} + \xi(x-a) + \frac{\xi^3}{3} + \frac{4}{3}na^{3/2})} g_F^n(t, \xi, \eta, h) \, d\xi \, d\eta, \tag{3.13}$$

where g_F^n are smooth functions compactly supported for η near 1 and where the relation between the amplitudes in the sum is dictated by the billiard ball maps. In this case, due to the presence of the translations in (y, t), the billiard ball maps have specific formulas

$$\delta_F^\pm(y, t, \eta, \tau) = \left(y \pm 4\left(\frac{\tau^2}{\eta^2} - 1\right)^{1/2} \pm \frac{8}{3}\left(\frac{\tau^2}{\eta^2} - 1\right)^{3/2}, t \mp 4\left(\frac{\tau^2}{\eta^2} - 1\right)^{1/2} \frac{\tau}{\eta}, \eta, \tau \right). \tag{3.14}$$

The associated Lagrangian sets are defined by

$$\Lambda_{F,n} := \left\{ \xi^2 + (x - a) = 0, \, y - t(1+a)^{1/2} + \xi(x-a) + \frac{\xi^3}{3} + \frac{4}{3}na^{3/2} = 0 \right\} \subset T^* X_F \setminus o.$$

Remark 3.8. Notice that, on the boundary, $\Lambda_{F,n}$ is the graph of the canonical transformation $(\delta_F^\pm)^n$ given by the formula

$$(\delta_F^\pm)^n(y, t, \eta, \tau) = \left(y \pm 4n\left(\frac{\tau^2}{\eta^2} - 1\right)^{1/2} \pm \frac{8}{3}n\left(\frac{\tau^2}{\eta^2} - 1\right)^{3/2}, t \mp 4n\left(\frac{\tau^2}{\eta^2} - 1\right)^{1/2} \frac{\tau}{\eta}, \eta, \tau \right). \tag{3.15}$$

We introduce the fold set $\Sigma_F = \{\xi = 0\}$ as the set of singular points of the canonical projection $\Lambda_{F,n} \to X_F$ and we define the caustic set to be the image of Σ_F through this canonical projection, hence the set $\{x - (\tau^2 - \eta^2)/\eta^2 = 0\} = \{\zeta_F(x, y, \eta, \tau) = 0\}$. Near the caustic set $\{\zeta_F = 0\}$ each solution $u_{F,h}^n$ "lives" essentially on a cusp defined by

$$y - t(1 + a)^{1/2} + \frac{4}{3}na^{3/2} = \pm(a - x)^{3/2}.$$

The key observation is that, if the parameter a is small enough, depending on the frequency, each such cusp type solution provides a loss in the Strichartz estimates.

3.2.1. Construction of an approximate solution in the model case. In this section we recall the construction from [10] of the symbols g_F^n of the cusp type parametrices $u_{F,h}^n$ in (3.13) for the model wave operator $\Box_F$. We take a of the form $a \simeq h^\alpha$ for some α to be chosen later as large as possible in the interval $(0, 2/3)$; notice that the case $a \simeq h^{2/3}$ corresponds to the whispering gallery modes dealt with in [10], while the case $a \simeq 1$ describes a wave transverse to the boundary.

Definition 3.9. Let $\lambda \geq 1$. For a given compact $K \subset \mathbb{R}$ we define the space $\mathcal{S}_K(\lambda)$ to consist of all functions $\varrho(z, \lambda) \in C^\infty(\mathbb{R})$ which satisfy

1. $\sup_{z \in \mathbb{R}, \lambda \geq 1} |\partial_z^\alpha \varrho(z, \lambda)| \leq C_\alpha$, where C_α are constants independent of λ,
2. if $\psi(z) \in C_0^\infty$ is a smooth function equal to 1 in a neighborhood of K and $0 \leq \psi \leq 1$ then $(1 - \psi)\varrho \in O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty})$.

Here $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing functions.

We define a new parameter $\lambda = a^{3/2}/h \gg 1$ and for some small $0 < c_0 \leq 3/8$, we set $K_0 = [-c_0, c_0]$. We take $\varrho(\cdot, \lambda) \in \mathcal{S}_{K_0}(\lambda)$ and set

$$g_F^0(t, \xi, \eta, h) = \varrho\left(\frac{t + 2(1 + a)^{1/2}\xi}{2(1 + a)^{1/2}a^{1/2}}, \lambda\right)\Psi(\eta),$$

where $\Psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ is supported in a small neighborhood of 1 and $0 \leq \Psi(\eta) \leq 1$. The boundary condition will help us determine the symbols g_F^n for every $0 \leq n \leq N$.

Proposition 3.10 ([10, Proposition 3.3, see also Lemma 3.2]). *On the boundary $u_{F,h}^0$ can be (modulo $O_{L^2}(\lambda^{-\infty})$) as a sum of two trace operators,*

$$u_{F,h}^0(0, y, t) = \sum_{\pm} \mathrm{Tr}_\pm(u_{F,h}^0)(y, t),$$

where

$$\mathrm{Tr}_\pm(u_{F,h}^0)(y, t) := h^{1/3} \int e^{\frac{i}{h}(y\eta - t(1+a)^{1/2}\eta \mp \frac{2}{3}a^{3/2}\eta)} \Psi(\eta)(\eta\lambda)^{-1/6}$$

$$\times I_\pm(\varrho(\cdot, \lambda))_\eta\left(\frac{t}{2(1 + a)^{1/2}a^{1/2}}, \lambda\right) d\eta, \quad (3.16)$$

with $I_\pm(\varrho(\cdot, \lambda))_\eta(z, \lambda)$ given by

$$I_\pm(\varrho(\cdot, \lambda))_\eta(z, \lambda) = \frac{\eta\lambda}{2\pi} \int e^{i\eta\lambda(w(z-z') \mp \frac{2}{3}((1-w)^{3/2}-1))} \kappa(w) a_\pm(w, \eta\lambda) \varrho(z', \lambda) \, dw.$$

(3.17)

Here κ is a smooth function supported for w in a small neighborhood of 0 such that $0 \le \kappa \le 1$ and $\kappa(w) = 1$ for w close to 0. In the integral (3.17), $a_\pm$ are given by the asymptotic expansions of the Airy function $\mathrm{Ai}(-(\lambda\eta)^{2/3}(1-w))$,

$$a_\pm(w, \eta, \lambda) \simeq e^{\pm i\pi/2 - i\pi/4}(1-w)^{-1/4} \sum_{j \ge 0} a_{\pm,j}(-1)^{-j/2}(1-w)^{-3j/2}(\eta\lambda)^{-j}.$$

Precisely, we used the decomposition $\mathrm{Ai}(z) = A_+(z) + A_-(z)$, where $A_\pm(z) = \mathrm{Ai}(e^{\mp 2\pi i/3} z)$. In particular, using the properties of the Airy functions $A_\pm$ it follows that the symbols $k(w)a_\pm(w, \eta\lambda)$ are elliptic at $w = 0$ (see [10, Appendix]).

Proposition 3.11 ([10, Lemma 3.4]). *Let $p \in \mathbb{Z}$ and $K_p = [-c_0 + p, c_0 + p]$. Then for η belonging to the support of Ψ we have*

$$I_{\pm,\eta} : \mathcal{S}_{K_p}(\lambda) \to \mathcal{S}_{K_{p\mp 1}}(\lambda).$$

Proposition 3.12. *Let η belong to the support of Ψ and let $J_{\pm,\eta}$ be the operators defined for some $\tilde\lambda \ge 1$ and $\breve\varrho \in \mathcal{S}_{K_{\mp 1}}(\tilde\lambda)$ by the formula*

$$J_\pm(\breve\varrho(\cdot, \tilde\lambda))_\eta(z', \lambda) := \frac{\eta\lambda}{2\pi} \int e^{i\eta\lambda((z'-z)w \pm \frac{2}{3}((1-w)^{3/2}-1))} b_\pm(w, \eta\lambda) \breve\varrho(z, \tilde\lambda) \, dz \, dw,$$

where $b_\pm(w, \eta\lambda) = \frac{k(w)}{a_\pm(w, \eta\lambda)}$ are asymptotic expansions in $(\eta\lambda)^{-1}$. Then

$$\breve\varrho(\cdot, \tilde\lambda) = I_\pm\big(J_\pm(\breve\varrho(\cdot, \tilde\lambda))_\eta(\cdot, \lambda)\big)_\eta(\cdot, \lambda) + O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) + O_{\mathcal{S}(\mathbb{R})}(\tilde\lambda^{-\infty}),$$

$$\varrho(\cdot, \tilde\lambda) = J_\pm\big(I_\pm(\varrho(\cdot, \tilde\lambda))_\eta(\cdot, \lambda)\big)_\eta(\cdot, \lambda) + O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) + O_{\mathcal{S}(\mathbb{R})}(\tilde\lambda^{-\infty}).$$

The construction of the operators $J_{\pm,\eta}$ is detailed in [10, Section 3.3.1].

Proposition 3.13 ([10, Proposition 3.6]). *Let $N \lesssim \lambda h^\epsilon$ for some small $\epsilon > 0$ and let $1 \le n \le N$. Let T_k denote the translation operator which to a given function $\varrho(z)$ associates $\varrho(z + k)$. Then for $\eta \in \mathrm{supp}(\Psi)$ we have*

$$(T_1 \circ J_+(\cdot)_\eta \circ I_-(\cdot)_\eta \circ T_1)^{\circ n} : \mathcal{S}_{K_0}(\lambda) \to \mathcal{S}_{K_0}(\lambda/n) \quad \text{uniformly in } n.$$

Notice that since $\lambda/n \ge h^{-\epsilon} \gg 1$, then $O_{\mathcal{S}(\mathbb{R})}(\lambda^{-\infty}) = O_{\mathcal{S}(\mathbb{R})}((\lambda/n)^{-\infty}) = O_{\mathcal{S}(\mathbb{R})}(h^\infty)$.

Moreover, the operator defined above can be written as a convolution

$$(T_1 \circ J_+(\cdot)_\eta \circ I_-(\cdot)_\eta \circ T_1)^{\circ n}(\varrho) = (F_{\eta\lambda})^{*n} * \varrho, *$$

where

$$(F_{\eta\lambda})^{*n}(z) = \frac{\eta\lambda}{2\pi} \int e^{i\eta\lambda(wz + n(2w + \frac{4}{3}((1-w)^{3/2}-1)))} \big(\kappa(w) a_+(w, \eta\lambda) b_-(w, \eta\lambda)\big)^n \, dw.$$

Definition 3.14. Let $\varrho(\cdot, \lambda) \in \mathcal{S}_{K_0}(\lambda)$ and $\eta \in \mathrm{supp}(\Psi)$. For $1 \leq n \leq N$, $N \lesssim \lambda h^\epsilon$ set

$$\varrho^n(z, \eta, \lambda) := (-1)^n (T_1 \circ J_+(\cdot)_\eta \circ I_-(\cdot)_\eta \circ T_1)^n (\varrho(\cdot, \lambda))(z), \qquad \varrho^0(z, \eta, \lambda) = \varrho(z, \lambda).$$

Remark 3.15. From Proposition 3.13 it follows that $\varrho^n(z, \eta, \lambda) \in \mathcal{S}_{K_0}(\lambda/n)$.

Definition 3.16. For $0 \leq n \leq N$ with $N \lesssim \lambda h^\epsilon$ define

$$g_F^n(t, \xi, \eta, h) := \varrho^n \left(\frac{t + 2(1+a)^{1/2}\xi}{2(1+a)^{1/2}a^{1/2}} - 2n, \eta, \lambda \right) \Psi(\eta).$$

Proposition 3.17. *For all $0 \leq n \leq N - 1$ we have*

$$\mathrm{Tr}_-(u_{F,h}^n)(y, t) + \mathrm{Tr}_+(u_{F,h}^{n+1})(y, t) = O_{L^2}(\lambda^{-\infty}).$$

Proposition 3.18. *If $0 \leq n \leq N$, $u_{F,h}^n(\cdot, y, t)$ is essentially supported for y and t in the interval*

$$I_n(c_0) := 2a^{1/2}(1+a)^{1/2} \times [2n - (1 + c_0), 2n + (1 + c_0)], \tag{3.18}$$

i.e. for y or t outside any neighborhood of $I_n(c_0)$ the contribution of $u_{F,h}^n$ is $O_{L^2}(h^\infty)$.

3.3. Construction of an approximate solution in the general case

In this section we construct an approximate solution to (2.4) satisfying the conditions of Theorem 2.2. It will be essentially based on the model construction and Theorem 3.4.

Inspired from [10], we construct superposition solutions u_h^n to (2.4) of the form

$$u_h^n(x, y, t) = \int e^{\frac{i}{h} \Phi^n(x, y, t, \xi, \eta, \tau)} g_h^n \, d\xi \, d\eta \, d\tau, \tag{3.19}$$

for some symbols g_h^n to be suitably chosen and where the phase functions are given by

$$\Phi^n(x, y, t, \xi, \eta, \tau) := \theta(x, y, t, \eta, \tau) + \eta^{1/3}\xi \zeta(x, y, \eta, \tau) + \eta \frac{\xi^3}{3} + \frac{4}{3} n(-\zeta_0)^{3/2}(\eta, \tau). \tag{3.20}$$

We determine the symbols g_h^n in (3.19) so that u_h^n is an approximate solution to (2.4) in a sense to be made precise. We start by defining their restriction to the boundary by requiring the Dirichlet condition to be fulfilled. We consider an operator J defined by

$$J(f)(y, t) := \frac{1}{(2\pi h)^2} \int e^{\frac{i}{h}\theta_0(y, t, \eta, \tau)} d_h(y, \eta, \tau) \widehat{f}(\eta/h, \tau/h) \, d\eta \, d\tau, \tag{3.21}$$

where $d_h(y, \eta, \tau) = d(y, \eta/h, \tau/h)$ for some elliptic symbol $d(y, \eta, \tau)$ of order 0 and type $(1, 0)$, compactly supported in a conic neighborhood of the glancing point $\pi(\rho_0, \vartheta_0)$. Here θ_0 denotes the restriction to the boundary of the phase θ introduced in Theorem 3.4.

The operator J defines an elliptic FIO in a neighborhood of $(\pi(\bar{\rho}_0, \bar{\vartheta}_0), \pi(\rho_0, \vartheta_0))$, with canonical relation χ_∂ given by the symplectomorphism generated by θ_0 which satisfies $\chi_\partial(\pi(\bar{\rho}_0, \bar{\vartheta}_0)) = \pi(\rho_0, \vartheta_0)$ (see the remarks following Theorem 3.4).

In what follows we compute $J \circ \mathrm{Tr}_\pm(u_{F,h}^n)$, where $0 \leq n \leq N$ for some N to be determined later. We keep the notation of Section 3.2.

Proposition 3.19. *On the boundary, $J \circ \mathrm{Tr}_\pm(u_{F,h}^n)$ reads*

$$J \circ \mathrm{Tr}_\pm(u_{F,h}^n)(y,t) = h^{1/3} \int e^{\frac{i}{h}(\theta_0(y,t,\eta,-\eta(1+a)^{1/2}) + \frac{2}{3}(2n\mp 1)(-\zeta_0)^{3/2}(\eta,-\eta(1+a)^{1/2}))}(\lambda\eta)^{-1/6}$$
$$\times I_\pm(g_h^n(\cdot,y,\eta))_\eta\left(\frac{\partial_\tau\theta_0(y,t,\eta,-\eta(1+a)^{1/2})}{2(1+a)^{1/2}a^{1/2}} - 2n, \lambda\right) d\eta,$$

where

$$g_h^n(z,y,\eta) \simeq \Psi(\eta)\left(\sum_{k\geq 0} h^{k/2}a^{-k/2}\mu_k(y,\eta,h)\partial_z^k \varrho^n(z,\eta,\lambda)\right). \qquad (3.22)$$

Here $\mu_k(y,\eta,h)$ are symbols of order 0 and type $(1,0)$ independent of n. Moreover, if $\eta \in \mathrm{supp}(\Psi)$ and $1 \leq n \leq N \lesssim \lambda h^\epsilon$ for some small $\epsilon > 0$ then $g_h^n(\cdot,y,\eta) \in \mathcal{S}_{K_0}(\lambda/n)$.

Proof. An explicit computation (using arguments from [10, Lemma 3.2]) gives

$$u_{F,h}^n(0,\bar{y},\bar{t}) = h^{1/3}\sum_\pm \int e^{\frac{i}{h}(\bar{y}\bar{\eta} - \bar{t}(1+a)^{1/2}\bar{\eta} \mp \frac{2}{3}a^{3/2}\bar{\eta} + \frac{4}{3}na^{3/2}\bar{\eta})}\Psi(\bar{\eta})(\bar{\eta}\lambda)^{-1/6}$$
$$\times I_\pm(\varrho^n(\cdot,\bar{\eta},\lambda))_{\bar{\eta}}\left(\frac{\bar{t}}{2(1+a)^{1/2}a^{1/2}} - 2n, \lambda\right) d\bar{\eta}, \qquad (3.23)$$

where $I_\pm(\varrho^n(\cdot,\bar{\eta},\lambda))_{\bar{\eta}}(z,\lambda)$ are defined in (3.17). The contributions corresponding to the $\pm$ signs in the right hand side of (3.23) are denoted $\mathrm{Tr}_\pm(u_{F,h}^n)(\bar{y},\bar{t})$.

We can now proceed to compute $J \circ \mathrm{Tr}_\pm(u_{F,h}^n)(\bar{y},\bar{t})$:

$$J \circ \mathrm{Tr}_\pm(u_{F,h}^n)(y,t) = \frac{h^{1/3}}{(2\pi h)^2} \int e^{\frac{i}{h}(\theta_0(y,t,\eta,\tau) - \bar{y}(\eta-\bar{\eta}) - \bar{t}(\tau + \bar{\eta}(1+a)^{1/2}) \mp \frac{2}{3}a^{3/2}\bar{\eta} + \frac{4}{3}na^{3/2}\bar{\eta})}\Psi(\bar{\eta})$$
$$\times I_\pm(\varrho^n(\cdot,\bar{\eta},\lambda))_{\bar{\eta}}\left(\frac{\bar{t}}{2(1+a)^{1/2}a^{1/2}} - 2n, \lambda\right)(\bar{\eta}\lambda)^{-1/6}\,d_h(y,\eta,\tau)\,d\bar{\eta}\,d\bar{y}\,d\bar{t}\,d\eta\,d\tau.$$

Since the symbol is independent of $\bar{y}$, integration in $\bar{y}$ gives $\eta = \bar{\eta}$. Now we are in a situation where the stationary phase theorem can be applied in the variables $(\bar{t},\tau)$. Consequently, $J \circ \mathrm{Tr}_\pm(u_{F,h}^n)$ admits the asymptotic expansion

$$J \circ \mathrm{Tr}_\pm(u_{F,h}^n)(y,t) \simeq h^{1/3} \int e^{\frac{i}{h}(\theta_0(y,t,\eta,-\eta(1+a)^{1/2}) + \frac{2}{3}(2n\mp 1)(-\zeta_0)^{3/2}(\eta,-\eta(1+a)^{1/2}))}\Psi(\eta)$$
$$\times \left[\sum_{k\geq 0} h^{k/2}a^{-k/2}\mu_k(y,\eta,h)\partial^k I_\pm(\varrho^n(\cdot,\eta,\lambda))_\eta\left(\frac{\partial_\tau\theta_0(y,t,\eta,-\eta(1+a)^{1/2})}{2(1+a)^{1/2}a^{1/2}} - 2n, \lambda\right)\right]$$
$$\times (\eta\lambda)^{-1/6}\,d\eta, \qquad (3.24)$$

where we set

$$\mu_k(y,\eta,h) = i^k 2^{-k} h^{k/2}(1+a)^{-k/2}\partial^k\left(e^{\frac{i}{h}r(y,t,\eta,\tau)}d_h(y,\eta,\tau)\right)\Big|_{\left\{\begin{array}{l}\tau = -\eta(1+a)^{1/2},\\ \bar{t} = \partial_\tau\theta_0(y,t,\eta,-\eta(1+a)^{1/2})\end{array}\right.}.$$

The main contribution of $\mu_{2\nu}$ is equal to $(\partial_\tau^2 r)^\nu d_h e^{\frac{i}{\hbar} r(\omega,\tau)}$ and that of $\mu_{2\nu-1}$ is $h(\partial_\tau^2 r)^\nu \partial_\tau d_h e^{\frac{i}{\hbar} r(\omega,\tau)}$, all the other terms in the sum defining μ_k being positive powers of h. Since d_h is a symbol of order 0 and type $(1,0)$, we deduce that μ_k is also a symbol of order 0 and type $(1,0)$.

Notice, moreover, that $I_\pm(\varrho^n)_\eta$ is a convolution product and consequently $\partial^k(I_\pm(\varrho^n)_\eta) = I_\pm(\partial^k \varrho^n)_\eta$. Since from Proposition 3.13 and Definition 3.14 the symbols $\varrho^n(\cdot, \eta, \lambda)$ belong to $\mathcal{S}_{K_0}(\lambda/n)$, where $K_0 = [-c_0, c_0]$, it follows that the sum

$$\Psi(\eta)\Big(\sum_{k \geq 0} h^{k/2} a^{-k/2} \mu_k(y, \eta, h) \partial^k \varrho^n(z, \eta, \lambda)\Big)$$

(denoted $g_h^n(z, y, \eta)$ in the statement of Proposition 3.19) also belongs to $\mathcal{S}_{K_0}(\lambda/n)$.   □

Lemma 3.20. *Let Φ^n be defined by* (3.20). *Then the integral curves of the vector field* $\langle 2d\Phi^n, d\cdot\rangle - \eta^{-1/3}\langle d\zeta, d\zeta\rangle \partial_\xi$ *are given by*

$$\eta^{-2/3}\zeta + \xi^2, \quad \partial_\tau \theta + \eta^{1/3}\xi \partial_\tau \zeta.$$

Now we can define u_h^n everywhere as follows:

Definition 3.21. *Let g_h^n be the symbol defined in* (3.22) *and for $0 \leq n \leq N \lesssim \lambda h^\epsilon$ let*

$$u_h^n(x, y, t) := \int e^{\frac{i}{\hbar}\Phi^n(x,y,t,\xi,\eta,-\eta(1+a)^{1/2})}$$

$$\times g_h^n\Big(\frac{\partial_\tau \theta + \eta^{1/3}\xi \partial_\tau \zeta}{2(1+a)^{1/2} a^{1/2}}(x, y, t, \eta, -\eta(1+a)^{1/2}) - 2n, y, \eta\Big) d\xi \, d\eta. \quad (3.25)$$

It remains to show that the restriction to $\partial\Omega$ of u_h^n defined in (3.25) coincides with the sum of the two terms in (3.24). We leave the proof of Proposition 3.22 below to the reader.

Proposition 3.22. *On the boundary $\partial\Omega$ we have, indeed,*

$$u_h^n(0, y, t) = \sum_\pm J(\mathrm{Tr}_\pm(u_{F,h}^n))(y, t, h).$$

Moreover,

$$J(\mathrm{Tr}_-(u_{F,h}^n))(y, t) + J(\mathrm{Tr}_+(u_{F,h}^{n+1}))(y, t) = O_{L^2}(h^\infty).$$

Applying the wave operator $\Box_g$ to u_h^n (defined by (3.25)) and using (3.8) yields

$$\Box_g u_h^n(x, y, t) = \int e^{\frac{i}{\hbar}\Phi^n(x,y,t,\xi,\eta,-\eta(1+a)^{1/2})}$$

$$\times \Big(\frac{i}{h}\Big(\langle 2d\Phi^n, dg_h^n\rangle - \eta^{-1/3}\langle d\zeta, d\zeta\rangle \partial_\xi g_h^n + (\Box_g \Phi^n)g_h^n\Big) + \Box g_h^n\Big) d\xi \, d\eta.$$

Using Lemma 3.20 and performing some elementary computations, we obtain:

Proposition 3.23. *The following estimates hold uniformly for $0 \leq n \leq N \lesssim \lambda h^\epsilon$:*

$$\|\Box_g u_h^n(\cdot, t)\|_{L^2(\Omega)} = O(h^{-1})\|u_h^n(\cdot, t)\|_{L^2(\Omega)}.$$

3.4. Main properties of the parametrix

In this section we state the main properties of the parametrix

$$U_h(x, y, t) := \sum_{n=0}^{N} u_h^n(x, y, t), \tag{3.26}$$

where $u_h^n(x, y, t)$ are introduced in (3.25) and where $N \lesssim \lambda h^\epsilon$ for some $\epsilon > 0$. Here ϵ is fixed at the beginning of Section 3 (see Remark 3.1). We first prove that each u_h^n is essentially supported for t in an interval of size $\simeq a^{1/2}$ and that $(u_h^n)_n$ have almost disjoint supports in time and in the tangential variable y.

In the first part of this section we choose N so that for all $0 \le n \le N$, each cusp type solution u_h^n preserves the same properties as u_h^0. We prove that this requires

$$4Na^{1/2} \lesssim Y. \tag{3.27}$$

Taking into account that we need to impose $N \lesssim \lambda h^\epsilon$ (since otherwise the construction in Section 3.2 may degenerate) and since we require the parameter a to be as small as possible, we must have

$$N \simeq \lambda h^\epsilon, \quad \text{where} \quad \lambda = a^{3/2}/h.$$

The last conditions yield $a \simeq \frac{1}{2} Y^{1/2} h^{(1-\epsilon)/2}$. We state a useful property of the parametrix u_h^n:

Proposition 3.24. *Let u_h^n be given by (3.25). Then the wave front set $\mathrm{WF}_h(u_h^n)$ of u_h^n is contained in the Lagrangian set Λ_{Φ^n} defined by*

$$\Lambda_n := \left\{ (x, y, t, \xi, \eta, -\eta(1+a)^{1/2}) \,\middle|\, \zeta(x, y, \eta, -\eta(1+a)^{1/2}) + \eta^{2/3}\xi^2 = 0, \right.$$

$$\left. (\partial_\eta \theta - (1+a)^{1/2}\partial_\tau \theta + \xi\zeta)(x, y, t, 1, -(1+a)^{1/2}) + \frac{\xi^3}{3} + \frac{4}{3}na^{3/2} = 0 \right\}.$$

In other words, outside any neighborhood of Λ_n the contribution of u_h^n is $O_{L^2}(h^\infty)$.

Proposition 3.24 follows from integrations by parts either in the variable ξ, using the operator $L_1 := \frac{h}{i} \frac{1}{\xi^2 + \eta^{-2/3}\zeta} \partial_\xi$, or in η, using $L_2 := \frac{h}{i} \frac{\partial_\eta \Phi^n}{|\partial_\eta \Phi^n|^2} \partial_\eta$. We need however to estimate the η-derivatives of g_h^n, and in order to do so we use the convolution type form of ϱ^n which follows from Proposition 3.13.

3.4.1. Number of iterations and time interval.
In what follows we choose the number N of iterated cusp type solutions u_h^n.

Proposition 3.25. *Let* $1 \leq N \leq C_0 Y a^{-1/2}$ *for some fixed* $C_0 > 0$. *If* C_0 *is chosen sufficiently small, then the operator* J *introduced in* (3.21) *is elliptic near the set*

$$\Big(\bigcup_{0 \leq n \leq N} I_n(c_0) \Big) \times \Big(\bigcup_{0 \leq n \leq N} I_n(c_0) \Big) \times (1, -(1+a)^{1/2})$$

and χ_∂ *is a diffeomorphism from a neighborhood of this set onto its image. We recall that* $I_n(c_0)$ *was introduced in* (3.18) *and* $0 < c_0 \leq 3/8$ *was fixed in Section* 3.2.1.

Proof. We first deal with the last statement. We use the properties of χ_∂ defined in Section 3.1.1 together with the assumption on $b(y)$ in Theorem 2.2. We can explicitly compute the gradient $\nabla_{y,t}(\chi_\partial^{-1})$ to be

$$\begin{pmatrix} \partial^2_{y,\eta} \theta_0(y, t, \eta, \tau) & 0 \\ \partial^2_{y,\tau} \theta_0(y, t, \eta, \tau) & 1 \end{pmatrix}. \tag{3.28}$$

Using the construction of the phases θ_0 and ζ_0 we obtain

$$\partial^2_{y,\eta} \theta_0(y, t, \eta, \tau) = b^{2/3}(y) + O(\zeta_0(\eta, \tau)) \tag{3.29}$$

and also

$$\partial^2_{y,\tau} \theta_0(y, t, \eta, \tau) = \Big(1 - \Big(\frac{\tau}{\eta}\Big)^{4/3} b^{2/3}(y)\Big) + O(\zeta_0(\eta, \tau)). \tag{3.30}$$

Notice also that the right hand side in both (3.29) and (3.30) is independent of t (which follows from the linearity in time of θ). Since $|b^{1/3}(y) - 1| \leq 1/10$ for $y \in [0, Y]$, if η/τ is close to 1 it follows that χ_∂ is a diffeomorphism from a small, fixed, conic neighborhood of $\pi(\bar{\varrho}_0, \bar{\vartheta}_0)$ into a small neighborhood of $\pi(\varrho_0, \vartheta_0)$.

Remark 3.26. In particular, the Jacobian of the restriction of χ_∂^{-1} to the values $(\eta, \tau) = (1, -(1+a)^{1/2})$ equals $\mathrm{Jac}(\chi_\partial^{-1}) = \partial^2_{y,\eta} \theta_0(y, t, 1, -(1+a)^{1/2})$ and is independent of t.

Now we proceed with the first part of Proposition 3.25. Recall that χ_∂ conjugates the billiard ball map $\delta^\pm$ to the normal form, $\chi_\partial \circ \delta_F^\pm = \delta^\pm \circ \chi_\partial$. Since we consider only positive time it is enough to work with δ^+, δ_F^+. For $n \geq 1$ and for $\pi(\bar{\varrho}, \bar{\vartheta})$ in a small, conic neighborhood of $\pi(\bar{\varrho}_0, \bar{\vartheta}_0)$ we have, writing $\pi(\varrho, \vartheta) = \chi_\partial(\pi(\bar{\varrho}, \bar{\vartheta}))$,

$$(\delta^+)^n(\pi(\varrho, \vartheta)) = (\delta^+)^n(\chi_\partial(\pi(\bar{\varrho}, \bar{\vartheta}))) = \chi_\partial\big((\delta_F^+)^n(\pi(\bar{\varrho}, \bar{\vartheta}))\big), \tag{3.31}$$

where $(\delta_F^+)^n$ is given by (3.15). We have assumed (without loss of generality) that $\pi(\varrho_0, \vartheta_0) = (0, 0, 1, -1)$. Modulo a translation we can also assume that $\pi(\bar{\varrho}_0, \bar{\vartheta}_0) = (0, 0, 1, -1)$. We now take $\bar{\varrho} = \bar{\varrho}_0$ and $\bar{\vartheta} = (1, -(1+a)^{1/2})$ in (3.31). Therefore, for a small, depending on h, $\pi(\bar{\varrho}, \bar{\vartheta})$ belongs to a conic neighborhood of $\pi(\bar{\varrho}_0, \bar{\vartheta}_0)$. Rewriting (3.31) at this point and using (3.15) yields

$$(\delta^+)^n(\pi(\varrho, \vartheta)) = \chi_\partial\Big(4na^{1/2}(1+a) - \frac{4}{3}na^{3/2}, 4na^{1/2}(1+a)^{1/2}, 1, -(1+a)^{1/2}\Big)$$

$$=: \chi_\partial(\bar{y}_n, \bar{t}_n, 1, -(1+a)^{1/2}),$$

where we set $\bar{y}_n = 4na^{1/2}(1+a) - \frac{4}{3}na^{3/2}$ and $\bar{t}_n = 4na^{1/2}(1+a)^{1/2}$. If $na^{1/2}$ belongs to a small but fixed neighborhood of 0 of size Y, we can write the right hand side in the last equation as

$$\chi_\partial(\bar{y}_n, \bar{t}_n, 1, -(1+a)^{1/2}) = \nabla_{y,t}\chi_\partial(\bar{y}, \bar{t}, 1, -(1+a)^{1/2})\binom{\bar{y}_n}{\bar{t}_n}, \qquad (3.32)$$

for some $(\bar{y}, \bar{t}) \in [0, \bar{y}_n] \times [0, \bar{t}_n]$. The matrix $\nabla_{y,t}\chi_\partial(\bar{y}, \bar{t}, 1, -(1+a)^{1/2})$ is independent of $\bar{t}$ (since it is given by the inverse of $\nabla_{y,t}(\chi_\partial^{-1})$ computed in (3.28)) and at $\bar{y} = 0$ it is close to the identity, therefore if $\bar{y}_n$ belongs to a small, fixed neighborhood of 0 of size y_0, the matrix $\nabla_{y,t}\chi_\partial(\bar{y}, \bar{t}, 1, -(1+a)^{1/2})$ remains close to the identity.

We can now estimate the number of iterations N that we will use in our construction. Since $\bar{y}_n = 4na^{1/2} + O(na^{3/2})$, choose $N \geq 1$ such that $Na^{1/2}$ is so small that for $0 \leq \bar{y} \leq \bar{y}_N$ the gradient matrix $\nabla_{y,t}(\chi_\partial)(\bar{y}, \cdot, 1, -(1+a)^{1/2})$ is close to the identity. This is possible due to the independence from $\bar{t}_N, \bar{t}$ of the gradient matrix and its uniform boundedness. Since from the initial assumption we have $|b^{2/3}(y) - 1| < 1/10$ for $y \in [0, Y]$, we can take

$$4N \simeq C_0 Y a^{-1/2} \qquad (3.33)$$

for some sufficiently small constant $0 < C_0 \leq 1$. $\qquad \square$

Remark 3.27. The operator J is elliptic in a small, fixed neighborhood of 0. If we take Y smaller if necessary, the symbol d of J will be elliptic for $y \in [0, Y]$, therefore the symbol of $J \circ \mathrm{Tr}_\pm(u_{F,h}^n)$ will remain elliptic for any $0 \leq n \leq N$ with N given by (3.33) if C_0 is small enough. This will be useful when computing the L^r norms of the cusps u_h^n, whose symbols will depend on d and therefore will be elliptic uniformly in $0 \leq n \leq N$.

In the following we set

$$a := \frac{\sqrt{C_0}}{2} Y^{1/2} h^{(1-\epsilon)/2}. \qquad (3.34)$$

In the remaining part of this section we estimate the time interval $[0, T]$ on which the norm of U_h will be evaluated. Using Proposition 3.28 below and N as in (3.33) we see that for $0 \leq n \leq N$, the cusp $u_h^n(\cdot, t)$ is essentially supported for (x, y, t) such that

$$\partial_\tau \theta(x, y, t, 1, -(1+a)^{1/2}) \in I_n(c_0),$$

where

$$I_n(c_0) = 2a^{1/2}(1+a)^{1/2} \times [2n - (1+c_0), 2n + (1+c_0)].$$

We shall choose T to belong to the essential support of u_h^N. From Proposition 3.28 below it follows that on the essential support of u_h^N the following should hold:

$$(\partial_\eta \theta - (1+a)^{1/2}\partial_\tau \theta)(x, y, t, 1, -(1+a)^{1/2})$$
$$= -\frac{4}{3}Na^{3/2} \pm \frac{2}{3}(-\zeta)^{3/2}(x, y, 1, -(1+a)^{1/2}).$$

We introduce the defining function for the caustic set, denoted $C(y, \eta, \tau)$, as follows:

$$-\zeta(x, y, \eta, \tau) = 0 \quad \text{if and only if} \quad x = C(y, \eta, \tau).$$

By the linearity of the phase $\theta(x, y, t, \eta, \tau)$ in time, it follows that the map

$$y \mapsto \partial_\eta \theta(C(y, 1, -(1+a)^{1/2}), y, \cdot, 1, -(1+a)^{1/2}) \tag{3.35}$$

is independent of time. Moreover, (3.35) is a diffeomorphism in a neighborhood of $y = 0$ since its derivative does not vanish.

Let $\bar{t}_N := 4Na^{1/2}(1+a)^{1/2}$ be the center of the interval $I_N(c_0)$. From the discussion above (and if C_0 in (3.33) is small) it follows that there exists a unique $y_N \in [0, Y]$ with

$$\partial_\eta \theta(C(y_N, 1, -(1+a)^{1/2}), y_N, \cdot, 1, -(1+a)^{1/2}) = (1+a)^{1/2} \bar{t}_N - \frac{4}{3} N a^{3/2} = \bar{y}_N. \tag{3.36}$$

We can now define T as the unique time value which satisfies

$$\partial_\tau \theta(C(y_N, 1, -(1+a)^{1/2}), y_N, T, 1, -(1+a)^{1/2}) = \bar{t}_N. \tag{3.37}$$

3.4.2. Localization properties of u_h^n. We now describe the essential supports of u_h^n and show that they are contained in almost disjoint intervals obtained by taking the image of $I_n(c_0)$ defined in Proposition 3.18 under the symplectomorphism χ_∂.

Proposition 3.28. *We have*

$$\text{ess-supp}(u_h^n) \subset \left\{ (x, y, t) \,\middle|\, \partial_\tau \theta(x, y, t, 1, -(1+a)^{1/2}) \in I_n(c_0) \text{ and} \right.$$

$$\left. \left(\partial_\eta \theta - (1+a)^{1/2} \partial_\tau \theta + \frac{4}{3} n a^{3/2} \mp \frac{2}{3}(-\zeta)^{3/2} \right)(x, y, t, 1, -(1+a)^{1/2}) = 0 \right\}.$$

By ess-supp(u_h^n) *we denote the closure of the set outside of which u_h^n is $O_{L^2}(h^\infty)$.*

Proof. The proposition follows using Proposition 3.19, which gives information about the essential support of the symbol $g_h^n(\cdot, y, \eta) \in \mathcal{S}_{[-c_0, c_0]}(\lambda/n)$. Let $c \in (0, 1)$ be such that

$$\left| \frac{\partial_\tau \theta(x, y, t, \eta, -\eta(1+a)^{1/2})}{2(1+a)^{1/2} a^{1/2}} - 2n \right| \geq 1 + c. \tag{3.38}$$

It is enough to show that on the essential support of u_h^n we must have $c \leq c_0$, which follows from Proposition 3.24. $\qquad\square$

Lemma 3.29. *Let*

$$J_n := \left\{ t \,\middle|\, \exists (x, y, t) \in \text{ess-supp}(u_h^n), \ \partial_\tau \theta(x, y, t, 1, -(1+a)^{1/2}) \in I_n(c_0/3 - 1) \right\} \tag{3.39}$$

and let $|J_n|$ denote its size. Then

$$|J_n| \geq c_0 a^{1/2}.$$

Moreover, if c_0 is sufficiently small and if (x, y, t) is such that $t \in J_n$, then

$$\frac{1}{2} a \leq x \leq \frac{3}{2} a. \tag{3.40}$$

The proof of Lemma 3.29 uses arguments similar to those in the proof of Proposition 3.28 and is left to the reader. In the same way we can show the following:

Lemma 3.30. *Let $k \geq 0$ and $t \in J_k$. Then*

$$U_h(x, y, t) = u_h^k(x, y, t) + O_{L^2}(h^\infty).$$

Remark 3.31. Lemma 3.30 shows that u_h^n have almost disjoint essential supports in the variables (y, t). Therefore, Proposition 3.23 applies to show that U_h defined in (3.26) is also an approximate solution to (2.4) in the sense that for $t \in [0, T]$ we have

$$\|\Box_g U_h(\cdot, t)\|_{L^2(\Omega)} \leq O(h^{-1})\|U_h(\cdot, t)\|_{L^2(\Omega)}.$$

In the rest of this section we prove that U_h satisfies the Dirichlet boundary condition.

Proposition 3.32. *The approximate solution U_h to (2.4) defined in (3.26) satisfies*

$$U_h|_{\partial\Omega \times [0,T]} = O(h^\infty).$$

Proof. From Proposition 3.22 we deduce

$$U_h(0, y, t) = J(\text{Tr}_+(u_{F,h}^0))(y, t) + J(\text{Tr}_-(u_{F,h}^N))(y, t). \tag{3.41}$$

The first term in the right hand side of (3.41) is essentially supported for t in a small interval that does not meet $[0, T]$, hence its contribution is clearly trivial for $t \in [0, T]$. We now deal with the second term in (3.41): it will be enough to show that

$$T \notin \text{ess-supp}(J(\text{Tr}_-(u_{F,h}^N))). \tag{3.42}$$

We argue by contradiction and assume that (3.42) fails to hold. Then on the support of g_h^N we must have, for some $y_T \in [0, Y]$,

$$\left| \frac{(\partial_\tau \theta_0 + \xi \partial_\tau \zeta_0)(y_T, T, 1, -(1+a)^{1/2})}{2(1+a)^{1/2}a^{1/2}} - 2n \right| \leq c_0. \tag{3.43}$$

From Proposition 3.24 it follows that $\xi = -(-\zeta_0)^{1/2}(1, -(1+a)^{1/2}) = -a$ in (3.43). Replacing this in (3.43) and using $\partial_\tau \zeta_0 = 2(1+a)^{1/2} + O(a)$ we find that

$$\partial_\tau \theta_0(y_T, T, 1, -(1+a)^{1/2}) \in 2a^{1/2}(1+a)^{1/2}[2N+1-c_0, 2N+1+c_0]. \tag{3.44}$$

From Proposition 3.28 we must also have

$$\partial_\eta \theta_0(y_T, T, 1, -(1+a)^{1/2}) = (1+a)^{1/2}\partial_\tau \theta_0(y_T, T, 1, -(1+a)^{1/2})$$
$$- \frac{4}{3}Na^{3/2} - \frac{2}{3}(-\zeta_0)^{3/2}(1, -(1+a)^{1/2}). \tag{3.45}$$

From the choice of T in (3.37) we will get a contradiction. Precisely, using (3.37) and (3.36) yields

$$\partial_\tau \theta_0(y_N, T, 1, -(1+a)^{1/2}) = \bar{t}_N + O(a), \tag{3.46}$$

$$\partial_\eta \theta_0(y_N, T, 1, -(1+a)^{1/2}) = \bar{y}_N + O(a), \tag{3.47}$$

where we recall that $\bar{T}_N = 4Na^{1/2}(1+a)^{1/2}$ and $\bar{y}_N = (1+a)^{1/2}\bar{t}_N - \frac{4}{3}Na^{3/2}$. Notice also that for N defined by (3.33) the contribution of $Na^{3/2}$ is $O(a)$.

Using (3.44)–(3.47) now gives

$$|\partial_\eta \theta_0(y_T, T, 1, -(1+a)^{1/2}) - \partial_\eta \theta_0(y_N, T, 1, -(1+a)^{1/2})|$$
$$= (1+a)^{1/2}|\partial_\tau \theta_0(y_T, T, 1, -(1+a)^{1/2}) - \partial_\tau \theta_0(y_N, T, 1, -(1+a)^{1/2})| + O(a)$$
$$\in [2a^{1/2}(1+a)^{1/2}(1-c_0), 2a^{1/2}(1+a)^{1/2}(1+c_0)] \quad (3.48)$$

(where the inclusion in the last line follows from (3.44) and (3.46)). Evaluating the terms in the first and second lines of (3.48) yields

$$3(1-c_0) \leq 1+c_0, \quad (3.49)$$

hence $c_0 \geq 1/2$, which is a contradiction since in Section 3.2.1 we have chosen $0 < c_0 \leq 3/8$. Therefore (3.42) does hold and in the same way we can see that ess-supp$(J(\mathrm{Tr}_-(u_{F,h}^N)))$ does not meet the interval $[0, T]$. $\qquad\square$

3.4.3. Strichartz estimates for the approximate solution U_h

Proposition 3.33. *Let $r > 4$ and $\epsilon > 0$ be the one fixed in Section 3. Define*

$$\beta(r) = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{1}{6}\left(\frac{1}{4} - \frac{1}{r}\right)$$

and let $\beta \leq \beta(r) - \epsilon$. Then the approximate solution U_h of the wave equation (2.4) satisfies

$$h^\beta \|U_h\|_{L^q([0,T],L^r(\Omega))} \geq h^{-7\epsilon/8}\|U_h|_{t=0}\|_{L^2(\Omega)} \gg \|U_h|_{t=0}\|_{L^2(\Omega)}.$$

Remark 3.34. Notice that the condition $\beta < \beta(r)$ shows that U_h cannot satisfy the Strichartz inequalities of the free case, a loss of at least $\frac{1}{6}\left(\frac{1}{4} - \frac{1}{r}\right)$ derivatives being unavoidable.

Proof. The key point here is that the u_h^n have almost disjoint supports in time and in the tangential variable, hence we can bound from below the $L^q([0, T])$ norm by a sum of integrals over small intervals of time J_k on which there will be only one cusp u_h^k to consider, the contribution from all the others being trivial. The intervals J_k will be the ones defined in (3.39) for which Lemma 3.29 applies. We have

$$\|U_h\|_{L^q([0,T],L^r(\Omega))}^q = \int_0^T \|U_h(\cdot, t)\|_{L^r(\Omega)}^q \, dt = \int_0^T \left\|\sum_{n=0}^N u_h^n(\cdot, t)\right\|_{L^r(\Omega)}^q \, dt$$

$$\geq \sum_{k \leq N} \int_{J_k} \left\|\sum_{n=0}^N u_h^n(\cdot, t)\right\|_{L^r(\Omega)}^q \, dt + O(h^\infty)$$

$$\simeq \sum_{k \leq N} |J_k| \, \|u_h^0(\cdot, 0)\|_{L^r(\Omega)}^q + O(h^\infty)$$

$$\simeq \frac{c_0}{4} Y \|u_h^0(\cdot, 0)\|_{L^r(\Omega)}^q + O(h^\infty). \quad (3.50)$$

Indeed, we have shown in Lemma 3.30 that for t belonging to small intervals of time J_k, only u_h^k has to be considered in the sum since the contribution from each u_h^n with $n \neq k$ is $O_{L^2}(h^\infty)$. In the last line of (3.50) we used Lemma 3.29 to estimate $|J_k|$ from below, uniformly in k, by $c_0 a^{1/2}$, where $c_0 \in (0, 3/8]$ is fixed, and the fact that $N \simeq \frac{Y}{4} a^{-1/2}$.

For $t \in J_k$, it follows from (3.40) that $x \geq a/2$, therefore on the essential support of $u_h^k(\cdot, t)$ the normal variable does not approach the boundary. Therefore the restrictions of u_h^k to J_k have disjoint supports.

Using similar arguments to [10, Prop. 6.7] we obtain the following result concerning the L^r norms of the phase integrals associated to a cusp type Lagrangian:

Proposition 3.35. *Let N be defined by* (3.33). *For $t \in J_n$ defined in* (3.39) *the $L^r(\Omega)$ norms of a cusp $u_h^n(\cdot, t)$ of the form* (3.25) *satisfy, uniformly for $n \in \{0, \dots, N\}$,*

(i) *for $2 \leq r < 4$,*

$$\|u_h^n(\cdot, t)\|_{L^r(\Omega)} \simeq h^{\frac{1}{r}+\frac{1}{2}} a^{\frac{1}{r}-\frac{1}{4}},$$

$$\|u_h^n(\cdot, 0)\|_{L^2(\Omega)} \simeq h a^{1/4};$$

(ii) *for $r > 4$,*

$$\|u_h^n(\cdot, t)\|_{L^r(\Omega)} \simeq h^{\frac{1}{3}+\frac{5}{3r}}.$$

Using Proposition 3.35 and (3.34) we deduce that there are constants $C = C(Y)$ independent of h such that for $r = 2$,

$$\|U_h|_{t=0}\|_{L^2(\Omega)} \simeq h \|\partial_t U_h|_{t=0}\|_{L^2(\Omega)} \simeq \|u_h^0(\cdot, 0)\|_{L^2(\Omega)} Y^{1/8} h^{1+(1-\epsilon)/8}.$$

For $r > 4$ we get, using (3.50),

$$\|U_h\|_{L^q([0,T],L^r(\Omega))} \geq C(Y) h^{\frac{1}{3}+\frac{5}{3r}}, \quad \text{where} \quad C(Y) = \left(\frac{c_0}{4} Y\right)^{1/q}.$$

We deduce that for $\beta \leq \beta(r) - \epsilon$,

$$h^\beta \|U_h\|_{L^q([0,T],L^r(\Omega))} \geq C(Y) h^{\beta(r)-\epsilon} h^{\frac{1}{3}+\frac{5}{3r}} \gg \|U_h|_{t=0}\|_{L^2(\Omega)} + h \|\partial_t U_h|_{t=0}\|_{L^2},$$

where we recall that Y was fixed, depending on b and, hence, on Ω only. □

3.4.4. End of proof of Theorem 2.2. We can now finish the proof of Theorem 2.2. Let $\epsilon > 0$ be as in Section 3 and let N be given by (3.27). Consider the L^2 normalized approximate solution to (2.4),

$$v_{h,\epsilon}^n(x, y, t) := \frac{1}{\|U_h(\cdot, 0)\|_{L^2(\Omega)}} u_h^n(x, y, t),$$

and set

$$\tilde{V}_{h,\epsilon}(x, y, t) := \sum_{n=0}^{N} v_h^n(x, y, t) = \frac{1}{\|U_h(\cdot, 0)\|_{L^2(\Omega)}} U_h(x, y, t).$$

We claim that $\tilde{V}_{h,\epsilon}$ and v_h^n satisfy the conditions of Theorem 2.2. Indeed, it follows from Proposition 3.35 that for $4 < r < \infty$, $v_{h,\epsilon}^n$ satisfy

$$\begin{cases} \|v_{h,\epsilon}^n(\cdot, t)\|_{L^r(\Omega)} \geq C h^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})-\frac{1}{6}(\frac{1}{4}-\frac{1}{r})+2\epsilon} & \text{for } t \in J_n, \\ \sup_{\epsilon>0} \|v_{h,\epsilon}^n(\cdot, t)\|_{L^2(\Omega)} \leq 1, \end{cases}$$

where in order to bound the L^2 norms uniformly we use the fact that for $t \in J_n$,

$$\|u_h^n(\cdot, t)\|_{L^2(\Omega)} \simeq \|u_h^0(\cdot, t)\|_{L^2(\Omega)} \simeq \|u_h^0(\cdot, 0)\|_{L^2(\Omega)} = \|U_h(\cdot, 0)\|_{L^2(\Omega)}.$$

From Proposition 3.28, the cusps $v_{h,\epsilon}^n$ have almost disjoint essential supports in the time and tangential variables and for the normal variable in an interval of size $a \simeq \frac{1}{2} Y^{1/2} h^{(1-\epsilon)/2}$. Moreover, the approximate solution $\tilde{V}_{h,\epsilon}$ is localized at spatial frequency $1/h$ and satisfies

$$\|\tilde{V}_{h,\epsilon}\|_{L^2(\Omega)} \lesssim 1, \qquad \|\partial_y \tilde{V}_{h,\epsilon}\|_{L^2(\Omega)} \lesssim \frac{1}{h}, \qquad \|\partial_y^2 \tilde{V}_{h,\epsilon}\|_{L^2(\Omega)} \lesssim \frac{1}{h^2},$$

with constants independent of ϵ, which follows from the spectral localization together with the uniform bounds of the derivatives of g_h^n with respect to y. From Proposition 3.23 and the almost orthogonality property of the supports in y we also obtain

$$\Box_g \tilde{V}_{h,\epsilon} = O_{L^2(\Omega)}(1/h).$$

Proposition 3.32 ensures that the Dirichlet condition holds on $\Omega \times [0, T]$.

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