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Skew-symmetric cluster algebras of finite mutation type

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Abstract. In the famous paper [FZ2] Fomin and Zelevinsky obtained a Cartan–Killing type classification of all cluster algebras of finite type, i.e. cluster algebras having only finitely many distinct cluster variables. A wider class of cluster algebras is formed by cluster algebras of finite mutation type, which have finitely many exchange matrices but are allowed to have infinitely many cluster variables. In this paper we classify all cluster algebras of finite mutation type with skew-symmetric exchange matrices. Besides cluster algebras of rank 2 and cluster algebras associated with triangulations of surfaces there are exactly 11 exceptional skew-symmetric cluster algebras of finite mutation type. More precisely, nine of them are associated with root systems E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$; the remaining two were found by Derksen and Owen in [DO]. We also describe a criterion which determines if a skew-symmetric cluster algebra is of finite mutation type, and discuss the growth rate of cluster algebras.

Contents

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in a series of papers [FZ1], [FZ2], [BFZ], [FZ3].

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We think of a cluster algebra as a subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ determined by generators ("cluster coordinates"). These generators are collected into n-element groups called *clusters* connected by local transition rules which are determined by an $n \times n$ skew-symmetrizable *exchange matrix* associated with each cluster. For precise definitions see Section 2.

In [FZ2], Fomin and Zelevinsky discovered a deep connection between cluster algebras of *finite type* (containing finitely many clusters) and the Cartan–Killing classification of simple Lie algebras. More precisely, they proved that there is a bijection between Cartan matrices of finite type and cluster algebras of finite type. The corresponding Cartan matrices can be obtained from exchange matrices by some symmetrization procedure.

Exchange matrices undergo *mutations* which are explicitly described locally. The collection of all exchange matrices of a cluster algebra is a *mutation class* of exchange matrices. In particular, the mutation class of a cluster algebra of finite type is finite. In this paper, we are interested in a larger class of cluster algebras, namely, cluster algebras whose exchange matrices form a finite mutation class. We will assume that exchange matrices are skew-symmetric.

Besides cluster algebras of finite type, there exist other series of algebras belonging to the class under consideration. One series of examples is provided by cluster algebras corresponding to Cartan matrices of affine Kac–Moody algebras with simply-laced Dynkin diagrams. It was shown in [BR] that these examples exhaust all cases of acyclic skew-symmetric cluster algebras of finite mutation type. Furthermore, Seven [S2] has shown that acyclic skew-symmetrizable cluster algebras of finite mutation type correspond to affine Kac–Moody algebras.

One more large class of infinite type cluster algebras of finite mutation type was studied in [FST], where, in particular, it was shown that the signed adjacency matrices of arcs of a triangulation of a bordered two-dimensional surface have finite mutation class.

In the same paper, Fomin, Shapiro and Thurston discussed the conjecture [FST, Problem 12.10] that besides the adjacency matrices of triangulations of bordered two-dimensional surfaces and matrices mutation-equivalent to one of the following nine types: E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ (see [FST, Section 12]), there exist finitely many skew-symmetric matrices of size at least 3×3 with finite mutation class. Notice that the first three types in the list correspond to cluster algebras of finite type.

In [DO] Derksen and Owen found two more skew-symmetric matrices (denoted by X_6 and X_7) with finite mutation class that are not included in the previous conjecture. The authors also ask if their list of 11 mutation classes contains all the finite mutation classes of skew-symmetric matrices of size at least 3×3 not corresponding to triangulations.

The main goal of this paper is to prove the conjecture by Fomin, Shapiro and Thurston by showing the completeness of the Derksen–Owen list, i.e. to prove the following:

Main Theorem (Theorem 6.1). *Any skew-symmetric $n \times n$ matrix, $n \geq 3$, with finite mutation class is either the adjacency matrix of a triangulation of a bordered two-dimensional surface or a matrix mutation-equivalent to a matrix of one of the following eleven types: E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$, X_6, X_7.*

Remark 1.1. The same approach that we use for skew-symmetric matrices is applicable (after small changes) for the more general case of skew-symmetrizable matrices. A complete list of skew-symmetrizable matrices with finite mutation class will be published elsewhere.

We also show a way to classify all minimal skew-symmetric $n \times n$ matrices with infinite mutation class. In particular, we prove that $n \leq 10$. This gives rise to the following criterion for a large skew-symmetric matrix to have finite mutation class:

Theorem 7.4. *A skew-symmetric $n \times n$ matrix B, $n \geq 10$, has finite mutation class if and only if the mutation class of every principal 10×10 submatrix of B is finite.*

As an application of the classification of skew-symmetric matrices of finite mutation type we characterize skew-symmetric cluster algebras of polynomial growth, i.e. cluster algebras for which the number of distinct clusters obtained from the initial one by n mutations grows polynomially in n.

The paper is organized as follows. In Section 2, we provide necessary background in cluster algebras, and reformulate the classification problem of skew-symmetric matrices in terms of *quivers* by assigning to every exchange matrix an oriented weighted graph.

In Section 3, we sketch the proof of the Main Theorem. We list all the key steps, and discuss the main combinatorial and computational ideas we use. Sections 4–6 contain the detailed proofs.

Section 4 is devoted to the technique of block-decomposable quivers. We recall the basic facts from [FST] and prove several properties we will heavily use. Section 5 contains the proof of the key theorem classifying minimal non-decomposable quivers. Section 6 completes the proof of the Main Theorem.

In Section 7 we provide a criterion for a skew-symmetric matrix to have finite mutation class. Section 8 is devoted to growth rates of cluster algebras.

Finally, in Section 9 we use the results of the previous sections to complete the description of mutation classes of quivers of order 3.

2. Cluster algebras, mutations, and quivers

We briefly recall the definition of a coefficient-free cluster algebra.

An integer $n \times n$ matrix B is called *skew-symmetrizable* if there exists an integer diagonal $n \times n$ matrix $D = \mathrm{diag}(d_1, \dots, d_n)$ such that DB is a skew-symmetric matrix, i.e., $d_i b_{i,j} = -b_{j,i} d_j$.

A seed is a pair (f, B), where $f = \{f_1, \dots, f_n\}$ isa collection of algebraically independent rational functions of n variables $x_1, \dots, x_n$, and B is a skew-symmetrizable matrix. The part f of (f, B) is called the *cluster*, the elements f_i are called the *cluster variables*, and B is called the *exchange matrix*.

Definition 2.1. For any k, $1 \le k \le n$, we define the *mutation* of the seed (f, B) in direction k as a new seed (f', B') in the following way:

$$B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k, \\ B_{ij} + \dfrac{|B_{ik}|B_{kj} + B_{ik}|B_{kj}|}{2} & \text{otherwise,} \end{cases} \tag{2.1}$$

$$f'_i = \begin{cases} f_i & \text{if } i \ne k, \\ \dfrac{\prod_{B_{ij}>0} f_j^{B_{ij}} + \prod_{B_{ij}<0} f_j^{-B_{ij}}}{f_i} & \text{otherwise.} \end{cases} \tag{2.2}$$

We write $(f', B') = \mu_k((f, B))$. Notice that $\mu_k(\mu_k((f, B))) = (f, B)$. We say that two seeds are *mutation-equivalent* if one is obtained from the other by a sequence of mutations. Similarly we define when two clusters or two exchange matrices are mutation-equivalent.

Notice that the exchange matrix (2.1) of the mutated seed depends only on the exchange matrix of the original seed. The collection of all matrices mutation-equivalent to a given matrix B is called the *mutation class* of B.

For any skew-symmetrizable matrix B we define the *initial seed* (x, B) as $(\{x_1, \dots, x_n\}, B)$, B is the *initial exchange matrix*, $x = \{x_1, \dots, x_n\}$ is the *initial cluster*.

The *cluster algebra* $\mathfrak{A}(B)$ associated with the skew-symmetrizable $n \times n$ matrix B is the subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by all cluster variables of the seeds mutation-equivalent to the initial seed (x, B).

The cluster algebra $\mathfrak{A}(B)$ is called *of finite type* if it contains only finitely many cluster variables. In other words, all clusters mutation-equivalent to the initial cluster contain in total only finitely many distinct cluster variables.

In [FZ2], Fomin and Zelevinsky proved a remarkable theorem that cluster algebras of finite type can be completely classified. More excitingly, this classification is parallel to the famous Cartan–Killing classification of simple Lie algebras.

Let B be an integer $n \times n$ matrix. Its *Cartan companion* $C(B)$ is the integer $n \times n$ matrix defined as follows:

$$C(B)_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -|B_{ij}| & \text{otherwise.} \end{cases}$$

Theorem 2.2 ([FZ2]). *There is a canonical bijection between the Cartan matrices of finite type and cluster algebras of finite type. Under this bijection, a Cartan matrix A of finite type corresponds to the cluster algebra $\mathfrak{A}(B)$, where B is an arbitrary skew-symmetrizable matrix with $C(B) = A$.*

The results by Fomin and Zelevinsky were further developed in [S1] and [BGZ], where effective criteria for cluster algebras to be of finite type were given.

A cluster algebra of finite type has only finitely many distinct seeds. Therefore, any cluster algebra that has only finitely many cluster variables involves only finitely many distinct exchange matrices. On the contrary, a cluster algebra with finitely many exchange matrices is not necessarily of finite type.

Definition 2.3. A cluster algebra with only finitely many exchange matrices is called *of finite mutation type*.

Example 2.4. One example of an infinite cluster algebra of finite mutation type is the Markov cluster algebra whose exchange matrix is

$$\begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

It was described in detail in [FZ1]. It is not of finite type, nor even finitely generated. Notice, however, that mutation in any direction leads simply to a sign change of the exchange matrix. Therefore, the Markov cluster algebra is clearly of finite mutation type.

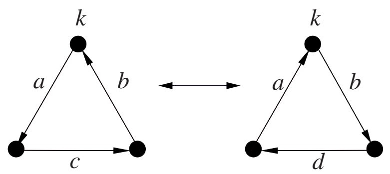
Remark 2.5. Since the orbit of an exchange matrix depends on the exchange matrix only, we may speak about skew-symmetrizable matrices of finite mutation type.

Therefore, the Main Theorem describes *all skew-symmetric integer matrices whose mutation class is finite*.

For our purposes it is convenient to encode an $n \times n$ skew-symmetric integer matrix B by a finite oriented multigraph without loops and 2-cycles called a quiver. More precisely, a *quiver* S is a finite 1-dimensional simplicial complex with oriented weighted edges, where weights are positive integers.

The vertices of S are labeled by $[1, \dots, n]$. If $B_{i,j} > 0$, we join vertices i and j by an edge directed from i to j and assign to this edge weight $B_{i,j}$. Vice versa, any quiver with integer positive weights corresponds to a skew-symmetric integer matrix. While drawing quivers, usually we draw edges of weight $B_{i,j}$ as edges of multiplicity $B_{i,j}$, but sometimes, when it is more convenient, we put the weight on a simple edge.

Mutations of exchange matrices induce *mutations of quivers*. If S is the quiver corresponding to a matrix B, and B' is the mutation of B in direction k, then we call the quiver S' associated to B' the *mutation of S at vertex k*. It is easy to see that mutation at vertex k changes weights of quivers in the way described in the following picture (see e.g. [K2]):



$c + d = ab$

Fig. 2.1. Mutations of quivers.

Clearly, for a given quiver the notion of *mutation class* is well-defined. We call a quiver *mutation-finite* if its mutation class is finite. Thus, we are able to reformulate the problem of classification of exchange matrices of finite type in terms of quivers: *find all mutation-finite quivers*.

The following criterion for a quiver to be mutation-finite is well-known (see e.g. [DO, Corollary 8]):

Theorem 2.6. *A quiver S of order at least 3 is mutation-finite if and only if any quiver in the mutation class of S contains no edges of weight greater than 2.*

One can use linear algebra tools to describe quiver mutations. Let $e_1, \dots, e_n$ be a basis of a vector space V over a field k equipped with a skew-symmetric form Ω. Denote by B the matrix of the form Ω with respect to the basis e_i, i.e. $B_{ij} = \Omega(e_i, e_j)$.

For each $i \in [1, n]$ we define a new basis $e'_1, \dots, e'_n$ in the following way:

$$\begin{aligned} e'_i &= -e_i, \\ e'_j &= e_j \quad \text{ if } \Omega(e_i, e_j) \geq 0, \\ e'_j &= e_j - \Omega(e_i, e_j) e_i \quad \text{ if } \Omega(e_i, e_j) < 0. \end{aligned}$$

Note that the matrix B' of the form Ω in the basis e'_k is the mutation of B in direction i.

From now on, we use the language of quivers only. Let us fix some notation we will use throughout the paper.

Let S be a quiver. A *subquiver* $S_1 \subset S$ is a subcomplex of S. The *order* $|S|$ is the number of vertices of S. If S_1 and S_2 are subquivers of S, we denote by $\langle S_1, S_2 \rangle$ the subquiver of S spanned by all the vertices of S_1 and S_2.

Let S_1 and S_2 be subquivers of S having no common vertices. We say that S_1 and S_2 are *orthogonal* ($S_1 \perp S_2$) if no edge joins vertices of S_1 and S_2.

We denote by $\mathrm{Val}_S(v)$ the unsigned valence of v in S (a double edge adds 2 to the valence). A *leaf* of S is a vertex joined to exactly one vertex in S.

3. Ideas of the proof

In this section we present all key steps of the proof.

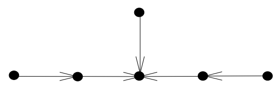
We need to prove that all mutation-finite quivers except some finite number of mutation classes have some special properties, namely they are block-decomposable (see Definition 4.1). In Section 5 we define a *minimal non-decomposable quiver* as a non-decomposable quiver minimal with respect to inclusion (see Definition 5.1). By definition, any non-decomposable quiver contains a minimal non-decomposable quiver as a subquiver. First, we prove the following theorem:

Theorem 5.2. *Any minimal non-decomposable quiver contains at most seven vertices.*

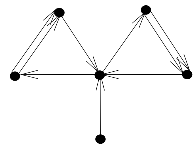
The proof of Theorem 5.2 contains the bulk of all the technical details in the paper. We assume that there exists a minimal non-decomposable quiver of order at least 8, and investigate the structure of block decompositions of proper subquivers of S. By an exhaustive case-by-case consideration we prove that S is also block-decomposable. The main tools are Propositions 4.6 and 4.8 which under some assumptions produce a block decomposition of S from block decompositions of proper subquivers of S.

The next step is to prove the following key theorem:

Theorem 5.11. *Any minimal non-decomposable mutation-finite quiver is mutation-equivalent to one of the two quivers X_6 and E_6 shown below.*



E_6



X_6

The proof is based on the fact that the number of mutation-finite quivers of order at most 7 is finite, and all such quivers can be easily classified. For that, we use an inductive procedure: we take one representative from each finite mutation class of quivers of order n and attach a vertex by edges of multiplicity at most 2 in all possible ways (here we use Theorem 2.6). For each quiver obtained we check if its mutation class is finite (by using Keller's applet for quiver mutations [K1]). In this way we get all the finite mutation classes of quivers of order $n + 1$. After collecting all finite mutation classes of order at most 7, we analyze whether they are block-decomposable. It turns out that all the classes except ones containing X_6 and E_6 are block-decomposable. The two quivers X_6 and E_6 are non-decomposable by [DO, Propositions 4 and 6].

Therefore, we prove that each mutation-finite non-decomposable quiver contains a subquiver mutation-equivalent to X_6 or E_6 (Corollary 5.13). This allows us to use the same inductive procedure to get all the finite classes of non-decomposable quivers. We attach a vertex to X_6 and E_6 by edges of multiplicity at most 2 in all possible ways. In this way we get all the finite mutation classes of non-decomposable quivers of order 7. More precisely, there are three of them, namely those containing X_7, E_7 and $\widetilde{E}_6$. Any mutation-finite non-decomposable quiver of order 8 should contain a subquiver mutation-equivalent to one of these three quivers due to the following lemma:

Lemma 6.4. *Let S_1 be a proper subquiver of S, and let S_0 be a quiver mutation-equivalent to S_1. Then there exists a quiver S' which is mutation-equivalent to S and contains S_0.*

Using the same procedure, we list one-by-one all the mutation-finite non-decomposable quivers of order 8, 9 and 10. The results are the entries from the list by Derksen–Owen (see Fig. 6.1). Applying the inductive procedure to a unique mutation-finite non-decomposable quiver $E_8^{(1,1)}$ of order 10, we obtain no mutation-finite quivers. Now we use the following statement:

Corollary 6.3. *Suppose that for some $d \geq 7$ there are no non-decomposable mutation-finite quivers of order d. Then the order of any non-decomposable mutation-finite quiver does not exceed $d - 1$.*

Corollary 6.3 implies that there is no non-decomposable mutation-finite quiver of order at least 11, which completes the proof of the Main Theorem.

4. Block decompositions of quivers

Let us start with the definition of block-decomposable quivers (we rephrase Definition 13.1 from [FST]).

Definition 4.1. A *block* is a quiver isomorphic to one of the quivers with black/white colored vertices shown in Fig. 4.1, or to a single vertex. Vertices marked white are called *outlets*. A connected quiver S is called *block-decomposable* if it can be obtained from a collection of blocks by identifying outlets of different blocks along some partial matching (matching of outlets of the same block is not allowed), where two edges with the same endpoints and opposite directions cancel out, and two edges with the same endpoints and same directions form an edge of weight 2. A non-connected quiver S is called block-decomposable if it either satisfies the definition above, or is a disjoint union of several mutually orthogonal quivers satisfying the definition above. If S is not block-decomposable then we call it *non-decomposable*. Depending on a block, we call it a *block of type* I, II, III, IV, V, or a *block of the n-th type*.

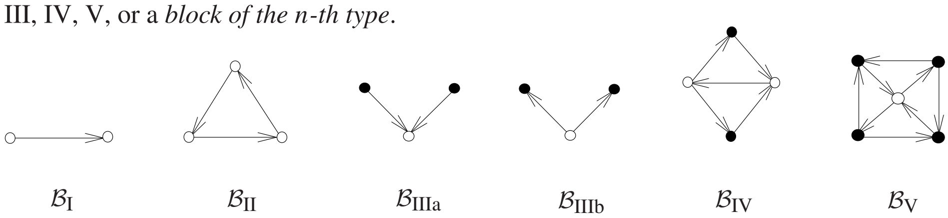


Fig. 4.1. Blocks. Outlets are colored white, dead ends are black.

We denote by $\mathcal{B}_{\mathrm{I}}$, $\mathcal{B}_{\mathrm{II}}$ etc. the isomorphism classes of blocks of types I, II, etc. respectively. For a block B we write $B \in \mathcal{B}_{\mathrm{I}}$ if B is of type I.

Remark 4.2. It is shown in [FST] that block-decomposable quivers have a nice geometrical interpretation: they are in one-to-one correspondence with the adjacency matrices of arcs of ideal (tagged) triangulations of bordered two-dimensional surfaces with marked points (see [FST, Section 13] for detailed explanations). Mutations of block-decomposable quivers correspond to flips of triangulations.

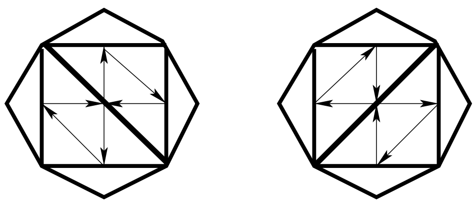


Fig. 4.2. Fat sides denote arcs of triangulations. Arrows form the corresponding quiver.

In particular, this description implies that the mutation class of any block-decomposable quiver is finite (indeed, the absolute value of an entry of the adjacency matrix cannot exceed 2). Another immediate corollary is that any subquiver of a block-decomposable quiver is block-decomposable too.

Remark 4.3. As the following example shows, a block decomposition of a quiver (if any) may not be unique.

Example 4.4. There are two ways to decompose an oriented triangle into blocks; see Fig. 4.3.

Fig. 4.3. Two different decompositions of an oriented triangle into blocks.

We say that a vertex of a block is a *dead end* if it is not an outlet.

Remark 4.5. Notice that if S is decomposed into blocks, and $u \in S$ is a dead end of some block B, then any edge of B incident to u cannot cancel with any other edge of another block and therefore must appear in S with weight 1.

We call a vertex $u \in S$ an *outlet of* S if S is block-decomposable and there exists a block decomposition of S such that u is contained in exactly one block B, and u is an outlet of B.

We use the following notation. For two vertices u_i, u_j of a quiver S we denote by (u_i, u_j) a directed arc connecting u_i and u_j which may or may not belong to S. It may be directed either way. By (u_i, u_j, u_k) we denote an oriented triangle with vertices u_i, u_j, u_k which is oriented either way and whose edges also may or may not belong to S. We use the standard notation $\langle u_i, u_j\rangle$ for an edge of S.

While drawing quivers, we keep the following notation:

- a non-oriented edge is used when orientation does not play any role in the proof;
- an edge $u \bullet \text{------} \bullet v$ is an edge of a block containing u and v, where u and v are not joined in the quiver. The figure assumes a fixed block decomposition;
- an edge $x \bullet \cdots\cdots \bullet a$ means that x is joined to a by some edge.

Proposition 4.6. *Let S be a connected quiver with n vertices, and let b be a vertex of S with the following properties:*

(0) *$S \setminus b$ is not connected;*
(1) *for any $u \in S$ the quiver $S \setminus u$ is block-decomposable;*
(2) *at least one connected component of $S \setminus b$ has at least 3 vertices;*
(3) *each connected component of $S \setminus b$ has at most $n-3$ vertices.*

Then S is block-decomposable.

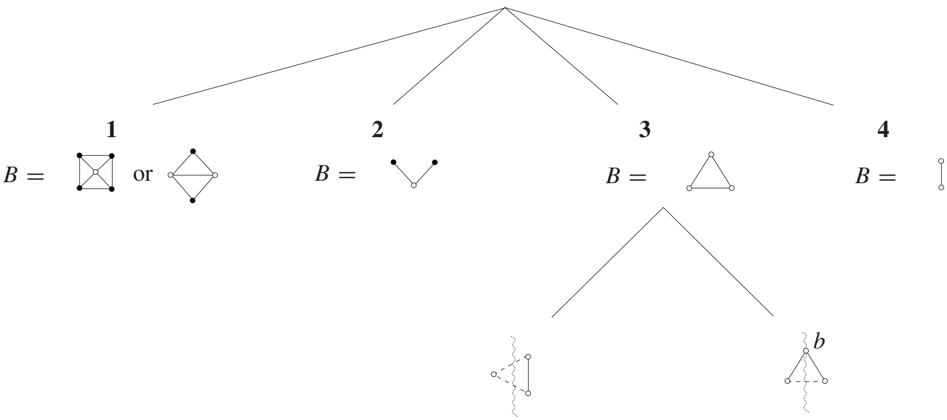
Proof. We divide $S \setminus b$ into two parts S_1 and S_2 in the following way: S_1 is any connected component of $S \setminus b$ with at least three vertices (it exists by assumption (2)), and $S_2 = S \setminus \langle b, S_1\rangle$. Notice that assumption (3) implies that $|S_2| \geq 2$.

Now choose vertices $a_1 \in S_1$ and $a_2 \in S_2$ satisfying the following conditions: $S \setminus a_i$ is connected, and $S \setminus a_i$ does not contain leaves connected to b and belonging to S_1. We always can take as a_2 a vertex of S_2 at maximal distance from b. To choose $a_1 \in S_1$, we look at the vertices at maximal distance from b in $\langle S_1, b\rangle$. If the maximal distance from b in $S_1 \cup b$ is greater than 2, then we may take as a_1 any vertex of S_1 at maximal distance from b: in this case $\langle S_1, b\rangle \setminus a_1$ does not contain leaves connected to b. If S_1 contains a leaf of S, then we can take as a_1 this leaf. This does not produce leaves of $\langle S_1, b\rangle \setminus a_1$ since S_1 is connected and $|S_1| \geq 3$. Finally, if the maximal distance from b in $S_1 \cup b$ is 2, and no vertex at distance 2 is a leaf, we take as a_1 any neighbor of b with minimal number of neighbors in S_1. Again, this does not produce leaves of $\langle S_1, b\rangle \setminus a_1$.

We will now prove that each $\langle S_i, b\rangle$ is block-decomposable with outlet b. Since b is the only common vertex of $\langle S_1, b\rangle$ and $\langle S_2, b\rangle$, this will imply that S is block-decomposable.

Consider the quiver $S \setminus a_2$. It is block-decomposable by assumption (1). Choose any such decomposition. Let us prove that for any block B either $B \cap S_1 = \emptyset$ or $B \cap (S_2 \setminus a_2) = \emptyset$. In particular, this will imply that S_1 is block-decomposable, and b is an outlet (since $|S_2| \geq 2$ and $S \setminus a_2$ is connected). Suppose that for some B both $B \cap S_1$ and $B \cap (S_2 \setminus a_2)$ are non-empty. We consider below all possible types of block B. Table 4.1 illustrates the plan of the proof.

Table 4.1. Proof of Proposition 4.6.



Case 1: $B \in \mathcal{B}_{\mathrm{IV}}$ or $B \in \mathcal{B}_{\mathrm{V}}$. At least one edge of B having a dead end runs from S_1 to $S_2 \setminus a_2$. By Remark 4.5 this edge appears in S, contradicting $S_1 \perp S_2$.

Case 2: $B \in \mathcal{B}_{\mathrm{III}}$. The unique outlet of B cannot coincide with b due to the way of choosing a_1 and a_2. Therefore, some edge of B joins vertices of S_1 and $S_2 \setminus a_2$, which is impossible since both edges of B have dead ends.

Case 3: $B \in \mathcal{B}_{\mathrm{II}}$. Suppose first that b is not a vertex of B. Then one vertex of B (say w) belongs to one part of $S \setminus a_2$ (i.e. either $S_2 \setminus a_2$ or S_1), and the remaining two (w_1 and w_2) to the other. Since S does not contain the edges (w, w_1) or (w, w_2), they must cancel out with edges from other blocks of the block decomposition. Since all these additional blocks contain w, the only way is to attach a block B_1 of the second type along all the three outlets of B. This results in three vertices w, w_1 and w_2 not joined to any other vertices of $S \setminus a_2$. In particular, $\langle S_1, b\rangle$ is not connected (since $|S_1| \geq 2$), which implies that S is not connected either.

Now suppose that b is a vertex of B. Then we may assume that the other vertices w_1 and w_2 of B belong to S_1 and $S_2 \setminus a_2$ respectively. S does not contain an edge (w_1, w_2). The only way to avoid it in S is to glue an edge (w_1, w_2) (which is a block B_1 of the first type) to B (since blocks of all other types are already prohibited by Cases 1, 2 and the above reasoning for this one). Then w_1 is a leaf of $\langle S_1, b\rangle$ attached to b, in contradiction with the way of choosing of a_2.

Case 4: $B \in \mathcal{B}_{\mathrm{I}}$. Let $B = (w_1, w_2)$, $w_1 \in S_1$ and $w_2 \in S_2 \setminus a_2$. The only way to avoid this edge in S is to glue another edge (w_1, w_2) (which is a block B_1 of the same type) to B (all other blocks are already prohibited by previous cases). Then $\langle S_1, b\rangle$ is not connected, which implies that S is not connected.

Having settled all the four cases, we deduce that S_1 is block-decomposable with outlet b. Considering $S \setminus a_1$ instead of $S \setminus a_2$, in a similar way we conclude that S_2 is also block-decomposable with outlet b. Gluing these decompositions together along b we obtain a block decomposition of S. □

Remark 4.7. In all the situations where we will apply Proposition 4.6, the connectedness of S and assumption (1) will be stated in advance. Usually it is sufficient only to point out the vertex b (in this case we say that S is block-decomposable by Proposition 4.6 applied to b) and all the assumptions are evidently satisfied. Only in the proofs of Lemma 5.7 and Theorem 5.2 does assumption (3) require additional explanations.

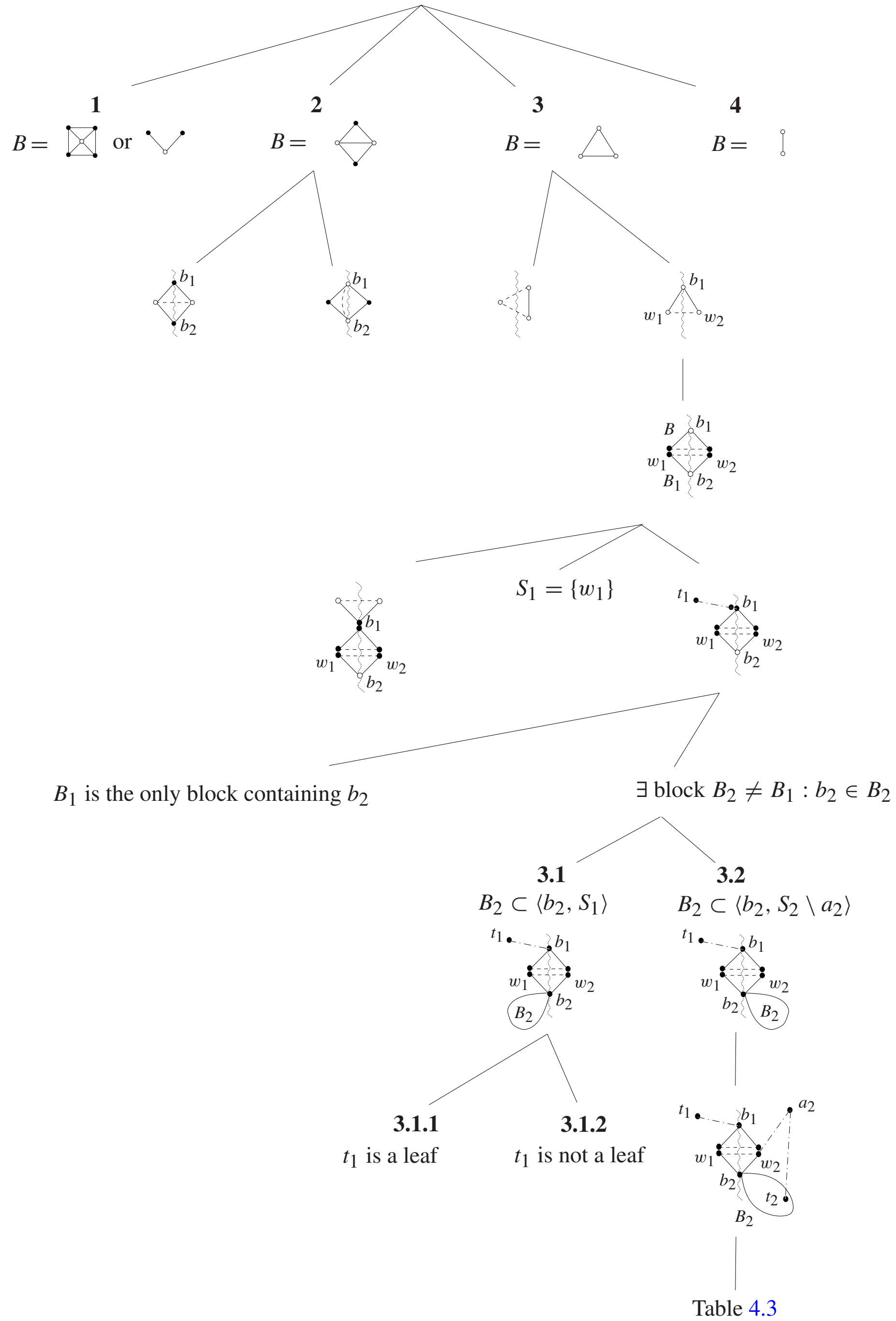
Proposition 4.8. *Let S be a connected quiver $S = \langle S_1, b_1, b_2, S_2\rangle$, where $S_1 \perp S_2$, and S has at least eight vertices. Suppose that*

(0) *b_1 and b_2 are not joined in S;*
(1) *for any $u \in S$ the quiver $S \setminus u$ is block-decomposable;*
(2) *there exist $a_1 \in S_1$, $a_2 \in S_2$ such that*
   (2a) *$S \setminus a_i$ is connected;*
   (2b) *either $\langle S_i, b_1, b_2\rangle \setminus a_i$ or $\langle S_j, b_1, b_2\rangle$ (for $i, j = 1, 2,\ j \neq i$) contains no leaves connected to b_1; similarly, either $\langle S_i, b_1, b_2\rangle \setminus a_i$ or $\langle S_j, b_1, b_2\rangle$ (for $j \neq i$) contains no leaves connected to b_2;*
   (2c) *if a_i is joined to b_j (for $i, j = 1, 2$), then there is another vertex $w_i \in S_i$ connected to b_j.*

Then S is block-decomposable.

Proof. The plan is similar to the proof of Proposition 4.6. The idea is to prove that each S_i together with those of b_1, b_2 which are connected to S_i is block-decomposable with outlets b_1 and b_2 (or just one of them if the second is not joined to S_i). Then we combine together these block decompositions to obtain a decomposition of S. First we show that for any block decomposition of $S \setminus a_2$ any block B is contained entirely either in $\langle S_1, b_1, b_2\rangle$ or in $\langle S_2, b_1, b_2\rangle \setminus a_2$. For this, we consider any block decomposition of $S \setminus a_2$,

Table 4.2. Proof of Proposition 4.8.



assuming that for a block B both $B \cap S_1$ and $B \cap (S_2 \setminus a_2)$ are non-empty. We consider all possible types of B (see Table 4.2) and obtain a contradiction for each type.

Notice that the assumption (2c) implies that $|S_i| \geq 2$. We will refer to this fact as assumption (3).

Case 1: $B \in \mathcal{B}_{\mathrm{V}}$ or $B \in \mathcal{B}_{\mathrm{III}}$. The proof is the same as in Proposition 4.6.

Case 2: $B \in \mathcal{B}_{\mathrm{IV}}$. Evidently, both $b_1, b_2 \in B$. We may assume that both b_1 and b_2 are either dead ends of B, or outlets of B (otherwise, by Remark 4.5, S contains a simple edge (b_1, b_2), contradicting assumption (0)). First suppose that b_1 and b_2 are dead ends of B. Then we may assume that the remaining vertices w_1, w_2 of B lie in S_1 and $S_2 \setminus a_2$ respectively. The edge (w_1, w_2) of B must be canceled out by an edge of another block B_1, otherwise (w_1, w_2) appears in S, contradicting $S_1 \perp S_2$. The block B_1 cannot be of type IV (otherwise none of its vertices is b_1, so the block directly connects S_1 and S_2). Therefore it is either of type II or I. If B_1 is of type II (i.e. a triangle with vertices w_1, w_2 and w, $w \neq b_i$), then the edges (w_1, w) and (w, w_2) are not canceled out by other blocks and appear in S. Therefore either (w_1, w) or (w, w_2) connects S_1 to S_2, which contradicts $S_1 \perp S_2$. If we glue an edge (w_1, w_2) as a block of the first type, then w_1 is not joined to any other vertex of S_1. Since b_1 and b_2 are dead ends of B, they are not joined to any other vertex of S_1 either. Thus, due to assumption (3) the subquiver $\langle S_1, b_1, b_2 \rangle$ is not connected, so S is not connected either.

Now suppose that b_1 and b_2 are outlets of B. Denote by $w_1 \in S_1$ and $w_2 \in S_2 \setminus a_2$ the other vertices of B. To avoid the edge (b_1, b_2) in S, some block should be glued along this edge. If we glue a block of the first or fourth type, then we obtain a quiver with respectively four and six vertices without outlets. Since S has at least eight vertices, the complement of this quiver in $S \setminus a_2$ is non-empty and $S \setminus a_2$ is not connected, contradicting the choice of a_2. If we glue a block of the second type (a triangle $b_1 b_2 w$), then we obtain a quiver with five vertices, and the only outlet is w. Therefore, none of the remaining vertices of $S \setminus a_2$ is joined to vertices of B. In particular, $w \in S_1$ (otherwise S_1 consists of w_1 only), so S_2 consists of w_2 and a_2. Hence, S is block-decomposable by Proposition 4.6 applied to w.

Case 3: $B \in \mathcal{B}_{\mathrm{II}}$. We may assume that the vertices of B are $b_1, w_1 \in S_1$ and $w_2 \in S_2 \setminus a_2$. Since the edge (w_1, w_2) is not in S it must be canceled by a block B_1. It is either of type I or II (since all other types are already excluded above). B_1 is not of the first type, otherwise w_1 and w_2 are leaves in S_1 and S_2, resp., connected to b_1, contradicting (2b). Therefore, B_1 is of type II, and the remaining vertex of B_1 is either b_1 or b_2. If it is b_1 then S is not connected. We conclude that $b_2 \in B_1$.

Any other block B'_1 with vertex b_1 is contained in either $\langle S_1, b_1 \rangle$ or $\langle b_1, S_2 \setminus a_2 \rangle$. (Otherwise, if B'_1 containing b_1 has non-empty intersection with both S_1 and $S_2 \setminus a_2$, then B'_1 is again of type II. As above there exists B'_2 containing b_2 completing B'_1 in such a way that $\langle B, B_1, B'_1, B'_2 \rangle$ is a six-vertex subquiver without outlets. Since $|S| \geq 8$, this implies that $S \setminus a_2$ is not connected.) Similarly, any block with vertex b_2 (other than B_1) is contained either in $\langle b_1, b_2, S_1 \rangle$ or in $\langle b_1, b_2, S_2 \setminus a_2 \rangle$. Further, no vertex of $S \setminus a_2$ except b_1, b_2 is joined to w_1 and w_2.

Therefore, $S \setminus a_2$ consists of $\langle b_1, b_2, w_1, w_2\rangle$, blocks connected to b_1 and b_2, and vertices not joined to $\langle b_1, b_2, w_1, w_2\rangle$. Combining that with assumption (3), we see that there is at least one vertex $t_1 \in S_1$ distinct from w_1 which is joined to at least one of b_1 and b_2 by a simple edge. We may assume that t_1 is connected to b_1. Since b_1 is contained in two blocks of $S \setminus a_2$, no vertex of S_2 except possibly a_2 is joined to b_1.

Suppose that no block except B_1 contains b_2. Then the subquiver $S \setminus \langle b_1, b_2, w_1, w_2, a_2\rangle$ is contained in S_1 and is joined to b_1 only. Applying Proposition 4.6 to b_1 we conclude that S is block-decomposable.

Let us consider the case when some block B_2 is connected to b_2. As we have already shown, B_2 is entirely contained either in $\langle S_1, b_1, b_2\rangle$ or in $\langle b_1, b_2, S_2 \setminus a_2\rangle$.

Case 3.1: B_2 is contained in $\langle S_1, b_2\rangle$. In this case S_2 consists of w_2 and a_2 only, so $|S_2| = 2$. Recall that w_1 is joined to b_1 and b_2 only, and define a new decomposition $S = \langle S_1', b_1, b_2, S_2'\rangle$, where $S_1' = S_1 \setminus w_1$ and $S_2' = \langle S_2, w_1\rangle$. We will show that this decomposition satisfies all the assumptions of Proposition 4.8. Since $|S_2| = 2$ and $|S| \geq 8$, this decomposition satisfies $|S_1'|, |S_2'| \geq 3$. Therefore we may avoid Case 3.1.

Clearly, assumptions (0) and (1) hold. We need to choose vertices a_1' and a_2' satisfying conditions (2a)–(2c). We keep $a_2' = a_2$. Recall that $t_1 \in S_1'$ is connected to b_1, and some vertex of B_2 (say t_2) is connected to b_2. So, if a_2 is not a leaf of S connected to exactly one of b_1 and b_2 we may choose as a_1' any vertex of S_1' different from both t_1 and t_2 and satisfying condition (2a); see Fig. 4.4.

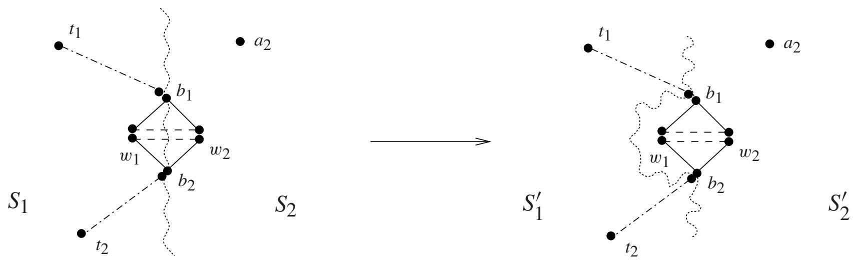


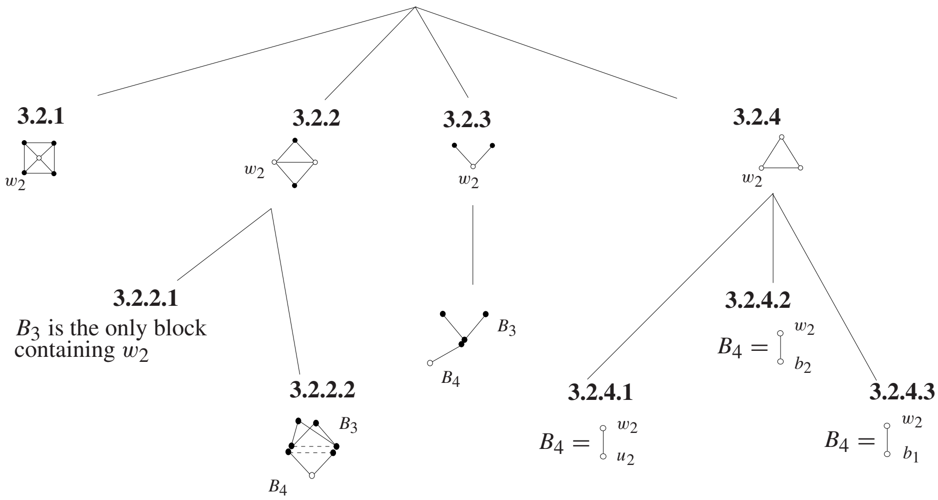
Fig. 4.4. Proof of Proposition 4.8, Case 3.1.

Suppose that a_2 is a leaf of S connected to exactly one of b_1 and b_2, say to b_1. Notice that there exists at most one leaf of S in S_1' connected to b_1, otherwise assumption (2b) does not hold for the initial decomposition of S. We may assume that if there is a leaf of S in S_1' connected to b_1, then it is t_1. Consider the following two cases.

Case 3.1.1: t_1 is a leaf of S. If there exists another vertex of S_1' connected to b_1, then $a_1' = t_1$ satisfies all the assumptions. So, t_1 is the only vertex of S_1' connected to b_1. Further, as we have seen above, $S \setminus \langle t_1, a_2\rangle$ is block-decomposable with outlet b_1, and the vertices t_1 and a_2 are joined to b_1 only. Therefore, depending on the orientation of the edges $\langle b_1, t_1\rangle$ and $\langle b_1, a_2\rangle$, we may glue either a block of the third type composed of b_1, t_1 and a_2, or a composition of a second type block with an extra edge (t_1, a_2) in appropriate direction, which implies that S is block-decomposable.

Case 3.1.2: t_1 is not a leaf of S. Denote by r_1 any vertex of S'_1 connected to t_1, and consider the quiver $S' = \langle r_1, t_1, b_1, w_1, w_2, a_2 \rangle$. As a proper subquiver of S, S' should be block-decomposable. However, using the Java applet [K1] by Keller one can easily check that the mutation class of S' is infinite, so S' is non-decomposable.

Table 4.3. Proof of Proposition 4.8, Case 3.2.



Case 3.2: B_2 is contained in $\langle b_2, S_2 \setminus a_2 \rangle$. If t_1 is not a leaf of S then applying Proposition 4.6 to b_1 we see that S is block-decomposable. Thus, we may assume that t_1 is a leaf of S. In this case S_1 consists of w_1 and t_1 only, so $|S_1| = 2$. If a_2 is joined to neither b_1 nor w_2, then S is block-decomposable by Proposition 4.6 applied to b_2. For the same reason, a_2 is joined to some vertex $t_2 \in S \setminus \langle t_1, b_1, w_1, w_2, b_2 \rangle$. If a_2 is not joined to w_2, then switching S_1 and S_2 leads us back to Case 3.1. Thus, we may assume that a_2 is connected to w_2 by a simple edge.

Now take any block decomposition of $S \setminus t_1$ and consider all possible types of blocks containing w_2 (see Table 4.3). Recall that $\mathrm{Val}_{S \setminus t_1}(w_2) = 3$.

Case 3.2.1: w_2 lies in a block B_3 of type V. In this case w_2 is a dead end of B_3 (due to its valence), one of b_1 and b_2 is a dead end of B_3, and the other is an outlet (since the orientations of $\langle w_2, b_1 \rangle$ and $\langle w_2, b_2 \rangle$ differ, see Fig. 4.5). Then the edge (b_1, b_2) in B_3 is

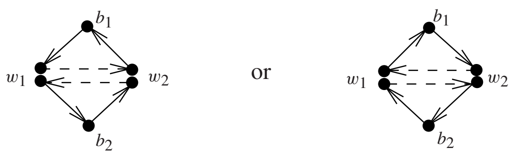


Fig. 4.5. Proof of Proposition 4.8, Case 3.2.1. The orientations of the edges $\langle w_2, b_1 \rangle$ and $\langle w_2, b_2 \rangle$ are different.

not canceled out by any other edge. Therefore, b_1 and b_2 are joined in S, contradicting assumption (0).

Case 3.2.2: w_2 is contained in a block B_3 of type IV. In this case w_2 is an outlet of B_3 (since $\mathrm{Val}_{S\setminus t_1}(w_2) = 3$). Consider two cases.

Case 3.2.2.1: w_2 is contained in the block B_3 only. Then one of b_1 and b_2 is a dead end of B_3, and the other is an outlet (since the orientations of $\langle w_2, b_1\rangle$ and $\langle w_2, b_2\rangle$ are different). Therefore, b_1 is joined to b_2, which contradicts assumption (0).

Case 3.2.2.2: w_2 is contained simultaneously in two blocks B_3 and B_4, $B_4 \neq B_3$. Since $\mathrm{Val}_{S\setminus t_1}(w_2) = 3$, B_4 is of the second type. Again, the orientations of $\langle w_2, b_1\rangle$ and $\langle w_2, b_2\rangle$ are different, so a_2 is a dead end of B_3. This implies that the valence of a_2 is 2, so only t_2 can be an outlet of B_3. The second dead end of B_3 should be joined to both t_2 and w_2. Since b_1 is not joined to t_2, b_2 is a dead end of B_3. But this contradicts the existence of an edge joining b_2 and w_1.

Case 3.2.3: w_2 is contained in a block B_3 of type III. Since $\mathrm{Val}_{S\setminus t_1}(w_2) = 3$, w_2 is the outlet of B_3. For the same reason at least one of b_1 and b_2 is a dead end of B_3, hence a leaf of $S \setminus t_1$. But neither b_1 nor b_2 is a leaf and such a B_3 does not exist.

Case 3.2.4: w_2 is contained in a block B_3 of type II. Recall that $\mathrm{Val}_{S\setminus t_1}(w_2) = 3$, so in this case w_2 is also contained in a block B_4 of the first type. There are exactly three ways to place w_2, b_1, b_2, a_2 in two blocks.

Case 3.2.4.1: $B_4 = (w_2, a_2)$. Then B_3 has an edge (b_1, b_2) which must cancel out in $S \setminus t_1$. Since b_1 is joined to w_1, the only way to cancel out (b_1, b_2) in $S \setminus t_1$ is to attach along this edge a second type block $B_5 = (b_1, b_2, w_1)$. Then $b_2 \in B_3 \cap B_5$, so it must be disjoint from B_2. Hence $B_2 = \emptyset$.

Case 3.2.4.2: $B_4 = (w_2, b_2)$. Then $B_3 = (w_2, b_1, a_2)$. Since w_1 is not joined to a_2 and $\mathrm{Val}_{S\setminus t_1}(b_1) \leq 3$, b_1 and w_1 form a block B_5 of the first type. Since $\mathrm{Val}_{S\setminus t_1}(w_1) = 2$, w_1 and b_2 also form a block B_6 of the first type. Again, this implies that $b_2 \in B_3 \cap B_6$, so $B_2 = \emptyset$.

Case 3.2.4.3: $B_4 = (w_2, b_1)$. Then $B_3 = (w_2, b_2, a_2)$. The proof is similar to the previous case. Since w_1 is not joined to a_2, and $\mathrm{Val}_{S\setminus t_1}(b_1) \leq 3$, b_1 and w_1 form a block B_5 of the first type. Since $\mathrm{Val}_{S\setminus t_1}(w_1) = 2$, w_1 and b_2 also form a block B_6 of the first type. This implies that $b_2 \in B_3 \cap B_6$, so again $B_2 = \emptyset$.

Case 4: $B \in \mathcal{B}_{\mathrm{I}}$. The proof is the same as in Proposition 4.6.

We call the connected component of $\langle S_i, b_1, b_2\rangle$ containing S_i the *closure* of S_i and denote it by $\widetilde{S}_i$. We proved above that any block in the decomposition of $S \setminus a_2$ is entirely contained in exactly one of $\widetilde{S}_1$ and $\widetilde{S}_2$. Consider the union of all the blocks with vertices from $\widetilde{S}_1$ only. They form a block decomposition of either $\widetilde{S}_1$ or $\widetilde{S}_1 \cup (b_1, b_2)$, i.e. $\widetilde{S}_1$ with edge (b_1, b_2). Due to assumption (2c), in both cases b_1 and b_2 are outlets. Similarly, considering a block decomposition of $S\setminus a_1$, we obtain a block decomposition of either $\widetilde{S}_2$, or $\widetilde{S}_2 \cup (b_1, b_2)$ where both b_1 and b_2 are outlets.

Suppose that in the way described above we got block decompositions of $\widetilde{S}_1$ and $\widetilde{S}_2$. Then we can glue these decompositions to obtain a block decomposition of S. Now we will prove that in all other cases S is also block-decomposable.

Now suppose that for one of $\widetilde{S}_1$ and $\widetilde{S}_2$ (say $\widetilde{S}_1$) we got a block decomposition of $\widetilde{S}_1$ with an edge (b_1, b_2). Consider the corresponding decomposition of $S \setminus a_2$. Clearly, there is a block B_1 in the decomposition of $\widetilde{S}_1$ with an edge (b_1, b_2) containing both b_1 and b_2 as outlets. Since b_1 and b_2 are not joined in S, there exists a block B_2 with vertices from $\widetilde{S}_2$ containing the edge (b_1, b_2), and again both b_1, b_2 are outlets. Notice that B_1 and B_2 are blocks of the second or fourth type (a block of the third or fifth type has one outlet only; if B_i is a block of the first type then no vertex of $S_i \setminus a_2$ can be connected to b_j, so $|S_i| \le 1$).

First, we prove that if S is not block-decomposable then $|S_1| = 2$. Indeed, no vertex of S_1 except the vertices of B_1 can be connected to b_1 or b_2. If B_1 is of the fourth type, then both its vertices belonging to S_1 are dead ends, so $|S_1| = 2$. If B_1 is of the second type with third vertex v_1, then S is block-decomposable by Proposition 4.6 applied to v_1 unless $|S_1| = 2$.

Finally, we look at the type of B_2. Again, no vertex of $S_2 \setminus a_2$ except vertices of B_2 can be connected to b_1 or b_2. If B_2 is of the fourth type, we see that $|S_2| \le 3$, so $|S| < 8$, which contradicts the assumptions of the proposition. Therefore, we may assume that B_2 is of the second type with third vertex v_2. In particular, v_2 is the only vertex of $S_2 \setminus a_2$ joined to b_1 and b_2. We will prove that S is block-decomposable.

If a_2 is joined to neither b_1 nor b_2, then S is block-decomposable by Proposition 4.6 applied to v_2. So, we may assume that a_2 is joined to one of b_1 and b_2, say b_1. If a_2 is joined to no vertex of $S_2 \setminus v_2$, then again S is block-decomposable by Proposition 4.6 applied to v_2. Hence, there is $t_2 \in S_2 \setminus v_2$ connected to a_2. Further, since $|S_1| = 2$ and $|S| \ge 8$, there exists a vertex $u_2 \in S_2 \setminus \langle v_2, a_2 \rangle$ joined to v_2 (see Fig. 4.6), otherwise S is block-decomposable by Proposition 4.6 applied to a_2.

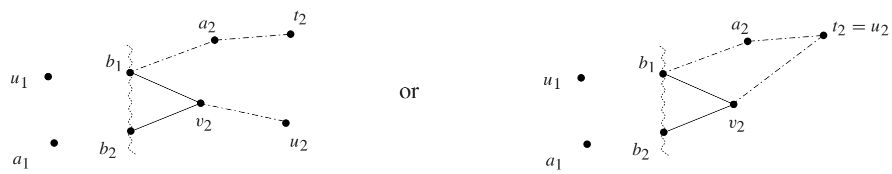


Fig. 4.6. Proof of Proposition 4.8.

Take $a_1 \in S_1$, and consider a block decomposition of $S \setminus a_1$. Denote by u_1 the remaining vertex of S_1, and consider all blocks containing b_1. Since $\mathrm{Val}_S(b_1) \le 4$ and no block contains vertices from S_1 and S_2 simultaneously, b_1 does not belong to a block of the fifth type. Since a_2 and v_2 are not leaves, b_1 does not belong to a block of the third type. Moreover, b_1 does not belong to a block of the fourth type: in this case a_2 and v_2 are dead ends, contradicting the existence of u_2.

A block (u_1, b_1, b_2) cannot exist, otherwise b_1 enters three blocks simultaneously. Therefore, (b_1, u_1) is a block of the first type while b_1, a_2, v_2 form a block of the second type (since $S \setminus a_1$ is connected). Notice that v_2 is the only vertex of $S_2 \setminus a_2$ joined to b_1

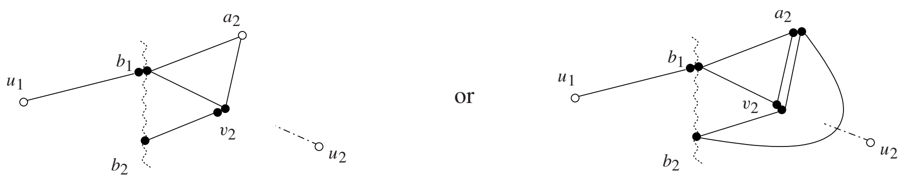


Fig. 4.7. Proof of Proposition 4.8; v_2 belongs to two blocks and cannot be joined to u_2.

and b_2, which implies that either (b_2, v_2) is a block of the first type, or (b_2, v_2, a_2) is a block of the second type (see Fig. 4.7). In both cases v_2 is contained in two blocks, so it cannot be connected to u_2. This contradiction completes the proof of the Proposition. □

Corollary 4.9. *Suppose that $S = \langle S_1, b_1, b_2, S_2\rangle$ satisfies all the assumptions of Proposition 4.8 except (2). Suppose also that $|S_1| \geq 2$, $|S_2| \geq 3$, and there exists $c_1 \in S_1$ such that the following holds:*

(a) *$S_1 \setminus c_1$ is connected;*
(b) *S_1 contains no leaves of S connected to b_1 or b_2, and $S_1 \setminus c_1$ contains no leaves of $S \setminus c_1$ connected to b_1 or b_2;*
(c) *$S_1 \setminus c_1$ is connected to both b_1 and b_2.*

Then S is block-decomposable.

Proof. We will show how to choose a_1 and a_2 to satisfy assumption (2) of Proposition 4.8.

As a_1 we can always take c_1. Clearly, assumptions (a)–(c) imply the corresponding assumptions (2a)–(2c) of Proposition 4.8 for a_1.

To choose a_2, we look how S_2 is connected to b_1 and b_2. If either S_2 is not connected to one of them (say b_2) or there is a vertex $v_2 \in S_2$ joined to both b_1 and b_2, then we take as a_2 any vertex of S_2 at maximal distance from b_1; otherwise, we fix two vertices $v_1, v_2 \in S_2$ joined to b_1 and b_2 respectively, and take as a_2 any vertex of $S_2 \setminus \langle v_1, v_2\rangle$ at maximal distance from b_1. □

5. Minimal non-decomposable quivers

Our aim is to prove that any non-decomposable quiver contains a subquiver of relatively small order which is also non-decomposable.

Definition 5.1. A *minimal non-decomposable quiver* is a quiver S such that

- S is non-decomposable;
- for any $u \in S$ the quiver $S \setminus u$ is block-decomposable.

Notice that a minimal non-decomposable quiver is connected. Indeed, if S is non-connected and non-decomposable, then at least one connected component of S is also non-decomposable.

Theorem 5.2. *Any minimal non-decomposable quiver contains at most seven vertices.*

The plan of the proof is the following. We assume that there exists a quiver S of order at least 8 satisfying the assumptions of Theorem 5.2, and show for each type of block that if a block-decomposable subquiver $S \setminus u$ contains a block of this type then S is also block-decomposable.

Throughout this section we assume that S satisfies the assumptions of Theorem 5.2. Here we emphasize that we do not assume the mutation class of S to be finite.

The *link* $L_S(v)$ of the vertex v in S is the subquiver of S spanned by all neighbors of v. If S is block-decomposable, we introduce for a given block decomposition a quiver $\Theta_S(v)$ obtained by gluing all blocks either containing v or having at least two points in common with $L_S(v)$. Notice that $\Theta_S(v)$ may not be a subquiver of S. Clearly, $L_S(v)$ is a subquiver of $\Theta_S(v)$ for any block decomposition of S.

Lemma 5.3. *For any $x \in S$ any block decomposition of $S \setminus x$ does not contain blocks of type* V.

To prove the lemma we use the following proposition.

Proposition 5.4. *Suppose that $S \setminus x$ contains a subquiver S_1 consisting of a block B of type* V *(with dead ends $v_1, \dots, v_4$ and outlet v) and a vertex t joined to v (and probably to some of v_i). Then for any $u \in S \setminus S_1$ and any block decomposition of $S \setminus u$ the subquiver $\langle v, v_1, \dots, v_4 \rangle$ is contained in one block of type* V. *In particular, t is connected to no v_i, $i = 1, \dots, 4$.*

Proof. Take any $u \in S \setminus S_1$ and consider any block decomposition of $S_2 = S \setminus u$. Since the valence of v in S_2 is at least 5, v is in exactly two blocks B_1 and B_2, at least one of which is of type V or IV. Suppose that none of them is of type V, and let B_1 be of type IV. Then for any choice of B_2 the number of vertices of S_2 which are neighbors of v and have valence at least three in S_2 does not exceed 3. However, there are at least four such vertices $v_1, \dots, v_4$, so the contradiction implies that we may assume B_1 to be of type V with outlet v.

Since the block B of $S \setminus x$ is of type V, the subquiver $\langle v_1, v_2, v_3, v_4 \rangle \subset S$ is a cycle. At the same time, the link $L_{S_2}(v)$ is a disjoint union of a cycle of order 4 (composed of dead ends of B_1) and another quiver with at most four vertices (composed of vertices of $B_2 \setminus v$). If we assume that v_1, v_2, v_3, v_4 are not contained in one block B_1 (or B_2) in S_2, then v_1, v_2, v_3, v_4 do not form a cycle, and we come to a contradiction. To complete the proof it is enough to notice that only a block of type V contains a chordless cycle of length 4. ◻

Proof of Lemma 5.3. Suppose that a block decomposition of $S \setminus x$ contains a block B of type V with dead ends $v_1, \dots, v_4$ and outlet v. Consider two cases: either $\mathrm{Val}_S(v) \geq 5$ or $\mathrm{Val}_S(v) = 4$.

Case 1: $\mathrm{Val}_S(v) \geq 5$. Then there exists $u \in S \setminus B$ that is joined to v. Denote $S_1 = \langle v, v_1, v_2, v_3, v_4, u \rangle$, and consider any block decomposition of $S_2 = S \setminus w_2$ for any $w_2 \notin S_1$. By Proposition 5.4, $\langle v, v_1, v_2, v_3, v_4 \rangle$ is a block B_1 of type V with outlet v. Therefore, no vertex of $\langle u, S_2 \setminus S_1 \rangle$ is joined to v_1, v_2, v_3, v_4. Since $|S| \geq 8$, we have $S \setminus \langle w_2, S_1 \rangle \neq \emptyset$.

Consider a block decomposition of $S_3 = S \setminus w_3$ for some $w_3 \in S \setminus \langle S_1, w_2 \rangle$. No vertex of $S_3 \setminus S_1$ is joined to v_1, v_2, v_3 or v_4. In particular, none of $w_2,\ w_3,\ u$ is joined to v_1, v_2, v_3 or v_4. Moreover, since w_2 and w_3 are arbitrary vertices of $S \setminus S_1$, this implies that no vertex of $\langle u, S \setminus S_1 \rangle$ is joined to v_1, v_2, v_3 or v_4. Thus, S is block-decomposable by Proposition 4.6 applied to v.

Case 2: $\mathrm{Val}_S(v) = 4$. Fix a block decomposition of $S \setminus x$ containing B. Since $\mathrm{Val}_S(v) = 4$ and $v_1, \dots, v_4$ are dead ends, no vertex of $S \setminus \{x \cup B\}$ is joined to vertices of B. Again, S is block-decomposable by Proposition 4.6 applied to x. □

Lemma 5.5. *For any $x \in S$ no block decomposition of $S \setminus x$ contains blocks of type* IV.

Proof. Consider any block decomposition of $S \setminus x$. Any vertex is contained in at most two blocks. By Lemma 5.3, no block decomposition of $S \setminus x$ contains blocks of type V. This implies that $\mathrm{Val}_{S \setminus x}(v) \le 6$ for any $v \in S \setminus x$. Thus, $\mathrm{Val}_S(v) \le 8$ (recall that any proper subquiver of S, and therefore S itself, does not contain edges of multiplicity greater than 2 due to Theorem 2.6).

Now let B be a block of type IV in some block decomposition of $S \setminus x$, denote by v_1 and v_2 the outlets of B, and assume that $\mathrm{Val}_{S \setminus x}(v_2) \le \mathrm{Val}_{S \setminus x}(v_1) \le 6$. If $\mathrm{Val}_{S \setminus x}(v_2) = \mathrm{Val}_{S \setminus x}(v_1)$, we assume that $\mathrm{Val}_S(v_2) \le \mathrm{Val}_S(v_1)$. We analyze the situation case by case with respect to the valence $\mathrm{Val}_{S \setminus x}(v_1)$ decreasing. Each case splits in two: either v_1 is joined to x or not (see Table 5.1).

Notice that x may be joined to v_1 by a simple edge only. Indeed, suppose that x is joined to v_1 by a double edge. Denote by w_1 and w_2 the dead ends of B. Since the subquiver $\langle x, v_1, w_1 \rangle$ is decomposable and its mutation class is finite, x is joined to w_1 and the triangle formed by x, v_1 and w_1 is oriented (this follows from the fact that the mutation class of a chain of a double edge and a simple edge is infinite independently of the orientations of the edges, and the mutation class of a non-oriented triangle containing a double edge is also infinite). For the same reason, v_2 is joined to both x and v_1, and the triangle fromed by x, v_1 and v_2 is oriented. Thus, the directions of the edges $\langle v_1, w_1 \rangle$ and $\langle v_1, v_2 \rangle$ induce opposite orientations of $\langle x, w_1 \rangle$ (see Fig. 5.1). The resulting contradiction shows that $\mathrm{Val}_S(v_1) \le \mathrm{Val}_{S \setminus x}(v_1) + 1$.

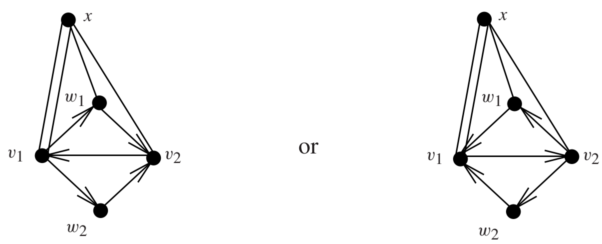
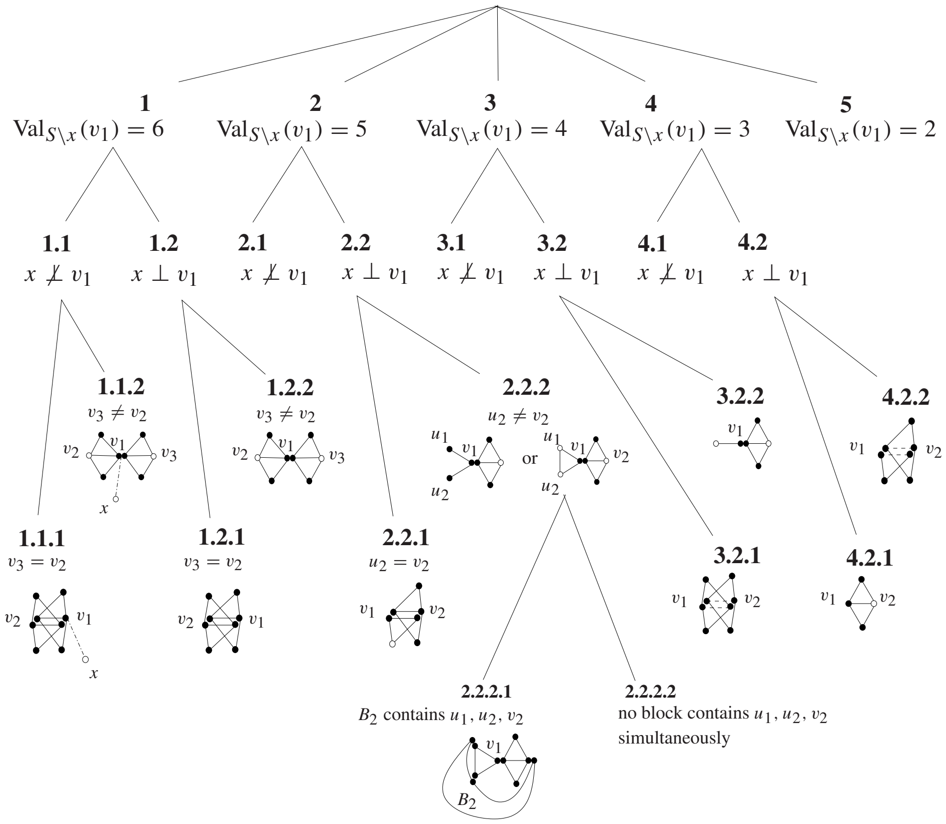


Fig. 5.1. Proof of Lemma 5.5. One of the triangles xv_1v_2 and xv_1w_1 is non-oriented.

Case 1: $\mathrm{Val}_{S \setminus x}(v_1) = 6$.

Case 1.1: $x \not\perp v$, hence $\mathrm{Val}_S(v_1) = 7$. Then v_1 is contained in a block B_1 of type IV, and v_1 is joined to x. Denote the dead ends of B_1 by w_3 and w_4, and the remaining outlet by v_3.

Table 5.1. Proof of Lemma 5.5.



Case 1.1.1: $v_3 = v_2$. Since $\mathrm{Val}_S(v_1) = 7$, v_1 and v_2 are joined by a double edge. Consider $S_1 = S \setminus w_1$ with some block decomposition. Clearly, $\mathrm{Val}_{S_1}(v_1) = 6$, and a subquiver $\langle v_1, v_2, x, w_2, w_3, w_4\rangle \subset S_1$ is obtained by gluing two blocks of the fourth type along (v_1, v_2). In particular, x is a dead end of one of these blocks, so x is joined to v_2 and is not joined to w_2, w_3, w_4. Similarly, considering $S_2 = S \setminus w_2$, we see that x is not joined to w_1 either.

Now consider the subquiver $S' = \langle v_1, x, w_1, w_2, w_3, w_4\rangle$. The only edges in S' are those joining v_1 to other vertices. Using Keller's applet [K1] we can check that the mutation class of S' is infinite. Recall that S' is a subquiver of a block-decomposable quiver, and hence is block-decomposable itself. Therefore, its mutation class must be finite, and this contradiction eliminates the case under consideration.

Case 1.1.2: v_3 does not coincide with v_2. Consider $S_1 = S \setminus v_3$. Since $\mathrm{Val}_{S_1}(v_1) = 6$, subquiver $\langle v_1, v_2, x, w_1, w_2, w_3, w_4\rangle \subset S_1$ is obtained by gluing two blocks of the fourth type at v_1. Since w_3 and w_4 are not joined, x is an outlet of the block $\langle v_1, x, w_3, w_4\rangle$. In particular, x is joined to w_3 and w_4 and is not joined to w_1 or w_2. Now, considering

$S_2 = S \setminus v_2$, we find in a similar way that x is joined to w_1 and w_2 and is not joined to w_3 or w_4, so we come to a contradiction.

Case 1.2: $x \perp v_1$, hence $\mathrm{Val}_S(v_1) = 6$. As in the previous case, v_1 is contained in a block B_1 of type IV. Denote the dead ends of B_1 by w_3 and w_4, and the remaining outlet by v_3.

Case 1.2.1: $v_3 = v_2$. Since $\mathrm{Val}_S(v_1) = 6$, v_1 and v_2 are joined by a double edge. The only outlets of B and B_1 are v_1 and v_2, and they are already contained in two blocks each. Therefore, the quiver spanned by B and B_1 has no outlets, so any other vertex of S except x is not joined to $S' = \langle v_1, v_2, w_1, w_2, w_3, w_4 \rangle$. Now take any vertex u distinct from x which is not contained in B or B_1, and consider any block decomposition of $S_1 = S \setminus u$. Since $\mathrm{Val}_{S_1}(v_1) = \mathrm{Val}_{S_1}(v_2) = 6$, the subquiver $S' \subset S_1$ is again obtained by gluing two blocks of the fourth type along the edge (v_1, v_2). By the reasons described above, no vertex of S except u is joined to vertices of S'. This implies that neither x nor u is connected to S', so no vertex of $S \setminus S'$ is connected to S', and S is not connected.

Case 1.2.2: $v_3 \neq v_2$. Recall that none of the vertices w_1, w_2, w_3, w_4 is joined to vertices of $S \setminus \{x \cup (B_1 \cap B_2)\}$. Let us prove that they are not joined to x either.

There are two options for the link $L_{S\setminus x}(v_1)$: either v_2 is joined to v_3 or not. Notice that if u and v are the dead ends of a block of type IV with outlet v_1 in some block decomposition of any quiver S', then u and v are leaves of the link $L_{S'}(v)$, and the distance between them equals two.

Consider the quiver $S_1 = S\setminus w_1$ with some block decomposition. Since $\mathrm{Val}_{S_1}(v_1) = 5$, a subquiver $\langle v_1, v_2, v_3, w_2, w_3, w_4 \rangle \subset S_1$ is obtained by gluing a block of the fourth type and a block of the second or third type at v_1. Looking at the link $L_{S_1}(v_1)$, we see that there is only one pair of leaves at distance two, namely w_3 and w_4. Therefore, v_1, v_3, w_3, w_4 are contained in one block of the fourth type. In particular, x is not joined to w_3 and w_4. Similarly, considering $S_2 = S \setminus w_3$, we find that x is not joined to w_1 and w_2.

Let us take another look at the block decomposition of S_1. Since w_2 is joined to v_2, the vertices v_1, v_2, w_2 are contained in a block of the second type, i.e. a triangle. Since none of w_1 and w_2 is joined to any vertex of S other than v_1 and v_2, we can replace the triangle $v_1 v_2 w_2$ by the block (v_1, v_2, w_1, w_2) of type IV to obtain a block decomposition of S.

Case 2: $\mathrm{Val}_{S\setminus x}(v_1) = 5$.

Case 2.1: $x \not\perp v_1$, $\mathrm{Val}_S(v_1) = 6$. Since $|S| \geq 8$, there exists $y \in S$ which is not joined to v_1. Since $\mathrm{Val}_{S\setminus y}(v_1) = 6$, in any block decomposition of $S \setminus y$ the vertex v_1 is an outlet of a block of type IV, so we may refer to Case 1.2.

Case 2.2: $x \perp v_1$, $\mathrm{Val}_S(v_1) = 5$. Then v_1 is contained in the block B and in a block B_1 of type II or III. The vertices w_1 and w_2 are dead ends of B, so they are joined in $S \setminus x$ to v_1 and v_2 only.

Denote by u_1 and u_2 the remaining vertices of B_1, and consider the quiver $\Theta_{S\setminus x}(v_1)$. Since the union of B and B_1 has at most three outlets, $\Theta_{S\setminus x}(v_1)$ consists of the blocks B, B_1 and probably some block B_2 containing at least two of the vertices v_2, u_1 and u_2. Consider the following three cases.

Case 2.2.1: $v_2 = u_2$. In this case v_1 and v_2 are joined by a double edge, B_1 is a block of the second type (which implies that u_1 is joined to v_2, so $\mathrm{Val}_S(v_2) \geq 5$), and the union of B and B_1 has a unique outlet u_1. Thus, $\langle L_S(v_1), v_1\rangle \setminus u_1$ may be joined to x only. Since $\mathrm{Val}_S(v_2) \leq \mathrm{Val}_S(v_1) = 5$, x is not joined to v_1 or v_2. If x is not joined to w_1 or w_2 either, then we can apply Proposition 4.6 to u_1. Therefore, we may assume that x is joined to at least one of w_1 and w_2, say w_1.

Now take any $y \notin \langle B, u_1, x\rangle$ and consider $S_1 = S \setminus y$ with some block decomposition. Recall that y cannot be joined to any vertex of B. Since $\mathrm{Val}_{S_1}(v_1) = 5$, v_1 is contained in some fourth type block of this decomposition together with v_2 and two of w_1, w_2, u_1. But w_1 is joined to x, so it cannot be a dead end of the block. Hence, v_1, v_2, w_2, u_1 form a block of type IV with outlets v_1, v_2 and dead ends w_2, u_1. In particular, neither w_2 nor u_1 is connected to x. If y is not joined to u_1, then we can apply Proposition 4.6 to w_1. Therefore, we may assume that y is joined to u_1.

By assumption, $|S| \geq 8$. Thus, we can take a vertex z which does not coincide with any of the preceding ones, and consider $S_2 = S \setminus z$ (see Fig. 5.2). As explained above, any block decomposition of the subquiver $\langle B, u_1\rangle$ is a union of two blocks with one outlet only. However, w_1 is joined to x, and u_1 is joined to y, so we come to a contradiction.

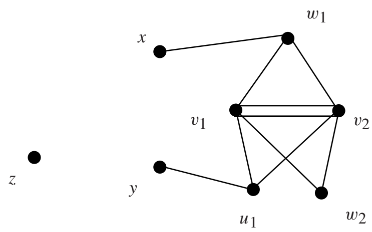


Fig. 5.2. Proof of Lemma 5.5, Case 2.2.1.

Case 2.2.2: Neither u_1 nor u_2 coincides with v_2. This case also splits into the following two.

Case 2.2.2.1: $\Theta_{S\setminus x}(v_1)$ consists of three blocks B, B_1, B_2, where B_2 is a triangle with vertices v_2, u_1 and u_2. In this case the quiver $\Theta_{S\setminus x}(v_1)$ has no outlets, so no vertex of $S \setminus \Theta_{S\setminus x}(v_1)$ except x is joined to $\Theta_{S\setminus x}(v_1)$.

Take any vertex $y \in S \setminus \Theta_{S\setminus x}(v_1)$ distinct from x, and consider the quiver $S_1 = S \setminus y$. The quiver $\Theta_{S\setminus y}(v_1)$ is spanned by $\Theta_{S\setminus x}(v_1)$ and probably x. Since $\mathrm{Val}_{S_1}(v_1) = 5$, three of the vertices w_1, w_2, v_2, u_1, u_2 should form a block B' of type IV together with v_1. Since none of w_1, w_2 is joined to any of u_1, u_2, that block contains v_2. The vertex v_1 is also contained in some block B'' of type II or III which contains the remaining two vertices of $\Theta_{S\setminus x}(v_1)$. Similarly, the same two vertices lie in some block B''' of type II or III containing v_2. In particular, all vertices of $L_S(v_1)$ are either dead ends of B', or are already contained in two blocks, so $x \notin \Theta_{S\setminus y}(v_1)$, and moreover x is not joined to $\Theta_{S\setminus y}(v_1)$, implying that S is not connected.

Case 2.2.2.2: No block contains v_2, u_1 and u_2 simultaneously. In this case $L_S(v_1)$ contains exactly two leaves at distance two from each other, namely w_1 and w_2 (see

Table 5.2. Proof of Lemma 5.5, Case 2.2.2.2.

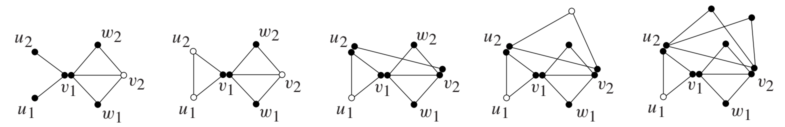
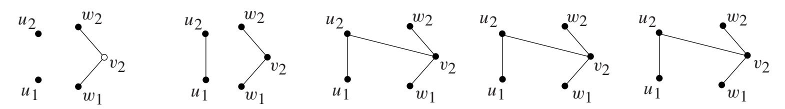
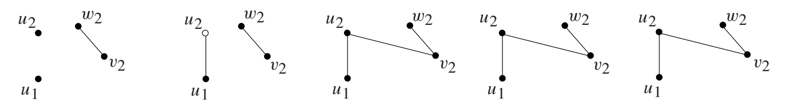
$\Theta_{S\setminus x}(v_1)$	
$L_{S\setminus x}(v_1)$	
$L_{S\setminus w_1}(v_1)$	

Table 5.2). This means that for any $y \notin L_S(v_1)$ distinct from v_1 and x and any block decomposition of $S_1 = S \setminus y$ the vertices v_1, v_2, w_1, w_2 form a block of the fourth type, which implies that x is not joined to w_1 or w_2.

Now consider $S_2 = S \setminus w_1$ with a block decomposition. Clearly, $\mathrm{Val}_{S_2}(v_1) = 4$. Looking at possible quivers $\Theta_{S\setminus x}(v_1)$ (see Table 5.2), one can notice that $L_{S_2}(v_1)$ does not contain a pair of leaves at distance two, so v_1 is contained in two blocks of the second or third type. Since w_2 is joined to v_1 and v_2 only, it is easy to see that w_2, v_1, v_2 form a block of type II. Now recall that neither w_1 nor w_2 is joined to any vertex of S except v_1 and v_2, so we can replace the triangle v_1, v_2, w_2 by the block v_1, v_2, w_1, w_2 of type IV to obtain a block decomposition of S.

Notice that we do not show in Table 5.2 the quivers $\Theta_{S\setminus x}(v_1)$ in which u_1 and u_2 are contained in two blocks simultaneously. The case of a triangle with vertices u_1, u_2 and v_2 is treated in Case 2.2.2.1, and it is easy to see that no others produce new leaves.

Case 3: $\mathrm{Val}_{S\setminus x}(v_1) = 4$.

Case 3.1: $x \not\perp v_1$, $\mathrm{Val}_S(v_1) = 5$. The proof is the same as in Case 2.1. Namely, since $|S| \geq 8$, there exists $y \in S$ which is not joined to v_1. Since $\mathrm{Val}_{S\setminus y}(v_1) = 5$, in any block decomposition of $S \setminus y$ the vertex v_1 is an outlet of a block of type IV, so we may refer to Case 2.2.

Case 3.2: $x \perp v_1$, $\mathrm{Val}_S(v_1) = 4$. In this case v_1 is contained in the block B and in a block B_1 of type I or IV. Consider these two cases separately.

Case 3.2.1: B_1 is of type IV. This case is similar to Case 1.2.1, the only difference is in the orientation of B_1: instead of getting a double edge, the edge (v_1, v_2) cancels out. The vertex v_2 is also contained in B_1; denote by w_3 and w_4 the dead ends of B_1. The only vertex joined to B and B_1 is x. We want to show that x is not joined to $L_S(v_1)$, which will imply that S is not connected.

Take any vertex $y \notin \langle L_S(v_1), v_1, x\rangle$ and consider $S_1 = S \setminus y$ with some block decomposition. If v_1 is contained in a block of the fourth type, then the second block containing v_1 is also of the fourth type (otherwise the remaining block is an edge, and the link $L_{S_1}(v_1) = L_S(v_1)$ contains at most three edges, contradicting the fact that $L_S(v_1)$ contains four edges), and we see that x is not joined to $L_S(v_1)$. Therefore, v_1 is contained in two blocks of type II or III. More precisely, since the valence of all neighbors of v_1 is at least two, v_1 is contained in two blocks B' and B'_1 of type II.

The block B' contains one of w_1, w_2 and one of w_3, w_4 (due to orientation, see Fig. 5.3(a)); assume that it contains w_1 and w_3. To avoid the edge (w_1, w_3) which does not appear in S, another block B'_1 should contain w_1 and w_3. Since w_1 and w_3 are joined to v_2, B'_1 is either of the second or of the fourth type. In the latter case v_2 is a dead end of B'_1, but v_2 is joined to w_2 and w_4 also. Hence, B'_1 is a triangle containing v_2, w_1, w_3 (see Fig. 5.3(b)). Similarly, w_2, w_4 and v_2 are contained in a block B''_1 of the second type. In particular, all the vertices of $L_S(v_1)$ are already contained in two blocks, so x is not joined to $L_S(v_1)$.

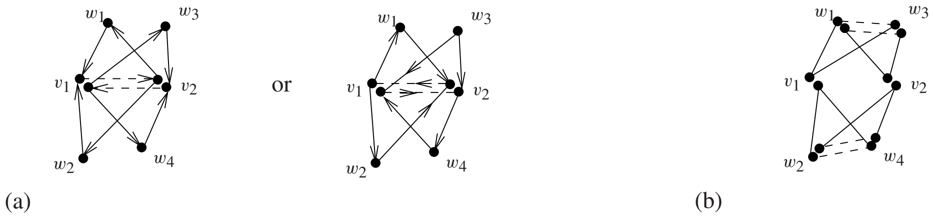


Fig. 5.3. Proof of Lemma 5.5, Case 3.2.1.

Case 3.2.2: B_1 is of type I. Denote by u the second vertex of B_1. If u coincides with v_2, then $\langle L_S(v_1), v_1\rangle$ has no outlets, so we can apply Proposition 4.6 to x. Now we may assume that $u \neq v_2$.

If u is joined to v_2, then the edge $\langle u, v_2\rangle$ must form a block of the first type, otherwise $\mathrm{Val}_S(v_2) > \mathrm{Val}_S(v_1)$. Therefore, no vertex except x is connected to $\langle L_S(v_1), v_1\rangle$. Since $|S| \geq 8$, in this case we can apply Proposition 4.6 to x. Thus, we may assume that u is not joined to v_2.

Take any vertex $y \notin \langle L_S(v_1), v_1, x\rangle$ and consider $S_1 = S \setminus y$ with some block decomposition. It is easy to see that v_1 should be contained in a block of type IV. Furthermore, looking at $L_S(v_1)$ one can notice that there is exactly one pair of leaves at distance two, namely w_1 and w_2, which implies that v_1, v_2, w_1, w_2 belong to one block, and w_1 and w_2 are dead ends. Therefore, x is not joined to w_1 or w_2, so only v_1 and v_2 are attached to w_1 and w_2.

Now we proceed as in Case 2.2.2.2. We consider the quiver $S_1 = S \setminus w_1$ with some block decomposition and show that w_2, v_1, v_2 form a block of type II. Since neither w_1 nor w_2 is joined to any vertex of S except v_1 and v_2, we can replace the triangle v_1, v_2, w_2 by the block v_1, v_2, w_1, w_2 of type IV to obtain a block decomposition of S.

Case 4: $\mathrm{Val}_{S\setminus x}(v_1) = 3$.

Case 4.1: $x \not\perp v_1$, $\mathrm{Val}_S(v_1) = 4$. Since $\mathrm{Val}_{S\setminus x}(v_2) \leq \mathrm{Val}_{S\setminus x}(v_1)$, we have $\mathrm{Val}_{S\setminus x}(v_2) = 3$. This means that vertices of B may be joined to x only. Thus, we may apply Proposition 4.6 to x.

Case 4.2: $x \perp v_1$, $\mathrm{Val}_S(v_1) = 3$. In this case v_1 is contained either in the block B only, or in the block B and in a second type block B_1. Consider the two cases.

Case 4.2.1: v_1 is contained in one block. This implies that v_1 and v_2 are joined in S, so no vertex except x is connected to B. We apply Proposition 4.6 to x.

Case 4.2.2: v_1 is contained in two blocks. In this case the second block B_1 is of type II, it contains v_1, v_2 and some vertex u. This case is similar to Case 2.2.1, the difference is in the orientation of B_1.

The union of B and B_1 has a unique outlet u. Thus, $\langle L_S(v_1), v_1\rangle \setminus u$ may be joined to x only. Further, x is not joined to v_1 or v_2 (since $\mathrm{Val}_S(v_2) \le \mathrm{Val}_S(v_1)$). If x is not joined to w_1 or w_2 either, then we can apply Proposition 4.6 to u. Therefore, we may assume that x is joined to one of w_1 and w_2, say w_1. Moreover, we may assume that some vertex $y \notin \langle B, u, x\rangle$ is joined to u, otherwise we can apply Proposition 4.6 to x.

Now take any $z \notin \langle B, u, x, y\rangle$ and consider $S_1 = S \setminus z$ with some block decomposition. The vertex v_1 is contained either in blocks of the fourth and second type, or in blocks of the second and first type. In the first case, due to orientations of edges (see Fig. 5.4) a block of type IV should contain all vertices of B, which is impossible since the dead end w_1 is joined to x. Hence, v_1 is contained in a triangle B' and an edge B'_1. Again, because of orientations of edges, w_1 and w_2 cannot belong to B' simultaneously, so B' contains u and w_i. To avoid the edge (w_i, u) these two vertices should be contained in some block B'_2. Since u is joined to v_2 and y, B'_2 contains v_2 and y also. Therefore, B'_2 is a block of the fourth type with outlets u, w_i and dead ends v_2, y. But this implies that y connects to w_i, which is impossible.

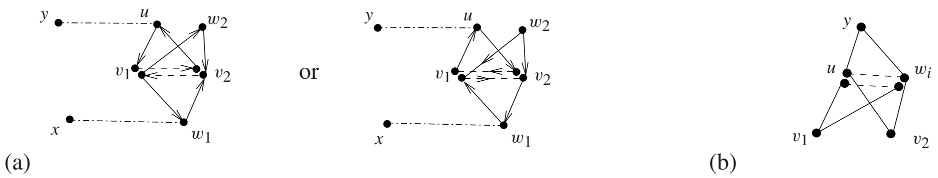


Fig. 5.4. Proof of Lemma 5.5, Case 4.2.2.

Case 5: $\mathrm{Val}_{S\setminus x}(v_1) = 2$. In this case v_1 is contained in a block B and a block B_1 of type I, where B_1 is an edge joining v_1 and v_2 with suitable orientation. The union of B and B_1 has no outlets, so we apply Proposition 4.6 to x.

Clearly, the valence of v_1 in $S \setminus x$ is at least two, so all cases have been studied and the lemma is proved. □

Corollary 5.6. *The valence of any vertex v of a minimal non-decomposable quiver S does not exceed* 4.

Proof. The proof is evident. Indeed, take any x which is not joined to v, and consider any block decomposition of $S \setminus x$. Then v is contained in at most two blocks, and the valence of any vertex of these blocks does not exceed 2. □

Lemma 5.7. *Let $v \in S$ be a vertex of valence* 4*. Then for any non-neighbor x of v and any block decomposition of $S \setminus x$ the vertex v is not contained in a block of the third type.*

Proof. Denote by v_1 and w_1 the dead ends of a block B of type III with outlet v, and denote by v_2 and w_2 the vertices of a block B_1 with outlet v. Clearly, v_1 and w_1 are not joined to v_2 or w_2. Denote $S' = \langle v_1, v_2, v, w_1, w_2 \rangle$. If no vertex of $S \setminus \langle S', x \rangle$ is joined to v_2 or w_2, then we apply Proposition 4.6 to x. Thus, we may assume that some vertex u_2 connects to one of v_2 and w_2, say v_2 (see Fig. 5.5(a)). In particular, this implies that B_1 is a block of type II.

Suppose that x is not joined to v_1 or w_1. Since $\mathrm{Val}_S(v) = 4$, the quiver $S \setminus v$ has at most four connected components. Two of them are v_1 and w_1. The remaining two (or one) contain at least five vertices (as $|S| \geq 8$), so at least one connected component has at least three vertices. Hence, we can apply Proposition 4.6 to v.

Therefore, we assume that x connects to at least one of v_1 and w_1, say w_1. We want to prove that S is block-decomposable by applying Proposition 4.8 to

$$S = \langle S_1 = \langle v_1, w_1 \rangle, b_1 = v, b_2 = x, S_2 = S \setminus \langle S_1, v, x \rangle \rangle$$

(see Fig. 5.5(b)). For this take $a_1 = v_1$, and try to choose a_2. The choice of a_2 will depend on $\Theta_{S \setminus x}(v)$.

If v_2 and w_2 are joined in S, then we choose from the vertices of S_2 not connected to x (if any) one which is at maximal distance from v in S. Clearly, such a vertex can be taken as a_2. If each vertex of S_2 is joined to x, we take as a_2 any vertex of $S_2 \setminus \langle v_2, w_2 \rangle$.

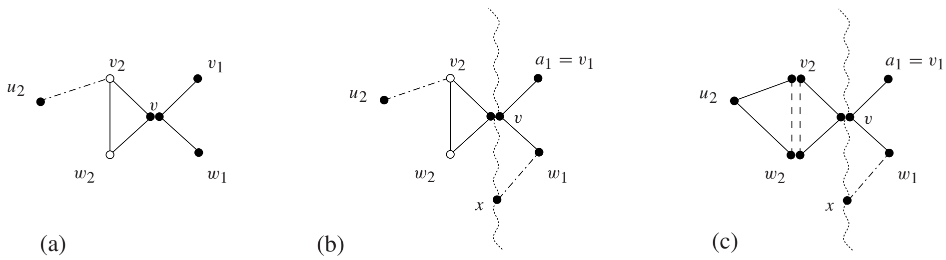


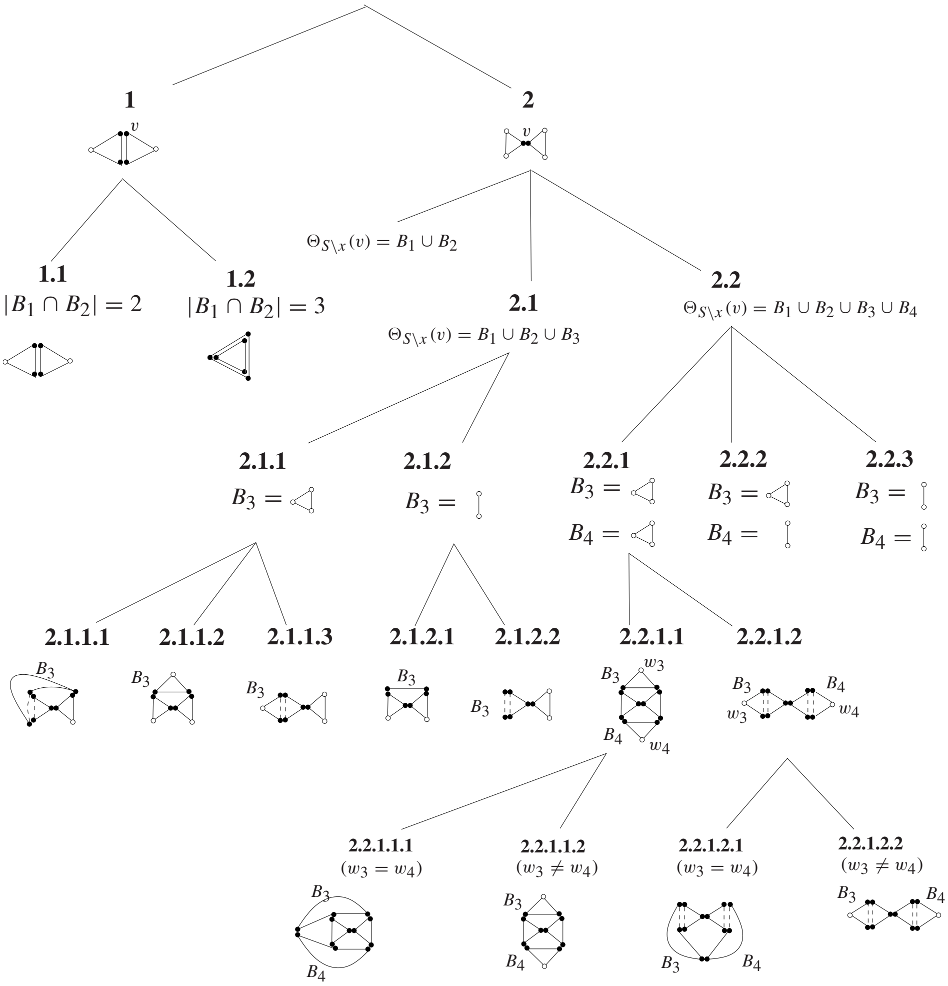
Fig. 5.5. Proof of Lemma 5.7.

Now suppose that v_2 and w_2 are not joined in S (in particular, $\Theta_{S \setminus x}(v)$ also contains some block B_2 formed by v_2, w_2 and probably some other vertex of S_2). B_2 cannot be of the first type since v_2 connects to u_2. Thus, w_2 is joined to u_2 (see Fig. 5.5(c)). Moreover, no vertex of $S \setminus x$ connects to any of v_2 and w_2 since both already belong to two blocks of the block decomposition of $S \setminus x$ under consideration. So, $S \setminus v_2$ is connected and we may take $a_2 = v_2$. □

Lemma 5.8. *Let $v \in S$ be a vertex of valence* 4. *Then for any non-neighbor x of v and any block decomposition of $S \setminus x$ the diagram $\Theta_{S \setminus x}(v)$ consists of exactly two blocks of type* II *having the only vertex v in common.*

Proof. Lemmas 5.7, 5.5, and 5.3 rule out blocks of types III, IV, and V from any decomposition of $S \setminus x$. The only possibility left is that v_1 is contained in two blocks B_1 and B_2 of the second type. Clearly, they have at most four outlets, so $\Theta_{S \setminus x}(v)$ may contain

Table 5.3. Proof of Lemma 5.8.



at most two additional blocks. Denote by v_1, u_1 and v_2, u_2 the remaining vertices of B_1 and B_2 respectively, and consider the following cases (see Table 5.3).

Case 1: B_1 and B_2 have at least two vertices in common.

Case 1.1: B_1 and B_2 have exactly two vertices in common. We may assume that $v_1 = v_2$. Then $\mathrm{Val}_S(v) = \mathrm{Val}_S(v_1) = 4$ is the maximal possible valence in S, so x is not connected to v and v_2. Consider all options for $\Theta_{S\setminus x}(v)$ (see Fig. 5.6).

If u_1 and u_2 are not joined (i.e. $\Theta_{S\setminus x}(v)$ consists of B_1 and B_2 only, see Fig. 5.6, left), then S is block-decomposable by Corollary 4.9 applied to

$$S = \langle S_1 = \langle v, v_2 \rangle, b_1 = u_1, b_2 = u_2, S_2 = S \setminus \langle S_1, u_1, u_2 \rangle \rangle \quad \text{with } c_1 = v.$$

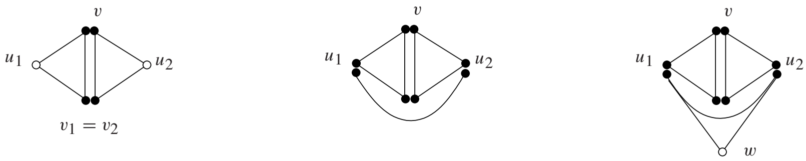


Fig. 5.6. Proof of Lemma 5.8, Case 1.1.

If $\langle u_1, u_2\rangle$ forms a block of the first type (see Fig. 5.6, middle), then u_1 and u_2 can be connected only to $x \in S$. Hence, all the vertices of $\Theta_{S\setminus x}(v)$ are dead ends, so S is block-decomposable by Proposition 4.6 applied to x.

If u_1 and u_2 are contained in a block of the second type with additional vertex w (see Fig. 5.6, right), then u_1 and u_2 have valence 4, so they are not joined to x. Therefore, S is block-decomposable by Proposition 4.6 applied to w.

Case 1.2: B_1 and B_2 have three vertices in common. In this case the union of B_1 and B_2 has no outlets, so S is block-decomposable by Proposition 4.6 applied to x.

Case 2: B_1 and B_2 intersect in v only. The quiver $\Theta_{S\setminus x}(v)$ consists of two, three, or four blocks. We are going to prove that there are no other blocks in $\Theta_{S\setminus x}(v)$ except B_1 and B_2; this will imply our lemma. For that we consider the remaining two cases and find a contradiction.

Case 2.1: $\Theta_{S\setminus x}(v)$ consists of three blocks B_1, B_2 and B_3. We go through different types of B_3 and the way it connects to B_1 and B_2.

Case 2.1.1: $B_3 \in \mathcal{B}_{\mathrm{II}}$.

Case 2.1.1.1: B_3 has three points in common with the union of B_1 and B_2. Let v_1, u_1, v_2 be the vertices of B_3. Either v_1 and u_1 are joined by a double edge or they are not joined at all. If they are joined by a double edge, then the valence of u_1 equals 4, so we are under the assumptions of Case 1, which implies that S is block-decomposable. Hence, we may assume that v_1 and u_1 are not joined in S. Thus, $\Theta_{S\setminus x}(v)$ is the quiver shown in Fig. 5.7. The only outlet is u_2. We may assume that some $y \notin \langle\Theta_{S\setminus x}(v), x\rangle$ is joined to u_2, otherwise S is block-decomposable by Proposition 4.6 applied to x.

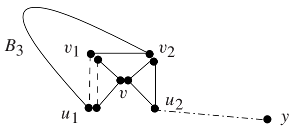


Fig. 5.7. Proof of Lemma 5.8, Case 2.1.1.1.

Consider any $z \notin \langle\Theta_{S\setminus x}(v), x, y\rangle$ and some block decomposition of $S \setminus z$. By Lemmas 5.7, 5.5, and 5.3, v is contained in two blocks of the second type. One of these blocks contains v, v_2 and one of u_1, u_2, v_1. The other one contains v and the remaining two vertices from u_1, u_2, v_1. Similarly, v_2 is also contained in a block of the second type with vertices v_2 and two of u_1, u_2, v_1. This implies that only two of u_1, u_2, v_1 are dead ends of $\langle v, v_1, v_2, u_1, u_2\rangle$, and only one of them is an outlet.

Note that no vertex other than u_1, u_2, v_1, v is joined to v_2, as otherwise $\mathrm{Val}_S(v_2) > 4$. If u_2 is an outlet, then x may be joined to u_2 only in $\Theta_{S\setminus x}(v)$, so S is block-decomposable by Proposition 4.6 applied to u_2.

If u_1 or v_1 is an outlet, then u_2 is a dead end, but y is connected to u_2, so we get a contradiction.

Case 2.1.1.2: B_3 has exactly one point in common with each of B_1 and B_2. We may assume that the vertices of B_3 are v_1, v_2 and w. Since $\mathrm{Val}_S(v_1) = \mathrm{Val}_S(v_2) = 4$, no vertex of $S \setminus \langle \Theta_{S\setminus x}(v), w\rangle$ is joined to v_1 or v_2.

Consider the quiver $S_1 = S \setminus v_1$ with a block decomposition. Since $\mathrm{Val}_{S_1}(v) = 3$ and v_2 is joined to u_2 in S, v is contained in one block of the second type and in one of the first type. But u_1 is joined neither to v_2 nor to u_2, so v, v_2, u_2 are vertices of one block, and v, u_1 form another one. Looking at v_2 we see that v_2 and w form a block of the first type, too. Replace the block v_2w by B_3, and the block vu_1 by B_1, and obtain a block decomposition of S.

Case 2.1.1.3: B_3 does not intersect one of B_1 and B_2. In this case we may assume that the vertices of B_3 are v_1, u_1 and w. Similarly to Case 2.1.1.1, we conclude that either this situation is already considered in Case 1, or v_1 and u_1 are not joined in S. Take any $y \notin \langle \Theta_{S\setminus x}(v), x\rangle$ and consider $S_1 = S \setminus y$. Since v_1 and u_1 are not joined to v_2 or u_2, the vertices v, v_2 and u_2 form one block (otherwise in order to cancel two edges joining $\langle v_1, u_1\rangle$ to $\langle v_2, u_2\rangle$ we have to glue in two blocks such that neither of them contains simultaneously u_2 and v_2; then both u_2 and v_2 are dead ends with no edge between them, contradicting the fact that u_2 and v_2 are joined in S). Therefore, v, v_1 and u_1 form a block, so v_1, u_1 and w also form a block. In particular, v_1 and u_1 are dead ends of $\langle u_1, w, v_1, v\rangle$, and x is not connected to v_1 or u_1. Recall also that no vertex except x, v, w could be connected to v_1 or u_1 since v_1 and u_1 are dead ends of $\Theta_{S\setminus x}(v)$. Hence, both u_1 and v_1 are joined to v and w only.

Consider $S_2 = S \setminus v_1$ with some block decomposition. Similarly to Case 2.1.1.2, it is easy to see that v, v_2, u_2 form one block of type II, and u_1, v form another block of type I. Since u_1 is not joined to any vertex except v and w, $\langle u_1, w\rangle$ is a block. Notice that the blocks $\langle u_1, w, v_1\rangle$ and $\langle u_1, v_1, v\rangle$ are oriented so that the side $\overrightarrow{v_1u_1}$ of one triangle cancels out with the side $\overrightarrow{u_1v_1}$ of the other. This implies that one of the edges $\langle u_1, w\rangle$ and $\langle u_1, v\rangle$ is directed towards u_1 while the other is directed from u_1. Replacing (u_1, w) by B_3, and (u_1, v) by B_1, we obtain a block decomposition of S.

Case 2.1.2: $B_3 \in \mathcal{B}_{\mathrm{I}}$. There are two possibilities to join B_3 to B_1 and B_2.

Case 2.1.2.1: B_3 has exactly one point in common with each of B_1 and B_2. We may assume that the vertices of B_3 are v_1 and v_2 (see Fig. 5.8(a)). If x is joined to neither v_1 nor v_2, then S is block-decomposable by Corollary 4.9 applied to $S = \langle S_1 = \langle v, v_1, v_2\rangle, b_1 = u_1, b_2 = u_2, S_2 = S \setminus \langle S_1, u_1, u_2\rangle\rangle$ with $c_1 = v_1$. So, we may assume that x is joined to at least one of v_1 and v_2, say v_1.

Take any $y \notin \langle \Theta_{S\setminus x}(v), x\rangle$ and consider $S_1 = S \setminus y$. The valence of v_1 is four, which means that v_1 is contained in two blocks of the second type. Since v_1 is joined to v, one of these blocks (call it B') contains both v_1 and v. The third vertex of B' is either v_2 or u_1

(since $\mathrm{Val}_S(v) = \mathrm{Val}_S(v_1) = 4$ is maximal possible and only u_1 and v_2 are joined to both v_1 and v).

If B' contains v_2, then the vertices u_1, v_1, x form one block. The second block containing v is composed of v, u_1, u_2. In particular, u_1 is joined to u_2, which contradicts the assumption (see Fig. 5.8(b)). Therefore, B' is composed of u_1, v_1 and v. The second block containing v is composed of v, v_2, u_2, and the second block containing v_1 is composed of v_1, v_2, x (see Fig. 5.8(c)). We obtain a quiver as considered above in Case 2.1.1.2.

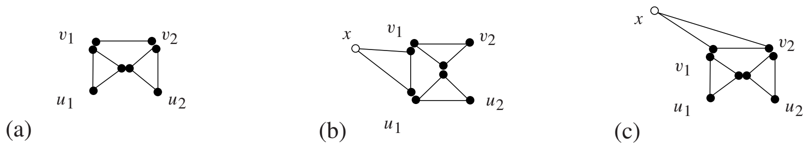


Fig. 5.8. Proof of Lemma 5.8, Case 2.1.2.1.

Case 2.1.2.2: B_3 does not intersect one of B_1 and B_2. The proof repeats the proof of Lemma 5.7.

Case 2.2: $\Theta_{S\setminus x}(v)$ consists of four blocks B_1, B_2, B_3, and B_4. We go through all the different types of B_3 and B_4 and all the ways to assemble $\Theta_{S\setminus x}(v)$ from them.

Case 2.2.1: Both $B_3 \in \mathcal{B}_{\mathrm{II}}$ and $B_4 \in \mathcal{B}_{\mathrm{II}}$. There are two possibilities to join B_3 and B_4 to B_1 and B_2.

Case 2.2.1.1: Each of B_3 and B_4 has exactly one vertex in common with each of B_1 and B_2. Denote by w_3 and w_4 the remaining vertices of B_3 and B_4 respectively, and consider two possibilities.

Case 2.2.1.1.1: $w_3 = w_4$. In this case the valences of all the six vertices of $\Theta_{S\setminus x}(v)$ are 4, showing that $\langle \Theta_{S\setminus x}(v)\rangle \perp (S \setminus \langle \Theta_{S\setminus x}(v)\rangle)$ and S is not connected.

Case 2.2.1.1.2: $w_3 \neq w_4$. We may assume that B_3 contains w_3, u_1 and u_2. Consider $S_1 = S \setminus u_1$ with some block decomposition. Since the valences of v, v_1, u_1, v_2, u_2 are 4, no vertex from $S\setminus\Theta_{S\setminus x}(v)$ connects to any of these five vertices. We want to prove that the edges $\langle w_3, u_2\rangle$ and $\langle v_1, v\rangle$ are blocks of the first type of S_1 (see Fig. 5.9). Then replacing (w_3, u_2) and (v_1, v) by B_3 and B_1 respectively we get a block decomposition of S.

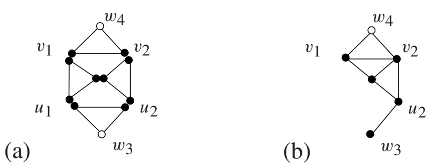


Fig. 5.9. Proof of Lemma 5.8, Case 2.2.1.1.2.

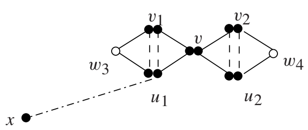
Since $\mathrm{Val}_{S_1}(u_2) = 3$, u_2 is contained in one block of the second type and one of the first type. The edge $\langle v_2, u_2\rangle$ belongs to the block of type II because $\mathrm{Val}_{S_1}(v_2) = 4$. For the same reason, w_3 and v_2 are not contained in one block. Therefore, v_2, v, u_2 form one block, and w_3, u_2 form another one. Looking at v_2 we see that the second block containing v_2 is spanned by v_2, v_1 and w_4. Recall that v and v_1 are not joined to any vertex of S_1 except w_3, v_2 and u_2. Thus, v and v_1 form a block, which completes the case.

Case 2.2.1.2: One of B_3 and B_4 (say B_3) does not intersect B_2 and the other (namely, B_4) does not intersect B_1. Denote by w_3 and w_4 the remaining vertices of B_3 and B_4 respectively. Taking into account Case 1, we may assume that u_2 is not joined to v_2 in S, and u_1 does not connect to v_1. We consider two possibilities.

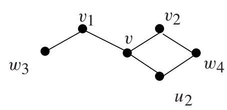
Case 2.2.1.2.1: $w_3 = w_4$. Notice that each vertex of $\Theta_{S\setminus x}(v)$ is already contained in two blocks, so no vertex of $S \setminus \langle\Theta_{S\setminus x}(v)\rangle$ except x is connected to $\langle\Theta_{S\setminus x}(v)\rangle$. To show that x is not joined to $\langle\Theta_{S\setminus x}(v)\rangle$ either, take any $y \notin \langle\Theta_{S\setminus x}(v), x\rangle$ (such a y does exist since $|S| \geq 8$) and consider $S \setminus y$ with some block decomposition. The valences of v and w_3 are 4, so they belong to two blocks of the second type each. Further, suppose that some pair of u_1, u_2, v_1, v_2 form a block with w_3. To avoid an edge between this pair of vertices, they should also form a block with v. Therefore, each vertex of $\Theta_{S\setminus x}(v)$ is contained in two blocks, so no vertex of $S \setminus \langle\Theta_{S\setminus x}(v)\rangle$ except y connects to $\langle\Theta_{S\setminus x}(v)\rangle$. In particular, $x \perp \langle\Theta_{S\setminus x}(v)\rangle$.

Thus, $S \setminus \langle\Theta_{S\setminus x}(v)\rangle \perp \langle\Theta_{S\setminus x}(v)\rangle$, and S is not connected.

Case 2.2.1.2.2: $w_3 \neq w_4$. Clearly, no vertex of $(S \setminus x) \setminus \Theta_{S\setminus x}(v)$ is joined to v or any of its neighbors (see Fig. 5.10(a)). Suppose that x is joined to a neighbor of v, say u_1. Consider $S_1 = S \setminus w_4$ with some block decomposition. Since u_1 does not connect to any neighbor of v and $\mathrm{Val}_{S_1}(u_1) = 3$, the edge $\langle u_1, v\rangle$ is not contained in a block of the second type. However, $\mathrm{Val}_{S_1}(v) = 4$ so $\langle u_1, v\rangle$ should be contained in some such block. The contradiction shows that $x \perp u_1$. Similarly, x is not joined to any other neighbor of v (or to v itself, of course).



(a)



(b)

Fig. 5.10. Proof of Lemma 5.8, Case 2.2.1.2.2.

Now consider $S_2 = S \setminus u_1$ (see Fig. 5.10(b)). Since $v_1 \perp v_2$, $v_1 \perp u_2$ and v is the only common neighbor of v_1 and v_2 (or v_1 and u_2), a block containing the edge $\langle v, v_1\rangle$ contains neither v_2 nor u_2. Therefore, v_1 and v form a block of the first type. As proved above, $\mathrm{Val}_S(v_1) = 2$ so v_1 and w_3 also form a block of the first type. Now replacing the edge (v_1, v) by B_1, and (v_1, w_3) by B_3, we obtain a block decomposition of S.

Case 2.2.2: $B_3 \in \mathcal{B}_{\mathrm{II}}$ and $B_4 \in \mathcal{B}_{\mathrm{I}}$. Denote by w_3 the remaining vertex of B_3. If B_4 does not intersect one of B_1 or B_2 (say B_1), then it must coincide with the edge $\langle u_2, v_2\rangle$ of B_2, and we get a situation described in Lemma 5.7 (notice that we did not use orientations of edges while proving Lemma 5.7). Hence, we may assume that B_4 intersects both B_1

and B_2. This implies that so does B_3. Let u_1 and u_2 be the vertices of B_4. Then $\Theta_{S\setminus x}(v)$ is a quiver shown in Fig. 5.11. Consider $\Theta_{S\setminus x}(v_2)$. Notice that $\mathrm{Val}_{S\setminus x}(v_2) = 4$ and $\Theta_{S\setminus x}(v_2)$ was treated in Case 2.1.1.2.

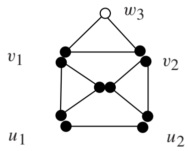


Fig. 5.11. Proof of Lemma 5.8, Case 2.2.2.

Case 2.2.3: $B_3, B_4 \in \mathcal{B}_{\mathrm{I}}$. In this case all the vertices of $\Theta_{S\setminus x}(v)$ are dead ends, so S is block-decomposable by Proposition 4.6 applied to x.

Since all possibilities are exhausted, the lemma is proved. □

Lemma 5.9. *Let $v \in S$ be a vertex of valence* 3*. Then for any non-neighbor x of v and any block decomposition of $S \setminus x$ the diagram $\Theta_{S\setminus x}(v)$ consists of exactly two blocks, one of type* II *and the other of type* I *having only the vertex v in common.*

Proof. Since the valence of v equals 3, v is contained in a block B_1 of the second or third type with two other vertices v_1 and u_1, and in a block B_2 of the first type with a second vertex v_2. We consider both types of B_1 and all possible quivers $\Theta_{S\setminus x}(v)$ below (see Table 5.4).

Case 1: $B_1 \in \mathcal{B}_{\mathrm{III}}$. In this case v_1 and u_1 are dead ends of the union of B_1 and B_2, so $\{v_1, u\} \perp (S \setminus \{x, v\})$, and $\Theta_{S\setminus x}(v)$ consists of B_1 and B_2 only. If x is not joined to v_1 or u_1, then S is block-decomposable by Proposition 4.6 applied to v_2. Therefore, we may assume that x is joined to at least one of v_1 and u_1, say u_1. If $v_2 \perp S \setminus \langle \Theta_{S\setminus x}(v), x\rangle$, then again S is block-decomposable by Proposition 4.6 applied to x. Thus, we can assume that v_2 is joined to some vertex distinct from v and x (see Fig. 5.12). Now we can apply Corollary 4.9 to $S = \langle S_1 = \langle u_1, v_1, v\rangle, b_1 = v_2, b_2 = x, S_2 = S \setminus \langle S_1, x, v_2\rangle\rangle$ with $c_1 = v_1$ to show that S is block-decomposable.

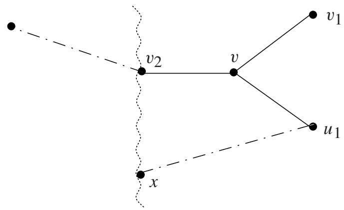
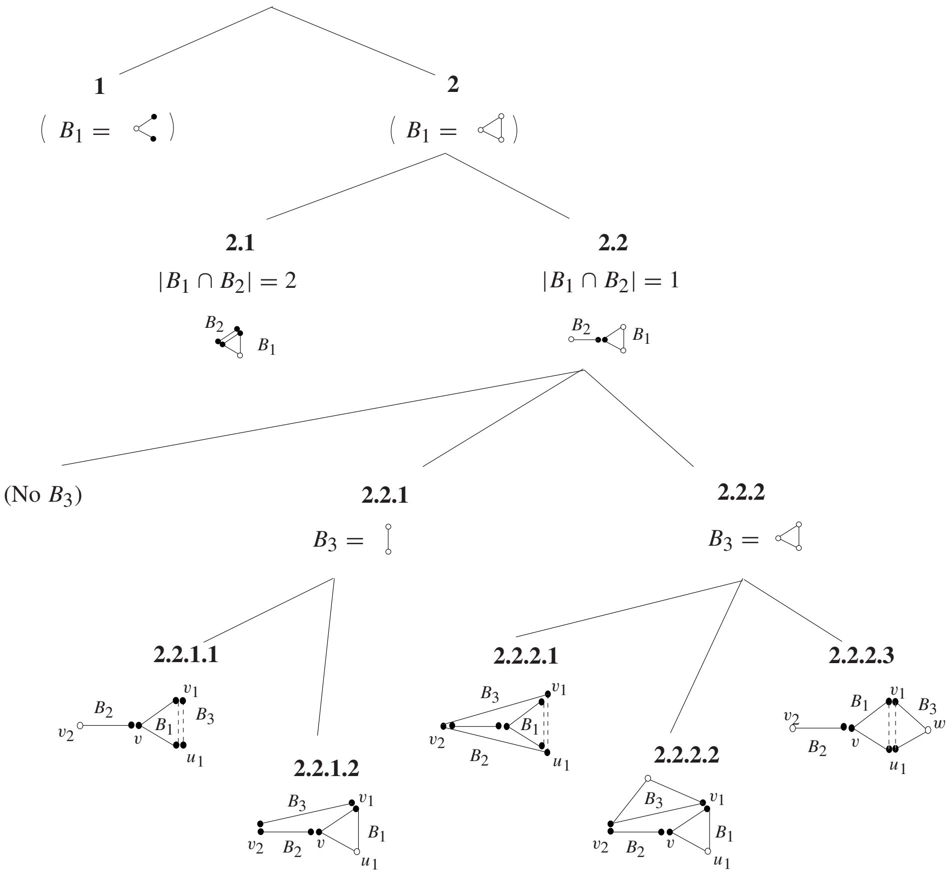


Fig. 5.12. Proof of Lemma 5.9, Case 1.

Case 2: $B_1 \in \mathcal{B}_{\mathrm{II}}$. Consider the following cases.

Case 2.1: B_1 and B_2 have two points in common. We may assume that $v_1 = v_2$. Since $\mathrm{Val}_S(v) = 3$ and $v \perp x$, v_2 and v are joined by a double edge. By Lemma 5.8, no vertex of valence 4 may be incident to a double edge. Thus, $S \setminus \langle u_1, v, v_2\rangle \perp \langle v, v_2\rangle$ and $S \setminus u_1$

Table 5.4. Proof of Lemma 5.9.



is not connected. Since the valence of u_1 in S does not exceed 4, at least one connected component of $S \setminus u_1$ has more than two vertices (as in the proof of Lemma 5.7). Therefore, S is block-decomposable by Proposition 4.6 applied to u_1.

Case 2.2: v is the only common vertex of B_1 and B_2. $\Theta_{S \setminus x}(v)$ may consist of two or three blocks. To prove the statement we need to exclude the option of three blocks. This is done in the remaining part of the proof. Assume that $\Theta_{S \setminus x}(v)$ contains an additional block B_3. Clearly, B_3 is either of the first or of the second type.

Case 2.2.1: $B_3 \in \mathcal{B}_{\mathrm{I}}$. There are two ways to join B_3 to B_1 and B_2.

Case 2.2.1.1: B_3 does not intersect B_2. In this case v_1 and u_1 are either joined by a double edge or are not joined in S at all. If they are not joined, then both v_1 and u_1 are dead ends of $\Theta_{S \setminus x}(v)$ and S is block-decomposable (as in Case 1). If they are joined by a double edge, then all three vertices v_1, u_1 and v have valence 3 in S, so $\langle v_1, u_1, v \rangle \perp x$, and S is block-decomposable by Proposition 4.6 applied to v_2.

Case 2.2.1.2: B_3 has one point in common with each of B_1 and B_2. We can assume that B_3 consists of v_1 and v_2. Since $L_S(v_1)$ contains a connected component of order at least 3, Lemma 5.8 yields $\mathrm{Val}_S(v_1) < 4$, so $\mathrm{Val}_S(v_1) = 3$. Hence, $S \setminus \langle \Theta_{S\setminus x}(v) \rangle \perp \langle v, v_1 \rangle$. We apply Corollary 4.9 to $S = \langle S_1 = \langle v, v_1 \rangle, u_1, v_2, S_2 = S \setminus \langle S_1, u_1, v_2 \rangle \rangle$ with $c_1 = v$ to show that S is block-decomposable.

Case 2.2.2: $B_3 \in \mathcal{B}_{\mathrm{II}}$. There are three ways to join B_3 to B_1 and B_2.

Case 2.2.2.1: B_3 contains all the three vertices v_1, u_1, v_2. Then all the vertices of $\Theta_{S\setminus x}(v)$ are dead ends, so S is block-decomposable by Proposition 4.6 applied to x.

Case 2.2.2.2: B_3 has one point in common with each of B_1 and B_2. We can assume that B_3 contains v_1. Then $\mathrm{Val}_S(v_1) = 4$, and $L_S(v_1)$ is connected, contradicting Lemma 5.8.

Case 2.2.2.3: B_3 does not intersect B_2. Denote by w the third vertex of B_3. Let us prove first that v_2 is not joined to w in S. If they are contained in a block of the first type in $S \setminus x$, then all the vertices of $\langle v, v_1, u_1, w, v_2 \rangle$ are dead ends, so S is block-decomposable by Proposition 4.6 applied to x. If w and v_2 are contained in a block of the second type, then $\mathrm{Val}_S(w) = 4$, but $L_S(w)$ consists of three connected components, contradicting Lemma 5.8. Therefore, $v_2 \perp w$. Observing that v_1 and u_1 are dead ends of $\Theta_{S\setminus x}(v)$, we see that they are joined only to v, w, and probably x.

Take any $y \notin \langle \Theta_{S\setminus x}(v), x \rangle$ and consider $S_1 = S \setminus y$ with a block decomposition. We will prove that x is joined to neither v_1 nor u_1. This implies that S is block-decomposable by Corollary 4.9 applied to $S = \big\langle S_1' = \langle v_1, u_1 \rangle, v, w, S_2' = S \setminus \langle S_1', v, w \rangle \big\rangle$ with $c_1 = v_1$.

Suppose that $x \not\perp v_1$. Since $\mathrm{Val}_S(v) = 3$ and $v \perp \langle x, w \rangle$, v is not contained in one block with any of x and w. Thus, v_1, x and w form a block of the second type and v, v_1 is a block of the first type. This implies that v, v_2, u_1 is a block of the second type.

Since $v_2 \perp u_1$ in S, in order to avoid the edge (v_2, u_1) there is another block containing v_2 and u_1. Since $u_1 \not\perp w$, this block should also contain w. But then $v_2 \not\perp w$, which is already proved to be false.

By exhausting all cases we have completed the proof of the lemma. □

Corollary 5.10. *A minimal non-decomposable quiver S does not contain double edges.*

Proof. Let v and u be joined by a double edge. Take any non-neighbor x of v and consider $S \setminus x$ with some block decomposition. By Lemmas 5.8 and 5.9, the valences of u and v do not exceed 2. Thus, they are joined to each other only and disconnected from the rest of S. □

Proof of Theorem 5.2. Consider a quiver S satisfying the assumptions of the theorem and having at least eight vertices. By Lemmas 5.3 and 5.5, the valence of any vertex of S does not exceed 4. By Lemmas 5.8 and 5.9, the link of any vertex of valence 4 consists of two disjoint edges, and the link of any vertex of valence 3 consists of one edge and one vertex. By Corollary 5.10, S does not contain double edges.

Now take all cycles of order 3 in S and color all their edges and vertices red (we assume all the remaining edges and vertices to be black). By Lemmas 5.8 and 5.9, each red edge belongs to a unique cycle of order 3. Each red vertex is contained either in four red edges, or in two red edges and at most one black edge. Notice also that due to Lemmas 5.8 and 5.9 each cycle of order 3 is cyclically oriented.

Denote by S_1 the quiver obtained by deleting all red edges from S. Let us show that S_1 is a forest. Indeed, S_1 does not contain vertices of valence 3 or more. Further, if S_1 contains a cycle C, then each vertex of this cycle is contained in two black edges, so the cycle does not contain any red vertex. This implies that no vertex of $S \setminus C$ is joined to C in S, so either S is not connected or $S = C$. In the latter case S is block-decomposable.

Take a block decomposition of S_1: any edge is a block. It is well defined since there are no vertices of valence 3 or more. Clearly, any red vertex is an outlet. Now consider each cycle of order 3 as a block of the second type, and glue it to S_1. We obtain a block decomposition of S, which contradicts the assumptions of the theorem. □

Now we are able to prove the main results of the section.

Theorem 5.11. *The only mutation-finite quivers satisfying the assumptions of Theorem 5.2 are ones mutation-equivalent to one of the two quivers X_6 and E_6 shown in Figure 5.13.*

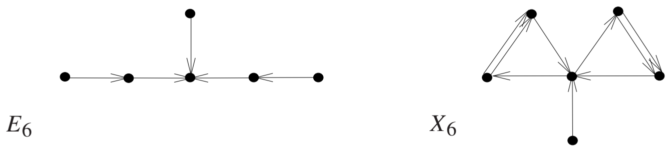


Fig. 5.13. Minimal non-decomposable mutation-finite quivers.

Remark 5.12. We recall that the property of a quiver S to be block-decomposable is preserved by mutations. Indeed, according to [FST], S is block-decomposable if and only if it corresponds to an ideal triangulation of a punctured bordered surface, and any mutation corresponds to a flip of the triangulation. Thus, any quiver mutation-equivalent to S arises from some triangulation, too. In particular, this implies that the set of non-decomposable quivers is invariant under mutations as well. At the same time, the property to be *minimal* non-decomposable may not be preserved by mutations *a priori*. However, while proving Theorem 5.11, we see that minimal non-decomposable quivers are invariant anyway.

Proof of Theorem 5.11. The two quivers in Figure 5.13 are mutation-finite and non-decomposable (see [DO, Propositions 4 and 6]). To prove the theorem, it is sufficient to show that all other mutation-finite quivers on at most seven vertices either are block-decomposable, or contain subquivers which are mutation-equivalent to one of X_6 and E_6. In particular, this will imply that all the quivers mutation-equivalent to X_6 or E_6 are also minimal non-decomposable.

Let S be a minimal non-decomposable mutation-finite quiver. By Theorem 5.2, $|S| \leq 7$. Since the mutation class of S is finite, multiplicities of edges of S do not exceed 2 (see Theorem 2.6). The number of quivers on at most seven vertices with bounded multiplicities of edges is finite. This means that we can use a computer to list all quivers, choose mutation-finite ones, and check which of them are block-decomposable. However, the number of quivers under consideration is large. To reduce the time required for computations, we organize the check as follows.

First, we list all mutation classes of connected mutation-finite quivers of order 3, there are three of them (see [DO, Theorem 7]), and choose one representative in each class. They are all block-decomposable. Clearly, any connected mutation-finite quiver of order 4 contains a proper subquiver mutation-equivalent to one of these three quivers of order three.

Next, we add a vertex and join it to each of the three quivers by edges of multiplicities at most 2 in all possible ways (we use the **C++** program of [FST1]). For each quiver obtained we check if its mutation class is finite, and choose one representative from each finite mutation class (here we use the Java applet for quiver mutations from [K1]). The resulting list contains five quivers of order 4, they all are block-decomposable.

Continuing in the same way we get seven finite mutation classes of order 5, again all block-decomposable. Then we get 13 classes of order 6, exactly two of which consist of non-decomposable quivers, namely quivers mutation-equivalent to X_6 and quivers mutation-equivalent to E_6. Since all mutation-finite quivers with at most five vertices are block-decomposable, all quivers mutation-equivalent to X_6 or E_6 are minimal non-decomposable. Attaching a vertex to representatives of all classes of order 6, we get 15 finite mutation classes of quivers of order 7, three of which consist of non-decomposable quivers, namely the classes containing E_7, $\widetilde{E}_6$, or X_7. These three mutation classes consist of 416, 132, and 2 quivers respectively. Each quiver from the first two classes contains a subquiver mutation-equivalent to E_6, each quiver from the third class contains a subquiver mutation-equivalent to X_6. Therefore, none of them is minimal non-decomposable. □

The following immediate corollary of Theorem 5.11 is the main tool in the classification of mutation-finite quivers.

Corollary 5.13. *Every non-decomposable mutation-finite quiver contains a subquiver mutation-equivalent to E_6 or to X_6.*

6. Classification of non-decomposable quivers

In this section we use Corollary 5.13 to classify all non-decomposable mutation-finite quivers.

Theorem 6.1. *A connected non-decomposable mutation-finite quiver of order greater than 2 is mutation-equivalent to one of the eleven quivers E_6, E_7, E_8, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, X_6, X_7, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ shown in Figure 6.1.*

All these quivers have finite mutation class [DO] and are non-decomposable since each contains a subquiver mutation-equivalent to E_6 or X_6. We only need to prove that this list is complete.

We prove several elementary preparatory statements first.

Lemma 6.2. *Let S be a non-decomposable quiver of order $d \geq 7$ with finite mutation class. Then S contains a non-decomposable mutation-finite subquiver S_1 of order $d-1$.*

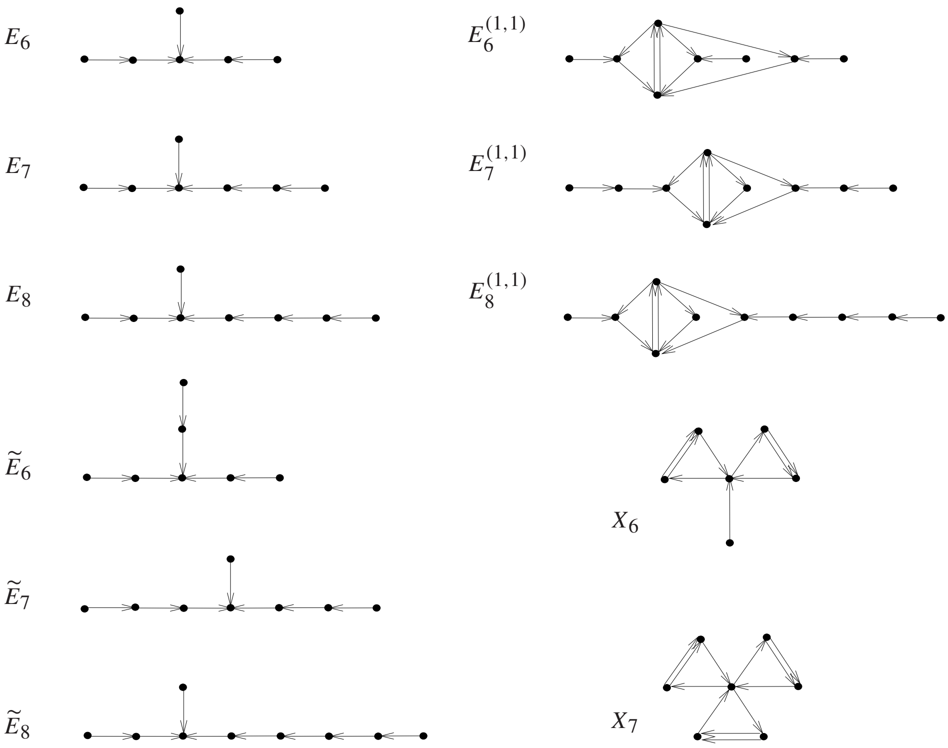


Fig. 6.1. Non-decomposable mutation-finite quivers of order at least 3.

Proof. According to Corollary 5.13, S contains a subquiver S_0 mutation-equivalent to E_6 or X_6. Let S_1 be any connected subquiver of S of order $d-1$ containing S_0. Clearly, S_1 is non-decomposable, and its mutation class is finite. □

Corollary 6.3. *Suppose that for some $d \geq 7$ there are no non-decomposable mutation-finite quivers of order d. Then the order of any non-decomposable mutation-finite quiver does not exceed $d-1$.*

The proof of the following lemma is evident.

Lemma 6.4. *Let S_1 be a proper subquiver of S, and let S_0 be a quiver mutation-equivalent to S_1. Then there exists a quiver S' which is mutation-equivalent to S and contains S_0.*

Proof of Theorem 6.1. According to Theorem 5.11, there are exactly two finite mutation classes of non-decomposable quivers of order 6, namely the classes of E_6 and X_6. Due to Corollary 5.13, all other non-decomposable mutation-finite quivers have at least seven vertices. By Lemma 6.4, in each finite mutation class of non-decomposable quivers there is a representative containing a subquiver E_6 or X_6.

Therefore, to find all finite mutation classes of non-decomposable quivers of order 7 we need to join a vertex to E_6 and X_6 in all possible ways (i.e. by edges of multiplicity at most two, with all orientations), and to choose amongst the resulting 7-vertex quivers all mutation-finite classes. This has been done by using the Java applet of [K1] and an elementary **C++** program [FST1] (in fact, the same algorithm was used in the proof of Theorem 5.11). In this way we get three finite mutation classes of quivers of order 7 with representatives X_7, E_7, and $\widetilde{E}_6$.

Now, Lemmas 6.4 and 6.2 allow us to continue the procedure. To list all finite mutation classes of non-decomposable quivers of order 8 we join a vertex to X_7, E_7, and $\widetilde{E}_6$ in all possible ways, and choose again all mutation-finite classes. The result is three mutation classes with representatives E_8, $\widetilde{E}_7$, and $E_6^{(1,1)}$.

In the same way we analyze the quivers of order 9 and obtain two mutation classes with representatives $\widetilde{E}_8$ and $E_7^{(1,1)}$. To find all finite mutation classes of non-decomposable quivers of order 10, we apply the same procedure to $\widetilde{E}_8$ and $E_7^{(1,1)}$. The result is a unique mutation class containing $E_8^{(1,1)}$.

Finally, the same procedure applied to $E_8^{(1,1)}$ gives no mutation-finite quivers at all. This implies that there are no non-decomposable mutation-finite quiver of order 11. Now Corollary 6.3 yields immediately the statement of the theorem. □

7. Minimal mutation-infinite quivers

The main goal of this section is to provide a criterion for a quiver to be mutation-finite (Theorem 7.5). For a given quiver S, this criterion allows one to check if S is mutation-finite in time polynomial in $|S|$.

Definition 7.1. A *minimal mutation-infinite quiver* is a quiver S such that

- S has infinite mutation class;
- any proper subquiver of S is mutation-finite.

Example 7.2. Any mutation-infinite quiver of order 3 is minimal. This is caused by the fact that any quiver of order at most 2 is mutation-finite.

Clearly, any minimal mutation-infinite quiver is connected. Notice that the property of being minimal mutation-infinite is not mutation invariant. Indeed, any mutation-infinite class contains quivers with arbitrarily large multiplicities of edges. If $|S| > 3$, then taking a connected subquiver of S of order 3 containing an edge of multiplicity greater than 2 we get a proper subquiver of S which is mutation-infinite (see Theorem 2.6). Note also that no minimal mutation-infinite quiver of order at least 4 contains edges of multiplicity greater than two.

We will deduce the criterion from the following lemma.

Lemma 7.3. *Any minimal mutation-infinite quiver contains at most ten vertices.*

The bound provided in the lemma is sharp: there exist numerous minimal mutation-infinite quivers of order 10. We show some of them below. One source of such examples

Fig. 7.1. Minimal mutation-infinite quivers of order 10 coming from Dynkin diagrams.

are simply-laced Dynkin diagrams of root systems of hyperbolic Kac–Moody algebras with any orientations of edges. There are two such diagrams of order 10 (e.g. see [K]); examples of corresponding quivers are shown in Fig. 7.1.

Remark 7.4. There is no general algorithm to determine if two infinite-mutational quivers are mutation-equivalent. However, for *acyclic* quivers (i.e., containing no oriented cycles) the following result is known (see [CK, Corollary 4]): if two acyclic quivers are mutation-equivalent, then there exists a sequence of mutations from one of them to the other via acyclic quivers only. In particular, this implies that the two quivers in Fig. 7.1 are not mutation-equivalent. Indeed, they both are trees, and it is easy to see that the only way to change the topological type of a tree by mutation is to create an oriented cycle.

Another series of examples can be obtained from the quiver shown in Fig. 7.2.

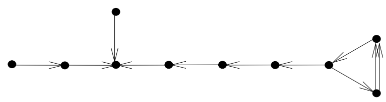


Fig. 7.2. Minimal mutation-infinite quiver of order 10.

Note that it is not clear whether the quiver of Fig. 7.2 is or is not mutation-equivalent to one of the quivers of Fig. 7.1.

Proof of Lemma 7.3. Let S be a minimal mutation-infinite quiver.

First, we prove a weaker statement, that $|S| \leq 11$. In fact, this follows immediately from Theorems 5.2 and 6.1. Indeed, either all the proper subquivers of S are block-decomposable, or S contains a proper mutation-finite non-decomposable subquiver of order $|S| - 1$ (we can assume that this quiver is connected: if it is not connected but non-decomposable, it contains a non-decomposable connected component S_0, and any connected subquiver of S of order $|S|-1$ containing S_0 is non-decomposable). In the former case $|S| \leq 7$ according to Theorem 5.2 (again, we emphasize that we did not require S to be mutation-finite in the assumptions of Theorem 5.2). In the latter case $|S|-1 \leq 10$ due to Theorem 6.1, which proves $|S| \leq 11$.

Now suppose that $|S| = 11$. Then S contains a proper mutation-finite non-decomposable subquiver S' of order 10. According to Theorem 6.1, S' is mutation-equivalent to $E_{10}^{(1,1)}$. The mutation class of $E_{10}^{(1,1)}$ consists of 5739 quivers, which can be easily computed using Keller's Java applet [K1]. In other words, S contains one of 5739 quivers of order 10 as a proper subquiver.

Hence, we can list all minimal mutation-infinite quivers of order 11 in the following way. To each of the 5739 quivers above we add one vertex in all possible ways (we can do that since the multiplicity of edges is bounded by two; the sources of the program can be found in [FST1]). For every quiver obtained we check whether all its proper subquivers of order 10 (and therefore all the others) are mutation-finite. However, the resulting set of the procedure above is empty: every quiver obtained has at least one mutation-infinite subquiver of order 10, so it is not minimal. □

As a corollary of Lemma 7.3, we get a criterion for a quiver to be mutation-finite.

Theorem 7.5. *A quiver S of order at least* 10 *is mutation-finite if and only if all subquivers of S of order* 10 *are mutation-finite.*

Proof. According to Definition 7.1, every mutation-infinite quiver contains some minimal mutation-infinite quiver as a subquiver. Thus, a quiver is mutation-finite if and only if it does not contain minimal mutation-infinite subquivers. By Lemma 7.3, this holds if and only if all subquivers of order at most 10 are mutation-finite. Since a subquiver of a mutation-finite quiver is also mutation-finite, the latter condition, in its turn, holds if and only if all subquivers of order 10 are mutation-finite, which completes the proof. □

8. Growth of skew-symmetric cluster algebras

We recall the definition of growth of cluster algebras [FST, Section 11].

Definition 8.1. A cluster algebra has *polynomial growth* if the number of distinct seeds which can be obtained from a fixed initial seed by at most n mutations is bounded from above by a polynomial function of n. A cluster algebra has *exponential growth* if the number of such seeds is bounded from below by an exponentially growing function of n.

In [FST, Proposition 11.1] a complete classification of block-decomposable quivers corresponding to algebras of polynomial growth is given. It turns out that the growth is polynomial if and only if the surface corresponding to the quiver is a sphere with at most three punctures and boundary components in total.

We prove the following theorem.

Theorem 8.2. *Any mutation-infinite skew-symmetric cluster algebra has exponential growth.*

Combining these two results, we see that to classify all cluster algebras of polynomial growth we need only to determine the growth of the 11 exceptional mutation-finite algebras listed in Theorem 6.1. Three of them, namely E_6, E_7 and E_8, are of finite type, so they have a finite number of seeds. Other three ($\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$) are of affine type, so they have linear growth (according to H. Thomas). Therefore, there remain five algebras for which the growth is unknown.

In other words, to complete the classification of cluster algebras by the growth rate it remains to ascertain the rates of growth in the five cases of X_6, X_7, $E_6^{(1,1)}$, $E_7^{(1,1)}$, and $E_8^{(1,1)}$.

A sequence of cluster transformations preserving a given exchange matrix defines *a group-like element*. The set of all group-like elements is the *generalized modular group*.

Using ideas similar to the proof of the famous Tits alternative, it can be proved that in all five cases the growth rate of the generalized modular group is exponential. More precisely, studying the attracting points of some induced action it can be proved that the generalized modular group contains the free group of rank two as a subgroup.

Details on the growth rates in exceptional cases will be published elsewhere.

Corollary 8.3. *A skew-symmetric cluster algebra of rank at least* 3 *has polynomial growth if and only if*

- *it is associated with a triangulation of either a sphere with three punctures, a disk with two punctures, an annulus with one puncture, or a pair of pants; or*
- *it is one of the following exceptional affine cases:* $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.

Remark 8.4. Note that by construction any cluster algebra of rank 2 is either of finite type or has linear (i.e., polynomial) growth.

The rest of this section is devoted to the proof of Theorem 8.2.

Remark 8.5. It is sufficient to prove Theorem 8.2 for cluster algebras corresponding to mutation-infinite quivers of order 3. Indeed, any mutation-infinite quiver S has a mutation-equivalent quiver S' with an edge of weight at least 3. According to Theorem 2.6, any connected subquiver $S_0 \subset S$ of order 3 containing that edge is mutation-infinite. Therefore, it is enough to show that the algebra corresponding to S_0 grows exponentially.

In fact, we prove a stronger result. Denote by $S^{(n)}$ the set of quivers which can be obtained from a quiver S by at most n mutations. According to Remark 8.5, the following lemma implies Theorem 8.2.

Lemma 8.6. *Let S be a mutation-infinite quiver of order* 3*. Then the order $|S^{(n)}|$ grows exponentially with respect to n.*

The proof of Lemma 8.6 splits into two steps. We start by proving the following lemma. By saying that a quiver is *oriented* we mean that all the cycles are oriented.

Lemma 8.7. *For any mutation-infinite quiver S of order* 3 *there exists a sequence of at most four mutations taking S to S' such that*

(1) *S' is oriented;*
(2) *all the weights of the edges of S' are greater than* 1*;*
(3) *an edge of maximal weight is unique.*

Proof. First, we make S oriented without empty edges. For this, we need at most two mutations. Indeed, if S is non-oriented without empty edges, then the mutation at a unique vertex which is neither a sink nor a source leads to an oriented quiver. If S has an empty edge, then by at most one mutation we put a sink and a source at the ends, and then, mutating at the middle vertex, we get the required quiver.

Thus, we can now assume that S is oriented without empty edges, and we have two mutations left to satisfy conditions (2) and (3). Denote the weights of S by (a, b, c)

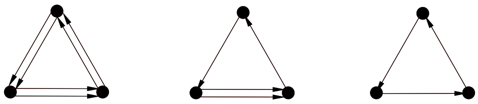


Fig. 8.1. Oriented mutation-finite quivers of order 3 without empty edges.

with $a \geq b \geq c$. Since S is mutation-infinite, it does not coincide with any of the three mutation-finite quivers of order 3 shown in Fig. 8.1. In particular, $a \geq b \geq 2$. We may assume that either $a = b$ or $c = 1$. If $c = 1$, then making a mutation preserving a and b, we obtain an oriented quiver with weights $(a, b, c' = ab - c)$. Clearly, $c' = ab - c \geq 3$, so condition (2) holds.

Now we have a quiver satisfying the first two conditions, and one mutation left to satisfy condition (3). Again, let the weights of S be (a, a, c) with $a \geq c$. If $c = 2$, then we make a mutation changing c to get a quiver with weights $(a, a, c' = a^2 - c)$. Since S is mutation-infinite, $a > 2 = c$, therefore $c' = a^2 - 2 > a$, so the third condition is satisfied. If $c > 2$, then mutating at any of the other two vertices, we get a quiver with weights $(a, b = ac - a, c)$. Clearly, $b = ac - a > a$ since $c > 2$. □

The last step in proving Lemma 8.6 is Lemma 8.8 below. We say that a sequence of mutations is *reduced* if it does not contain two consecutive mutations at the same vertex. Note also that we distinguish quivers with the same weights but different orientations.

Lemma 8.8. *Let S be a mutation-infinite quiver of order* 3 *satisfying conditions* (1)–(3) *of Lemma* 8.7*, and denote by* (a, b, c) *the weights of the edges of S, $a > b \geq c$. Let S_1 and S_2 be quivers obtained from S by different reduced sequences of mutations, such that the first mutation in each sequence preserves the weight a of S. Then S_1 and S_2 are distinct.*

Clearly, Lemma 8.8 together with Lemma 8.7 imply Lemma 8.6. Before proving Lemma 8.8, we provide the following auxiliary statement.

Lemma 8.9. *Let S satisfy the assumptions of Lemma* 8.8*. Suppose that S' is obtained from S by a mutation preserving the weight a. Then*

- *the maximal weight of S' is greater than a;*
- *S' satisfies conditions* (1)–(3) *of Lemma* 8.7.

Proof. To prove the first statement, compute the weights of S'. If we preserve the weights a and b, then the weights of S' are $(a, b, c' = ab - c)$, so $c' > a$ since $b \geq 2$ and $c < a$. If we preserve the weights a and c, then the weights of S' are $(a, b' = ac - b, c)$, so $b' > a$ since $c \geq 2$ and $b < a$.

Now the second statement is evident. □

The following immediate corollary of Lemma 8.9 is a particular case of Lemma 8.8.

Corollary 8.10. *Let S satisfy the assumptions of Lemma* 8.8*, and let $\mu_n, \dots, \mu_1$ be a reduced sequence of mutations, where μ_1 preserves the maximal weight of S. Denote by S_i the quiver $\mu_i \dots \mu_1 S$. Then all the quivers $S, S_1, \dots, S_n$ are distinct.*

Proof of Lemma 8.8. Suppose S_1 and S_2 coincide. We may assume that any two quivers S_1' and S_2' in the sequences of quivers from S to S_1 and S_2 respectively are distinct. Consider the next-to-last quivers S_1' and S_2' in the sequences of quivers from S to S_1 and S_2. By Lemma 8.9, both S_1 and S_2 have the following property: the edge of the maximal weight is opposite to the vertex in which the last mutation was made. Therefore, S_1' and S_2' coincide also. This contradiction proves the lemma. $\square$

9. Quivers of order 3

The structure of mutation classes of quivers of order 3 was described in [ABBS] and [BBH]. These papers provide a complete classification of mutation classes containing quivers without oriented cycles given in different terms.

Define the *total weight* of a quiver as the sum of the weights of all edges. It is proved in [ABBS] (see also [BBH, Lemma 2.1]) that if a mutation class does not contain quivers without oriented cycles, then it contains a unique (up to duality) quiver of minimal total weight, and any other mutation-equivalent quiver can be reduced to that one in a unique way.

We complete the description of mutation classes containing quivers without oriented cycles by a similar statement. We use the terminology of [BBH]: a quiver S of order 3 is called *cyclic* if it is an oriented cycle, and *acyclic* otherwise; S is called *cluster-cyclic* if any quiver mutation-equivalent to S is cyclic, and *cluster-acyclic* otherwise.

Theorem 9.1. *Let S be a connected cluster-acyclic quiver of order* 3*. Then*

- *the mutation class of S contains a unique (up to changing orientations of edges) quiver S_0 without oriented cycle;*
- *the total weight of S_0 is minimal in the mutation class;*
- *any sequence of mutations decreasing total weight at each step applied to S ends in S_0.*

We use the following notation. By $(a, b, c)^-$ we denote a non-oriented cycle with weights (a, b, c). We need not fix orientations of edges since any two such quivers are mutation-equivalent under mutations at sources or sinks only. Similarly, $(a, b, c)^+$ is an oriented cycle with weights (a, b, c).

Proof of Theorem 9.1. Consider a connected acyclic quiver $S_0 = (a,b,c)^-$ with $a \geq b \geq c$. Denote by $\mathcal{S}$ the set of quivers satisfying conditions (1)–(3) of Lemma 8.7.

Suppose first that none of the weights is equal to 1. If $c = 0$, then the only quiver we can get by one mutation different from S_0 is $(a, b, ab)^+$, which is contained in $\mathcal{S}$. If $c \geq 2$, then we can obtain three possibly different quivers $(a, b, ab + c)^+$, $(a, ac + b, c)^+$ and $(bc + a, b, c)^+$ all belonging to $\mathcal{S}$. According to Lemma 8.9, all the other quivers from the mutation class of S_0 also belong to $\mathcal{S}$, which proves the first statement. Moreover, Lemmas 8.8 and 8.9 imply that for any quiver from the mutation class of S_0 belonging to $\mathcal{S}$ there is a unique reduced sequence of mutations decreasing the total weight at each step, and the minimal element is in $\mathcal{S}$ if and only if the entire mutation class is contained in $\mathcal{S}$ (this is also proved in [BBH, Lemma 2.1]). Since S_0 is the only quiver not contained in S, all the statements are proved.

Now suppose that at least one of the weights of S_0 is 1. We may assume that S_0 is mutation-infinite (there are exactly two acyclic mutation-finite quivers of order 3, namely $\widetilde{A}_2$ and A_3, and the theorem is evident for them). Our aim is to show that almost all quivers mutation-equivalent to S_0 belong to S; then looking at the remaining quivers all the statements become evident. Indeed, S_0 is of one of the following three types: $(a,1,0)^-$, $(a,1,1)^-$, or $(a,b,1)^-$, where $a \geq b \geq 2$. We list the quivers which can be obtained from those three by mutations.

The only way to change $(a,1,0)^-$ is to obtain $(a,a,1)^+$, from which we may get $(a,a,a^2-1)^+$ only, which is in $\mathcal{S}$.

The quiver $(a,1,1)^-$ can be mutated into $(a+1,1,1)^+$ and $(a+1,a,1)^+$. The first one can then be mutated into the second one only, and the latter into the first one or into $(a,a+1,a^2+a-1)^+ \in \mathcal{S}$.

The quiver $(a,b,1)^-$ can be mutated either into $(a,b,ab+1)^+ \in \mathcal{S}$, or into $(a+b,b,1)^+$ or $(a+b,a,1)^+$. These two can be mutated either into each other or into quivers belonging to $\mathcal{S}$.

Therefore, in the mutation class of S_0 there are at most three quivers not belonging to $\mathcal{S}$, and S_0 has minimal total weight amongst them. Applying the same arguments as in the first case, we complete the proof. □

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