J. Eur. Math. Soc. 15, 145–155

DOI 10.4171/JEMS/357

Hernán Castro · Juan Dávila · Hui Wang

A Hardy type inequality for $W_0^{m,1}(\Omega)$ functions

Received May 11, 2011

Abstract. We consider functions $u \in W_0^{m,1}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, and $m \geq 2$ is an integer. For all $j \geq 0$ and $1 \leq k \leq m-1$ such that $1 \leq j+k \leq m$, we prove that $\partial^j u(x)/d(x)^{m-j-k} \in W_0^{k,1}(\Omega)$ with

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)},$$

where d is a smooth positive function which coincides with $\operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$, and ∂^l denotes any partial derivative of order l.

Keywords. Hardy inequality, Sobolev spaces

1. Introduction

In [4, Theorem 1.2], the following one-dimensional Hardy type inequality was proven: Suppose that $u \in W^{2,1}(0,1)$ satisfies $u(0) = u'(0) = 0$. Then $u(x)/x \in W^{1,1}(0,1)$ with $u(x)/x|_0 = 0$ and

$$\left\| \left(\frac{u(x)}{x} \right)' \right\|_{L^1(0,1)} \leq \|u''\|_{L^1(0,1)}. \tag{1}$$

As explained in [4], this inequality is somehow unexpected because one can construct a function $u \in W^{2,1}(0,1)$ such that $u(0) = u'(0) = 0$ and neither $u'(x)/x$ nor $u(x)/x^2$ belongs to $L^1(0,1)$; however, as (1) shows, for such a u, the difference $u'(x)/x - u(x)/x^2 = (u(x)/x)'$ is in fact an L^1 function, reflecting a "magical" cancelation of the non-integrable terms.

H. Castro: Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA; e-mail: castroh@math.rutgers.edu

J. Dávila: Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile; e-mail: jdavila@dim.uchile.cl

H. Wang: Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA and Department of Mathematics, Technion, Israel Institute of Technology, 32000 Haifa, Israel; e-mail: huiwang@math.rutgers.edu

With estimate (1) already proven, it was natural to raise the following question: Assume Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and let u be in $W_0^{2,1}(\Omega)$. For $x \in \Omega$, denote by $\delta(x) = d(x, \partial\Omega)$ the distance from x to the boundary of Ω, and let $d : \Omega \to (0, \infty)$ be a smooth function such that $d(x) = \delta(x)$ near $\partial\Omega$. Is it true that $u/d \in W_0^{1,1}(\Omega)$? If so, can one obtain the corresponding Hardy-type estimate

$$\int_\Omega \left| \nabla \left(\frac{u(x)}{d(x)} \right) \right| dx \leq C \|\nabla^2 u\|_{L^1(\Omega)},$$

for some constant C?

The purpose of this work is to give a positive answer to the above question. In fact, this is a special case of the following:

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Given $x \in \Omega$, we denote by $\delta(x)$ the distance from x to the boundary $\partial\Omega$. Let $d : \Omega \to (0, \infty)$ be a smooth function such that $d(x) = \delta(x)$ near $\partial\Omega$. Suppose $m \geq 2$ and let j, k be non-negative integers such that $1 \leq k \leq m - 1$ and $1 \leq j + k \leq m$. Then for every $u \in W_0^{m,1}(\Omega)$, we have $\partial^j u(x)/d(x)^{m-j-k} \in W_0^{k,1}(\Omega)$ with*

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{m,1}(\Omega)}, \tag{2}$$

where ∂^l denotes any partial derivative of order l, and $C > 0$ is a constant depending only on Ω and m.

The rest of this paper is organized into three sections. In Section 2 we introduce the notation and give some preliminary results. In order to present the main ideas used to prove Theorem 1, we begin in Section 3 with the proof for the special case $m = 2$; then in Section 4 we provide the proof for the general case $m \geq 2$.

2. Notation and preliminaries

Throughout, we denote by $\mathbb{R}_+^N := \{(y_1, \ldots, y_{N-1}, y_N) \in \mathbb{R}^N : y_N > 0\}$ the upper half-space, and $B_r^N(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < r\}$; also, when $x_0 = 0$, we write $B_r^N := B_r^N(0)$.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Given $x \in \Omega$, we denote by $\delta(x)$ the distance from x to the boundary $\partial\Omega$, that is,

$$\delta(x) := \operatorname{dist}(x, \partial\Omega) = \inf\{|x - y| : y \in \partial\Omega\}.$$

For $\epsilon > 0$, the tubular neighborhood of $\partial\Omega$ in Ω is the set

$$\Omega_\epsilon := \{x \in \Omega : \delta(x) < \epsilon\}.$$

The following well known result (see e.g. Lemma 14.16 in [5]) shows that δ is smooth in some neighborhood of $\partial\Omega$.

Lemma 2.1. *Let Ω and $\delta : \Omega \to (0, \infty)$ be as above. Then there exists $\epsilon_0 > 0$, only depending on Ω, such that $\delta|_{\Omega_{\epsilon_0}} : \Omega_{\epsilon_0} \to (0, \infty)$ is smooth. Moreover, for every $x \in \Omega_{\epsilon_0}$ there exists a unique $y_x \in \partial\Omega$ so that*

$$x = y_x + \delta(x)\nu_{\partial\Omega}(y_x),$$

where $\nu_{\partial\Omega}$ denotes the unit inward normal vector field on $\partial\Omega$.

Since $\partial\Omega$ is smooth, for fixed $\tilde{x}_0 \in \partial\Omega$, there exists a neighborhood $\mathcal{V}(\tilde{x}_0) \subset \partial\Omega$, a radius $r > 0$ and a map

$$\tilde{\Phi} : B_r^{N-1} \to \mathcal{V}(\tilde{x}_0) \tag{3}$$

which defines a smooth diffeomorphism. Define

$$\mathcal{N}_+(\tilde{x}_0) := \{x \in \Omega_{\epsilon_0} : y_x \in \mathcal{V}(\tilde{x}_0)\}, \tag{4}$$

where ϵ_0 and y_x are as in Lemma 2.1. We denote by $\Phi : B_r^{N-1} \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^N$ the map defined by

$$\Phi(\tilde{y}, t) := \tilde{\Phi}(\tilde{y}) + y_N \cdot \nu_{\partial\Omega}(\tilde{\Phi}(\tilde{y})), \tag{5}$$

where $\tilde{y} = (y_1, \ldots, y_{N-1})$, and we write

$$\mathcal{N}(\tilde{x}_0) := \Phi(B_r^{N-1} \times (-\epsilon_0, \epsilon_0)). \tag{6}$$

Lemma 2.2. *The map $\Phi|_{B_r^{N-1}\times(0,\epsilon_0)}$ is a diffeomorphism and*

$$\mathcal{N}_+(\tilde{x}_0) = \Phi(B_r^{N-1} \times (0, \epsilon_0)).$$

Proof. This is a direct corollary of the definition and Lemma 2.1. □

Remark 2.1. The map $\Phi|_{B_r^{N-1}\times(0,\epsilon_0)}$ gives a local coordinate chart which straightens the boundary near $\tilde{x}_0$. This type of coordinates are sometimes called *flow coordinates* (see e.g. [3] and [6]).

From now on, $C > 0$ will always denote a constant only depending on Ω and possibly the integer $m \geq 2$. The following is a direct, but very useful, corollary.

Corollary 2.1. *Let $f \in L^1(\mathcal{N}_+(\tilde{x}_0))$ and Φ be given by (5). Then*

$$\begin{aligned}\frac{1}{C}\int_{B_r^{N-1}}\int_0^{\epsilon_0} |f(\Phi(\tilde{y}, y_N))|\, dy_N\, d\tilde{y} &\leq \int_{\mathcal{N}_+(\tilde{x}_0)} |f(x)|\, dx \\ &\leq C\int_{B_r^{N-1}}\int_0^{\epsilon_0} |f(\Phi(\tilde{y}, y_N))|\, dy_N\, d\tilde{y}.\end{aligned}$$

Proof. Since $\Phi|_{B_r^{N-1}\times(0,\epsilon_0)}$ is a diffeomorphism, we know that for all $(\tilde{y}, y_N) \in B_r^{N-1} \times (0,\epsilon_0)$ we have

$$1/C \le |\det D\Phi(\tilde{y}, y_N)| \le C.$$

The result then follows from the change of variables formula. □

The following lemma provides us with a partition of unity in $\mathbb{R}^N$, constructed from the neighborhoods $\mathcal{N}(\tilde{x}_0)$. Consider the open cover of $\partial\Omega$ given by $\{\mathcal{V}(\tilde{x}) : \tilde{x} \in \partial\Omega\}$, where $\mathcal{V}(\tilde{x}) \subset \partial\Omega$ is defined in (3). By the compactness of $\partial\Omega$, there exist $\{\tilde{x}_1, \dots, \tilde{x}_M\} \subset \partial\Omega$ so that $\partial\Omega = \bigcup_{l=1}^M \mathcal{V}(\tilde{x}_l)$. Notice that by the definition of $\mathcal{N}(\tilde{x}_0)$ in (6), $\bigcup_{l=1}^M \mathcal{N}(\tilde{x}_l)$ is also an open cover of $\partial\Omega$ in $\mathbb{R}^N$. The following is a classical result (see e.g. [2, Lemma 9.3] and [1, Theorem 3.15]).

Lemma 2.3 (partition of unity). *There exist functions $\rho_0, \rho_1, \dots, \rho_M \in C^\infty(\mathbb{R}^N)$ such that*

(i) $0 \le \rho_l \le 1$ *for all* $l = 0, 1, \dots, M$ *and* $\sum_{l=0}^M \rho_i(x) = 1$ *for all* $x \in \mathbb{R}^N$*,*
(ii) $\operatorname{supp}\rho_l \subset \mathcal{N}(\tilde{x}_l)$ *for all* $l = 1, \dots, M$*,*
(iii) $\rho_0|_\Omega \in C_0^\infty(\Omega)$.

In order to simplify notation, we will denote by ∂^l any partial differential operator of order l where l is a positive integer.[1] Also, ∂_i will denote the partial derivative with respect to the i-th variable, and $\partial_{ij}^2 = \partial_i \circ \partial_j$.

Remark 2.2. We conclude this section by showing that, to prove Theorem 1, it is enough to prove estimate (2) for smooth functions with compact support. Suppose $u \in W_0^{m,1}(\Omega)$. Then there exists a sequence $\{u_n\} \subset C_0^\infty(\Omega)$ so that $\|u - u_n\|_{W^{m,1}(\Omega)} \to 0$ as $n \to \infty$. In particular, after maybe extracting a subsequence, one can assume that

$$\partial^l u_n \to \partial^l u \quad \text{a.e. in } \Omega \text{ for all } 0 \le l \le m.$$

Since d is smooth, the above implies that for a.e. $x \in \Omega$ and all $j \ge 0$ and $1 \le k \le m-1$ with $1 \le j+k \le m$,

$$\begin{aligned}\partial^k\left(\frac{\partial^j u(x)}{d(x)^{m-j-k}}\right) &= \frac{\partial^{j+k}u(x)}{d(x)^{m-j-k}} + \partial^j u(x)\partial^k\left(\frac{1}{d(x)^{m-j-k}}\right)\\ &= \lim_{n\to\infty} \frac{\partial^{j+k}u_n(x)}{d(x)^{m-j-k}} + \partial^j u_n(x)\partial^k\left(\frac{1}{d(x)^{m-j-k}}\right)\\ &= \lim_{n\to\infty} \partial^k\left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}}\right).\end{aligned}$$

Therefore, Fatou's Lemma applies and we obtain

$$\left\|\partial^k\left(\frac{\partial^j u(x)}{d(x)^{m-j-k}}\right)\right\|_{L^1(\Omega)} \le \liminf_{n\to\infty}\left\|\partial^k\left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}}\right)\right\|_{L^1(\Omega)}.$$

[1] In general, one would say: "For a given multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$, we denote by ∂^α the partial derivative of order $l = |\alpha| = \alpha_1 + \cdots + \alpha_N$". Since we only care about the order of the operator, it makes sense to abuse the notation and identify α with its order $|\alpha| = l$.

Once (2) has been proven for $u_n \in C_0^\infty(\Omega)$, we get

$$\left\| \partial^k \left(\frac{\partial^j u_n(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \|u_n\|_{W^{m,1}(\Omega)},$$

and thus we can conclude that

$$\left\| \partial^k \left(\frac{\partial^j u(x)}{d(x)^{m-j-k}} \right) \right\|_{L^1(\Omega)} \leq C \liminf_{n\to\infty} \|u_n\|_{W^{m,1}(\Omega)} = C\|u\|_{W^{m,1}(\Omega)}.$$

Finally estimate (2) together with the fact that $\partial^j u_n(x)/d(x)^{m-j-k} \in C_0^\infty(\Omega)$ and $\overline{C_0^\infty(\Omega)}^{W^{k,1}(\Omega)} = W_0^{k,1}(\Omega)$ implies that $\partial^j u(x)/d(x)^{m-j-k} \in W_0^{k,1}(\Omega)$.

3. The case $m = 2$

We begin this section by proving estimate (2) in Theorem 1 for $\Omega = \mathbb{R}^N_+$, $m = 2$, $j = 0$ and $k = 1$.

Lemma 3.1. *Suppose that $u \in C_0^\infty(\mathbb{R}^N_+)$. Then for all $i = 1, \dots, N$,*

$$\left\| \partial_i \left(\frac{u(y)}{y_N} \right) \right\|_{L^1(\mathbb{R}^N_+)} \leq 2\|u\|_{W^{2,1}(\mathbb{R}^N_+)}.$$

Proof. Consider first the case $i = N$. This is similar to (1), but for completeness, we provide the proof. Notice that we can write

$$\frac{\partial}{\partial y_N}\left(\frac{u(\tilde{y}, y_N)}{y_N} \right) = \frac{1}{y_N^2} \int_0^{y_N} \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) t \, dt,$$

hence by integrating we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial}{\partial y_N}\left(\frac{u(\tilde{y}, y_N)}{y_N} \right) \right| dy_N \, d\tilde{y} &\leq \int_{\mathbb{R}^{N-1}} \int_0^\infty \frac{1}{y_N^2} \int_0^{y_N} \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t \, dt \, dy_N \, d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t \int_t^\infty \frac{1}{y_N^2} \, dy_N \, dt \, d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| t \int_t^\infty \frac{1}{y_N^2} \, dy_N \, dt \, d\tilde{y} \\
&= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left| \frac{\partial^2}{\partial y_N^2} u(\tilde{y}, t) \right| dt \, d\tilde{y},
\end{aligned}$$

hence

$$\int_{\mathbb{R}^N_+} \left| \frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N} \right) \right| dy \leq \int_{\mathbb{R}^N_+} \left| \frac{\partial^2 u(y)}{\partial y_N^2} \right| dy. \tag{7}$$

When $1 \le i \le N-1$, we need to estimate $\int_{\mathbb{R}^N_+} \frac{1}{y_N}\left|\frac{\partial u}{\partial y_i}(y)\right| dy$. To do so, consider the change of variables $y = \Psi(x)$, where

$$\Psi(x_1, \ldots, x_i, \ldots, x_N) = (x_1, \ldots, x_i + x_N, \ldots, x_N). \tag{8}$$

Notice that $\det D\Psi(x) = 1$, hence

$$\int_{\mathbb{R}^N_+} \frac{1}{y_N}\left|\frac{\partial u(y)}{\partial y_i}\right| dy = \int_{\mathbb{R}^N_+} \frac{1}{x_N}\left|\frac{\partial u}{\partial y_i}(\Psi(x))\right| dx.$$

Observe that if we let $v(x) = u(\Psi(x))$, we can write

$$\frac{1}{x_N}\frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N}\left(\frac{v(x)}{x_N}\right) - \frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N}\right)\bigg|_{y=\Psi(x)}. \tag{9}$$

Applying estimate (7) to u and v yields

$$\begin{aligned}
\int_{\mathbb{R}^N_+} \frac{1}{x_N}\left|\frac{\partial u}{\partial y_i}(\Psi(x))\right| dx &\le \int_{\mathbb{R}^N_+}\left|\frac{\partial}{\partial x_N}\left(\frac{v(x)}{x_N}\right)\right| dx + \int_{\mathbb{R}^N_+}\left|\frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N}\right)\bigg|_{y=\Psi(x)}\right| dx \\
&= \int_{\mathbb{R}^N_+}\left|\frac{\partial}{\partial x_N}\left(\frac{v(x)}{x_N}\right)\right| dx + \int_{\mathbb{R}^N_+}\left|\frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N}\right)\right| dy \\
&\le \int_{\mathbb{R}^N_+}\left|\frac{\partial^2 v(x)}{\partial x_N^2}\right| dx + \int_{\mathbb{R}^N_+}\left|\frac{\partial^2 u(y)}{\partial y_N^2}\right| dy.
\end{aligned}$$

Finally, notice that

$$\frac{\partial^2 v(x)}{\partial x_N^2} = \frac{\partial^2 u(y)}{\partial y_N^2}\bigg|_{y=\Psi(x)} + 2\frac{\partial^2 u(y)}{\partial y_i \partial y_N}\bigg|_{y=\Psi(x)} + \frac{\partial^2 u(y)}{\partial y_i^2}\bigg|_{y=\Psi(x)}. \tag{10}$$

Thus, after reversing the change of variables when needed, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N_+} \frac{1}{y_N}\left|\frac{\partial u(y)}{\partial y_i}\right| dy &= \int_{\mathbb{R}^N_+} \frac{1}{x_N}\left|\frac{\partial u}{\partial y_i}(\Psi(x))\right| dx \\
&\le 2\int_{\mathbb{R}^N_+}\left|\frac{\partial^2 u(y)}{\partial y_N^2}\right| dy + 2\int_{\mathbb{R}^N_+}\left|\frac{\partial^2 u(y)}{\partial y_i \partial y_N}\right| dy + \int_{\mathbb{R}^N_+}\left|\frac{\partial^2 u(y)}{\partial y_i^2}\right| dy \\
&\le 2\|u\|_{W^{2,1}(\mathbb{R}^N_+)}.
\end{aligned}$$

□

Recall (see Section 2) that for every $\tilde{x}_0 \in \partial\Omega$, we have the neighborhood $\mathcal{N}_+(\tilde{x}_0) \subset \Omega$ given by (4) and the diffeomorphism $\Phi : B_r^{N-1} \times (0, \epsilon_0) \to \mathcal{N}_+(\tilde{x}_0)$ given by (5). Moreover, we know that $\delta(x)$ is smooth over $\mathcal{N}_+(\tilde{x}_0)$.

Lemma 3.2. *Let $\tilde{x}_0 \in \partial\Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ be given by (4), and suppose $u \in C_0^\infty(\mathcal{N}_+(\tilde{x}_0))$. Then for all $i = 1, \ldots, N$,*

$$\left\|\partial_i\left(\frac{u(x)}{\delta(x)}\right)\right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \le C\|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}.$$

Proof. We first use Corollary 2.1 to obtain

$$\int_{\mathcal{N}_+(\tilde{x}_0)} \left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \right| dx \le C \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} \right| dy_N \, d\tilde{y}.$$

Let $v(\tilde{y}, y_N) = u(\Phi(\tilde{y}, y_N))$. We claim that

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} \right| dy_N \, d\tilde{y} \le C \sum_{j=1}^N \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N} \right) \right| dy_N \, d\tilde{y}. \qquad (11)$$

We will prove (11) at the end, so that we can conclude the argument. Since $v \in C_0^\infty(B_r^{N-1} \times (0, \epsilon_0)) \subset C_0^\infty(\mathbb{R}_+^N)$, we can apply Lemma 3.1 to obtain

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N} \right) \right| dy_N \, d\tilde{y} \le C \|v\|_{W^{2,1}(B_r^{N-1} \times (0, \epsilon_0))}.$$

Notice that the chain rule and the fact that Φ is a diffeomorphism imply that for all $1 \le i, j \le N$,

$$|\partial_{ij}^2 v(\tilde{y}, y_N)| \le C \Big(\sum_{p,q=1}^N |\partial_{pq}^2 u(x)|_{x=\Phi(\tilde{y}, y_N)}| + \sum_{p=1}^N |\partial_p u(x)|_{x=\Phi(\tilde{y}, y_N)}| \Big),$$

so with the aid of Corollary 2.1, we can write

$$\begin{aligned} \|v\|_{W^{2,1}(B_r^{N-1} \times (0, \epsilon_0))} &\le C \int_{B_r^{N-1}} \int_0^{\epsilon_0} \Big(\sum_{p,q} |\partial_{pq}^2 u|_{x=\Phi(\tilde{y}, y_N)}| + \sum_p |\partial_p u|_{x=\Phi(\tilde{y}, y_N)}| \Big) dy_N \, d\tilde{y} \\ &\le C \int_{\mathcal{N}_+(\tilde{x}_0)} \Big(\sum_{p,q} |\partial_{pq}^2 u(x)| + \sum_p |\partial_p u(x)| \Big) dx \le C \|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_0))}. \end{aligned}$$

To conclude, we need to prove (11). To do so, notice that $u(x) = v(\Phi^{-1}(x))$, and $\delta(x) = c(\Phi^{-1}(x))$, where $c(\tilde{y}, y_N) = y_N$. Thus, by using the chain rule we obtain

$$\partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} = \sum_{j=1}^N \partial_j \left(\frac{v(y)}{c(y)} \right) \Big|_{y=(\tilde{y}, y_N)} \cdot \partial_i (\Phi^{-1})_j (\Phi(\tilde{y}, y_N)),$$

and since Φ is a diffeomorphism, we obtain

$$\left| \partial_i \left(\frac{u(x)}{\delta(x)} \right) \Big|_{x=\Phi(\tilde{y}, y_N)} \right| \le C \sum_{j=1}^N \left| \partial_j \left(\frac{v(y)}{c(y)} \right) \Big|_{y=(\tilde{y}, y_N)} \right|.$$

Estimate (11) then follows by integrating the above inequality. □

We end this section with the proof of the main result when $m = 2$.

Proof of Theorem 1 when $m = 2$. When $j = 1$ and $k = 1$ the estimate (2) is trivial. Taking into account Remark 2.2, we only need to prove

$$\left\| \partial_i \left(\frac{u(x)}{d(x)} \right) \right\|_{L^1(\Omega)} \leq C \|u\|_{W^{2,1}(\Omega)} \tag{12}$$

for $u \in C_0^\infty(\Omega)$ and $i = 1, \ldots, N$. To do so, we use the partition of unity given by Lemma 2.3 to write $u(x) = \sum_{l=0}^{M} u_l(x)$ on Ω where $u_l(x) := \rho_l(x) u(x)$, $l = 0, 1, \ldots, M$. Now, without loss of generality, we can assume that $d(x) = \delta(x)$ for all $x \in \Omega_{\epsilon_0}$, and that $d(x) \geq C > 0$ for all $x \in \operatorname{supp} \rho_0 \cap \Omega$. Notice that in $\operatorname{supp} \rho_0 \cap \Omega$, we have

$$u_0/d \in C^\infty(\overline{\operatorname{supp} \rho_0 \cap \Omega}) \quad \text{with} \quad \|u_0/d\|_{W^{1,1}(\operatorname{supp} \rho_0 \cap \Omega)} \leq C \|u_0\|_{W^{1,1}(\sup \rho_0 \cap \Omega)}.$$

To take care of the boundary part, notice that $u_l \in C_0^\infty(\mathcal{N}_+(\tilde{x}_l))$ for $l = 1, \ldots, M$, so Lemma 3.2 applies and we obtain

$$\left\| \partial_i \left(\frac{u_l(x)}{\delta(x)} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_l))} \leq C \|u_l\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))} \quad \text{for all } l = 1, \ldots, M.$$

To conclude, notice that $\partial_i(u(x)/d(x)) = \sum_{l=1}^{M} \partial_i(u_l(x)/\delta(x)) + \partial_i(u_0(x)/d(x))$ on Ω and that $|\rho_l(x)|$, $|\partial_i \rho_l(x)|$ and $|\partial_{ij}^2 \rho_l(x)|$ are uniformly bounded for all $l = 0, 1, \ldots, M$, therefore

$$\begin{aligned}
\left\| \partial_i \left(\frac{u(x)}{d(x)} \right) \right\|_{L^1(\Omega)} &\leq \sum_{l=1}^{M} \left\| \partial_i \left(\frac{u_l(x)}{\delta(x)} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_l))} + \left\| \partial_i \left(\frac{u_0(x)}{d(x)} \right) \right\|_{L^1(\operatorname{supp} \rho_0 \cap \Omega)} \\
&\leq C \Big(\sum_{l=1}^{M} \|u_l\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))} + \|u_0\|_{W^{1,1}(\operatorname{supp} \rho_0 \cap \Omega)} \Big) \\
&\leq C \Big(\sum_{l=1}^{M} \|u\|_{W^{2,1}(\mathcal{N}_+(\tilde{x}_l))} + \|u\|_{W^{1,1}(\operatorname{supp} \rho_0 \cap \Omega)} \Big) \leq C \|u\|_{W^{2,1}(\Omega)},
\end{aligned}$$

thus completing the proof. □

4. The general case $m \geq 2$

To prove the general case, we need to generalize Lemma 3.1 in the following way

Lemma 4.1. *Suppose $u \in C_0^\infty(\mathbb{R}_+^N)$. Then for all $m \geq 1$ and $i = 1, \ldots, N$ we have*

$$\left\| \partial_i \left(\frac{u(y)}{y_N^{m-1}} \right) \right\|_{L^1(\mathbb{R}_+^N)} \leq C \|u\|_{W^{m,1}(\mathbb{R}_+^N)}.$$

Proof. The case $m = 1$ is a trivial statement, whereas $m = 2$ is exactly what we proved in Lemma 3.1. So from now on we suppose $m \geq 3$. We first notice that when $i = N$, the result follows from the proof of [4, Theorem 1.2] when $j = 0$ and $k = 1$. We refer the reader to [4] for the details.

When $1 \le i \le N-1$, we can proceed as in the proof of Lemma 3.1. Define $v(x) = u(\Psi(x))$ where Ψ is given by (8). Notice that when $m \ge 3$, instead of equation (9) we have

$$\frac{1}{x_N^{m-1}} \frac{\partial u}{\partial y_i}(\Psi(x)) = \frac{\partial}{\partial x_N}\left(\frac{v(x)}{x_N^{m-1}}\right) - \frac{\partial}{\partial y_N}\left(\frac{u(y)}{y_N^{m-1}}\right)\bigg|_{y=\Psi(x)},$$

and instead of (10) we have

$$\frac{\partial^m v(x)}{\partial x_N^m} = \sum_{l=0}^{m} \binom{m}{l} \frac{\partial^m u(y)}{\partial y_i^{m-l} \partial y_N^l}\bigg|_{y=\Psi(x)}.$$

Hence the estimate is reduced to the already proven result for $i = N$. We omit the details. □

We also have the analog of Lemma 3.2.

Lemma 4.2. *Let $\tilde{x}_0 \in \partial\Omega$ and $\mathcal{N}_+(\tilde{x}_0)$ as in Lemma 3.2. Let $u \in C_0^\infty(\mathcal{N}_+(\tilde{x}_0))$. Then for all $m \ge 1$ and $i = 1, \dots, N$ we have*

$$\left\| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right\|_{L^1(\mathcal{N}_+(\tilde{x}_0))} \le C \|u\|_{W^{m,1}(\mathcal{N}_+(\tilde{x}_0))}.$$

Proof. The proof involves only minor modifications from the proof of Lemma 3.2, which we provide in the next few lines. Corollary 2.1 gives

$$\int_{\mathcal{N}_+(\tilde{x}_0)} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right| dx \le C \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \bigg|_{x=\Phi(\tilde{y}, y_N)} \right| dy_N \, d\tilde{y}.$$

If $v(\tilde{y}, y_N) = u(\Phi(\tilde{y}, y_N))$, then

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \bigg|_{x=\Phi(\tilde{y}, y_N)} \right| dy_N \, d\tilde{y} \le C \sum_{j=1}^{N} \int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N^{m-1}} \right) \right| dy_N \, d\tilde{y}. \quad (13)$$

Just as for (11), estimate (13) follows from the fact that Φ is a smooth diffeomorphism. Since $v \in C_0^\infty(B_r^{N-1} \times (0, \epsilon_0)) \subset C_0^\infty(\mathbb{R}_+^N)$, we can apply Lemma 4.1 and obtain

$$\int_{B_r^{N-1}} \int_0^{\epsilon_0} \left| \partial_j \left(\frac{v(\tilde{y}, y_N)}{y_N^{m-1}} \right) \right| dy_N \, d\tilde{y} \le C \|v\|_{W^{m,1}(B_r^{N-1} \times (0, \epsilon_0))}.$$

Notice that by the chain rule and the fact that Φ is a smooth diffeomorphism, we get

$$|\partial^m v(\tilde{y}, y_N)| \le C \sum_{l \le m} |\partial^l u(x)|_{x=\Phi(\tilde{y}, y_N)}|,$$

where the left hand side is a fixed m-th order partial derivative, and on the right hand side the summation contains all partial derivatives operators of order $l \leq m$. Again with the aid of Corollary 2.1, we can write

$$\|v\|_{W^{m,1}(B_r^{N-1}\times(0,\epsilon_0))} \leq C \sum_{l\leq m} \int_{B_r^{N-1}} \int_0^{\epsilon_0} (|\partial^l u|_{x=\Phi(\tilde{y},y_N)}|)\, dy_N\, d\tilde{y}$$
$$\leq C \sum_{l\leq m} \int_{\mathcal{N}_+(\tilde{x}_0)} |\partial^l u(x)|\, dx \leq C\|u\|_{W^{m,1}(\mathcal{N}_+(\tilde{x}_0))}. \qquad \square$$

And of course we have

Lemma 4.3. *Suppose $u \in C_0^\infty(\Omega)$. Then for all $m \geq 1$ and $i = 1, \ldots, N$ we have*

$$\left\| \partial_i \left(\frac{u(x)}{\delta(x)^{m-1}} \right) \right\|_{L^1(\Omega)} \leq C\|u\|_{W^{m,1}(\Omega)}.$$

We omit the proof, because it is almost a line by line copy of the proof of the estimate (12) in Section 3 using the partition of unity. We are now ready to prove Theorem 1.

Proof Theorem 1. For any fixed integer $m \geq 3$, just as in the case $m = 2$, it is enough to prove the estimate (2) for $u \in C_0^\infty(\Omega)$. Notice that since

$$\|\partial^j u\|_{W^{m-j,1}(\Omega)} \leq \|u\|_{W^{m,1}(\Omega)} \qquad \text{for all } 0 \leq j \leq m,$$

it is enough to show

$$\left\| \partial^k \left(\frac{u(x)}{d(x)^{m-k}} \right) \right\|_{L^1(\Omega)} \leq C\|u\|_{W^{m,1}(\Omega)} \tag{14}$$

for $u \in C_0^\infty(\Omega)$ and $1 \leq k \leq m-1$. We proceed by induction on k. The case $k = 1$ corresponds exactly to Lemma 4.3. If one assumes the result for k, then we have to estimate, for $i = 1, \ldots, N$,

$$\partial_i \partial^k \left(\frac{u(x)}{d(x)^{m-k-1}} \right) = \partial^k \left(\frac{\partial_i u(x)}{d(x)^{m-k-1}} \right) - (m-k-1)\partial^k \left(\frac{u(x)\partial_i d(x)}{d(x)^{m-k}} \right).$$

Using the induction hypothesis for $\tilde{m} = m - 1$ yields

$$\left\| \partial^k \left(\frac{\partial_i u(x)}{d(x)^{(m-1)-k}} \right) \right\|_{L^1(\Omega)} \leq C\|\partial_i u\|_{W^{m-1,1}(\Omega)} \leq C\|u\|_{W^{m,1}(\Omega)};$$

on the other hand, by using the induction hypothesis and the fact that d is smooth in $\overline{\Omega}$, we obtain

$$\left\| \partial^k \left(\frac{u(x)\partial_i d(x)}{d(x)^{m-k}} \right) \right\|_{L^1(\Omega)} \leq C\|u\partial_i d\|_{W^{m,1}(\Omega)} \leq C\|u\|_{W^{m,1}(\Omega)}.$$

Therefore

$$\left\|\partial_i \partial^k\left(\frac{u(x)}{d(x)^{m-k-1}}\right)\right\|_{L^1(\Omega)} \leq C\|u\|_{W^{m,1}(\Omega)},$$

thus concluding the proof. □

Acknowledgments. We would like to thank M. Marcus for suggesting the use of flow coordinates. Also we thank H. Brezis for his valuable suggestions on the elaboration of this article.

J. D. was partially supported by CAPDE-Anillo ACT-125 and Fondo Basal CMM.

H. W. was supported by the European Commission under the Initial Training Network-FIRST, agreement No. PITN-GA-2009-238702.

References

[1] Adams, R. A., Fournier, J. J. F.: Sobolev Spaces. 2nd ed., Pure Appl. Math. 140, Elsevier/Academic Press, Amsterdam (2003) Zbl 1098.46001 MR 2424078

[2] Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York (2011) Zbl 1220.46002 MR 2759829

[3] Brezis, H., Marcus, M.: Hardy's inequalities revisited. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25**, 217–237 (1997) Zbl 1011.46027 MR 1655516

[4] Castro, H., Wang, H.: A Hardy type inequality for $W^{m,1}(0,1)$ functions. Calc. Var. Partial Differential Equations **39**, 525–531 (2010) Zbl 1204.26024 MR 2729310

[5] Gilbarg, D., Trudinger, N. S.: Elliptic Partial Differential Equations of Second Order. Classics Math., Springer, Berlin (2001) (reprint of the 1998 edition) Zbl 1042.35002 MR 1814364

[6] Marcus, M., Véron, L.: Removable singularities and boundary traces. J. Math. Pures Appl. (9) **80**, 879–900 (2001) Zbl 1134.35365 MR 1865379