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Dynamics of one-resonant biholomorphisms

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Abstract. Our first main result is a construction of a simple formal normal form for holomorphic diffeomorphisms in $\mathbb{C}^n$ whose differentials have a one-dimensional family of resonances in the first m eigenvalues, $m \le n$ (but more resonances are allowed for other eigenvalues). Next, we provide invariants and give conditions for the existence of basins of attraction. Finally, we give applications and examples demonstrating the sharpness of our conditions.

Keywords. Discrete dynamics, resonances, normal forms, basins of attraction

1. Introduction

Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing the origin 0 with diagonalizable differential. The dynamical behavior of the sequence of iterates $\{F^{\circ q}\}_{q\in\mathbb{N}}$ of F in a neighborhood of 0 is determined at the first order by the dynamics of its differential dF_0. In fact, depending on the eigenvalues $\lambda_1, \dots, \lambda_n$ of dF_0, in some cases both dynamics are the same.

In the hyperbolic case (namely when none of the eigenvalues is of modulus 1) the map is topologically conjugate to its differential (by the Hartman–Grobman theorem [19], [13], [14]) and the dynamics is clear. Moreover, if the eigenvalues have modulus either all strictly smaller than one or all strictly greater than one, then the origin is an attracting or respectively repelling fixed point for an open neighborhood of 0. Also, by the stable/unstable manifold theorem, there exists a holomorphic (germ of) manifold invariant under F and tangent to the sum of the eigenspaces of those λ_j's such that $|\lambda_j| < 1$ (resp. $|\lambda_j| > 1$) which is attracted to (resp. repelled from) 0. However, already in the case when all eigenvalues have modulus different from 1, holomorphic linearization is not always possible due to the presence of resonances among the eigenvalues (see, for instance, [4, Chapter IV]).

The case where some eigenvalue has modulus 1 is the most "chaotic" and interesting, since it presents a plethora of possible scenarios. For instance, if those eigenvalues of

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modulus 1 are not roots of unity and satisfy some Bryuno-type conditions, then there exist Siegel-type invariant submanifolds (see [21], [32]) on which the map is (holomorphically) linearizable. If the map is tangent to the identity, it has been proved by Écalle [11] and Hakim [18] that generically there exist "petals", also called "parabolic curves", namely, one-dimensional F-invariant analytic discs having the origin in their boundary and on which the dynamics is of parabolic type. Later, Abate [1] (see also [3]) proved that such petals always exist in dimension two.

On the other hand, Hakim [17] (basing on the previous work by Fatou [12] and Ueda [29], [30] in $\mathbb{C}^2$, see also Takano [27]) studied the so-called *semi-attractive* case, with one eigenvalue equal to 1 and the others of modulus less than 1. She proved that either there exists a curve of fixed points, or there exist attracting open petals. That result was later generalized by Rivi [23].

The *quasi-parabolic* case of a germ in $\mathbb{C}^2$, i.e. having one eigenvalue 1 and the other of modulus 1, but not a root of unity, has been studied in [7] and it has been proved that, under a certain generic hypothesis called "dynamical separation", there exist petals tangent to the eigenspace of 1. That result has been generalized to higher dimensions by Rong [24], [25]. We refer the reader to the survey papers [2] and [5] for a more detailed review of existing results.

In the case of diffeomorphisms with *unipotent* linear part, it was shown by Takens [28] (see also [16, Chapter 1]) that such a diffeomorphism can be embedded in the flow of a formal vector field. Therefore, in this case the dynamics of the diffeomorphism, at least at the formal level, is related to that of a (formal) associated vector field. For instance, using the Camacho–Sad theorem on the existence of separatrices for vector fields [9], Brochero, Cano and Hernanz [8] gave another proof of Abate's theorem. On the other hand, when the linear part of the diffeomorphism is not unipotent, the authors are not aware of any general result about embedding such a diffeomorphism into the flow of a formal vector field. In fact, one encounters somewhat unexpected differences between the dynamics of diffeomorphisms and that of vector fields (see Raissy [22]).

The aim of the present paper is to study the normal forms and the dynamics of germs of holomorphic diffeomorphisms having a one-dimensional family of resonances among only certain eigenvalues (that we call here *partially one-resonant* diffeomorphisms). It should be mentioned that (fully) one-resonant vector fields have been studied by Stolovitch [26], who also obtained a normal form for vector fields up to multiplication by a unit. In the case of diffeomorphisms considered here, there is no natural analogue of multiplying by a unit and thus we are led to seek a normal form for the original diffeomorphism only under conjugations.

More precisely, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the linear part of a biholomorphic diffeomorphism germ F at 0. We say that F is *one-resonant with respect to the first m eigenvalues* $\{\lambda_1, \dots, \lambda_m\}$ $(1 \le m \le n)$ (or *partially one-resonant*) if there exists a fixed multi-index $\alpha = (\alpha_1, \dots, \alpha_m, 0, \dots, 0) \neq 0 \in \mathbb{N}^n$ such that for $s \le m$, the resonances $\lambda_s = \prod_{j=1}^n \lambda_j^{\beta_j}$ are precisely of the form $\lambda_s = \lambda_s \prod_{j=1}^m \lambda_j^{k\alpha_j}$, where $k \ge 1 \in \mathbb{N}$ is arbitrary. We stress that, since arbitrary resonances are allowed for $s > m$, such a condition is much weaker (see Example 2.4) than the one-resonance condition normally found in the literature corresponding here to the case $m = n$ (see e.g. [15, 26]). The main advantage of the new notion of partial one-resonance is that it can be applied to the subset of all eigenvalues of modulus 1; it is natural to treat it differently from the rest of the eigenvalues.

In the case of partial one-resonance, the classical Poincaré–Dulac theory implies that, whenever F is not formally linearizable in the first m components, it is formally conjugate to a map whose first m components are of the form $\lambda_j z_j + a_j z^{\alpha k} z_j + R_j(z)$, $j = 1, \dots, m$, where, the number $k \in \mathbb{N}$ is an invariant, called the *order of F with respect to* $\{\lambda_1, \dots, \lambda_m\}$, the vector $(a_1, \dots, a_m) \neq 0$ is invariant up to a scalar multiple and the R_j's contain only resonant terms of higher degree. The number

$$\Lambda = \Lambda(F) := \sum_{j=1}^{m} \frac{a_j \alpha_j}{\lambda_j}$$

is an invariant up to a scalar multiple, and the map F is said to be *non-degenerate* provided $\Lambda \neq 0$.

We show that (partially) one-resonant non-degenerate diffeomorphisms have a simple formal *normal form* (see Theorem 3.6) in which the first m components are of the form

$$\lambda_j z_j + a_j z^{k\alpha} z_j + \mu \alpha_j \overline{\lambda_j}^{-1} z^{2k\alpha} z_j, \qquad j = 1, \dots, m.$$

Although none of the eigenvalues λ_j, $j = 1, \dots, m$, can be a root of unity, such a normal form is the exact analogue of the formal normal form for parabolic germs in $\mathbb{C}$. In fact, a one-resonant germ acts as a parabolic germ on the space of leaves of the formal invariant foliation $\{z^\alpha = \text{const}\}$ and that is the reason for this parabolic-like behavior.

Let F be a one-resonant non-degenerate diffeomorphism with respect to the eigenvalues $\{\lambda_1, \dots, \lambda_m\}$. We say that F is *parabolically attracting* with respect to $\{\lambda_1, \dots, \lambda_m\}$ if

$$|\lambda_j| = 1, \qquad \mathsf{Re}(a_j {\lambda_j}^{-1} \Lambda^{-1}) > 0, \qquad j = 1, \dots, m.$$

Again, such a condition is invariant, and its inequality part is vacuous in dimension 1 or whenever $m = 1$ and $|\lambda_1| = 1$ since in that case $\alpha = (\alpha_1, 0, \dots, 0)$ with $\alpha_1 > 0$. Our main result is the following:

Theorem 1.1. *Let F be a holomorphic diffeomorphism germ at* 0 *that is one-resonant, non-degenerate and parabolically attracting with respect to* $\{\lambda_1, \dots, \lambda_m\}$. *Suppose that* $|\lambda_j| < 1$ *for $j > m$. Let $k \in \mathbb{N}$ be the order of F with respect to* $\{\lambda_1, \dots, \lambda_m\}$. *Then F has k disjoint basins of attraction having* 0 *on the boundary.*

The different basins of attraction for F (that may or may not be connected) project via the map $z \mapsto u = z^\alpha$ onto different petals of the germ $u \mapsto u + \Lambda(F) u^{k+1} + o(|u|^{k+1})$.

Theorem 1.1 has many consequences. For instance, we recover a result of Hakim (see Corollary 6.1) since non-formally linearizable semi-attractive germs are always one-resonant, non-degenerate and parabolically attracting. Also, we apply our machinery to the case of quasi-parabolic germs, providing "fat petals" in the quasi-parabolic dynamically separating and attracting cases (see Subsection 6.2). Another area of application of Theorem 1.1 concerns elliptic germs which, in dimension greater than 1, might exhibit some, maybe unexpected, parabolic-like behavior (see Subsection 6.3). Finally, we present examples of one-resonant degenerate as well as non-degenerate but not parabolically attracting germs which have no basins of attraction at 0, demonstrating sharpness of the assumptions of Theorem 1.1 (see Subsections 6.4 and 6.5).

The outline of the paper is as follows. In Section 2 we briefly recall the one-dimensional theory of parabolic germs and define one-resonant germs in higher dimensions. In Section 3 we construct a formal normal form for non-degenerate partially one-resonant germs. In Section 4 we study the dynamics of normal forms, as a motivation for the subsequent Section 5, where we give the proof of Theorem 1.1. Finally, in Section 6 we apply our theory to the semi-attractive case, quasi-parabolic case, elliptic case and provide examples of diffeomorphisms with no basins of attraction.

2. One-resonant diffeomorphisms

2.1. Preliminaries on germs tangent to the identity in $\mathbb{C}$

For the material of this subsection, see e.g. [10]. Let

$$h(u) := u + Au^{k+1} + O(|u|^{k+2}) \tag{2.1}$$

for some $A \neq 0$ and $k \geq 1$, be a germ at 0 of a holomorphic self-mapping of $\mathbb{C}$.

The *attracting directions* $\{v_1, \dots, v_k\}$ for h are given by the k-th roots of $-|A|/A$. These are precisely the directions v such that the vector Av^{k+1} points in the direction opposite to v. An *attracting petal* P for h is a simply-connected domain such that $0 \in \partial P$, $h(P) \subseteq P$ and $\lim_{m\to\infty} h^{\circ m}(z) = 0$ for all $z \in P$, where $h^{\circ m}$ denotes the mth iterate of h.

We state here (a part of) the Leau–Fatou flower theorem. We write $a \sim b$ whenever there exist constants $0 < c < C$ such that $ca \leq b \leq Ca$.

Theorem 2.1 (Leau–Fatou). *Let $h(u)$ be as in (2.1) and v an attracting direction for h at 0. Then there exists an attracting petal P for h (said to be* centered at v*) such that for each $z \in P$ the following hold:*

(1) $h^{\circ m}(z) \neq 0$ *for all m and* $\lim_{m\to\infty} h^{\circ m}(z)/|h^{\circ m}(z)| = v$,
(2) $|h^{\circ m}(z)|^k \sim 1/m$.

Moreover, the petals centered at the attracting direction v can be chosen to be connected components of the set

$$\{z \in \mathbb{C} : |Az^k + \delta| < \delta\},$$

where $0 < \delta \ll 1$.

By property (1), petals centered at different attracting directions must be disjoint.

Remark 2.2. Property (1) of Theorem 2.1 is a part of the standard statement of the Leau–Fatou theorem (see, *e.g.*, [2] or [6]). Property (2) follows from the construction of the so-called Leau–Fatou coordinate. We sketch it briefly here for the reader's convenience. Up to a dilation one can assume $A = -1/k$ and $v = 1$. Let $H := \{w \in \mathbb{C} : \mathrm{Re}\, w > 0,\ |w| > C\}$ and $\Psi(w) := w^{-1/k}$ for $w \in H$ with the k-th root chosen so that $1^{1/k} = 1$. By the Leau–Fatou construction (see, e.g., [6, pp. 19–22]), if $C > 0$ is sufficiently large then the set $P := \Psi(H)$ is h-invariant and the map $\varphi := \Psi^{-1} \circ h \circ \Psi \colon H \to H$ satisfies

$$\varphi(w) = w + 1 + O(|w|^{-1}), \quad w \in H.$$

From this both (1) and (2) follow easily.

2.2. *Partially one-resonant germs*

Let $\mathrm{Diff}(\mathbb{C}^n; 0)$ denote the space of germs of holomorphic diffeomorphisms of $\mathbb{C}^n$ fixing 0. Write $\mathbb{N} = \{0, 1, \ldots\}$. Given a set $\{\lambda_1, \ldots, \lambda_n\}$ of complex numbers, recall that a *resonance* is a pair (j, l), where $j \in \{1, \ldots, n\}$ and $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$ is a multi-index with $|l| \geq 2$ such that $\lambda_j = \lambda^l$ (where $\lambda^l := \lambda_1^{l_1} \cdots \lambda_n^{l_n}$).

In all the rest of the paper, and without mentioning it explicitly, we shall consider only germs of diffeomorphisms whose differential is diagonal.

Definition 2.3. For $F \in \mathrm{Diff}(\mathbb{C}^n; 0)$, assume that the differential dF_0 has eigenvalues $\lambda_1, \ldots, \lambda_n$. We say that F is *one-resonant with respect to the first m eigenvalues* $\{\lambda_1, \ldots, \lambda_m\}$ $(1 \leq m \leq n)$ if there exists a fixed multi-index $\alpha = (\alpha_1, \ldots, \alpha_m, 0, \ldots, 0) \neq 0 \in \mathbb{N}^n$ such that the resonances (j, l) with $j \in \{1, \ldots, m\}$ are precisely of the form $(j, \alpha k + e_j)$, where $e_j \in \mathbb{N}^n$ is the unit vector with 1 at the jth place and 0 elsewhere and where $k \in \mathbb{N} \setminus \{0\}$ is arbitrary. (In particular, it follows that the relation $\lambda_1^{\alpha_1} \cdots \lambda_m^{\alpha_m} = 1$ holds and generates all other relations $\lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n} = 1$ with $\beta_s \geq 0$ for all s.) The multi-index α is called the *index of resonance*. If F is one-resonant with respect to $\{\lambda_1, \ldots, \lambda_n\}$ (i.e. $m = n$) we simply say that F is one-resonant.

The notion of one-resonance for $m = n$ has been known in the literature (see e.g. [15, 26]). However, its generalization for $m < n$ given here seems to be new. The following class of examples illustrates the difference.

Example 2.4. Let $F \in \mathrm{Diff}(\mathbb{C}^3; 0)$ be any diffeomorphism with eigenvalues λ, μ, ν of dF_0 such that λ is a root of unity, $|\mu| < 1$ and $\nu = \mu^s$ for some natural number $s \geq 1$. Then F is one-resonant with respect to λ but has resonances of the form $(3, se_2)$, showing that it is not one-resonant with respect to all the eigenvalues.

Remark 2.5. It follows directly from the definition that, if F is one-resonant with respect to $\{\lambda_1, \ldots, \lambda_m\}$, then $\lambda_j \neq \lambda_s$ for any $j \in \{1, \ldots, m\}$ and $s \in \{1, \ldots, n\}$ with $j \neq s$. Indeed, otherwise one would have resonances of type $(j, k\alpha + e_s)$, which are not of the required form $(j, k\alpha + e_j)$.

Example 2.6. The same diffeomorphism can be considered one-resonant with respect to different groups of eigenvalues. For instance, consider $F(z,w) = (z + z^3, e^{2\pi i\theta}w + zw)$, where θ is irrational. Then F is one-resonant with respect to $\lambda_1 = 1$ with index of resonance $(1,0)$. But F is also one-resonant (with respect to $\{\lambda_1, \lambda_2\} = \{1, e^{2\pi i\theta}\}$ with the same index of resonance $(1,0)$). (Note that the higher order terms of F play no role here but will be used later in Example 3.4.)

As the previous example shows, there may exist "non-maximal" sets of "one-resonant eigenvalues". However, it is easy to see from the definition that any set of "one-resonant eigenvalues" is contained in the unique maximal set and contains the unique minimal set. Namely, let F be one-resonant with respect to $\{\lambda_1, \dots, \lambda_m\}$ and assume that the index of resonance is $\alpha = (\alpha_1, \dots, \alpha_m, 0, \dots, 0)$. Since the relation $\lambda_1^{\alpha_1} \cdots \lambda_m^{\alpha_m} = 1$ holds and generates all other relations $\lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n} = 1$ with $\beta_s \geq 0$ for all s, it follows that any other resonant set of eigenvalues corresponds to the same index α. Then it follows directly from the definition that every set of one-resonant eigenvalues contains the minimal set L of all λ_j with $\alpha_j \neq 0$ and the set L itself is one-resonant. On the other hand, let $\widetilde{L}$ be the set of all λ_j such that any resonance (j,l) is of the required form $(j, k\alpha + e_j)$. Then $\widetilde{L}$ is the maximal one-resonant set that contains any other one-resonant set of eigenvalues.

The choice of the set of eigenvalues with respect to which the map is considered one-resonant depends on the problem one is facing; in our main result Theorem 1.1 it is natural to consider one-resonance with respect to the set of all eigenvalues of modulus 1.

3. Normal form for non-degenerate one-resonant diffeomorphisms

Let $F \in \mathrm{Diff}(\mathbb{C}^n; 0)$ be one-resonant with respect to $\{\lambda_1, \dots, \lambda_m\}$ with index of resonance α. Using Poincaré–Dulac theory (see, e.g., [4, Chapter IV]), one can formally conjugate F to a germ $G = (G_1, \dots, G_n)$ such that

$$G_j(z) = \lambda_j z_j + a_j z^{\alpha k} z_j + R_j(z), \qquad j = 1, \dots, m, \tag{3.1}$$

where either $a = (a_1, \dots, a_m) \neq 0$ and $R_j(z)$ contains only resonant monomials $a_{js} z^{\alpha s} z_j$ with $s > k$, or $a_j = 0$ and $R_j \equiv 0$ for all $j = 1, \dots, m$. Note that the second case occurs precisely when F is *formally linearizable* in the first m variables.

Definition 3.1. Let $F \in \mathrm{Diff}(\mathbb{C}^n; 0)$ be one-resonant with respect to $\{\lambda_1, \dots, \lambda_m\}$ such that

$$F_j(z) = \lambda_j z_j + a_j z^{\alpha k} z_j + O(|z|^{|\alpha|k+2}), \qquad j = 1, \dots, m, \tag{3.2}$$

with $k \geq 1$ and $a = (a_1, \dots, a_m) \neq 0$, where α is the index of resonance. Set

$$\Lambda = \Lambda(F) := \sum_{j=1}^{m} \frac{a_j \alpha_j}{\lambda_j}. \tag{3.3}$$

We say that F is *non-degenerate* if $\Lambda \neq 0$.

Remark 3.2. The integer k in (3.2) is invariant under conjugations preserving the form (3.2), and the vector $a = (a_1, \dots, a_m)$ is invariant up to multiplication by a scalar. In particular, the non-degeneracy condition given by Definition 3.1 is invariant. Indeed, if conjugation with a map $\psi = (\psi_1, \dots, \psi_n) \in \mathrm{Diff}(\mathbb{C}^n; 0)$ preserves the form (3.2) (possibly changing a), then $\psi_j(z) = b_j z_j + O(|z|^2)$, $b_j \in \mathbb{C}^*$, for any $j = 1, \dots, m$, in view of Remark 2.5. Conjugating with the linear part of ψ, we see that for any such j, a_j is replaced by $a_j b^{\alpha k}$. Assume now that $\psi(z) = z + O(|z|^2)$. Then by Poincaré–Dulac theory, since ψ preserves (3.2), all terms of order less than $|\alpha|k + 2$ that ψ has in its first m components must be resonant, and therefore a is invariant.

Definition 3.3. We call the invariant number k the *order* of F with respect to $\lambda_1, \dots, \lambda_m$.

Example 3.4. Let F be the germ in Example 2.6. Then F is non-degenerate when regarded as a one-resonant germ with respect to the eigenvalue 1 (with $k = 2$ and $a = a_1 = 1$). But it becomes degenerate when regarded as a one-resonant germ (with respect to both eigenvalues $\{1, e^{2\pi i\theta}\}$), because in that case $a = (a_1, a_2) = (0, 1)$ and the index of resonance is $(1, 0)$, thus $\Lambda(F) = 0$. The main reason here is the change of the order k.

Note that, more generally, for a germ of the form $(z + \cdots, e^{2\pi i\theta} w + \cdots)$ with θ irrational, the condition of being non-degenerate with respect to $\{1, e^{2\pi i\theta}\}$ is equivalent to F being dynamically separating in the terminology of [7] (see Subsection 6.2).

As illustrated by the latter example, if one passes from a smaller set of one-resonant eigenvalues to a larger one, the order k may drop, in which case the corresponding non-degeneracy conditions are not related, i.e. F can be non-degenerate with respect to the smaller set but not the larger one or vice versa. On the other hand, if the order k is the same for both sets, since both sets contain the set of all λ_j with $\alpha_j \neq 0$, the (non-)degeneracies with respect to the smaller and larger sets are clearly equivalent.

Remark 3.5. If F is one-resonant with respect to $\{\lambda_1\}$, then λ_1 is a root of unity. Moreover, in this case F is non-degenerate if and only if it is not formally linearizable in the first component.

We have the following *normal form* for non-degenerate partially one-resonant diffeomorphisms.

Theorem 3.6. *Let* $\in \mathrm{Diff}(\mathbb{C}^n; 0)$ *be one-resonant and non-degenerate with respect to* $\lambda_1, \dots, \lambda_m$ *with index of resonance* α. *Then there exist* $k \in \mathbb{N}$ *and numbers* $\mu, a_1, \dots, a_m \in \mathbb{C}$ *such that* F *is formally conjugate to the map* $\hat{F}(z) = (\hat{F}_1(z), \dots, \hat{F}_n(z))$, *where*

$$\hat{F}_j(z) = \lambda_j z_j + a_j z^{k\alpha} z_j + \mu \alpha_j \overline{\lambda}_j^{-1} z^{2k\alpha} z_j, \qquad j = 1, \dots, m, \tag{3.4}$$

and the components $\hat{F}_j(z)$ *for* $j = m+1, \dots, n$ *contain only resonant monomials.*

Proof. By Poincaré–Dulac theory, we may assume that $F_j(z)$ for $j = m+1, \dots, n$ contain only resonant monomials and

$$F_j(z) = \lambda_j z_j + a_j z^{k\alpha} z_j + \sum_{l \geq 1} a_{jl} z^{(l+k)\alpha} z_j, \qquad j = 1, \dots, m. \tag{3.5}$$

With the notation $F' := (F_1, \dots, F_m)$, $\lambda' := \mathrm{diag}(\lambda_1, \dots, \lambda_m)$, $z' := (z_1, \dots, z_m)$, we can rewrite (3.5) in the more compact form

$$F'(z) = \lambda' z' + z^{k\alpha} \sum_j a_j z_j e_j + \sum_{k'>k} z^{k'\alpha} \sum_j a_{k'j} z_j e_j, \tag{3.6}$$

where the summation over j is from 1 to m and e_j is the standard unit vector.

We now study the conjugation $\widetilde{F} = \Theta \circ F \circ \Theta^{-1}$ under the map

$$\Theta(z) = z + \theta(z), \quad \theta(z) = \Big(z^{l\alpha} \sum_j b_j z_j e_j, 0\Big) = (b_1 z^{l\alpha} z_1, \dots, b_m z^{l\alpha} z_m, 0, \dots, 0), \tag{3.7}$$

for an integer $l \geq 1$ and a vector $b = (b_1, \dots, b_m) \in \mathbb{C}^m$. We also use the notation

$$F(z) = \lambda z + f(z), \quad \widetilde{F}(z) = \lambda z + \widetilde{f}(z), \quad f, \widetilde{f} = O(|z|^2),$$

and the Taylor expansions

$$\widetilde{f}(z+h) = \widetilde{f}(z) + \sum_{r\geq 1} \frac{1}{r!} \widetilde{f}^{(r)}(z)(h), \quad \theta(z+h) = \theta(z) + \sum_{r \geq 1} \frac{1}{r!} \theta^{(r)}(z)(h), \tag{3.8}$$

where the derivatives $\widetilde{f}^{(r)}(z)(h)$ and $\theta^{(r)}(z)(h)$ are regarded as n-tuples of homogeneous polynomials of degree r in h. We use (3.8) to rewrite the identity

$$\widetilde{F}(\Theta(z)) = \Theta(F(z)) \tag{3.9}$$

as

$$\widetilde{f}(z) + \lambda\theta(z) + \sum_{r\geq 1} \frac{1}{r!} \widetilde{f}^{(r)}(z)(\theta(z)) = \theta(\lambda z) + f(z) + \sum_{r\geq 1} \frac{1}{r!} \theta^{(r)}(\lambda z)(f(z)). \tag{3.10}$$

In view of the resonance relations, we have $\lambda\theta(z) = \theta(\lambda z)$ and hence (3.10) is equivalent to

$$\widetilde{f}(z) - f(z) = \sum_{r\geq 1} \frac{1}{r!} \big(\theta^{(r)}(\lambda z)(f(z)) - \widetilde{f}^{(r)}(z)(\theta(z))\big). \tag{3.11}$$

Now identifying terms of order up to $k|\alpha| + 1$ in (3.11), we conclude by induction on the order that

$$\widetilde{f}'(z) = f'(z) + O(|z|^{k|\alpha|+2}) = z^{k\alpha} \sum_j a_j z_j e_j + O(|z|^{k|\alpha|+2}), \tag{3.12}$$

where $\widetilde{f}' = (\widetilde{f}_1, \dots, \widetilde{f}_m)$. Next, identifying terms of order up to $(k+l)|\alpha| + 1$, we obtain

$$\widetilde{f}'(z) - f'(z) = \theta'^{(1)}(\lambda z)(f(z)) - \widetilde{f}'^{(1)}(z)(\theta(z)) + O(|z|^{(k+l)|\alpha|+2}).$$

Substituting f', $\widetilde{f}'$ from (3.12) and θ from (3.7), we find

$$\widetilde{f}'(z) - f'(z) = z^{(k+l)\alpha} \sum_{j,s} a_j b_s \big((l\alpha_j \lambda^{l\alpha - e_j + e_s} + \delta_{js}\lambda^{l\alpha}) z_s e_s - (k\alpha_s + \delta_{js}) z_j e_j \big) + O(|z|^{(k+l)|\alpha|+2}). \tag{3.13}$$

By the resonance conditions, $\lambda^{l\alpha} = 1$. In particular, the terms with δ_{js} cancel each other and we obtain

$$\begin{aligned} \widetilde{f}'(z) - f'(z) &= z^{(k+l)\alpha} \sum_{j,s} a_j b_s (l\alpha_j \lambda^{e_s - e_j} z_s e_s - k\alpha_s z_j e_j) + O(|z|^{(k+l)|\alpha|+2}) \\ &= z^{(k+l)\alpha} bAZ + O(|z|^{(k+l)|\alpha|+2}), \end{aligned} \tag{3.14}$$

where $b = (b_1, \dots, b_m)$, Z is the diagonal matrix with entries $z_1, \dots, z_m$, and A is the $m \times m$ matrix given by

$$A = l(aL^{-1}\alpha^t)L - k\alpha^t a.$$

Here α^t is the transpose of α and L is the diagonal matrix with entries $\lambda_1, \dots, \lambda_m$. Note that the expression in parentheses is a scalar. Then

$$A = CL, \qquad C = l(aL^{-1}\alpha^t)\,\mathrm{id} - k\alpha^t aL^{-1}.$$

Since the matrix $k\alpha^t aL^{-1}$ is of rank one, it has at most one non-zero eigenvalue equal to its trace $kaL^{-1}\alpha^t$. By our non-degeneracy assumption, this trace is actually different from zero. The first matrix in the expression of C is scalar with all its diagonal entries equal to $laL^{-1}\alpha^t$. Since $aL^{-1}\alpha^t \neq 0$, we conclude that C is invertible if and only if $l \neq k$. Since f has the form (3.5), given any $l \neq k$, it follows from (3.14) that there exists a (unique) vector b such that $\widetilde{f}' = f' + O(|z|^{(k+l)|\alpha|+1})$ and the terms of $\widetilde{f}'$ of order $(k+l)|\alpha| + 1$ all vanish.

On the other hand, in case $l = k$, C has rank $m - 1$. In this case we use the identity

$$bC\alpha^t = kb(aL^{-1}\alpha^t)\alpha^t - kb(\alpha^t aL^{-1})\alpha^t = k(aL^{-1}\alpha^t)b\alpha^t - kb\alpha^t(aL^{-1}\alpha^t) = 0,$$

where we have used that both b and $b\alpha^t$ commute with the scalar $aL^{-1}\alpha^t$. Hence α^t annihilates the image of the map $b \mapsto bC$. Since C has rank $m - 1$, its image is precisely the orthogonal complement of α (with respect to the standard hermitian scalar product on $\mathbb{C}^m$, note that $\overline{\alpha} = \alpha$). Then the image of the map $b \mapsto bA$ is precisely the orthogonal complement of $\alpha\overline{L}^{-1} = (\alpha_1\overline{\lambda}_1^{-1}, \dots, \alpha_m\overline{\lambda}_m^{-1})$. It now follows from (3.14) that, choosing a suitable b, we can arrange that the term of $\widetilde{f}'$ of order $2k|\alpha| + 1$ equals

$$z^{2k\alpha}\mu\alpha\overline{L}^{-1}Z = z^{2k\alpha}\mu \sum_j \alpha_j \overline{\lambda}_j^{-1} z_j e_j \tag{3.15}$$

for some $\mu \in \mathbb{C}$.

We now apply inductively the above procedure for each $l \geq 1$, either to eliminate the corresponding term in (3.5) or normalize it as in (3.15), by conjugating with a suitable map (3.7) for that l. At each step we may create non-resonant terms whose order must be greater than $l|\alpha| + 1$ in view of (3.10). Those terms can be eliminated inductively according to Poincaré–Dulac theory by conjugation with further maps $\Theta(z) = z + \theta(z)$ with $\theta(z)$ being suitable monomials of order greater than $l|\alpha| + 1$. Again using (3.10) we see that those additional conjugations do not affect the normalized terms of order $l|\alpha|+1$. Thus by induction on l, we obtain the desired normalization (3.4). □

Remark 3.7. It is clear from the proof of Theorem 3.6 that for any given $t \in \mathbb{N}$ there exists a holomorphic (polynomial) change of coordinates which transforms F into $\hat{F} + O(t)$, where $\hat{F}$ satisfies (3.4) and $O(t)$ denotes a function vanishing of order $\geq t$ at 0.

4. Dynamics of normal forms

Motivated by Theorem 3.6, we shall first study the dynamics of a one-resonant diffeomorphism $G \in \mathsf{Diff}(\mathbb{C}^n; 0)$ of the form $G(z) = (G_1(z), \dots, G_n(z))$ with

$$G_j(z) = \lambda_j z_j + a_j z^{k\alpha} z_j + b_j z^{2k\alpha} z_j, \qquad j = 1, \dots, n, \tag{4.1}$$

and $\Lambda = \Lambda(G) \neq 0$, where $\Lambda(G)$ is as in Definition 3.1. We consider the singular foliation $\mathcal{F}$ of $\mathbb{C}^n$ given by $\{z^\alpha = \text{const}\}$.

Lemma 4.1. *The foliation $\mathcal{F}$ is G-invariant.*

Proof. Let $\pi(z) = z^\alpha$. Then

$$\pi(G(z)) = z^\alpha \prod_{j=1}^{n} (\lambda_j + a_j z^{k\alpha} + b_j z^{2k\alpha})^{\alpha_j},$$

where the right-hand side is clearly a holomorphic function of z^α. Hence G maps leaves of $\mathcal{F}$ into (possibly different) leaves of $\mathcal{F}$ and the desired conclusion follows. □

Let $\mathcal{L}$ denote the space of leaves of $\mathcal{F}$. Let $\pi : \mathbb{C}^n \to \mathcal{L}$ be the projection given by $(z_1, \dots, z_n) \mapsto z^\alpha$. Clearly, $\mathcal{L} \simeq \mathbb{C}$. Let $u = z^\alpha = \pi(z)$. The action of G on $\mathcal{L}$ is given by

$$\Phi(u) := G_1(z)^{\alpha_1} \cdots G_n(z)^{\alpha_n} = u + \Lambda(G) u^{k+1} + O(|u|^{k+2}), \tag{4.2}$$

where we have used that $\lambda^\alpha = 1$. Note that $\Phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is locally biholomorphic. Let $v_1, \dots, v_k$ be the attracting directions for Φ, and $P_j \subset \mathbb{C}$, $j = 1, \dots, k$, attracting petals centered at v_j (see Section 2.1). Set

$$U_j := \pi^{-1}(P_j) \subset \mathbb{C}^n.$$

Since $\mathcal{F}$ is G-invariant, the domains U_j are also G-invariant.

Let $z \in U_j$. Then $\Phi^{\circ m}(\pi(z)) \to 0$ as $m \to \infty$. In order to understand the dynamics of G, it is then sufficient to understand the "motion" along the leaves of $\mathcal{F}$. As a matter of notation, let $p_j(z_1, \dots, z_n) = z_j$.

Proposition 4.2. *Let* $G \in \mathrm{Diff}(\mathbb{C}^n; 0)$ *be in the normal form* (4.1) *with* $\Lambda = \Lambda(G) \neq 0$. *Fix* $1 \le j \le n$ *and* $1 \le t \le k$.

(1) *If* $|\lambda_j| < 1$, *then* $\lim_{m\to\infty} p_j \circ G^{\circ m}(z) = 0$ *for all* $z \in U_t$.
(2) *If* $|\lambda_j| > 1$, *then* $\lim_{m\to\infty} p_j \circ G^{\circ m}(z) = \infty$ *for all* $z \in U_t$ *with* $z_j \neq 0$.
(3) *If* $|\lambda_j| = 1$ *and* $\mathrm{Re}(a_j \lambda_j^{-1} \Lambda^{-1}) > 0$, *then* $\lim_{m\to\infty} p_j \circ G^{\circ m}(z) = 0$ *for all* $z \in U_t$.
(4) *If* $|\lambda_j| = 1$ *and* $\mathrm{Re}(a_j \lambda_j{}^{-1} \Lambda^{-1}) < 0$, *then* $\lim_{m\to\infty} p_j \circ G^{\circ m}(z) = \infty$ *for all* $z \in U_t$ *with* $z_j \neq 0$.

Proof. Note that by construction $\Phi(\pi(z)) = \pi(G(z))$. We can write, for $j = 1, \dots, n$,

$$G_j(z) = (\lambda_j + a_j u^k + b_j u^{2k}) z_j$$

and, letting $u_l := \Phi^{\circ l}(u) = \pi(G^{\circ l}(z))$,

$$p_j \circ G^{\circ m}(z) = \lambda_j^m \prod_{l=1}^{m} \left(1 + \frac{a_j}{\lambda_j} u_l^k + \frac{b_j}{\lambda_j} u_l^{2k}\right) z_j.$$

We examine the asymptotical behavior of the infinite product

$$\prod_{l=1}^{\infty} \left(1 + \frac{a_j}{\lambda_j} u_l^k + \frac{b_j}{\lambda_j} u_l^{2k}\right). \tag{4.3}$$

Let $z \in U_t$, therefore $u = \pi(z) \in P_t$. By Theorem 2.1(2), it follows that $|u_l^k| = |u_l|^k \sim 1/l$.

Let $A_l := (a_j/\lambda_j) u_l^k + (b_j/\lambda_j) u_l^{2k}$. We examine the behavior of $\prod_{l=1}^m |1 + A_l|$. Taking the logarithm we have

$$\log\left(\prod_{l=1}^{m} |1 + A_l|\right) = \frac{1}{2} \sum_{l=1}^{m} \log(1 + |A_l|^2 + 2\,\mathrm{Re}\, A_l).$$

For $l \gg 1$, $\big||A_l|^2 + 2\,\mathrm{Re}\, A_l\big| \sim l^{-c}$ for some $c \in \mathbb{N}^*$. Hence, $\log(1 + |A_l|^2 + 2\,\mathrm{Re}\, A_l) \sim |A_l|^2 + 2\,\mathrm{Re}\, A_l$ since for $l \gg 1$,

$$\frac{1}{2} \sum_{l=1}^{\infty} |\log(1 + |A_l|^2 + 2\,\mathrm{Re}\, A_l)| \sim \frac{1}{2} \sum_{l=1}^{\infty} l^{-c}.$$

It follows that the infinite product (4.3) either converges or diverges much slower than $|\lambda_j|^m$ in case $|\lambda_j| \neq 1$. Thus (1) and (2) follow.

As for (3) and (4), we need a better estimate. By Theorem 2.1(1), it follows that $u_l^k/|u_l|^k \to v_t^k = -|\Lambda|\Lambda^{-1}$ as $l \to \infty$. Hence

$$\lim_{l\to\infty} \mathrm{Re}\left(\frac{a_j}{\lambda_j} \frac{u_l^k}{|u_l|^k}\right) = \mathrm{Re}\left(\frac{a_j}{\lambda_j} v_t^k\right) = -\,\mathrm{Re}\left(\frac{a_j}{\lambda_j} \frac{|\Lambda|}{\Lambda}\right).$$

Therefore in case (3), for l large, $|A_l|^2 + 2\,\mathrm{Re}\, A_l \sim -l^{-1}$. Hence

$$\log\left(\prod_{l=1}^{\infty} |1 + A_l|\right) = \frac{1}{2} \sum_{l=1}^{\infty} \log(1 + |A_l|^2 + 2\,\mathrm{Re}\, A_l) \sim \frac{1}{2} \sum_{l=1}^{m} \frac{-1}{l} = -\infty,$$

and thus (3) follows. Statement (4) is similar. □

5. Dynamics of non-degenerate one-resonant maps

Definition 5.1. Let $F \in \mathsf{Diff}(\mathbb{C}^n; 0)$ be one-resonant and non-degenerate with respect to $\{\lambda_1, \dots, \lambda_m\}$. Let $k \in \mathbb{N}$ be the order of F with respect to $\{\lambda_1, \dots, \lambda_m\}$ (see Definition 3.3). Choose coordinates such that (3.2) holds. We say that F is *parabolically attracting* with respect to $\{\lambda_1, \dots, \lambda_m\}$ if

$$|\lambda_j| = 1, \quad \mathsf{Re}(a_j \lambda_j^{-1} \Lambda^{-1}) > 0, \quad j = 1, \dots, m, \tag{5.1}$$

where $\Lambda = \Lambda(F)$ is given by (3.3).

Remark 5.2. The condition of being parabolically attracting is independent of the coordinates chosen. To see this, let ψ be a transformation which preserves (3.2), and let $\tilde{F} := \psi \circ F \circ \psi^{-1}$. In view of Remark 3.2, it suffices to check the invariance of (5.1) for ψ linear with $\psi_j(z) = b_j z_j$, $b_j \in \mathbb{C}^*$, for any $j = 1, \dots, m$. Then, a_j is replaced by $\tilde{a}_j := a_j b^{\alpha k}$ and $\Lambda(\tilde{F}) = \Lambda(F) b^{\alpha k}$, from which the claim follows.

Remark 5.3. If F is one-resonant and non-degenerate with respect to $\{\lambda_1\}$ (with $|\lambda_1| = 1$), then it is always parabolically attracting. Indeed, in that case, $\Lambda = a_1 \alpha_1 {\lambda_1}^{-1}$ and

$$\mathsf{Re}(a_1 \lambda_1^{-1} \Lambda^{-1}) = \alpha_1^{-1} > 0.$$

Definition 5.4. Let $F \in \mathsf{Diff}(\mathbb{C}^n; 0)$. A *basin of attraction for F at* 0 is a non-empty (not necessarily connected) open set $U \subset \mathbb{C}^n$ with $0 \in \overline{U}$ for which there exists a neighborhood basis $\{\Omega_j\}$ of 0 such that $F(U \cap \Omega_j) \subset U \cap \Omega_j$ and $F^{\circ m}(z) \to 0$ as $m \to \infty$ whenever $z \in U \cap \Omega_j$ for some j.

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Denote $u := z^\alpha$. In view of Theorem 3.6, up to biholomorphic conjugation we can assume that $F(z) = (F_1(z), \dots, F_n(z))$ with

$$F_j(z) = \begin{cases} (\lambda_j + a_j u^k + \mu \lambda_j \alpha_j u^{2k}) z_j + O(|z|^l), & j = 1, \dots, m, \\ \lambda_j z_j + O(|z|^2), & j = m+1, \dots, n, \end{cases} \tag{5.2}$$

for any fixed l to be chosen later. Also, acting with a dilation (cf. Remark 3.2) we can assume that $\Lambda = \Lambda(F) = -1/k$. Then, since F is parabolically attracting, we have

$$\mathsf{Re}(a_j \lambda_j^{-1}) < 0, \quad j = 1, \dots, m. \tag{5.3}$$

Let $R > 0$ be a number we will suitably choose later. Let

$$\Delta_R := \left\{ u \in \mathbb{C} : \left| u^k - \frac{1}{2R} \right| < \frac{1}{2R} \right\}.$$

Note that Δ_R has exactly k connected components corresponding to different branches of the k-th root. The desired basins of attraction will be constructed by means of the projection $z \mapsto z^\alpha$ over sectors contained in such connected components.

We first construct a basin of attraction based on a sector centered at the direction 1, namely,

$$S_R(\varepsilon) := \{u \in \Delta_R : |\mathrm{Arg}\, u| < \varepsilon\}, \tag{5.4}$$

for some small $\varepsilon > 0$ to be chosen later.

Let $\beta > 0$ be such that $\beta|\alpha| < 1$ and let

$$B := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < |u|^\beta,\ j = 1, \dots, m,\\ |(z_{m+1}, \dots, z_n)| < |u|^\beta,\ u := z^\alpha \in S_R(\varepsilon)\}.$$

First of all, $B \neq \emptyset$ and $0 \in \partial B$. Indeed, it is easy to see that $z_r = (r, \dots, r) \in B$ for $r > 0$ sufficiently small. Moreover, since the map $z \mapsto z^\alpha$ is open and 0 is not in the interior of $S_R(\varepsilon)$, it follows that $0 \notin B$, i.e. $0 \in \partial B$. Finally, the set B is obviously open.

Next, we prove that B is F-invariant. Let $z \in B$ and let $u := z^\alpha$. Let

$$\Phi(u, z) := F_1^{\alpha_1}(z) \cdots F_m^{\alpha_m}(z) = u - \frac{1}{k}u^{k+1} + h_1(u) + h_2(z),$$

where we consider Φ as a function of the variables $z, u = z^\alpha$ and where $h_1(u) = O(|u|^{k+2})$ and $h_2(z) = O(|z|^l)$. We make the change of coordinates $U = u^{-k}$ and write

$$\widetilde{\Phi}(U, z) := \Phi(U^{-1/k}, z)^{-k}$$

for the map Φ in the new coordinates. Note that $u \in S_R(\varepsilon)$ if and only if $U \in H_R(\varepsilon)$, where

$$H_R(\varepsilon) := \{w \in \mathbb{C} : \mathrm{Re}\, w > R,\ |\mathrm{Arg}\, w| < k\varepsilon\}.$$

Since $\mathrm{Re}(a_j \lambda_j^{-1}) < 0$, it is easy to see that, choosing R sufficiently large and ε sufficiently small, we obtain

$$\left|1 + \frac{a_j}{\lambda_j}\frac{1}{w} + \mu\alpha_j \frac{1}{w^2}\right| < 1 - \frac{c}{|w|} \tag{5.5}$$

for some $c > 0$ and for all $w \in H_R(\varepsilon)$.

Now fix $0 < \delta < 1/2$ such that

$$H_R(\varepsilon) + 1 + \tau \subset H_R(\varepsilon) \quad \text{whenever} \quad |\tau| < \delta, \tag{5.6}$$

Note that δ depends on ε but not on R. Fix $0 < c' < c$. By choosing $\beta < 1/2$ sufficiently small, we can assume that

$$\beta(\delta + 1) - c'k < 0 \tag{5.7}$$

and choose $l > 1$ such that

$$\beta l > k + 1. \tag{5.8}$$

After a direct computation we find

$$\widetilde{\Phi}(U, z) = U\left(\frac{1}{1 - \frac{1}{kU} + U^{1/k}h_1(U^{-1/k}) + U^{1/k}h_2(z)}\right)^k. \tag{5.9}$$

Since $|z| < n|u|^\beta$ in B, there exists $K > 0$ such that

$$|U|^{1/k}|h_1(U^{-1/k})| \leq K|U|^{1/k}|U|^{-(k+2)/k} = K|U|^{-1-1/k}$$

and

$$|U|^{1/k}|h_2(z)| \leq K|U|^{1/k}|u|^{\beta l} = K|U|^{(1-\beta l)/k}.$$

Therefore, if R is sufficiently large and $z \in B$ (hence $U \in H_R(\varepsilon)$), we have

$$\widetilde{\Phi}(U, z) = U + 1 + \nu(U, z) \quad \text{with } |\nu(U, z)| < \delta, \tag{5.10}$$

where we have used (5.8). In particular, $U_1 := \tilde{\Phi}(U, z) \in H_R(\varepsilon)$ in view of (5.6) and $\operatorname{Re} U_1 \geq \operatorname{Re} U + 1/2$. Therefore we have proved that

$$z \in B \ \Rightarrow \ u_1 := \Phi(u, z) \in S_R(\varepsilon). \tag{5.11}$$

Moreover, by the same token, setting by induction $u_{m+1} := \Phi(u_m, F^{\circ m}(z))$, it follows that

$$\lim_{m\to\infty} u_m = 0. \tag{5.12}$$

Now we examine the components F_j for $j = m + 1, \ldots, n$. Set $x := (z_1, \ldots, z_m)$ and $y := (z_{m+1}, \ldots, z_n)$. Then

$$y_1 = My + h(z)z,$$

where M is the $(n - m) \times (n - m)$ diagonal matrix with entries λ_j ($j = m + 1, \ldots, n$) and h is a holomorphic $(n - m) \times n$ matrix valued function in a neighborhood of 0 such that $h(0) = 0$. If $z \in B$, then $|y| < |u|^\beta$. Moreover, since $|\lambda_j| < 1$ for $j = m + 1, \ldots, n$, it follows that there exists $a < 1$ such that $|My| < a|y| < a|u|^\beta$. Also, let $0 < b < 1-a$. Then, for R sufficiently large, it follows that $|h(z)| \leq b/n$ if $z \in B$. Hence, letting $p = a + b < 1$, we obtain

$$|y_1| \leq |My| + |h(z)|\,|z| < a|u|^\beta + \frac{b}{n}n|u|^\beta = (a + b)|u|^\beta = p|u|^\beta. \tag{5.13}$$

Now, we claim that for R sufficiently large,

$$|u| \leq \frac{1}{p^{1/\beta}}|u_1|, \tag{5.14}$$

where $u_1 = \Phi(u, z)$. Indeed, (5.14) is equivalent to $|U_1| \leq p^{-k/\beta}|U|$ and hence to

$$\frac{|U + 1 + \nu(U, z)|}{|U|} \leq p^{-k/\beta}.$$

But the limit as $|U| \to \infty$ of the left-hand side is 1 and the right-hand side is > 1, thus (5.14) holds for R sufficiently large.

Hence, by (5.13) and (5.14) we obtain

$$|y_1| \leq |u_1|^\beta. \tag{5.15}$$

Now we examine the components F_j for $j = 1, \dots, m$. Let $z \in B$ and, as before, let $U = u^{-k} \in H_R(\varepsilon)$. By (5.2) and by (5.3) we have

$$F_j(z) = \lambda_j\left(1 + \frac{a_j}{\lambda_j}\frac{1}{U} + \mu\alpha_j\frac{1}{U^2}\right)z_j + R_l(z) \tag{5.16}$$

with $R_l(z) = O(|z|^l)$. If $z \in B$ and R is sufficiently large, one has

$$|R_l(z)| < |z|^{l-1} < (n|u|^\beta)^{l-1}. \tag{5.17}$$

From (5.16), and since $U \in H_R(\varepsilon)$, we now obtain, using (5.5) and (5.17),

$$|F_j(z)| \le \left(1 - \frac{c}{|U|} + n^{l-1}|u|^{\beta(l-2)}\right)|u|^\beta = \left(1 - \frac{c}{|U|} + \frac{n^{l-1}}{|U|^{\beta(l-2)/k}}\right)|u|^\beta.$$

Then $\beta(l-2) > k + 1 - 2\beta > k$ by (5.8) and thus $\beta(l-2)/k > 1$. Hence, if R is sufficiently large,

$$p(u) := 1 - \frac{c}{|U|} + \frac{n^{l-1}}{|U|^{\beta(l-2)/k}} < 1. \tag{5.18}$$

Hence

$$|F_j(z)| \le p(u)|u|^\beta. \tag{5.19}$$

Now we claim that, setting $u_1 := \Phi(u, z)$, we obtain

$$|u| \le \frac{1}{|p(u)|^{1/\beta}}|u_1|. \tag{5.20}$$

Indeed, (5.20) is equivalent to $|U_1| \le p(u)^{-k/\beta}|U|$ and hence, in view of (5.10), to

$$\frac{|U + 1 + \nu(U, z)|}{|U|} \le p(u)^{-k/\beta}. \tag{5.21}$$

Note that, since $0 < c' < c$ (recall that c' is chosen before (5.7)), taking R sufficiently large, $(1 - c'/|U|)^{-k/\beta} \le p(u)^{-k/\beta}$. Also, by (5.10) we have $|U + 1 + \nu(U, z)|/|U| \le 1 + (1 + \delta)/|U|$, hence (5.21) holds if we can show that

$$1 + \frac{1+\delta}{|U|} \le \left(1 - \frac{c'}{|U|}\right)^{-k/\beta}. \tag{5.22}$$

But

$$\left(1 - \frac{c'}{|U|}\right)^{-k/\beta} = 1 + \frac{k}{\beta}\frac{c'}{|U|} + o\left(\frac{1}{|U|}\right),$$

and thus (5.22) holds whenever $\delta + 1 - c'k/\beta < 0$ (which is ensured by (5.7)) and R is sufficiently large. Hence, (5.20) holds. Putting together (5.19) and (5.20) we have

$$|F_j(z)| \le |u_1|^\beta, \qquad j = 1, \dots, m. \tag{5.23}$$

Inequalities (5.15) and (5.23) imply that $F(B) \subseteq B$. Moreover, by induction, for all $z \in B$, denoting by $\rho_j(z) := z_j$ the projection on the j-th component, we have

$$|\rho_j \circ F^{\circ m}(z)| \le |u_m|^\beta,$$

hence $F^{\circ m}(z) \to 0$ as $m \to \infty$ by (5.12). Therefore B is a basin of attraction for F at 0.

To end the proof, we note that the previous argument can be repeated by considering in (5.4) the sectors $S_R^j(\varepsilon)$, $j = 1, \dots, k$, of the form

$$S_R^j(\varepsilon) := \{u \in \Delta_R : |\operatorname{Arg} u - 2\pi(j-1)/k| < \varepsilon\}.$$

Let B_j be the basin of attraction constructed over $S_R^j(\varepsilon)$, namely

$$B_j := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < |u|^\beta,\ j = 1, \dots, m, \\ |(z_{m+1}, \dots, z_n)| < |u|^\beta,\ u := z^\alpha \in S_R^j(\varepsilon)\}.$$

Then clearly $B_1, \dots, B_k$ are disjoint and the proof is complete. □

Remark 5.5. Let F be as in Theorem 1.1 and let $B_1, \dots, B_k$ be its basins of attraction at 0 constructed in the proof. If $S_1, \dots, S_k$ denote the k petals for the induced germ $u \mapsto u + \Lambda u^{k+1} + O(|u|^{k+2})$ (see (4.2)) then, up to relabeling, $\pi(B_j) \subset S_j$ for $j = 1, \dots, k$, where $\pi : \mathbb{C}^n \ni z \mapsto z^\alpha \in \mathbb{C}$. In particular, if $\alpha = (q, 0, \dots, 0)$ for some $q \ge 1$, then each B_k has q connected components.

6. Applications and examples

6.1. Semi-attractive case

One-resonant diffeomorphisms with respect to one eigenvalue (which is necessarily a root of unity) are either formally linearizable in the associated eigendirection, or non-degenerate and parabolically attracting (see Remark 5.3). Thus, in particular, we recover Hakim's theorem on semi-attractive germs (cf. [17, Thm. 1.1] for $q = 1$):

Corollary 6.1. *Let $F \in \mathrm{Diff}(\mathbb{C}^n; 0)$. Let $\{\lambda_1, \dots, \lambda_n\}$ be the eigenvalues of dF_0. Suppose that $\lambda_1^q = 1$ for some $q \in \mathbb{N} \setminus \{0\}$ and $\lambda_1^l \neq 1$ for $l = 1, \dots, q-1$, and that $|\lambda_j| < 1$ for $j = 2, \dots, m$. In particular, F is one-resonant with respect to $\{\lambda_1\}$. Let k be the order of F with respect to λ_1. Then either*

(1) *$k < \infty$ and there exist k basins of attraction for F at 0, each having q connected components which are cyclically permuted by F, or*
(2) *$k = \infty$ and F is formally linearizable in the first component. This is the case if and only if there exists a holomorphic germ of a non-singular curve of fixed points of $F^{\circ q}$ passing through 0.*

Proof. (1) If $k < \infty$ then F is non-degenerate with respect to $\{\lambda_1\}$ and with index of resonance $(q, 0, \dots, 0)$ (see Remark 3.5). By Remark 5.3, F is parabolically attracting with respect to $\{\lambda_1\}$ and hence Theorem 1.1 applies, yielding k basins of attraction. Let $B_1, \dots, B_k$ be the basins of attraction constructed in the course of proof of Theorem 1.1. By Remark 5.5, each B_j has q connected components.

Fix one such basin of attraction $B = B_j$ and let $D_0, \dots, D_{q-1}$ be its connected components. By Remark 5.5, the image of B in $\mathbb{C}$ via the map $\mathbb{C}^n \ni z \mapsto z_1^q$ belongs to a petal S of $u \mapsto u + \Lambda(F)u^{k+1}$. Let $w \in \mathbb{C}$ be such that $w^q \in S$. In view of the construction in the proof of Theorem 1.1, assuming w is sufficiently small, we have $Q_p := (\lambda_1^p w, 0, \dots, 0) \in B$ for $p = 0, \dots, q-1$. Moreover, the Q_p's belong to different connected components of B. We can assume $Q_p \in D_p$ for $p = 0, \dots, q-1$. Now $F_1(Q_p) = \lambda_1^{p+1} w + o(|w|)$ and hence $F(Q_p)$ belongs to D_{p+1} (where $D_q = D_0$), proving the statement.

(2) We note that by Definition 3.3, the orders of F and of $F^{\circ q}$ with respect to λ_1 coincide. Furthermore, $k = \infty$ if and only if F is formally linearizable in the first component, and hence if and only if $F^{\circ q}$ is formally linearizable in the first component. Therefore we can assume $q = 1$. One direction being clear, we only show that if $k = \infty$ then F has a holomorphic non-singular curve of fixed points through 0. Write $z = (z, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$, $\lambda' = (\lambda_2, \dots, \lambda_n)$ and

$$F(z, z') = (f(z, z'), \lambda' z' + g(z, z')) \in \mathbb{C} \times \mathbb{C}^{n-1},$$

where $f(z, z') = z + o(|(z, z')|)$ and $g(z, z') = o(|(z, z')|)$. We look for a curve given by $\psi : \zeta \mapsto (\zeta, v(\zeta))$ where $v : U \to \mathbb{C}^{n-1}$ is a germ at 0 of holomorphic map defined in some open set $U \subset \mathbb{C}$ such that $v(0) = 0$ and such that $F(\psi(\zeta)) = \psi(\zeta)$ for all $\zeta \in U$. We decouple the latter condition as

$$f(\zeta, v(\zeta)) = \zeta, \tag{6.1}$$

$$\lambda' v(\zeta) + g(\zeta, v(\zeta)) = v(\zeta). \tag{6.2}$$

Since $k = +\infty$, Poincaré–Dulac theory shows that F is formally conjugate to a map of the type $\hat{F}(z, z') = (z, \lambda' z' + h(z, z'))$, where each monomial in the expansion of $h(z, z')$ is divisible by z_j for some $j = 2, \dots, n$. Clearly $\hat{F}$ has a unique curve of fixed points tangent to e_1, namely $z' = 0$. Hence, F has a unique formal solution to (6.1) and (6.2). It is enough to show that such a solution is actually holomorphic. To this end, we let $G(x, y) := (\lambda' - \mathrm{id})y + g(x, y)$ with $x \in \mathbb{C}$ and $y \in \mathbb{C}^{n-1}$. Since the Jacobian matrix $\{\frac{\partial G_j(x,y)}{\partial y_k}|_0\}_{j,k=1}^{n-1} = \lambda' - \mathrm{id}$ has maximal rank, by the (holomorphic) implicit function theorem there exists a unique function $v(x)$ defined and holomorphic near $x = 0$ such that $G(x, v(x)) \equiv 0$, and the proof is complete. □

6.2. *Quasi-parabolic germs*

A germ of holomorphic diffeomorphism of $\mathbb{C}^2$ at 0 of the form $F(z, w) = (z + \cdots, e^{2\pi i\theta} w + \cdots)$ with $\theta \in \mathbb{R}$ is called *quasi-parabolic*. In particular, if $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then F is one-resonant. We shall restrict to this case here.

Using Poincaré–Dulac theory, since all resonances are of the type $(1,(m,0))$, $(2,(m,1))$, the map F is formally conjugate to a map of the form

$$\hat{F}(z,w) = \Big(z + \sum_{j=\nu}^{\infty} a_j z^j,\, e^{2\pi i\theta} w + \sum_{j=\mu}^{\infty} b_j z^j w\Big), \tag{6.3}$$

where we assume that either $a_\nu \neq 0$ or $\nu = \infty$ if $a_j = 0$ for all j. Similarly for b_μ.

As proved in [7], the number $\nu(F) := \nu$ is a formal invariant of F. Moreover, it is proved that, in case $\nu < \infty$, the sign of $\Theta(F) := \nu - \mu - 1$ is a formal invariant. The map F is called *dynamically separating* if $\nu < \infty$ and $\Theta(F) \le 0$.

An argument similar to that in the proof of Corollary 6.1(2) yields:

Proposition 6.2. *Let F be a quasi-parabolic germ of diffeomorphism of $\mathbb{C}^2$ at 0. Then $\nu(F) = +\infty$ if and only if there exists a germ of (holomorphic) curve through 0 that consists of fixed points of F.*

In case $\nu(F) < \infty$, we note that the index α is $(1,0)$ and therefore $\Lambda(F)$ equals either $a_{\nu(F)}$ or 0, depending on whether $\nu \le \mu + 1$ or $\nu > \mu + 1$. Hence F is dynamically separating if and only if it is non-degenerate with respect to $\{1, e^{2\pi i\theta}\}$. In case F is non-degenerate, $k := \nu(F) - 1$ is the order of F with respect to $\{1, e^{2\pi i\theta}\}$.

In [7] it is proven that if F is a quasi-parabolic dynamically separating germ of diffeomorphism at 0 then there exist $\nu(F) - 1$ petals for F at 0. A direct computation shows that if F is dynamically separating, then it is parabolically attracting if and only if

$$\mathsf{Re}\left(\frac{b_{\nu-1}}{e^{2\pi i\theta} a_\nu}\right) > 0. \tag{6.4}$$

Then as a consequence of Theorem 1.1 and Remark 5.5 we have:

Corollary 6.3. *Let F be a dynamically separating quasi-parabolic germ, formally conjugate to (6.3). If (6.4) holds, then there exist $\nu(F) - 1$ disjoint connected basins of attraction for F at 0.*

6.3. An example of an elliptic germ with parabolic dynamics

Let $\lambda = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$F(z,w) = (\lambda z + az^2 w + \cdots, \lambda^{-1} w + bzw^2 + \cdots)$$

with $|a| = |b| = 1$. Then F is one-resonant with index of resonance $(1,1)$ and for each choice of (a,b) such that the germ is non-degenerate (i.e. $a\lambda^{-1} + b\lambda \neq 0$), there exists a basin of attraction for F at 0. Indeed, it can be checked that the non-degeneracy condition implies that F is parabolically attracting with respect to $\{\lambda, \lambda^{-1}\}$ and hence Theorem 1.1 applies.

A similar argument can be applied to F^{-1}, producing a basin of repulsion for F at 0. Hence we have a parabolic type dynamics for F.

On the other hand, suppose further that there exist $c > 0$ and $N \in \mathbb{N}$ such that $|e^{2\pi q i\theta} - 1| \geq cq^{-N}$ for all $q \in \mathbb{N} \setminus \{0\}$ (this holds for θ in a full measure subset of the unit circle). Since $\lambda^q \neq \lambda$ for all $q \in \mathbb{N}$, it follows from [21, Theorem 1] that there exist two analytic discs through 0, tangent at the origin to the z-axis and to the w-axis respectively, which are F-invariant and such that the restriction of F to each disc is conjugate to $\zeta \mapsto \lambda\zeta$ or $\zeta \mapsto \lambda^{-1}\zeta$ respectively. Thus, in that case, the elliptic and parabolic dynamics mix, although the spectrum of dF_0 is only of elliptic type.

6.4. *Examples of one-resonant degenerate germs with no basins of attraction*

Set

$$F(z, w) = \left(\lambda z\left(1 - \frac{zw}{\lambda}\right)^{-1}, \frac{w}{\lambda}\left(1 - \frac{zw}{\lambda}\right)\right), \tag{6.5}$$

with $|\lambda| = 1$ and λ not a root of unity. Then F is one-resonant with index of resonance $\alpha = (1, 1)$ but it is degenerate because

$$\Lambda(F) = \frac{1}{\lambda} - \frac{1}{\lambda^2} \cdot \lambda = 0.$$

Note also that the order of F is $k = 1$. Set $u = zw$ and

$$\Phi(u) = F_1(z, w) \cdot F_2(z, w) = u.$$

We claim that F has no basins of attraction at 0. Indeed, suppose $F^{\circ n}(z, w) \to 0$ as $n \to \infty$ for some (z, w). Then it follows that $\Phi^{\circ n}(zw) \to 0$ as $n \to \infty$, which implies that $zw = 0$. The latter cannot hold on a non-empty open set.

A less trivial example demonstrating this phenomenon is the following. Set

$$F(z, w) = \left(\lambda z + z^2 w, \frac{1}{\lambda} w - \frac{1}{\lambda^2} z w^2\right), \tag{6.6}$$

where $|\lambda| = 1$ and λ is not a root of unity. As before, F is one-resonant with index of resonance $(1, 1)$ and $\Lambda(F) = 0$. The order of F is 1 and for $u = zw$ we obtain

$$\Phi(u) = F_1(z, w) \cdot F_2(z, w) = u - \frac{1}{\lambda^2} u^3.$$

The order of Φ at $u = 0$ is 2. Now the attracting directions of Φ at 0 are $v = \pm\lambda$. The map Φ is a polynomial, with two attracting *maximal* petals $P(\lambda)$ and $P(-\lambda)$ at the origin. The maximal petals $P(\lambda)$ and $P(-\lambda)$ are disjoint and obtained as unions of all preimages under $F^{\circ n}$, $n = 1, 2, \dots$, of two fixed local petals.

Let J be the Julia set of Φ. Set $\widehat{J} := \{(z, w) : zw \in J\}$. Then $\widehat{J}$ has empty interior since J does (see e.g. [10]). We claim that if $(z, w) \notin \widehat{J}$, then $F^{\circ n}(z, w) \nrightarrow 0$ as $n \to \infty$.

It is well-known that, if $u_0 \notin J$ and $\Phi^{\circ n}(u_0) \to 0$ as $n \to \infty$, then $u_0 \in P(\lambda) \cup P(-\lambda)$. Therefore if $(z, w) \notin \widehat{J}$ is such that $zw \notin P(\lambda) \cup P(-\lambda)$ then $\{\Phi^{\circ n}(zw)\}$ cannot converge to 0 and therefore $F^{\circ n}(z, w) \nrightarrow 0$.

Now, let $(z, w) \notin \widehat{J}$ be such that $zw \in P(\lambda) \cup P(-\lambda)$. Then, setting $u_l = \Phi^{\circ l}(zw)$, we have

$$F^{\circ m}(z, w) = \left(\lambda^m z \prod_{l=1}^{m}\left(1 + \frac{u_l}{\lambda}\right), \frac{w}{\lambda^m} \prod_{l=1}^{m}\left(1 - \frac{u_l}{\lambda}\right)\right),$$

so that the behavior of $F^{\circ m}(z, w)$ depends only on the behavior of the infinite products $\prod_{l=1}^{m}(1 \pm u_l/\lambda)$.

The sequence $\{u_l\}$ tends to 0 with speed

$$|u_l|^2 \sim 1/l, \tag{6.7}$$

while $u_l/|u_l| \to \pm\lambda$ depending on whether $zw \in P(\pm\lambda)$. But

$$\left|1 \pm \frac{u_l}{\lambda}\right| = \sqrt{1 + |u_l|^2 \pm 2|u_l| \operatorname{Re} \frac{u_l}{\lambda |u_l|}} \sim 1 \pm \frac{1}{\sqrt{l}} \operatorname{Re} \frac{v}{\lambda},$$

where $v = \pm\lambda$. Therefore, if $zw \in P(\lambda)$, i.e. $v = \lambda$, then the behavior of $p_1 \circ F^{\circ m}(z, w)$ (here $p_1(z, w) = z$) depends on the infinite product

$$\prod\left(1 + \frac{1}{\sqrt{l}}\right),$$

which diverges to ∞, while the behavior of $p_2 \circ F^{\circ m}(z, w)$ (here $p_2(z, w) = w$) depends on the infinite product

$$\prod\left(1 - \frac{1}{\sqrt{l}}\right),$$

which converges to 0. If $zw \in P(-\lambda)$ the situation is reversed. In both cases $F^{\circ n}(z, w) \not\to 0$. Hence F has no basin of attraction at 0.

6.5. Example of a one-resonant non-degenerate (but not parabolically attracting) germ with no basins of attraction

Consider the germ given by

$$F(z, w) = (z - z^2, \lambda w + \lambda z w),$$

where $|\lambda| = 1$ and λ is not a root of unity. Then F is one-resonant with index of resonance $(1, 0)$. Furthermore $\Lambda = -1$, hence F is non-degenerate. On the other hand, F is not parabolically attracting, in fact $\operatorname{Re}(a_2 \lambda_2^{-1} \Lambda^{-1}) = -1 < 0$. Thus Theorem 1.1 does not apply and, in fact, F has no basin of attraction.

Indeed, if $F^{\circ n}(z_0, w_0) \to 0$ as $n \to \infty$, then z_0 must belong to the maximal petal of the map $\varphi(z) = z - z^2$. Setting $z_n := \varphi^{\circ n}(z_0)$, we have

$$F^{\circ n}(z_0, w_0) = \left(z_n, \lambda^n w \prod_{l=1}^{n}(1 + z_l)\right).$$

In view of Theorem 2.1,

$$\left|\prod_{l=1}^{n}(1+z_l)\right| \geq \prod_{l=1}^{n}\left(1+\frac{\varepsilon}{l}\right) = +\infty$$

for suitable $\varepsilon > 0$. Hence the only possibility for $F^{\circ n}(z_0, w_0) \to 0$ is when $w_0 = 0$. Thus we cannot have an (open) basin of attraction.

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