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Reeb vector fields and open book decompositions

Received March 28, 2009

Abstract. We determine parts of the contact homology of certain contact 3-manifolds in the framework of open book decompositions, due to Giroux. We study two cases: when the monodromy map of the compatible open book is periodic and when it is pseudo-Anosov. For an open book with periodic monodromy, we verify the Weinstein conjecture. In the case of an open book with pseudo-Anosov monodromy, suppose the boundary of a page of the open book is connected and the fractional Dehn twist coefficient c equals k/n , where n is the number of prongs along the boundary. If $k \geq 2$, then there is a well-defined linearized contact homology group. If $k \geq 3$, then the linearized contact homology is exponentially growing with respect to the action, and every Reeb vector field of the corresponding contact structure admits an infinite number of simple periodic orbits.

Keywords. Tight, contact structure, open book decomposition, mapping class group, Reeb dynamics, pseudo-Anosov, contact homology

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Mathematics Subject Classification (2010): Primary 57M50; Secondary 53C15

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1. Background and introduction

About ten years ago, Emmanuel Giroux [Gi] described a 1-1 correspondence between isotopy classes of contact structures and equivalence classes of open book decompositions (in any odd dimension). This point of view has been extremely fruitful, particularly in dimension three. Open book decompositions were the conduit for defining the contact invariant in Heegaard–Floer homology (due to Ozsváth–Szabó [OSz]). This contact invariant has been studied by Lisca–Stipsicz [LS] and others with great success, and has contributed considerably to the understanding of tight contact structures on Seifert fibered spaces. It was also open book decompositions that enabled the construction of concave symplectic fillings for any contact 3-manifold (due to Eliashberg [El] and Etnyre [Et1]); this in turn was the missing ingredient in Kronheimer–Mrowka’s proof of Property P for knots [KM]. In higher dimensions, the full potential of the open book framework is certainly not yet realized, but we mention Bourgeois’ existence theorem for contact structures on any odd-dimensional torus T^{2n+1} [Bo2].

The goal of this paper is to use the open book framework to calculate parts of the contact homology $HC(M, \xi)$ of a contact manifold (M, ξ) adapted to an open book decomposition, in dimension three. Giroux already indicated that there exists a Reeb vector field R which is in a particularly nice form with respect to the open book: R is transverse to the interior of each page S , and is tangent to and agrees with the orientation of the binding ∂S of the open book. (Here the orientation of ∂S is induced from S .) The difficulty that we encounter is that this Reeb vector field is not *nice enough* in general, e.g., it is not easy to see whether the contact homology is cylindrical, and boundary maps are

difficult to determine. (Some results towards understanding $HC(M, \xi)$ were obtained by Yau [Y2, Y3].)

In this paper we prove that, for large classes of tight contact 3-manifolds, $HC(M, \xi)$ is cylindrical, and moreover that $HC(M, \xi) \neq 0$. What enables us to get a handle on the contact homology is a better understanding of tightness in the open book framework. The second author, together with Kazez and Matić [HKM], showed a contact manifold (M, ξ) is tight if and only if all its compatible open books have *right-veering* monodromy. We will see that there is a distinct advantage to restricting our attention to right-veering monodromy maps.

In this section we review some notions around open book decompositions in dimension three.

1.1. Fractional Dehn twist coefficients

Let S be a compact oriented surface with nonempty boundary ∂S . Fix a reference hyperbolic metric on S so that ∂S is geodesic. (This excludes the cases where S is a disk or an annulus, which we understand well.) Suppose that ∂S is connected. Let $h : S \rightarrow S$ be a diffeomorphism for which $h|_{\partial S} = \text{id}$. If h is not reducible, then h is freely homotopic to a homeomorphism ψ of one of the following two types:

- (1) A *periodic* diffeomorphism, i.e., there is an integer $n > 0$ such that $\psi^n = \text{id}$.
- (2) A *pseudo-Anosov* homeomorphism.

Let $H : S \times [0, 1] \rightarrow S$ be the free isotopy from $h(x) = H(x, 0)$ to its periodic or pseudo-Anosov representative $\psi(x) = H(x, 1)$. We can then define $\beta : \partial S \times [0, 1] \rightarrow \partial S \times [0, 1]$ by sending $(x, t) \mapsto (H(x, t), t)$, i.e., β is the trace of the isotopy H along ∂S . Form the union of $\partial S \times [0, 1]$ and S by gluing $\partial S \times \{1\}$ and ∂S . By identifying this union with S , we construct the homeomorphism $\beta \cup \psi$ on S which is isotopic to h relative to ∂S . We will assume that $h = \beta \cup \psi$, although ψ is usually just a homeomorphism in the pseudo-Anosov case. (More precisely, ψ is smooth away from the singularities of the stable/unstable foliations.)

If we choose an oriented identification $\partial S \simeq \mathbb{R}/\mathbb{Z}$, then we can define an orientation-preserving homeomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ as follows: lift $\beta : \mathbb{R}/\mathbb{Z} \times [0, 1] \rightarrow \mathbb{R}/\mathbb{Z} \times [0, 1]$ to $\tilde{\beta} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$ and set $f(x) = \tilde{\beta}(x, 1) - \tilde{\beta}(x, 0) + x$. We then call β a *fractional Dehn twist* by an amount $c \in \mathbb{Q}$, where c is the *rotation number* of f , i.e., $c = \lim_{n \rightarrow \infty} (f^n(x) - x)/n$ for any x . In case ψ is periodic, c is simply $f(x) - x$ for any x . In the pseudo-Anosov case, c can be described as in the next paragraphs.

A pseudo-Anosov homeomorphism ψ is equipped with a pair of laminations—the *stable and unstable measured geodesic laminations* (Λ^s, μ^s) and (Λ^u, μ^u) —which satisfy $\psi(\Lambda^s, \mu^s) = (\Lambda^s, \tau\mu^s)$ and $\psi(\Lambda^u, \mu^u) = (\Lambda^u, \tau^{-1}\mu^u)$ for some $\tau > 1$. (Here Λ^s and Λ^u are the laminations and μ^s and μ^u are the transverse measures.) The lamination Λ ($= \Lambda^s$ or Λ^u) is *minimal* (i.e., does not contain any sublaminations), does not have closed or isolated leaves, is disjoint from the boundary ∂S , and every component of $S - \Lambda$ is either an open disk or a semi-open annulus containing a component of ∂S . In particular, every leaf of Λ is dense in Λ .

Now the connected component of $S - \Lambda^s$ containing ∂S is a semi-open annulus A whose metric completion \hat{A} has geodesic boundary consisting of n infinite geodesics $\lambda_1, \dots, \lambda_n$. Suppose that the λ_i are numbered so that i increases (modulo n) in the direction given by the orientation on ∂S . Now let $P_i \subset A$ be a semi-infinite geodesic which begins on ∂S , is perpendicular to ∂S , and runs parallel to γ_i and γ_{i+1} (modulo n) along the “spike” that is “bounded” by γ_i and γ_{i+1} . These P_i will be referred to as the *prongs*. Let $x_i = P_i \cap \partial S$ be the endpoint of P_i on ∂S . We may assume that ψ permutes (rotates) the prongs, and in particular there exists an integer k so that $\psi : x_i \mapsto x_{i+k}$ for all i . It then follows that c is a lift of $k/n \in \mathbb{R}/\mathbb{Z}$ to \mathbb{Q} .

If ∂S is not connected, then one can similarly define a fractional Dehn twist coefficient c_i for the i th boundary component of S .

1.2. Open book decompositions and tightness

In this paper, the ambient 3-manifold M is oriented and the contact structure ξ is cooriented.

Let (S, h) be a pair consisting of a compact oriented surface S and a diffeomorphism $h : S \xrightarrow{\sim} S$ which restricts to the identity on ∂S , and let K be a link in a closed oriented 3-manifold M . An *open book decomposition* for M with *binding* K is a homeomorphism between $((S \times [0, 1])/\sim_h, (\partial S \times [0, 1])/\sim_h)$ and (M, K) . The equivalence relation \sim_h is generated by $(x, 1) \sim_h (h(x), 0)$ for all $x \in S$ and $(y, t) \sim_h (y, t')$ for all $y \in \partial S$ and $t, t' \in [0, 1]$. We will often identify M with $(S \times [0, 1])/\sim_h$; with this identification $S_t = S \times \{t\}$, $t \in [0, 1]$, is called a *page* of the open book decomposition and h is called the *monodromy map*. Two open book decompositions are *equivalent* if there is an ambient isotopy taking binding to binding and pages to pages. We will denote an open book decomposition by (S, h) , although, strictly speaking, an open book decomposition is determined by the triple (S, h, K) . There is a slight difference between the two—if we do not specify $K \subset M$, we are referring to isomorphism classes of open books instead of isotopy classes.

Every closed 3-manifold has an open book decomposition, but it is not unique. One way of obtaining inequivalent open book decompositions is to perform a *positive* or *negative stabilization*: (S', h') is a *stabilization* of (S, h) if S' is the union of the surface S and a band B attached along the boundary of S (i.e., S' is obtained from S by attaching a 1-handle along ∂S), and h' is defined as follows. Let γ be a simple closed curve in S' “dual” to the cocore of B (i.e., γ intersects the cocore of B at exactly one point) and let $\text{id}_B \cup h$ be the extension of h by the identity map to $B \cup S$. Also let R_γ be the positive (or right-handed) Dehn twist about γ . Then for a *positive* stabilization, $h' = R_\gamma \circ (\text{id}_B \cup h)$, and for a *negative* stabilization, $h' = R_\gamma^{-1} \circ (\text{id}_B \cup h)$. It is well-known that if (S', h') is a positive (resp. negative) stabilization of (S, h) , and (S, h) is an open book decomposition of (M, K) , then (S', h') is an open book decomposition of (M, K') where K' is obtained by a Murasugi sum of K (also called the *plumbing* of K) with a positive (resp. negative) Hopf link.

According to Giroux [Gi], a contact structure ξ is *supported* by the open book decomposition (S, h, K) if there is a contact 1-form α which:

- (1) induces a symplectic form $d\alpha$ on each page S_i ;
- (2) K is transverse to ξ , and the orientation on K given by α is the same as the boundary orientation induced from S coming from the symplectic structure.

In the 1970's, Thurston and Winkelnkemper [TW] showed that (in Giroux's terminology) any open book decomposition (S, h, K) of M supports a contact structure ξ . Moreover, the contact planes can be made arbitrarily close to the tangent planes of the pages, away from the binding.

The following result is the converse (and more), due to Giroux [Gi].

Theorem 1.1 (Giroux). *Every contact structure (M, ξ) on a closed 3-manifold M is supported by some open book decomposition (S, h, K) . Moreover, two open book decompositions (S, h, K) and (S', h', K') which support the same contact structure (M, ξ) become equivalent after applying a sequence of positive stabilizations to each.*

Akbulut–Ozbagci [AO] and Giroux (independently) also clarified the role of Stein fillability, inspired by the work of Loi–Piergallini [LP]:

Corollary 1.2 (Loi–Piergallini, Akbulut–Ozbagci, Giroux). *A contact structure ξ on M is holomorphically fillable if and only if ξ is supported by some open book (S, h, K) with h a product of positive Dehn twists.*

The second author, together with Kazez and Matic [HKM], partially clarified the role of tightness in the open book framework. In particular, the following theorem was obtained:

Theorem 1.3. *A contact structure (M, ξ) is tight if and only if all of its open book decompositions (S, h) have right-veering h .*

We will briefly describe the notion of *right-veering*. Let α and β be isotopy classes, rel endpoints, of properly embedded oriented arcs $[0, 1] \rightarrow S$ with a common initial point $\alpha(0) = \beta(0) = x \in \partial S$. Assume $\alpha \neq \beta$. Choose representatives a, b of α, β so that they intersect transversely (this include the endpoints) and *efficiently*, i.e., with the fewest possible number of intersections. Then we say β is *strictly to the right of α* if the tangent vectors $(\dot{b}(0), \dot{a}(0))$ define the orientation on S at x . A monodromy map h is *right-veering* if for every choice of basepoint $x \in \partial S$ and every choice of arc α based at x , $h(\alpha)$ is α or is strictly to the right of α .

Theorem 1.4. *Suppose h is freely homotopic to ψ which is periodic or pseudo-Anosov, and c_i is the fractional Dehn twist coefficient corresponding to the i th boundary component of S .*

- (1) *If ψ is periodic, then h is right-veering if and only if all c_i are nonnegative. Hence (M, ξ) is overtwisted if some c_i is negative.*
- (2) *If ψ is pseudo-Anosov, then h is right-veering if and only if all c_i are positive. Hence (M, ξ) is overtwisted if some c_i is nonnegative.*

2. Main results

In this article, we prove the existence and nontriviality of cylindrical contact homology for a contact structure (M, ξ) given by an open book decomposition (S, h) with periodic or pseudo-Anosov monodromy, under favorable conditions. Here S is a compact, oriented surface with nonempty boundary ∂S (often called a “bordered surface”), and $h : S \xrightarrow{\sim} S$ is an orientation-preserving diffeomorphism which restricts to the identity on the boundary.

One of the motivating problems in 3-dimensional contact geometry is the following Weinstein conjecture:

Conjecture 2.1 (Weinstein conjecture). *Let (M, ξ) be a contact 3-manifold. Then for any contact form α with $\ker \alpha = \xi$, the corresponding Reeb vector field $R = R_\alpha$ admits a periodic orbit.*

During the preparation of this paper, Taubes [Ta] gave a complete proof of the Weinstein conjecture in dimension three. Our methods are completely different from those of Taubes, who uses Seiberg–Witten Floer homology instead of contact homology. In some situations (i.e., Theorem 2.3 and Corollary 2.6), we prove a better result which guarantees an infinite number of simple periodic orbits.

Prior to the work of Taubes, the Weinstein conjecture in dimension three was verified for contact structures which admit planar open book decompositions [ACH] (also see related work of Etnyre [Et2]), for certain Stein fillable contact structures [Ch, Ze], and for certain universally tight contact structures on toroidal manifolds [BC]. We also refer the reader to the survey article by Hofer [H2].

2.1. The periodic case

Our first result is the following:

Theorem 2.2. *The Weinstein conjecture holds when (S, h) has periodic monodromy.*

Proof. By the work of Hofer [H1], the Weinstein conjecture holds for overtwisted contact structures, on S^3 and manifolds which are covered by S^3 , and on manifolds for which $\pi_2(M) \neq 0$. If any of the fractional Dehn twist coefficients c_i are negative, then (M, ξ) is overtwisted by Theorem 1.4. If any c_i is zero, then $h = \text{id}$. In this case, M is a connected sum of $(S^1 \times S^2)$'s, and has $\pi_2(M) \neq 0$.

When all the c_i are positive and the universal cover of M is \mathbb{R}^3 , then the cylindrical contact homology is well-defined and nontrivial by Theorem 4.4. \square

In the periodic case, we also prove Theorem 4.2, which states that (M, ξ) is tight if and only if h is right-veering. Moreover, the tight contact structures are S^1 -invariant and also Stein fillable.

2.2. The pseudo-Anosov case

We now turn our attention to the pseudo-Anosov case. For simplicity, suppose S has only one boundary component. Then h gives rise to two invariants: the fractional Dehn twist coefficient c and the pseudo-Anosov homeomorphism ψ which is freely homotopic to h .

If $c \leq 0$, then the contact manifold (M, ξ) is overtwisted by Theorem 1.4. Hence we restrict our attention to $c > 0$. Let us write $c = k/n$, where n is the number of prongs about the unique boundary component ∂S . Our main theorem is the following:

Theorem 2.3. *Suppose ∂S is connected and $c = k/n$ is the fractional Dehn twist coefficient.*

- (1) *If $k \geq 2$, then any chain group $(\mathcal{A}(\alpha, J), \partial)$ of the full contact homology admits an augmentation ε . Hence there is a well-defined linearized contact homology group $HC^\varepsilon(M, \alpha, J)$.*
- (2) *If $k \geq 3$, then the linearized contact homology group $HC^\varepsilon(M, \alpha, J)$ has exponential growth with respect to the action. (In particular, $HC^\varepsilon(M, \alpha, J)$ is nontrivial.)*

For the notions of *full contact homology*, *linearized contact homology*, and *augmentations*, see Section 3. The *action* $A_\alpha(\gamma)$ of a closed orbit γ with respect to a contact 1-form α is $\int_\gamma \alpha$. The linearized contact homology group $HC^\varepsilon(M, \alpha, J)$ with respect to the contact 1-form α and adapted almost complex structure J on the symplectization is said to have *exponential growth with respect to the action* if there exist constants $c_1, c_2 > 0$ so that the number of linearly independent generators in $HC^\varepsilon(M, \alpha, J)$ which are represented by $\sum_i a_i(\gamma_i - \varepsilon(\gamma_i))$, $a_i \in \mathbb{Q}$, with $A_\alpha(\gamma_i) < L$ for all i is greater than $c_1 e^{c_2 L}$. Here the contact homology groups are defined over \mathbb{Q} . The notion of exponential growth with respect to the action is independent of the choice of contact 1-form α and almost complex structure J in the following sense: Given another $\mathcal{A}(\alpha', J')$, there is a chain map $\Phi : \mathcal{A}(\alpha', J') \rightarrow \mathcal{A}(\alpha, J)$ which pulls back the augmentation ε on $\mathcal{A}(\alpha, J)$ to $\Phi^* \varepsilon$ on $\mathcal{A}(\alpha', J')$, so that $HC^{\Phi^* \varepsilon}(M, \alpha', J') \simeq HC^\varepsilon(M, \alpha, J)$ and $HC^{\Phi^* \varepsilon}(M, \alpha', J')$ has exponential growth if and only if $HC^\varepsilon(M, \alpha, J)$ does.

Theorem 2.3, together with Theorem 1.4, implies the following:

Corollary 2.4. *The Weinstein conjecture holds for (M, ξ) which admits an open book with pseudo-Anosov monodromy if either $c \leq 0$ or $c \geq 3/n$.*

In [CH2], we prove that every open book (S, h) can be stabilized (after a finite number of stabilizations) to (S', h') so that h' is freely homotopic to a pseudo-Anosov homeomorphism and $\partial S'$ is connected. This proves that “almost all” contact 3-manifolds satisfy the Weinstein conjecture.

Remark 2.5. With our approach, it remains to prove the Weinstein conjecture for $c = 1/n$ and $2/n$. The $c = 1/n$ case is fundamentally different, and requires a different strategy; the $c = 2/n$ case might be possible by a more careful analysis of Conley–Zehnder indices.

We also have the following:

Corollary 2.6. *Let α be a contact 1-form for (M, ξ) which admits an open book with pseudo-Anosov monodromy and $c \geq 3/n$. Then the corresponding Reeb vector field R_α admits an infinite number of simple periodic orbits.*

In the corollary we do not require that the contact 1-form α be nondegenerate. The proof of Corollary 2.6 will be given in Section 11.3.

The main guiding philosophy of the paper is that a Reeb flow is not too unlike a pseudo-Anosov flow on a 3-manifold, since both types of flows are transversely area-preserving. The difference between the two will be summarized briefly in Section 6.1 and discussed more thoroughly in the companion paper [CHL].

We are also guided by the fundamental work of Gabai–Oertel [GO] on essential laminations, which we now describe as it pertains to our situation. Suppose S is hyperbolic with geodesic boundary. Suspending the stable geodesic lamination Λ^s of ψ , for example, we obtain a codimension 1 lamination \mathcal{L} on M , which easily satisfies the conditions of an *essential lamination*, provided $k > 1$. In particular, the universal cover \tilde{M} of M is \mathbb{R}^3 and each leaf of \mathcal{L} has fundamental group which injects into $\pi_1(M)$.

The following is an immediate corollary of the proof of Theorem 2.3:

Corollary 2.7. *A contact structure (M, ξ) supported by an open book with pseudo-Anosov monodromy with $k > 1$ is universally tight with universal cover \mathbb{R}^3 .*

2.3. Growth rates of contact homology

Theorem 2.3 opens the door to questions about the growth rates of linearized contact homology groups on various contact manifolds.

Example 1. Consider the standard tight contact structure on S^3 . Modulo taking direct limits, there is a contact 1-form with two simple periodic orbits, both of elliptic type. The two simple orbits, together with their multiple covers, generate the cylindrical contact homology group. Hence the growth is linear with respect to the action.

Example 2. Consider the unique Stein fillable tight contact structure (T^3, ξ) , given by $\alpha = \sin(2\pi z)dx - \cos(2\pi z)dy$ on $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with coordinates (x, y, z) . Modulo direct limits, the closed orbits are in 2-1 correspondence with $\mathbb{Z}^2 - \{(0, 0)\}$. Hence the cylindrical contact homology grows quadratically with respect to the action.

Example 3. The set of periodic orbits of the geodesic flow on the unit cotangent bundle of a closed hyperbolic surface Σ is in 1-1 correspondence with the set of closed geodesics of Σ . Hence the cylindrical contact homology of the corresponding contact structure grows exponentially with respect to the action.

Question 2.8. *Which contact manifolds (M, ξ) have linearized contact homology with exponential growth? Are there contact manifolds which have linearized contact homology with polynomial growth, where the degree of the polynomial is greater than 2?*

A special case of the question is:

Question 2.9. *What happens to contact structures on circle bundles over closed hyperbolic surfaces Σ with Euler number between 0 and $2g(\Sigma) - 3$, which are transverse to the fibers? Here $g(\Sigma)$ is the genus of Σ .*

Euler number $2g(\Sigma) - 2$ corresponds to the unit cotangent bundle case; on the other hand, Euler number ≤ -1 corresponds to the S^1 -invariant case, and has linear growth.

We also conjecture the following:

Conjecture 2.10. *Universally tight contact structures on hyperbolic manifolds have exponential growth.*

Organization of the paper. The notions of *contact homology* will be described in Section 3. In particular we quickly review the notions of augmentations and linearizations. Section 4 is devoted to the periodic case. In particular, we show that a periodic (S, h) is tight if and only if h is right-veering (Theorem 4.2); moreover, a tight (S, h) with periodic monodromy is Stein fillable. In Section 5, we present the Rademacher function and its generalizations, adapted to periodic and pseudo-Anosov homeomorphisms. In Section 6 we construct the desired Reeb vector field R which closely hews to the suspension lamination. This section is the technical heart of the paper, and is unfortunately rather involved. Section 7 is devoted to some discussions on perturbing the contact form to make it nondegenerate. Then in Sections 8 and 9, we give restrictions on the holomorphic disks and cylinders. In particular, we prove Theorem 8.1, which states that for any $N \gg 0$ there is a contact 1-form α for a contact structure which is supported by an open book with pseudo-Anosov monodromy and fractional Dehn twist coefficient $c > 1/n$, so that none of the closed orbits γ of action $\leq N$ are positive asymptotic limits of (holomorphic) finite energy planes \tilde{u} . The actual calculation of contact homology with such contact 1-forms will involve direct limits, discussed in Section 10. We then prove Theorem 2.3(1) in Section 10.2. Finally, we discuss the growth rate of periodic points of a pseudo-Anosov homeomorphism and use it to conclude the proofs of Theorem 2.3(2) and Corollary 2.6 in Section 11.

3. Contact homology

In this section we briefly describe the full contact homology, its linearizations, and Morse–Bott theory. Contact homology theory is part of the symplectic field theory of Eliashberg–Givental–Hofer [EGH]. For a readable account, see Bourgeois’ lecture notes [Bo3].

Disclosure. The full details of contact homology have not yet appeared. In particular, the gluing argument and, more importantly, the treatment of multiply-covered orbits are not written anywhere. However, various portions of the theory are available. For the asymptotics, refer to Hofer–Wysocki–Zehnder [HWZ1]. Compactness was explained in [BEHWZ]. The Fredholm theory and transversality (for non-multiply-covered curves) were treated by Dragnev [Dr]. For the Morse–Bott approach, refer to [Bo1]. Examples of contact homology calculations were done by Bourgeois–Colin [BC], Ustilovsky [U] and Yau [Y1].

3.1. Definitions

Let (M, ξ) be a contact manifold, α a contact form for ξ , and $R = R_\alpha$ the corresponding Reeb vector field, i.e., $i_R d\alpha = 0$ and $i_R \alpha = 1$. Consider the symplectization $(\mathbb{R} \times M, d(e^t \alpha))$, where t is the coordinate for \mathbb{R} . We will restrict our attention to almost

complex structures J on $\mathbb{R} \times M$ which are *adapted to the symplectization*: If we write $T_{(t,x)}(\mathbb{R} \times M) = \mathbb{R}\partial/\partial t \oplus \mathbb{R}R \oplus \xi$, then J maps ξ to itself and sends $\partial/\partial t \mapsto R$ and $R \mapsto -\partial/\partial t$.

Let γ be a closed orbit of R with period T . The closed orbit γ is *nondegenerate* if the derivative $\xi_{\gamma(0)} \rightarrow \xi_{\gamma(T)}$ of the first return map does not have 1 as an eigenvalue. A Reeb vector field R is said to be *nondegenerate* if all its closed orbits γ are nondegenerate. Suppose α is a contact 1-form for which R is nondegenerate.

A closed orbit is said to be *good* if it does not cover a simple orbit γ an even number of times, where the first return map $\xi_{\gamma(0)} \rightarrow \xi_{\gamma(T)}$ has an odd number of eigenvalues in the interval $(-1, 0)$. Here T is the period of the orbit γ . Let $\mathcal{P} = \mathcal{P}_\alpha$ be the collection of *good* closed orbits of $R = R_\alpha$. We emphasize that \mathcal{P} includes multiple covers of simple periodic orbits, as long as they are good.

In dimension three, a closed orbit γ has *even parity* (resp. *odd parity*) if the derivative of the first return map is of hyperbolic type with positive eigenvalues (resp. is either of hyperbolic type with negative eigenvalues or of elliptic type). The *Conley–Zehnder index* is a lift of the parity from $\mathbb{Z}/2\mathbb{Z}$ to \mathbb{Z} . If γ is a contractible periodic orbit which bounds a disk D , then we trivialize $\xi|_D$ and define the Conley–Zehnder index $\mu(\gamma, D)$ to be the Conley–Zehnder index of the path of symplectic maps $\{d\phi_t : \xi_{\gamma(0)} \rightarrow \xi_{\gamma(t)}, t \in [0, T]\}$ with respect to this trivialization, where ϕ_t is the time t flow of the Reeb vector field R . In our cases of interest, $\pi_2(M) = 0$, so $\mu(\gamma)$ is independent of the choice of D . We will also sometimes write $|\gamma| = \mu(\gamma) - 1$. If $\gamma, \gamma'_1, \dots, \gamma'_m \in \mathcal{P}$ and $[\gamma] = [\gamma'_1] + \dots + [\gamma'_m] \in H_1(M; \mathbb{Z})$, then let Z be a surface whose boundary is $\gamma - \gamma'_1 - \dots - \gamma'_m$. Trivialize $\xi|_Z$ and define the *Conley–Zehnder index* $\mu_{[Z]}(\gamma, \gamma'_1, \dots, \gamma'_m)$ with respect to the relative homology class $[Z] \in H_2(M, \gamma \cup \bigcup_i \gamma'_i)$ to be the Conley–Zehnder index of γ minus the sum from $i = 1$ to m of the Conley–Zehnder indices of γ'_i , all calculated with respect to the trivialization on Z .

Now, we fix a point m_γ , called a *marker*, on each simple periodic orbit γ . Also, an *asymptotic marker* at $z \in S^2$ is a ray r originating from z .

Define $\text{Hol}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)$ to be the set of all holomorphic maps

$$\tilde{u} = (a, u) : (\Sigma = S^2 - \{x, y_1, \dots, y_m\}, j) \rightarrow (\mathbb{R} \times M, J),$$

together with asymptotic markers r at x and r_i at $y_i, i = 1, \dots, m$, subject to the following:

- $\lim_{\rho \rightarrow 0} u(\rho, \theta) = \gamma(\theta)$ near x ;
- $\lim_{\rho \rightarrow 0} u(\rho, \theta) = \gamma'_i(\theta)$ near y'_1, \dots, y'_m ;
- the limit of u as $\rho \rightarrow 0$ along r is m_γ ;
- the limit of u as $\rho \rightarrow 0$ along r_i is $m_{\gamma'_i}$;
- $\lim_{\rho \rightarrow 0} a(\rho, \theta) = +\infty$ near x ;
- $\lim_{\rho \rightarrow 0} a(\rho, \theta) = -\infty$ near y'_1, \dots, y'_m .

Here, $x, y_1, \dots, y_m \in S^2, j$ is a complex structure on Σ, u is in the class $[Z]$, we are using polar coordinates (ρ, θ) near each puncture, and $\gamma(\theta)$ and $\gamma'_i(\theta), i = 1, \dots, m$, refer to some parametrization of the trajectories γ and γ'_i . The convergence for $u(\rho, \theta)$

and $a(\rho, \theta)$ is in the C^0 -topology. In the current situation, the punctures x, y_1, \dots, y_m , the complex structure j , and the asymptotic markers r, r_1, \dots, r_m are allowed to vary, while the markers m_γ stay fixed.

Next, two curves $\tilde{u} : (\Sigma = S^2 - \{x, y_1, \dots, y_m\}, j) \rightarrow (\mathbb{R} \times M, J)$ and $\tilde{u}' : (\Sigma = S^2 - \{x', y'_1, \dots, y'_m\}, j') \rightarrow (\mathbb{R} \times M, J)$ in $\text{Hol}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)$ are equivalent if $\tilde{u}' = \tilde{u} \circ H$, where H is a biholomorphism of Σ which takes the asymptotically marked punctures

$$((x', r'), (y'_1, r'_1), \dots, (y'_m, r'_m)) \rightarrow ((x, r), (y_1, r_1), \dots, (y_m, r_m)).$$

We define the moduli space $\mathcal{M}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)$ to be the quotient of $\text{Hol}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)$ under the above equivalence relation. The moduli space $\mathcal{M}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)$ supports an \mathbb{R} -action (in the target), obtained by translating a curve along the \mathbb{R} -direction of $\mathbb{R} \times M$. Assuming sufficient transversality, $\mathcal{M}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)/\mathbb{R}$ is endowed with the structure of a weighted branched manifold with rational weights. For that one can use the Kuranishi perturbation theory of Fukaya–Ono [FO] or the multi-valued perturbation of Liu and Tian [LT]; see also McDuff [McD]. In that case, $\mathcal{M}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)/\mathbb{R}$ is a union of manifolds with corners along a codimension one branching locus, each piece having the expected dimension $\mu_{[Z]}(\gamma, \gamma'_1, \dots, \gamma'_m) - (1 - m) - 1$. When this dimension is 0, we find a finite collection of points, according to the “Gromov compactness theorem” due to [BEHWZ].

We now define the *full contact homology* groups $FHC(M, \alpha, J)$. The contact homology groups are necessarily defined over \mathbb{Q} , since we must treat multiply covered orbits. The chain group is the supercommutative \mathbb{Q} -algebra $\mathcal{A} = \mathcal{A}(\alpha, J)$ with unit, which is freely generated by the elements of \mathcal{P} . Here supercommutative means that γ_1 and γ_2 commute if one of them has odd parity (and hence even degree $|\cdot|$) and anticommute otherwise. Now the boundary map $\partial : \mathcal{A} \rightarrow \mathcal{A}$ is given on elements $\gamma \in \mathcal{P}$ by

$$\partial\gamma = \sum \frac{n_{\gamma, \gamma'_1, \dots, \gamma'_m}}{(i_1)! \dots (i_l)! \kappa(\gamma'_1) \dots \kappa(\gamma'_m)} \gamma'_1 \dots \gamma'_m,$$

where the sum is over all unordered tuples $\overline{\gamma'} = (\gamma'_1, \dots, \gamma'_m)$ of orbits of \mathcal{P} and homology classes $[Z] \in H_2(M, \gamma \cup \bigcup_i \gamma'_i)$ so that, for any given ordering $\gamma'_1, \dots, \gamma'_m$ of $\overline{\gamma'}$, the expected dimension of the moduli space $\mathcal{M}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)/\mathbb{R}$ is zero. Here $\kappa(\gamma)$ is the multiplicity of γ . The integers i_1, \dots, i_l denote the number of occurrences of each orbit γ'_i in the list $\gamma'_1, \dots, \gamma'_m$. Also, we denote by $n_{\gamma, \gamma'_1, \dots, \gamma'_m}$ the signed weighted count of points in $\mathcal{M}_{[Z]}(\gamma, \gamma'_1, \dots, \gamma'_m)/\mathbb{R}$, for the corresponding ordering of $\overline{\gamma'}$, following a coherent orientation scheme as given in [EGH]. This definition does not depend on the ordering of $\overline{\gamma'}$, since if we permute γ'_i and γ'_{i+1} , the coefficient $n_{\gamma, \gamma'_1, \dots, \gamma'_m}$ is multiplied by $(-1)^{|\gamma'_i| |\gamma'_{i+1}|}$, which is annihilated by the sign coming from the supercommutativity of \mathcal{A} . If $\gamma, \gamma'_1, \dots, \gamma'_m$ are multiply covered, then each non-multiply-covered holomorphic curve $\tilde{u} \in \mathcal{M}_{[Z]}(\gamma, \gamma'_1, \dots, \gamma'_m)/\mathbb{R}$ contributes $\pm \kappa(\gamma) \kappa(\gamma'_1) \dots \kappa(\gamma'_m)$ to $n_{\gamma, \gamma'_1, \dots, \gamma'_m}$. This is due to the fact that, for the puncture x (resp. y_i), there are $\kappa(\gamma)$ (resp. $\kappa(\gamma'_i)$) possible positions for the asymptotic marker r (resp. r_i). If \tilde{u} is a cover of a somewhere injective

holomorphic curve, then it is counted as $\pm \frac{1}{k}(\kappa(\gamma)\kappa(\gamma'_1) \dots \kappa(\gamma'_m))$, where k is the number of automorphisms of the cover, since this group of automorphisms acts freely on the set of asymptotic markers and thus allows us to identify different positions. The coefficient $i_1! \dots i_l!$ takes into account the following overcounting: if, for example, y_1, \dots, y_{i_1} go to γ'_1 , then, for any permutation of these indices, the corresponding permutation of the punctures will give rise to different maps in $\mathcal{M}_{[Z]}(\gamma, \gamma'_1, \dots, \gamma'_m)/\mathbb{R}$. The definition of ∂ is then extended to all of \mathcal{A} using the graded Leibniz rule.

Theorem 3.1 (Eliashberg–Givental–Hofer). (1) $\partial^2 = 0$, so that $(\mathcal{A}(\alpha, J), \partial)$ is a differential graded algebra.
 (2) $FHC(M, \alpha, J) = H_*(\mathcal{A}(\alpha, J), \partial)$ does not depend on the choice of the contact form α for ξ , the complex structure J and the multi-valued perturbation.

The action $A_\alpha(\gamma) = \int_\gamma \alpha$ of a closed orbit γ with respect to the 1-form α gives rise to a filtration, which we call the *action filtration*. Define the action of $\gamma_1 \dots \gamma_m$ to be $A_\alpha(\gamma_1 \dots \gamma_m) = \sum_{i=1}^m A_\alpha(\gamma_i)$. The boundary map is action-decreasing, since every non-trivial holomorphic curve has positive $d\alpha$ -energy. There is a second filtration which comes from an open book decomposition, which we call the *open book filtration*, and is given by the number of times an orbit intersects a given page. This will be described in more detail in Section 9.

3.2. Linearized contact homology

In this subsection we discuss augmentations, as well as linearizations of contact homology induced by the augmentations. We have learned what is written here from Tobias Ekholm [Ek]. Details of the assertions are to appear in [BEE]. The notion of an augmentation first appeared in [Chk], in the context of Legendrian contact homology.

Let $(\mathcal{A} = \mathcal{A}(\alpha, J), \partial)$ be the chain group for the full contact homology as defined above. An *augmentation* for \mathcal{A} is a \mathbb{Q} -algebra homomorphism $\varepsilon : \mathcal{A} \rightarrow \mathbb{Q}$ which is also a chain map. (Here we are assuming that the boundary map ∂' for \mathbb{Q} satisfies $\partial'a = 0$ for all $a \in \mathbb{Q}$. This means that $\varepsilon\partial = 0$.) In this paper, we will assume that $\varepsilon(a) = 0$ if a not contractible or if a is contractible but $|a| \neq 0$. (Recall that $\pi_2(M) = 0$ in this paper.) Let $\text{Aug}(\mathcal{A}, \partial)$ denote the set of augmentations of (\mathcal{A}, ∂) .

An augmentation ε for (\mathcal{A}, ∂) induces a “change of coordinates” $a \mapsto \bar{a} = a - \varepsilon(a)$ of \mathcal{A} , where $a \in \mathcal{P}$. Then $\partial\bar{a}$ has the property that it does not have any constant terms when expressed in terms of sums of words in \bar{a}_i . (Proof by example: Suppose $\partial a = 1 + a_1 + a_2a_3$. Then

$$\partial\bar{a} = 1 + (\bar{a}_1 + \varepsilon(a_1)) + (\bar{a}_2 + \varepsilon(a_2))(\bar{a}_3 + \varepsilon(a_3)) = \varepsilon(1 + a_1 + a_2a_3) + \text{h.o.} = \text{h.o.}$$

Here ‘h.o.’ means higher order terms in the word length filtration.) In other words, with respect to the new generators \bar{a} , ∂ is nondecreasing with respect to the word length filtration, i.e., $\partial = \partial_1 + \partial_2 + \dots$, where ∂_j is the part of the boundary map which counts words of length j in the \bar{a}_i 's. Therefore it is possible to define the *linearized contact homology*

group $HC^\varepsilon(M, \alpha, J)$ with respect to ε to be the homology of $(\mathcal{A}_1, \partial_1)$, where \mathcal{A}_1 is the \mathbb{Q} -vector space generated by the \bar{a}_i for all $a_i \in \mathcal{P}$.

Example 1: cylindrical contact homology. When ∂a does not have a constant term for all $a \in \mathcal{P}$, then it admits the *trivial augmentation* ε which satisfies $\varepsilon(1) = 1$ and $\varepsilon(a) = 0$ for all $a \in \mathcal{P}$. The linearized contact homology with respect to the trivial augmentation ε is usually called *cylindrical contact homology*, and will be denoted by $HC(M, \alpha, J)$. If we restrict to the class of nondegenerate Reeb vector fields R_α with trivial augmentations, then $HC(M, \alpha, J)$ does not depend on α (or on J) and will be written as $HC(M, \xi = \ker \alpha)$.

We make two remarks about cylindrical contact homology. First, the trivial augmentation does not always exist. Second, it is possible to have finite energy planes which asymptotically limit to a at the positive end and still have ∂a without a constant term, as long as the total signed count is zero.

Example 2: augmentations from cobordisms. Suppose (X^4, ω) is an exact symplectic cobordism with (M, α) at the positive end and (M', α') at the negative end, and J be a compatible almost complex structure on (X^4, ω) . If $\mathcal{A}(M', \alpha', J|_{\ker \alpha'})$ admits an augmentation

$$\varepsilon' : \mathcal{A}(M', \alpha', J|_{\ker \alpha'}) \rightarrow \mathbb{Q},$$

then we can compose it with the chain map

$$\Phi_{(X,J)} : \mathcal{A}(M, \alpha, J|_{\ker \alpha}) \rightarrow \mathcal{A}(M', \alpha', J|_{\ker \alpha'})$$

to obtain the pullback augmentation $\varepsilon = \Phi_{(X,J)}^* \varepsilon' = \varepsilon' \circ \Phi_{(X,J)}$. Moreover, we have an induced map

$$HC^\varepsilon(M, \alpha) \rightarrow HC^{\varepsilon'}(M', \alpha')$$

between the linearized contact homology groups.

Two augmentations $\varepsilon_0, \varepsilon_1 : (\mathcal{A}, \partial) \rightarrow \mathbb{Q}$ are said to be *homotopic* if there is a derivation $K : (\mathcal{A}, \partial) \rightarrow (\mathcal{A}, \partial)$ of degree 1 satisfying

$$\varepsilon_1 = \varepsilon_0 \circ e^{\partial \circ K + K \circ \partial}.$$

Theorem 3.2 (Bourgeois–Ekholm–Eliashberg). (1) *If $\varepsilon_0, \varepsilon_1$ are homotopic augmentations of $(\mathcal{A}(M, \alpha, J), \partial)$, then*

$$HC^{\varepsilon_0}(M, \alpha, J) \simeq HC^{\varepsilon_1}(M, \alpha, J).$$

(2) *Given a 1-parameter family of compatible almost complex structures $J_t, t \in [0, 1]$, on the exact symplectic cobordism (X^4, ω) from (M, α) to (M', α') which agrees with J on M and J' on M' , and an augmentation ε' on $\mathcal{A}(M', \alpha', J')$, the pullback augmentations $\varepsilon_0 = \varepsilon' \circ \Phi_{(X,J_0)}$ and $\varepsilon_1 = \varepsilon' \circ \Phi_{(X,J_1)}$ are homotopic and induce the same map*

$$HC^{\varepsilon_i}(M, \alpha, J) \rightarrow HC^{\varepsilon'}(M', \alpha', J').$$

(3) *The set*

$$\{HC^\varepsilon(M, \alpha, J) \mid \varepsilon \in \text{Aug}(\mathcal{A}(M, \alpha, J), \partial)\}$$

of linearized contact homologies up to isomorphism is an invariant of $(M, \xi = \ker \alpha)$.

3.3. Morse–Bott theory

We briefly describe how to compute the cylindrical contact homology of a degenerate contact form of Morse–Bott type. For more details, we refer the reader to Bourgeois’ thesis [Bo2]. Again, let ϕ_t be the time t flow of the Reeb vector field R_α of α .

A contact form α is of *Morse–Bott type* if:

- (1) the *action spectrum* $\sigma(\alpha) = \{A_\alpha(\gamma) \mid \gamma \text{ periodic orbit}\}$ is discrete;
- (2) the union N_T of fixed points of ϕ_T is a closed submanifold of M ;
- (3) the rank of $d\alpha|_{N_T}$ is locally constant and $T_p N_T = \ker(d\phi_T(p) - I)$.

The submanifold N_T is foliated by orbits of R_α . In the case where $\dim N_T = 3$, the manifold N_T is a Seifert fibered space. The quotient space S_T is thus an orbifold, whose singularities with singularity groups $\mathbb{Z}/m\mathbb{Z}$ are the projections of orbits of actions T/m .

Choose a complex structure J on ξ which is invariant under the S^1 -action on N_T induced by the flow ϕ_t . Now, for each T , pick a Morse function $f_T : S_T \rightarrow \mathbb{R}$ so that the downward gradient trajectory of f_T with respect to the metric induced from $d\alpha(\cdot, J\cdot)$ (by quotienting out the S^1 -direction) is of Morse–Smale type. We also assume that, if $S_T \subset S_{kT}$, then f_{kT} extends the function f_T so that f_{kT} has positive definite Hessian in the normal directions to S_T .

Let $\gamma \in S_T$. We choose a trivialization of $\xi|_\gamma$. As before, define the Conley–Zehnder index $\mu(\gamma)$ to be the Conley–Zehnder index of the path

$$\{d\phi_t(\gamma(0)) : \xi_{\gamma(0)} \rightarrow \xi_{\gamma(T)}, t \in [0, T]\}$$

with respect to the trivialization, using the Robbin–Salamon definition [RS]. (Note that the value 1 belongs to the spectrum of the map $d\phi_T(\gamma(0))$, with eigenspace isomorphic to the tangent space of S_T .)

If $\gamma \in S_T$ is a critical point of f_T , then define the grading $|\gamma|$ of γ as

$$|\gamma| = \mu(\gamma) - \frac{1}{2} \dim S_T + \text{index}_\gamma(f_T) - 1.$$

The parity of $|\gamma|$ does not depend on the choice of framing of ξ along γ . Also, if $\gamma \in S_T$, then a choice of framing for ξ along γ induces a framing along any $\gamma' \in S_T$, by isotoping through fibers. The index $\mu(\gamma)$ then does not depend on $\gamma \in S_T$ for this particular family of framings. If $\gamma \in S_T$, then let $m\gamma$ denote the m -fold cover of γ in S_{mT} . A critical point γ of f_T is *bad* if it is an even multiple $2k\gamma'$ of a point γ' whose parity differs from the one of γ , and is *good* otherwise.

Let $MBC(\alpha, J)$ be the free \mathbb{Q} -vector space generated by the good critical points of f_T , for all $T \in \sigma(\alpha)$. We now briefly describe the differential ∂ on MBC . Let γ^+ and γ^- be good critical points of $\bigsqcup_T f_T$. The coefficient $\langle \partial\gamma^+, \gamma^- \rangle$ of γ^- in the differential $\partial\gamma^+$ is a signed count of points of 0-dimensional moduli spaces of generalized holomorphic cylinders from γ^+ to γ^- . A *generalized holomorphic cylinder* from γ^+ to γ^- is a finite collection $\{C_1, \dots, C_k\}$ of J -holomorphic cylinders, together with downward gradient trajectories $\{a_0, a_1, \dots, a_{k+1}\}$ of f_T on S_T , satisfying the following:

- The holomorphic cylinder C_i , $i = 1, \dots, k$, is asymptotic to γ_i^+ at $+\infty$ and to γ_i^- at $-\infty$.

- $\gamma^+ = \gamma_0^-$ and $\gamma^- = \gamma_{k+1}^+$.
- The orbits γ_i^- and $\gamma_{i+1}^+, i = 0, \dots, k$, lie on the same component of S_T , and a_i connects γ_i^- to γ_{i+1}^+ .

The map ∂ is extended linearly to all of $MBC(\alpha, J)$. The main theorem of Bourgeois' thesis is the following:

Theorem 3.3. *If no orbit of R_α is the asymptotic limit of a finite energy plane, then $(MBC(\alpha, J), \partial)$ is a chain complex and its homology is isomorphic to $HC(M, \xi)$.*

Example. If M is fibered by Reeb periodic orbits of action T , then S_T is a smooth surface. Since all the orbits in $N_T = M$ have the same action T , the only generalized cylinders between orbits in S_T are the gradient flow lines of f_T . Thus, if $\gamma, \gamma' \in S_T$, then $\langle \partial\gamma, \gamma' \rangle$ will be the same as that given by the Morse differential.

Example. If M is a Seifert fibered space with singular fibers of orders s_1, s_2, \dots, s_n so that all its fibers are Reeb orbits, then all regular orbits have the same action T , and the singular orbits have actions $T/s_1, \dots, T/s_n$. If γ_i denotes the singular fiber of action T/s_i , then S_{T/s_i} is γ_i . Moreover, $\mu(\gamma_i)$ is odd since the regular fibers rotate about the singular fiber. Hence $|\gamma_i| = \mu(\gamma_i) - 1$ is even.

The above examples will be explored in more detail in Section 4.

4. The periodic case

Suppose the contact 3-manifold (M, ξ) admits an open book decomposition (S, h) with periodic monodromy. Let c_i be the fractional Dehn twist coefficient of the i th boundary component and ψ be the periodic representative of h .

Theorem 4.1. *If all the c_i are positive, then (M, ξ) is an S^1 -invariant contact structure which is transverse to the S^1 -fibers.*

A transverse contact structure ξ (= transverse to the fibers) on a Seifert fibered space M with base B and projection map $\pi : M \rightarrow B$ is said to be S^1 -invariant if there is a Reeb vector field R of ξ so that (i) each fiber $\pi^{-1}(p)$ is an orbit of R and (ii) a neighborhood of a singular fiber is a $\mathbb{Z}/m\mathbb{Z}$ -quotient of $S^1 \times D^2$ with the standard contact form $dt + \beta$, where t is the coordinate for S^1 , β is rotationally invariant and independent of t , $d\beta$ is an area form on D^2 , and the Reeb vector field is $\partial/\partial t$.

Proof of Theorem 4.1. Suppose (S, h) is periodic. Let β be a 1-form on S satisfying $d\beta > 0$. We additionally require that, along each component of ∂S , $\beta = (C/2\pi)d\phi$, where ϕ is the angular coordinate of the boundary component equipped with the boundary orientation, and $C > 0$ is a constant. If the periodic representative ψ of h has order n (here $c_i = k_i/n$, where k_i and n are relatively prime), we average β by taking

$$\bar{\beta} = \frac{1}{n} \sum_{i=0}^{n-1} (\psi^i)^* \beta.$$

Consider the $[0, 1]$ -invariant contact 1-form $\alpha = dt + \bar{\beta}$ on $S \times [0, 1]$. Here t is the $[0, 1]$ -coordinate. By construction, α descends to a contact form on the manifold $N = (S \times [0, 1]) / (x, 1) \sim (\psi(x), 0)$ and the corresponding Reeb vector field R on $S \times [0, 1]$ is $\partial/\partial t$. The manifold N is a Seifert fibered space whose fibers are closed orbits of R . Observe that a nonsingular fiber intersects $S \times \{0\}$ at n points. Although R is probably the most natural Reeb vector field, it is highly degenerate, i.e., for each $p \in S$ there is a corresponding closed orbit $\{p\} \times S^1$. Hence we are in the Morse–Bott situation.

We then extend the contact 1-form to the neighborhood $N(K) \simeq S^1 \times D^2 = \mathbb{R}/\mathbb{Z} \times D^2$ of each binding component K . Let us use cylindrical coordinates $(z, (r, \theta))$ on $N(K)$, so that the pages restrict to $\theta = \text{const}$. From the construction of α on N , $\partial N(K)$ is (i) linearly foliated by the Reeb vector field R of slope c_i ; and (ii) linearly foliated by the characteristic foliation of ξ of slope $-1/C$, $C > 0$. Here we are using coordinates $(\theta/2\pi, z)$ to identify $\partial N(K) \simeq \mathbb{R}^2/\mathbb{Z}^2$.

Start with $[0, 1] \times D^2$ with coordinates $(z, (r, \theta))$ and contact form $\alpha = dz + \frac{1}{2}r^2d\theta$. Here $\frac{1}{2}r^2d\theta$ is the primitive of an area form for D^2 and is invariant under rotation by $\theta = \theta_0$. Moreover, $R = \partial/\partial z$. Now glue $\{1\} \times D^2$ to $\{0\} \times D^2$ via a diffeomorphism ϕ which sends $(r, \theta) \mapsto (r, \theta + \theta_0)$ for some constant θ_0 . The Reeb vector field R will then have slope $2\pi/\theta_0$; pick θ_0 so that $c_i = 2\pi/\theta_0$. Furthermore, if we adjust the size of the disk D^2 to have a suitable radius, then the characteristic foliation on $\partial(S^1 \times D^2)$ would have slope $-1/C$. (For another, more or less equivalent, construction, see Section 6.2.3.)

By taking the n -fold cover of $S^1 \times D^2$ we obtain a transverse contact structure on $S^1 \times D^2$ which is fibered by Reeb vector fields and which does not have any singular fibers. This completes the proof of Theorem 4.1. □

Theorem 4.2. *If (S, h) has periodic monodromy, then (M, ξ) is tight if and only if h is right-veering. Moreover, the tight contact structures are Stein fillable.*

Let M be a Seifert fibered space over an oriented closed surface of genus g and with r singular fibers, whose Seifert invariants are $\beta_1/\alpha_1, \dots, \beta_r/\alpha_r$. Then the Euler number $e(M)$ is $\sum_{i=1}^r \beta_i/\alpha_i$.

Some of the contact structures will be (universally) tight contact structures on lens spaces, which we know are Stein fillable.

Proof of Theorem 4.2. By Theorem 1.4, h is right-veering if and only if all the c_i are nonnegative. Moreover, if any coefficient c_i is negative, then (M, ξ) is overtwisted. If some c_i is 0, then h must be the identity, since ψ is periodic. In this case (M, ξ) is the standard Stein fillable contact structure on $\#(S^1 \times S^2)$.

Hence it remains to consider the case where all c_i are positive. According to Theorem 4.1, (M, ξ) is S^1 -invariant. According to a result of Lisca and Matic [LM], a Seifert fibered space M carries an S^1 -invariant transverse contact structure if and only if $e(M) < 0$. It is not hard to see that this S^1 -invariant contact structure is symplectically fillable and universally tight.

Neumann and Raymond [NR, Corollary 5.3] have shown that, if $e(M) < 0$, then M is the link of an isolated surface singularity with a holomorphic \mathbb{C}^* -action. Hence M is the oriented, strictly pseudoconvex boundary of a compact complex surface (with

a singularity). Let ξ' be the complex tangencies $TM \cap J(TM)$. The holomorphic \mathbb{C}^* -action on the complex surface becomes an S^1 -action on M . The vector field X on M generated by the S^1 -action is transverse to ξ' , since JX is transverse to M . Hence X is a Reeb vector field for ξ' , and ξ' is an S^1 -invariant transverse contact structure. Now, by Bogomolov [Bog] (also Bogomolov–de Oliveira [Bd, Theorem (2')]), (M, ξ') is also a strictly pseudoconvex boundary of a smooth Stein surface.

It remains to identify the S^1 -invariant transverse contact structures ξ and ξ' on M . By Lemma 4.3 below, there is a unique S^1 -invariant horizontal contact structure on M up to isotopy, once the fibering is fixed. By Hatcher [Hat, Theorem 4.3], Seifert fiberings of closed orientable Seifert fibered spaces over orientable bases are unique up to isomorphism, with the exception of S^3 , $S^1 \times S^2$, and lens spaces. (The other items on Hatcher’s list consist of M with boundary or identifications with Seifert fibered spaces over nonorientable bases.) All the tight contact structures on S^3 , $S^1 \times S^2$, and lens spaces are Stein fillable. \square

Lemma 4.3. *For any Seifert fibered space M with a fixed fibering, any two S^1 -invariant transverse contact structures are isotopic.*

Proof. Let $\pi : M \rightarrow B$ be a fixed fibering and let ξ, ξ' be S^1 -invariant transverse contact structures on M . Given any point p in B (p may be a singular fiber), there exist small neighborhoods $U, U' \subset B$ of p so that the holonomy of the characteristic foliation of ξ on $\pi^{-1}(\partial U)$ and ξ' on $\pi^{-1}(\partial U')$ agree. By taking a diffeomorphism of U to U' , we may assume that $U = U'$. Writing $\pi^{-1}(U) = S^1 \times U$ with fibers $S^1 \times \{\text{pt}\}$ and coordinates $(t, (x, y))$, we may modify $t \mapsto t + f(x, y)$ in a neighborhood of ∂U so that $\xi = \xi'$ along $S^1 \times \partial U$. The case of a singular fiber is similar.

The rest of the argument is similar to that which appears in Giroux [Gi2]. We now have S^1 -invariant transverse contact structures ξ and ξ' on $S^1 \times B'$, where B' is a surface with boundary and $\xi = \xi'$ on $S^1 \times \partial B'$. We may then write $\xi = \ker(dt + \beta)$ and $\xi' = \ker(dt + \beta')$, where $\beta = \beta'$ on $S^1 \times \partial B'$. Here β and β' are 1-forms on B' which are independent of t . We simply interpolate by taking $\alpha_s = dt + (1 - s)\beta + s\beta'$. Since $d\beta$ and $d\beta'$ are area forms on B' , α_s is a contact form. \square

Theorem 4.4. *If all the c_i are positive, then the cylindrical contact homology is well-defined. If the universal cover of M is \mathbb{R}^3 , then the cylindrical contact homology is non-trivial.*

Proof. If $M \rightarrow B$ is the Seifert fibration by orbits of the Reeb vector field, then view the closed, oriented base B as an orbifold. Since we are disallowing the case when the page S is D^2 , B is always a good orbifold in the sense of Scott [Sc, Theorem 2.3], and admits a finite covering which is a closed surface with no orbifold singularities. Now, if we view a Seifert fibered space M as an orbifold circle bundle, then the pullback bundle of an orbifold cover $\pi : B' \rightarrow B$ is a genuine covering space M' of M [Sc, Lemma 3.1]. Taking a closed surface B' with no singularities, we see that M' is a circle bundle over B' . The Euler number $e(M)$ lifts to the Euler number $e(M')$, which is $e(M)$ times the degree of the cover. Since $e(M) < 0$, it follows that $e(M') < 0$.

Suppose first that $B' \simeq S^2$. Then the universal cover \tilde{M} of M must be S^3 and the Reeb fibration becomes the Hopf fibration. In particular, there can be no contractible periodic orbit γ of M with Conley–Zehnder index $\mu(\gamma) = 2$, since there is none in \tilde{M} . On the other hand, if $g(B') \geq 1$, then $\tilde{M} \simeq \mathbb{R}^3$. Since every fiber of M lifts to \mathbb{R} , there are no contractible periodic orbits γ of M . In either case, the cylindrical contact homology is well-defined.

Next we prove that if γ' winds m' times around a regular fiber and γ'' winds m'' times around a regular fiber, then there are no holomorphic cylinders in the symplectization from γ' to γ'' , provided $m' \neq m''$. If there is such a holomorphic cylinder in $\mathbb{R} \times M$, then there would be a cylinder from γ' to γ'' in M . Since the cylinder has the homotopy type of, say, γ' , it can be lifted to M' since regular fibers are not expanded under bundle pullbacks. Now $e(M') < 0$, so the homology class of a regular fiber is a generator of $\mathbb{Z}/|e(M')|\mathbb{Z} \subset H_1(M; \mathbb{Z})$. If we take an n -fold cover B'' of B' , then the pullback M'' satisfies $e(M'') = n \cdot e(M')$, and we can distinguish γ' from γ'' homologically, provided n is sufficiently large. Analogous statements can also be made for multiple covers of singular fibers, by simply viewing a singular fiber as a suitable fraction of a regular fiber F .

Now suppose that $\tilde{M} = \mathbb{R}^3$. First suppose that B does not have any orbifold singular points. Then the orbits of smallest action are simple orbits around the S^1 -fibers, parametrized by the base B . Therefore, the portion of $HC(M, \xi)$ with the least action is $H_*(B; \mathbb{Q})$, by Bourgeois' Morse–Bott theory sketched in Section 3.3. Next suppose that the orbifold singularities of S/ψ have orders s_1, \dots, s_m , arranged in nonincreasing order (these are the “interior” singularities). The orbifold singularities coming from the binding all have order n , where $c_i = k_i/n$ as before. Hence the simple Reeb orbits corresponding to the singular fibers are $1/s_1, \dots, 1/s_m, 1/n$ of a regular fiber F . They are all elliptic orbits and have even parity, so there are no holomorphic cylinders amongst them. Hence simple orbits around the singular fibers correspond to nontrivial classes in $HC(M, \xi)$. \square

Remark 4.5. The techniques involved in proving Theorem 4.4 are sufficient to completely determine the cylindrical contact homology groups of the relevant contact structures.

5. Rademacher functions

We now define the Rademacher function and its generalizations. The usual Rademacher function is a beautiful function on the Farey tessellation, which admits an interpretation as a bounded cohomology class in $H_b^2(SL(2, \mathbb{Z}))$. For more details, see [BG, GG1, GG2]. The (generalized) Rademacher functions are used to measure certain types of “lengths” of arcs in the universal cover \tilde{S} of a compact hyperbolic surface S with geodesic boundary.

In this section we do not make any assumptions about the number of boundary components of S .

5.1. The usual Rademacher function

Let S be a compact hyperbolic surface with geodesic boundary. We first triangulate S with geodesic arcs which begin and end on ∂S . Here the boundary of each “triangle” consists

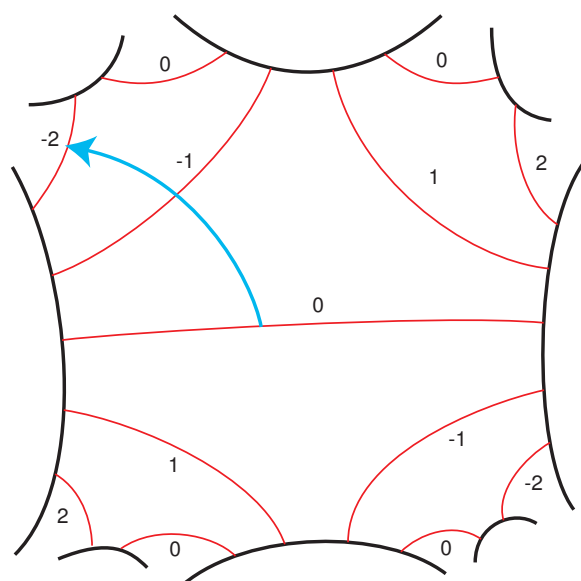


Fig. 1. The tessellation of the universal cover \tilde{S} of S and values of the Rademacher function on the tessellation (given right next to each edge).

of three geodesic arcs (which may happen to coincide), together with subarcs of ∂S . (Henceforth we omit the quotes when referring to *triangles*. In general, when we refer to an n -gon, we will not be counting the subarcs of ∂S .) Let τ be the set of geodesic arcs of the triangulation that are not subarcs of ∂S . Also let $\tilde{\tau} = \pi^{-1}(\tau)$, where $\pi : \tilde{S} \rightarrow S$ is the universal covering map.

The *Rademacher function* Φ is a function $\tilde{\tau} \rightarrow \mathbb{Z}$, defined as follows: Pick a reference arc $a \in \tilde{\tau}$, and set $\Phi(a) = 0$. Given $a' \in \tilde{\tau}$, take an *oriented* geodesic arc δ in \tilde{S} from a to a' . Then $\Phi(a')$ is the number of right turns taken minus the number of left turns taken along the path δ from a to a' . In other words, if $a', a'', a''' \in \tilde{\tau}$ form a triangle in \tilde{S} , where the edges are in counterclockwise order around the triangle, and we have inductively defined $\Phi(a')$ but not $\Phi(a'')$ and $\Phi(a''')$, then we set $\Phi(a'') = \Phi(a') + 1$ and $\Phi(a''') = \Phi(a') - 1$. Here the induction is on the distance of the triangle from the reference arc a .

Let us also define $\Phi(\gamma)$, where γ is an oriented arc with endpoints on $a', a'' \in \tilde{\tau}$, to be the number of right turns minus the number of left turns of a geodesic representative of γ . We will write γ^{-1} for γ with reversed orientation, and $\gamma\gamma'$ for the concatenation of γ , followed by γ' .

The Rademacher function has the following useful properties:

Lemma 5.1. *Let $a', a'', a''' \in \tilde{\tau}$, γ be a geodesic arc from a' to a'' and γ' be a geodesic arc from a'' to a''' . Then:*

- (1) $\Phi(\gamma^{-1}) = -\Phi(\gamma)$.
- (2) $\Phi(\gamma) + \Phi(\gamma') = \Phi(\gamma\gamma') + 3\varepsilon$, where $\varepsilon = -1, 0, 1$, depending on the angles made by γ and γ' .

Proof. (1) is immediate—a right turn becomes a left turn when traveling in the other direction.

To prove (2), suppose first that a', a'', a''' form a triangle in \tilde{S} . If a', a'', a''' are in counterclockwise order, then $\Phi(\gamma) + \Phi(\gamma') = \Phi(\gamma\gamma') + 3$; if a', a'', a''' are in clockwise order, then $\Phi(\gamma) + \Phi(\gamma') = \Phi(\gamma\gamma') - 3$. We can then reduce to the above situation by applying (1) to subarcs of γ and γ' that cancel. Observe that $\varepsilon = 0$ happens when either (i) the concatenation of γ and γ' is already efficient with respect to $\tilde{\tau}$, i.e., $\gamma\gamma'$ and its geodesic representative intersect $\tilde{\tau}$ in the same number of times, or (ii) the sequence of arcs of $\tilde{\tau}$ intersecting γ' is exactly the reverse of those intersecting γ (or vice versa). \square

5.2. Rademacher functions for periodic diffeomorphisms

We initially envisioned a more complicated proof of Theorem 4.4 which involved Rademacher functions. Although no longer logically necessary, in this subsection we describe Rademacher functions which are adapted to periodic diffeomorphisms.

Let ψ be a periodic diffeomorphism on S and let S' be the orbifold obtained by quotienting S by the action of ψ . (For more details on 2-dimensional orbifolds, see [Sc].) The orbifold S' will have the same number of boundary components as S , and m orbifold singularities. Assume S' is not a disk with $m = 1$. Then cut up S' using (not always geodesic) arcs from $\partial S'$ to itself which do not pass through any orbifold singularities, so that the complementary regions are either (1) triangles which do not contain any singularities, or (2) monogons containing exactly one singularity. Denote the union of such arcs on S' by τ' , their preimage on S by τ , and the preimage in the universal cover \tilde{S} by $\tilde{\tau}$. The connected components of $\tilde{S} - \tilde{\tau}$ are s -gons, where $s > 1$ (no monogons!). In particular, if we have a connected component of $\tilde{S} - \tilde{\tau}$ which projects to a monogon containing a singularity of order s , then the component is an s -gon.

We now define the (generalized) Rademacher function Φ on the oriented geodesic arcs γ of \tilde{S} which have endpoints on $\tilde{\tau}$. The function Φ will now take values in \mathbb{Q} instead of \mathbb{Z} . We define $\Phi(\gamma)$ to be the sum, over the set of s -gons P intersecting γ in their interior, of $\Phi(\gamma|_P)$, so we may assume γ to be an arc in P . Order the edges of P in τ in counterclockwise order to be a_0, a_1, \dots, a_{s-1} . If γ goes from a_0 to a_i , then define

$$\Phi(\gamma|_P) = 3 \frac{s - 2 - 2(i - 1)}{s} = 3 - \frac{6i}{s}.$$

Observe this formula agrees with the previous definition of the Rademacher function when $\tilde{\tau}$ consists only of triangles. Also, it is possible that $s = 2$, in which case $\Phi(\gamma|_P) = 0$. [It is instructive to compute $\Phi(\gamma|_P)$ if γ connects a_0 to a_i and $s = 7$. In that case, the values are, in counterclockwise order, $15/7, 9/7, 3/7, -3/7, -9/7, -15/7$.]

It is not difficult to see that the generalized Φ also satisfies Lemma 5.1. Also observe that Φ is *invariant* under ψ .

5.3. Rademacher functions for pseudo-Anosov homeomorphisms

Let S be a compact oriented surface endowed with a hyperbolic metric so that ∂S is geodesic, and let ψ be a pseudo-Anosov homeomorphism of S . The reader is referred

to [FLP] for the stable/unstable foliation perspective and to [Bn, CB] for the lamination perspective.

We will first explain the Rademacher function Φ from the lamination perspective, and later rephrase the definition in the language of singular foliations. The well-definition of the Rademacher function and its properties are easier to see in the lamination context, whereas the characteristic foliations that we construct in Section 6 will closely hew to the stable foliation.

In the geodesic lamination setting, Φ of an oriented arc $\gamma : [0, 1] \rightarrow S$ is defined as follows: Let $\Lambda = \Lambda^s$ be the stable lamination and \mathcal{S} be the union of all the prongs of S . Then isotop γ relative to its endpoints so that γ is geodesic, or at least intersects $\mathcal{S} \cup \Lambda$ efficiently (assuming γ is not contained in $\mathcal{S} \cup \Lambda$). Also let $\tilde{\Lambda}$ and $\tilde{\mathcal{S}}$ be the preimages of Λ and \mathcal{S} in the universal cover $\pi : \tilde{S} \rightarrow S$, and $\tilde{\gamma}$ be any lift of γ . Now consider the (open) intervals of $\text{Im}(\tilde{\gamma}) - (\tilde{\mathcal{S}} \cup \tilde{\Lambda})$. Then $\Phi(\gamma)$ is a signed count of intervals, both of whose endpoints lie on $\tilde{\mathcal{S}}$. (We throw away all other intervals!) The sign is positive if the interval is oriented in the same direction as $\partial\tilde{S}$, and negative otherwise. Although there are infinitely many intervals of $\text{Im}(\tilde{\gamma}) - (\tilde{\mathcal{S}} \cup \tilde{\Lambda})$, the sum $\Phi(\gamma)$ is finite. In fact, if Q is a connected component of $S - (\mathcal{S} \cup \Lambda)$ which nontrivially intersects ∂S , and \tilde{Q} is its lift to the universal cover, then the distance between two lifts $\tilde{P}_j, \tilde{P}_{j+1}$ of prongs on $\partial\tilde{Q}$ is bounded below. See Figure 2 for a sample calculation of $\Phi(\gamma)$. We will usually blur the distinction between arcs and isotopy classes of arcs.

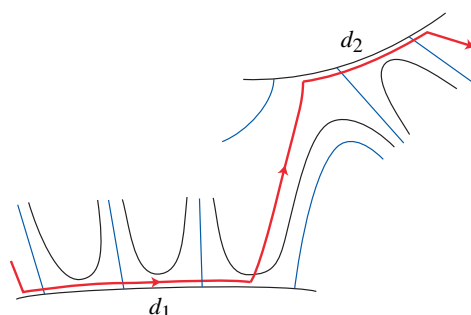


Fig. 2. The Rademacher function Φ on the given arc is 1, with a contribution of 2 from the component d_1 of $\partial\tilde{S}$ and a contribution of -1 from the component d_2 of $\partial\tilde{S}$. Here the blue arcs are the lifts of the prongs.

Proposition 5.2. *The Rademacher function Φ satisfies the following:*

- (1) Φ is invariant under ψ .
- (2) $\Phi(\gamma^{-1}) = -\Phi(\gamma)$.
- (3) $\Phi(\gamma\gamma') = \Phi(\gamma) + \Phi(\gamma') + \varepsilon$, where $\varepsilon = -1, 0$ or 1 .
- (4) Let γ be an arc which parametrizes a component $(\partial S)_i$ of ∂S , i.e., $\gamma(0) = \gamma(1)$ and γ wraps once around $(\partial S)_i$, in the direction of the boundary orientation of S . If $\gamma(0) \in \mathcal{S}$, then $\Phi(\gamma) = n_i$, where n_i is the number of prongs along $(\partial S)_i$, and if $\gamma(0) \notin \mathcal{S}$, then $\Phi(\gamma) = n_i - 1$.

Proof. (2) and (4) are straightforward.

(1) Recall that $\psi(\Lambda) = \Lambda$. This implies that ψ maps complementary regions of Λ , i.e., connected components of $S - \Lambda$, to complementary regions of Λ . In particular, an interior n -gon is either mapped to itself (its edges might be cyclically permuted) or mapped to another interior n -gon (with the same n). A semi-open annulus region A , i.e., an n -gon with a disk removed, is also mapped to itself, and its edges cyclically permuted. We may also assume that the prongs along the boundary component $(\partial S)_i$ are cyclically permuted.

Therefore, if γ is a geodesic arc, then $\psi(\gamma)$ is not necessarily geodesic, but at least intersects $\mathcal{S} \cup \Lambda$ efficiently. Moreover, ψ is type-preserving: intervals of $\text{Im}(\tilde{\gamma}) - (\tilde{\mathcal{S}} \cup \tilde{\Lambda})$ with endpoints on $\tilde{\mathcal{S}}$ get mapped to intervals with endpoints on $\tilde{\mathcal{S}}$, and intervals without both endpoints on $\tilde{\mathcal{S}}$ get mapped to intervals without both endpoints on $\tilde{\mathcal{S}}$. This proves (1).

(3) First isotop γ and γ' relative to their endpoints so that they are geodesic. Then lift γ, γ' and $\gamma\gamma'$ to \tilde{S} . We abuse notation and omit tildes, with the understanding that the terminal point of γ is the initial point of γ' , even in the universal cover.

Suppose γ and γ' can be factored into $\gamma_0\gamma_1$ and $\gamma'_0\gamma'_1$, respectively, where the initial point of γ_1 and the terminal point of γ'_0 lie on the same leaf \tilde{L} of $\tilde{\Lambda}$. In that case we may contract $\gamma_1\gamma'_0$ to a point on \tilde{L} , using (2) in the process. By successively shortening γ and γ' if possible, we are reduced to the cases (i), (ii), (iii), or (iv) below.

Let \tilde{Q} be a connected component of $\tilde{S} - (\tilde{\mathcal{S}} \cup \tilde{\Lambda})$ which nontrivially intersects $\partial\tilde{\mathcal{S}}$, and let \tilde{Q}' be a lift of an interior m -gon.

(i) Suppose γ and γ' are arcs in \tilde{Q}' . There are no contributions from interior m -gons, so $\Phi(\gamma), \Phi(\gamma')$, and $\Phi(\gamma\gamma')$ are all zero.

(ii) Suppose γ, γ' are arcs in \tilde{Q} , and all the endpoints of γ, γ' lie on $\partial\tilde{Q} \cap (\tilde{\mathcal{S}} \cup \tilde{\Lambda})$. If $\gamma(0)$ and $\gamma'(1)$ lie on the same leaf of $\tilde{\mathcal{S}} \cup \tilde{\Lambda}$, then $\Phi(\gamma\gamma') = \Phi(\gamma) + \Phi(\gamma')$. Hence we may assume that $\gamma(0)$ and $\gamma'(1)$ lie on distinct leaves. Depending on whether $\gamma(0), \gamma(1) = \gamma'(0), \gamma'(1)$ are in counterclockwise order or not, we have $\Phi(\gamma\gamma') = \Phi(\gamma) + \Phi(\gamma') \pm 1$.

(iii) Suppose γ, γ' are arcs in \tilde{Q} , and $\gamma(0), \gamma'(1)$ lie on $\partial\tilde{Q} \cap (\tilde{\mathcal{S}} \cup \tilde{\Lambda})$, but $\gamma(1) = \gamma'(0)$ does not. Then $\Phi(\gamma) = \Phi(\gamma') = 0$, and $\Phi(\gamma\gamma')$ is 0 or 1. (A similar consideration holds if two of $\gamma(0), \gamma(1) = \gamma'(0), \gamma'(1)$ lie on $\partial\tilde{Q} \cap (\tilde{\mathcal{S}} \cup \tilde{\Lambda})$.)

(iv) Suppose γ, γ' are arcs in \tilde{Q} , and $\gamma(0), \gamma'(1)$ do not lie on $\partial\tilde{Q} \cap (\tilde{\mathcal{S}} \cup \tilde{\Lambda})$. Then $\Phi(\gamma), \Phi(\gamma')$, and $\Phi(\gamma\gamma')$ are all zero. □

Next we translate the definition of Φ into the singular foliation language. Let \mathcal{F}^s (resp. \mathcal{F}^u) be the invariant stable (resp. unstable) foliation of S with respect to ψ . We will take $\mathcal{F} = \mathcal{F}^s$. The boundary of S is tangent to \mathcal{F} , and \mathcal{F} has n_i singular points of saddle type along the i th component $(\partial S)_i$ of ∂S . Here n_i is also the number of prongs that end on $(\partial S)_i$ in the lamination picture. Let \mathcal{S} be the union of the separatrices of the saddle points on ∂S that are not tangent to ∂S . (This set corresponds to the union of the prongs in the lamination picture.) Then $\tilde{\mathcal{S}} = \pi^{-1}(\mathcal{S})$ can be decomposed into a disjoint union of sets $\tilde{\mathcal{S}}_d$, where d is a component of $\partial\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}_d$ is the union of the components of $\tilde{\mathcal{S}}$ which intersect d .

Given an oriented arc $\gamma : [0, 1] \rightarrow S$, we isotop it, relative to its endpoints, to an oriented arc γ' so that (i) γ' has efficient intersection with \mathcal{S} , (ii) γ' is piecewise smooth, and (iii) each smooth piece is either transverse to \mathcal{F} away from the interior singularities, or is contained in ∂S . Such an arc γ' is called a *quasi-transverse arc*, in the terminology of [FLP, Exposé 5 (I.7)]. The proof of the existence of the isotopy is given in [FLP, Exposé 12 (Lemma 6)]. To pass from the geodesic lamination Λ to the foliation \mathcal{F} , we collapse the interstitial regions of Λ . If we take a geodesic representative γ'' of γ , then the desired quasi-transverse arc γ' is the image of γ'' under the collapsing operation.

We now rephrase $\Phi(\gamma)$ with respect to \mathcal{F} . Given the arc γ , choose a quasi-transverse representative γ' with the same endpoints, and let $\tilde{\gamma}'$ be any lift of γ' to \tilde{S} . Then $\Phi(\gamma)$ is the sum, over all components d of $\partial\tilde{S}$, of the signed number of intervals of $\text{Im}(\tilde{\gamma}') - \tilde{S}_d$ that do not contain an endpoint of $\tilde{\gamma}'$. The signs of the intervals are assigned as follows: positive if $\tilde{\gamma}'$ is oriented in the same direction as $\partial\tilde{S}$ along the interval, and negative otherwise. Alternatively, $\Phi(\gamma)$ is the sum of the signed number of intersections of \tilde{S}_d with $\tilde{\gamma}'$ minus one, if we have at least one intersection.

There are also slight variants of Φ described above. The simplest modification is to use the unstable lamination instead of the stable one. Also, we can take the universal cover of $S - \bigcup_i D_i$, where D_i are small disks removed from interior n -gons; this version then also counts contributions along interior n -gons. However, our Φ and its variants are “fake” Rademacher functions, which only register boundary rotations and discard all other intersections with n -gons. We close this section with a question:

Question 5.3. *Is there a “genuine” Rademacher function $\Phi(\gamma)$ which is adapted to a stable geodesic lamination Λ^s in the sense that it actually somehow sums the “left turn” and “right turn” contributions of γ , where the sum is over all the intervals $\gamma - \Lambda^s$?*

6. Construction of the Reeb vector field

6.1. First return maps

Let S be a compact oriented surface with nonempty boundary, ω be an area form on S , and h be an area-preserving diffeomorphism of (S, ω) . Suppose for the moment that $h|_{\partial S}$ is not necessarily id, but does not permute the boundary components.

Consider the mapping torus $\Sigma(S, h)$ of (S, h) , which we define as $(S \times [0, 1]) / (x, 1) \sim (h(x), 0)$. Here (x, t) are coordinates on $S \times [0, 1]$. If there is a contact form α on $\Sigma(S, h)$ for which $d\alpha|_{S \times \{0\}} = \omega$ and the corresponding Reeb vector field R_α is directed by ∂_t , then we say h is the *first return map* of R_α .

We are interested in the realizability of a given pseudo-Anosov ψ as the first return map of some R_α , after possibly perturbing ψ near the singular points to make ψ smooth. We summarize the following results from [CHL]:

Fact 1. *If $h^* - \text{id} : H^1(S; \mathbb{R}) \rightarrow H^1(S; \mathbb{R})$ is invertible, then h can be realized as the first return map of some R_α . Hence, a pseudo-Anosov homeomorphism ψ (after a small perturbation near its singular points) isotopic to such an h can be realized as the first return map of some R_α .*

Fact 2. *On the other hand, there exist pseudo-Anosov homeomorphisms ψ which (even after a small perturbation near its singular points) cannot be realized as the first return map of any R_α .*

If ψ is realizable, then we can use R_α and avoid the technicalities of the rest of Section 6. If ψ is not realizable as a first return map of a Reeb vector field R_α , then there are two strategies: (1) Enlarge the class of vector fields to the class of *stable Hamiltonian* ones, as described in [BEHWZ]. The drawback is that one needs to prove invariance of the generalized contact homology groups using the bifurcation strategy, instead of the continuation method which is usually used in contact homology. (2) Carefully construct a Reeb vector field for which we have some control over the periodic orbits. The drawback of this approach is that the construction is rather complicated. Since the details of the bifurcation strategy do not exist in the literature at this moment, we opt for (2). This will occupy the rest of the section.

6.2. Preliminary constructions

6.2.1. *Construction of a contact 1-form on $S \times [0, 1]$.* Let S be a compact oriented surface with nonempty boundary. Consider $S \times [0, 1]$ with coordinates (x, t) . We will first construct a contact 1-form α and the corresponding Reeb vector field R on $S \times [0, 1]$.

Lemma 6.1. *Given 1-forms β_0, β_1 on S which agree near ∂S and which satisfy $d\beta_i > 0$, $i = 0, 1$, there exist contact 1-forms $\alpha = \alpha_\varepsilon$ and Reeb vector fields $R = R_\varepsilon$ on $S \times [0, 1]$, depending on $\varepsilon > 0$ sufficiently small, which satisfy the following properties:*

- (1) $\alpha = dt + \varepsilon\beta_t$, where $\beta_t, t \in [0, 1]$, is a 1-form on S which varies smoothly with t .
- (2) R is directed by $\partial/\partial t + Y$, where $Y = Y_\varepsilon$ is tangent to $\{t = \text{const}\}$.
- (3) $Y = 0$ in a neighborhood of $(\partial S) \times [0, 1]$.
- (4) At points $x \in S$ where β_0 and β_1 have the same kernel, Y is tangent to $\ker \beta_0 = \ker \beta_1$.
- (5) The direction of the Reeb vector field R_ε does not depend on the choice of $\varepsilon > 0$, as long as ε is sufficiently small to satisfy the contact condition.
- (6) By taking $\varepsilon > 0$ sufficiently small, R_ε can be made arbitrarily close to $\partial/\partial t + Y$.

Proof. Let $\chi : [0, 1] \rightarrow [0, 1]$ be a smooth map for which $\chi(0) = 0, \chi(1) = 1, \chi'(0) = \chi'(1) = 0$, and $\chi'(t) > 0$ for $t \in (0, 1)$. Consider the form

$$\beta_t = (1 - \chi(t))\beta_0 + \chi(t)\beta_1.$$

Let us write $\omega_t = (1 - \chi(t))d\beta_0 + \chi(t)d\beta_1$. Observe that ω_t is an area form on S .

We then compute

$$\begin{aligned} d\alpha &= \varepsilon((1 - \chi(t))d\beta_0 + \chi(t)d\beta_1 + \chi'(t)dt \wedge (\beta_1 - \beta_0)) \\ &= \varepsilon(\omega_t + \chi'(t)dt \wedge (\beta_1 - \beta_0)), \\ \alpha \wedge d\alpha &= \varepsilon dt \wedge \omega_t - \varepsilon^2 \chi'(t)dt \wedge \beta_0 \wedge \beta_1. \end{aligned}$$

If ε is small enough, then α satisfies the contact condition $\alpha \wedge d\alpha > 0$.

The Reeb vector field R for α is collinear to $\partial/\partial t + Y$, where Y is tangent to the levels $\{t = \text{const}\}$ and satisfies

$$i_Y \omega_t = \chi'(t)(\beta_0 - \beta_1). \tag{6.2.1}$$

(Verification:

$$i_{\partial/\partial t + Y} d\alpha = \varepsilon \cdot i_{\partial/\partial t + Y}(\omega_t + \chi'(t)dt \wedge (\beta_1 - \beta_0)) = 0$$

implies that

$$i_Y \omega_t + \chi'(t)(\beta_1 - \beta_0) - dt \cdot \chi'(t)(\beta_1 - \beta_0)(Y) = 0.$$

This separates into two equations

$$i_Y \omega_t + \chi'(t)(\beta_1 - \beta_0) = 0, \quad \chi'(t)(\beta_1 - \beta_0)(Y) = 0.$$

The first is (6.2.1), and the second follows from the first.) Observe that (3), (4), and (5) are consequences of (6.2.1). To prove (6), observe that $\alpha(\partial/\partial t + Y) = 1 + \varepsilon\beta_t(Y)$. Then $R_\varepsilon = \frac{\partial/\partial t + Y}{|1 + \varepsilon\beta_t(Y)|}$, which approaches $\partial/\partial t + Y$ as $\varepsilon \rightarrow 0$. \square

6.2.2. *Construction of a contact 1-form on $\Sigma(S, g)$.* For notational simplicity, assume that ∂S is connected. Let β be a 1-form on S satisfying $d\beta > 0$. We say that β exits ∂S uniformly with respect to a diffeomorphism $g : S \xrightarrow{\sim} S$ if there exists a small annular neighborhood $A = S^1 \times [0, 1]$ of ∂S with coordinates (θ, y) so that

- (1) $\partial S = S^1 \times \{0\}$ and $\beta = (C - y)d\theta$, where C is a constant $\gg 0$.
- (2) g restricts to a rotation $(\theta, y) \mapsto (\theta + C', y)$ on $S^1 \times [0, 1]$, where C' is some constant.

Suppose β exits ∂S uniformly with respect to g . The easiest construction of a contact 1-form on $\Sigma(S, g)$ would be to set $\beta_0 = g_*\beta = (g^{-1})^*\beta$ and $\beta_1 = \beta$, and glue up the contact 1-form from Lemma 6.1. However, in this paper we will use a slightly more complicated 1-form, given below.

Construction. Let $\beta_0 = g_*(f_{\varepsilon'}\beta) = f_{\varepsilon'}(g_*\beta)$, $\beta_{1/2} = \beta$, and $\beta_1 = f_{\varepsilon'}\beta$, where $\varepsilon' > 0$ is a sufficiently small constant. Here, $f_{\varepsilon'} : S \rightarrow \mathbb{R}$ is ε' outside the small annular neighborhood A of ∂S , and, inside A , is independent of θ , equals 1 for $y \in [0, \varepsilon'']$, and satisfies $\partial f_{\varepsilon'}/\partial y < 0$ for $y \in (\varepsilon'', 1)$. We can easily verify that $df_{\varepsilon'} \wedge \beta \geq 0$; hence $f_{\varepsilon'}\beta$ is a primitive of an area form on S . Then let β_t be the interpolation between β_0 and $\beta_{1/2}$ for $t \in [0, 1/2]$, given by

$$\beta_t = (1 - \chi_0(t))\beta_0 + \chi_0(t)\beta_{1/2},$$

where $\chi_0 : [0, 1/2] \rightarrow [0, 1]$ is a smooth map for which $\chi_0(0) = 0$, $\chi_0(1/2) = 1$, $\chi_0'(0) = \chi_0'(1/2) = 0$ and $\chi_0'(t) > 0$ for $t \in (0, 1/2)$. Similarly define the interpolation β_t between $\beta_{1/2}$ and β_1 for $t \in [1/2, 1]$ by

$$\beta_t = (1 - \chi_1(t))\beta_{1/2} + \chi_1(t)\beta_1,$$

where $\chi_1 : [1/2, 1] \rightarrow [0, 1]$ is a smooth map for which $\chi_1(1/2) = 0$, $\chi_1(1) = 1$, $\chi_1'(1/2) = \chi_1'(1) = 0$ and $\chi_1'(t) > 0$ for $t \in (1/2, 1)$. Then we set $\alpha_{\varepsilon, \varepsilon'} = dt + \varepsilon\beta_t$ as in Lemma 6.1. It induces a contact form $\alpha_{\varepsilon, \varepsilon'}$ on $\Sigma(S, g)$.

Let $\omega_t = d_2\beta_t$, where d_2 indicates the exterior derivative in the S -direction. Then the Reeb vector field $R = R_{\varepsilon, \varepsilon'}$ for $\alpha = \alpha_{\varepsilon, \varepsilon'}$ is collinear to $\partial/\partial t + Y$, where $Y = Y_{\varepsilon'}$ is tangent to the levels $\{t = \text{const}\}$ and satisfies

$$i_Y\omega_t = -\dot{\beta}_t. \tag{6.2.2}$$

Here a dot means d/dt . Observe that Y does not depend on ε (by (5) of Lemma 6.1) and the direction of $R_{\varepsilon, \varepsilon'}$ does not depend on ε . By taking ε sufficiently small as in (6) of Lemma 6.1, we can make R as close to $\partial/\partial t + Y$ as we like.

Description of $R_{\varepsilon, \varepsilon'}$. Let Z be a vector field which directs $\ker \beta$. Fix a small neighborhood $U \subset S$ of the singular set of β . Also let $A' \subset A$ be the set $\{0 \leq y \leq \varepsilon''\}$.

Lemma 6.2. *The Reeb vector field $R_{\varepsilon, \varepsilon'}$ is directed by and is arbitrarily close to $\partial/\partial t + Y_{\varepsilon'}$, provided $\varepsilon > 0$ is sufficiently small. The vector field $Y_{\varepsilon'}$ satisfies the following:*

- (1) $Y_{\varepsilon'} = 0$ on $A' \times [0, 1]/\sim$. In particular, $R_{\varepsilon, \varepsilon'}$ is tangent to $\partial\Sigma(S, g)$.
- (2) $Y_{\varepsilon'} = 0$ when $t = 0$ and $t = 1/2$.
- (3) On $(S - A' - U) \times (0, 1/2)$, $Y_{\varepsilon'}(x, t)/|Y_{\varepsilon'}(x, t)| \rightarrow -Z(x)/|Z(x)|$ uniformly, as $\varepsilon' \rightarrow 0$.
- (4) On $(S - A') \times (1/2, 1)$, $Y_{\varepsilon'}$ is parallel to and in the same direction as Z .

One can think of the vertical projections Y of R as what happens in a ‘‘puffer machine’’: Between $t = 0$ and $t = 1/2$, Y flows away from ∂S and is sucked towards the singularities of β along $\ker \beta$ (with some error), and, between $t = 1/2$ and $t = 1$, Y flows away from the singularities of β towards ∂S along $\ker \beta$ (with no error).

Proof. This follows from (6.2.2). First suppose $t \in [0, 1/2]$. Then

$$\dot{\beta}_t = \chi'_0(t)(\beta_{1/2} - \beta_0) = \chi'_0(t)(\beta - f_{\varepsilon'}(g_*\beta)). \tag{6.2.3}$$

(1) follows from $\dot{\beta}_t = 0$ by observing that $\beta_0 = \beta_{1/2}$ on A' . Similarly, (2) follows from $\dot{\beta}_t = 0$ by observing that $\chi'_0(t) = 0$ when $t = 0$ or $t = 1/2$. We now prove (3). The vector field $Y_{\varepsilon'}$ directs the kernel of $f_{\varepsilon'}(g_*\beta) - \beta$, since $\chi'_0(t) > 0$ when $t \in (0, 1/2)$. In order for $Y_{\varepsilon'}(x, t)/|Y_{\varepsilon'}(x, t)|$ to make sense, we need $Y_{\varepsilon'}(x, t)$ to be nonzero—this is achieved by making ε' sufficiently small and restricting to $(S - A' - U) \times (0, 1/2)$. The uniform convergence of $f_{\varepsilon'}(g_*\beta) - \beta$ to $-\beta$ on $(S - A - U) \times (0, 1/2)$ as $\varepsilon' \rightarrow 0$ (note that we wrote A instead of A') implies the uniform convergence of $Y_{\varepsilon'}(x, t)/|Y_{\varepsilon'}(x, t)|$ to $-Z(x)/|Z(x)|$ on $(S - A - U) \times (0, 1/2)$. On the other hand, on $(A - A') \times (0, 1/2)$, $Y_{\varepsilon'}(x, t)/|Y_{\varepsilon'}(x, t)| = -Z(x)/|Z(x)|$ already.

The situation $t \in [1/2, 1]$ is similar and is left to the reader. □

6.2.3. Extension to the binding. Let S_0 be the surface obtained by gluing an annulus $A_0 = S^1 \times [-1, 0]$ to S so that $S^1 \times \{0\}$ is identified with ∂S . Let $h : S_0 \xrightarrow{\sim} S_0$ be a diffeomorphism which restricts to the identity on ∂S_0 . Suppose $h = h_0 \cup g$, where g is the diffeomorphism on S as above and $h_0 : S^1 \times [-1, 0] \xrightarrow{\sim} S^1 \times [-1, 0]$ maps

$(\theta, y) \mapsto (\theta + C'(y + 1), y)$, where C' is the positive constant which records the rotation about the boundary and $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$.

Fix a 1-form β on S so that $d\beta > 0$ and β exits ∂S uniformly with respect to g . Let $\alpha = \alpha_{\varepsilon, \varepsilon'}$ and $R = R_{\varepsilon, \varepsilon'}$ be the contact 1-form and Reeb vector field constructed on $\Sigma(S, g)$ as in the previous subsection. According to Lemma 6.2, $R = \partial/\partial t$ on $\partial\Sigma(S, g)$; hence $\partial\Sigma(S, g)$ is linearly foliated by R . Also the characteristic foliation of α along $\partial\Sigma(S, g)$ is linearly foliated by leaves which are close to ∂S .

We now extend α and R to the closed 3-manifold M which corresponds to the open book (S_0, h) , by gluing a neighborhood $N(K)$ of the binding to $\partial\Sigma(S, g)$. Endow $N(K) \simeq \mathbb{R}/\mathbb{Z} \times D^2$ with cylindrical coordinates $(z, (r, \theta))$ so that $D^2 = \{r \leq 1\}$. The fibration of the open book is given on $N(K)$ by $(z, r, \theta) \mapsto \theta$. If we use coordinates $(\theta/2\pi, z)$ to identify $\partial N(K) \simeq \mathbb{R}^2/\mathbb{Z}^2$, then R has slope $C' > 0$ and $\xi|_{\partial(S^1 \times D^2)}$ has slope $-1/C_\varepsilon$ for $C_\varepsilon \gg 0$. We extend the contact form $\alpha_{\varepsilon, \varepsilon'}$ to $N(K)$ by an equation of the form $a_\varepsilon(r)dz + b_\varepsilon(r)d\theta$, where $a_\varepsilon(r) > 0$ and $b_\varepsilon(r) \geq 0$. The characteristic foliation on $\{r = r_0\}$ will then be directed by $a_\varepsilon(r_0)\partial/\partial\theta - b_\varepsilon(r_0)\partial/\partial z$. The contact condition is given by the inequality $a_\varepsilon b'_\varepsilon - a'_\varepsilon b_\varepsilon > 0$. It expresses the fact that the plane curve $(a_\varepsilon(r), b_\varepsilon(r))$ is transverse to the radial foliation of the plane, and rotates in the counterclockwise direction about the origin. The Reeb vector field is given by

$$R_\varepsilon = \frac{1}{a_\varepsilon b'_\varepsilon - a'_\varepsilon b_\varepsilon} \left(b'_\varepsilon \frac{\partial}{\partial z} - a'_\varepsilon \frac{\partial}{\partial \theta} \right).$$

The boundary condition uniquely determines the values $a_\varepsilon(1), b_\varepsilon(1), a'_\varepsilon(1)$ and $b'_\varepsilon(1)$. In particular, these values depend smoothly on ε . For all ε , $(a_\varepsilon(1), b_\varepsilon(1))$ is in the interior of the first quadrant. Also, we require that $(a_\varepsilon(r), b_\varepsilon(r))$ lie on a line segment that starts on the positive θ -axis and ends at $(a_\varepsilon(1), b_\varepsilon(1))$, and is directed by $(a'_\varepsilon(1), b'_\varepsilon(1))$. We can then extend a_ε and b_ε on $[0, 1]$, so that $a_\varepsilon(r) = C_{0,\varepsilon} - C_{1,\varepsilon}r^2$ and $b_\varepsilon(r) = r^2$ near $r = 0$ (where $C_{0,\varepsilon}, C_{1,\varepsilon}$ are appropriate positive constants which depend on ε) and so that they depend smoothly on ε . By construction R_ε will linearly foliate the level tori $\{r = \text{const} > 0\}$ so that the slope remains constant ($= C'$). In particular, R will be transverse to the pages $S \times \{t\}$, except along the binding γ_0 , which is a closed orbit of R .

In the remaining subsections of this section, we will construct a suitable diffeomorphism $g = \psi'$ which is freely homotopic to a pseudo-Anosov homeomorphism ψ , and a 1-form β which is adapted to ψ .

6.3. Main proposition

Let M be a closed, oriented 3-manifold and ξ be a cooriented contact structure. Suppose that ξ is carried by an open book with page S and monodromy $h : S \xrightarrow{\sim} S$. Recall that $h|_{\partial S} = \text{id}$. For notational simplicity, assume that ∂S is connected.

Suppose h is freely homotopic to a pseudo-Anosov homeomorphism ψ with fractional Dehn twist coefficient $c = k/n$. Let $(\mathcal{F}, \mu) = (\mathcal{F}^s, \mu^s)$ be the stable foliation on S , and $\lambda > 1$ be the constant such that $\psi_*\mu = \lambda\mu$. The foliation \mathcal{F} has saddle type singularities on ∂S , and the singular points of \mathcal{F} on ∂S are denoted by x_1, \dots, x_n . (Here the subscript i increases in the direction given by the orientation of ∂S .) Denote the interior singularities

of \mathcal{F} by y_1, \dots, y_q . The homeomorphism ψ is a diffeomorphism away from these singular points. Also let P_i be the prong emanating from x_i , and let $Q_{j1}, Q_{j2}, \dots, Q_{jm_j}$ be the prongs emanating from y_j , arranged in counterclockwise order about y_j .

6.3.1. $N(\partial S)$. Let $N(\partial S) \subset S$ be a neighborhood of ∂S with a particular shape (see Figure 3):

- (1) $\partial(N(\partial S)) - \partial S$ is a concatenation of smooth arcs which are alternately tangent to \mathcal{F}^u (the *vertical* arcs a_1, \dots, a_n , since they are transverse to \mathcal{F}) and tangent to \mathcal{F} (the *horizontal* arcs b_1, \dots, b_n). Here b_i is between a_i and a_{i+1} , where the indices are taken modulo n , and there is a prong P_i starting at x_i which exits $N(\partial S)$ through a_i .
- (2) Each transverse arc a_i is divided into two subarcs by the prong P_i starting at x_i . We pick $N(\partial S)$ so that all these subarcs have the same transverse measure $\delta \ll 1$. (This becomes important later on!)
- (3) No horizontal arc b_i is contained in any prong P_j or Q_{jl} . (This can be achieved by observing that the intersection between a_i and any prong is countable, and by shrinking δ if necessary.)

Let P'_i be the first component of $P_i \cap (S - \text{int}(N(\partial S)))$ that can be reached from the singular point x_i , traveling inside P_i . By (3), P'_i is a compact arc with endpoints on $\text{int}(a_i)$ and some $\text{int}(a_{i'})$. Similarly, let Q'_{jl} be the component of $Q_{jl} \cap (S - \text{int}(N(\partial S)))$ that begins at y_j and ends on some $\text{int}(a_{j'})$.

Next, endow each a_i with the boundary orientation of $S - \text{int}(N(\partial S))$. For each a_i , define a parametrization $p_i : [-\delta, \delta] \rightarrow a_i$ so that $p_i(-\delta)$ is the initial point of a_i , $p_i(\delta)$ is the terminal point, and the μ -measure from $p_i(-\delta)$ to $p_i(s)$ is $s + \delta$. Let $\varepsilon > 0$ be a sufficiently small constant so that all the leaves of $\mathcal{F}|_{S - \text{int}(N(\partial S))}$ which start from $p_i([-\delta, -\delta + \varepsilon])$ exit $S - \text{int}(N(\partial S))$ together along some $a_{i'}$ and also are disjoint from $P'_{i'}$. Also, for each i , define the map $q_i : [-\delta, \delta] \rightarrow \partial S$ so that $q_i(s)$ is the point on ∂S which is closest to $p_i(s)$ with respect to the fixed hyperbolic metric. In particular, $q_i(0) = x_i$ and the geodesic through $p_i(0)$ and $q_i(0)$ agrees with the prong P_i , since the prong P_i is perpendicular to ∂S . Also let $p_i(s)q_i(s)$ be the shortest geodesic between $p_i(s)$ and $q_i(s)$.

6.3.2. *Walls*. Let W be a properly embedded, oriented arc of S so that $W \cap N(\partial S)$ consists of exactly two components. The component containing the initial point is the *initial arc* of W , the component containing the terminal point is the *terminal arc* of W , and $W \cap (S - \text{int}(N(\partial S)))$ is the *middle arc* of W . We now define the *walls* $W_{i,L}, W_{i,R}$ for $i = 1, \dots, n$. The wall $W_{i,L}$ (resp. $W_{i,R}$) is a properly embedded, oriented arc of S which intersects $N(\partial S)$ in two components. The initial arc of $W_{i,L}$ (resp. $W_{i,R}$) is the geodesic arc $q_i(-\delta + \varepsilon)p_i(-\delta + \varepsilon)$ (resp. $q_i(\delta - \varepsilon)p_i(\delta - \varepsilon)$), the terminal arc of $W_{i,L}$ is $p_{i'}(s_{i,L})q_{i'}(s_{i,L})$ (resp. $p_{i'}(s_{i,R})q_{i'}(s_{i,R})$), and the middle arc is a leaf of $\mathcal{F}|_{S - \text{int}(N(\partial S))}$. (We may need to take a C^0 -small modification of the geodesic arcs, so that the walls become smooth. From now on, we assume that such smoothings have taken place, with the tacit understanding that the arcs $p_i(s)q_i(s)$ are only “almost geodesic”.) It is conceivable that, a priori, $W_{i,L} = W_{i',R}$ for $i \neq i'$, with opposite orientations. In that case, perturb ε so that the walls are pairwise disjoint.

6.3.3. $N(y_j)$. Now, for each j , we define $N(y_j)$ to be a sufficiently small neighborhood of $Q'_{j1} \cup \dots \cup Q'_{jm_j}$ in $S - \text{int}(N(\partial S))$ so that $\partial N(y_j)$ is a union of smooth arcs which are alternately vertical and horizontal, and so that $N(y_j)$ has the following properties:

- (1) $N(y_j)$ is disjoint from $N(y_{j'})$ for $j' \neq j$.
 - (2) $N(y_j)$ does not intersect any P'_i .
 - (3) Each vertical arc of $\partial N(y_j)$ is contained in some $\text{int}(a_i)$ and is disjoint from $p_i([-\delta, -\delta + \varepsilon])$ and $p_i([\delta - \varepsilon, \delta])$.
- (3) is possible since the horizontal arc b_i is disjoint from all the prongs.

6.3.4. S' and S'' . We now define the subsets $S'' \subset S' \subset S$:

$$S' = S - \bigcup_{1 \leq j \leq q} \text{int}(N(y_j)) - \text{int}(N(\partial S)),$$

$$S'' = S' - \bigcup_{1 \leq i \leq n} \text{int}(N(P'_i)).$$

Here we take $N(P'_i)$ to be a plaque of \mathcal{F} , each of whose vertical boundary components is sufficiently short to be contained in the interior of some vertical arc in $\partial S'$, and is disjoint from $p_i([-\delta, -\delta + \varepsilon])$, $p_i([\delta - \varepsilon, \delta])$.

6.3.5. *Modified diffeomorphism ψ'* . Finally, we describe the diffeomorphism $\psi' : S \xrightarrow{\sim} S$, which is derived from and freely homotopic to the pseudo-Anosov homeomorphism ψ , and agrees with ψ outside a small neighborhood of $N(\partial S) \cup \psi(N(\partial S))$. First consider the restriction $\psi : S - \text{int}(N(\partial S)) \rightarrow S$ with image $S - \psi(\text{int}(N(\partial S)))$. Let g_1 be a flow on S which is parallel to the stable foliation, pushes $\psi(N(\partial S))$ into $N(\partial S)$, and maps $\psi(a_i)$ inside a_{i+k} . (Recall k/n is the fractional Dehn twist coefficient.) Next, let g_2 be a flow on S which is parallel to the unstable foliation and maps $g_1 \circ \psi(N(\partial S))$ to $N(\partial S)$. Observe that $g_2 \circ g_1$ can be taken to be supported in a neighborhood of $N(\partial S) \cup \psi(N(\partial S))$. Now, $\psi' = g_2 \circ g_1 \circ \psi$ is a diffeomorphism from $S - \text{int}(N(\partial S))$ to itself, and, by choosing g_1 and g_2 judiciously, we can ensure that the restrictions $\psi' : a_i \rightarrow a_{i+k}$ and $\psi' : b_i \rightarrow b_{i+k}$ are transverse measure-preserving diffeomorphisms. Hence we can extend ψ' to $N(\partial S)$ by a rigid rotation about ∂S which takes $p_i(s)q_i(s)$ to $p_{i+k}(s)q_{i+k}(s)$. In the case $k = 0$, $\psi'|_{N(\partial S)}$ is the identity.

6.3.6. *Statement of proposition*. Let γ_1 and γ_2 be two properly embedded oriented arcs of S with the same initial point $x \in \partial S$. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be lifts to the universal cover \tilde{S} , starting at the same point \tilde{x} . We say that γ_1 is *setwise to the left of* γ_2 , and write $\gamma_1 \leq \gamma_2$, if $\tilde{\gamma}_1$ does not intersect the component of $\tilde{S} - \tilde{\gamma}_2$ whose boundary orientation is opposite that of the orientation of $\tilde{\gamma}_2$. We make this definition to distinguish from the notion of $[\gamma_1]$ being to the left of $[\gamma_2]$, where $[\gamma_i]$, $i = 1, 2$, is the isotopy class of γ_i rel endpoints. Having said that, *in the rest of the paper, "to the left" will always mean "setwise to the left"*. (The same definition can be made when only one of the γ_i is a properly embedded arc of S , and the other is an arc which just starts at x .)

We are now ready to state the main proposition of this section.

Proposition 6.3. *There exists a 1-form β on S with $d\beta > 0$ such that the walls $W_{i,L}, W_{i,R}, i = 1, \dots, n$, are integral curves of $\ker \beta$, and satisfy the following properties:*

- (1) *Each wall contains exactly one singularity of β . It is an elliptic singularity, and is in the interior of the initial arc or in the interior of the terminal arc.*
- (2) *$W_{i,L}$ (resp. $W_{i,R}$) is to the left (resp. to the right) of the prong P_i .*
- (3) *$W_{i,L}$ (resp. $W_{i,R}$) is to the left of $\psi'(W_{i-k,L})$ (resp. to the right of $\psi'(W_{i-k,R})$).*
- (4) *β exits ∂S uniformly with respect to ψ' .*
- (5) *The initial and terminal arcs of $(\psi')^{-1}(W_{i,L})$ and $(\psi')^{-1}(W_{i,R})$ are integral arcs of $\ker \beta$.*
- (6) *Suppose ψ' maps an initial (resp. terminal) arc of $(\psi')^{-1}(W_{i,L})$ to an initial (resp. terminal) arc of $W_{i,L}$. If both arcs contain elliptic singularities, then ψ' matches the germs of these elliptic singularities. The same holds for $(\psi')^{-1}(W_{i,R})$ and $W_{i,R}$.*

When comparing $W_{i,L}$ with P_i , we concatenate P_i with a small arc from $q_i(-\delta + \varepsilon)$ to x_i , and assume $W_{i,L}$ and P_i have the same initial points (and similarly for $W_{i,R}$ and P_i).

The proof of Proposition 6.3 occupies the next two subsections. In Subsection 6.4 we construct the 1-form β , and in Subsection 6.5 we verify the properties satisfied by the walls.

6.4. Construction of β

Step 1 (Construction of β on S''). The surface S'' has corners and carries the nonsingular line field $\mathcal{F}|_{S''}$.

Claim 6.4. *The restriction of \mathcal{F} to S'' is orientable.*

Proof. All the corners of S'' are convex, except those which are also corners of $N(\partial S)$. Each concave corner $p_i(\pm\delta)$ is always adjacent to a convex one of type $p_i(s)$, obtained by removing the neighborhood of some P'_j or Q'_{jl} ; moreover, such an assignment defines an injective map from the set of concave corners to the set of convex corners. Let C be a connected component of S'' . Since the foliation \mathcal{F} is either tangent to or transverse to every smooth subarc of $\partial S''$, if we smooth ∂C , then for every boundary component c_i of ∂C , the degree of \mathcal{F} along ∂c_i is

$$\deg(\mathcal{F}, \partial c_i) = \frac{1}{4}(\#\{\text{concave corners}\} - \#\{\text{convex corners}\}) \leq 0.$$

Therefore we see that $\chi(C) \geq 0$, and C is either a disk or an annulus. Moreover, in the case of an annulus, the degree of \mathcal{F} must be zero on the two boundary components. The claim follows. □

From now on, fix an orientation of $\mathcal{F}|_{S''}$. The transverse measure for $\mathcal{F}|_{S''}$ on S'' is now given by a closed 1-form ν which vanishes on $\mathcal{F}|_{S''}$ and satisfies $\psi_*\nu = \lambda\nu$. The measure of an arc transverse to $\mathcal{F}|_{S''}$ is given by the absolute value of the integral of ν along this arc. We stipulate that $\nu(Y) > 0$ if (X, Y) is an oriented basis for TS at a point and X is tangent to and directs $\mathcal{F}|_{S''}$.

The surface S'' can be covered by interiors of finitely many Markov charts of \mathcal{F} in $S - \text{int}(N(\partial S))$. (Here we are using the topology induced from $S - \text{int}(N(\partial S))$.) It suffices to consider charts of type $R = [-1, 1] \times [-\delta_0, \delta_0]$ and $R' = ([-1, 1] \times [-\delta_0, \delta_0]) - ([-1/3, 1/3] \times [-\delta_0, 0])$, where $\delta_0 > 0$ is small and we use coordinates (x, y) for both types of charts. Here we take $\nu = dy$; in particular, this means $\mathcal{F} = \{dy = 0\}$. Moreover, we require $\{\pm 1\} \times [-\delta_0, \delta_0]$ to be subarcs of vertical arcs of $\partial S'$ for both R and R' ; $\{\pm 1/3\} \times [-\delta_0, 0]$ to be subarcs of vertical arcs of $\partial(N(\partial S))$ for R' ; and $[-1/3, 1/3] \times \{0\}$ to be a horizontal arc of $\partial(N(\partial S))$ for R' .

On each chart $U = R$ or R' , let f_U be a function $R \rightarrow \mathbb{R}^{\geq 0}$ (resp. $R' \rightarrow \mathbb{R}^{\geq 0}$) which satisfies:

- (1) $f_U = 0$ on $[-1, 1] \times \{-\delta_0, \delta_0\}$ (resp. $([-1, 1] \times \{-\delta_0, \delta_0\}) \cap R'$).
- (2) $\frac{\partial f_U}{\partial x}(x, y) > 0$ for $y \in (-\delta_0, \delta_0)$.

If we sum the forms $f_U dy$ over all the charts U , we obtain a form ν' on S'' with $d\nu' > 0$. Now consider the form $\beta = \nu + \varepsilon_0 \nu' = F\nu$, where $\varepsilon_0 > 0$. We have $d\beta > 0$, since $d\nu = 0$ and $d\nu' > 0$. Observe that β is defined in a slight enlargement of S'' , and the desired β is $\beta|_{S''}$.

Step 2 (Extension of β to S in the absence of interior singularities). Suppose there are no interior singularities. We first state and prove a useful lemma.

Lemma 6.5. *Consider the rectangle $R = [-1, 1] \times [-\delta_1, \delta_1]$ with coordinates (x, y) . Let $\beta = Fdy$, $F > 0$, be the germ of a 1-form on ∂R which satisfies $\partial F/\partial x > 0$. Then β admits an extension to R with the properties that $d\beta > 0$ and $\beta(\partial/\partial y) > 0$ if and only if $\int_{\partial R} \beta > 0$.*

Proof. The condition $\int_{\partial R} \beta > 0$ is clearly necessary by Stokes' theorem. We check that it is sufficient. Let $\phi_{-1} : [-\delta_1, \delta_1] \rightarrow [-\delta_1, \delta_1]$ be an orientation-preserving diffeomorphism for which $\phi_{-1}^* \beta(-1, y) = c_{-1} dy$, where $c_{-1} > 0$ is a constant. Similarly define ϕ_1 so that $\phi_1^* \beta(1, y) = c_1 dy$ with $c_1 > c_{-1}$. Take an isotopy ϕ_x rel endpoints between ϕ_{-1} and ϕ_1 , and define $\phi(x, y) = (x, \phi_x(y))$. There exists a 1-form Gdy near ∂R which agrees with $\phi^*(\beta)$ and we may extend G to all of R with $\partial G/\partial x > 0$. Now let $\beta = \phi_*(Gdy)$. □

Next consider the walls $W_{i,L}, W_{i,R}, i = 1, \dots, n$. The walls are pairwise disjoint, and are disjoint from P'_i , as well as the portions of $P'_i \cap N(\partial S)$ of type $p'_i(0)q'_i(0)$, for all i' . Moreover, we may assume that ψ' leaves the union of the initial arcs of $W_{i,L}$ (resp. $W_{i,R}$), $i = 1, \dots, n$, invariant.

Step 2A. Consider the region B_{i-1} of $N(\partial S)$ which is bounded by the initial arcs of $W_{i-1,R}$ and $W_{i,L}$, the arc b_{i-1} , portions of a_i, a_{i-1} , and an arc of ∂S . Assume without loss of generality that, with respect to the orientation on $\mathcal{F}|_{S''}$, b_{i-1} is oriented from a_{i-1} to a_i . Our strategy is as follows: Start with $\mathcal{F}'|_{S''} = \mathcal{F}|_{S''}$, extend \mathcal{F}' to a singular Morse–Smale characteristic foliation on B_{i-1} , and construct a 1-form β with $d\beta > 0$ so that $\ker \beta = \mathcal{F}'$.

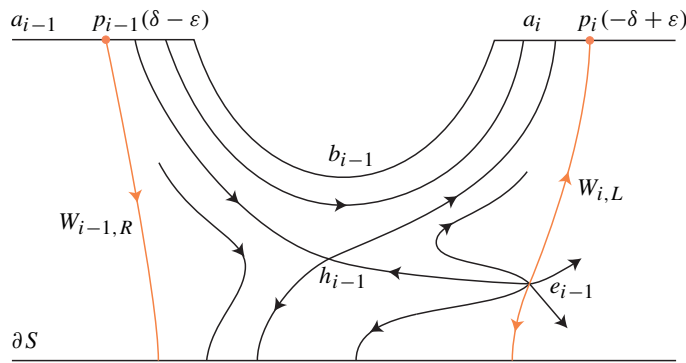


Fig. 3. The kernel of the 1-form β in the region B_{i-1} between the walls $W_{i-1,R}$ and $W_{i,L}$. The walls are the orange arcs (see the pdf file).

Consider a characteristic foliation \mathcal{F}' on a small neighborhood of B_{i-1} with the following properties (see Figure 3):

- The initial arc of $W_{i-1,R}$ is a nonsingular leaf which points out of ∂S .
- The initial arc of $W_{i,L}$ is an integral curve with one positive elliptic singularity e_{i-1} on it. We place the e_{i-1} so that Property (1) of Proposition 6.3 holds.
- $\text{int}(B_{i-1})$ contains a positive hyperbolic singularity h_{i-1} , where the stable separatrices come from e_{i-1} and from $p_{i-1}(\delta - \varepsilon_2)$ on a_{i-1} , and the unstable separatrices go to ∂S and $p_i(-\delta + \varepsilon_3)$ on a_i .
- By making $\varepsilon_2 > 0$ small enough, we have

$$\int_{[\delta - \varepsilon_2, \delta]} p_{i-1}^* \beta < \int_{[-\delta + \varepsilon_3, -\delta]} p_i^* \beta. \tag{6.4.1}$$

- The foliation points out of S along ∂S .

We now explain how to extend β to B_{i-1} so that $\ker \beta$ is the above characteristic foliation and $d\beta > 0$. This procedure follows Giroux’s construction in [Gi2, Proposition 2.6]. The form β can be defined in a neighborhood of h_{i-1} and e_{i-1} , and, provided β is sufficiently large on the boundary of the neighborhood of h_{i-1} , the extension to small neighborhoods of the separatrices and initial arcs of $W_{i-1,R}$ and $W_{i,L}$ is immediate. The complementary regions are all foliated rectangles. All but one have one vertical edge either on ∂S or on $\partial N(e_{i-1})$, and easily satisfy the conditions of Lemma 6.5. The condition that β exit uniformly with respect to ψ' is also easily met, on the portion that is defined. The remaining component is a rectangle whose vertical edges are small retractions of $p_{i-1}([\delta - \varepsilon_2, \delta])$ and $p_i([-\delta + \varepsilon_3, -\delta])$. Observe that the conditions of Lemma 6.5 also hold for the remaining rectangle, thanks to (6.4.1). Now we can apply Lemma 6.5, and extend β to all the rectangles.

Step 2B. Next, for each i , we extend the horizontal arcs of $\partial(N(P'_i))$ inside $N(\partial S)$ by geodesic arcs to ∂S . More precisely, let $p_i(s_1), p_i(s_2), p_{i'}(s_3), p_{i'}(s_4)$ be the four corners of $N(P'_i)$, where $s_1 < s_2$ and $s_3 < s_4$. Extend the horizontal arc $p_i(s_1)p_{i'}(s_4)$ by geodesic

arcs $p_i(s_1)q_i(s_1)$, $p_{i'}(s_4)q_{i'}(s_4)$ to obtain the arc $d_{i,L}$ which is properly embedded in S . Similarly, extend $p_i(s_2)p_{i'}(s_3)$ by geodesic arcs $p_i(s_2)q_i(s_2)$, $p_{i'}(s_3)q_{i'}(s_3)$ to obtain $d_{i,R}$. Let $A_i \subset S$ be the strip which lies between $d_{i,L}$ and $d_{i,R}$. We extend $\mathcal{F}' = \ker \beta$ to A_i . There are two cases:

(1) The orientations of $\mathcal{F}|_{S''}$ agree on $d_{i,L} \cap \partial(N(P'_i))$ and $d_{i,R} \cap \partial(N(P'_i))$. This situation is given in Figure 4. In this case, we extend the characteristic foliation \mathcal{F}' to $N(P'_i)$ so that it coincides with \mathcal{F} on $N(P'_i)$ and the orientation agrees with that of $\mathcal{F}|_{S''}$ along $d_{i,L} \cap \partial(N(P'_i))$ and $d_{i,R} \cap \partial(N(P'_i))$. There are two remaining rectangles R_{to} and R_{from} in A_i to be foliated. The rectangle R_{to} (resp. R_{from}) has a vertical edge in common with $N(P'_i)$, along which \mathcal{F}' exits (resp. enters) $N(P'_i)$. On R_{to} , the foliation \mathcal{F}' consists of geodesic arcs from $p_{i'}(s)$ to $q_{i'}(s)$, for $s \in [s_3, s_4]$, by switching i, i' if necessary. On R_{from} , we place a positive elliptic singularity on each of $p_i(s_1)q_i(s_1)$ and $p_i(s_2)q_i(s_2)$ so that the two geodesics become integral curves. Next we place a positive hyperbolic singularity in the interior of R_{from} , so that both stable separatrices come from the two elliptic points and the unstable separatrices exit through ∂S and $\partial N(P'_i)$. Also, we arrange so that \mathcal{F}' exits from S along $\partial S \cap A_i$. The extension of β to A_i as a 1-form with kernel \mathcal{F}' subject to the condition $d\beta > 0$ follows from Lemma 6.5 and the considerations in Step 2A.

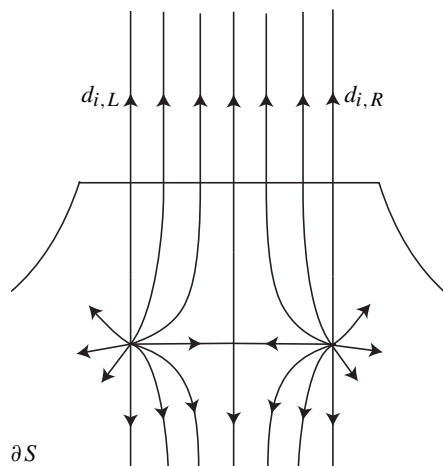


Fig. 4. Construction of \mathcal{F}' near R_{from} .

(2) The orientations of $\mathcal{F}|_{S''}$ on $d_{i,L} \cap \partial(N(P'_i))$ and $d_{i,R} \cap \partial(N(P'_i))$ are opposite. Without loss of generality assume that \mathcal{F}' is oriented from $p_i(s_1)$ to $p_{i'}(s_4)$. Place an elliptic singularity between $p_i(s_1)$ and $q_i(s_1)$, and between $p_{i'}(s_3)$ and $q_{i'}(s_3)$, so that $d_{i,L}$ and $d_{i,R}$ are integral curves of \mathcal{F}' . Next, place a hyperbolic singularity in the interior of one of the components of $A_i - N(P'_i)$. Its stable separatrices come from the elliptic singularities on $d_{i,L}$ and $d_{i,R}$, and its unstable separatrices exit S along the two distinct components of $A_i \cap \partial S$. We can extend the foliation \mathcal{F}' to all of A_i without adding any extra singularities

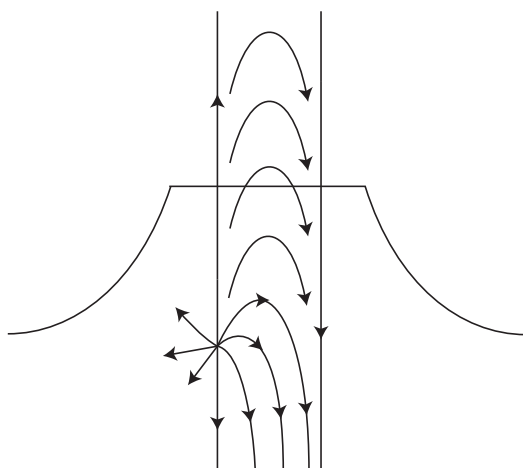


Fig. 5

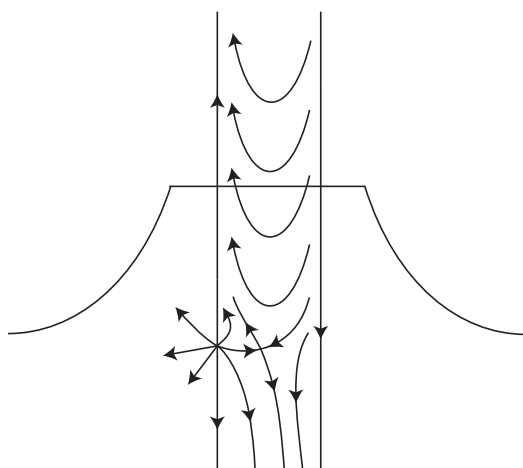


Fig. 6

and so that $\mathcal{F}'|_{N(P_i)}$ is a Reeb component. See Figures 5 and 6 for the ends of A_i . Finally, extend β to A_i as before.

Step 2C. After Steps 2A and 2B, we are left with rectangles \mathcal{R} in $N(\partial S)$ whose vertical edges are on a_i and on ∂S and whose horizontal edges are of type $p(s)q(s)$. We subdivide the rectangles by adding horizontal edges so that the $W_{i,L}$, $W_{i,R}$ and $(\psi')^{-1}(W_{i,L})$, $(\psi')^{-1}(W_{i,R})$ become integral arcs of $\ker \beta$, i.e., Property (5) of Proposition 6.3 is satisfied. Observe that the initial arcs of $W_{i,L}$ and $W_{i,R}$ are already integral arcs by Step 2A, and the initial arcs of $(\psi')^{-1}(W_{i,L})$, $(\psi')^{-1}(W_{i,R})$ are the same as the initial arcs of $W_{i-k,L}$ and $W_{i-k,R}$ by the definition of ψ' . Hence we only consider the terminal arcs.

Let $p_{\phi_L(i)}(s_{i,L})$, $p_{\phi_R(i)}(s_{i,R})$, $(\psi')^{-1}(p_{\phi_L(i)}(s_{i,L}))$, and $(\psi')^{-1}(p_{\phi_R(i)}(s_{i,R}))$ be the initial points of the terminal arcs of $W_{i,L}$, $W_{i,R}$, $(\psi')^{-1}(W_{i,L})$, and $(\psi')^{-1}(W_{i,R})$. Here ϕ_L, ϕ_R are some functions. By construction, $p_{\phi_L(i)}(s_{i,L})$ is on the boundary of some rectangle \mathcal{R} . If the orientation of \mathcal{F}' at $p_{\phi_L(i)}(s_{i,L})$ points into \mathcal{R} , then extend \mathcal{F}' and β so that \mathcal{F}' is tangent to and nonsingular along the horizontal edge $p_{\phi_L(i)}(s_{i,L})q_{\phi_L(i)}(s_{i,L})$. If the orientation points out of \mathcal{R} , then extend \mathcal{F}' and β so that $p_{\phi_L(i)}(s_{i,L})q_{\phi_L(i)}(s_{i,L})$ is an integral curve containing an elliptic singularity. Next, if $p_{\phi_L(i)-k}(s') = (\psi')^{-1}(p_{\phi_L(i)}(s_{i,L}))$ is on the boundary of some rectangle \mathcal{R} , then \mathcal{F}' and β' can be extended similarly. If $p_{\phi_L(i)-k}(s')$ is inside some $A_{i''}$, then let W be a properly embedded arc in S obtained by concatenating $q_{\phi_L(i)-k}(s')p_{\phi_L(i)-k}(s')$, the leaf of $\mathcal{F}|_{S-\text{int}(N(\partial S))}$ through $p_{\phi_L(i)-k}(s')$, and a terminal arc of type $p_{i''}(s'')q_{i''}(s'')$. We then modify \mathcal{F}' by erasing $\mathcal{F}'|_{A_{i''}}$, extending \mathcal{F}' to W so that W is an integral curve which contains an elliptic singularity, splitting $A_{i''}$ into two annuli along W , and applying the procedure in Step 2B to each of the two annuli. The cases of $p_{\phi_R(i)}(s_{i,R})$ and $\psi'(p_{\phi_R(i)}(s_{i,R}))$ are treated similarly.

Finally, the extensions of \mathcal{F}' and β to the interiors of the rectangles are identical to the extensions to R_{to} and R_{from} in Case (1) of Step 2B.

We remark that the extension of β can be chosen so that β exits ∂S uniformly with respect to ψ' .

Step 3 (Extension of β to S in the presence of interior singularities). We now explain how to extend β to $N(y_j)$. Let us denote the vertical boundary components of $N(y_j)$ by c_1, \dots, c_{m_j} and the horizontal components by d_1, \dots, d_{m_j} (both ordered in a counterclockwise manner), where the prong Q_{jl} is between d_l and d_{l+1} and intersects c_l . Here we orient d_l by \mathcal{F}' . For a fixed j , extend d_l by geodesics to ∂S as before, and denote them by d'_l . Now let $C_j \supset N(y_j)$ be the subsurface of S bounded by d'_1, \dots, d'_{m_j} . Refer to Figure 7.

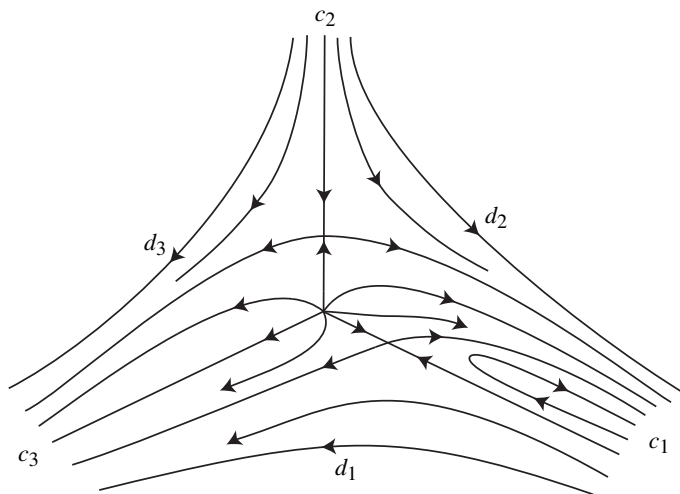


Fig. 7. Description of \mathcal{F}' on $N(y_j)$.

We extend \mathcal{F}' to C_j as follows: Place an elliptic singularity at y_j . Next, place a hyperbolic singularity on the prong Q'_{jl} emanating from y_j , if and only if at least one of d_l, d_{l+1} enters the region $S - \text{int}(N(\partial S))$ along c_l . In this case, the prong Q'_{jl} is contained in the union of the stable separatrices. If both d_l and d_{l+1} enter $S - \text{int}(N(\partial S))$ along c_l , then the unstable separatrices exit $N(y_j)$ along c_{l-1} and c_{l+1} . Otherwise, one unstable separatrix exits along c_l and the other exits along c_{l-1} (resp. c_{l+1}) if d_l (resp. d_{l+1}) enters $S - \text{int}(N(\partial S))$ along c_l . Finally, we complete \mathcal{F}' and β on $N(y_j)$ without adding extra singularities, and then extend to C_j using the models of R_{to} and R_{from} (Figure 4) from Case (1) of Step 2B, and Figure 5 from Case (2) of Step 2B.

As in Step 2C, if there is an initial point of a terminal arc of $(\psi')^{-1}(W_{i,L})$ or $(\psi')^{-1}(W_{i,R})$ which lies in $N(y_j)$, then we may need to insert extra arcs of type W and redo the construction of \mathcal{F}' and β .

This completes the construction of β on S .

6.5. Verification of the properties

In this subsection we prove Properties (1)–(6) of Proposition 6.3. Properties (1) and (4)–(6) are clear from the construction.

(2) We compare $W_{i,L}$ and P_i . The wall $W_{i,L}$ is initially to the left of P_i . (More precisely, $p_i(-\delta + \varepsilon)q_i(-\delta + \varepsilon)$ is to the left of $p_i(0)q_i(0)$.) On $S - N(\partial S)$, $W_{i,L}$ and P'_i are leaves of \mathcal{F} , and they do not cross. If there is some prong P'_j or Q'_{jk} that intersects $p_i([-\delta + \varepsilon, 0])$, then $W_{i,L}$ and P_i bifurcate in the universal cover \tilde{S} and never reintersect. Otherwise, $W_{i,L} \cap (S - N(\partial S))$ and $P_i \cap (S - N(\partial S))$ are parallel paths in \tilde{S} . Let $p_{i'}(s)$ be the “other” endpoint of $W_{i,L} \cap (S - N(\partial S))$, i.e., the one that is not $p_i(-\delta + \varepsilon)$.

If $s > 0$, then we claim that the prong $P'_{i'}$ is between $W_{i,L}$ and P'_i . Indeed, first observe that the transverse distance between $W_{i,L}$ and P'_i is $\delta - \varepsilon$. Now, in Section 6.3.1, $\varepsilon > 0$ was defined so that all the leaves of \mathcal{F} which start from $p_i([-\delta, -\delta + \varepsilon])$ exit together along some $a_{i'}$ and also avoid $P'_{i'}$. Hence $s \leq \delta - \varepsilon$ and $s \neq 0$. This means that P'_i intersects $a_{i'}$ at $p_{i'}(s')$ with $s' < 0$, and the prong $P'_{i'}$ is between $W_{i,L}$ and P'_i .

Therefore $s < 0$, and P_i continues to the right while $W_{i,L}$ enters $N(\partial S)$ and exits along ∂S .

(3) To compare $W_{i,L}$ and $\psi'(W_{i-k,L})$, we compare $W = W_{i,L} \cap (S - N(\partial S))$ and $W' = \psi''(W_{i-k,L} \cap (S - N(\partial S)))$, where $\psi'' = g_1 \circ \psi$ and g_1 is as given in Section 6.3.5. The initial point of W is $p_i(-\delta + \varepsilon)$ and the initial point of W' is $p_i(\frac{1}{\lambda}(-\delta + \varepsilon))$. As in (2), if there is a prong P'_j or Q'_{jk} that intersects $p_i([-\delta + \varepsilon, \frac{1}{\lambda}(-\delta + \varepsilon)])$, then W and W' bifurcate in \tilde{S} , implying (3).

Otherwise W and W' fellow-travel. Let $p_{i'}(s) \in a_{i'}$ be other endpoint of W and let $p_{i'}(s'')$ be the corresponding intersection of W' with $a_{i'}$. (If W' is shorter than W , then (3) is immediate.) We observe that $s = s'' + (1 - \frac{1}{\lambda})(\delta - \varepsilon)$. If s'' is in the interval $[-\frac{1}{\lambda}(\delta - \varepsilon), \frac{1}{\lambda}(\delta - \varepsilon)]$, then $p_{i'}(s'')$ is the other endpoint of W' and (3) follows. If $s'' \geq 0$, then we necessarily have $s'' \in [0, \frac{1}{\lambda}(\delta - \varepsilon)]$, since $s - s'' = (1 - \frac{1}{\lambda})(\delta - \varepsilon)$. On the other hand, (3) is immediate if $s'' < 0$.

This concludes the proofs of Properties (2) and (3), and also the proof of Proposition 6.3.

6.6. Calculation of Φ

First observe the following:

Lemma 6.6.

$$P_{i-1} \leq W_{i-1,R} \leq (\psi')^{-1}(W_{i-1+k,R}) \leq (\psi')^{-2}(W_{i-1+2k,R}) \leq \cdots \\ \cdots \leq (\psi')^{-2}(W_{i+2k,L}) \leq (\psi')^{-1}(W_{i+k,L}) \leq W_{i,L} \leq P_i.$$

Recall that $a \leq b$ means a is to the left of b . Also, $c = k/n$ is the fractional Dehn twist coefficient, where n is the number of prongs.

Proof. By Proposition 6.3, $P_{i-1} \leq W_{i-1,R}$ and $\psi'(W_{i-1,R}) \leq W_{i-1+k,R}$. Hence $W_{i-1,R} \leq (\psi')^{-1}(W_{i-1+k,R})$, and the first row of inequalities holds. Similarly, the second row of inequalities holds. Next, $W_{i-1,R} \leq W_{i,L}$. (Reason: $W_{i-1,R}$ is initially to the left of $W_{i,L}$. In order for them to reintersect, $W_{i-1,R} \cap (S - \text{int}(N(\partial S)))$ and $W_{i,L} \cap (S - \text{int}(N(\partial S)))$ must both have endpoints on the same a_i . This implies the existence of a monogon, which is a contradiction.) Repeated application of $(\psi')^{-1}$ gives $(\psi')^{-j}(W_{i-1+jk,R}) \leq (\psi')^{-j}(W_{i+jk,L})$. \square

Consider a trajectory Q of the type

$$Q = \gamma_1((\psi')^{-1}(\gamma_2)) \cdots ((\psi')^{-m+2}(\gamma_{m-1}))((\psi')^{-m+1}(\gamma_m)). \quad (6.6.1)$$

The trajectory Q is said to be an *ideal trajectory* if, for each i , $\gamma_i : [0, 1] \rightarrow S$ is tangent to \mathcal{F}' and does not pass through a singular point, and $\gamma_i(1) = (\psi')^{-1}(\gamma_{i+1}(0))$.

Proposition 6.7. *If Q is an ideal trajectory, then $\Phi(Q) = 0$.*

Proof. Assume without loss of generality that $\psi(x_i) = x_i$. (This is just for ease of indexing.)

Towards a contradiction, suppose that $\Phi(Q) < 0$. (The case $\Phi(Q) > 0$ is similar.) We lift to the universal cover $\pi : \tilde{S} \rightarrow S$. A tilde placed over a curve will indicate an appropriate lift to \tilde{S} . Then a lift \tilde{Q} of Q in \tilde{S} must intersect two consecutive prongs \tilde{P}_i and \tilde{P}_{i-1} which start from the same component L of $\partial\tilde{S}$, in that order. Also choose a lift $\tilde{\psi}' : \tilde{S} \rightarrow \tilde{S}$ of ψ' which fixes L pointwise.

Assume $\tilde{\gamma}_1(0)$ is strictly to the right of the lift $\tilde{W}_{i,L}$, whose initial point is between the initial points of \tilde{P}_i and \tilde{P}_{i-1} on L . (The modifier ‘‘strictly’’ means $\tilde{\gamma}_1(0)$ is not on $\tilde{W}_{i,L}$.) Then, since the trajectory is ideal, $\tilde{\gamma}_1(1)$ is also strictly to the right of $\tilde{W}_{i,L}$. Next, $(\tilde{\psi}')^{-1}(\tilde{W}_{i,L}) \leq \tilde{W}_{i,L}$, so $\tilde{\gamma}_1(1) = (\tilde{\psi}')^{-1}(\tilde{\gamma}_2(0))$ is strictly to the right of $(\tilde{\psi}')^{-1}(\tilde{W}_{i,L})$. Again, since the trajectory is ideal, $(\tilde{\psi}')^{-1}(\tilde{\gamma}_2(1))$ is strictly to the right of $(\tilde{\psi}')^{-1}(\tilde{W}_{i,L})$. Eventually we prove that $(\tilde{\psi}')^{-m+1}(\tilde{\gamma}_m(1))$ is strictly to the right of $(\tilde{\psi}')^{-m+1}(\tilde{W}_{i,L})$. Since $\tilde{P}_{i-1} \leq (\tilde{\psi}')^{-m+1}(\tilde{W}_{i,L})$ by Lemma 6.6, it follows that \tilde{Q} cannot cross from \tilde{P}_i to \tilde{P}_{i-1} , a contradiction.

An equivalent proof (the one we use in the general case) is to consider the sequence $\tilde{\gamma}_1, \dots, \tilde{\gamma}_m$, where $\tilde{\psi}'(\tilde{\gamma}_i(1)) = \tilde{\gamma}_{i+1}(0)$. First, $\tilde{\gamma}_1$ is strictly to the right of $\tilde{W}_{i,L}$. Hence $\tilde{\psi}'(\tilde{\gamma}_1)$ is strictly to the right of $\tilde{\psi}'(\tilde{W}_{i,L})$ and also strictly to the right of $\tilde{W}_{i,L}$. This implies that $\tilde{\gamma}_2$ is strictly to the right of $\tilde{W}_{i,L}$. Repeating the procedure, $\tilde{\gamma}_m$ is strictly to the right of $\tilde{W}_{i,L}$. Shifting back by $(\tilde{\psi}')^{-m+1}$, $(\tilde{\psi}')^{-m+1}(\tilde{\gamma}_m(1))$ is strictly to the right of $(\tilde{\psi}')^{-m+1}(\tilde{W}_{i,L})$, a contradiction. \square

Let β be the 1-form on S constructed in Section 6.4, and let $\alpha_{\varepsilon, \varepsilon'}$ and $R_{\varepsilon, \varepsilon'}$ be the contact 1-form and Reeb vector field constructed in Section 6.2.2 using β . Define a *genuine trajectory* Q to be a concatenation of the type given by (6.6.1), where each γ_i is the image of a trajectory of $R_{\varepsilon, \varepsilon'}$ from $t = 0$ to $t = 1$, under the projection $\pi : S \times [0, 1] \rightarrow S$ onto the first factor.

Proposition 6.8. *Given $N \gg 0$, for sufficiently small $\varepsilon, \varepsilon' > 0$, any genuine trajectory Q of $R_{\varepsilon, \varepsilon'}$ with $m \leq N$ satisfies $\Phi(Q) = 0$.*

During the proof, a *leaf* of a singular foliation is understood to be a maximal integral submanifold which *does not contain a singular point*.

Proof. Let Q be a genuine trajectory. Suppose each $\gamma_i : [0, 1] \rightarrow S$ is parametrized by t . We prove that a lift \tilde{Q} of Q cannot cross \tilde{P}_i and \tilde{P}_{i-1} , as in Proposition 6.7.

Case 1. Suppose that ψ' matches the germ of an elliptic point on $(\psi')^{-1}(W_{i,L})$ to the germ of an elliptic point on $W_{i,L}$. Recall that, by Property (6) of Proposition 6.3, the germs of the elliptic singularities are matched by ψ' if the initial arcs (or terminal arcs) of $(\psi')^{-1}(W_{i,L})$ and $W_{i,L}$ both contain elliptic singularities.

We first recall Lemma 6.2. Let U be a small neighborhood of the singular set of $\ker \beta$. On $(S - U) \times [0, 1/2]$, given $\delta_0 > 0$ small, there exists $\varepsilon' > 0$, so that, at points where $Y_{\varepsilon'}$ is nonzero, $|Y_{\varepsilon'} / |Y_{\varepsilon'}| + Z| \leq \delta_0$, where Z is a unit vector field which directs $\ker \beta$. On $S \times [1/2, 1]$, $Y_{\varepsilon'}$ directs $\ker \beta$, at points where $Y_{\varepsilon'}$ is nonzero.

Let $U_{i,L}$ be the connected component of U which, after possibly shrinking U , satisfies the following:

- $U_{i,L}$ is a small disk which is centered at the elliptic point of $W_{i,L}$.
- $\ker \beta = \ker (\psi')_* \beta$ on $U_{i,L}$.

The second condition holds since ψ' matches the germs of the elliptic singularities of $(\psi')^{-1}(W_{i,L})$ and $W_{i,L}$. This implies that an arc $\gamma : [0, 1] \rightarrow U_{i,L}$ which is tangent to $Y_{\varepsilon'}$ does not jump from one leaf of $\ker \beta|_{U_{i,L}}$ to another leaf.

Next let $N_{i,L} = ([0, 1] \cup [2, 3]) \times [-\tau, \tau] \subset S - U$ be a foliated neighborhood of $W_{i,L} \cap (S - U)$ with coordinates (x, y) , so that $y = \text{const}$ are leaves of $\ker \beta$ and $y = 0$ is $W_{i,L} \cap (S - U)$. See Figure 8. Since the lengths of leaves of $\ker \beta|_{N_{i,L}}$ are bounded, there exists a constant K (independent of $\varepsilon, \varepsilon'$) so that any arc $\gamma : [0, 1] \rightarrow S - U$ which is tangent to $Y_{\varepsilon'}$ and passes through $\{y = y_0\}$ must be contained in $\{\max(y_0 - K\delta_0, -\tau) \leq y \leq \min(y_0 + K\delta_0, \tau)\} \cup (S - N_{i,L} - U)$. We then take δ_0 sufficiently small so that $NK\delta_0 < \tau$. In other words, $N_{i,L}$ is the protective layer of $W_{i,L}$ which makes it hard to cross $W_{i,L}$ when ε' is small.

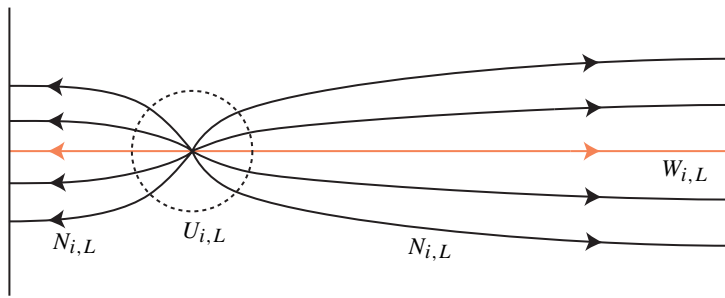


Fig. 8. The neighborhood of the wall $W_{i,L}$.

The initial point $\tilde{\gamma}_1(0)$ must be strictly to the right of \tilde{P}_i and also disjoint from $\tilde{N}_{i,L}$ corresponding to $\tilde{W}_{i,L}$. By the considerations of the previous two paragraphs, $\tilde{\gamma}_1([0, 1]) \cap \tilde{N}_{i,L} \subset \{\tau - K\delta_0 \leq y\}$, where we are taking $y > 0$ to be to the right of $\tilde{W}_{i,L}$. Observe that, since $\tilde{\gamma}_1|_{[1/2, 1]}$ is tangent to $\ker \beta$, it does not jump leaves.

Next, $\tilde{\gamma}_2(0) = \tilde{\psi}'(\tilde{\gamma}_1(1))$. Since $\tilde{\psi}'(\tilde{W}_{i-k,L})$ is to the right of $\tilde{W}_{i,L}$, $\tilde{\gamma}_2(0)$ is further to the right of $\tilde{\gamma}_1(1)$. In particular, if $\tilde{\gamma}_2(0)$ is in $\tilde{N}_{i,L}$, then its y -coordinate is greater than or equal to that of $\tilde{\gamma}_1(1)$. Now apply the same considerations to $\tilde{\gamma}_2$ to deduce that $\tilde{\gamma}_2([0, 1]) \cap \tilde{N}_{i,L} \subset \{\tau - 2K\delta_0 \leq y\}$. Continuing in this manner, we find that $\tilde{\gamma}_m([0, 1]) \cap \tilde{N}_{i,L} \subset \{\tau - mK\delta_0 \leq y\}$. If $m \leq N$, then $\tau - mK\delta_0 > 0$. Hence the entire trajectory \tilde{Q} must be to the right of $(\tilde{\psi}')^{-m+1}(\tilde{W}_{i+(m-1)k,L})$, which, in turn, is to the right of \tilde{P}_{i-1} .

Case 2. Suppose that the initial arc of $W_{i,L}$ contains an elliptic point, whereas the initial arc of $(\psi')^{-1}(W_{i,L})$ does not.

The difference with the previous case is that γ_i can now switch leaves inside $U_{i,L}$. Consider coordinates (x, y) on $U_{i,L}$ so that $W_{i,L} \cap U_{i,L} = \{y = 0\}$, $W_{i,L}$ is directed from $x > 0$ to $x < 0$, $\beta = xdy - ydx$, and $f_{\varepsilon'}\psi'_*\beta = \varepsilon'(u(x)dy)$, with $\partial u/\partial x > 0$ and $u > 0$. We also suppose that being locally to the right of $W_{i,L}$ means $y > 0$. We compute that

$$f_{\varepsilon'}\psi'_*\beta - \beta = (-x + \varepsilon'u(x))dy + ydx,$$

which is directed by $Y' = (-x + \varepsilon'u(x))\partial/\partial x - y\partial/\partial y$. If ε' is sufficiently small, then the elliptic point of $f_{\varepsilon'}\psi'_*\beta - \beta$ is contained inside $U_{i,L}$. Suppose $\gamma_i|_{[0, 1/2]}$ enters $U_{i,L}$ at $x < 0, y > 0$. Comparing with the vector field $x\partial/\partial x + y\partial/\partial y$ which directs $\ker \beta$, we see that $\gamma_i|_{[1/2, 1]}$ exits $U_{i,L}$ along a leaf of $\ker \beta$ which is further to the right of $W_{i,L}$, since $-x + \varepsilon'u(x) > -x$. If $x > 0$ and $y > 0$, then $\gamma_i|_{[1/2, 1]}$ will exit $U_{i,L}$ along a leaf of $\ker \beta$ which is closer to $W_{i,L}$. However, this does not present any problem, since the initial arc of $W_{i,L}$ is an integral arc of both β and $(\psi')_*\beta$, and Lemma 6.1(4) implies that γ_i cannot cross the wall $W_{i,L}$ along the initial arc.

The same proof holds when the terminal arc of $W_{i,L}$ contains an elliptic point and the terminal arc of $(\psi')^{-1}(W_{i,L})$ does not. □

7. Nondegeneracy of Reeb vector fields

In this section we collect some results on perturbing the contact 1-form to make the corresponding Reeb vector field nondegenerate.

We start with the proof of the following well-known fact (see for example [HWZ6, Proposition 6.1]).

Lemma 7.1. *Let α be a contact form on a closed 3-manifold M . The set of smooth functions $g : M \rightarrow (0, +\infty)$ for which the form $g\alpha$ is nondegenerate is a dense subset of $C^\infty(M, (0, +\infty))$ in the C^∞ -topology.*

Proof. Fix $N > 0$. We consider the set G_N of functions $g : M \rightarrow (0, +\infty)$ for which all the orbits of the Reeb vector field $R_{g\alpha}$ of $g\alpha$ of period $\leq N$ are nondegenerate.

We first claim that G_N is open. First observe that the union $\Gamma_{\leq N}$ of all closed orbits of R_α of action $\leq N$ is closed, and hence compact. Next, any closed orbit which comes sufficiently close to an orbit γ , all of whose multiple covers with period $\leq N$ are nondegenerate, is a long orbit, i.e., has action $> N$. (A sequence of closed orbits of period $\leq N$ converging to γ implies that the return map for some multiple cover of γ before time N has 1 as eigenvalue.) It is therefore possible to cover $\Gamma_{\leq N}$ by finitely many sufficiently small disjoint solid tori U_1, \dots, U_k , together with security neighborhoods V_1, \dots, V_k , so that, if $D(r)$ is the disk of radius r centered at the origin and $S^1 = \mathbb{R}/\mathbb{Z}$, then:

- (1) For all $1 \leq i \leq k$, $V_i \simeq S^1 \times D(2)$ and $V_i \supset U_i \simeq S^1 \times \text{int}(D(1))$.
- (2) $S^1 \times \{(0, 0)\}$ is a nondegenerate periodic orbit of R_α of action $\leq N$, and is the only periodic orbit of action $\leq N$ inside V_i .
- (3) R_α is transverse to the foliation by horizontal disks on V_i .
- (4) For any $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$, all the orbits which start from $\{\theta\} \times D(1)$ stay inside V_i at least for an amount of time greater than $N + 1$ and thus at least until the first return to $\{\theta\} \times D(2)$, which occurs before time $N + \varepsilon$.

Any sufficiently small perturbation $R_{g\alpha}$ will still have a single nondegenerate orbit, amongst those that start from $\{\theta\} \times D(1)$. This follows from the transversality of the graphs of the return maps with the diagonal of $(\{\theta\} \times D(1)) \times (\{\theta\} \times D(2))$. (The same considerations also hold for multiple covers of the nondegenerate orbit of period $\leq N$.) If $Z = M - \bigcup_{i=1}^k U_i$, then there is a small constant $\tau > 0$ so that no orbit $\delta : [0, N] \rightarrow M$ which is strictly contained in Z has two points t_1, t_2 sufficiently far apart, so that $\delta(t_1)$ is within τ -distance of $\delta(t_2)$. This implies that a sufficiently small perturbation will not create any new periodic orbits of action $\leq N$. This proves the claim.

Next we prove that G_N is dense in $C^\infty(M, (0, +\infty))$. It suffices to show that there exists a sequence of functions f_n going to zero, so that $1 + f_n \in G_N$ for all $n \in \mathbb{N}$. Let $\{U_i\}_{i=1}^k$ be an open cover of $\Gamma_{\leq N}$ and V_i be the security neighborhood of U_i , satisfying (1), (3), (4) of the previous paragraph. As in the previous paragraph, if we make a sufficiently small perturbation of α , we do not create orbits of action less than N outside $\bigcup_{i=1}^k U_i$.

We first modify α on V_1 . Let $C > 0$ so that the first return occurs at a time strictly greater than C . We take an embedding $[0, C] \times D(3/2) \rightarrow V_1$ so that $\{0\} \times D(3/2)$ is

mapped to the horizontal disk $\{0\} \times D(3/2) \subset S^1 \times D(2)$, and so that, if t is the coordinate of the $[0, C]$ factor, $\partial/\partial t = R$. We may also assume that $\alpha = dt + \beta$, where β is a 1-form on $D(3/2)$ which is independent of t , $\beta(0, 0) = 0$, and β is small on $D(3/2)$ (by taking the V_i to be sufficiently small to start with). If f is a function on M with support on $[0, C] \times D(3/2)$, then

$$d((1 + f)\alpha) = \left(d_2 f - \frac{\partial f}{\partial t} \beta \right) \wedge dt + (1 + f)d\beta,$$

where d_2 means the exterior derivative in the direction of $D(3/2)$. Provided $d_2 f \gg \frac{\partial f}{\partial t} \beta$ on $[0, C] \times D(1)$ (which is the case for our specific choice of f below), $R_{(1+f)\alpha}$ is parallel to $\partial/\partial t + X$, which approximately satisfies

$$i_X d\beta = -\frac{d_2 f}{1 + f}.$$

Now we can look at a family of deformations corresponding to functions which are zero outside of $[0, C] \times D(3/2)$ and given by

$$f_{a,b}(x, y, t) = \chi_1(t)\chi_2\left(\sqrt{x^2 + y^2}\right)(ax + by)$$

inside, where (t, x, y) are coordinates on $[0, C] \times D(3/2)$, χ_1 and χ_2 are positive cut-off functions such that the 2-jet of χ_1 is 0 at $t = 0$ and $t = C$, $\chi_2([0, 1]) = 1$, and $\chi_2 = 0$ near $3/2$. This gives a sequence of families $(F_{a,b}^l)_{1 \leq l \leq p}$, with $1 \leq p \leq N/C + 1$, of l th return maps which are transverse to the diagonal in $(\{0\} \times D(1)) \times (\{0\} \times D(2))$ as families: the transversality is obtained by looking at derivatives of $F_{a,b}^l$ in (a, b) variables at $(a, b) = (0, 0)$.

The transversality of the families $F_{a,b}^l$ implies that the fixed points of $F_{a,b}^l$ are nondegenerate for a generic $(a, b) \in \mathbb{R}^2$. Thus we can find a function $f_{a,b}$ as small as we want so that the periodic orbits of period less than N of $R_{g_1\alpha}$, $g_1 = 1 + f_{a,b}$, which are contained in U_1 , are nondegenerate. Next, we deform $g_1\alpha$ on V_2 , by multiplying by a function g_2 which is very close to 1, so that the orbits in U_1 of period $\leq N$ remain nondegenerate and all the periodic orbits inside U_2 of period $\leq N$ become nondegenerate. The density of G_N follows by induction.

Now the proof of the lemma follows by looking at $\bigcap_{N \in \mathbb{N}} G_N$, which is a dense G_δ -set. □

Lemma 7.2. *Given $N \gg 0$, there is a C^∞ -small perturbation of $\alpha_{\varepsilon, \varepsilon'}$ so that the perturbed Reeb vector field R satisfies the following:*

- (1) *All the closed orbits of R with action $\leq N$ are nondegenerate.*
- (2) *The binding γ_0 and its multiple covers are nondegenerate periodic orbits of R .*
- (3) *All other orbits of R are positively transverse to the pages of the open book.*
- (4) *All the orbits near γ_0 lie on the boundary of a solid torus whose core curve is γ_0 and have irrational slope on the solid torus.*

Proof. Recall the functions $a_\varepsilon(r) = C_{0,\varepsilon} - C_{1,\varepsilon}r^2$ and $b_\varepsilon(r) = r^2$ from Section 6.2.3. We can slightly modify $a_\varepsilon(r)$ near $r = 0$ by replacing $C_{1,\varepsilon} \mapsto C_{1,\varepsilon} + \tau$, where τ is a small number so that the ratio $b'_\varepsilon(r) : \frac{a'_\varepsilon(r)}{2\pi}$ becomes irrational. Hence (2) and (4) are satisfied. Also, (1) and (3) are satisfied for orbits near γ_0 . Finally, the procedure in Lemma 7.1 can be applied to $M - N(\gamma_0)$ to yield (1) and (3). \square

The following lemma is used in the proof of Corollary 2.6.

Lemma 7.3. *Let α be a contact 1-form on a closed 3-manifold M . If α is degenerate and has a finite number of simple orbits, then for any $N \gg 0$ there exists a smooth function g_N which is C^∞ -close to 1 so that all the periodic orbits of $R_{g_N\alpha}$ of action $\leq N$ are nondegenerate and lie in small neighborhoods of the periodic orbits of R_α (and hence are freely homotopic to multiple covers of the periodic orbits of R_α).*

Proof. Let U_1, \dots, U_k be the neighborhoods of the simple orbits $S^1 \times \{(0, 0)\}$ and V_1, \dots, V_k be the security neighborhoods as in Lemma 7.1, taken so they are sufficiently small and mutually disjoint. As before, on $Z = M - \bigcup_{i=1}^k U_i$, any sufficiently small perturbation will not create any new orbits of action $\leq N$. The perturbations inside the V_i will make the Reeb vector field nondegenerate. \square

8. Holomorphic disks

Let h be a diffeomorphism which is freely homotopic to a pseudo-Anosov representative ψ and let S be a page of the open book decomposition. For each boundary component $(\partial S)_i$ of ∂S , there is an associated fractional Dehn twist coefficient $c_i = k_i/n_i$, where n_i is the number of prongs. Let $\alpha_{\varepsilon,\varepsilon'}$ be the contact 1-forms and $R_{\varepsilon,\varepsilon'}$ be the corresponding Reeb vector fields constructed in Section 6. Recall that the direction of $R_{\varepsilon,\varepsilon'}$ does not depend on ε , provided ε is small enough that the contact condition is satisfied. In what follows we assume that $\alpha_{\varepsilon,\varepsilon'}$ is nondegenerate, by applying the C^∞ -small perturbation given in Lemma 7.2. Let $(\mathbb{R} \times M, d(e^t \alpha_{\varepsilon,\varepsilon'}))$ be the symplectization of $(M, \xi_{\varepsilon,\varepsilon'} = \ker \alpha_{\varepsilon,\varepsilon'})$ and $J_{\varepsilon,\varepsilon'}$ be an almost complex structure which is adapted to the symplectization. We have the following theorem:

Theorem 8.1. *Suppose the fractional Dehn twist coefficient $c_i \geq 2/n_i$ for each boundary component $(\partial S)_i$. Given $N \gg 0$, for sufficiently small $\varepsilon, \varepsilon' > 0$, no closed orbit γ of $R_{\varepsilon,\varepsilon'}$ with action $\int_\gamma \alpha_{\varepsilon,\varepsilon'} \leq N$ is the positive asymptotic limit of a finite energy plane \tilde{u} with respect to $J_{\varepsilon,\varepsilon'}$.*

We will usually say that γ which is the positive asymptotic limit of a finite energy plane \tilde{u} bounds \tilde{u} . Theorem 8.1 implies that:

Corollary 8.2. *Suppose $c_i \geq 2/n_i$ for each boundary component $(\partial S)_i$. Given $N \gg 0$, for sufficiently small $\varepsilon, \varepsilon' > 0$, the cylindrical contact homology group $HC_{\leq N}(M, \alpha_{\varepsilon,\varepsilon'})$ is well-defined.*

Outline of proof of Theorem 8.1. Without loss of generality assume that ∂S is connected. Fix $N \gg 0$. By Proposition 6.8, for sufficiently small $\varepsilon, \varepsilon' > 0$, any genuine trajectory Q of $R_{\varepsilon, \varepsilon'}$ which intersects a page at most N times satisfies $\Phi(Q) = 0$. Also, for ε small, the number of intersections of a closed orbit γ with a page of the open book is approximately the same as the action $A_{\alpha_{\varepsilon, \varepsilon'}}(\gamma)$, provided γ lies in $\Sigma(S, \psi')$. Fix sufficiently small constants $\varepsilon, \varepsilon' > 0$ so that the above hold, and write $\alpha = \alpha_{\varepsilon, \varepsilon'}$, $R = R_{\varepsilon, \varepsilon'}$, $\xi = \xi_{\varepsilon, \varepsilon'}$, and $J = J_{\varepsilon, \varepsilon'}$. We will prove that no closed orbit of R in $\Sigma(S, \psi')$ which intersects a page at most N times bounds a finite energy plane, and that no closed orbit in $M - \Sigma(S, \psi')$ bounds a finite energy plane.

Towards a contradiction, suppose there exists a nondegenerate orbit γ of R which bounds a finite energy plane $\tilde{u} = (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times M$. Assume in addition that γ is not a cover of the binding γ_0 , oriented as the boundary of a page. By construction, γ is transverse to the pages of the open book. After perturbing u if necessary, we may take u to be positively transverse to γ_0 . The holomorphicity of \tilde{u} ensures that there are no negative intersections. Now let $N(\gamma_0)$ be a sufficiently small tubular neighborhood of γ_0 , one which depends on γ . By restricting to $M - N(\gamma_0)$ and reparametrizing, we view u as a map \bar{u} from a planar surface P to $M - N(\gamma_0)$. Here P is obtained from a unit disk D by excising small disks which consist of points whose images lie in $N(\gamma_0)$. Next identify $M - N(\gamma_0) \simeq (S'_0 \times [0, 1]) / (x, 1) \sim (h(x), 0)$, where S'_0 is a small retraction of the page S_0 . Cut $M - N(\gamma_0)$ open along $S'_0 \times \{0\}$ and project to S'_0 via the projection $\pi : S'_0 \times [0, 1] \rightarrow S'_0$ onto the first factor. Then we obtain the map $\pi \circ \bar{u} : \mathcal{D} \rightarrow S'_0$, where \mathcal{D} is a disk obtained from P by making cuts along arcs as given in Figure 9. The cutting-up/normalization process will be done in detail in Section 8.1. In Section 8.2 we prove Proposition 8.6, which states that $\pi(\bar{u}(\partial\mathcal{D}))$ cannot be contractible if $c \geq 2/n$. This is proved using the Rademacher function Φ which is adapted to \mathcal{F} , and relies on the fact from Proposition 6.8 that Φ of a genuine trajectory Q is zero. This gives us the desired contradiction. The case when γ is a multiple cover of γ_0 is similar. \square

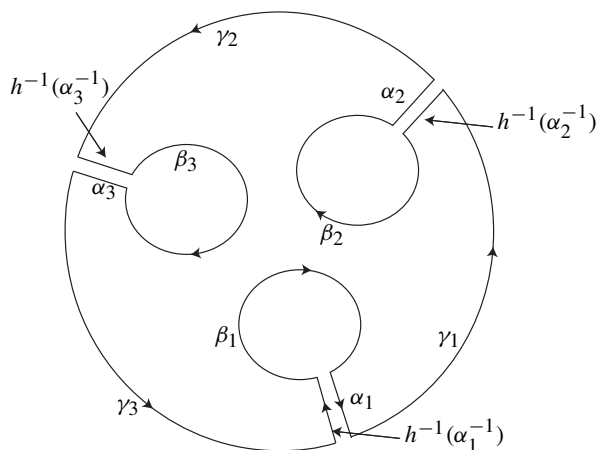


Fig. 9. The disk \mathcal{D} . The labelings indicate the image of the given boundary arc under the map \bar{u} .

8.1. Cutting up the finite energy plane

The type of argument we are using first appeared in [CHI, Annexe]. The discussion in this subsection does not depend on the specific diffeomorphism type of h .

Suppose there exists a nonconstant finite energy plane $\bar{u} = (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times M$ which is bounded by γ . We will slightly modify u to get a map $\bar{u} : D^2 = \{|z| \leq 1\} \rightarrow M$. Let w be the coordinate for \mathbb{C} and z be the coordinate for D^2 . Also let $\Pi : TM = \mathbb{R}R \oplus \xi \rightarrow \xi$ be the projection to ξ .

The following summarizes the results of Hofer–Wysocki–Zehnder [HWZ1], tailored to our needs:

Proposition 8.3. *There exists a smooth map $\bar{u} : D^2 \rightarrow M$ which is bounded by γ and satisfies the following:*

- $\bar{u}|_{\text{int}(D^2)}$ is immersed away from a finite number of points in $\text{int}(D^2)$.
- $\bar{u}(z) = u(Rz)$ for large $R > 0$ and $|z| \leq r_0 < 1$. Hence $(a(Rz), \bar{u}(z))$ is holomorphic on the subdisk $\{|z| \leq r_0\}$. Moreover, $\{|w| \leq Rr_0\}$ contains the set of nonimmersed points of u .
- At points where \bar{u} is immersed, \bar{u} is positively transverse to R .
- $\bar{u}(z) \notin \text{Im}(\gamma)$ for $r_0 < |z| < 1$.

Proof. Let $\mathbb{R}/\mathbb{Z} \times D_\delta^2$ be a small neighborhood of $\text{Im}(\gamma)$, where D_δ^2 is a disk of radius $\delta > 0$ and γ maps to $\mathbb{R}/\mathbb{Z} \times \{0\}$. Let k_0 be the multiplicity of γ , i.e., the number of times γ covers a simple orbit. When restricted to $|w| \gg 0$, $u(w)$ maps to $\mathbb{R}/\mathbb{Z} \times D_\delta^2$ and has components $(u_0(w), u_1(w))$. Then, according to [HWZ1, Theorem 1.4],

- $u_0(re^{2\pi it})$ asymptotically approaches $k_0 t$, with error term $O(e^{-Cr})$. The same holds for all the derivatives of u_0 .
- $u_1(re^{2\pi it}) = e^{\int_{r_1}^r \mu(\tau) d\tau} [E(e^{2\pi it}) + F(re^{2\pi it})]$, where $\mu : [r_1, \infty) \rightarrow \mathbb{R}$ is a smooth function which limits to $\lambda < 0$, $E(e^{2\pi it})$ is a nowhere vanishing function with values in \mathbb{R}^2 , and $F(re^{2\pi it})$ is the remainder term which approaches 0 uniformly in t for all derivatives, as $r \rightarrow \infty$. (The function $E(e^{2\pi it})$ is an eigenfunction of a suitable operator with eigenvalue λ .)

We note that some care is required in choosing the coordinates on $\mathbb{R}/\mathbb{Z} \times D_\delta^2$.

We now reparametrize $u : \mathbb{C} \rightarrow M$. Consider the map $\phi : \text{int}(D^2) \rightarrow \mathbb{C}$, $(r, \theta) \mapsto (f(r), \theta)$, where $f(r) = Rr$ for $r \leq r_0$, $f'(r) > 0$, and $f(r) = 1/(1-r)$ near $r = 1$. Then let $\bar{u} = u \circ \phi$ on $\text{int}(D^2)$ and $\bar{u}(e^{2\pi it}) = (k_0 t, 0)$. The above asymptotics guarantee the smoothness of $\bar{u} : D^2 \rightarrow M$.

The first and last statements follow from [HWZ1, Theorem 1.5], which states that (i) $\Pi \circ u_*$ is nonzero (and hence u is an immersion) away from a finite number of points and that (ii) u intersects γ at finitely many points. □

The map \bar{u} , restricted to $\text{int}(D^2)$, either intersects γ_0 transversely and positively or intersects γ_0 at a point where $\Pi \circ \bar{u}_* = 0$. The following lemma allows us to restrict to the former situation.

Lemma 8.4. *There exists a perturbation \bar{v} of \bar{u} with small support inside $\text{int}(D^2)$ so that \bar{v} is positively transverse to R away from isolated complex branch points and is positively transverse to γ_0 .*

We emphasize that \bar{u} and \bar{v} are no longer holomorphic everywhere.

Proof. Suppose without loss of generality that $\Pi \circ \bar{u}_*(0) = 0$ and $\bar{u}(0) \in \gamma_0$. Let $[-1, 1] \times D^2 \subset M$ be a small neighborhood of $\bar{u}(0) = (0, 0)$, where the Reeb orbits are $[-1, 1] \times \{\text{pt}\}$ and $[-1, 1] \times \{0\}$ is a subarc of γ_0 . Now restrict the domain of \bar{u} to a small neighborhood $D_\varepsilon^2 = \{|z| \leq \varepsilon\}$ of 0 and write $\bar{u} = (\bar{u}_0, \bar{u}_1)$, where \bar{u}_0 is the component that maps to $[-1, 1]$ and \bar{u}_1 is the component that maps to D^2 . Define a smooth function $f : D_\varepsilon^2 \rightarrow \mathbb{R}$ which satisfies the following:

- $f(z) = \delta$ on $|z| \leq \varepsilon''$.
- $f(z) = 0$ on $|z| \geq \varepsilon'$.
- $|f'|$ is small on D_ε^2 . (This means that δ must be a very small positive number.)

Here $0 < \varepsilon'' < \varepsilon' < \varepsilon$. Then define $\bar{v}(z) = (\bar{u}_0(z), \bar{u}_1(z) + f(z))$. On $|z| \leq \varepsilon''$, we are simply translating the holomorphic disk; this does not affect the transversality with R . Now, provided $|f'|$ is sufficiently small, the transversality on $\varepsilon'' \leq |z| \leq \varepsilon'$ is unaffected. The point near $z = 0$ which intersects γ_0 is distinct from the point $z = 0$ at which $\Pi \circ \bar{v}_* = 0$. □

The map \bar{v} from Lemma 8.4 will be renamed as \bar{u} .

Suppose that γ does not cover γ_0 . By this we mean $\gamma \neq \gamma_0$ and γ is not a multiple cover of γ_0 . Since γ_0 intersects \bar{u} transversely, there is a small neighborhood $N(\gamma_0)$ of γ_0 so that $\bar{u}(D^2) \cap \partial(M - N(\gamma_0))$ is a union of circles, each of which intersects $\partial(S'_0 \times \{pt\})$ exactly once. Let P be the planar subsurface of D^2 obtained by excising all $z \in D^2$ such that $\bar{u}(z) \in \text{int}(N(\gamma_0))$. We write $\partial P = \partial_0 P + \partial_1 P$ where $\partial_0 P$ maps to γ and the components of $\partial_1 P$ map to $\partial(M - N(\gamma_0))$.

Now take $S'_0 = S'_0 \times \{0\}$ and consider the intersection of S'_0 and $\bar{u}(P)$. Observe that $\bar{u}|_P$ is already transverse to $S'_0 \times \{t\}$ in a neighborhood of ∂P , for all t . Next, by Sard's theorem, there exists $S'_0 \times \{\varepsilon\}$ which is transverse to $\bar{u}|_P$ with ε arbitrarily small. By renaming the t -variable (i.e., translating $t \mapsto t - \varepsilon$), we may assume that $S'_0 = S'_0 \times \{0\}$ and $\bar{u}|_P$ intersect transversely. We now have the following:

Lemma 8.5. *The intersection $P \cap \bar{u}^{-1}(S'_0)$ is a union of properly embedded arcs and embedded closed curves in P which satisfy the following:*

- (1) *The embedded closed curves bound disks in P .*
- (2) *There is an arc a_i which is the unique arc to connect the i th component of $\partial_1 P$ to $\partial_0 P$.*

Order the a_i so that their endpoints in $\partial_0 P$ are in counterclockwise order, and order the components of $\partial_1 P$ using the a_i . Also, a_i is oriented from $\partial_1 P$ to $\partial_0 P$.

Proof. Let δ be a closed curve of $P \cap \bar{u}^{-1}(S'_0)$. Then δ cuts off a planar subsurface P_0 whose boundary consists of δ , together with components of $\partial_1 P$. Now consider the intersection pairing with S'_0 . Since $\langle \bar{u}(\delta), S'_0 \rangle = 0$ but each component of $\bar{u}(\partial_1 P)$ intersects S'_0 negatively, it follows that $\partial P_0 = \delta$.

Now if $\langle \gamma, S'_0 \rangle = m > 0$, then there must be m endpoints of $P \cap \bar{u}^{-1}(S'_0)$ on $\partial_0 P$ and one endpoint on each of the m components of $\partial_1 P$. If the arc a_i which begins at the i th component of $\partial_1 P$ ends on another component of $\partial_1 P$, then there must be an arc from $\partial_0 P$ to itself. This would contradict $\langle \gamma, S'_0 \rangle = m$. The lemma follows. \square

Take an embedded closed curve of $P \cap \bar{u}^{-1}(S'_0)$ in P which bounds an innermost disk D_0 . Then $\bar{u}(D_0)$ can be pushed across S'_0 by either flowing forwards or backwards along R (depending on the situation). In this way we can eliminate all the embedded closed curves of $P \cap \bar{u}^{-1}(S'_0)$ in P .

Now cut P along the union of the arcs in $P \cap \bar{u}^{-1}(S'_0)$ to obtain a disk \mathcal{D} . We now have a map $\bar{u} : \mathcal{D} \rightarrow S'_0 \times [0, 1]$. After cutting open at one point, $\bar{u}(\partial \mathcal{D})$ is given by

$$\bar{u}(\partial \mathcal{D}) = h^{-1}(\alpha_1^{-1})\beta_1\alpha_1\gamma_1 \dots h^{-1}(\alpha_m^{-1})\beta_m\alpha_m\gamma_m, \tag{8.1.1}$$

where α_i refers to $\alpha_i \times \{0\}$, α_i^{-1} is α_i with the opposite orientation, $h^{-1}(\alpha_i^{-1})$ refers to $h^{-1}(\alpha_i^{-1}) \times \{1\}$, γ_i are components of the Reeb orbit γ cut along $S'_0 \times \{0\}$, and β_i are arcs of the type $\{\text{pt}\} \times [0, 1]$ where $\text{pt} \in \partial S'_0$. See Figure 9.

Next we compose \bar{u} with the projection $\pi : S'_0 \times [0, 1] \rightarrow S'_0$ onto the first factor. Then the curve $\pi(\bar{u}(\partial \mathcal{D})) \subset S'_0$ is decomposed into consecutive arcs:

$$\pi(\bar{u}(\partial \mathcal{D})) = h^{-1}(\alpha_1^{-1})\alpha_1\gamma_1 \dots h^{-1}(\alpha_m^{-1})\alpha_m\gamma_m, \tag{8.1.2}$$

where the γ_i are actually $\pi(\gamma_i)$. Also note that the β_i project to points.

Rewrite $\pi(\bar{u}(\partial \mathcal{D}))$ as

$$\pi(\bar{u}(\partial \mathcal{D})) = h^{-1}(\xi_1^{-1})\xi_1 h^{-1}(\xi_2^{-1})\xi_2 \dots h^{-1}(\xi_m^{-1})\xi_m Q', \tag{8.1.3}$$

where $Q' = h^{m-1}(\gamma_1)h^{m-2}(\gamma_2) \dots \gamma_m$ is the projection $S'_0 \times [0, m] \rightarrow S'_0$ onto the first factor of a lift of γ to $S'_0 \times [0, m]$ and

$$\begin{aligned} \xi_1 &= \alpha_1, \\ \xi_2 &= \alpha_2 h(\gamma_1^{-1}), \\ \xi_3 &= \alpha_3 h(\gamma_2^{-1})h^2(\gamma_1^{-1}), \\ &\vdots \end{aligned}$$

8.2. Noncontractibility

Let $h : S_0 \xrightarrow{\sim} S_0$ be a diffeomorphism with $h|_{\partial S_0} = \text{id}$, fractional Dehn twist coefficient $c = k/n$, and pseudo-Anosov representative ψ . Writing $S_0 = A_0 \cup S$, we may assume that $h|_S = \psi'$ and $h|_{A_0}$ is a rotation/fractional Dehn twist by c . Also let $h_0 : S_0 \xrightarrow{\sim} S_0$ be a homeomorphism which is isotopic to h relative to ∂S_0 , so that $h_0|_S = \psi$.

In this subsection we prove the following proposition:

Proposition 8.6. *Suppose γ does not cover γ_0 . If $k \geq 2$, then γ does not bound a finite energy plane.*

Suppose $\gamma \subset \Sigma(S, \psi')$. If γ bounds a finite energy plane \tilde{u} , then we can cut up the finite energy plane to obtain \mathcal{D} which satisfies (8.1.3). If we apply h^{-m+1} to (8.1.3), then we obtain

$$\Gamma := h^{-m+1}(\pi(\tilde{u}(\partial\mathcal{D}))) = h^{-1}(\zeta_1^{-1})\zeta_1 h^{-1}(\zeta_2^{-1})\zeta_2 \dots h^{-1}(\zeta_m^{-1})\zeta_m Q. \tag{8.2.1}$$

Here $\zeta_i = h^{-m+1}(\xi_i)$ and

$$Q = \gamma_1 h^{-1}(\gamma_2) \dots h^{-m+1}(\gamma_m) = \gamma_1 (\psi')^{-1}(\gamma_2) \dots (\psi')^{-m+1}(\gamma_m)$$

is a concatenation of the type appearing in (6.6.1). The goal is to prove that Γ is not contractible.

The key ingredient to proving the noncontractibility of Γ is the Rademacher function Φ with respect to the stable foliation \mathcal{F} . Let (θ, y) be coordinates on $A_0 = S^1 \times [-1, 0]$ so that $S^1 \times \{0\}$ is identified with ∂S . Pick a retraction $\rho : S_0 \rightarrow S$ which sends $(\theta, y) \mapsto (\theta, 0)$. If τ is an arc in S_0 , then we define $\Phi(\tau) = \Phi(\rho(\tau))$.

Lemma 8.7. *Let η be an arc on S_0 . Then $\Phi(h_0^{-1}(\eta^{-1})\eta) = k - 1$ or k .*

When we compute Φ values, we often suppress ρ and write τ to mean $\rho(\tau)$.

Proof. First observe that the arc $h_0^{-1}(\eta^{-1})\eta$ is isotopic, relative to its endpoints, to the concatenation $\psi^{-1}(\eta^{-1})\delta\eta$, where δ is a subarc of ∂S .

Next lift $\psi^{-1}(\eta^{-1})\delta\eta$ to the universal cover \tilde{S} . We place a tilde to indicate a lift. Let $\tilde{\psi}$ be an appropriate lift of ψ so that $\tilde{\psi}^{-1}(\tilde{\eta}^{-1})\tilde{\delta}\tilde{\eta}$ is the chosen lift of $\psi^{-1}(\eta^{-1})\delta\eta$. Let d be the component of $\partial\tilde{S}$ which contains $\tilde{\delta}$. Recall that \tilde{S}_d is the union of prongs \tilde{P}_i that emanate from d . We orient each component \tilde{P}_i of \tilde{S}_d so that it points into \tilde{S} .

If necessary, we perturb the initial point of $\tilde{\eta}$ (and hence the terminal point of $\tilde{\psi}^{-1}(\tilde{\eta}^{-1})$) so that the endpoints of $\tilde{\delta}$ do not lie on \tilde{S}_d . In that case, $\Phi(\delta) = k - 1$, since Φ of an oriented arc on d is the oriented intersection number with \tilde{S}_d minus 1 when the contribution is positive, and plus 1 when it is negative. If the terminal point of $\tilde{\eta}$ lies on \tilde{S}_d , then the whole of $\tilde{\psi}^{-1}(\tilde{\eta}^{-1})\tilde{\delta}\tilde{\eta}$ can be isotoped onto d via an isotopy which constrains the endpoints to lie on \tilde{S}_d . In this case, $\Phi(h_0^{-1}(\eta^{-1})\eta) = k$. Assume otherwise. Then we can isotop $\tilde{\eta}$ while fixing one endpoint and constraining the other to lie on d , so that $\tilde{\eta}$ becomes disjoint from \tilde{S}_d . We may also assume that $\tilde{\eta}$ is a quasi-transverse arc. By the $\tilde{\psi}$ -invariance of \tilde{S}_d , it follows that $\tilde{\psi}^{-1}(\tilde{\eta}^{-1})$ is also disjoint from \tilde{S}_d . Hence the contribution of d to $\Phi(\psi^{-1}(\eta^{-1})\delta\eta)$ is $k - 1$. Moreover, there is no concatenation error if we use the $\tilde{\eta}$ as normalized above. Since $\Phi(\psi^{-1}(\eta^{-1})) = -\Phi(\eta)$ by Proposition 5.2, it follows that $\Phi(h_0^{-1}(\eta^{-1})\eta) = k - 1$. \square

Let x be the initial point of Q . Then (8.2.1) can be written as

$$\Gamma = R'h_0^{-1}(\eta_1^{-1})\eta_1 h_0^{-1}(\eta_2^{-1})\eta_2 \dots h_0^{-1}(\eta_m^{-1})\eta_m Q, \tag{8.2.2}$$

where R' is the path $g_1 h_0^{-1}(g_2) \dots h_0^{-m+1}(g_m)$ which joins $h^{-m}(x)$ to $h_0^{-m}(x)$ for some x , and

$$\begin{aligned} \eta_m &= \zeta_m, \\ \eta_{m-1} &= \zeta_{m-1} g_m, \\ \eta_{m-2} &= \zeta_{m-2} g_{m-1} h_0^{-1}(g_m), \\ &\vdots \end{aligned}$$

Here g_i is a path such that $h^{-1}(\zeta_i^{-1}) = g_i h_0^{-1}(\zeta_i^{-1})$.

In what follows, we pass to the universal cover \tilde{S}_0 of S_0 . Choose a lift of $\pi(\bar{u}(\partial\mathcal{D}))$. Let $\widetilde{h_0^{-m}}$ be the lift of h_0^{-m} which sends the terminal point of the lift of $h_0^{-1}(\eta_1^{-1})\eta_1 h_0^{-1}(\eta_2^{-1})\eta_2 \dots h_0^{-1}(\eta_m^{-1})\eta_m$ to its initial point. We may decompose it as

$$\widetilde{h_0^{-m}} = \widetilde{h_{0,m}^{-1}} \circ \widetilde{h_{0,m-1}^{-1}} \circ \dots \circ \widetilde{h_{0,1}^{-1}},$$

where the $\widetilde{h_{0,i}^{-1}}$ is the lift of h_0^{-1} which sends the terminal point of $\tilde{\eta}_i$ to the terminal point of $\tilde{\eta}_{i-1}$. Also let $\widetilde{h^{-m}} = \widetilde{h_m^{-1}} \circ \widetilde{h_{m-1}^{-1}} \circ \dots \circ \widetilde{h_1^{-1}}$ be the lift of h^{-m} which coincides with $\widetilde{h_0^{-m}}$ near $\partial\tilde{S}_0$, and let $(\widetilde{\psi'}^{-m}) = (\widetilde{\psi'_m})^{-1} \circ \dots \circ (\widetilde{\psi'_1})^{-1}$ and $\widetilde{\psi^{-m}} = \widetilde{\psi_m^{-1}} \circ \dots \circ \widetilde{\psi_1^{-1}}$ be the restrictions of $\widetilde{h^{-m}}$ and $\widetilde{h_0^{-m}}$ to \tilde{S} . Now the arc \tilde{R}' is an arc which joins $\widetilde{h^{-m}}(\tilde{\eta}_m(1))$ to $\widetilde{h_0^{-m}}(\tilde{\eta}_m(1))$ in \tilde{S}_0 .

Assume that $\psi(x_i) = x_i$, again for ease of indexing.

Lemma 8.8. *Suppose $\widetilde{\psi^{-m}}$ and $(\widetilde{\psi'}^{-m})$ are isotopic lifts of ψ^{-m} and $(\psi')^{-m}$, with $m \geq 1$. If $p \in \tilde{S}$ and $\widetilde{\psi^{-m}}(p)$ is to the left of a lift \tilde{P}_i of P_i , then $(\widetilde{\psi'}^{-m})(p)$ is strictly to the left of the lift of $(\psi')^{-m}(W_{i,R})$ which starts near \tilde{P}_i on the same component of $\partial\tilde{S}$.*

The same holds if we replace all occurrences of “left” by “right”.

Proof. If $\widetilde{\psi^{-m}}(p)$ is to the left of a lift \tilde{P}_i of P_i , then p is to the left of $\widetilde{\psi^m}(\tilde{P}_i)$. Since $\widetilde{\psi^m}(\tilde{P}_i)$ is a prong, p is strictly to the left of the lift $\tilde{W}_{i,R}$ of $W_{i,R}$ which starts near it (as $\widetilde{\psi^m}(\tilde{P}_i) \leq \tilde{W}_{i,R}$). Applying $(\widetilde{\psi'}^{-m})$ to p and $\tilde{W}_{i,R}$ gives the lemma. \square

Corollary 8.9. *Suppose $\widetilde{\psi^{-m}}$ and $(\widetilde{\psi'}^{-m})$ are isotopic lifts of ψ^{-m} and $(\psi')^{-m}$, with $m \geq 1$. For any $p \in \tilde{S}$, the path $g_{-m,p}$ from $\widetilde{\psi^{-m}}(p)$ to $(\widetilde{\psi'}^{-m})(p)$ satisfies $\Phi(g_{-m,p}) = 0$.*

Proof. Suppose $\Phi(g_p) \neq 0$. Then $g_{-m,p}$ needs to cross two consecutive prongs \tilde{P}_i and \tilde{P}_{i+1} which emanate from the same component of $\partial\tilde{S}$. Suppose, without loss of generality, that $\widetilde{\psi^{-m}}(p)$ is to the left of \tilde{P}_i and $(\widetilde{\psi'}^{-m})(p)$ is to the right of \tilde{P}_{i+1} . This contradicts Lemma 8.8, since $(\widetilde{\psi'}^{-m})(\tilde{W}_{i,R}) \leq \tilde{P}_{i+1}$. \square

We now prove Proposition 8.6.

Proof of Proposition 8.6. Suppose that $\gamma \subset \Sigma(S, \psi')$. By Lemma 8.7, $\Phi(h_0^{-1}(\eta_j^{-1})\eta_j) \geq k - 1$ for all j . By (3) of Proposition 5.2, we deduce that

$$\Phi(h_0^{-1}(\eta_1^{-1})\eta_1 h_0^{-1}(\eta_2^{-1})\eta_2 \dots h_0^{-1}(\eta_m^{-1})\eta_m) \geq m(k - 1) - (m - 1).$$

Since $k \geq 2$, the right-hand side of the inequality is ≥ 1 . Hence we know that there exist consecutive lifts \tilde{P}_i and \tilde{P}_{i+1} starting on the same component d of ∂S , so that the initial point and the terminal point of a lift of the arc $h_0^{-1}(\eta_1^{-1})\eta_1 \dots h_0^{-1}(\eta_m^{-1})\eta_m$ are respectively strictly to the left of \tilde{P}_i and strictly to the right of \tilde{P}_{i+1} .

As we saw in the proof of Proposition 6.8, the endpoint of \tilde{Q} (= the endpoint of the lift $\tilde{\Gamma}$ of Γ) is then strictly to the right of $(\psi')^{-m+1}(\tilde{W}_{i+1,L})$, which starts on d between \tilde{P}_i and \tilde{P}_{i+1} , provided $0 < m \leq N$. Now, by Lemma 8.8, the initial point of \tilde{R}' (= the initial point of $\tilde{\Gamma}$) is strictly to the left of $(\psi')^{-m}(\tilde{W}_{i,R})$, which starts on d between \tilde{P}_i and \tilde{P}_{i+1} . Since $(\psi')^{-m}(\tilde{W}_{i,R}) \leq (\psi')^{-m+1}(\tilde{W}_{i+1,L})$, it follows that Γ is not contractible, which is a contradiction.

Next suppose that γ lies in $M - \Sigma(S, \psi')$. In this case, we retract Γ using $\rho : S_0 \rightarrow S$; this time the endpoints of η_i are moved to ∂S . The rest of the argument is the same. This concludes the proof of Proposition 8.6. \square

Case when γ covers γ_0 . Finally consider the case when γ covers γ_0 . Let $N(\gamma_0)$ be a small tubular neighborhood of γ_0 so that $(\Pi \circ \bar{u}_*)(q) \neq 0$ for all q with $\bar{u}(q) \in N(\gamma_0)$. Also, by Lemma 7.2, we may take $\partial N(\gamma_0)$ to be foliated by Reeb orbits of irrational slope c , where $\partial N(\gamma_0)$ is identified with $\mathbb{R}^2/\mathbb{Z}^2$ so that the meridian has slope 0 and $\partial S'_0$ has slope ∞ . Consider $M - N(\gamma_0) = S'_0 \times [0, 1]/\sim$ as before. As $\bar{u}|_{\text{int}(D^2)}$ is transverse to the Reeb vector field R away from finitely many branch points, it follows that the component δ of $\bar{u}(D^2) \cap N(\gamma_0)$, parallel to and oriented in the same direction as γ in $\bar{u}(D^2)$, would have slope $1/m_0$ which satisfies $m_0 < 1/c$. On the other hand, all the other components of $\bar{u}(D^2) \cap \partial N(\gamma_0)$ —those that bound meridian disks in $N(\gamma_0)$ and are oriented as $\partial(\bar{u}(D^2) \cap (M - N(\gamma_0)))$ —intersect S'_0 negatively. Since the oriented intersection number of $\partial(\bar{u}(D^2) \cap (M - N(\gamma_0)))$ with S'_0 is zero, it follows that $m_0 \geq 0$. Now, $m_0 = 0$ is impossible, since δ could then be homotoped to ∂S_0 and $\bar{u}(D^2)$ to a disk in S_0 . If $m_0 > 0$, then we can apply the analysis of this section with a slightly smaller disk whose boundary maps to δ . This time, $\Phi(Q)$ will contribute positively, making $\Phi(\pi(\bar{u}(\partial D)))$ more positive. This concludes the proof of Theorem 8.1.

9. Holomorphic cylinders

In this section we give restrictions on holomorphic cylinders between closed Reeb orbits. We say that there is a holomorphic cylinder from γ to γ' if there is a holomorphic cylinder in the symplectization $\mathbb{R} \times M$ whose asymptotic limit at the positive end is γ and whose asymptotic limit at the negative end is γ' .

Let $\mathcal{P}_{\varepsilon, \varepsilon'}$ be the set of good orbits of $R_{\varepsilon, \varepsilon'}$. A periodic orbit γ which is an m_0 -fold cover of the binding γ_0 will be written as $m_0\gamma_0$. Let $\mathcal{P}_{\varepsilon, \varepsilon'}^{>0}$ be the set of good orbits which

are not $m_0\gamma_0$ for any m_0 . In other words, they nontrivially intersect the pages of the open book. We now define the *open book filtration* $F : \mathcal{P}_{\varepsilon, \varepsilon'}^{>0} \rightarrow \mathbb{N}$, which maps γ to the number of intersections with a given page. (This filtration was pointed out to the authors by Denis Auroux.) Denote by γ_b any periodic orbit in $\mathcal{P}_{\varepsilon, \varepsilon'}^{>0}$ such that $F(\gamma_b) = b$. The following lemma shows that the boundary map is filtration nonincreasing.

Lemma 9.1. *There are no holomorphic cylinders from γ_b to $\gamma_{b'}$ if $b < b'$.*

Proof. The holomorphic cylinders intersect the binding positively. (We may need to perturb the holomorphic cylinder to also make it intersect the binding transversely.) If there is a holomorphic cylinder \tilde{u} from γ_b to $\gamma_{b'}$, then there is a map $\bar{u} : P \rightarrow M - N(\gamma_0)$, obtained by a cutting-up process given in Section 9.1. By examining the intersection number of $\partial\bar{u}(P)$ with $S'_0 \times \{0\}$, we see that $b \geq b'$. \square

The main theorem of this section is the following:

Theorem 9.2. *Suppose $c_i \geq 3/n_i$ for each boundary component $(\partial S)_i$. Given $N \gg 0$, for sufficiently small $\varepsilon, \varepsilon' > 0$, there are no holomorphic cylinders from γ to γ' if $\int_\gamma \alpha_{\varepsilon, \varepsilon'}, \int_{\gamma'} \alpha_{\varepsilon, \varepsilon'} \leq N$, and one of the following holds:*

- (1) $\gamma = \gamma_b, \gamma' = \gamma_{b'}$, and $b \neq b'$;
- (2) $\gamma = \gamma_b$ and $\gamma' = m_0\gamma_0$;
- (3) $\gamma = m_0\gamma_0$ and $\gamma' = \gamma_b$;
- (4) $\gamma = m_0\gamma_0, \gamma' = m_1\gamma_0$, and $m_0 \neq m_1$.

The rest of this section is devoted to proving Theorem 9.2. The basic idea is exactly the same as in the proof of Theorem 8.1.

9.1. Cutting up the holomorphic cylinder

Suppose that $\gamma = \gamma_b$ and $\gamma' = \gamma_{b'}$. By Lemma 9.1, $b < b'$ is not possible, so assume that $b > b'$.

Suppose there is a holomorphic cylinder $\tilde{u} = (a, u) : S^1 \times \mathbb{R} \rightarrow \mathbb{R} \times M$ from γ to γ' . Again, with the aid of the asymptotics from [HWZ1], we view u as a smooth map $\bar{u} : S^1 \times [0, 1] \rightarrow M$ where:

- $\bar{u}(S^1 \times \{1\}) = \gamma$ and $\bar{u}(S^1 \times \{0\}) = \gamma'$. Moreover, the orientation on $u(S^1 \times \{1\})$ induced from $S^1 \times [0, 1]$ agrees with that of γ and the orientation on $u(S^1 \times \{0\})$ induced from $S^1 \times [0, 1]$ is opposite to that of γ' .
- $\bar{u}|_{\text{int}(S^1 \times [0, 1])}$ is immersed away from a finite number of points in $\text{int}(S^1 \times [0, 1])$.
- At points where \bar{u} is immersed, \bar{u} is positively transverse to R .
- $\bar{u}(z) \notin \text{Im}(\gamma) \cup \text{Im}(\gamma')$ for $z \in S^1 \times ([0, r_0] \cup [1 - r_0, 1])$, where r_0 is a small positive number.

As before, perturb \bar{u} so that \bar{u} is still positively transverse to R away from $\partial(S^1 \times [0, 1])$ and a finite set F in $\text{int}(S^1 \times [0, 1])$, points in F are complex branch points, and $(\Pi \circ \bar{u}_*)(z) \neq 0$ for all z with $\bar{u}(z)$ in a sufficiently small neighborhood $N(\gamma_0)$ of γ_0 . Let P be the

planar subsurface of $S^1 \times [0, 1]$ obtained by excising $z \in S^1 \times [0, 1]$ such that $\bar{u}(z) \in \text{int}(N(\gamma_0))$. We write $\partial P = \partial_0 P + \partial_1 P + \partial_2 P$, where $\partial_0 P$ maps to γ , $\partial_1 P$ maps to $-\gamma'$, and the components of $\partial_2 P$ map to $\partial N(\gamma_0)$.

We now consider the intersection of $S'_0 = S'_0 \times \{0\}$ and $\bar{u}(P)$, which we may assume to be a transverse intersection. Then the set of points of P which map to S'_0 under \bar{u} are properly embedded arcs and embedded closed curves. By the positivity of intersection of $\partial_0 P$, $-\partial_1 P$, and each component of $-\partial_2 P$ with S'_0 , we find that the embedded closed curves bound disks in P , hence can be isotoped away as before.

Therefore, the holomorphic cylinder \tilde{u} from γ_b to $\gamma_{b'}$ gives rise to a map $\bar{u} : P \rightarrow M - N(\gamma_0)$, where P is a planar surface and $N(\gamma_0)$ a small tubular neighborhood of the binding γ_0 , such that:

- \bar{u} is an immersion away from a finite number of points;
- $\bar{u}(\partial P) = \gamma_b \cup \gamma_{b'}^{-1} \cup \bar{u}(\partial_2 P)$, where $\bar{u}(\partial_2 P) \subset \partial N(\gamma_0)$ and consists of $b - b'$ closed curves which are parallel to and oriented in the opposite direction from the boundary of the meridian disk of $N(\gamma_0)$ which intersects γ_0 positively;
- $\bar{u}(P) \cap (S'_0 \times \{0\})$ consists of b properly embedded arcs. Here b' arcs begin on $\gamma_{b'}$, $b - b'$ begin on $\partial_2 P$, and all end on γ_b .

The arcs of $\bar{u}(P) \cap (S'_0 \times \{0\})$ cut P into b' disks. See Figure 10.

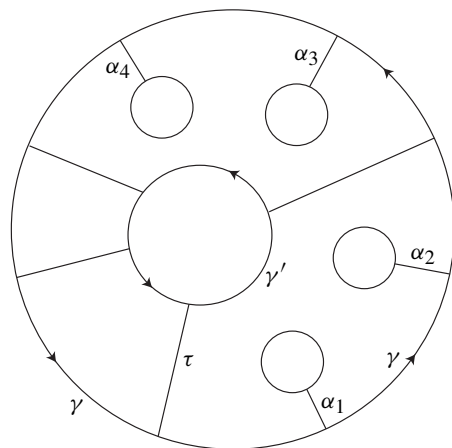


Fig. 10.

We select one arc amongst the b' arcs beginning on $\gamma_{b'}$, and let τ be the image of the arc under \bar{u} . Consider the disk \mathcal{D} obtained by cutting P along (the arc that maps to) τ as well as along the $b - b'$ arcs from $\bar{u}(\partial_1 P)$ to γ_b . Also denote the images of the $b - b'$ arcs from $\partial_2 P$ to γ_b under \bar{u} by $\alpha_1, \dots, \alpha_{b-b'}$, in counterclockwise order along γ_b . Here, α_1 is the first arc reached from τ , traveling counterclockwise along γ_b . The disk $\bar{u}(\mathcal{D})$ can be thought of as living in $S'_0 \times [0, b']$. Consider the projection $\pi : S'_0 \times [0, b'] \rightarrow S'_0$ onto the first factor.

We consider the curve $\pi(\bar{u}(\partial\mathcal{D}))$ and obtain a contradiction by showing that it is not contractible in S'_0 , in a manner analogous to Theorem 8.1. Let Γ (resp. Γ') be the subarc of $\pi(\bar{u}(\partial\mathcal{D}))$ which maps to $\pi(\gamma_b)$ (resp. $\pi(\gamma_{b'})$) in S'_0 . (By $\pi(\gamma_b)$ we mean the projection of an appropriate lift of γ_b to $S'_0 \times [0, b']$.)

9.2. Completion of proof of Theorem 9.2

We now complete the proof of Theorem 9.2. Suppose $k \geq 3$.

(1) Suppose $\gamma = \gamma_b, \gamma' = \gamma_{b'}$, and $\gamma, \gamma' \subset \Sigma(S, \psi')$. The case when at least one of $\gamma, \gamma' \not\subset \Sigma(S, \psi')$ is similar. As in the proof of Theorem 8.1, we can write

$$\pi(\bar{u}(\partial\mathcal{D})) = h^{-1}(\xi_1^{-1})\xi_1 h^{-1}(\xi_2^{-1})\xi_2 \dots h^{-1}(\xi_{b-b'}^{-1})\xi_{b-b'} \Gamma h^{-b'}(\tau^{-1})(\Gamma')^{-1}\tau. \tag{9.2.1}$$

Here we are writing τ for $\pi(\tau), \Gamma = h^{b-1}(\gamma_1) \dots \gamma_b$ and $\Gamma' = h^{b'-1}(\gamma'_1) \dots \gamma'_{b'}$.

Next, we apply h^{-b+1} and rewrite our equation as

$$h^{-b+1}(\pi(\bar{u}(\partial\mathcal{D}))) = R'\Gamma_1(h^{-b+1}(\Gamma))h^{-b'}(\kappa^{-1})(h^{-b+1}(\Gamma')^{-1})\kappa, \tag{9.2.2}$$

where

$$\Gamma_1 = h_0^{-1}(\eta_1^{-1})\eta_1 h_0^{-1}(\eta_2^{-1})\eta_2 \dots h_0^{-1}(\eta_{b-b'}^{-1})\eta_{b-b'},$$

and R' is of the type which appears in (8.2.2).

We will apply the retraction $\rho : S_0 \rightarrow S$ if necessary, without further mention, and work on S . By taking a sufficiently large cover of the holomorphic cylinder from γ to γ' (and replacing γ and γ' by $K\gamma$ and $K\gamma'$), we may assume that $b - b'$ is sufficiently large. Hence,

$$\Phi(\Gamma_1) \geq (b - b')(k - 1) - (b - b') + 1 \gg 0.$$

Next we note that $\Phi(R') = 0$ by Corollary 8.9. Also, by Proposition 6.8, $\Phi((\psi')^{-b+1}(\Gamma)) = 0$, since $(\psi')^{-b+1}(\Gamma) = \gamma_1 \dots h^{-b+1}(\gamma_b)$. Although $(\psi')^{-b+1}(\Gamma') = h^{b'-b}(\gamma'_1) \dots h^{-b+1}(\gamma'_{b'})$ is not quite a concatenation of type Q , the same proof shows that $\Phi((\psi')^{-b+1}(\Gamma')) = 0$. Finally, $\Phi(\kappa) = -\Phi(\psi^{-b'}(\kappa^{-1}))$, and the difference between $\psi^{-b'}(\kappa^{-1})$ and $(\psi')^{-b'}(\kappa^{-1})$ is two arcs of the type R' . Since these two arcs of type R' have $\Phi = 0$, we have

$$\Phi(h^{-b+1}(\pi(\bar{u}(\partial\mathcal{D})))) \approx \Phi(\Gamma_1) \gg 0.$$

This is a contradiction.

(2) Suppose $\gamma = \gamma_b$ and $\gamma' = m_0\gamma_0$. As in Section 8, let $N(\gamma_0)$ be a small tubular neighborhood of γ_0 so that $(\Pi \circ \bar{u}_*)(q) \neq 0$ for all q with $\bar{u}(q) \in N(\gamma_0)$. We retract the cylinder so that γ is fixed but γ' is now on $\partial N(\gamma_0)$. Since $\partial N(\gamma_0)$ is foliated by Reeb orbits of irrational slope c and we require that the cylinder be positively transverse to the Reeb vector field, it follows that γ' has slope m_0/m_1 which satisfies $m_1/m_0 > 1/c$. Therefore, $\Phi(\Gamma')$ contributes negatively, and hence $\Phi((\Gamma')^{-1})$ contributes positively, which is in our favor.

(3) Suppose $\gamma = m_0\gamma_0$ and $\gamma' = \gamma_b$. Consider $N(\gamma_0)$ as in (2). This time, we retract the cylinder so that γ is on $\partial N(\gamma_0)$ and γ' is fixed. Then γ has slope m_0/m_1 which satisfies $m_1/m_0 < 1/c$, and $\Phi(\Gamma)$ contributes more positively, which is in our favor.

(4) This just combines the observations made in (2) and (3).

This completes the proof of Theorem 9.2.

10. Direct limits

10.1. Direct limits in contact homology

Let α and α' be contact 1-forms for the same contact structure ξ , with nondegenerate Reeb vector fields $R_\alpha, R_{\alpha'}$. Denote by $\mathcal{A}_{\leq L}(\alpha)$ the supercommutative \mathbb{Q} -algebra with unit generated by $\mathcal{P}_\alpha^{\leq L}$, the set of good orbits of R_α with action $\int \alpha \leq L$. The boundary map ∂ is the restriction of $\partial : \mathcal{A}(\alpha) \rightarrow \mathcal{A}(\alpha)$ to $\mathcal{A}_{\leq L}(\alpha) \subset \mathcal{A}(\alpha)$.

Write $\alpha' = f_0(x)\alpha$, where $f_0(x)$ is a positive function. If K is a constant satisfying $K > \sup_{x \in M} f_0(x)$, then a sufficient condition for the existence of a chain map

$$\Phi_{\alpha\alpha'} : \mathcal{A}_{\leq L}(\alpha) \rightarrow \mathcal{A}_{\leq L'}(\alpha')$$

is that $L' > KL$, as will be explained in the next paragraph.

Consider $\mathbb{R} \times M$ with coordinates (t, x) . We define a function $f(t, x)$ for which (i) $\partial f/\partial t > 0$, (ii) $f(t, x) \rightarrow K$ as $t \rightarrow +\infty$, (iii) $f(t, x) \rightarrow f_0(x)$ as $t \rightarrow -\infty$, (iv) $f(t, x)$ does not depend on x for t large positive, and (v) $f(t, x) = g(t)f_0(x)$ for t large negative (this means that $g(t)$ is a function which approaches 1 as $t \rightarrow -\infty$ and has small positive derivative dg/dt). Then define the symplectization $d(f(t, x)\alpha)$. Let J be an almost complex structure which is adapted to the symplectization at both ends. Take the collections $\mathcal{P}_\alpha, \mathcal{P}_{\alpha'}$ of the good orbits for α and α' , respectively. Let $\mathcal{M}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)$ be the moduli space of J -holomorphic rational curves with (asymptotically marked) punctures which limit to $\gamma \in \mathcal{P}_\alpha$ at the positive end and to $\gamma'_1, \dots, \gamma'_m \in \mathcal{P}_{\alpha'}$ at the negative end. Then define

$$\Phi_{\alpha\alpha'}(\gamma) = \sum \frac{n_{\gamma, \gamma'_1, \dots, \gamma'_m}}{(i_1)! \dots (i_l)! \kappa(\gamma'_1) \dots \kappa(\gamma'_m)} \gamma'_1 \dots \gamma'_m,$$

where the sum is over all unordered tuples $\overline{\gamma'} = (\gamma'_1, \dots, \gamma'_m)$ and homology classes $[Z] \in H_2(M, \gamma \cup \gamma')$ so that the moduli space $\mathcal{M}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)$ is 0-dimensional. Here $n_{\gamma, \gamma'_1, \dots, \gamma'_m}$ is a signed count of points in $\mathcal{M}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)$, $\kappa(\gamma)$ is the multiplicity of γ , and i_1, \dots, i_l denote the number of occurrences of each orbit γ'_i in the list $\gamma'_1, \dots, \gamma'_m$. By Stokes' theorem applied to $d(f(t, x)\alpha)$, we see that if there is a holomorphic curve from γ to $\gamma'_1, \dots, \gamma'_m$, then $K \int_\gamma \alpha \geq \sum_{i=1}^m \int_{\gamma'_i} f_0 \alpha$. Hence if $L' > KL$, then $\Phi_{\alpha\alpha'}$ is well-defined. Moreover, $\Phi_{\alpha\alpha'}$ is a chain map, as can easily be seen by analyzing the breaking of 1-dimensional moduli spaces $\mathcal{M}_{[Z]}(J, \gamma, \gamma'_1, \dots, \gamma'_m)$. There is an induced map on the full contact homology:

$$\Phi_{\alpha\alpha'} : FHC_{\leq L}(M, \alpha) \rightarrow FHC_{\leq L'}(M, \alpha').$$

In this paper we will use the same notation $\Phi_{\alpha\alpha'}$ for the map on the chain level and on the level of homology; it should be clear from the context which we are referring to.

We now discuss direct limits. Fix a nondegenerate contact 1-form α for (M, ξ) . Let $\{\alpha_i = f_i\alpha\}$, $i = 1, 2, \dots$, be a collection of contact 1-forms and let $M_i = \sup\{f_i(x), 1/f_i(x) \mid x \in M\}$. We say that the sequence (α_i, L_i) is *exhaustive* if there is a sequence $L_i \rightarrow \infty$ such that

$$L_{i+1} > CM_iM_{i+1}L_i,$$

where $C > 1$ is a constant.

Proposition 10.1. *Suppose the sequence (α_i, L_i) is exhaustive. Then the direct limit*

$$\lim_{i \rightarrow \infty} FHC_{\leq L_i}(\alpha_i)$$

exists. Moreover,

$$\Phi : FHC(\alpha) \xrightarrow{\sim} \lim_{i \rightarrow \infty} FHC_{\leq L_i}(\alpha_i).$$

This implies that the direct limit calculates the full contact homology of (M, ξ) , and is independent of the particular choice of nondegenerate contact 1-form.

Proof. Suppose (α_i, L_i) is exhaustive. Then the chain maps $\Phi_{\alpha_i\alpha_{i+1}} : \mathcal{A}(\alpha_i) \rightarrow \mathcal{A}(\alpha_{i+1})$ restrict to

$$\Phi_{\alpha_i\alpha_{i+1}} : \mathcal{A}_{\leq L_i}(\alpha_i) \rightarrow \mathcal{A}_{\leq L_{i+1}}(\alpha_{i+1}),$$

since $L_{i+1} > (\sup_{x \in M} f_{i+1}(x)/f_i(x)) \cdot L_i$ by the exhaustive condition. Hence the direct limit exists.

Next we show that for any $N \gg 0$ there exists a pair (α_i, L_i) so that

$$\Phi_{\alpha\alpha_i} : \mathcal{A}_{\leq N}(\alpha) \rightarrow \mathcal{A}_{\leq L_i}(\alpha_i),$$

obtained by counting rigid marked rational holomorphic curves, is a chain map. Since $L_i/M_i > L_{i-1}$ and $L_{i-1} \rightarrow \infty$ as $i \rightarrow \infty$, there is a symplectization from α to α_i so that γ with $A_\alpha(\gamma) \leq N$ is mapped to $\overline{\gamma'}$ with $A_{\alpha_i}(\overline{\gamma'}) \leq L_i$. In fact, we simply need $L_i/M_i > N$.

Now, $\Phi_{\alpha_i\alpha_{i+1}} \circ \Phi_{\alpha\alpha_i}$ and $\Phi_{\alpha\alpha_{i+1}}$ are chain homotopic by the usual argument, so the collection of maps $\Phi_{\alpha\alpha_i}$ induces the map

$$\Phi_{\leq N} : FHC_{\leq N}(\alpha) \rightarrow \lim_{i \rightarrow \infty} FHC_{\leq L_i}(\alpha_i)$$

on the level of homology. Now, it is easy to see that $FHC(\alpha) = \lim_{i \rightarrow \infty} FHC_{\leq N_i}(\alpha)$, provided $N_i \rightarrow \infty$ (and is increasing). By the usual chain homotopy argument, $\Phi_{\leq N_i}$ is equal to the composition of $FHC_{\leq N_i}(\alpha) \rightarrow FHC_{\leq N_{i+1}}(\alpha)$ followed by $\Phi_{\leq N_{i+1}}$. Hence, by the universal property of direct limits, we obtain the map

$$\Phi : FHC(\alpha) \rightarrow \lim_{i \rightarrow \infty} FHC_{\leq L_i}(\alpha_i).$$

Finally, to prove that Φ is an isomorphism, we use the usual chain homotopy argument. Given (α_i, L_i) , there exist N_i and i' so that $L_i M_i < N_i < L_{i'}/M_{i'}$. Hence we have maps $\mathcal{A}_{\leq L_i}(\alpha_i) \rightarrow \mathcal{A}_{\leq N_i}(\alpha)$ and $\mathcal{A}_{\leq N_i}(\alpha) \rightarrow \mathcal{A}_{\leq L_{i'}}(\alpha_{i'})$, and their composition is chain homotopic to

$$\Phi_{\alpha_i \alpha_{i'}} : \mathcal{A}_{\leq L_i}(\alpha_i) \rightarrow \mathcal{A}_{\leq L_{i'}}(\alpha_{i'}).$$

Therefore, the composition

$$\lim_{i \rightarrow \infty} FHC_{\leq L_i}(\alpha_i) \xrightarrow{\Psi} FHC(\alpha) \xrightarrow{\Phi} \lim_{i \rightarrow \infty} FHC_{\leq L_i}(\alpha_i)$$

is equal to the identity map. This gives a right inverse of Φ ; the left inverse is argued similarly. \square

10.2. Verification of the exhaustive condition

The goal of this subsection is to show the existence of an exhaustive sequence (α_i, L_i) , where the α_i are all adapted to the same open book (S, h) with pseudo-Anosov monodromy and fractional Dehn twist coefficient $c \geq 2/n$, so that the direct limit process can be applied. Let $C_{\leq L_i}(\alpha_i)$ be the \mathbb{Q} -vector space generated by $\mathcal{P}_{\alpha_i}^{\leq L_i}$.

Let $\alpha_{\varepsilon, \varepsilon'}$ be the contact 1-form defined in Section 6. In what follows, we perturb $\alpha_{\varepsilon, \varepsilon'}$ with respect to a suitable large constant $N \gg 0$, as in Lemma 7.2. For simplicity of notation, we will still call the perturbed 1-form $\alpha_{\varepsilon, \varepsilon'}$.

Proposition 10.2. *Given a sequence $L_i, i = 1, 2, \dots$, going to ∞ , there is a sequence of contact 1-forms $\alpha_{\varepsilon_i, \varepsilon'_i}, i \in \mathbb{N}$, with $\varepsilon_i, \varepsilon'_i \rightarrow 0$, so that:*

- (1) *The chain groups $C_{\leq L_i}(\alpha_{\varepsilon_i, \varepsilon'_i})$ are cylindrical.*
- (2) *There exists an isotopy $(\varphi_s^i)_{s \in [0, 1]}$ of M so that $(\varphi_1^i)^* \alpha_{\varepsilon_i, \varepsilon'_i} = G_i \alpha_{\varepsilon_0, \varepsilon'_0}$ and $1/4^i \leq G_i \leq 4^i$.*

We now apply Proposition 10.2 to obtain an exhaustive sequence: In our situation $M_i = 4^i$. Pick L_i so that the exhaustive condition is satisfied. By Proposition 10.2, there exist $\alpha_i = (\varphi_1^i)^* \alpha_{\varepsilon, \varepsilon'} = G_i \alpha_{\varepsilon_0, \varepsilon'_0}$ so that (α_i, L_i) is exhaustive.

We first prove some preparatory lemmas, which are proved for the *unperturbed* $\alpha_{\varepsilon, \varepsilon'}$; however, the same results are also true for the perturbed $\alpha_{\varepsilon, \varepsilon'}$, since the perturbation is a C^∞ -small one.

Lemma 10.3. *On $\Sigma(S, \psi')$, the quantity $|\beta_t(Y_{\varepsilon'})|$ is bounded from above by a constant which is independent of $0 < \varepsilon' < 1$ (and of course independent of ε).*

Proof. For technical reasons, we specialize the function $f_{\varepsilon'} : S \rightarrow \mathbb{R}$, defined in Section 6.2.2, on the region $A = S^1 \times [0, 1]$. In particular, we require that $\partial f_{\varepsilon'} / \partial y \leq -1$ when $y \in [y_0, y_1]$, where $f_{\varepsilon'}(y_0) > 1/2$ and $f_{\varepsilon'}(y_1) = 2\varepsilon'$.

We first restrict to the subset (away from $\partial \Sigma(S, \psi')$) where $f_{\varepsilon'} \leq 2\varepsilon'$. Suppose $t \in [0, 1/2]$. Recall that $i_{Y_{\varepsilon'}} \omega_t = -\dot{\beta}_t$ (Equation (6.2.2)) and $\dot{\beta}_t = \chi'_0(t)(\beta - f_{\varepsilon'}(g_*\beta))$

(Equation (6.2.3)). The quantity $|\dot{\beta}_t|$ is bounded above by a constant independent of ε and ε' , since $|\chi'_0(t)|$, $|\beta|$, $|g_*\beta|$, and $|f_{\varepsilon'}|$ are all bounded above. Next,

$$\omega_t = (1 - \chi_0(t))d(f_{\varepsilon'}(g_*\beta)) + \chi_0(t)d\beta.$$

Clearly, $|d\beta|$ is bounded from below. On the other hand, $d(f_{\varepsilon'}g_*\beta) = \varepsilon'd(g_*\beta)$ on $S - A$ and $(f_{\varepsilon'} - \frac{\partial f_{\varepsilon'}}{\partial y}(C - y))d\theta dy$ on $S^1 \times [y_1, 1]$. Hence $|d(f_{\varepsilon'}g_*\beta)|$ is bounded below by ε' times a positive constant. This means that $|Y_{\varepsilon'}(t)|$ is bounded above by

$$\frac{C_0}{(1 - \chi_0(t))\varepsilon'C_1 + \chi_0(t)C_2},$$

where $C_0, C_1, C_2 > 0$ are constants. Since $\beta_t = (1 - \chi_0(t))f_{\varepsilon'}(g_*\beta) + \chi_0(t)\beta$, we obtain

$$|\beta_t(Y_{\varepsilon'})| \leq \frac{(1 - \chi_0(t))\varepsilon'C_3 + \chi_0(t)C_4}{(1 - \chi_0(t))\varepsilon'C_5 + \chi_0(t)C_6},$$

where $C_3, \dots, C_6 > 0$ are constants. The expression on the right-hand side is bounded above by a constant independent of ε' . The situation $t \in [1/2, 1]$ is treated similarly.

Now, on the subset $f_{\varepsilon'} \geq 2\varepsilon'$, $d(f_{\varepsilon'}(g_*\beta)) = (f_{\varepsilon'} - \frac{\partial f_{\varepsilon'}}{\partial y}(C - y))d\theta dy$, and $\partial f_{\varepsilon'}/\partial y < -1$ or $f_{\varepsilon'} > 1/2$. Hence $|d(f_{\varepsilon'}(g_*\beta))|$ is bounded below by a positive constant which is independent of ε' . Hence $|Y_{\varepsilon'}|$ is bounded above by a constant independent of ε' , and the conclusion follows easily. \square

As a consequence of Lemma 10.3, if ε is sufficiently small, then $\alpha_{\varepsilon, \varepsilon'}(\partial/\partial t + Y_{\varepsilon'}) = 1 + \varepsilon\beta_t(Y_{\varepsilon'})$ is bounded from below by a positive constant.

Now we recall Moser's method: Let $(\alpha_s)_{s \in [0,1]}$ be a path of contact 1-forms. We are looking for an isotopy $(\phi_s)_{s \in [0,1]}$ such that $\phi_s^*\alpha_s = H_s\alpha_0$. If X_s is a time-dependent vector field which generates ϕ_s , then it satisfies the equation

$$\phi_s^*(\dot{\alpha}_s + \mathcal{L}_{X_s}\alpha_s) = \dot{H}_s\alpha_0,$$

where the dot means the derivative in the s -variable (at time s). Using the relation $\phi_s^*\alpha_s = H_s\alpha_0$, this can be rewritten as

$$\dot{\alpha}_s + \mathcal{L}_{X_s}\alpha_s = G_s\alpha_s, \tag{10.2.1}$$

where $G_s = (\frac{d}{ds} \log H_s) \circ \phi_s^{-1}$. It will be convenient to choose X_s to be in $\ker \alpha_s$, in which case $\mathcal{L}_{X_s}\alpha_s = i_{X_s}d\alpha_s$.

Lemma 10.4. *For every $0 < \varepsilon' < 1$, one can find $\delta_1(\varepsilon') > 0$ so that, for every $0 < \varepsilon_1 < \varepsilon_0 < \delta_1(\varepsilon')$, the 1-forms $\alpha_{\varepsilon_0, \varepsilon'}$ and $\alpha_{\varepsilon_1, \varepsilon'}$ are contact forms and there exists an isotopy $(\phi_s)_{s \in [0,1]}$ of M starting from the identity such that $\phi_1^*\alpha_{\varepsilon_1, \varepsilon'} = H\alpha_{\varepsilon_0, \varepsilon'}$ with $1/2 \leq H \leq 2$.*

Proof. We first work on $\Sigma(S, \psi')$. Apply Moser's method to the path of contact 1-forms given by $\alpha_{\varepsilon_s, \varepsilon'}$, where $\varepsilon_s = (1 - s)\varepsilon_0 + s\varepsilon_1$ and $s \in [0, 1]$. The infinitesimal generator $X_s \in \ker \alpha_{\varepsilon_s, \varepsilon'}$ of the isotopy ϕ_s satisfies $\dot{\alpha}_{\varepsilon_s, \varepsilon'} + i_{X_s} d\alpha_{\varepsilon_s, \varepsilon'} = G_s \alpha_{\varepsilon_s, \varepsilon'}$. If we evaluate this equation on $\partial/\partial t + Y_{\varepsilon'}$, we obtain

$$(\varepsilon_1 - \varepsilon_0)\beta_t(Y_{\varepsilon'}) = G_s(1 + \varepsilon_s\beta_t(Y_{\varepsilon'})).$$

By Lemma 10.3, $|\beta_t(Y_{\varepsilon'})|$ is bounded above and $|1 + \varepsilon_s\beta_t(Y_{\varepsilon'})|$ is bounded below by a positive constant, provided we take ε_0 and ε_1 small enough. Hence $|G_s|$ and $|\frac{d}{ds} \log H_s|$ are bounded above by a small constant. This implies that $1/C < H_s < C$ for $C > 1$, say $C = 2$.

In the neighborhood $N(K) = \mathbb{R}/\mathbb{Z} \times D^2$ of the binding, $\alpha_{\varepsilon, \varepsilon'}$ is of the form $a_\varepsilon(r)dz + b_\varepsilon(r)d\theta$, according to Section 6.2.3. For sufficiently small $\varepsilon_0, \varepsilon_1, \alpha_{\varepsilon_0, \varepsilon'}$ and $\alpha_{\varepsilon_1, \varepsilon'}$ are close, and so are $R_{\varepsilon_0, \varepsilon'}$ and $R_{\varepsilon_1, \varepsilon'}$. Hence, the left-hand side of (10.2.1), evaluated on the Reeb vector field R_s , can be made arbitrarily small. Hence we conclude that H is arbitrarily close to 1 near the binding. \square

Lemma 10.5. *For every $0 < \varepsilon'_1 < \varepsilon'_0 < 1$, there exists $\delta_2(\varepsilon'_1) > 0$ so that, for every $0 < \varepsilon_1 < \delta_2(\varepsilon'_1)$, there exists an isotopy $(\phi'_s)_{s \in [0, 1]}$ of M so that $(\phi'_1)^* \alpha_{\varepsilon_1, \varepsilon'_1} = H' \alpha_{\varepsilon_1, \varepsilon'_0}$ with $1/2 \leq H' \leq 2$.*

Proof. We can concentrate our attention on $\Sigma(S, \psi')$, since $\alpha_{\varepsilon, \varepsilon'}$ does not depend on ε' on $N(K)$. Given $0 < \varepsilon'_0, \varepsilon'_1 < 1$ and $s \in [0, 1]$, let $\varepsilon'_s = (1 - s)\varepsilon'_0 + s\varepsilon'_1$. By (10.2.1) applied to the path $\alpha_{\varepsilon_1, \varepsilon'_s}$, we obtain

$$\varepsilon_1 \dot{\beta}_t(Y_{\varepsilon'_s}) = G_s(1 + \varepsilon_1\beta_t(Y_{\varepsilon'_s})),$$

where the dot is the derivative in the s -variable. As before, we see that if ε_1 is small enough, then $|1 + \varepsilon_1\beta_t(Y_{\varepsilon'_s})|$ is bounded below by a positive constant. Since $\dot{\beta}_t$ and $Y_{\varepsilon'_s}$ do not depend on ε_1 , $|G_s|$ is bounded above by a small constant, provided $\delta_2(\varepsilon'_1)$ is sufficiently small. Again, this implies that $1/C < H'_s < C$ for $C > 1$, say $C = 2$. \square

We are now ready to prove Proposition 10.2.

Proof of Proposition 10.2. First use Theorem 8.1 to choose sequences $(\varepsilon_i)_{i \in \mathbb{N}}$ and $(\varepsilon'_i)_{i \in \mathbb{N}}$ so that no closed orbit of the Reeb vector field $R_{\varepsilon_i, \varepsilon'_i}$ which intersects a page $\leq L_i$ times bounds a finite energy plane. (Here we are using the perturbed $\alpha_{\varepsilon_i, \varepsilon'_i}$.) Next, after shrinking ε_i if necessary, suppose $\varepsilon_0 < \delta_1(\varepsilon'_0)$ and $(\varepsilon_i)_{i \in \mathbb{N}}$ is a decreasing sequence which satisfies $\varepsilon_i < \inf\{\delta_1(\varepsilon'_{i-1}), \delta_1(\varepsilon'_i), \delta_2(\varepsilon'_i)\}$. By Lemma 6.2, we may also assume that ε_i is sufficiently small so that $R_{\varepsilon_i, \varepsilon'_i}$ is arbitrarily close to $\partial/\partial t + Y_{\varepsilon'_i}$ on $\Sigma(S, \psi')$ and the action is almost the same as the number of intersections with a page. Hence $C_{\leq L_i}(\alpha_{\varepsilon_i, \varepsilon'_i})$ is cylindrical. Now, if we compose the two isotopies given by Lemmas 10.4 and 10.5, we find an isotopy whose time 1 map pulls $\alpha_{\varepsilon_{i+1}, \varepsilon'_{i+1}}$ back to $H_i \alpha_{\varepsilon_i, \varepsilon'_i}$ with $1/4 \leq H_i \leq 4$. The composition of all these isotopies produces an isotopy whose time 1 map pulls $\alpha_{\varepsilon_i, \varepsilon'_i}$ back to $G_i \alpha_{\varepsilon_0, \varepsilon'_0}$ with $1/4^i \leq G_i \leq 4^i$. \square

10.3. Proof of Theorem 2.3(1)

Suppose ∂S is connected. Let (S, h) be the open book, where h is freely homotopic to the pseudo-Anosov ψ and has fractional Dehn twist coefficient $c = k/n$.

Suppose $k \geq 2$. By Proposition 10.2 there exists an exhaustive sequence $\{(\alpha_i, L_i)\}_{i=1}^\infty$ adapted to (S, h) , so that each \mathbb{Q} -vector space $C_{\leq L_i}(\alpha_i)$ is cylindrical. There are chain maps

$$\Phi_{\alpha_i \alpha_{i+1}}^{\text{cyl}} : C_{\leq L_i}(\alpha_i) \rightarrow C_{\leq L_{i+1}}(\alpha_{i+1}),$$

which count rigid holomorphic cylinders in the symplectization from α_i to α_{i+1} . Let $\lim_{i \rightarrow \infty} HC_{\leq L_i}(\alpha_i)$ be the direct limit.

Next consider the chain maps

$$\Phi_{\alpha_i \alpha_{i+1}} : \mathcal{A}(\alpha_i) \rightarrow \mathcal{A}(\alpha_{i+1})$$

which count rigid punctured rational curves in the symplectization from α_i to α_{i+1} . We claim that no orbit γ of $\mathcal{P}_{\alpha_i}^{\leq L_i}$ bounds a finite energy plane in the symplectization from α_i to α_{i+1} . The argument is identical to that of Theorem 8.1, by observing that the almost complex structure J can be chosen so that \mathbb{R} times the binding γ_0 is J -holomorphic. (This is possible since we can make the binding an orbit of the Reeb vector field for each $f(t_0, x)\alpha$ with t_0 fixed.) Therefore, under the maps $\Phi_{\alpha_i \alpha_{i+1}}$, the trivial augmentation ε_{i+1} on $\mathcal{A}_{\leq L_{i+1}}(\alpha_{i+1})$ pulls back to the trivial augmentation ε_i of $\mathcal{A}_{\leq L_i}(\alpha_i)$.

We now prove that $\mathcal{A}(\alpha)$ admits an augmentation ε . Define $\Phi_{\alpha \alpha_1}$ in the same way as $\Phi_{\alpha_i \alpha_{i+1}}$. Take $\gamma \in \mathcal{A}(\alpha)$. If we let

$$\Phi_{\alpha_i} = \Phi_{\alpha_{i-1} \alpha_i} \circ \dots \circ \Phi_{\alpha_1 \alpha_2} \circ \Phi_{\alpha \alpha_1},$$

then, for sufficiently large i , each term of $\Phi_{\alpha_i}(\gamma)$ has α_i -action $\leq L_i$ by the exhaustive condition. (Here $A_{\alpha_i}(a\gamma_1 \dots \gamma_m) = \sum_j A_{\alpha_i}(\gamma_j)$, where $a \in \mathbb{Q}$.) We then define $\varepsilon(\gamma) = \varepsilon_i \circ \Phi_{\alpha_i}(\gamma)$. The definition of $\varepsilon(\gamma)$ does not depend on the choice of sufficiently large i , due to the fact that ε_{i+1} pulls back to ε_i under $\Phi_{\alpha_i \alpha_{i+1}} : \mathcal{A}_{\leq L_i}(\alpha_i) \rightarrow \mathcal{A}_{\leq L_{i+1}}(\alpha_{i+1})$.

It remains to see that $HC^\varepsilon(\alpha) \simeq \lim_{i \rightarrow \infty} HC_{\leq L_i}(\alpha_i)$. We use the same argument as in Proposition 10.1. Given L_i , there exist N_i and i' so that there are maps $\Psi_i : \mathcal{A}_{\leq L_i}(\alpha_i) \rightarrow \mathcal{A}_{\leq N_i}(\alpha)$ and $\Phi_{i'} : \mathcal{A}_{\leq N_i}(\alpha) \rightarrow \mathcal{A}_{\leq L_{i'}}(\alpha_{i'})$ so that $\Phi_{i'}^* \varepsilon_{i'} = \varepsilon$. Hence we have

$$HC_{\leq L_i}^{\Psi_i^* \varepsilon}(\alpha_i) \xrightarrow{\Psi_i} HC_{\leq N_i}^\varepsilon(\alpha) \xrightarrow{\Phi_{i'}} HC_{\leq L_{i'}}(\alpha_{i'}),$$

whose composition is the map $HC_{\leq L_i}(\alpha_i) \xrightarrow{\Phi_{\alpha_i \alpha_{i'}}} HC_{\leq L_{i'}}(\alpha_{i'})$ by Theorem 3.2, since $\Psi_i^* \varepsilon$ is homotopic to the trivial augmentation and $HC_{\leq L_i}(\alpha_i) \simeq HC_{\leq L_i}^{\Psi_i^* \varepsilon}(\alpha_i)$. The direct limit of the right-hand side yields

$$\Phi : HC^\varepsilon(\alpha) \rightarrow \lim_{i \rightarrow \infty} HC_{\leq L_i}(\alpha_i).$$

As before, we have a right inverse of Φ , and a left inverse exists similarly.

11. Exponential growth of contact homology

11.1. Periodic points of pseudo-Anosov homeomorphisms

We collect some known facts about the dynamics of pseudo-Anosov homeomorphisms. Let Σ be a closed oriented surface and ψ be a pseudo-Anosov homeomorphism on Σ . The homeomorphism ψ is smooth away from the singularities of the stable/unstable foliations.

A pseudo-Anosov homeomorphism ψ admits a Markov partition $\{R_1, \dots, R_l\}$ of Σ , where $R_i = [0, 1] \times [0, 1]$ are “birectangles” with coordinates (x, y) , where $y = \text{const}$ are leaves of the unstable foliation \mathcal{F}^u and $x = \text{const}$ are leaves of the stable foliation \mathcal{F}^s . (See [FLP, Exposé 10] for details, including the definition of a Markov partition.)

The Markov partition gives rise to a graph G as follows: the set of vertices is $\{R_1, \dots, R_l\}$ and there is a directed edge from R_i to R_j if $\text{int}(\psi(R_i)) \cap \text{int}(R_j) \neq \emptyset$. The periodic orbits of ψ of order m are in 1-1 correspondence with cycles of G of length m . (The singular points of \mathcal{F}^s or \mathcal{F}^u are omitted from this consideration.) In particular, the orbits which multiply cover a simple orbit correspond to cycles of G which multiply cover a “simple” cycle of G . As a corollary, we have the following exponential growth property:

Theorem 11.1. *There exist constants $A, B > 0$ so that the number of periodic orbits of ψ of period m is greater than Ae^{Bm} . The same is true for simple periodic orbits or good periodic orbits, i.e., orbits which are not even multiple covers of hyperbolic orbits with negative eigenvalues.*

Next we transfer this property to an arbitrary diffeomorphism h of Σ which is homotopic to ψ , using *Nielsen classes*. Let f, g be homotopic homeomorphisms of Σ . If x is a periodic point of f and y is a periodic point of g , both of order m , then we write $(f, x) \sim (g, y)$ if there exist lifts \tilde{x}, \tilde{y} of x, y and lifts \tilde{f}, \tilde{g} of f, g to the universal cover $\tilde{\Sigma}$ such that $d(\tilde{f}^k(\tilde{x}), \tilde{g}^k(\tilde{y})) \leq K$ for all $k \in \mathbb{Z}$. Here $K > 0$ is a constant and d is some equivariant metric on $\tilde{\Sigma}$. Elements (f, x) and (g, y) which satisfy $(f, x) \sim (g, y)$ are said to be in the same *Nielsen class*. Since the periodic points of ψ belong to different Nielsen classes, we have the following:

Theorem 11.2. *For each periodic (ψ, x) , there exists at least one (h, y) in the same Nielsen class. Hence the number of periodic points h of period m is greater than or equal to the number of periodic points of ψ of period m .*

The above theorem is stated by Thurston in [Th]. A proof can be found in [Hn].

Given a diffeomorphism $f : \Sigma \rightarrow \Sigma$ and a nondegenerate fixed point x of Σ , its ± 1 contribution to the Lefschetz fixed point formula is calculated by the sign of $\det(df(x) - \text{id})$. More precisely, if $df(x)$ is of hyperbolic type with positive eigenvalues, then $\det(df(x) - \text{id}) < 0$ and the contribution is -1 ; if $df(x)$ is of hyperbolic type with negative eigenvalues or of elliptic type, then $\det(df(x) - \text{id}) > 0$ and the contribution is $+1$. They correspond to even and odd parity, respectively, in the contact homology setting. For a pseudo-Anosov ψ , the sum of contributions (in the Lefschetz fixed point theorem) of a periodic orbit x of period m in a particular Nielsen class is ± 1 if the orbit does not pass through a singular point of the stable/unstable foliation, since there is only

one orbit in its Nielsen class. On the other hand, if the orbit passes through a singular point, then the sum of contributions is still nonzero, but the orbit is counted with multiplicity. Now, the same holds for the sum of contributions from all the (h, y) that are in the same Nielsen class as (ψ, x) .

So far the discussion has been for Σ closed. In our case, the surface S has nonempty boundary. Let $f : S \rightarrow S$ be the first return map of the Reeb vector field $R = R_{\varepsilon, \varepsilon'}$ constructed above. By construction, $f|_{\partial S} = \text{id}$ and is homotopic to h and ψ . We cap off S by attaching disks D_i to obtain a closed surface Σ and extend f to Σ by extending by the identity map. Since we want to compare f on Σ to ψ on Σ , we extend ψ to Σ (as well as the stable and unstable foliations). The extension of ψ to Σ is pseudo-Anosov, provided the number of prongs on the boundary is not $n = 1$. (Boundary monogons could exist, although interior monogons do not.) We can avoid monogons by passing to a ramified cover which is ramified at the singular point of the monogon. The nondegeneracy of R implies the nondegeneracy on the cover. Also, there is an at most finite-to-one correspondence between periodic points on the cover of S and periodic points on S . Hence, the number of Nielsen classes of f with period m grows exponentially with respect to m . We will then discard the fixed points of f in the same Nielsen class as (f, x) , where $x \in \partial S$.

11.2. Proof of Theorem 2.3(2)

Suppose $k \geq 3$. We prove that the direct limit $\lim_{i \rightarrow \infty} HC_{\leq L_i}(\alpha_i)$ has exponential growth with respect to the action. Recall we already proved the isomorphism between $HC^\varepsilon(\alpha)$ and $\lim_{i \rightarrow \infty} HC_{\leq L_i}(\alpha_i)$, during the proof of Theorem 2.3(1).

Let $C' = C'_{\leq L_i}(\alpha_i)$ be the subspace of $C = C_{\leq L_i}(\alpha_i)$ generated by the orbits that are not covers of the binding. Also let $C'' = C''_{\leq L_i}(\alpha_i)$ be the subspace of C generated by the orbits which are covers of the binding. Then $C = C' \oplus C''$ and $\partial = \partial' + \partial''$, where $\partial' : C' \rightarrow C'$ and $\partial'' : C'' \rightarrow C''$, in view of Theorem 9.2. Here $\partial, \partial', \partial''$ only count holomorphic cylinders. Also C' is filtered by the open book filtration (i.e., the number of times an orbit intersects a page). Let \mathcal{F}_j be the subspace of C' generated by orbits which intersect a page exactly j times, and let $\mathcal{F}_{j,(\psi,x)}$ be the subspace of \mathcal{F}_j generated by orbits in the same Nielsen class as (ψ, x) . Suppose (ψ, x) is *good*, i.e., it is not an even multiple of an orbit which has negative eigenvalues. The set of such good Nielsen classes grows exponentially with respect to j , provided $j < L_i$. (Recall that we can take the contact form so that the action is arbitrarily close to the number of intersections with the binding.) By Theorem 9.2, the boundary map $\partial : C \rightarrow C$, restricted to $\mathcal{F}_{j,(\psi,x)}$, has image in $\mathcal{F}_{j,(\psi,x)}$. Since $\mathcal{F}_{j,(\psi,x)}$ can be split into even and odd parity subspaces, and they have dimensions that differ by one by Euler characteristic reasons, it follows that the homology of $(\mathcal{F}_{j,(\psi,x)}, \partial)$ has dimension at least one. This proves the exponential growth of $HC_{\leq L_i}(\alpha_i)$ with respect to the action, provided we stay with orbits of action $\leq L_i$. (Alternatively, one can say that the E_1 -term of the spectral sequence given by the open book filtration which converges to $HC_{\leq L_i}(\alpha_i)$ has exponential growth with respect to the action, and, moreover, the higher differentials of the spectral sequence vanish.)

Let $\mathcal{F}_{j,(\psi,x)}(\alpha_i)$ be $\mathcal{F}_{j,(\psi,x)}$ for α_i . Suppose (ψ, x) is good. We claim that the map

$$\Phi_{\alpha_i\alpha_{i+1}} : C_{\leq L_i}(\alpha_i) \rightarrow C_{\leq L_{i+1}}(\alpha_{i+1})$$

sends $\mathcal{F}_{j,(\psi,x)}(\alpha_i)$ to $\mathcal{F}_{j,(\psi,x)}(\alpha_{i+1})$. In other words, no holomorphic cylinder from a generator γ of $\mathcal{F}_{j,(\psi,x)}(\alpha_i)$ to a generator γ' of $\mathcal{F}_{j,(\psi,x)}(\alpha_{i+1})$ intersects $\mathbb{R} \times \gamma_0$. This follows from applying the same argument as in the proof of Theorem 9.2.

Finally we show that, by choosing sufficiently large L_i , there is a sequence $N_i \rightarrow \infty$ so that the map

$$\Phi_{\alpha_i\alpha_{i+1}} : H(\mathcal{F}_{j,(\psi,x)}(\alpha_i)) \rightarrow H(\mathcal{F}_{j,(\psi,x)}(\alpha_{i+1})) \tag{11.2.1}$$

on the level of homology is injective, if $j \leq N_i$. This is sufficient to guarantee the exponential growth for the direct limit. Recall that the orbits of R_{α_i} of action K map to orbits of $R_{\alpha_{i+1}}$ of action $\leq M_i M_{i+1} K$ under $\Phi_{\alpha_i\alpha_{i+1}}$, and the orbits of $R_{\alpha_{i+1}}$ of action K' map to orbits of R_{α_i} of action $\leq M_i M_{i+1} K'$ under $\Phi_{\alpha_{i+1}\alpha_i}$. Hence, in order to compose $\Phi_{\alpha_{i+1}\alpha_i} \circ \Phi_{\alpha_i\alpha_{i+1}}$ in the cylindrical regime, we need $j \leq L_i / (M_i M_{i+1})^2$. Provided this holds, the usual chain homotopy proof shows that $\Phi_{\alpha_i\alpha_{i+1}}$, restricted to $H(\mathcal{F}_{j,(\psi,x)}(\alpha_i))$, has a left inverse and hence is injective. Therefore, we choose L_i so that, in addition to the exhaustive condition, $N_i = L_i / (M_i M_{i+1})^2$ is strictly increasing to ∞ .

This completes the proof of Theorem 2.3(2).

11.3. Proof of Corollary 2.6

Suppose α is nondegenerate. By Theorem 2.3(1), there is a linearized contact homology for any nondegenerate α . Observe that, if R_α only has finitely many simple orbits, then $HC^\varepsilon(M, \alpha)$ will have at most polynomial growth for any augmentation ε . The corollary then follows from Theorem 2.3(2).

Suppose α is degenerate and has a finite number of simple orbits $\gamma_1, \dots, \gamma_l$. Then, according to Lemma 7.3, for any $N \gg 0$ there exists a C^∞ -small perturbation α_N of α so that the only periodic orbits of action $\leq N$ are isotopic to multiple covers of γ_i . This means that the only free homotopy classes which could possibly have generators in the linearized contact homology group $\lim_{i \rightarrow \infty} HC_{\leq L_i}(\alpha_i)$ are multiples of the l simple orbits. This contradicts the fact, sketched in the next two paragraphs, that there are infinitely many simple free homotopy classes in M which have generators in $\lim_{i \rightarrow \infty} HC_{\leq L_i}(\alpha_i)$.

We now sketch the proof, following Gabai–Oertel [GO, Lemma 2.7]. If γ and γ' are closed orbits of the suspension flow of ψ , then γ and γ' are both tangent to the suspension lamination \mathcal{L} , which is an essential lamination if $c > 1/n$. Let $u : \mathbb{R} \times S^1 \rightarrow M$ be an immersed cylinder from γ to γ' . Then the lamination on $\mathbb{R} \times S^1$, induced by pulling back \mathcal{L} via u , cannot have any 0-gons or monogons, after normalizing/simplifying as in [GO, Lemma 2.7]. Since, by Euler characteristic reasons, an m -gon with $m > 2$ implies the existence of a 0-gon or a monogon, it implies that m -gons with $m > 2$ also do not exist. Hence the only complementary regions of $u^{-1}(\mathcal{L})$ are annuli $S^1 \times [0, 1]$ and $\mathbb{R} \times [0, 1]$.

Now, if $c > 2/n$, then it is possible to replace u by u' which does not intersect the binding γ_0 : Let $v : S^1 \times [0, 1] \rightarrow M$ be an immersion whose interior maps to the

connected component V of $M - \mathcal{L}$ that contains γ_0 and such that $S^1 \times \{0, 1\}$ maps to \mathcal{L} . The map v is the restriction of u to the closure of one connected component of $u^{-1}(M - \mathcal{L})$. It is not hard to see that v can be replaced by v' so that they agree on $\partial(S^1 \times [0, 1])$ and v' is disjoint from γ_0 . The same technique works for $v : \mathbb{R} \times [0, 1] \rightarrow M$. Therefore, γ and γ' are freely homotopic in M if and only if they correspond to the same Nielsen class.

Remark 11.3. The above argument gives an easy proof of Theorem 2.3 if ψ is realized as a first return map of a Reeb vector field.

Acknowledgments. First and foremost, we thank Tobias Ekholm for his help throughout this project. We are also extremely grateful to Francis Bonahon, Frédéric Bourgeois, Dragomir Dragnev, Yasha Eliashberg, François Laudenbach, and Bob Penner for invaluable discussions. We thank Paolo Ghiggini and Michael Hutchings for encouraging us to understand the relationship between periodic monodromy and S^1 -invariant contact structures. KH wholeheartedly thanks Will Kazez and Gordana Matić—this work would not exist without their collaboration in [HKM]. KH also thanks l'Université de Nantes for its hospitality during his visit in the summer of 2005.

VC was supported by the Institut Universitaire de France and the ANR Symplexe. KH was supported by an Alfred P. Sloan Fellowship and an NSF CAREER Award (DMS-0237386) and NSF Grant DMS-0805352.

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